UNIVERSAL R-MATRIX FOR QUANTUM SUPERGROUPS

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For quantum deformation of classical finite-dimensional Lie superalgebras we give an explicit formula for the universal R-matrix. This formula generalizes the analogous formulae for classical quantum groups obtained by M. Rosso, A.N. Kirillov and N. Reshetikhin, Ya. Soibelman and S. Levendorskii. Our approach is based on careful analysis of rank two algebras, a combinatorial structure of the root systems and algebraic properties of q-exponential functions. We don't use quantum Weyl group.

1. Introduction

V.G. Drinfeld [1] and M. Jimbo [2] introduced the notion of quantum group that gives a number of examples for solutions of Yang-Baxter (YB) equation. Later, Drinfeld [3,4] defined the quasitriangular Hopf algebras with the universal solution of YB equation. Namely, quasitriangular Hopf algebra is a Hopf algebra A with an additional element $R \in A\otimes A$ such that

$$\Delta'(x) = R\Delta(x)R^{-1}, \qquad x \in A , \tag{1.1}$$

$$(\Delta\otimes id)R = R^{13}R^{23}, \qquad (id\otimes\Delta)R = R^{13}R^{12}. \tag{1.2}$$

This element R satisfies the YB equation and is called "the universal R-matrix". The method of construction the quasitriangular Hopf algebras is based on the quantum double notion [3]. If A is any Hopf algebra then the quantum double $W(A)$ is a quasitriangular Hopf algebra ($\sim A\otimes A'$ as a vector space) with the canonical R-matrix

$$R = \sum e_i \otimes e^i , \tag{1.3}$$

where e_i and e^i are dual bases in A and A'. For any quantum group $U_q(g)$ (the Drinfeld-Jimbo deformation of Kac-Moody algebra g) there exists an epimorphism to $U_q(g)$ from quantum double of the corresponding Borel subalgebra: $W(U_q(b_+)) \to U_q(g)$. Thus any quantum group $U_q(g)$ is a quasitriangular Hopf algebra.

The problem is to obtain an explicit expression for the universal R-matrix directly in terms of $U_q(g)$. General form of such an expression was found by Drinfeld [3,4]. M.Rosso [5] obtained the explicit factorized expression of the universal R-matrix for $U_q(sl(n))$ by examining the identification of $U_q(sl(n))$ with quantum double of $U_q(b_+)$. This formula was generalized in [6,7] to quantum deformation of semisimple Lie algebras using q-Weyl group.

We deduce the analogous formula for quantum supergroups (q-deformation of finite-dimensional simple Lie superalgebras). Our proof is different to that of [5-7]. We don't use quantum Weyl group. Our approach is based on careful analysis of rank two algebras, a combinatorial structure of root systems and algebraic properties of q-exponential functions.

2. The Cartan-Weyl basis for quantum supergroups

Let $g(A,\theta)$ be a contragredient finite-dimensional superalgebra with a symmetrizable Cartan matrix A (i.e., $A = DA^{(s)}$, where $A^{(s)}=(a_{ij}^{(s)})$ is a symmetric Cartan matrix, and $D=Diag(d_1,\dots,d_n)$, $d_i\neq 0)$ and with a parity function θ: {simple roots}$\to\{0,1\}$. We define the quantum supergroup $U_q(g(A,\theta))=U_q(g)$ as the Drinfeld-Jimbo deformation of $U(g)$. The definition differs from that of [3,4] by replacing the Lie brackets [,] with supercommutator $[a,b] = ab - (-1)^{\theta(a)\theta(b)}ba$ and supercommutativity of tensor product [8]. For the comultiplication we use the following formulas:

$$\Delta_q(e_{\alpha_i}) = e_{\alpha_i}\otimes I + q^{-h_i}\otimes I, \quad \Delta_q(e_{-\alpha_i}) = e_{-\alpha_i}\otimes q^{h_i} + I\otimes e_{-\alpha_i}. \quad (2.1)$$

To define the Cartan-Weyl basis in $U_q(g)$ we choose a normal order $\Sigma_+^{\to}$ in the reduced system of positive roots Σ_+ and define root vectors on induction as follows [8]: If $\gamma=\alpha+\beta$, $\alpha<\gamma<\beta$ and there are no other positive roots α', β' between α and β such that $\gamma=\alpha'+\beta'$ then we set

$$e_\gamma = [e_\alpha,e_\beta]_q := e_\alpha e_\beta - (-1)^{\theta(\alpha)\theta(\beta)}q^{(\alpha,\beta)}e_\beta e_\alpha\,, \quad (2.2a)$$

$$e_{-\gamma} = [e_{-\beta},e_{-\alpha}]_{\bar q} := e_{-\beta}e_{-\alpha} - (-1)^{\theta(\alpha)\theta(\beta)}q^{-(\alpha,\beta)}e_{-\alpha}e_{-\beta}\,. \quad (2.2b)$$

We have the following properties of the Cartan-Weyl generators.

Proposition 1. [8] (i) *For any $\gamma\in\Sigma_+$ following relation is valid*

$$[e_\gamma,e_{-\gamma}] = (q^{h_\gamma} - q^{-h_\gamma})/a_\gamma(q), \quad (2.3)$$

where $a_\gamma(q)$ is a function of q.

(ii) *For any $\alpha,\beta\in\Sigma_+$, $\alpha<\beta$, we have*

$$[e_\alpha,e_\beta]_q = \sum_{\alpha<\gamma_1\ldots\gamma_n<\beta} C_{k_i,\gamma_i}\, e_{\gamma_1}^{k_1} e_{\gamma_2}^{k_2} \ldots e_{\gamma_n}^{k_n}\,, \tag{2.4}$$

where $\sum_i k_i\gamma_i = \alpha+\beta$.

3. The universal factorized R-matrix for quantum supergroups

We set

$$exp_q(x) := \sum_{n\geqslant 0} x^n/(n)_q!\,, \tag{3.1a}$$

where

$$(n)_q! \equiv (1)_q\,(2)_q \cdots (n)_q\,, \quad (k)_q \equiv (1-q^n)/(1-q). \tag{3.1b}$$

For any $\gamma\in\Sigma_+$ we set also

$$\tilde{R}_\gamma := exp_{q_\gamma}\left[a_\gamma(q)e^2_{-\gamma}e^1_\gamma\right]\,, \tag{3.2}$$

where $e^1_\gamma = e_\gamma\otimes I$, $e^2_{-\gamma} = I\otimes e_{-\gamma}$, $q_\gamma = (-1)^{\theta(\gamma)}q^{-(\gamma,\gamma)}$.

Theorem. *For any quantum supergroup* $U_q(g)$ *the universal R-matrix can be written in the following form*

$$R = \Big(\prod_{\gamma\in\vec{\Sigma}_+}\tilde{R}_\gamma\Big)\, q^{\Sigma d_{ij}h_i\otimes h_j}\,, \tag{3.3}$$

where the order in the product coincides with the chosen normal order in Σ_+, $d_{ij}=(a^{(s)}_{ij})^{-1}$ *is the inverse to symmetric Cartan matrix.*

Proof. Let $\tilde{R}=\prod_\gamma\tilde{R}_\gamma$ and φ: $U_q(n_+)\to U_q(n_-)$, $(g=n_-\oplus h\oplus n_+)$, be isomorphism defined by $\varphi(e_{\alpha_i})=e_{-\alpha_i}$ for simple root vectors. We can prove the following lemma by direct computations in rank two supergroups.

Lemma. *For any rank two quantum supergroup we have:*

(i) $[\Delta'_{\bar q}(e_\alpha),(\tilde{R}_\alpha)^{-1}\tilde{R}] = [\Delta'_q(e_\beta),\tilde{R}\,(\tilde{R}_\beta)^{-1}] = 0$, $\quad(\bar q\equiv q^{-1})$, (3.4)

if the normal order Σ_+ *is* $(\alpha,\gamma_1,\ldots,\gamma_n,\beta)$,

(ii) *the equation system*

$$\Delta'_q(e_{\alpha_i})\,X = X\,\Delta'_{\bar q}(e_{\alpha_i}),\quad i=1,\ldots,n \tag{3.5}$$

has the unique solution in the space $(I\otimes\varphi)\,U_q(n_+)$.

Here the comultiplication Δ'_q $(\Delta'_{\bar q})$ is opposite to Δ_q $(\Delta_{\bar q})$. As a corollary we prove that the factorized R-matrix satisfies the equation (1.1) $\Delta'=R\Delta R^{-1}$ and does not depend on the normal ordering of the root system. Note that the statements of the lemma are valid for any quantum supergroup and we may consider the reduced R-matrix

$$\tilde{R} = R\,q^{-\Sigma d_{ij}h_i\otimes h_j}. \tag{3.6}$$

as the unique solution of equations (3.5) in the species $(I\otimes\varphi)\ U_q(n_+)$. The remaining properties (1.2) of R-matrix we are able to check directly for quantum supergroups of A-type using the following properties of q-exponential functions.

Proposition 2. *Let x and y are $\bar{q}$-commuting variables, $xy=\bar{q}yx$, $\bar{q}\equiv q^{-1}$, then*

$$exp_q(x+y) = exp_q(x)\ exp_q(y), \qquad (exp_q(x))^{-1} = exp_{\bar{q}}(-x) \qquad (3.7a,b)$$

Proposition 3. *(q-Analog of H'Adamar identity) Let x is any element of A then*

$$exp_q(x)\ y\ (exp_q(x))^{-1} \equiv Ad\ exp_q(x)\ (y) =$$
$$= exp_q(ad_q x)\ (y) \equiv y + \Big[\sum_{n\geqslant 1} \frac{1}{(n)_q!}\, ad_q^n\ x\Big](y), \qquad (3.8)$$

where

$$ad_q^1\ x\ (y) \equiv [x,\ y], \qquad ad_q^2\ x\ (y) \equiv [x,[x,y]]_q,\ \dots ,$$
$$ad_q^{n+1} x\ (y) = [x, ad_q^n\ x\ (y)]_{q^n}, \qquad ([x,z]_{q^k} \equiv xz - q^k zx). \qquad (3.9)$$

Using q-analog of the H'Adamar formula we show that (1.1) is just an additive property of q-exponents for $\bar{q}$-commuting variables. For other type of quantum supergroups we prove the equality

$$\Delta'_q(e_\gamma) \prod_{\alpha<\gamma} \tilde{R}_\alpha = \prod_{\alpha<\gamma} \tilde{R}_\alpha\ (1 \otimes e_\gamma + e_\gamma \otimes q^{-h_\gamma}) \qquad (3.10)$$

in induction on the height of root $\gamma\in\vec{\Sigma}_+$ and then repeat Rosso's quantum double arguments [5],[7].

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