Formal and Applied AdS/CFT

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Abstract

The gauge/gravity duality is a powerful mathematical tool that relates strongly-interacting gauge theories with large numbers of colors to classical gravitational theories with negative cosmological constant. This thesis uses the gauge/gravity duality in two ways.

The first half of the thesis explores the notion of a holographic p-wave superconductor/superfluid. On the gauge theory side there is an $SU(2)$ global symmetry that is explicitly broken to $U(1)$ by turning on a charge density. This $U(1)$ symmetry is in turn spontaneously broken when the ratio between temperature and charge density is smaller than a critical value. The spontaneous breaking of the $U(1)$ symmetry is accompanied by a spontaneous breaking of rotational symmetry. On the gravity side the $SU(2)$ and $U(1)$ symmetries are gauged, and the symmetry-broken backgrounds are charged black branes surrounded by clouds made of off-diagonal gauge bosons. The gauge/gravity duality is used to compute various critical exponents and transport coefficients related to the phase transition between the $U(1)$ symmetry-broken and symmetry-restored phases.

The second half of this thesis builds on the recent progress on using the technique of localization for computing supersymmetry-protected quantities in gauge theories with $\mathcal{N} \geq 2$ supersymmetry on the three-sphere. Using this technique, the infinite-dimensional path integrals of these theories were reduced to finite-dimensional multi-matrix integrals. In the second half of this thesis these multi-matrix integrals are computed approximately for the case of effective gauge theories on M2-branes probing various Calabi-Yau singularities. The answers match the predictions of the gauge/gravity duality. In particular, they reproduce the $N^{3/2}$ scaling of the number of degrees of freedom on N coincident M2-branes.

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Contents

Chapter 1

Introduction

The gauge/gravity duality provides the closest connection to date between string theory and the observable world. At the same time, it makes a rich playground for enhancing our theoretical understanding of strongly-interacting quantum systems, gravity, and ultimately string theory itself. Even though it was born out of string theory, in the past few years this duality has started a life of its own as an effective description of strongly-interacting quantum systems. Such an effective description forgets about the stringy origin of the duality and focuses on some of its properties that are believed to be universal to many other strongly-interacting systems with or without a stringy origin. Within this context, the duality has been used extensively to describe phenomena similar to the ones encountered in heavy-ion collisions and in superconducting materials. The first half of this thesis takes such an approach and investigates a gauge/gravity description of what would be a p-wave superconductor if realized in the lab. The second half of this thesis goes back to the stringy origin of the duality, and building on recent advances in explicit descriptions of certain supersymmetric field theories, it solves a long-standing puzzle related to the number of degrees of freedom in these theories. The methods used in the two halves of this thesis are very different from each other because the first half contains calculations on the gravity side of the correspondence, whereas the second half deals with the gauge theory side. The two halves may therefore seem disconnected, and they may indeed be so, showing the versatility of the gauge/gravity correspondence as a tool for solving a wide spectrum of problems. They are however connected in that they both aim to describe strongly-interacting quantum systems and renormalization group flows, both in the language of quantum field theory and in that of the dual gravitational description.

1.1 Gauge/gravity phenomenology

One can go some way towards understanding the insights behind the gauge/gravity duality without mentioning string theory at all. On one side of the duality there is a strongly-interacting gauge theory with a large number of colors N in $d+1$ spacetime dimensions, whose Lagrangian description we may not even know. What we do know about this gauge theory are the scaling dimensions and global symmetry charges of a few of its gauge-invariant operators. These operators could be scalars, spinors, vectors, etc. All theories contain a universal symmetric tensor operator, namely the stress-energy tensor $T_{\mu\nu}$. For each global symmetry, if any, there is a conserved current J_μ. That the gauge theory is strongly interacting means that quantum effects are important and cannot be studied using perturbation theory. The precise limit considered in the gauge/gravity duality is the 't Hooft limit where N is taken to infinity while keeping the 't Hooft coupling $g^2_{\mathrm{YM}}N$ fixed, g_{YM} being the Yang-Mills coupling. If one further takes the strong coupling limit $g^2_{\mathrm{YM}}N \gg 1$, the gauge theory becomes simpler because typically the dimensions of most operators grow with $g^2_{\mathrm{YM}}N$, and many operators decouple from the dynamics.

On the other side of the duality there is a classical gravitational theory with negative cosmological constant in $d+2$ spacetime dimensions. That the gravity side is classical means that quantum effects are negligible, but it is assumed that this classical theory is in fact a limit of a more general theory of quantum gravity. The gravitational theory is described by an action written in terms of fields of various spin. While the precise action and field content depend on the gravitational theory in question, one of these fields, namely the metric $g_{\mu\nu}$, is universal. Some of these gravitational theories could also contain gauge sectors with gauge fields A_μ. Since quantum effects are assumed to be small, the Euler-Lagrange equations derived from the action provide a good approximation to the dynamics. The static solutions of these Euler-Lagrange equations are typically black holes or black branes, but one can also obtain in special cases smooth spacetimes without a black hole or black brane horizon by taking the size of the horizon to zero.

An instance of the gauge/gravity duality is a pair of a $(d+1)$-dimensional strongly-interacting large N quantum field theory and a $(d+2)$-dimensional classical gravitational theory, where quantum states in the field theory correspond to solutions of the equations of motion in gravity, and gauge-invariant operators in the field theory correspond to fields on the gravity side. An important example of a quantum state is that described by a thermal density matrix with temperature T and entropy S. On the gravity side, this state corresponds to a background with a black hole horizon whose Hawking temperature is T and Bekenstein-Hawking entropy is S. As an example of the correspondence between operators and fields, the field theory stress-energy tensor $T_{\mu\nu}$ corresponds to the metric $g_{\mu\nu}$. A conserved current J_μ in the field theory corresponds to a gauge field A_μ on the gravity side. In general, it may be hard to determine which field theory operator corresponds to which field in gravity.

Small excitations around a given field theory state are dual to perturbations of the gravitational background, where these perturbations satisfy equations of motion derived from the gravitational action. In particular, correlation functions in the field theory, which can be interpreted as the response of the system to small excitations, can be computed from the perturbations of the gravitational solution. The field theory can be thought of as living on the boundary of the gravitational spacetime in the sense that it is the behavior of the gravitational perturbations close to this boundary that encode the field theory correlators. For concreteness, let us denote the field theory coordinates by x_m, with m ranging from 0 to the number d of spatial dimensions, and the extra "radial" coordinate that exists only on the gravity side by r. Let the boundary of the gravitational spacetime be at $r = \infty$. If one perturbs the boundary field theory by some operator $\mathcal{O}_\phi(x)$, this perturbation affects the boundary conditions at $r = \infty$ for the dual bulk field ϕ and determines a unique causal solution for ϕ. From the large r asymptotic behavior of ϕ at another point x', one can extract the correlation function $\langle \mathcal{O}_\phi(x)\mathcal{O}_\phi(x')\rangle$. At short distances or large energies, when the points x and x' are close together, the asymptotics of ϕ at x' are determined mostly by what the gravity background is like close to boundary, at large r. When x and x' are widely separated, the large r asymptotics of ϕ at x', and hence also the correlation function $\langle \mathcal{O}_\phi(x)\mathcal{O}_\phi(x')\rangle$, probe regions of the geometry that are typically far from the boundary. The coordinate r can therefore be interpreted as an energy scale in the field theory; the farther we are from the boundary, the lower the energy scale. The gravitational background can be interpreted as a "holographic RG flow," with the ultraviolet (UV) at large r and the infrared (IR) at small r. The word "holographic" means that the bulk gravitational dynamics is determined by what happens in the boundary field theory, and vice versa.

A dictionary of how to relate correlation functions in the field theory to perturbations of the gravitational solution is known in detail for the case where the gravitational solution is asymptotically anti-de Sitter (AdS) at large r. Form the holographic RG flow interpretation, one can infer that as we move towards the UV, a field theory dual to an asymptotically anti-de Sitter background bears a closer and closer resemblance to a theory dual to anti-de Sitter space. Field theories dual to anti-de Sitter space are special because there is no RG flow. They are invariant at the quantum level under conformal transformations, which are rescalings of the field theory metric by a position-dependent factor. This entire dissertation deals with conformal field theories and RG flows caused by relevant deformations thereof, as well as their dual gravitational descriptions. It is therefore befitting to now review a few general properties of conformal field theories and of anti-de Sitter space in an arbitrary number of spacetime dimensions, as well as the relation between the two. This relation is known as the Anti-de Sitter / Conformal Field Theory (AdS/CFT) correspondence. The results in this section are of course not new. The original papers where the AdS/CFT correspondence was proposed are [1–3]. For a review, see for example [4].

Before we start, it should be mentioned that in most well-understood examples of AdS/CFT the gauge theories are supersymmetric and can be realized as effective field theories on the intersection of various types of branes in string theory or M-theory. Supersymmetry is important because it places very strong constraints on various quantities that can then be computed and compared with the gravity side even though the gauge theory is strongly-interacting. In the second half of this thesis we will see an example of such a quantity. In some sense, the first half of this thesis is on a less firm footing than the second half because it uses AdS/CFT more loosely. The models used in the first half are not derived explicitly from a brane construction, and little is known about the dual gauge theories. AdS/CFT is used to define and explore aspects of a broad class of models.

1.1.1 Conformal field theories on $\mathbb{R}^{d,1}$

The precise description of the conformal group depends on the manifold where the field theory is defined. On $\mathbb{R}^{d,1}$ with $d > 1$, the conformal group consists of spacetime translations $x^m \to x^m + t^m$, Lorentz transformations $x^m \to \Lambda^m{}_n x^n$, dilatations $x^m \to$

λx^m, and special conformal transformations

$$x^m \to \frac{x^m + a^m x^2}{1 + 2x^n a_n + a^2 x^2} \,. \tag{1.1}$$

Here, indices are raised and lowered with the standard metric on $\mathbb{R}^{3,1}$, namely $\eta_{mn} = \mathrm{diag}\{-1, 1, 1, 1\}$. Let the (anti-Hermitian) generators of these transformations be P_m, M_{mn}, D, and K_m, respectively, normalized so that the commutation relations are

$$\begin{gathered}
[M_{mn}, P_r] = -(\eta_{mr} P_n - \eta_{nr} P_m) \,, \qquad [M_{mn}, K_r] = -(\eta_{mr} K_n - \eta_{nr} K_m) \,, \\
[P_m, K_n] = 2M_{mn} - 2\eta_{mn} D \,, \qquad [D, P_m] = -P_m \,, \qquad [D, K_m] = K_m \,, \\
[M_{mn}, M_{rs}] = -\eta_{mr} M_{ns} - \eta_{rn} M_{sm} - \eta_{ns} M_{mr} - \eta_{sm} M_{rn} \,,
\end{gathered} \tag{1.2}$$

with all other commutators being zero. These commutation relations can be checked, for example, in the differential representation where the generators of the conformal group are

$$\begin{aligned}
P_m &= \partial_m \,, \qquad M_{mn} = x_m \partial_n - x_n \partial_m \,, \\
D &= x^m \partial_m \,, \qquad K_m = x^2 \partial_m - 2x_m x^n \partial_n \,.
\end{aligned} \tag{1.3}$$

The differential representation is the representation under which functions on $\mathbb{R}^{d,1}$ transform. For example, under dilatations we have $f(x) \to f(\lambda x)$, so under infinitesimal dilatations we have $f(x) \to f(x) + \epsilon x^m \partial_m f(x) + \mathcal{O}(\epsilon^2)$, where we wrote $\lambda = 1 + \epsilon$ and expanded in small ϵ. The generator of dilatations is therefore $x^m \partial_m$ in this representation, as in (1.3).

In general, operators in the field theory transform in more complicated representations of the conformal group. We see from (1.2) that the only conformal group generator that commutes with the Lorentz generators M_{mn} is D, so one can choose a basis of operators that transform in a finite-dimensional representation of the Lorentz group and are eigenfunctions of the dilatation operator D. That an operator $\mathcal{O}$ is an eigenfunction of D with eigenvalue $-\Delta$ means that under the rescaling $x \to \lambda x$ it transform as $\mathcal{O}(x) \to \lambda^{-\Delta}\mathcal{O}(\lambda x)$. Equivalently, D acts on $\mathcal{O}$ as

$$[D, \mathcal{O}(x)] = (-\Delta + x^m \partial_m)\, \mathcal{O}(x)\,. \tag{1.4}$$

From an AdS/CFT point of view, perhaps the most important property of the conformal group on $\mathbb{R}^{d,1}$ is that it is isomorphic to $SO(d,2)$, which as we will see in the next section is also the isometry group of AdS_{d+2}. That the conformal group is isomorphic to $SO(d,2)$ can be seen at the level of the generators by defining

$$\begin{aligned} J_{mn} &= M_{mn}\,, \qquad & J_{m(d+1)} &= \frac{1}{2}\left(K_m - P_m\right)\,, \\ J_{(d+2)(d+1)} &= D\,, \qquad & J_{m(d+2)} &= \frac{1}{2}\left(K_m + P_m\right)\,. \end{aligned} \tag{1.5}$$

An explicit computation using (1.2) shows that the J_{MN} satisfy the commutation relations

$$[J_{MN}, J_{RS}] = -\eta_{MR}J_{NS} - \eta_{RN}J_{SM} - \eta_{NS}J_{MR} - \eta_{SM}J_{RN}\,, \tag{1.6}$$

with $\eta_{MN} = \mathrm{diag}\{-1, 1, 1, \ldots, 1, -1\}$. These commutation relations are those of the $SO(d,2)$ algebra with the signature given by η_{MN}.

1.1.2 Anti-de Sitter space

AdS_{d+2} is a hyperboloid in $\mathbb{R}^{d+1,2}$ with the metric η_{MN} introduced above. If X^M are the coordinates in $\mathbb{R}^{d+1,2}$, and the radius of AdS_{d+2} is L, then the embedding of AdS_{d+2} in $\mathbb{R}^{d+1,2}$ is

$$\eta_{MN}X^M X^N = -L^2\,. \tag{1.7}$$

It is not hard to see that the isometry group of AdS_{d+2} is $SO(d,2)$, because both the metric η_{MN} on the ambient space $\mathbb{R}^{d+1,2}$ and the embedding equation (1.7) are invariant under $SO(d,2)$ transformations. Each J_{MN} generates rotations in the $X_M X_N$ plane.

While visualizing AdS_{d+2} as the hyperboloid (1.7) has its merits, for describing the holographic dual of a theory on $\mathbb{R}^{d,1}$ it is preferable to parameterize AdS_{d+2} by the $d+1$ field theory coordinates x_m and the extra coordinate r mentioned earlier. One can obtain such a parameterization by writing

$$\begin{aligned} X^m &= x^m e^r \,, \qquad X^d = \frac{x^2 e^r}{2L} - L \sinh r \,, \\ X^{d+1} &= \frac{x^2 e^r}{2L} + L \cosh r \,, \end{aligned} \tag{1.8}$$

where r and x^m are unrestricted. These coordinates parameterize only half of the hyperboloid (1.7). In these coordinates, the metric on AdS_{d+2} induced from η_{MN} is

$$ds^2 = e^{2r} dx_m dx^m + L^2 dr^2 \,, \tag{1.9}$$

and the Killing vectors satisfying the commutation relations (1.2) are

$$\begin{aligned} P_m &= \partial_m \,, \qquad M_{mn} = x_m \partial_n - x_n \partial_m \,, \\ D &= -\partial_r + x^m \partial_m \,, \qquad K_m = x^2 \partial_m - 2 x_m x^n \partial_n + L^2 e^{-2r} \partial_m + 2 x_m \partial_r \,. \end{aligned} \tag{1.10}$$

We mentioned above that the field theory should be thought of as being defined on the boundary of AdS, which is at $r = \infty$. More precisely, the field theory spacetime $\mathbb{R}^{d,1}$ should be identified, up to a conformal transformation, with a constant r slice of the background (1.9) as r is taken to infinity. We see quite nicely that in this limit the Killing vectors of AdS_{d+2} reduce to the conformal Killing vectors on $\mathbb{R}^{d,1}$ given in eq. (1.3).

1.1.3 Asymptotically AdS spacetimes

In addition to anti-de Sitter space, it is common to consider other asymptotically AdS backgrounds. Anti-de Sitter space is an extremum of the Einstein-Hilbert action

$$S = \frac{1}{2\kappa_{d+2}^2} \int d^{d+2}x \sqrt{-g}\,(R - 2\Lambda) \ , \tag{1.11}$$

where Λ is a negative cosmological constant related to the radius L of AdS_{d+2} through $\Lambda = -d(d+1)/(2L^2)$, and κ_{d+2} is the gravitational constant in $d+2$ dimensions. The gravitational constant is related to the Newton constant G_{d+2} through $\kappa_{d+2}^2 = 8\pi G_{d+2}$. The action (1.11) has other extrema too. One of them is the black-brane metric[1]

$$ds^2 = e^{2r}\left[-f(r)dt^2 + dx_i dx^i\right] + L^2 \frac{dr^2}{f(r)} \, , \qquad f(r) \equiv 1 - e^{(d+1)(r_h - r)} \, , \tag{1.12}$$

where $r \geq r_h$ for some r_h, and we denoted $x^0 = t$. At $r = r_h$ the blackening function f has a simple zero and the spacetime has a flat event horizon. This event horizon has an associated Hawking temperature T, which can be computed by requiring that the Euclidean continuation of the metric (1.12) should not have a conical singularity at $r = r_H$. This requirement implies

$$T = \frac{(d+1)e^{r_h}}{4\pi L} \, . \tag{1.13}$$

The AdS-Schwarzschild metric is an example of an asymptotically-AdS metric, because at very large r, the blackening function f approaches unity, and the metric approaches (1.9). This background is dual to a finite-temperature state of a CFT with temperature T. One can also compute the entropy S as the area of the black

[1] A perhaps more familiar form of this metric can be obtained by sending $r \to \log(r/L)$. Then $ds^2 = \frac{r^2}{L^2}(-f dt^2 + dx_i dx^i) + \frac{L^2}{r^2 f} dr^2$ and $f = 1 - \left(\frac{r_h}{r}\right)^{d+1}$.

hole horizon divided by $4G_{d+2}$. One obtains

$$S = \frac{2\pi}{\kappa_{d+2}^2} e^{dr_h} V_d \,, \tag{1.14}$$

where V_d is the d-dimensional coordinate volume.

If in addition to the Einstein-Hilbert action (1.11) one also adds matter fields, for example a scalar ϕ with some potential $V(\phi)$, it may be possible to find extrema of the action of the form

$$ds^2 = e^{2a(r)} dx_m dx^m + L^2 dr^2 \,, \tag{1.15}$$

with $a(r) \sim r$ as $r \to \infty$. Such a spacetime would be another example of an asymptotically-AdS metric, and it would describe a holographic RG flow.

1.1.4 Rough AdS/CFT dictionary

What does it take for an operator $\mathcal{O}_\phi$ of conformal dimension Δ to be dual to some bulk field ϕ in an asymptotically AdS_{d+2} geometry? The first requirement would be that both $\mathcal{O}_\phi$ and ϕ should transform in the same representation of the Lorentz group, so the field ϕ should have an index structure in the field theory directions that mimics that of $\mathcal{O}_\phi$. A second requirement would be that both $\mathcal{O}_\phi$ and ϕ transform in the same way under dilatations, and hence under the whole conformal group. While the action of the generator D on $\mathcal{O}_\phi$ is given in (1.4), D acts on ϕ through a Lie derivative with respect to the AdS Killing field (1.10). If s is the number of lower indices minus the number of upper indices of ϕ in the x^m directions, an explicit calculation gives

$$\mathcal{L}_D \phi = \left(x^m \partial_m - \partial_r + s\right) \phi \,. \tag{1.16}$$

From comparing (1.16) with (1.4), we see that it is possible for the field ϕ to be dual to $\mathcal{O}_\phi$ while preserving conformal invariance asymptotically at large r only if $(-\partial_r + s)\phi = \Delta\phi$, or in other words if $\phi(x_m, r) \sim e^{-(\Delta-s)r}$ at large r. The coefficient of $e^{-(\Delta-s)r}$ can be identified up to normalization with the expectation value $\langle\mathcal{O}_\phi(x)\rangle$ because both of these quantities have the same transformation properties under the conformal group.

Perturbations of the field theory action with a source J_ϕ for $\mathcal{O}_\phi$ should also have a dual gravity description. Adding the source J_ϕ means that the field theory action changes by

$$\delta S = \int d^{d+1}x J_\phi(x)\mathcal{O}_\phi(x)\,, \tag{1.17}$$

where all the Lorentz indices of $\mathcal{O}_\phi$ are contracted with those of J_ϕ. One can assign J_ϕ conformal dimension $d-\Delta$. Of course, since J_ϕ is a fixed external source, it might seem silly to say that it transforms in a particular way under dilatations. However, if one keeps J_ϕ arbitrary, then the perturbed theory is classically invariant under dilatations provided that J_ϕ transforms with conformal dimension $d-\Delta$. In order for $J_\phi\mathcal{O}_\phi$ to be a Lorentz scalar, the number of lower indices minus the number of upper indices of J_ϕ must be $-s$. J_ϕ therefore transforms in the same way as the coefficient of $e^{-(d-\Delta+s)r}$ of a bulk field $\tilde{\phi}$ whose index structure matches that of J_ϕ. In simple cases, ϕ and $\tilde{\phi}$ are related by raising and lowering indices with the bulk metric. In particular, if $s = 0$ one could have $\phi = \tilde{\phi}$, and the coefficient of e^{-dr} would correspond to $\langle\mathcal{O}_\phi\rangle$, while the coefficient of $e^{(\Delta-d)r}$ would correspond to J_ϕ.

For small values of ϕ, only the quadratic action is important because it gives a linear equation of motion, and the corrections coming from higher order terms in ϕ are small. If ϕ is a real scalar field one can normalize it such that the quadratic action is

$$S_\phi = \int d^{d+2}x \sqrt{-g} \left[\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 \right] , \tag{1.18}$$

for some constant m, which is the mass of the scalar field. The equation of motion takes the asymptotic form at large r:

$$\frac{1}{e^{(d+1)r}} \partial_r (e^{(d+1)r} \partial_r \phi) - m^2 L^2 \phi = 0 \, . \tag{1.19}$$

The solution is given in terms of two integration constants $A_\phi(x_m)$ and $B_\phi(x_m)$:

$$\phi(x_m, r) = B_\phi(x_m) e^{(\Delta - d)r} \left[1 + \cdots \right] + A_\phi(x_m) e^{-\Delta r} \left[1 + \cdots \right] , \tag{1.20}$$

where the dots stand for corrections coming from the fact that (1.19) is only an approximation at large r. The constant Δ satisfies the equation

$$\Delta(\Delta - d) = m^2 L^2 \, . \tag{1.21}$$

The integration constants $A_\phi(x_m)$ and $B_\phi(x_m)$ are not independent if one requires $\phi(x_m, r)$ to be regular at the other end of the integration region. Up to normalization issues that will be made precise in the following section, one can interpret $A_\phi(x_m)$ as the expectation value $\langle \mathcal{O}_\phi(x_m) \rangle$ in the presence of the source $J_\phi(x_m) = B_\phi(x_m)$.

1.1.5 More precise AdS/CFT dictionary

The precise statement of the AdS/CFT correspondence is stated more clearly in Euclidean signature for a reason that will be made clear shortly. The statement is that the generating functional for connected correlators equals minus the on-shell gravitational action:

$$W[J_\phi] = -S_{\text{on-shell}}[J_\phi]\,, \qquad W[J_\phi] = \log \left\langle \exp \int d^{d+1}x\, J_\phi(x)\mathcal{O}_\phi(x) \right\rangle \,, \tag{1.22}$$

both computed as functionals of the source J_ϕ. As above, on the gravity side the source J_ϕ is the coefficient of $e^{(\Delta-s-d)r}$ in the large r expansion of the dual field ϕ. The gravity action is typically UV divergent, but the divergences can be subtracted systematically through a procedure called holographic renormalization (for a review, see [5]). Once one constructs a regularized action that is finite on-shell for any J_ϕ, connected correlation functions of $\mathcal{O}_\phi$ can be computed by taking functional derivatives of the on-shell action with respect to J_ϕ.

It is easier to formulate the prescription for computing correlation functions in Euclidean signature because, at finite temperature for example, the gravitational background has only one boundary at $r = \infty$, and once one specifies a source J_ϕ at this boundary there is a unique solution for ϕ that is regular everywhere away from the boundary. In Minkowski signature the presence of an event horizon complicates the boundary conditions one needs to use. The boundary conditions used in this dissertation are so that ϕ looks like an infalling wave at the event horizon. Such boundary conditions are appropriate for computing retarded two-point functions, which are the two-point functions corresponding to causal field theory response to perturbations. For a more detailed discussion, see [6–9].

1.1.6 Superconductors and superconducting black holes

Superconductors and superfluids

The first half of this dissertation contains applications of the ideas presented above to superconductivity. A superconductor is a material where the electric current flows with no resistance, or in other words where the DC conductivity is infinite. In addition to being a perfect conductor, superconductors exhibit the Meissner effect, which consists of expelling magnetic field lines from their interior. Many materials, such as aluminum or lead, go superconducting below a certain temperature. As described in [10], many properties of superconductors, such as the infinite DC conductivity or the Meissner effect, can be attributed to the spontaneous breaking of the gauge symmetry. That the gauge symmetry is spontaneously broken means that for small excitations around the superconducting state the photon acquires a mass. Suppose there is a vacuum with no external electromagnetic fields and zero current. The system responds to small fluctuations of the gauge field by generating a current

$$J_m(\omega, \vec{k}) = G^R_{nm}(\omega, \vec{k}) A^n(\omega, \vec{k})\,, \tag{1.23}$$

for some function $G^R_{nm}(\omega, \vec{k})$. Since the symmetry is broken spontaneously, as opposed to softly, this current is conserved, so $k^m G^R_{mn}(\omega, \vec{k}) = 0$, where $k^m = (\omega, \vec{k})$. One can think of this response as coming from an effective action of the form $\int \delta J_m A^m$, so the fact that the gauge field acquires a mass translates into a nonzero limit of $G^R_{mn}(\omega, \vec{k})$ as ω and $\vec{k}$ are taken to zero.

In order to study the conductivity, one needs to take $\vec{k} \to 0$ first. One can apply an external electric field $Ee^{i\omega t}$ in the ith direction by setting $A^n(\omega, 0) = \frac{1}{i\omega}\delta^n{}_i E e^{i\omega t}$. Eq. (1.23) predicts a current equal to $J_j(\omega, 0) = \frac{i}{\omega} G^R_{ij}(\omega, 0) E e^{i\omega t}$. In the low-frequency limit $\omega \to 0$, we see that as long as $G^R_{ij}(0, 0)$ does not vanish, a DC electric field creates an infinite current, indicating an infinite DC conductivity. If the material is spatially isotropic, then $G^R_{ij}(0, 0)$ is proportional to the identity matrix, and the conductivity is the same in all directions. The superconductor has an s-wave symmetry in this case. In chapters 2 and 3 in this dissertation we will find examples of superconductors where this rotational symmetry is broken by having a preferred direction. These would be p-wave superconductors.

In order to study the Meissner effect, one needs to take $\omega \to 0$ first. For simplicity, let's consider the case of s-wave superconductors, in which case the condition $k^m G^R_{mn} = 0$ together with the rotational symmetry in the spatial directions implies

$$G^R_{ij}(0, \vec{k}) = G(\vec{k}^2)\left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2}\right) . \tag{1.24}$$

From eq. (1.23), the current is

$$J_i(0, \vec{k}) = -G(\vec{k}) A_i^{\perp}(\vec{k}) \, , \qquad A_i^{\perp}(\vec{k}) \equiv \left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2}\right) A_i(\vec{k}) \, . \tag{1.25}$$

In the low wavelength limit where $G(\vec{k}) \approx G(0)$, this equation is called the London equation. In chapter (3) we will encounter the London equations corresponding to the case of an anisotropic p-wave superconductor. The London equation almost shows that there is a Meissner effect if $G(0) \neq 0$: taking the antisymmetrized derivative w.r.t. x^i and using the Maxwell equation $\partial^m F_{mn} = J_n$, one obtains the differential equation $(\nabla^2 - G(0))F_{ij} = 0$. If $G(0) \neq 0$, the solutions of this differential equations are exponentially decaying or increasing, so the magnetic field F_{ij} is expelled from the interior of a superconductor.

The above derivation of the Meissner effect sheds light on another issue: a superconductor can also be thought of as a superfluid, and so can a system with a spontaneously broken global as opposed to gauge symmetry. Suppose we have a non-relativistic fluid with a $U(1)$ global or gauge symmetry described by some Hamiltonian H in a frame S where in the ground state the fluid is at rest. Boosting the system to a frame S' moving with velocity $\vec{v}$ is equivalent to coupling the system to a pure gauge external field $\vec{A} = \vec{v}$, where the charge now plays the role of the mass of each particle. Since the change in the effective action is $\int \vec{J} \cdot \vec{A}$, the energy difference between the ground state energies in the two frames is

$$\Delta E \propto v^2 G(0)\,. \tag{1.26}$$

Now if we boost the wavefunction corresponding to the ground state of H, Galilean transformations show that its H' eigenvalue would be larger than its H eigenvalue by $Mv^2/2$, where M is the total mass of the system. It follows from (1.26) that the ground state is preserved under the boost only if $G(0) \neq 0$. When $G(0) = 0$ the ground state of H is actually an excited state of H'. Typically, under the effect of any static perturbations in S' this excited state decays into states with lower energy and the fluid eventually stops. This case corresponds to a normal fluid. For a superfluid, the vacuum is invariant under boosts, so one necessarily has $G(0) \neq 0$.

Superconducting black holes

An interesting observation was made in [11] that charged black branes and black holes in asymptotically AdS spaces can also break the gauge symmetry spontaneously. The simplest example of this sort is the Abelian Higgs model coupled to gravity with negative cosmological constant. The dynamical fields are the metric, a gauge field A_μ with field strength $F_{\mu\nu}$, and a complex scalar field ψ that has charge q under A_μ. The action is

$$S = \frac{1}{2\kappa_{d+2}^2} \int d^{d+2}x \sqrt{-g} \left[R - 2\Lambda - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} - |D_\mu \psi|^2 - m^2 |\psi|^2 \right] , \tag{1.27}$$

where $D_\mu = \partial_\mu - iqA_\mu$ is the gauge covariant derivative. Suppose we want to study static solutions of the equations of motion following from this action that are asymptotically AdS, have translational and rotational symmetry in d spatial directions, and are electrically charged, meaning that asymptotically there is a nonzero electric flux.

The ansatz appropriate for describing these solutions is

$$ds^2 = e^{2a(r)}(-h(r)dt^2 + d\vec{x}^2) + L^2\frac{dr^2}{h(r)}\,, \qquad A_\mu dx^\mu = \Phi(r)dt\,, \qquad \psi = \psi(r)\,, \tag{1.28}$$

where a, h, Φ, and ψ are functions of the radial coordinate r that satisfy equations of motion derived from the action (1.27). Note that we take only the time component of the gauge field to be nonzero, indicating that the black branes are electrically, and not magnetically, charged. On the field theory side ψ is dual to an operator $\mathcal{O}$ of conformal dimension Δ, where $\Delta(\Delta - d) = m^2L^2$, and the gauge field A_μ is dual to a conserved current J_m. It is always the case in AdS/CFT that global symmetries on the boundary correspond to gauge symmetries in the bulk.

The Reissner-Nordström Anti-de Sitter (RNAdS) black branes are some of the extrema of the action of the form (1.28) with

$$\begin{aligned} a(r) &= r\,, \qquad h(r) = 1 + Q^2e^{-(2d-2)(r-r_h)} - (1+Q^2)e^{-d(r-r_h)}\,, \\ \psi &= 0\,, \qquad \Phi = \mu + \rho e^{-d-2r}\,, \end{aligned} \tag{1.29}$$

where we defined

$$Q^2 \equiv \frac{d-2}{2(d-1)}\rho^2 e^{-(2d-2)r_h}\,. \tag{1.30}$$

These backgrounds do not break the gauge symmetry. If the charge q of the scalar field is large enough, it was noticed in [11] in the AdS_4 case that at low enough temperatures there are also solutions of the form (1.28) where ψ is nonzero and its behavior close to the boundary of AdS corresponds to an expectation value $\langle\mathcal{O}\rangle$ as opposed to a source. These "hairy" black hole solutions break the Abelian gauge symmetry spontaneously because gauge transformations act on ψ by multiplying it by a phase.

The field theory interpretation is as follows. The solutions (1.28) where $\psi = 0$ correspond to states in a CFT at nonzero temperature and charge density. The temperature is nonzero because the backgrounds (1.29) have a black hole horizon at $r = r_h$ with some nonzero Hawking temperature T. The charge density is nonzero because, as discussed earlier, the ρe^{-d-2r} term in Φ corresponds up to normalization to an expectation value of the dual operator, namely the time component J_t of the conserved current dual to A_μ. The solutions with nonzero ψ that exist only at small enough temperatures correspond to states in the CFT where in addition to a nonzero temperature and charge density there is also a nonzero expectation value of $\langle \mathcal{O} \rangle$. The operator $\mathcal{O}$ is charged under the global $U(1)$ symmetry generated by the current J_m, so these states break the global symmetry spontaneously. As per the discussion above, such states are usually referred to as superfluid states.

One can imagine weakly gauging the $U(1)$ global symmetry in the boundary theory, and thinking of this weakly gauged $U(1)$ symmetry as describing electromagnetism. As we have seen above, the states where the $U(1)$ gauge symmetry is broken then describe a superconductor because the DC conductivity is infinite and there is a Meissner effect. In [12, 13] the AC conductivity $\sigma(\omega)$ was computed holographically, and it was checked that it indeed exhibits a delta-function peak at $\omega = 0$.

Chapters 2 and 3 contain another example of a gravity Lagrangian that exhibits spontaneous gauge symmetry breaking. Instead of ψ and the $U(1)$ gauge field A_μ, these examples contain an $SU(2)$ gauge field. We consider black holes that are charged only with respect to a $U(1)$ subgroup of $SU(2)$. The off-diagonal gauge bosons combine into a complex vector field that is charged under this $U(1)$ with a charge proportional to the gauge coupling. At very low temperatures, it is these gauge bosons that play the role of ψ and condense. Since the field that condenses is now a vector, the resulting charged black branes also break the $SO(d)$ rotational symmetry. As we will see, the conductivity matrix will be anisotropic in this case and describe a superconductor with p-wave symmetry.

1.1.7 Free energies in theories dual to AdS_4

Before we move on to the study of explicit examples of theories dual to anti-de Sitter space let us calculate from the gravity side of the correspondence two quantities that measure the field theory number of degrees of freedom. From examining the regime of validity of these computations, we will conclude that field theories with gravity duals are rather special in that they contain a parametrically large number of degrees of freedom.

Thermal free energy

One way of measuring the number of degrees of freedom is by calculating the thermal free energy at some given temperature T. The finite temperature state is given by the black-brane metric (1.12). Eliminating r_h between the eqs. (1.13) and (1.14), one obtains

$$S = \frac{(4\pi)^{d+1}}{(d+1)^d} \frac{L^d}{2\kappa_{d+2}^2} V_d T^d \,. \tag{1.31}$$

Invariance of a conformal field theory under dilatations implies that the expectation value of the trace of the stress-energy tensor vanishes, so the energy density ϵ is related to the pressure p through $\epsilon = dp$. Extensivity of the energy as a function of S and V implies that $E = TS - pV_d$, so in a conformal theory it must be true that $E = \frac{d}{d+1} TS$. The thermal free energy $F_T = E - TS$ therefore equals $-\frac{1}{d+1} TS$. Combining this expression with (1.31), one obtains

$$F_T = -\frac{1}{2} \left(\frac{4\pi}{d+1} \right)^{d+1} \frac{L^d}{\kappa_{d+2}^2} V_d T^{d+1} \,. \tag{1.32}$$

The case of interest in most of this dissertation is $d = 2$, corresponding to conformal field theories dual to AdS_4. In this case eq. (1.32) reduces to

$$F_T = -\frac{32\pi^3}{27}\frac{L^2}{\kappa_4^2}V_2T^3\,. \tag{1.33}$$

Apart from the dependence on V_d and T, which follows from the extensivity of the thermal free energy and dimensional analysis, we see that what measures the number of degrees of freedom in a theory dual to AdS_{d+2} is the dimensionless ratio L^d/κ_{d+2}^2. It is reasonable to expect that Einstein gravity should provide a good approximation to a more general theory of quantum gravity as long as gravity is weak. In other words, the length scale over which the geometry doesn't change significantly should be large compared to the Planck length in $(d+2)$ dimensions. Since L is the radius of curvature of AdS_{d+2}, the requirement that gravity should be weak implies that $L^d/\kappa_{d+2}^2 \gg 1$. From eq. (1.33) it follows that the field theories dual to AdS_{d+2} have a parametrically large number of degrees of freedom.

Free energy on S^3

There is another way of estimating the number of degrees of freedom for theories dual to AdS_4 that will be relevant for the second half of this thesis. It was suggested recently that a good measure of the number of degrees of freedom in this case might be minus the logarithm of the path integral of the Euclidean field theory on S^3 [14–18]. We will call this quantity F, and we will refer to it also as a free energy.[2] While the field theories we have been considering are defined on Minkowski space, one can go to Euclidean signature and use conformal invariance to map these theories to the three-sphere, which is possible because the three-sphere is conformally equivalent to

[2] A seemingly different measure of the number of degrees of freedom in a 3-d CFT was proposed in [19,20]; it is the entanglement entropy between the two hemispheres in the CFT on $\mathbb{R}\times S^2$. In [21] it was shown that this quantity, which is the same as the entanglement entropy between a circle and its complement on a plane, is also equal to minus the free energy of the Euclidean theory on S^3.

$\mathbb{R}^3$ as can be seen by using the stereographic projection. From an AdS standpoint, going to Euclidean signature is achieved by changing the metric η_{MN} used to define the hyperboloid (1.7) to $\eta_{MN} = \text{diag}\{1, 1, 1, 1, -1\}$. The resulting space

$$(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 - (X^4)^2 = -L^2 \tag{1.34}$$

is the four-dimensional hyperbolic space $\mathbb{H}^4$. Using the parameterization

$$X^m = L(\sinh\rho)\Omega^m \,, \qquad X^{d+1} = L(\cosh\rho)\,, \tag{1.35}$$

with Ω^m a unit vector in $\mathbb{R}^4$ parameterizing S^3, the metric on $\mathbb{H}^4$ becomes

$$ds^2 = L^2 d\rho^2 + L^2 \sinh^2\rho\, d\Omega_3^2\,, \tag{1.36}$$

where $d\Omega_3^2$ is the standard line element on the three-sphere. In order to cover the hyperboloid (1.34) only once, one should take $\rho \geq 0$. The metric (1.36) gives a foliation of $\mathbb{H}^4$ where the constant ρ leaves are conformally equivalent to S^3. One can therefore think of this metric as being dual to the vacuum state of a Euclidean theory on S^3, where the field theory lives at $\rho = \infty$, in precisely the same way as the metric (1.9) was appropriate for describing a field theory on $\mathbb{R}^{2,1}$ that lives at $r = \infty$.

The AdS/CFT correspondence, in particular eq. (1.22), implies that the free energy on S^3 equals the on-shell action in Euclidean signature. The Ricci scalar on $\mathbb{H}^{d+2}$ is $R = -d(d-1)/L^2$, so $R - 2\Lambda = -2(d+1)/L^2$. For $d = 2$, one can straightforwardly calculate F by evaluating formally the on-shell action (1.11):

$$F = \frac{1}{2\kappa_4^2}\frac{6}{L^2}\,\text{Vol}(\mathbb{H}^4)\,. \tag{1.37}$$

The volume of $\mathbb{H}^4$ is of course divergent and it requires regularization. In the field theory, one should also perform a similar regularization of UV divergences to define F. The regularization of $\mathrm{Vol}(\mathbb{H}^4)$ is done by imposing a hard cutoff ρ_0 on the radial coordinate ρ and computing the volume of the space with $0 \leq \rho \leq \rho_0$:

$$L^4 \,\mathrm{Vol}(S^3) \int_0^{\rho_0} d\rho \, \sinh^3 \rho = 2\pi^2 L^4 \left[\frac{e^{3\rho_0}}{24} - \frac{3e^{\rho_0}}{8} + \frac{2}{3} + \mathcal{O}(e^{-\rho_0}) \right] . \tag{1.38}$$

Keeping only the finite part, it follows that the regularized volume of $\mathbb{H}^4$ is $\mathrm{Vol}(\mathbb{H}^4) = 4L^4\pi^2/3$. The free energy is therefore [15, 22, 23]

$$F = \frac{4\pi^2 L^2}{\kappa_4^2} = \frac{\pi L^2}{2G_4} . \tag{1.39}$$

Again, we see that the number of degrees of freedom is measured by L^2/κ_4^2, which must be large in order for the Einstein gravity approximation to make sense.

1.2 AdS/CFT and M-theory

I will now describe examples of $(2+1)$-dimensional conformal field theories with gravity duals. The AdS/CFT correspondence was originally proposed for $(3+1)$-dimensional gauge theories derived from type IIB string theory backgrounds describing coincident D3-branes probing Calabi-Yau singularities (see, for example, [1, 2, 24, 25]). In this case, the ten-dimensional metric close to the branes looks like $AdS_5 \times Y^5$, where Y is a five-dimensional Einstein manifold. In the limit where the number N of D3-branes is large and stringy effects are negligible, the effective field theory on the D3-branes is dual to the low-energy string theory excitations around the $AdS_5 \times Y^5$ background. Explicit Lagrangian descriptions of many of these conformal field theories are known, the simplest such example being the $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang-Mills theory, for which Y^5 is the five-dimensional round sphere S^5. Very similar constructions exist in M-theory: if one places N M2-branes at a Calabi-Yau singularity, then close to the branes the metric becomes $AdS_4 \times Y^7$ at large N, where Y^7 is a seven-dimensional Einstein space. Explicit field theory Lagrangians in $2+1$ dimensions describing these M2-brane theories have been written down only in the past few years, starting with the work of Bagger and Lambert [26–28]. In this section I will not talk about D3-branes or $(3+1)$-dimensional field theories at all, but instead review some of these more recent developments related to M2-brane theories. I will focus mostly on the simplest large N construction worked out by Aharony, Bergman, Jafferis, and Maldacena (ABJM) [29], where the space Y^7 is a particular orbifold of the round seven-sphere S^7. The following discussion is drawn mostly from [29–33].

1.2.1 Supersymmetric Chern-Simons matter theories in three dimensions

To construct explicit Lagrangians for 3-d supersymmetric theories, it is important to first understand the field content of the supersymmetry multiplets at our disposal. Any theory with $\mathcal{N} \geq 2$ supersymmetry can be described in terms of $\mathcal{N} = 2$ multiplets, which are nothing but the dimensional reduction of the corresponding $\mathcal{N} = 1$ multiplets from four dimensions. An $\mathcal{N} = 2$ vector multiplet $\mathcal{V}$ in three dimensions consists of a gauge field A_μ, a scalar field σ that comes from the component of the four-dimensional gauge field along the direction we're reducing, a two-component Dirac spinor χ, and an auxiliary scalar D, all valued in the adjoint representation of the gauge group. An $\mathcal{N} = 2$ chiral multiplet Φ consists of a complex scalar ϕ, a two-component Dirac spinor ζ, and an auxiliary complex scalar field F, all valued in the same representation of the gauge group.

The ingredients for constructing actions with at least $\mathcal{N} = 2$ supersymmetry relevant to us are: the supersymmetric Chern-Simons action for a vector multiplet $\mathcal{V}$, the action for a chiral multiplet transforming in a given representation of the gauge group, and a superpotential interaction term between the matter fields. The supersymmetric Chern-Simons action at level k is

$$S_{\mathcal{V}}(k) = \frac{k}{4\pi}\int d^3x \, \mathrm{tr}\left(\epsilon^{mnr}\left(A_m\partial_n A_r + \frac{2}{3}A_m A_n A_r\right) + i\bar{\chi}\chi - 2D\sigma\right) . \tag{1.40}$$

Invariance of this action under large gauge transformations requires the Chern-Simons level to be quantized. If the gauge group is $SU(N)$ or $U(N)$ with $N > 1$, the Chern-Simons level takes integer values if the trace is in the fundamental representation. The action of a chiral multiplet transforming in the representation R of the gauge group is

$$\begin{aligned} S_\Phi = \int d^3x \Big(& -\mathcal{D}_m\phi^\dagger\mathcal{D}^m\phi - i\zeta^\dagger \not{\mathcal{D}}\zeta - F^\dagger F + \phi^\dagger D\phi - \phi^\dagger\sigma^2\phi \\ & -\zeta^\dagger\sigma\zeta + i\phi^\dagger\bar{\chi}\zeta - i\zeta^\dagger\chi\phi\Big) . \end{aligned} \tag{1.41}$$

The notation in this equation needs some explanation. The gauge-covariant derivative $\mathcal{D}_m$ is defined as $\mathcal{D}_m = \partial_m + iA_m^\alpha T^\alpha$, where the $(\dim R) \times (\dim R)$ matrices T^α are the generators of the gauge group in the representation R. For example, if the gauge group is $U(1)$ and Φ has charge q, then one can take $T^1 = q$; if the gauge group is $SU(2)$ and Φ transforms in the fundamental representation, then T^α are the Pauli matrices. The chiral superfield Φ and all its components carry an index a that ranges from 1 to the dimension of R. The notation $F^\dagger F$ is short for $F_a^* F_a$; the notation $\zeta^\dagger\sigma\zeta$ is short for $\zeta_a^\dagger\sigma_\alpha T_{ab}^\alpha\zeta_b$, etc. In addition to (1.41), there could be a superpotential interaction between the matter fields: if the superpotential is $W(\Phi_i)$, then

$$S_W = -\int d^3x \sum_i \left|\frac{\partial W}{\partial\Phi_i}\right|^2 + \text{fermionic terms} , \tag{1.42}$$

where the sum runs over all chiral superfields.

Using the ingredients presented above one can construct many $\mathcal{N} = 2$ supersymmetric Chern-Simons matter theories in three dimensions. All the field theories considered in the second half of this thesis are so-called quiver gauge theories, whose field content can be read off from a quiver diagram. These theories contain a number of gauge groups, each with a Chern-Simons action of the form (1.40), and each corresponding to a dot in the quiver diagram. The chiral superfields are represented by arrows. Each arrow corresponds to chiral superfield transforming in the anti-fundamental representation of the gauge group where the arrow starts and the fundamental representation of the gauge group where the arrow ends. The quiver diagram does not specify the field theory uniquely—it just specifies the field content. In the examples we will be looking at we will also need to specify the Chern-Simons levels for each gauge group, as well as the superpotential.

In general, the gauge theories constructed this way are not conformal. In order for the gauge theory to be conformal, the matter content and the gauge groups need to be chosen appropriately. These are the examples we will focus on.

1.2.2 ABJM theory

The simplest superconformal gauge theory with a gravity dual is the one discovered by ABJM [29]. This theory has $\mathcal{N} = 6$ supersymmetry, but one can describe its field content in $\mathcal{N} = 2$ language. There are two $U(N)$ gauge groups with vector multiplets $\mathcal{V}$ and $\tilde{\mathcal{V}}$, one with Chern-Simons level k and the other one $-k$, as well as two bifundamental chiral superfields $\mathcal{W}_A$, $A = 1, 2$ transforming in the $(\mathbf{N}, \overline{\mathbf{N}})$ representation of $U(N)\times U(N)$, and two bifundamental chiral superfields $\mathcal{Z}^A$, $A = 1, 2$ transforming in $(\overline{\mathbf{N}}, \mathbf{N})$. The superpotential is

$$W = \frac{2\pi}{k}\epsilon_{AC}\epsilon^{BD}\,\mathrm{tr}\left(\mathcal{Z}^A\mathcal{W}_B\mathcal{Z}^C\mathcal{W}_D\right)\,. \tag{1.43}$$

The action is therefore $S_{\mathcal{V}}(k) + S_{\tilde{\mathcal{V}}}(-k) + \sum_{A=1}^{2}\left(S_{\mathcal{Z}^A} + S_{\mathcal{W}_A}\right) + S_W$.

This theory has a manifest $SU(2) \times SU(2)$ global symmetry, where under the first $SU(2)$ factor the $\mathcal{Z}^A$ transform as a doublet, and under the second $SU(2)$ factor the $\mathcal{W}_A$ transform as a doublet. There is also a $U(1)_R$ symmetry under which $\mathcal{Z}^A$ and $\mathcal{W}_A$ get multiplied by the same phase. Actually, one can check that because the coefficient $2\pi/k$ in the superpotential was chosen just right, the R-symmetry is enhanced to $SU(2)_R$, under which $(\mathcal{Z}^1, \mathcal{W}^{\dagger 1})$ and $(\mathcal{Z}^2, \mathcal{W}^{\dagger 2})$ form doublets. Because this $SU(2)_R$ does not commute with the global $SU(2) \times SU(2)$ symmetry mentioned earlier, and together the two symmetries generate an $SU(4)$ symmetry, it must be that the R-symmetry is $SU(4)_R \cong SO(6)_R$. Under $SU(4)_R$, $(\mathcal{Z}^1, \mathcal{Z}^2, \mathcal{W}^{\dagger 1}, \mathcal{W}^{\dagger 2})$ transform in the fundamental representation. In three dimensions, the R-symmetry of a field theory with $\mathcal{N}$ supersymmetries is $SO(\mathcal{N})_R$, so ABJM theory has $\mathcal{N} = 6$ supersymmetry. Later on, we will encounter theories where the superpotential will not allow an enhancement of $U(1)_R$ to $SU(2)_R$, so those theories will just have $\mathcal{N} = 2$ supersymmetry. We will also encounter theories with $\mathcal{N} = 3$ supersymmetry where there is no additional enhancement to $\mathcal{N} = 6$.

In addition to the $SO(6)_R$ global symmetry, ABJM theory has a $U(1)_b$ global symmetry that commutes with $SO(6)_R$. If one were to examine the $SU(N) \times SU(N)$ theory, this $U(1)_b$ would have been a "baryonic" global symmetry under which $\mathcal{Z}^A$ and $\mathcal{W}_A$ transform with opposite phases. In the $U(N) \times U(N)$ theory, this baryonic symmetry is gauged. However, in three dimensions, every Abelian gauge symmetry with gauge connection A_m and field strength F_{mn} has an associated global symmetry generated by the current $J_m = \epsilon_{mnr} F^{nr}$. The conservation of this current follows from the Bianchi identity for F_{mn}. One could then ask why ABJM theory does not actually have two conserved currents of this sort? One of these currents would be generated by $J_m = \epsilon_{mnr} \operatorname{tr} F^{nr}$ and one by $\tilde{J}_m = \epsilon_{mnr} \operatorname{tr} \tilde{F}^{nr}$, where F_{mn} and $\tilde{F}_{mn}$ are the field strengths of the gauge fields A_m and $\tilde{A}_m$ in the two vector multiplets. The answer is that the equations of motion for the two gauge fields imply that $J_m = \tilde{J}_m$, so the two currents are actually equal. Ignoring $\mathbb{Z}_2$ factors, the full symmetry group is therefore

$SO(6)_R \times U(1)_b$. When $k = 1$ or 2, this symmetry is further enhanced to $SO(8)_R$, and the field theory has the maximal amount of supersymmetry in three dimensions, $\mathcal{N} = 8$. It is not easy to check explicitly that the supersymmetry is indeed enhanced to $\mathcal{N} = 8$, and we will comment on this issue later on.

As a step towards understanding the gravity dual of ABJM theory, one can compute the moduli space of vacua. Let us examine the case $N = 1$ first. The bosonic part of the action is

$$\begin{aligned} S = \int d^3x \Bigg[&\frac{k}{4\pi}\epsilon^{mnr}(A_m\partial_n A_r - \tilde{A}_m\partial_n\tilde{A}_r) - \frac{k}{2\pi}(D\sigma - \tilde{D}\tilde{\sigma}) \\ &- \left|(\partial_m + iA_m - i\tilde{A}_m)W_A\right|^2 + W^{A\dagger}(D - \tilde{D})W_A - W^{A\dagger}(\sigma - \tilde{\sigma})^2 W_A \\ &- \left|(\partial_m - iA_m + i\tilde{A}_m)Z^A\right|^2 + Z^\dagger_A(\tilde{D} - D)Z^A - Z^\dagger_A(\sigma - \tilde{\sigma})^2 Z^A \Bigg] , \end{aligned} \tag{1.44}$$

where we integrated out the auxiliary F and $\tilde{F}$ fields. Integrating out D and $\tilde{D}$ sets

$$\sigma = \tilde{\sigma} = \frac{2\pi}{k}\left(|W_1|^2 + |W_2|^2 - \left|Z^1\right|^2 - \left|Z^2\right|^2\right) . \tag{1.45}$$

The vacua of the theory are determined by the classical solutions of the equations of motion for which the scalar fields have constant expectation values. Plugging (1.45) back into the action we see that the equations of motion are satisfied provided that $A_m = \tilde{A}_m$. One can think of the condition $A_m = \tilde{A}_m$ as being related to (1.45) by supersymmetry. The resulting potential for W_A and Z^A is identically zero, suggesting that all these vacua are supersymmetric. One might conclude that the there is a $\mathbb{C}^4$ worth of supersymmetric vacua parameterized by the four complex numbers W_A and Z^A, but this would be a little too quick, as some of these vacua may be gauge equivalent.

Gauge transformations act as

$$\begin{aligned} A_m &\to A_m + \partial_m \Lambda\,, \qquad & \tilde{A}_m &\to \tilde{A}_m + \partial_m \tilde{\Lambda}\,, \\ W_A &\to W_A e^{i(\Lambda - \tilde{\Lambda})}\,, \qquad & Z^A &\to Z^A e^{i(\tilde{\Lambda} - \Lambda)}\,. \end{aligned} \tag{1.46}$$

Under these gauge transformations, the action changes by the boundary term

$$\delta S = \frac{k}{2\pi} \int_{S^2} \left(\Lambda F - \tilde{\Lambda} \tilde{F} \right)\,, \tag{1.47}$$

where the two-sphere we are integrating over should be taken to have a very large radius. Since $A_m = \tilde{A}_m$, this boundary term simplifies to

$$\delta S = \frac{k}{2\pi} (\Lambda - \tilde{\Lambda}) \int_{S^2} F\,. \tag{1.48}$$

It is sensible to require that the gauge fields A_m and $\tilde{A}_m$ go to zero at infinity and that the sources for F are localized. The second condition implies that at sufficiently large radii, the integral of F over a two-sphere of radius r is independent of r. The first condition implies that the gauge parameters Λ and $\tilde{\Lambda}$ must be required to approach constant values asymptotically at large r. However, not all such gauge transformations should be allowed, because in a well-defined quantum theory the quantity e^{iS} should be invariant under all gauge transformations. Since Dirac quantization implies that the integral of F over any two-manifold must be quantized in units of 2π, the requirement that e^{iS} should not change under gauge transformations implies that the only allowed gauge transformations are those where

$$\Lambda - \tilde{\Lambda} \to \frac{2\pi \ell}{k} \qquad \text{as } r \to \infty \tag{1.49}$$

for some $\ell \in \mathbb{Z}$. As far as the moduli space is concerned, only constant gauge transformations are relevant because we are looking at solutions of the equations of motion where $A_m = \tilde{A}_m$ and where W_A and Z^A are constant. It follows that on the moduli space, we have the identifications

$$W_A \sim W_A e^{2\pi i \ell/k}\,, \qquad Z^A \sim Z^A e^{-2\pi i \ell/k}\,, \tag{1.50}$$

for any integer ℓ. The moduli space is therefore $\mathbb{C}^4/\mathbb{Z}_k$.

The moduli space of vacua of the $U(N) \times U(N)$ theory for $N > 1$ is rather complicated, as there are now many more ways of satisfying the classical equations of motion. While the moduli space has several branches of various dimensions, it can be argued that the branch of moduli space with the highest dimension is the Nth symmetric power of the $N = 1$ moduli space $\mathbb{C}^4/\mathbb{Z}_k$ [29, 32].

1.2.3 The gravity dual of ABJM theory

It is believed that the $\mathcal{N} = 6$ ABJM theory is the effective IR theory on N coincident M2-branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ orbifold singularity. When N is large and k small (the precise condition being $N \gg k^5$ as we will see shortly), this brane configuration can be described reliably within eleven-dimensional supergravity. The 11-d supergravity action is

$$S = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-G} \left(R - \frac{1}{2} |F_4|^2 \right) - \frac{1}{12\kappa_{11}^2} \int A_3 \wedge F_4 \wedge F_4\,, \tag{1.51}$$

where κ_{11} is the gravitational coupling constant in eleven dimensions, which is related to the Planck length ℓ_p by

$$2\kappa_{11}^2 = (2\pi)^8 \ell_p^9\,, \tag{1.52}$$

and A_3 is the three-form gauge potential for the four-form $F_4 = dA_3$. In a little more generality than we presently need, one can construct solutions to the equations of motion following from (1.51) describing N coincident M2-branes at the tip of an eight-dimensional Calabi-Yau cone X (in our case $X = \mathbb{C}^4/\mathbb{Z}_k$) of the form:

$$\begin{aligned} ds^2 &= H^{-2/3} dx_\mu dx^\mu + H^{1/3} ds_X^2 \,, \\ F_4 &= dH^{-1} \wedge dx^0 \wedge dx^1 \wedge dx^2 \,, \end{aligned} \tag{1.53}$$

where $dx_\mu dx^\mu = (-dx^0)^2 + (dx^1)^2 + (dx^2)^2$, and H is a harmonic function on X away from the tip of the cone. Let Y be the seven-dimensional base of the cone (in our case $Y = S^7/\mathbb{Z}_k$). If the metric on Y is normalized so that $R_{mn} = 6g_{mn}$, the cone metric takes the standard form

$$ds_X^2 = dr^2 + r^2 ds_Y^2 \,, \tag{1.54}$$

where r is a radial coordinate. The simplest harmonic function on X is one that depends only on the radial coordinate r, and the most general such function is a linear combination of a constant function and $1/r^6$, as in eight-dimensional flat space: so

$$H = \alpha + \frac{(2L)^6}{r^6} \,, \tag{1.55}$$

for some constants α and L. If we want the solution (1.53) to asymptote to $\mathbb{R}^{2,1} \times X$ at large r, we should take $\alpha = 1$.

The solution (1.53) describes a stack of M2-branes extended along the 012 directions and located at the tip of the cone at $r = 0$. One can relate L to the number of branes and the Planck length by a modified version of Gauss's Law: M2-branes are electric sources for F_4 with charge equal to their tension $\tau_{\rm M2} = 2\pi/(2\pi\ell_p)^3$, so the integral of $*_{11}F_4$ over any seven-dimensional Gaussian surface enclosing the branes should equal N times $2\kappa_{11}^2 \tau_{\mathrm{M2}} = (2\pi \ell_p)^6$. The extra factor of $2\kappa_{11}^2$ in this formula comes from the normalization of the action (1.51). Choosing the Gaussian surface to be a section through the cone at a fixed radius r, we obtain

$$\int_Y *_{11} F_4 = -\int_Y *_8 dH = 384 L^6 \,\mathrm{Vol}(Y)\,. \tag{1.56}$$

Therefore

$$\left(\frac{L}{\ell_p}\right)^6 = \frac{\pi^6 N}{6\,\mathrm{Vol}(Y)}\,. \tag{1.57}$$

The background (1.53) simplifies further if we look close to the stack of M2-branes. Indeed, at small r one can neglect the constant α in (1.55). Setting $\alpha = 0$ still yields a solution to the equations of motion because α was allowed to be arbitrary in (1.55). Close to the branes, the metric becomes

$$ds^2 = \frac{r^4}{(2L)^4} dx_\mu dx^\mu + \frac{(2L)^2}{r^2} dr^2 + (2L)^2 ds_Y^2\,. \tag{1.58}$$

Changing variables from r to a new radial coordinate $\rho = 2\log(r/2L)$, this expression becomes

$$ds^2 = e^{2\rho} dx_\mu dx^\mu + L^2 d\rho^2 + (2L)^2 ds_Y^2\,. \tag{1.59}$$

In the first two terms one recovers the metric (1.9) on AdS_4. The metric near a stack of N coincident M2-branes at the tip of a Calabi-Yau cone is therefore a direct product between AdS_4 with radius L and the base Y of the cone with radius $2L$.

For now we are interested only in the particular case $X = \mathbb{C}^4/\mathbb{Z}_k$, where $Y = S^7/\mathbb{Z}_k$. An important question that we have postponed so far is for what values of N and k is the $AdS_4 \times S^7/\mathbb{Z}_k$ classical background a reliable approximation to the quantum-mechanical M-theory dynamics? Quite generally, quantum effects become important in a gravitational theory when the geometry changes significantly over distances of the order of the Planck length. One could then say that since L is the only length scale in the solution (1.59), we should require it to be much larger than the Planck length. Since $\mathrm{Vol}(S^7/\mathbb{Z}_k) = \pi^4/(3k)$, from the quantization condition (1.57) we would conclude that $Nk \gg 1$. This is true, but not restrictive enough. The smallest length scale in the geometry is the length of the circle along which the $\mathbb{Z}_k$ isometry acts. The length of this circle is proportional to L/k, so we should require L/k to be much larger than the Planck length. Eq. (1.57) then gives $N \gg k^5$. This is the most restrictives range of N and k for which quantum corrections to the $AdS_4 \times S^7/\mathbb{Z}_k$ background of M-theory are suppressed.

1.2.4 Why it works

Moduli space of vacua

One reason for believing that ABJM theory is the dual to $AdS_4 \times S^7/\mathbb{Z}_k$ is that the moduli space of vacua of ABJM theory is the N-th symmetric power of $\mathbb{C}^4/\mathbb{Z}_k$. Indeed, in M-theory parallel M2-branes are BPS objects and there is no force between them, so one can take any number of M2-branes from the stack and move them around on $\mathbb{C}^4/\mathbb{Z}_k$ at no energy cost. Each such M2-brane configuration would correspond to a vacuum of the theory on coincident M2-branes where the position of an M2-brane away from the orbifold singularity would correspond to a vacuum expectation value (VEV) of a certain scalar operator. Since there are N indistinguishable M2-branes in the stack, one therefore expects that the manifold of vacua of this theory should be the Nth symmetric power of $\mathbb{C}^4/\mathbb{Z}_k$.

Type IIB brane construction

Another reason why ABJM theory is believed to be the IR fixed point of the theory on N coincident M2-branes probing $\mathbb{C}^4/\mathbb{Z}_k$ is based on a type IIB brane construction. This brane construction engineers ABJM theory and, upon T-duality and lift to M-theory, yields precisely N M2-branes probing the orbifold singularity. The type IIB brane construction consists of an NS5-brane stretching along the 012345 directions, a $(1, k)$-fivebrane[3] stretching along the 012 directions and sitting at angle $\theta = \arg(1+ik)$ in the 37, 48, and 59-planes, and N D3-branes stretched along the 0126 directions. The NS5-brane and the $(1, k)$-brane sit at fixed locations in the 6th direction, which is taken to be compact. This configuration is known to preserve 6 supercharges, corresponding to $\mathcal{N} = 3$ supersymmetry from point of view of the 3-d theory on the intersection of the three types of branes. The $SO(3)_R$ symmetry corresponds to simultaneous rotations in the 345 and 789 subspaces.

At short distances, on each of the two D3-brane segments between the fivebranes there is a $U(N)$ $\mathcal{N} = 4$ theory in four dimensions, which reduces to an $\mathcal{N} = 8$ theory in three dimensions, which, in $\mathcal{N} = 2$ language, consists of a vector multiplet containing the open string gauge field on the D3-brane and three chiral multiplets whose bottom components represent the position of the brane in the six transverse directions. If instead of an NS5-brane and a $(1, k)$-brane we had two parallel NS5-branes in the 012345 directions, then two chiral multiplets on each D3-brane segment would become massive, the remaining two massless ones, Φ and $\tilde{\Phi}$, as well as the scalars σ and $\tilde{\sigma}$ in the vector multiplets corresponding to the motion of the D3-branes along the NS5's. This configuration preserves $\mathcal{N} = 4$ SUSY in three dimensions, the $SO(4)_R \cong SU(2)_1 \times SU(2)_2$ R-symmetry corresponding to $SO(3)_1 \cong SU(2)_1$ rotations in the 345 subspace and $SO(3)_2 \cong SU(2)_2$ rotations in the 789 subspace.

[3]We adopt the convention where a (p, q)-fivebrane carries p units of NS5-charge and q units of D5-charge.

Turning one of the NS5-branes into a $(1,k)$-brane can be thought of as adding k D5-branes in the 012789 directions intersecting that NS5, and then separating out the D5-brane half-planes from each side of the NS5-brane by equal amounts in the 3, 4, and 5 directions. As we separate out the D5-brane half-planes, it is energetically favorable to form a fivebrane bound state carrying $(1,k)$ charge making an angle $\theta = \arg(1+ik)$ with the 3, 4, and 5 axis in the 37, 48, and 59 planes. Introducing the k D5-branes does not break any additional supersymmetry and gives rise to $4k$ D3-D5 strings, k of them connecting each D3-brane segment to each D5-brane segment. They correspond to chiral multiplets q_i and $\tilde{q}^i$ transforming in $\mathbf{N}$ and $\overline{\mathbf{N}}$ of one of the gauge groups and Q_i and $\tilde{Q}^i$ transforming in $\mathbf{N}$ and $\overline{\mathbf{N}}$ of the other gauge group, where i runs from 1 to k. Separating out the two D5-brane half-planes and forming the $(1,k)$ brane gives superpotential masses proportional to k to Φ and $\tilde{\Phi}$ and real mass terms proportional to k to the auxiliary scalars σ and $\tilde{\sigma}$ from the vector multiplets. One can think of these mass terms as arising from integrating out the fundamental and antifundamental chiral fields; these mass terms are proportional to k because there are k fundamental and antifundamental chirals of each type. When integrating out the fermions in the chiral multiplets there is also a Chern-Simons term being generated through the parity anomaly mechanism [34] with Chern-Simons coefficient k and $-k$ for the two gauge fields. (Each fermion generates a CS term with coefficient 1/2 times the sign of the fermion's mass [34], and for each gauge group there are $2k$ such fermions that we integrate out.) The separation of the D5-brane half-planes is what breaks SUSY from $\mathcal{N}=4$ to $\mathcal{N}=3$. The remaining $SO(3)_R$ symmetry corresponds to simultaneous $SO(3)$ rotations in the 345 and 789 subspaces.

The degrees of freedom that we discussed so far are two vector multiplets with Chern-Simons levels k and $-k$ and two adjoint chirals with superpotential mass terms $\delta W \sim k\Phi^2 - k\tilde{\Phi}^2$. In addition to the vector multiplets, at low energies we also have massless excitations consisting of strings connecting the two segments of D3-brane across the NS5 and $(1,k)$-branes. The strings stretching across the NS5-brane give rise to the $\mathcal{Z}^1$ and $\mathcal{W}_1$ chiral multiplets, while the strings stretching across the $(1,k)$-brane give rise to $\mathcal{Z}^2$ and $\mathcal{W}_2$. Under $SO(3)_R$, $\mathcal{Z}^1$ and $\mathcal{W}^{\dagger 1}$ get mixed together, and so do $\mathcal{Z}^2$ and $\mathcal{W}^{\dagger 2}$. These bifundamental fields have the standard superpotential interactions $\delta W \sim \mathrm{tr}(\mathcal{W}_A \tilde{\Phi} \mathcal{Z}^A) + \mathrm{tr}(\mathcal{Z}^A \Phi \mathcal{W}_A)$. Integrating out Φ and $\tilde{\Phi}$ at low energies produces a quartic superpotential proportional to $1/k$, as in (1.43). The R-symmetry enhancement in the IR from $SO(3)_R$ to $SO(6)_R$ comes from the fact that at low energies it does not matter that the fivebranes are separated, so $\mathcal{W}_1$ and $\mathcal{W}_2$ become indistinguishable, and so do $\mathcal{Z}^1$ and $\mathcal{Z}^2$.

This configuration can be T-dualized along the 6th direction and lifted to M-theory. Upon T-duality, the D3-branes become D2-branes, the NS5-brane becomes a Kaluza-Klein (KK) monopole, while the $(1,k)$-brane becomes a bound state between a KK monopole and k D6-branes. When this configuration is lifted to M-theory, the D2-branes become M2-branes, and everything else lifts to Taub-NUT spaces. Such M-theory backgrounds corresponding to M2-branes located at isolated singularities of a superposition of Taub-NUT spaces were found in [35], and they typically preserve 3/16 supersymmetry, or $\mathcal{N}=3$ SUSY from a three-dimensional perspective. It can be checked [29] that the specific background corresponding to an NS5-brane and a $(1,k)$-brane in type IIB describes M2-branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ singularity in M-theory, where $\mathbb{Z}_k$ acts on the coordinates z_A of $\mathbb{C}^4$ by the identification $(z_1, z_2, w_1, w_2) \sim (z_1 e^{-2\pi i/k}, z_2 e^{-2\pi i/k}, w_1 e^{2\pi i/k}, w_2 e^{2\pi i/k})$.

There is a subtlety in the discussion of how the type IIB brane construction can be T-dualized and lifted to M-theory, because in the supergravity approximation T-duality works only for backgrounds with a $U(1)$ isometry along the T-duality direction. In reducing the M-theory background of [35] to type IIA and T-dualizing it to type IIB, one does not obtain a configuration with localized NS5 and $(1,k)$ branes in the 6th direction. The resulting type IIB backgrounds contain smeared NS5 and

$(1, k)$-branes in the 6th direction. As long as the supergravity approximation is appropriate, the smeared solution is however a good approximation to the IR dynamics of the theory on the intersection of the D3-branes with the localized fivebranes, since at very low energies it does not matter where the fivebranes are. In eleven dimensions, the supergravity approximation is reliable as long as the stack of M2-branes produces a weakly curved geometry. As discussed in the previous section, this happens as long as $N \gg k^5$.

Operator matching

A third reason why ABJM theory is thought to be dual to $AdS_4 \times S^7/\mathbb{Z}_k$ is that the correspondence between fields on the gravity side and operators in the field theory is at least partly understood. The operators that correspond to fluctuations of the $AdS_4 \times S^7/\mathbb{Z}_k$ background are gauge invariant single-trace operators. A subclass of such operators consists of the chiral primaries, which are in one-to-one correspondence with homogeneous holomorphic functions on the $\mathbb{C}^4/\mathbb{Z}_k$ cone [36]. The homogeneous holomorphic functions on $\mathbb{C}^4/\mathbb{Z}_k$ are of the form $z_1^{a_1} z_2^{a_2} w_1^{b_1} w_2^{b_2}$ with the condition that $b_1 + b_2 - a_1 - a_2 = mk$ for some integer m that should be imposed in order to make these functions invariant under the $\mathbb{Z}_k$ action. The exponents a_A and b_A should of course be nonnegative integers.

Let us focus on the $U(1) \times U(1)$ gauge theory. It is tempting to say that the functions $z_1^{a_1} z_2^{a_2} w_1^{b_1} w_2^{b_2}$ introduced above should correspond to operators of the form $\mathcal{Z}_1^{a_1} \mathcal{Z}_2^{a_2} \mathcal{W}_1^{b_1} \mathcal{W}_2^{b_2}$, but it is not hard to see that unless $b_1 + b_2 = a_1 + a_2$ this operator is not gauge-invariant. In addition to the $\mathcal{W}_A$ and $\mathcal{Z}^A$ fields, the gauge theory contains other operators that are important in constructing low-dimension chiral operators. These are called monopole operators and they create states with magnetic flux through a two-sphere surrounding the insertion point. The monopole operators that can be used to construct operators corresponding to homogeneous holomorphic functions on the moduli space are the "diagonal" ones T^m. These operators create m units of magnetic flux through each $U(1)$ gauge group. We have seen above that on the moduli space we must have $A_m = \tilde{A}_m$, which is why only diagonal monopole operators are relevant.

A monopole operator T^m has $\int F = \int \tilde{F} = 4\pi m$, so the Chern-Simons term in the action becomes

$$\frac{k}{4\pi}\int\left(A\wedge F-\tilde{A}\wedge\tilde{F}\right)=km\int A-km\int\tilde{A}\,. \tag{1.60}$$

This action looks like that of a point particle with charge km under the first gauge group and $-km$ under the second one. Recalling that the $\mathcal{Z}_A$ have gauge charges $(-1,1)$ and the $\mathcal{W}_A$ have gauge charges $(1,-1)$, it is therefore possible to construct operators of the form $T^{-m}\mathcal{Z}_1^{a_1}\mathcal{Z}_2^{a_2}\mathcal{W}_1^{b_1}\mathcal{W}_2^{b_2}$, which can be made gauge invariant provided that $b_1+b_2-a_1-a_2=mk$. These are all the operators corresponding to holomorphic functions on $\mathbb{C}^4/\mathbb{Z}_k$.

Since we introduced the monopole operators, it is worth noting that they play an important role in the supersymmetry enhancement from $\mathcal{N}=6$ to $\mathcal{N}=8$ when $k=1$ or 2. When $k>2$ the R-symmetry is $SO(6)\cong SU(4)$, so there should be an R-symmetry current transforming in the adjoint representation of $SU(4)$. To write down the 15 linearly-independent currents, it is useful to combine the bottom components W_A and Z^A of $\mathcal{W}_A$ and $\mathcal{Z}^A$, respectively, into the four component vector

$$Y^A=\left(Z^1,Z^2,W^{\dagger 1}W^{\dagger 2}\right)\,,\qquad Y_A^{\dagger}=\left(Z_1^{\dagger},Z_2^{\dagger},W_1,W_2\right) \tag{1.61}$$

because now Y^A transforms in the fundamental representation of $SU(4)_R$, and moreover all the Y^A have the same charges under the two $U(1)$ gauge groups. The $SU(4)$

R-symmetry current is then

$$J_m{}^A{}_B \sim Y^A \partial_m Y_B^\dagger - (\partial_m Y^A) Y_B^\dagger + \text{fermionic terms}\,. \tag{1.62}$$

When $k = 1$ or 2 it should be possible to construct an R-symmetry current transforming in the adjoint representation of $SO(8)$, so there should be 28 linearly-independent currents. 15 of them are the ones in (1.62). One of them is the $U(1)_b$ symmetry current under which Y^A and $Y_A^\dagger$ have opposite charges:

$$J_m \sim Y^A \partial_m Y_A^\dagger - (\partial_m Y^A) Y_A^\dagger + \text{fermionic terms}\,. \tag{1.63}$$

The remaining 12 are of the form

$$J_m{}^{AB} \sim Y^A \partial_m Y^B - (\partial_m Y^A) Y^B + \text{fermionic terms} \tag{1.64}$$

and their complex conjugates. While (1.62) and (1.63) are gauge-invariant expressions, the currents in (1.64) need to be combined with monopole operators with two indices in each gauge group in order to produce a gauge-invariant expression. As we discussed, at CS level k monopole operators that create m units of flux through each $U(1)$ gauge group have gauge charges $(km, -km)$. Monopole operators with gauge charge $(2, -2)$ needed to make (1.64) gauge-invariant therefore exist only when $k = 1$ or 2. The extra conserved currents (1.64) therefore simply do not exist when $k > 2$ and there is only $\mathcal{N} = 6$ SUSY in that case.

1.2.5 Other M2-brane theories

Starting from the type IIB brane construction, there is a natural generalization of ABJM theory that will be relevant in Chapter 4. The brane construction for ABJM theory consists of N D3-branes in the 0126 directions and two $(1, q_a)$-fivebranes, with

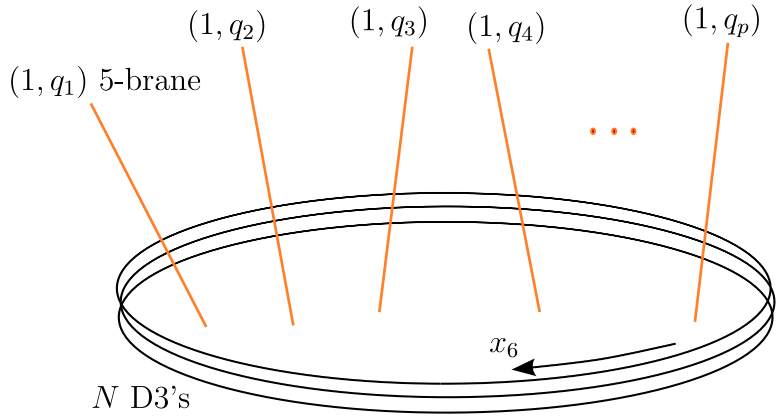

Figure 1.1: A schematic picture of the brane construction. The N D3-branes span the 0126 direction, and the $(1, q_a)$ 5-branes span the 012 directions as well as the lines in the 37, 48, and 59 planes that make angles $\theta_a = \arg(1 + iq_a)$ with the 3, 4, and 5 axes, respectively. The three-dimensional $\mathcal{N} = 3$ theories considered in this dissertation live on the 012 intersection of these branes.

$q_1 = 0$ and $q_2 = k$, filling the 012 directions and sitting at angles in the 37, 48, and 59 planes that are determined by their brane charges and the requirement that the whole brane construction should preserve $\mathcal{N} = 3$ supersymmetry. The generalization of this construction is to have an arbitrary number p of $(1, q_a)$-fivebranes (see figure 1.1). To preserve supersymmetry, each $(1, q_a)$ brane should make an angle $\theta_a = \arg(1 + iq_a)$ with the 3, 4, and 5 axes in the 37, 48, and 59 planes, respectively. It is possible to generalize this construction further and allow the brane charges to be (p_a, q_a) with $p_a \neq 1$, but in that case the field theory description is more complicated [37].

The field content of the IR limit of these field theories can be read off from the brane construction, just like for ABJM theory. There are p $U(N)$ gauge groups corresponding to the p D3-brane segments between the fivebranes, and p pairs of bifundamental fields A_a and B_a corresponding to strings connecting the D3-branes and that are stretched across the $(1, q_a)$-brane. The quiver diagram resembles a necklace (see figure 1.2).

In addition to the quiver diagram, we should specify the Chern-Simons levels and the superpotential. As in the brane construction for ABJM theory, one can think

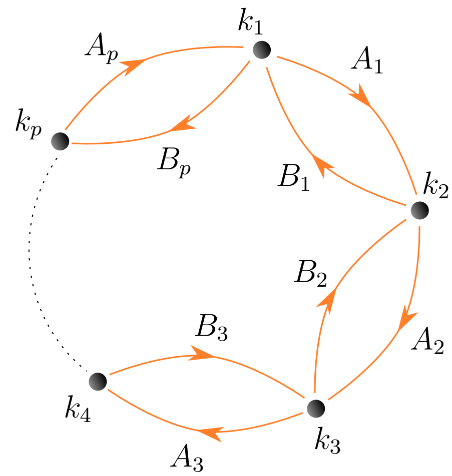

Figure 1.2: Necklace quiver diagrams for $U(N)^p$ Chern-Simons gauge theories.

of a $(1, q_a)$ brane as a bound state between an NS5-brane and q_a D5-branes. The Chern-Simons levels arise from integrating out the fermions in the chiral multiplets corresponding to the D3-D5 strings. For each gauge group a, there are q_{a+1} such strings from the $(1, q_{a+1})$ brane and q_a strings from the $(1, q_a)$ brane. Thinking carefully about the signs of the masses of these fermions, one concludes that the resulting CS levels are

$$k_a = q_{a+1} - q_a \,. \tag{1.65}$$

In particular, the CS levels satisfy $\sum_{a=1}^{p} k_a = 0$. The superpotential is just a generalization of the ABJM superpotential in eq. (1.43):

$$W = -\sum_{a=1}^{p} \frac{2\pi}{k_a} (B_{a-1}A_{a-1} - A_a B_a)^2 \,. \tag{1.66}$$

For $N = 1$ the moduli space of each 3-d model was calculated in [31, 38] and shown to be given by a certain hyper-Kähler cone in four complex dimensions [39–41]. It was therefore conjectured that such a model describes the low-energy dynamics of N coincident M2-branes placed at the tip of this cone.

For a general set of Chern-Simons levels such a $p > 2$ gauge theory has $\mathcal{N} = 3$ superconformal invariance, but in the special case where p is even and the CS levels are $(k, -k, k, -k, \ldots)$ the supersymmetry is enhanced to $\mathcal{N} = 4$ [30, 38, 42].

1.2.6 The number of degrees of freedom on coincident M2-branes

In section 1.1.7 we computed two quantities that measure the number of degrees of freedom in a field theory dual to AdS_4: the thermal free energy F_T and the free energy F of the Euclidean theory on S^3. Both of these quantities were proportional to the ratio L^2/κ_4^2. Let's end the discussion of M2-brane theories by expressing this ratio in terms of the number N of coincident M2-branes at the tip of a Calabi-Yau cone X and the volume of the base Y of this cone.

It is possible to relate the effective four-dimensional gravitational constant κ_4 to the gravitational constant in eleven dimensions by performing the integral over the compact directions in the M-theory action (1.51):

$$\frac{1}{2\kappa_4^2} = \frac{1}{2\kappa_{11}^2}(2L)^7 \,\mathrm{Vol}(Y)\,. \tag{1.67}$$

Combining this relation with the quantization condition (1.57), one obtains

$$\frac{L^2}{\kappa_4^2} = \frac{\pi N^{3/2}}{6\sqrt{6\,\mathrm{Vol}(Y)}}\,. \tag{1.68}$$

The thermal free energy and the free energy of the Euclidean theory on S^3 become

$$F_T = -\frac{2^{7/2}\pi^4 N^{3/2}}{3^{9/2}\sqrt{\mathrm{Vol}(Y)}}T^3 V_d\,, \qquad F = N^{3/2}\sqrt{\frac{2\pi^6}{27\,\mathrm{Vol}(Y)}}\,. \tag{1.69}$$

For ABJM theory, we have further that $\mathrm{Vol}(Y) = \mathrm{Vol}(S^7)/k = \pi^4/(3k)$, and

$$F_T = -\frac{2^{7/2}}{81}\pi^2 k^{1/2} N^{3/2} T^3 V_d \,, \qquad F = \frac{\pi\sqrt{2}}{3} k^{1/2} N^{3/2} \,. \tag{1.70}$$

The $N^{3/2}$ scaling of the thermal free energy of coincident M2-branes was known even before the discovery of the AdS/CFT duality [43]. Until recently, this $N^{3/2}$ scaling had been quite puzzling because one expected the effective field theory on coincident M2-branes to involve $N \times N$ matrices whose number of degrees of freedom scales as N^2. The discovery of ABJM and related theories did not solve this puzzle at first, as these theories were indeed written in terms of $N \times N$ matrices. However, as one takes N to infinity, the Chern-Simons level k should be held fixed, which means that the effective coupling N/k also becomes large. At strong coupling it is possible that the field theory intuition that gave N^2 may break down.

It turns out that it is possible to compute the free energy on S^3 exactly in the field theory [44] using the method of localization originally developed by Pestun for four-dimensional gauge theories [45]. Quite remarkably, this computation does reproduce the $N^{3/2}$ scaling, as was noticed initially [23] for ABJM theory and later on in [46] through a different method that applies to all necklace quiver gauge theories with $\mathcal{N} = 3$ supersymmetry introduced in the previous section. In addition to the $N^{3/2}$ dependence, the calculation of F also reproduces the $1/\sqrt{\mathrm{Vol}(Y)}$ dependence on the volumes of the internal spaces Y. Chapter 4 of this dissertation contains this calculation. Chapter 5 contains similar checks but now in theories with $\mathcal{N} = 2$ supersymmetry, where we can also study RG flows between various field theories.

Chapter 2

The gravity dual of a *p*-wave superconductor

This chapter is an edited version of ref. [47] written in collaboration with Steve Gubser.

2.1 Introduction

This chapter explores a way in which black holes in asymptotically AdS space can break a gauge symmetry spontaneously. An example of this sort was already mentioned in section 1.1.6 in the context of the Abelian Higgs model coupled to gravity with a negative cosmological constant. In that example the symmetry breaking occurs because there are solutions of the equations of motion where a charged scalar field has a nonvanishing profile outside the black hole horizon. The superconducting layer of charged scalar hair floats above the horizon because the horizon is also charged. Electrostatic repulsion overcomes the gravitational attraction that ordinarily would suck the superconducting layer into the horizon. If the spacetime were asymptotically flat, then (barring some special interactions such as considered in [48]) one expects that electrostatic repulsion would cause the superconducting layer to be blown off to infinity. But asymptotically anti-de Sitter geometries prevent this. Massive particles, no matter how strongly repelled from a horizon, cannot reach the boundary of anti-de Sitter space. So they instead condense near the horizon, where "near" means that the field profile is normalizable, carrying finite charge if the horizon is finite, or finite charge density if the horizon has infinite extent. An analogy with the classic two-fluid model of superconductors is possible: the charged horizon describes the normal component, and the condensate above it is the superconducting component. See figure 2.1. In this analogy, it is important to recall that in the gauge-string duality [1–3], the extra dimension r is not an additional flat dimension transverse to the sample; instead, it is a way of organizing energy scales in the dual field theory, which is strictly $2+1$-dimensional and non-gravitational. Thus, although the condensate is "above" the horizon in the gravity picture, it interpenetrates the normal state in the field theory picture.

The moduli space of black hole solutions includes Reissner-Nordström anti-de Sitter black holes (hereafter RNAdS), which describe the normal state,[1] joining continuously onto a branch of symmetry-breaking solutions. The simplest argument supporting this picture is based on studying linear perturbations of the charged field around an RNAdS solution. They obey an equation of the form

$$(\Box - m_{\text{eff}}^2)\psi = 0\,, \tag{2.1}$$

where $\Box$ is an appropriate covariant wave operator and

$$m_{\text{eff}}^2 = m^2 + g^{tt} q^2 \Phi^2\,. \tag{2.2}$$

[1] It has been suggested [49,50] that RNAdS black holes are dual to a close analog of the pseudogap state of high T_c materials. In the context of our constructions, this does not seem quite right, because the fraction of charge in the condensate goes to zero near T_c, scaling as $T_c - T$, whereas the transition from superconductivity to the pseudogap state appears to take place while this fraction is finite and non-zero.

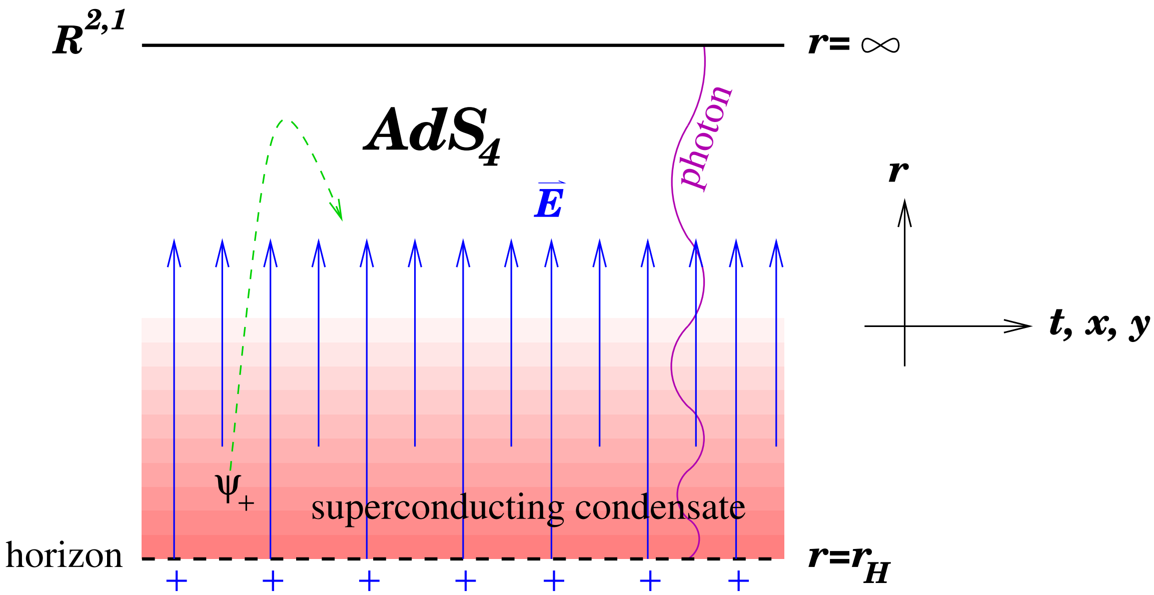

Figure 2.1: A superconducting condensate floats above a black hole horizon because of a balance of gravitational and electrostatic forces. The condensate carries a finite fraction of the total charge density, so there is more electric flux above the condensate than there is right at the horizon. A massive charged particle, labeled ψ_+, may be driven upward by the electrostatic force, but because of the warped geometry of AdS_4, its trajectory cannot reach the boundary. So ψ_+ must participate in the condensate if it doesn't fall into the horizon. The frequency-dependent conductivity can be found by calculating an on-shell amplitude for a photon propagating straight down into the geometry.

Here q is the charge of a quantum of the charged bosonic field ψ. Φ is the electrostatic potential, which vanishes at the horizon but grows quickly outside it if the electric field is strong. The metric component g^{tt} is negative in the conventions we use, and it diverges to $-\infty$ at the horizon, so (2.2) implies that ψ is tachyonic near the horizon if q is big enough and m is small enough, provided also that the horizon carries sufficient charge.[2] It is a matter of calculation to determine when (2.1) admits a static solution. When it does, one may reasonably assume that it signifies the joining of a branch of symmetry breaking solutions onto the RNAdS solutions. And one may calculate a critical temperature T_c where the joining occurs. It does not necessarily follow that T_c is the temperature of a second order phase transition: it could be that the solutions which only slightly break the symmetry are thermodynamically disfavored, and that a first order transition to solutions with finite symmetry breaking occurs at a different temperature.

A similar example of spontaneous symmetry breaking near a black hole horizon was proposed in [51] in the case of the Einstein-Yang-Mills action with a negative cosmological constant:

$$S = \frac{1}{2\kappa^2} \int d^4x \left[R - \frac{1}{4}(F^a_{\mu\nu})^2 + \frac{6}{L^2} \right] , \tag{2.3}$$

where $F^a_{\mu\nu}$ is the field strength of an $SU(2)$ gauge field. In [51], a $U(1)$ subgroup of $SU(2)$ was regarded as the gauge group of electromagnetism,[3] and off-diagonal gauge bosons, which are charged under this $U(1)$, were observed to condense outside the black hole horizon at low enough temperatures. The action (2.3) is almost completely

[2]Actually, the most commonly considered cases have $m^2 < 0$ in the case of scalars, or $m = 0$ for non-abelian gauge bosons. The argument about massive particles' trajectories never reaching the boundary of anti-de Sitter space then no longer holds up, but it is replaced by standard notions of boundary conditions in anti-de Sitter space which again lead to normalizable condensates.

[3]More precisely, the boundary theory has a global $SU(2)$ symmetry, and adding electromagnetism means weakly gauging this $U(1)$ in the boundary theory. By contrast, the gauging of the full $SU(2)$ symmetry in the gravity theory encodes aspects of the $SU(2)$ current algebra dynamics in the boundary theory.

dictated at the two-derivative level by local diffeomorphism symmetry and $SU(2)$ gauge symmetry.

Part of the goal in this chapter is to construct superconducting extrema of (2.3) different from the ones in [51] and argue that they are thermodynamically preferred over the ones of [51]. The difference between the two types of superconducting solutions is in what components of the gauge field condense and how much symmetry is preserved by the condensate. The bulk $SU(2)$ field in the action 2.3 corresponds to an $SU(2)$ conserved current in the boundary theory J_m^a where m is a spacetime index and a is an adjoint $SU(2)$ index. Let's assume that the $U(1)$ of electromagnetism corresponds to the 3rd isospin direction. Turning on a nonzero charge density in this $U(1)$ breaks the $SU(2)$ symmetry in the bulk to an $SO(2)$ that acts by rotations in the 12 isospin plane and the Lorentz symmetry $SO(2,1)$ also to an $SO(2)$ of rotations in the xy plane. In [51] this remaining $SO(2) \times SO(2)$ symmetry was broken to the diagonal $SO(2)$ by nonzero expectation values of equal magnitudes of J_x^1 and J_y^2. We will refer to this state and the dual gravity background as having $p+ip$-wave symmetry. In this chapter, the $SO(2) \times SO(2)$ symmetry is broken completely by a nonzero expectation value for just J_x^1, which would correspond to a superconducting state with p-wave symmetry.

In [51] it was shown that there is a second order transition, with mean field theory exponents, between a non-superconducting state at high temperatures, where all the charge is in the normal component, and the $p+ip$ superconducting state at low temperatures. In addition to making a comparison of the free energies of the normal and superconducting states, one must also ask whether the symmetry breaking solution is stable under small perturbations. At least for a certain range of parameters, we will show in section 2.4.2 that the solutions of [51] are unstable against a perturbation that seems likely to turn them into p-wave solutions of the form described in section 2.2. We have not yet found an unstable perturbation of the

p-wave solutions.

The outline of the rest of this chapter is as follows. In section 2.2 we describe the background solutions of interest. In section 2.3 we study the electromagnetic response, along the lines of [12]. We find a frequency-dependent conductivity that depends strongly on the polarization of the applied electric field. The low-frequency behavior is suggestive of quasi-particle excitations whose dissipative mechanisms are entirely due to finite-temperature effects. In section 2.4 we provide numerical evidence that the p-wave backgrounds are stable against small perturbations that turn on a $p + ip$ gap. In section 2.4.2 we provide numerical evidence that the $p + ip$-wave backgrounds of [11] are unstable against small perturbations that turn them into the p-wave backgrounds described in section 2.2. Our numerical explorations are far from covering the full range of parameters, but the simplest scenario consistent with them is that $p + ip$-wave backgrounds are always unstable, and that p-wave backgrounds represent the thermodynamically preferred phase for T less than a critical temperature T_c.

2.2 The backgrounds

We follow the conventions of [51] for the metric and gauge field, except for denoting the spatial directions of $\mathbb{R}^{2,1}$ as x and y rather than x^1 and x^2. We will restrict attention to the limit of large g, where the metric is simply AdS_4-Schwarzschild,

$$ds^2 = \frac{r^2}{L^2}\left[-\left(1 - \frac{r_H^3}{r^3}\right)dt^2 + dx^2 + dy^2\right] + \frac{L^2}{r^2}\frac{dr^2}{1 - r_H^3/r^3}\,. \tag{2.4}$$

The gauge field ansatz is

$$A = \Phi(r)\tau^3 dt + w(r)\tau^1 dx\,. \tag{2.5}$$

It is convenient to define

$$\tilde{\Phi} = gL^2\Phi \,, \qquad \tilde{w} = gL^2 w \,. \tag{2.6}$$

If one also fixes a scale by setting $r_H = 1$, then the relevant Yang-Mills equations are

$$\begin{aligned} \tilde{\Phi}'' + \frac{2}{r}\tilde{\Phi}' - \frac{1}{r(r^3-1)}\tilde{w}^2\tilde{\Phi} = 0 \,, \\ \tilde{w}'' + \frac{1+2r^3}{r(r^3-1)}\tilde{w}' + \frac{r^2}{(r^3-1)^2}\tilde{\Phi}^2\tilde{w} = 0 \,, \end{aligned} \tag{2.7}$$

where primes denote d/dr. These equations are similar to (B4) of [51] because the ansatz (2.5) is also similar. But in [51], the symmetry breaking term takes the form $w(\tau^1 dx + \tau^2 dy)$, which corresponds to wrapping the part of the gauge group generated by τ^3—call it $U(1)_3$—around the rotational symmetry group $SO(2)$ that acts on x and y. Choosing instead $w(\tau^1 dx - \tau^2 dy)$ corresponds to wrapping $U(1)_3$ the other way around $SO(2)$. We think of (2.5) heuristically as a superposition of the two different wrapping solutions, in the same way that linearly polarized light is a superposition of left-handed and right-handed polarizations. This analogy has limited utility because the Yang-Mills equations governing the different "polarizations" are non-linear.

In addition to breaking $U(1)_3$, the condensate $w\tau^1 dx$ picks out the x direction as special. Therefore, if back-reaction of the Yang-Mills field on the metric were included, then we would not expect to be able to set $g_{xx} = g_{yy}$, as we did in (2.4). The wrapping condensate $w(\tau^1 dx + \tau^2 dy)$ is simpler in this regard, because although it breaks $U(1)_3$ and $SO(2)$ separately, it preserves a diagonal subgroup which makes the stress tensor isotropic in the x and y directions.

The temperature of the horizon is

$$T = \frac{3}{4\pi L^2} \,, \tag{2.8}$$

where, as before, we have set $r_H = 1$. The total charge density ρ is proportional to the τ^3 part of the electric field at infinity: if

$$\Phi = p_0 + \frac{p_1}{r} + O\left(\frac{1}{r^2}\right) , \tag{2.9}$$

then

$$\rho = -\frac{p_1}{L\kappa^2} . \tag{2.10}$$

The charge density ρ_n in the normal component is proportional to the τ^3 part of the electric field at the horizon: if

$$\Phi = \Phi_1(r-1) + O[(r-1)^2] , \tag{2.11}$$

then

$$\rho_n = \frac{\Phi_1}{L\kappa^2} . \tag{2.12}$$

Far-field and near-horizon expansions for the rescaled fields $\tilde{\Phi}$ and $\tilde{w}$ take the form

$$\text{Far field:} \qquad \begin{cases} \tilde{\Phi} = \tilde{p}_0 + \dfrac{\tilde{p}_1}{r} + \ldots \\[2ex] \tilde{w} = \dfrac{\tilde{W}_1}{r} + \ldots , \end{cases} \tag{2.13}$$

$$\text{Near horizon:} \qquad \begin{cases} \tilde{\Phi} = \tilde{\Phi}_1(r-1) + \ldots \\[2ex] \tilde{w} = \tilde{w}_0 + \tilde{w}_2(r-1)^2 + \ldots , \end{cases} \tag{2.14}$$

and it is convenient to introduce rescaled versions of the total and normal component charge densities:

$$\begin{aligned} \tilde{\rho} &\equiv \kappa^2 g L^2 \rho = -\frac{\tilde{p}_1}{L} , \\ \tilde{\rho}_n &\equiv \kappa^2 g L^2 \rho_n = \frac{\tilde{\Phi}_1}{L} . \end{aligned} \tag{2.15}$$

We also define the superconducting charge density as $\tilde{\rho}_s = \tilde{\rho} - \tilde{\rho}_n$. A natural choice

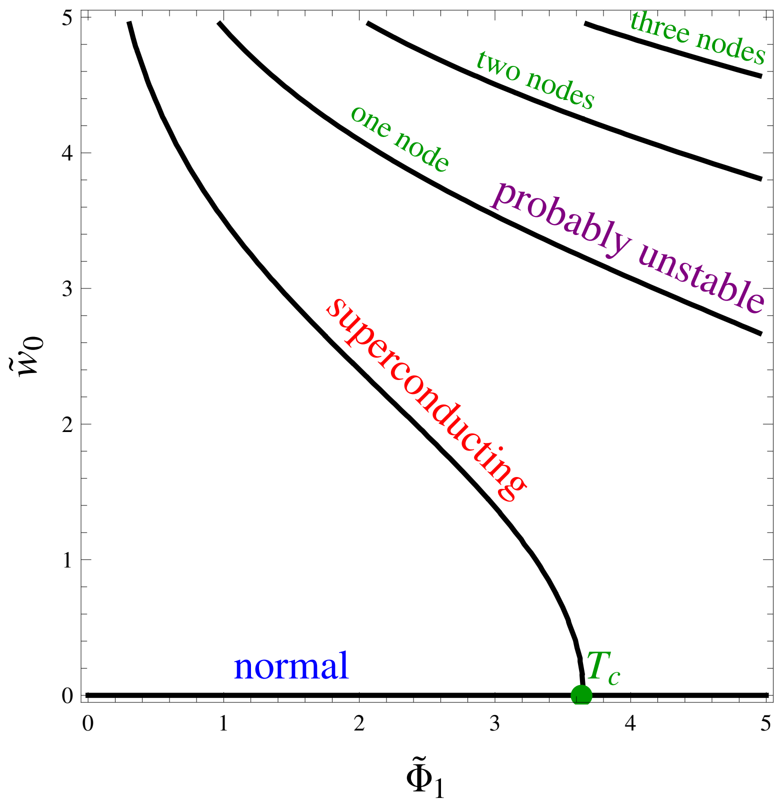

Figure 2.2: Each point along the contours plotted represents a solution to the non-linear boundary value problem specified by (2.7), (2.13), and (2.14). Points on the line labeled "normal" are RNAdS solutions, and if charge density is held fixed, temperature rises as one moves to the left. Points on the curve labeled "superconducting" break the abelian gauge symmetry generated by $U(1)_3$. Points on the other curves also break the abelian gauge symmetry but are expected to be unstable. The point where the superconducting solutions join onto the normal solutions is labeled T_c because the simplest scenario is for there to be a second order phase transition at this point.

of order parameter is $\tilde{W}_1$, because the $SU(2)$ currents J^a_m dual to the gauge bosons A^a_μ have a symmetry-breaking expectation value

$$\langle J^a_i \rangle \propto \tilde{W}_1 \delta^1_i \delta^a_1 \,. \tag{2.16}$$

There is a one-parameter family of "allowed" solutions to the Yang-Mills equations (2.7), where allowed means that the far-field and near-horizon asymptotic forms, (2.13) and (2.14), are satisfied. Thus (2.7), (2.13), and (2.14) specify a non-linear boundary value problem. To understand why there is only a one-parameter family of solutions, let us examine the far-field and near-horizon expansions separately. The generic solution to (2.7) includes a constant term $\tilde{W}_0$ in the far-field expansion of $\tilde{w}$, and this is disallowed because it corresponds to deforming the field theory lagrangian by some multiple of J_1^1. Another way to describe why $\tilde{W}_0$ is disallowed is that if it is non-zero, then the condensate is not normalizable. In the expansion (2.13), all three parameters shown explicitly are independent, which matches a simple counting argument: four integration constants (for two second order differential equations) minus one (for the constraint $\tilde{W}_0 = 0$) equals three. Requiring that the gauge field is smooth and well-defined at the horizon leads to the expansions (2.14). The parameters $\tilde{\Phi}_1$ and $\tilde{w}_0$ are independent, but $\tilde{w}_2$ and higher coefficients can be determined in terms of them. Having only two independent parameters in the near-horizon expansion (i.e. $\tilde{\Phi}_1$ and $\tilde{w}_0$) means that there are two constraints at the horizon. Generically, these two constraints will be independent of the far-field constraint $\tilde{W}_0 = 0$. So there are three constraints total on four integration constants, leading indeed to a one-parameter family of solutions. At special points, one of the horizon constraints may become degenerate with the far-field constraint, and this is when one finds two branches of solutions joining together.

Solutions to the boundary value problem discussed in the previous paragraph can be generated using a "shooting" procedure. First one guesses numerical values of $\tilde{\Phi}_1$ and $\tilde{w}_0$. Then one uses the near-horizon expansion (2.14) to seed a finite-element differential equation solver, such as Mathematica's `NDSolve`. Next one matches the numerical solution to the far-field expansion (2.13), augmented by a constant term $\tilde{W}_0$. In this way one finds $\tilde{W}_0$ as a function of $\tilde{\Phi}_1$ and $\tilde{w}_0$. The zeroes of this function correspond to the solutions of the boundary value problem: see figure 2.2. Hereafter we restrict attention to solutions where $\tilde{w}(r)$ has no nodes. There are additional solu-

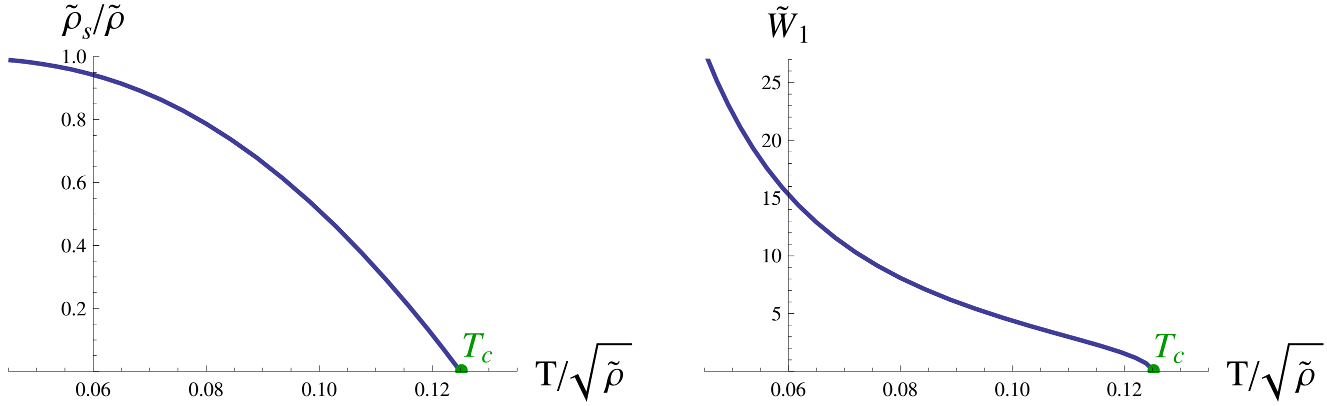

Figure 2.3: The fraction $\tilde{\rho}_s/\tilde{\rho}$ of the charge carried by the superconducting condensate and the order parameter $\tilde{W}_1$ are plotted against the rescaled temperature $T/\sqrt{\tilde{\rho}}$. At T_c, $\tilde{\rho}_s/\tilde{\rho}$ vanishes linearly, while $\tilde{W}_1$ vanishes as $\sqrt{T_c - T}$.

tions with nodes, but one generally expects them to be thermodynamically disfavored, because radial oscillations in $\tilde{w}$ cost energy.

Thermodynamic quantities for solutions along the node-free symmetry-breaking branch labeled "superconducting" in figure 2.2 are plotted in figure 2.3.

2.3 Electromagnetic perturbations

The quantity of primary interest in understanding the electromagnetic response is the conductivity,

$$\sigma_{ij}(\omega) = \frac{i}{\omega} G^R_{ij}(\omega, 0)\,, \tag{2.17}$$

where

$$G^R_{mn}(\omega, \vec{k}) = -i \int d^3x\, e^{i\omega t - i\vec{k}\cdot\vec{x}} \theta(t) \langle [J_m(t, \vec{x}), J_n(0,0)] \rangle \tag{2.18}$$

is the retarded Green's function of the electromagnetic current J_m. The angle brackets in (2.18) denote expectation values at finite temperature, namely

$$\langle \mathcal{A} \rangle \equiv \frac{1}{Z} \operatorname{tr} e^{-\beta H} \mathcal{A}\,, \qquad Z \equiv \operatorname{tr} e^{-\beta H} \tag{2.19}$$

for any operator $\mathcal{A}$. The hermitian part of σ_{ij} is dissipative, while the anti-hermitian part is reactive.[4] According to a spectral decomposition, the hermitian part of σ_{ij} should be positive semi-definite. To see this, first note that the spacetime dependence of the hermitian operators $J_i(t,\vec{x})$ is found through

$$J_i(t,\vec{x}) = e^{iHt - i\vec{P}\cdot\vec{x}} J_i(0,0) e^{-iHt + i\vec{P}\cdot\vec{x}} . \tag{2.20}$$

Introducing a complete set of states between the two operators in (2.18) and integrating over t and $\vec{x}$ one obtains

$$G^R_{ij}(\omega,0) = \frac{1}{Z}\sum_{n,m}(2\pi)^2\delta^{(2)}(\vec{P}_{nm})J_i^{nm}J_j^{mn}\frac{e^{-\beta E_n} - e^{-\beta E_m}}{\omega + E_{nm} + i0} , \tag{2.21}$$

where

$$J_i^{nm} = \langle n|J_i(0,0)|m\rangle , \qquad \vec{P}_{nm} = \vec{P}_n - \vec{P}_m , \qquad E_{nm} = E_n - E_m , \tag{2.22}$$

$\vec{P}_n$ and E_n being the eigenvalues of $\vec{P}$ and H in the state $|n\rangle$. Plugging (2.21) into (2.17) one straightforwardly obtains

$$\begin{aligned} \frac{1}{2}(\sigma_{ij} + \sigma_{ji}^*) &= \sum_{n,m} J_i^{nm}J_j^{mn}A_{nm} , \\ A_{nm} &= \frac{1}{Z}(2\pi)^3\delta^{(2)}(\vec{P}_{nm})\delta(\omega + E_{nm})e^{-\beta\frac{E_n+E_m}{2}}\frac{\sinh\frac{\beta\omega}{2}}{\omega} . \end{aligned} \tag{2.23}$$

[4] In the theory of AC circuits it is standard to consider the complex power $S = \int d^2x\, E_i^* j_i = \int d^2x\, E_i^*\sigma_{ij}E_j$, whose real and imaginary parts are the real and reactive powers, respectively. The real power P can therefore be expressed in terms of the hermitian part of σ_{ij} through $P = \int d^2x\, E_i^*\frac{1}{2}(\sigma_{ij} + \sigma_{ji}^*)E_j$, while the reactive power $Q = \int d^2x\, E_i^*\frac{1}{2i}(\sigma_{ij} - \sigma_{ji}^*)E_j$ corresponds to the anti-hermitian part.

Formally, $A_{nm} \geq 0$, so multiplying (2.23) by an arbitrary column vector v_j to the right and by its adjoint v_i^* to the left yields

$$v_i^* \frac{1}{2}(\sigma_{ij} + \sigma_{ji}^*) v_j = \sum_{n,m} |v_i^* J_i^{nm}|^2 A_{nm} \geq 0 \,, \tag{2.24}$$

proving that, indeed, the hermitian part of σ_{ij} is positive semi-definite.

The conductivity $\sigma_{ij}(\omega)$ characterizes the response to light of frequency ω which is incident on the superconductor in a direction normal to the $\mathbb{R}^{2,1}$ that the sample occupies. So it is perhaps intuitive that to calculate $\sigma_{ij}(\omega)$ for the black hole, one should send photons down from the boundary of AdS_4 and inquire how they are absorbed or reflected by the condensate and the horizon. More precisely, one uses the gauge-string duality to extract two-point functions from tree-level propagation of photons. The prescription for computing such Green's functions was first articulated in [2, 3]. An adaptation of it to thermal backgrounds was correctly guessed in [52] and then derived from the original prescription of [2,3] in [6] using Schwinger-Keldysh contours.[5] In the case of two-point functions, the gauge-string prescription is closely related to D-brane black hole absorption amplitudes computed in a long series of papers beginning with [53]. If one expresses an asymptotically AdS_4 background as

$$ds^2 = \frac{r^2}{L^2}(-dt^2 + dx^2 + dy^2) + \frac{L^2}{r^2} dr^2 + (\text{corrections}) \,, \tag{2.25}$$

where the terms shown explicitly are the leading large r behavior, then a complexified photon perturbation polarized in the x direction can be expanded for large r as

$$A_x = e^{-i\omega t} \left[A_x^{(0)} + \frac{A_x^{(1)}}{r} + O\left(\frac{1}{r^2}\right) \right] \,, \tag{2.26}$$

[5]Modulo some issues related to behavior near $\omega = 0$, this prescription can also be justified [7] by the fact that if one analytically continues $G_{ij}^R(\omega)$ to the upper half-plane, then at the Matsubara frequencies $\omega_n = 2\pi n T$ with $n > 0$ it agrees with the corresponding Fourier mode of the Euclidean correlator computed from the prescription proposed in [2, 3].

and the retarded Green's function is given simply by

$$G^R_{xx}(\omega, 0) = -\frac{2}{\kappa^2}\frac{A_x^{(1)}}{A_x^{(0)}}\,, \tag{2.27}$$

where $\kappa = \sqrt{8\pi G_N}$ is the gravitational coupling, and it is assumed that the photon wave-function is purely infalling at the horizon. More sophisticated examples have been discussed, for example, in [54].

In the superconducting phase of the black holes constructed using the Abelian Higgs model, $\sigma_{xx} = \sigma_{yy}$ and $\sigma_{xy} = 0$ because the order parameter is a scalar, breaking gauge invariance but not rotational invariance. There is a delta-function spike in $\mathrm{Re}\,\sigma_{xx}(\omega)$ at $\omega = 0$, and an associated pole in $\mathrm{Im}\,\sigma_{xx}(\omega)$. For non-zero ω and T not too close to T_c, $\mathrm{Re}\,\sigma_{xx}(\omega)$ is very small up to a finite frequency, which can be denoted $\omega_g = 2\Delta$ in order to evoke a comparison with BCS theory: Δ is then to be compared with the quantity denoted by the same letter in BCS, whose physical interpretation is the minimal energy of a single normal-component quasi-particle excitation. Above ω_g, $\mathrm{Re}\,\sigma_{xx}(\omega)$ rises quickly to a plateau and then asymptotes to a constant as $\omega \to \infty$. One can argue, along the lines of [10], that the delta-function spike at $\omega = 0$ had to be there because of the broken gauge invariance. But the existence of a gap is additional information, revealed by the calculations of [12] but apparently not necessitated by symmetry principles. In BCS theory, the gap arises because of a pairing mechanism of otherwise nearly free quasi-particle excitations of a Fermi surface. No such mechanism is manifest in the gravity description; instead, the simplest way to characterize the gravity calculation is that photons with frequency less than 2Δ are very unlikely to penetrate through the condensate and be absorbed by the horizon. There is clearly something in common between BCS theory and the gravitational calculation, because the horizon represents the dynamics of the uncondensed charge carriers (i.e. the normal component), and absorption of a photon with $\omega > 0$ is associated with an excitation of these carriers. The obvious difference is that the charge carriers in the gravitational calculation (or, more precisely, the charge carriers in the appropriate holographic dual description) are strongly coupled even when they are in the normal state.

The strong coupling inherent in gauge-string duals in the gravity approximation raises the appealing possibility that black hole constructions might provide useful physical analogies to the mysterious dynamics of electrons in high T_c materials that go beyond traditional ideas based on quasi-particle excitations of Fermi surfaces. But the black holes we study provide anything but a microscopic understanding of superconductivity: the gravity description is more like Landau-Ginzburg theory, and the dual field theory would be the venue for some attempt at a microscopic theory comparable to BCS.

Rather than presenting superconducting black holes as an incipient theory of high T_c, we prefer the viewpoint that they are a new theoretical laboratory, seemingly divorced from traditional perturbative concepts, but capable of exhibiting assorted phenomena reminiscent of real materials. Perhaps a sufficiently comprehensive understanding of their dynamics will suggest new ideas which can also be applied successfully to real materials.

The purpose of the present chapter is to narrow the gap between black hole constructions and interesting high T_c materials by introducing black holes with a p-wave gap. Although it is apparently a d-wave gap that controls the dynamics of the cuprates, d-wave and p-wave are similar in that excitations of the normal component can be probed using low-frequency photons.

By making the black hole charged under the gauge symmetry $U(1)_3$ generated by τ^3, we explicitly break the $SU(2)$ gauge group down to $U(1)_3$. As mentioned earlier, we interpret $U(1)_3$ as the gauge group of electromagnetism, which means that we plan to consider a weak gauging of this group in the boundary theory.[6] As discussed in the introduction, the linear response to electromagnetic probes is described by the two-point function of the $U(1)_3$ current, and in the dual black hole, this means that we want to know how linear perturbations of the τ^3 component of the gauge field propagate. We persist in choosing the spatial momenta $k_i = 0$ in the x and y directions, so the photon is directed straight down into AdS_4, as illustrated in figure 2.1.

As a warmup, we work out in section 2.3.1 the conductivity in two examples where it can be done analytically, including the normal state, where the condensate $\tilde{w}$ is set to 0. In section 2.3.2 we explain how to set up the perturbation equations in the more difficult case of a symmetry-breaking background as described in section 2.2. In sections 2.3.3 and 2.3.4 we present results of a numerical study of $\sigma_{xx}(\omega)$ and $\sigma_{yy}(\omega)$ which reveal a p-wave gap.

2.3.1 Analytical calculations

The simplest case to start with is pure AdS_4, corresponding to zero temperature, zero charge density, and no symmetry breaking. At the linearized level, the gauge coupling of $SU(2)$ doesn't enter, so we will pass to free Maxwell theory in AdS_4: that is,

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[R - \frac{1}{4} F_{\mu\nu}^2 + \frac{6}{L^2} \right] . \tag{2.28}$$

The perturbation calculation is simple because the background geometry is conformally flat:

$$ds^2 = \frac{L^2}{z^2} \left(-dt^2 + dx^2 + dy^2 + dz^2 \right) . \tag{2.29}$$

[6]This situation is analogous to the Hubbard model, which has (at least) a global $U(2)$ symmetry. The central $U(1)$ is identified as the electromagnetic gauge symmetry, but electromagnetism is not explicitly part of the model.

Equivalently, we may consider the line element (2.4) with $r_H = 0$: it is the same as (2.29) if one sets

$$z = L^2/r \,. \tag{2.30}$$

Conformal flatness is special because Maxwell's equations respect conformal symmetry. Thus the complexified photon perturbation is a plane wave:

$$A_x = e^{-i\omega(t-z)} \,. \tag{2.31}$$

We chose the plane wave solution that moves in the positive z direction: that is, it moves away from the conformal boundary at $z = 0$ and toward the degenerate Killing horizon of the Poincaré patch, at $z = \infty$. (In figure 2.1, the positive z direction is downward.) This choice means that we will wind up computing the retarded Green's function rather than the advanced one. The Green's function can be read off from an expansion near the conformal boundary:

$$A_x = e^{-i\omega t}(1 + i\omega z + \ldots) = e^{-i\omega t}\left(1 + \frac{i\omega L^2}{r} + \ldots\right) \,. \tag{2.32}$$

Comparing the last expression in (2.32) to (2.26), and using (2.17) and (2.27), one finds

$$\sigma_{xx} = \sigma_\infty \equiv \frac{2L^2}{\kappa^2} \,. \tag{2.33}$$

Because of rotation invariance, $\sigma_{yy} = \sigma_{xx}$ and $\sigma_{xy} = 0$. Hereafter we will normalize all conductivities against σ_∞ by defining

$$\tilde{\sigma}_{ij} = \frac{\sigma_{ij}}{\sigma_\infty} \,. \tag{2.34}$$

Putting (2.17), (2.27), (2.33), and (2.34) together, one has

$$\tilde{\sigma}_{xx} = -\frac{i}{\omega L^2} \frac{A_x^{(1)}}{A_x^{(0)}} \,. \tag{2.35}$$

A surprising result of [55] is that $\tilde{\sigma}_{xx} = \tilde{\sigma}_{yy} = 1$ for the AdS_4-Schwarzschild solution (2.4), for all ω and T. In the approximation where the gauge field (2.5) doesn't back-react on the metric, this result persists so long as $\tilde{w} = 0$. The quickest way to derive it is to compute directly the linearized equation of motions for complexified gauge field perturbations of the background (2.4)–(2.5): that is, $A \to A + a$, where

$$a = e^{-i\omega t} a_x^3(r) \tau^3 dx \,. \tag{2.36}$$

The result of plugging this perturbation into the linearized Yang-Mills equations is

$$\left[\partial_r^2 + \frac{2r^3+1}{r(r^3-1)} \partial_r + \frac{\omega^2 L^4 r^2}{(r^3-1)^2} \right] a_x^3 = 0 \,, \tag{2.37}$$

where we have set $r_H = 1$ as usual. Because the rotational symmetry is unbroken in the absence of the condensate, the same equation governs a_y^3 perturbations. The solution to (2.37) describing gauge bosons falling into the horizon at $r = 1$ is

$$a_x^3 = (r-1)^{-i\omega/4\pi T} (r^2 + r + 1)^{i\omega/8\pi T} \left(\frac{r + \frac{1}{2} + \frac{i\sqrt{3}}{2}}{r + \frac{1}{2} - \frac{i\sqrt{3}}{2}} \right)^{\sqrt{3}\omega/8\pi T} , \tag{2.38}$$

where we have used (2.8). The behavior $a_x^3 \propto (r-1)^{-i\omega/4\pi T}$ is typical of solutions falling into a finite-temperature horizon. The expansion of (2.38) near the conformal boundary is the same as (2.32) through order $1/r$, so the conductivity is the same, as claimed.

2.3.2 Electromagnetic perturbations of the superconducting phase

In the presence of the condensate $\tilde{w}\tau^1 dx$, perturbations of the form (2.36) mix with other components at the level of linearized equations. An ansatz which is sufficiently general to obtain consistent linearized equations is $A \to A + a$, where

$$a = e^{-i\omega t}\left[(a_t^1\tau^1 + a_t^2\tau^2)dt + a_x^3\tau^3 dx + a_y^3\tau^3 dy\right] . \tag{2.39}$$

All the a_m^a are functions of r. Plugging the perturbation (2.39) into the linearized Yang-Mills equations, one finds that the a_y^3 mode obeys an equation of motion decoupled from the others:

$$\left[\partial_r^2 + \frac{2r^3+1}{r(r^3-1)}\partial_r + \frac{\omega^2 L^4 r^2}{(r^3-1)^2} - \frac{\tilde{w}^2}{r(r^3-1)}\right] a_y^3 = 0\,. \tag{2.40}$$

This is identical to (13) of [12], except that the last term has slightly different radial dependence. Unsurprisingly, the rescaled complex conductivity $\tilde{\sigma}_{yy}$ exhibits similar gapped behavior to what was found in [12]: see figure 2.4. Because the analysis is so similar to [12], we will not discuss it further here, but simply present the results in sections 2.3.3 and 2.3.4.

The linearized Yang-Mills equations mix a_x^3 with a_t^1 and a_t^2, resulting in three second order equations of motion,

$$\begin{aligned}
\left[\partial_r^2 + \frac{2r^3+1}{r(r^3-1)}\partial_r + \frac{\omega^2 L^4 r^2}{(r^3-1)^2}\right] a_x^3 - \frac{r^2\tilde{\Phi}\tilde{w}}{(r^3-1)^2}a_t^1 - \frac{i\omega L^2 r^2}{(r^3-1)^2}a_t^2 &= 0\,, \\
\left[\partial_r^2 + \frac{2}{r}\partial_r\right] a_t^1 + \frac{\tilde{\Phi}\tilde{w}}{r(r^3-1)}a_x^3 &= 0\,, \\
\left[\partial_r^2 + \frac{2}{r}\partial_r - \frac{\tilde{w}^2}{r(r^3-1)}\right] a_t^2 - \frac{i\omega L^2\tilde{w}}{r(r^3-1)}a_x^3 &= 0\,,
\end{aligned} \tag{2.41}$$

and two first-order constraints,

$$
\begin{aligned}
i\omega L^2 (a_t^1)' + \tilde{\Phi}(a_t^2)' - \tilde{\Phi}' a_t^2 = 0\,, \\
i\omega L^2 (a_t^2)' - \tilde{\Phi}(a_t^1)' + \tilde{\Phi}' a_t^1 - \left(1 - \frac{1}{r^3}\right) \left[\tilde{w}\partial_r - \tilde{w}'\right] a_x^3 = 0\,,
\end{aligned}
\tag{2.42}
$$

where, as before, primes mean d/dr. The constraints are not independent of the equations of motion: if one takes the r derivative of each constraint, the resulting second order equation follows algebraically from the equations of motion and the undifferentiated constraints. It takes six constants of integration to specify a solution to the equations of motion, but two of them are used up in satisfying the constraints, leaving four independent solutions. Of these, two can be found in closed form and are related to residual gauge invariance, as we will discuss in more detail below. There is also a solution describing gauge bosons falling into the horizon, and another describing gauge bosons coming out.

Let's focus on the infalling solution, which determines a retarded Green's function, as we have seen in easier examples above. Near the horizon,

$$
\begin{aligned}
a_x^3 &= (r-1)^{-i\omega/4\pi T}\left[1 + a_x^{3(1)}(r-1) + a_x^{3(2)} + \ldots\right]\,, \\
a_t^1 &= (r-1)^{-i\omega/4\pi T}\left[a_t^{1(2)}(r-1)^2 + a_t^{1(3)}(r-1)^3 + \ldots\right]\,, \\
a_t^2 &= (r-1)^{-i\omega/4\pi T}\left[a_t^{2(1)}(r-1) + a_t^{2(2)}(r-1)^2 + \ldots\right]\,,
\end{aligned}
\tag{2.43}
$$

and all the coefficients $a_m^{a(s)}$ can be determined once the background and ω are specified. Near the conformal boundary, a generic solution to the equations of motion takes the form

$$
\begin{aligned}
a_x^3 &= A_x^{3(0)} + \frac{A_x^{3(1)}}{r} + \ldots\,, \\
a_t^1 &= A_t^{1(0)} + \frac{A_t^{1(1)}}{r} + \ldots\,, \\
a_t^2 &= A_t^{2(0)} + \frac{A_t^{2(1)}}{r} + \ldots\,,
\end{aligned}
\tag{2.44}
$$

and the constraints impose the relations

$$
\begin{aligned}
i\omega L^2 A_t^{1(1)} + \tilde{p}_0 A_t^{2(1)} - \tilde{p}_1 A_t^{2(0)} &= 0\,, \\
i\omega L^2 A_t^{2(1)} - \tilde{p}_0 A_t^{1(1)} + \tilde{p}_1 A_t^{1(0)} + \tilde{W}_1 A_x^{3(0)} &= 0\,,
\end{aligned}
\tag{2.45}
$$

where the coefficients $\tilde{p}_s$ and $\tilde{W}_1$ are the ones appearing in (2.13). The infalling solution is unique up to an overall scaling, which is fixed once we choose the leading behavior of a_x^3 to be $(r-1)^{-i\omega/4\pi T}$ as in the first line of (2.43). Thus the far-field coefficients $A_m^{a(s)}$ are in principle known as functions of ω once the background is specified. We claim that

$$
\tilde{\sigma}_{xx} = -\frac{i}{\omega L^2 A_x^{3(0)}} \left(A_x^{3(1)} + \tilde{W}_1 \frac{i\omega L^2 A_t^{2(0)} + \tilde{p}_0 A_t^{1(0)}}{\tilde{p}_0^2 - \omega^2 L^4} \right) . \tag{2.46}
$$

The first term in parentheses is the expected result based on the considerations of (2.17)–(2.27). The second term has to do with solutions to (2.41) and (2.42) which are pure gauge outside the horizon, as we will now explain.

An infinitesimal gauge transformation of the $SU(2)$ gauge field takes the form $\delta A = D\alpha$, where $D = d + gA$ is the gauge-covariant derivative and α is an adjoint scalar gauge function. Let's consider the case

$$
\alpha = e^{-i\omega t} \alpha^a \tau^a\,. \tag{2.47}
$$

After performing the split $A \to A + a$ of the gauge field into background and fluctuating parts, we can view the infinitesimal gauge transformation as acting only on $a = e^{-i\omega t} a_\mu^a \tau^a dx^\mu$:

$$
\delta(e^{-i\omega t} a_\mu^a) = \partial_\mu (e^{-i\omega t} \alpha^a) + g \epsilon^{abc} A_\mu^b e^{-i\omega t} \alpha^c\,. \tag{2.48}
$$

If any α^a depends on r, then the gauge-transformed perturbations will include components a_r^a which weren't present in the original ansatz (2.39). Setting these components to zero amounts to choosing a form of axial gauge, and the gauge transformations that preserve axial gauge are the ones where α^a doesn't depend on r. Dependence on x^1 and x^2 is excluded because we are always considering modes with zero spatial momentum. We also set $\alpha^3 = 0$ because it would introduce components of the perturbations that are not present in the ansatz (2.39). To summarize: we are interested in infinitesimal gauge transformations of the form (2.48) where α^1 and α^2 are constant and $\alpha^3 = 0$. The explicit form of this restricted set of gauge transformations is

$$\begin{aligned} \delta a_x^3 &= \tilde{w}\tilde{\alpha}^2 \,, \\ \delta a_t^1 &= -i\omega L^2 \tilde{\alpha}^1 - \tilde{\Phi}\tilde{\alpha}^2 \,, \\ \delta a_t^2 &= -i\omega L^2 \tilde{\alpha}^2 + \tilde{\Phi}\tilde{\alpha}^1 \,, \end{aligned} \tag{2.49}$$

where in order to simplify notation we have defined $\tilde{\alpha}^a = \alpha^a / L^2$. It is readily checked that the expressions in (2.49) solve the equations of motion (2.41) and the constraints (2.42). This had to happen because (2.41)–(2.42) came from the gauge-invariant Yang-Mills equations. These are the two closed-form solutions which we mentioned in the text following (2.42).

Up to an overall scaling, there is a unique linear combination of a_x^3, a_t^1, and a_t^2 which is invariant under the gauge transformation (2.49):

$$\hat{a}_x^3 = a_x^3 + \tilde{w}\frac{i\omega L^2 a_t^2 - \tilde{\Phi} a_t^1}{\tilde{\Phi}^2 - \omega^2 L^4} \,. \tag{2.50}$$

The conductivity $\tilde{\sigma}_{xx}$ captures some gauge-invariant information about the bulk theory, and as such it must be expressible in terms of $\hat{a}_x^3$. If one expands

$$\hat{a}_x^3 = \hat{A}_x^{3(0)} + \frac{\hat{A}_x^{3(1)}}{r} + \ldots \tag{2.51}$$

near the conformal boundary, then the unique extension of (2.35) that respects the gauge invariance is

$$\tilde{\sigma}_{xx} = -\frac{i}{\omega L^2} \frac{\hat{A}_x^{3(1)}}{\hat{A}_x^{3(0)}} \,. \tag{2.52}$$

This is precisely the result (2.46) that we claimed earlier. In appendix 5.2 we describe how $\tilde{\sigma}_{xx}$ fits into a 3×3 matrix of conductivities which can all be determined in terms of $A_x^{3(0)}$ and $A_x^{3(1)}$.

2.3.3 Results of numerics

Let us review the structure of the problem before discussing results. The gauge field background (2.5) is constructed by numerically solving the Yang-Mills equations (2.7) in a fixed gravitational background, (2.4), subject to constraints near the conformal boundary and near the horizon, (2.14) and (2.13) respectively. From a numerical solution, one can pick out the parameters $\tilde{p}_0$, $\tilde{p}_1$, and $\tilde{W}_1$ appearing in (2.46). A symmetry-breaking background with $\tilde{w} > 0$ everywhere is labeled uniquely by the value of $T/\sqrt{\tilde{\rho}}$, which has a maximum value of approximately 0.125. It is interesting that this value is within numerical error of 1/8, but we don't see any reason why it should be exactly 1/8. With a numerically constructed background in hand, one chooses a value of ω, initializes a finite-element differential equation solver close to the horizon using the series expansion (2.43), solves (2.41), and extracts the coefficients $A_x^{3(0)}$, $A_x^{3(1)}$, $A_t^{1(0)}$, and $A_t^{2(0)}$ appearing in (2.46) by comparing the far-field behavior of the numerical solution with the expansions (2.44). It is important to note that ω and L appear in the differential equations (2.41) and the conductivity formula (2.46) only in the combination

$$\omega L^2 = \frac{3}{4\pi} \frac{\omega}{T} \,, \tag{2.53}$$

where we have used (2.8). (Recall that we have set $r_H = 1$. If we had not, the left hand side of (2.53) would be instead $\omega L^2 / r_H$, because then $T = 3 r_H / 4\pi L^2$.) Thus it is more precise to say that one chooses a numerical value for the dimensionless quantity ω/T and determines $\tilde{\sigma}_{xx}$, which is also dimensionless, in terms of it. One expects that for large enough ω/T, $\tilde{\sigma}_{xx} \to 1$. This is because the condensate involves dynamics with a characteristic energy scale, which turns out to be $\sqrt{\tilde{\rho}}$. Provided we avoid the extreme limit $T \to 0$, $\sqrt{\tilde{\rho}}$ and T are comparable. If $\omega \gg \sqrt{\tilde{\rho}}$, the propagation of the gauge bosons should be largely insensitive to the condensate: instead, its wave function approximately takes the form (2.31) that we found for photons in pure AdS_4, and $\sigma_{xx} \approx \sigma_\infty$.

Numerical computations can only detect the continuous part of $\tilde{\sigma}_{xx}(\omega)$, but there is also a distributional part with some interesting structure. Because $\tilde{\sigma}_{xx}(\omega)$ is proportional to a retarded Green's function, it is analytic on the upper half-plane of complex ω. It therefore satisfies the Kramers-Kronig relations:

$$\begin{aligned} \mathrm{Re}[\tilde{\sigma}_{xx}(\omega) - 1] &= \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} d\omega' \, \frac{\mathrm{Im}\,\tilde{\sigma}_{xx}(\omega')}{\omega' - \omega} \,, \\ \mathrm{Im}\,\tilde{\sigma}_{xx}(\omega) &= -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} d\omega' \, \frac{\mathrm{Re}[\tilde{\sigma}_{xx}(\omega') - 1]}{\omega' - \omega} \,. \end{aligned} \tag{2.54}$$

The reason that $\tilde{\sigma}_{xx} - 1$ appears in (2.54) rather than $\tilde{\sigma}_{xx}$ itself is that it is $\tilde{\sigma}_{xx} - 1$ which vanishes as $\omega \to \infty$, and such vanishing is a necessary condition in order to obtain (2.54) from a contour integral in the upper half-plane. $\mathcal{P}$ denotes the principle part of the integral. Evidently, a simple pole in $\mathrm{Im}\,\tilde{\sigma}_{xx}(\omega)$ at $\omega = \omega_0$ implies a delta-function contribution $\delta(\omega - \omega_0)$ to $\mathrm{Re}\,\tilde{\sigma}_{xx}(\omega)$. The positivity constraint on the real part of conductivities applies separately to the continuous and delta-function parts of $\mathrm{Re}\,\tilde{\sigma}_{xx}(\omega)$, so any pole of $\mathrm{Im}\,\tilde{\sigma}_{xx}(\omega)$ on the real axis must have positive residue.

Plots of $\tilde{\sigma}_{xx}(\omega)$ and $\tilde{\sigma}_{yy}(\omega)$ are shown in figure 2.4. The conspicuous features are:

1. $\tilde{\sigma}_{xx}$ and $\tilde{\sigma}_{yy}$ both approach 1 as ω becomes large, as expected on general grounds.

2. $\tilde{\sigma}_{yy}$ displays gapped dependence similar to the findings of [12], with $\Delta \approx \frac{1}{2}\sqrt{\tilde{\rho}}$. That is, $\mathrm{Re}\,\sigma$ is very small in the infrared, then rises quickly at $\omega = 2\Delta \equiv \omega_g \approx$

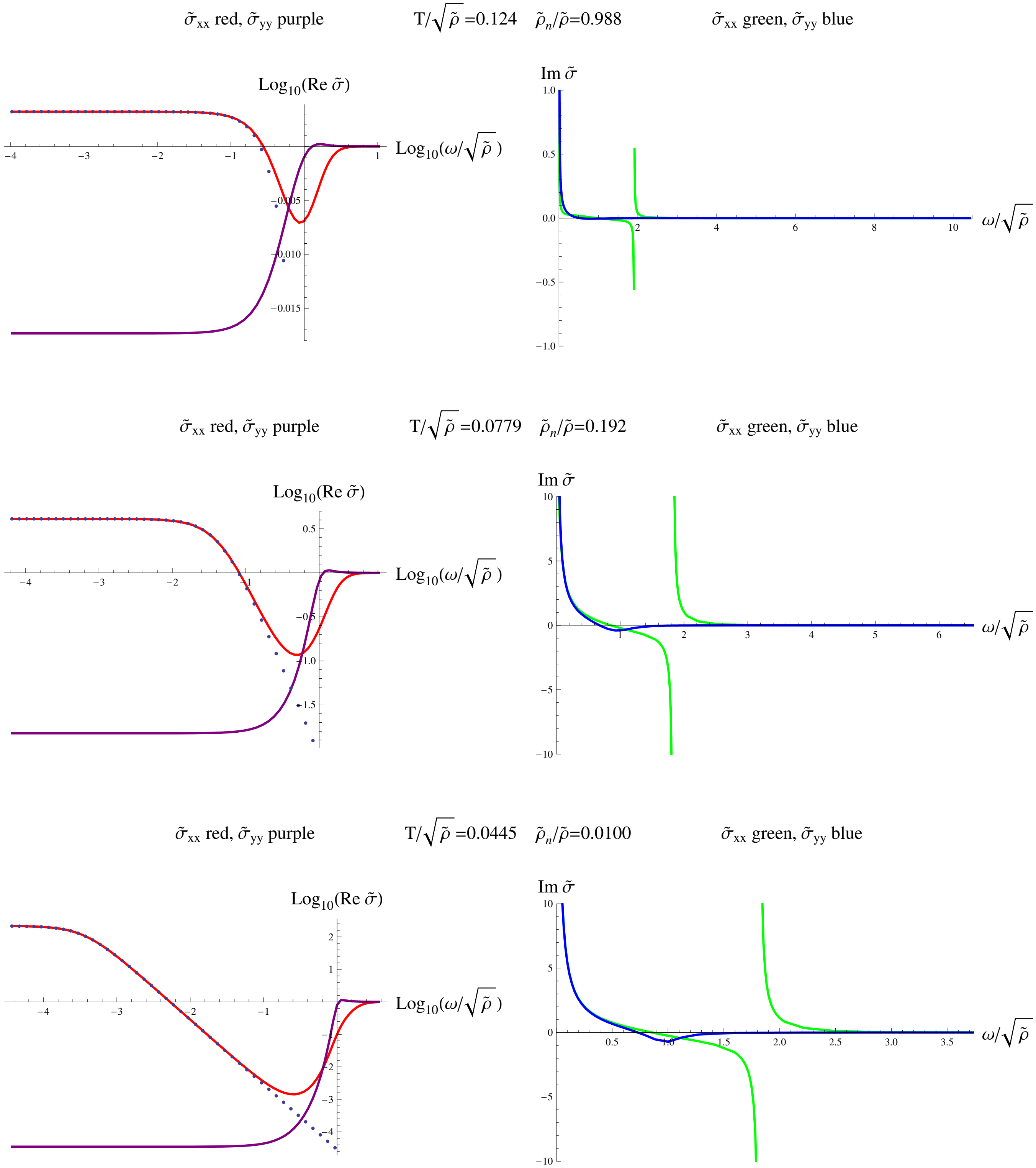

Figure 2.4: Rescaled conductivities $\tilde{\sigma}_{xx}$ and $\tilde{\sigma}_{yy}$ as functions of frequency. The dotted curves are the best fits of the Drude model prediction (2.56) to $\mathrm{Re}\,\tilde{\sigma}_{xx}(\omega)$ at low ω.

$\sqrt{\tilde{\rho}}$, with a slight "bump" a little above ω_g that is reminiscent of the behavior expected for fermionic pairing. We use the notation ω_g even though it's not clear that $\operatorname{Re}\tilde{\sigma}_{yy}$ is strictly zero for $0 < \omega < \omega_g$.

3. There is a pole in $\operatorname{Im}\tilde{\sigma}_{xx}$ at $\omega = \omega_0 \approx 1.8\sqrt{\tilde{\rho}}$. Its residue becomes small as one approaches T_c. It's clear from (2.46) that this pole had to arise, with residue proportional to the order parameter $\tilde{W}_1$: it comes from the denominator of the second term, and
$$\omega_0 = \frac{4\pi}{3}\tilde{p}_0 T\,. \tag{2.55}$$
As discussed following (2.54), there is a delta-function contribution to $\operatorname{Re}\tilde{\sigma}_{xx}$ at $\omega = \omega_0$, whose coefficient is proportional to the residue of this pole. This resonance is perfectly stable even at finite temperature, but perhaps if we relax some of the limits we have taken (like large N) it would acquire a width.

4. $\operatorname{Re}\tilde{\sigma}_{xx}$ never goes as low as $\operatorname{Re}\tilde{\sigma}_{yy}$, and its rise toward 1 happens more gradually and at a somewhat larger value of ω, on order ω_0.

5. The small ω behavior of $\operatorname{Re}\tilde{\sigma}_{xx}$ can be parameterized very accurately in terms of the Drude model, which predicts
$$\operatorname{Re}\sigma_{\text{Drude}} = \frac{\sigma_0}{1+\omega^2\tau^2}\,, \tag{2.56}$$
where $\sigma_0 = ne^2\tau/m$ is a constant related to the density of charge carriers, and τ is the scattering time.

We are especially interested in the low-frequency dependence of the conductivities. Our numerical results make it plausible but not certain that σ_{yy} is strictly zero below a finite value of ω when $T = 0$. However, neglecting the back-reaction of the gauge field may not be a valid approximation for very low temperatures. On the other hand, the narrow Drude peak in $\tilde{\sigma}_{xx}$ suggests conductivity due to quasi-particles whose scattering time diverges as $T \to 0$. Putting the behavior of $\tilde{\sigma}_{xx}$ and $\tilde{\sigma}_{yy}$ together suggests a very special type of "node in the gap," namely one which is infinitely narrow as a function of angle in Fourier space.[7]

2.3.4 Fits of temperature-dependent quantities

In order to extract some simple quantitative information from our numerical results, we considered the dependence of various dimensionless quantities on the rescaled temperature $T/\sqrt{\tilde{\rho}}$. Our findings can be summarized briefly as follows:

$$\begin{aligned}
\frac{\tilde{\rho}}{\tilde{\rho}_n} &\approx \exp\left\{0.303\frac{\sqrt{\tilde{\rho}}}{T} - 2.20\right\} , \\
\frac{\tilde{W}_1}{\tilde{\rho}} &\approx 1 - 167\left(\frac{T}{\sqrt{\tilde{\rho}}}\right)^{3.05} , \\
\frac{\tilde{\rho}_n^2}{\tilde{\rho}T}\tau &\approx 4.5 \,, \\
\frac{\tilde{\rho}}{\tilde{\rho}_n^2\tau^2} \lim_{\omega\to 0} \operatorname{Re}\tilde{\sigma}_{xx}(\omega) &\approx 0.302 \,, \\
\left(\frac{\tilde{\rho}}{\tilde{\rho}_n}\right)^2 \lim_{\omega\to 0} \operatorname{Re}\tilde{\sigma}_{yy}(\omega) &\approx 0.34 \,, \\
\lim_{\omega\to 0} \frac{\omega}{\sqrt{\tilde{\rho}}} \operatorname{Im}\tilde{\sigma}_{xx}(\omega) &\approx 0.52 \,, \\
\lim_{\omega\to 0} \frac{\omega}{\sqrt{\tilde{\rho}}} \operatorname{Im}\tilde{\sigma}_{yy}(\omega) &\approx 0.55 \,, \\
\lim_{\omega\to \omega_0} \frac{\omega - \omega_0}{\sqrt{\tilde{\rho}}} \operatorname{Im}\tilde{\sigma}_{xx}(\omega) &\approx 0.28 \,.
\end{aligned} \tag{2.57}$$

The approximately equalities in (2.57) are in some cases quite close over a substantial range of $\sqrt{\tilde{\rho}}/T$, and in others represent no more than a $T \to 0$ extrapolation: see figure 2.5. None of the relations (2.57) should be taken too seriously, because they were made over intervals where $T/\sqrt{\tilde{\rho}}$ varied only by a factor of 5. A particularly challenging case is the quantity $\lim_{\omega\to 0} \frac{\omega}{\sqrt{\tilde{\rho}}} \operatorname{Im}\tilde{\sigma}_{xx}(\omega)$. The $\omega \to 0$ limit converges

[7] We thank D. Huse and P. Ong for discussions that led to the picture of an infinitely narrow node described here.

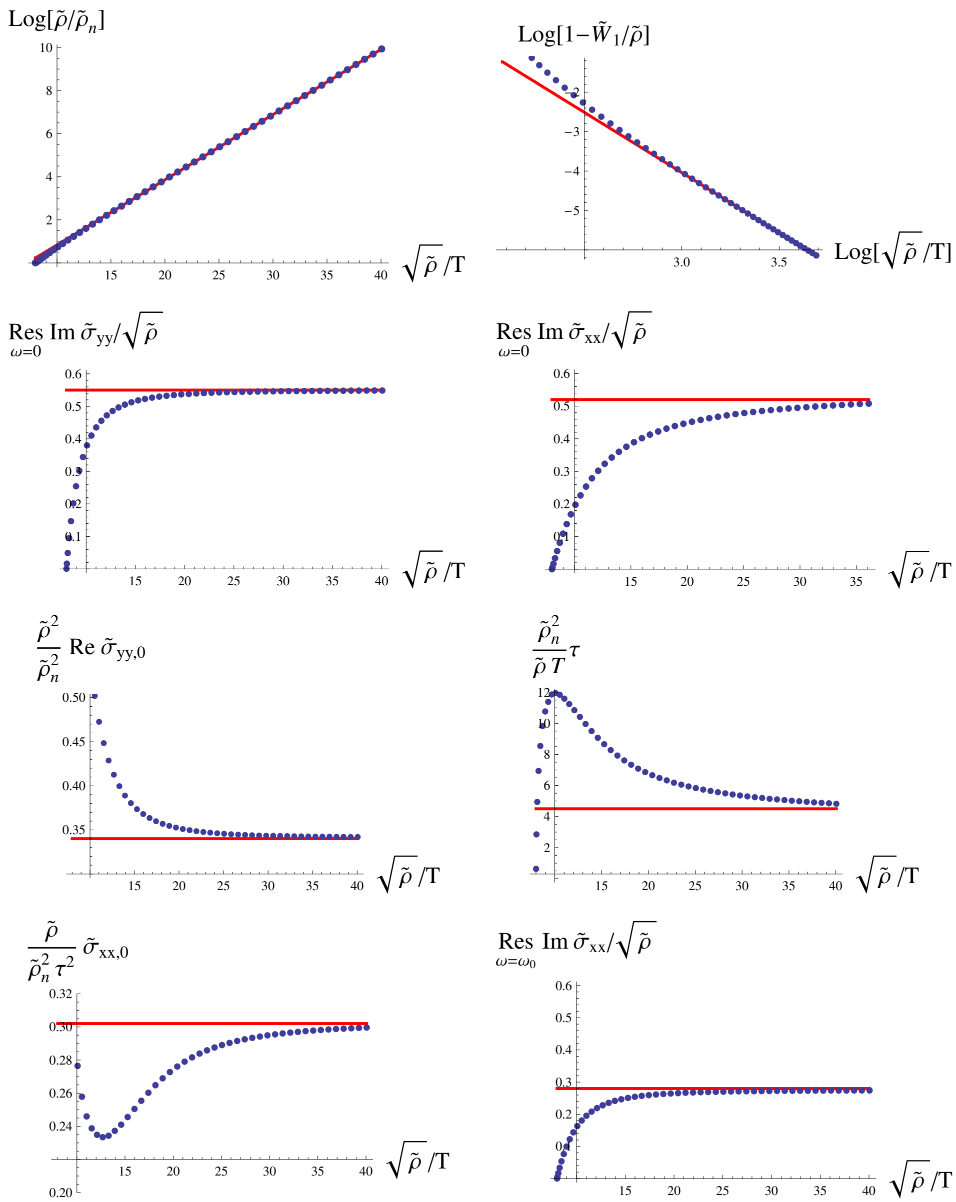

Figure 2.5: Temperature-dependent quantities and approximate fits, as explained in (2.57) and the surrounding text. We have defined $\tilde{\sigma}_{xx,0} = \lim_{\omega\to 0} \operatorname{Re}\tilde{\sigma}_{xx}(\omega)$, $\tilde{\sigma}_{yy,0} = \lim_{\omega\to 0} \operatorname{Re}\tilde{\sigma}_{yy}(\omega)$, $\operatorname{Res}_{\omega=0} \operatorname{Im}\tilde{\sigma}_{xx}/\sqrt{\tilde{\rho}} = \lim_{\omega\to 0} \frac{\omega}{\sqrt{\tilde{\rho}}} \operatorname{Im}\tilde{\sigma}_{xx}(\omega)$, $\operatorname{Res}_{\omega=0} \operatorname{Im}\tilde{\sigma}_{yy}/\sqrt{\tilde{\rho}} = \lim_{\omega\to 0} \frac{\omega}{\sqrt{\tilde{\rho}}} \operatorname{Im}\tilde{\sigma}_{yy}(\omega)$, and $\operatorname{Res}_{\omega=\omega_0} \operatorname{Im}\tilde{\sigma}_{xx}/\sqrt{\tilde{\rho}} = \lim_{\omega\to\omega_0} \frac{\omega-\omega_0}{\sqrt{\tilde{\rho}}} \operatorname{Im}\tilde{\sigma}_{xx}(\omega)$.

slowly because of a "shelf effect:" for values in a region around $\omega \sim 1/\tau$, we observed $\frac{\omega}{\sqrt{\bar{\rho}}} \operatorname{Im} \tilde{\sigma}_{xx} \approx 0.55$ at low temperatures, which is the same value as we find in the $\omega \to 0$ limit for $\frac{\omega}{\sqrt{\bar{\rho}}} \operatorname{Im} \tilde{\sigma}_{yy}$. But for $\omega \lesssim 1/50\tau$, we observed instead the value some 6% smaller quoted in (2.57). Our numerical algorithms aren't optimized for extremely small T and ω, and it's possible that this shelf effect goes away at very small T, so that the residues of $\operatorname{Im} \tilde{\sigma}_{xx}$ and $\operatorname{Im} \tilde{\sigma}_{yy}$ agree in this limit. But the balance of evidence from our numerical exploration is that this does not happen, or happens very slowly as T is decreased.

2.4 Stability calculations

We expected that the p-wave backgrounds (2.5) would be unstable against small perturbations that would eventually turn them into backgrounds of the type studied in [51]. These backgrounds display behavior analogous to a $p + ip$ gap.[8] But the opposite seems to be true: numerical explorations of quasinormal modes close to T_c show that it is the $p + ip$-wave backgrounds that are unstable, and it seems that they evolve toward pure p-wave backgrounds, which are stable. In section 2.4.1 we exhibit the equations describing the perturbations of the pure p-wave backgrounds that we thought would be unstable and explain how the lowest-lying quasinormal modes exhibit stability instead, close to T_c. In section 2.4.2, we show that similar perturbations of the backgrounds studied in [51] exhibit an instability slightly below T_c.

[8] The analogy to a $p+ip$ gap is apt because the combination $\tau^1 dx + \tau^2 dy$ distinguishes an orientation on $\mathbb{R}^2$ and implies a spontaneous magnetization. To see this, note first that the positive charge of the black hole under $U(1)_3$ privileges τ^3 over $-\tau^3$. The structure constants ϵ^{abc} of $SU(2)$ then privilege the ordering (τ^1, τ^2) over (τ^2, τ^1), because having distinguished the positive τ^3 direction in the Lie algebra lets us set $c = 3$. Finally, $\tau^1 dx + \tau^2 dy$ "locks" this orientation in the Lie algebra to an orientation $dx \wedge dy$ on $\mathbb{R}^2$. More physically, a contribution $w(\tau^1 dx + \tau^2 dy)$ to A means that there is a term $w^2 \tau^3 dx \wedge dy$ in F, representing a spontaneous magnetization that again picks out an orientation $dx \wedge dy$ in $\mathbb{R}^2$. In any case, the symmetries of this state are clearly those of a $p + ip$ gap whose ip component is of identical magnitude to its p component, so that the gap is uniform in magnitude but has a phase that rotates by 2π as one goes once around the Fermi surface.

2.4.1 Quasinormal frequencies of p-wave backgrounds

Let us begin by explaining why we thought p-wave backgrounds would be unstable. At $T = T_c$, both the $\tau^1 dx$ mode and the $\tau^2 dy$ directions exhibit marginally stable modes. So a natural expectation is that both become unstable for $T < T_c$. Yet the p-wave backgrounds described in section 2.2 involve only $\tau^1 dx$, whereas the $p+ip$-wave backgrounds of [51] involve the combination $\tau^1 dx + \tau^2 dy$. In the latter case we are taking advantage of both directions of instability, and it seems reasonable that such a configuration should be preferred. But this reasoning ignores the non-linearities of the Yang-Mills equations. It turns out that condensing in the $\tau^1 dx$ direction stabilizes against condensation in the $\tau^2 dy$ direction—at least, close to T_c. That stabilization is what we are going to address in this section.

Starting from the backgrounds (2.5), we want to study $\tau^2 dy$ perturbations, which is to say a_y^2. At the linearized level, a_y^2 couples with a_y^1, so we are forced to examine the combined perturbation $A \to A + a$, where

$$a = e^{-i\omega t}\left(a_y^1 \tau^1 + a_y^2 \tau^2\right) dy\,. \tag{2.58}$$

The equations of motion read

$$\begin{aligned}\left[\partial_r^2 + \frac{2r^3+1}{r(r^3-1)}\partial_r + \frac{r^2(\omega^2 L^4 + \tilde{\Phi}^2)}{(r^3-1)^2}\right] a_y^1 - \frac{2i\omega L^2 r^2 \tilde{\Phi}}{(r^3-1)^2} a_y^2 &= 0\,,\\ \left[\partial_r^2 + \frac{2r^3+1}{r(r^3-1)}\partial_r + \frac{r^2(\omega^2 L^4 + \tilde{\Phi}^2)}{(r^3-1)^2} - \frac{\tilde{w}^2}{r(r^3-1)}\right] a_y^2 + \frac{2i\omega L^2 r^2 \tilde{\Phi}}{(r^3-1)^2} a_y^1 &= 0\,.\end{aligned} \tag{2.59}$$

The appropriate boundary conditions for quasinormal modes are that a_y^1 and a_y^2 should vanish at the boundary of AdS_4 and that a should be a function only of the infalling coordinate $t + \frac{1}{4\pi T}\log(r-1)$ at the black hole horizon (where, as usual, $r_H = 1$). These conditions can be simultaneously satisfied only for certain complex quasinormal frequencies ω. Since we assumed $e^{-i\omega t}$ time dependence, quasinormal frequencies with

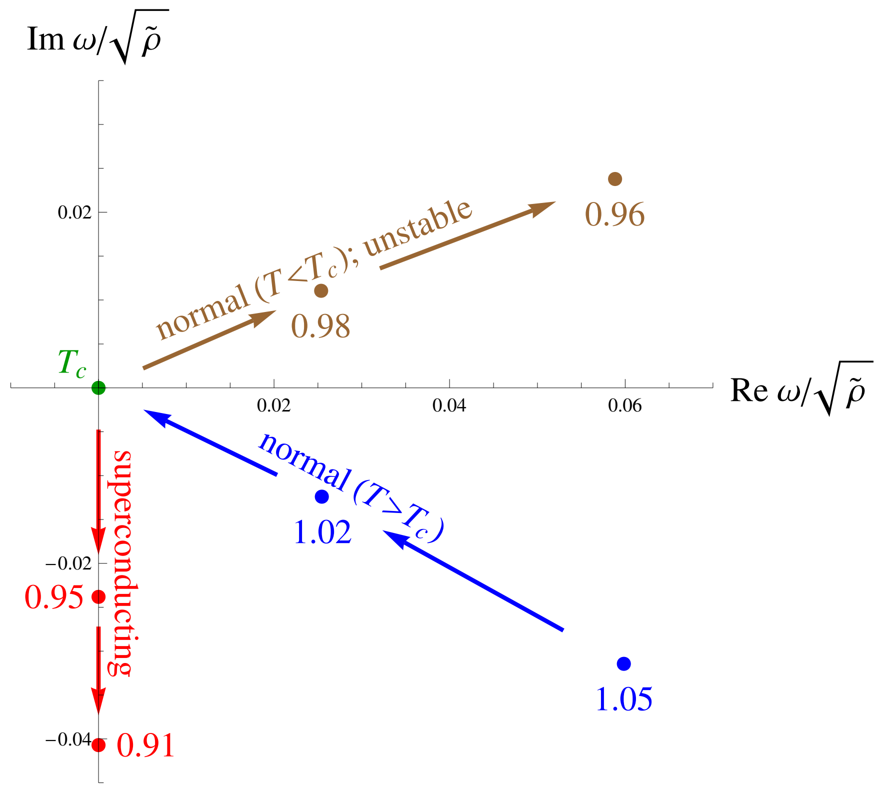

Figure 2.6: Quasinormal frequencies corresponding to the perturbation (2.59) of the p-wave superconducting background (2.5) near the critical temperature. The quasinormal mode spectrum is symmetric about the imaginary axis, and we are only showing the quasinormal frequencies with non-negative real parts. The arrows are in the direction of decreasing temperature, and the number displayed next to each quasinormal frequency represents T/T_c. The blue points correspond to backgrounds with no condensate above T_c; the brown points correspond to backgrounds with no condensate below T_c; and the red points correspond to superconducting backgrounds below T_c. The superconducting backgrounds also have a quasinormal mode at $\omega = 0$ (see main text) which is not displayed. The backgrounds with no condensate below T_c have quasinormal frequencies with positive imaginary parts, indicating an instability. The other backgrounds (namely normal state above T_c and superconducting below T_c) appear to be stable.

negative imaginary parts correspond to stable modes, while those with positive imaginary parts correspond to unstable modes. Solutions with purely real ω correspond to true normal modes of the system. From the symmetries of the equations (2.59) and of the boundary conditions described above, it follows that if ω is a quasinormal frequency, then so is $-\omega^*$. So let's restrict attention to quasinormal frequencies with $\mathrm{Re}\,\omega \geq 0$. Figure 2.6 shows how the lowest-lying quasinormal frequencies behave as functions of temperature close to T_c. Above T_c, the normal state is stable, and the quasinormal modes come in degenerate pairs with the same ω. As we mentioned earlier, there are two quasinormal modes that become marginally stable at T_c: their frequencies go to zero. One of these modes, involving only a_y^1, stays right at $\omega = 0$ below T_c on the superconducting branch. It is a Goldstone mode describing spatial rotations of the condensate. The other mode is stable on the superconducting branch below T_c. What makes it stable is the $\frac{-\tilde{w}^2}{r(r^3-1)}$ term in the second equation of (2.59). This term is like a positive, r-dependent contribution to the mass term of the gauge boson. Dropping this term amounts to passing to the normal state below T_c, and our normal investigation showed that this state is unstable. So the $\frac{-\tilde{w}^2}{r(r^3-1)}$ term is the advertised stabilization mechanism, and it is evidently due to the non-linearities of the Yang-Mills equations of motion.

2.4.2 Quasinormal frequencies of $p + ip$-wave backgrounds

We now wish to show that the large gL limit of the $p + ip$ backgrounds studied in [51] are unstable, at least for T close to T_c. The instability decreases the ip component of the gap and appears likely to lead the system into a p-wave state like (2.5). Our strategy is to find out what happens to the modes which are marginally stable at T_c as we go slightly away from the critical temperature on the superconducting and normal branches.

At large g, the gauge field ansatz for the circularly polarized backgrounds is

$$A = \Phi(r)\tau^3 dt + w(r)\left(\tau^1 dx + \tau^2 dy\right) , \tag{2.60}$$

and it is again convenient to define

$$\tilde{\Phi} = gL^2\Phi \,, \qquad \tilde{w} = gL^2 w \,. \tag{2.61}$$

In the large g limit there is no back-reaction on the metric, so the metric is simply (2.4). The equations of motion for $\tilde{\Phi}$ and $\tilde{w}$ are similar to (2.7). They are given explicitly in (B4) of [51], and we will not reproduce them here.

There are many ways in which one can perturb the background (2.60), but the perturbations that might show an instability towards converting $p + ip$ into p should be of the form

$$a = e^{-i\omega t} a_1(\tau^1 dx - \tau^2 dy) + e^{-i\omega t} a_2(\tau^2 dx + \tau^1 dy) \,. \tag{2.62}$$

The a_1 perturbation changes the relative magnitude of the p and ip components of the background ansatz (2.60). Nothing in the ansatz (2.60) picks out whether $\tau^1 dx$ or $\tau^2 dy$ is the p-wave part (as opposed to the ip part) so changing the relative size of these two components with a linear perturbation can be interpreted as decreasing the ip component without loss of generality. The a_2 component is a $90°$ spatial rotation of the a_1 component. The linearized equations for a_1 and a_2 are

$$\begin{aligned} \left[\partial_r^2 + \frac{2r^3+1}{r(r^3-1)}\partial_r + \frac{r^2(\omega^2 L^4 + \tilde{\Phi}^2)}{(r^3-1)^2} + \frac{\tilde{w}^2}{r(r^3-1)}\right] a_1 + \frac{2i\omega L^2 r^2 \tilde{\Phi}}{(r^3-1)^2} a_2 = 0 \,, \\ \left[\partial_r^2 + \frac{2r^3+1}{r(r^3-1)}\partial_r + \frac{r^2(\omega^2 L^4 + \tilde{\Phi}^2)}{(r^3-1)^2} + \frac{\tilde{w}^2}{r(r^3-1)}\right] a_2 - \frac{2i\omega L^2 r^2 \tilde{\Phi}}{(r^3-1)^2} a_1 = 0 \,. \end{aligned} \tag{2.63}$$

The perturbations should take the form of infalling waves close to the horizon and

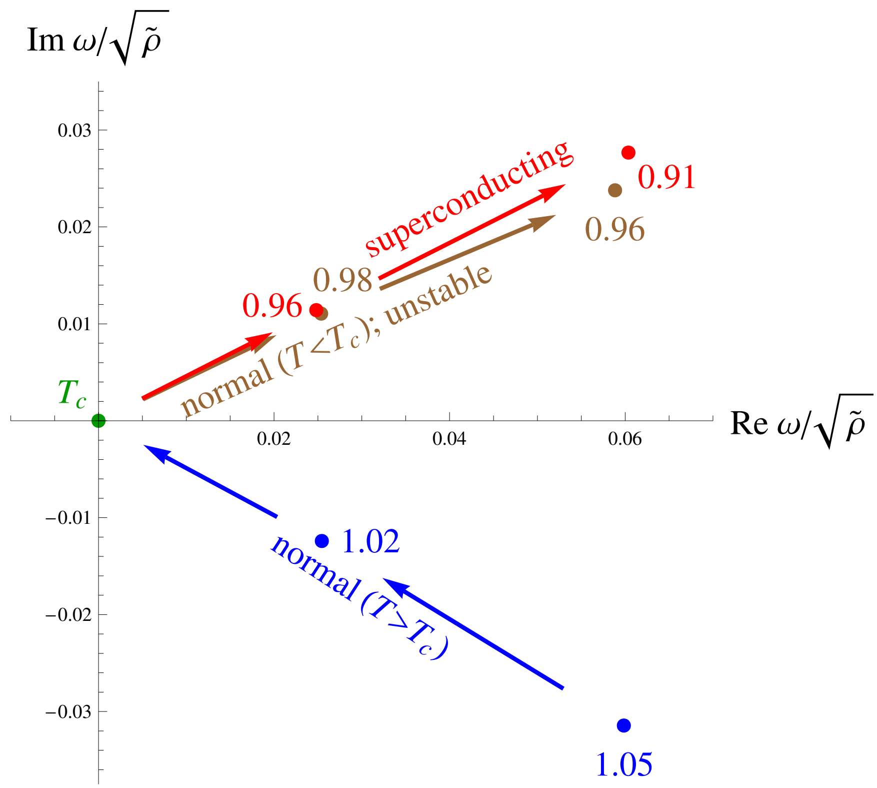

Figure 2.7: Quasinormal frequencies corresponding to the perturbation (2.62) of the $p+ip$-wave background (2.60) near the critical temperature. The spectrum of quasinormal modes is symmetric about the imaginary axis, and we are only showing the quasinormal frequencies with non-negative real parts. The arrows are in the direction of decreasing temperature, and the number displayed next to each quasinormal frequency represents T/T_c. The blue points correspond to backgrounds with no condensate above T_c; the brown points correspond to backgrounds with no condensate below T_c; and the red points correspond to superconducting backgrounds below T_c. The backgrounds with no condensate above T_c, as well as the superconducting ones below T_c, have quasinormal frequencies with positive imaginary parts, indicating an instability. The backgrounds with no condensate above T_c are likely to be stable.

should vanish at the boundary, as in the case of the linearly polarized backgrounds examined in the previous section. Only for discretely many quasinormal frequencies are these boundary conditions satisfied.

When $\tilde{w} = 0$, equations (2.63) are the same as the equations for a_y^1 and a_y^2 given in (2.59), so at zero condensate the quasinormal modes coincide with the ones displayed in figure 2.6. When $T < T_c$ we find an instability whether or not there is a condensate: see figure 2.7. This result could perhaps have been anticipated by noting that the $\tilde{w}^2/r(r^3-1)$ terms in (2.63) enter with the opposite sign from the way they entered (2.59). So instead of tending to stabilize perturbations, they tend to destabilize them. It's worth noting, however, that $\tilde{w}$ is the coefficient of $\tau^1 dx$ in (2.59), whereas it is the coefficient of $\tau^1 dx + \tau^2 dy$ in (2.63). So, the functional forms of $\tilde{w}$ will differ in the two cases, becoming equal only in the limit $T \to T_c$.

2.5 Discussion

The distinguishing feature of the superconducting black holes constructed in this chapter is that the condensate is anisotropic, in the sense of picking out the x direction as preferred. This is in contrast to earlier constructions [11, 12, 48, 51]. What is special about the x directions is that the conductivity in this direction, σ_{xx}, becomes large at small but non-zero ω. So far, the situation is similar to p-wave superconductors. But in real materials, impurity scattering would keep σ_{xx} finite for small non-zero ω, whereas in our setup, the only upper bound comes from the effects of finite temperature. The biggest difference from real materials—from the perspective of the electromagnetic response—is that σ_{yy} displays gapped dependence, similar to what was found for an s-wave construction in [12]. In real p-wave materials, the gap vanishes at $\theta = 0$ and $\theta = \pi$ but has finite slope there. Gapped σ_{yy} suggests instead an infinitely narrow node in the gap: the slope of Δ as a function of θ is infinite at $\theta = 0$ and π. To put it another way, the states which usually occupy a Dirac cone near a p-wave gap have been squeezed into a purely one-dimensional structure, at least in the limit of low energy. We emphasize that this picture of an infinitely narrow node in the gap is entirely heuristic, given that we do not have a microscopic description of the condensate in the language of a dual CFT. What we can say most clearly in the CFT language is that there is an $SU(2)$ current algebra, and when there is a strong enough chemical potential for the charge density J_t^3, the component J_x^1 develops an expectation value. We are tempted to conjecture that $J_m^a \sim \bar{\psi}_i \gamma_m \tau_{ij}^a \psi_i$ for some fermion fields ψ_i in a representation of $SU(2)$. Then the condensate is composed of fermion pairs created by J_x^1, which have one unit of angular momentum.

Our results are preliminary in various ways:

1. We didn't consider back-reaction of the gauge field on the metric. Back-reaction can be suppressed by taking the gauge coupling large, but this limit is non-uniform in that as $T \to 0$, the A_x^1 component of the gauge field gets larger and larger at the horizon, demanding a bigger value of the gauge coupling to justify the neglect of back-rection.

2. Our conductivity calculations do not allow for spatial momentum. In other words, we calculated a retarded two-point function $G_R(\omega, 0)$ of J_i^3 at non-zero frequency but zero spatial momentum. A study of the electromagnetic response at non-zero k might help consolidate the heuristic Fermi-surface picture we have offered, or it might invalidate it and suggest a different interpretation.

3. We encountered some curious numerical coincidences, ranging from $T_c/\sqrt{\tilde{\rho}} \approx 1/8$ to the scaling of the "scattering rate" $1/\tau$ and the small ω limits of $1/\tilde{\sigma}_{xx}$ and $\tilde{\sigma}_{yy}$ approximately as $\tilde{\rho}_n^2$ rather than some fractional power of $\tilde{\rho}_n$. The latter coincidence evokes the idea that the behavior of quantities like the scattering rate are largely controlled by kinematic factors of two incoming quasi-particles.

It would be interesting if some of these numerical coincidences could be understood in terms of exact solutions to the Yang-Mills equations, or in terms of some systematic approximation scheme rather than brute-force numerics.

4. The scope of our stability calculations is very restricted: not only have we limited ourselves to the no-back-reaction limit, but we also stayed close to T_c. Moreover, we do not claim to have considered every possible perturbation, only the ones that seemed obvious candidates for exhibiting instabilities. It would clearly be desirable to be more thorough.

5. We have limited ourselves entirely to classical configurations, excluding any discussion of fluctuations. This would seem problematic in two spatial dimensions because of infrared divergences, but fluctuations are suppressed when the radius of AdS_4 is much larger than the Planck scale, corresponding to a large N limit in the dual CFT. But to understand the condensate's contribution to the specific heat, presumably one should consider fluctuations.

Chapter 3

The Second Sound of SU(2)

This chapter is a lightly edited version of ref. [56], which was written in collaboration with Chris Herzog.

3.1 Introduction

As mentioned already, the gauge/gravity duality gives a new perspective on the physics of superfluids and superconductors by equating the superconducting phase transition with the instability of a charged black hole to develop charged scalar hair [11, 12]. Recalling that the AdS/CFT correspondence maps a strongly interacting field theory to a classical gravity description, this new perspective holds promise for deepening our understanding of superconductivity in strongly interacting regimes where BCS theory [57] is inadequate.[1]

This chapter is in some sense a continuation of the previous one, where we found superconducting solutions of the Einstein-Yang-Mills theory with negative cosmological constant in four spacetime dimensions. Here, we study the same model in five spacetime dimensions. While going to five dimensions might seem like a mere rewrit-

[1]See ref. [58] for a review of the limits of BCS theory when confronted with high temperature superconductivity.

ing of the previous chapter, there is actually a considerable benefit: in five dimensions one can obtain analytical solutions close to the second order phase transition, and one can compute many quantities related to the phase transition analytically. Generally, the differential equations that describe holographic systems that exhibit spontaneous gauge symmetry breaking are nonlinear, and analytical solutions do not appear to be available in most cases. As in the previous chapter, one usually has to resort to the use of numerics to see the phase transition and to calculate the conductivities and critical exponents. Analytical results, for example the low temperature approximation of the conductivity in ref. [12], are scarce.[2] In the case of an $SU(2)$ gauge field in AdS_5, however, it was noticed in ref. [61] that the zero mode responsible for the existence of the superconducting branch of solutions has a simple analytic form. From this zero mode, one can extract a long list of properties near the phase transition:

1. The speed of second sound near the phase transition.

2. That the phase transition is second order.

3. The conductivity and in particular the residue of the pole in the imaginary part of the conductivity.

4. The system satisfies a London type equation that implies a Meissner effect.

5. A large selection of current-current Green's functions in the hydrodynamic limit, and that they satisfy the appropriate non-Abelian Ward identities.

The title of this chapter makes reference to the fact that in a two-component fluid there are typically two propagating collective modes. The first mode corresponds to ordinary sound in which the two components move in phase. The second mode corresponds to second sound in which the two components move out of phase. Typically,

[2] See refs. [59, 60] for other nice analytic results for this class of models.

ordinary sound can be produced by pressure oscillations while second sound couples much more strongly to temperature oscillations [62].

As in the previous chapter, we introduce by hand a chemical potential in the third isospin direction, which induces a charge density, $\langle j_3^t \rangle \neq 0$, that breaks both the global $SU(2)$ symmetry to a $U(1)$ sugroup and also Lorentz invariance. There is a superconducting phase transition at a critical temperature T_c, below which a current develops orthogonal to the third isospin direction that completely breaks the residual $U(1)$ symmetry and also breaks the remaining rotational symmetry of the system to $U(1)$. For convenience, we take this current to be in the direction $\langle j_x^1 \rangle$, leaving a rotational symmetry in the yz-plane.

The fact that rotational symmetry is broken in the superconducting phase makes the physics of this model rich and complicated. This $SU(2)$ model appears to be a holographic realization of the type of scenario described from a formal perturbative field theoretic point of view in ref. [63]. Transport coefficients such as the speed of second sound and conductivities will depend on which direction we decide to look. Such a breaking of rotational invariance is not unheard of in real world materials. To pick a particularly simple example, a ferromagnet will break rotational symmetry when the spins align. We emphasized in the previous chapter a possible connection of this $SU(2)$ model to a p-wave superconductor, where the order parameter for the phase transition is a vector.

Of real world materials, superfluid liquid helium-3 perhaps comes closest in approximating the physics of the $SU(2)$ model. Liquid helium-3 at very low temperatures is a p-wave superfluid. Two fermionic helium-3 atoms pair up to form a loosely bound bosonic molecule with weak interaction between the orbital and spin degrees of freedom of the electrons [64]. The orbital and spin angular momenta are both equal to one, and the order parameter is often written A_{ai} where a indexes the spin angular momentum and i the orbital angular momentum, in surprisingly close analogy with our j^a_μ. There are many stable phases of superfluid helium-3, depending on the pressure, temperature, and applied magnetic field. The A phases are known to break rotational symmetry.

Despite plausible similarities between the symmetries of our model and various real world materials, there is one crucial difference. While the order parameters for these real world materials may have vector or tensor structure, they are not currents, and the signature of the phase transition is not the production of a persistent current. In contrast, the model investigated in this chapter has $\langle j^x_1 \rangle \neq 0$.

The summary of this chapter is as follows. We begin in Section 2 with a review of the $SU(2)$ model and the probe limit, this time in an arbitrary number of dimensions, generalizing the results of Chapter 2. As in Chapter 2, we choose to work in a limit in which gravity is weak and the non-Abelian field does not back-react on the metric. Thus, at heart, in this paper we will be solving the classical $SU(2)$ non-Abelian Yang-Mills equations in a fixed background spacetime, that of a Schwarzschild black hole in AdS_5.[3] In Section 3, we find a solution to the Yang-Mills equations near the phase transition. This power series solution in the order parameter and superfluid velocities allows us to demonstrate that the phase transition is second order and to calculate the speed of second sound from thermodynamic identities. In Section 4, we make some formal remarks about the current-current correlation functions for our model. We discuss the Ward identities that these Green's functions satisfy and some of their discrete symmetries. We also review how to calculate these two-point functions using the AdS/CFT correspondence. In Section 5, through a study of fluctuations about our solution near the phase transition, we extract the current-current correlation functions in the hydrodynamic limit. From the location of the poles, we independently confirm the speed of second sound calculated in Section 3. We are also able to calculate various damping coefficients and see explicitly that the Green's functions satisfy the

[3] Attempts to solve the full set of coupled equations for a non-Abelian black hole go back many years [65, 66]. See refs. [67, 68] for reviews.

non-Abelian Ward identities. In the last part of Section 5, we consider the $\omega \to 0$ and $k \to 0$ limits. From these limits we extract conductivities and also demonstrate that the system obeys a type of London equation.

3.2 The Model

The Einstein-Yang-Mills action coupled to gravity with a negative cosmological constant Λ in $d+1$ spacetime dimensions is:

$$S = \frac{1}{2\kappa^2} \int d^{d+1}x\sqrt{-g}\,(R - 2\Lambda) - \frac{1}{4g^2} \int d^{d+1}x\sqrt{-g}\, F^a_{AB}F^{aAB} \,. \tag{3.1}$$

The field strength F^{aAB} can be written in terms of a gauge connection A^a_B as follows:

$$F^a_{AB} = \partial_A A^a_B - \partial_B A^a_A + f^a{}_{bc} A^b_A A^c_B \,, \tag{3.2}$$

where $f^a{}_{bc}$ are the structure constants for the Lie algebra $\mathfrak{g}$ with generators T_a such that $[T_a, T_b] = i f_{ab}{}^c T_c$. Taking $\mathfrak{g} = \mathrm{su}(2)$, the generators are $T_a = \sigma_a/2$, σ_a being the Pauli spin matrices, and the structure constants are $f_{abc} = \epsilon_{abc}$.[4]

The equations of motion for the gauge field that follow from the action (3.1) are $D_A F^{aAB} = 0$, which can be expanded as

$$\nabla_A F^{aAB} + f^a{}_{bc} A^b_A F^{cAB} = 0 \,. \tag{3.3}$$

Einstein's equations are

$$R_{AB} + \left(\Lambda - \frac{1}{2}R\right) g_{AB} = \frac{\kappa^2}{2g^2}\left(2F^a_{CA}F^{aC}{}_B - \frac{1}{2}F^a_{CD}F^{aCD} g_{AB}\right) . \tag{3.4}$$

[4]The $\mathfrak{g} = \mathrm{su}(2)$ indices $a, b, c, \ldots$ are raised and lowered with the Kronecker delta δ^a_b. The capital indices $A, B, C \ldots$ are raised and lowered with the five dimensional space time metric g_{AB}. We will also shortly introduce Greek indices $\mu, \nu, \ldots$ which will be raised and lowered with the four dimensional Minkowski tensor $\eta^{\mu\nu} = (-{+}{+}{+})$.

A solution to these equations in the case of a negative cosmological constant, $\Lambda = -d(d-1)/2$, is a $(d+1)$-dimensional Reissner-Nordström black hole with anti-de Sitter space asymptotics:[5]

$$ds^2 = \frac{1}{u^2}\left[-f(u)dt^2 + d\vec{x}^2 + \frac{du^2}{f(u)}\right], \qquad A = \left(\mu + \rho u^{d-2}\right) T_3 dt \tag{3.5}$$

where $d\vec{x}^2 = dx^2 + dy^2 + dz^2$ and the warp factor is

$$f(u) = 1 + Q^2 \left(\frac{u}{u_h}\right)^{2d-2} - (1+Q^2)\left(\frac{u}{u_h}\right)^{d}, \tag{3.6}$$

with the charge Q being defined as

$$Q^2 \equiv \frac{\kappa^2}{g^2}\frac{d-2}{d-1}\tilde{\rho}^2 u_h^{2d-2}. \tag{3.7}$$

This black hole solution requires only an abelian gauge symmetry, and we chose that Abelian symmetry to be the $U(1)$ subgroup of $SU(2)$ generated by T_3. In slightly different conventions, this background also appears in eq. (1.29). The horizon is located at $u = u_h$, and the Hawking temperature is

$$T_H = \frac{d-(d-2)Q^2}{4\pi u_h}. \tag{3.8}$$

The gauge potential (3.5) is well defined globally, at both the horizon and the boundary, provided

$$\rho = -\mu/u_h^{d-2}. \tag{3.9}$$

As in the previous chapter, one can consider the probe limit $\kappa^2/g^2 \to 0$ where the gauge field does not back-react on the metric. The metric remains that of an

[5]We set the radius of curvature $L = 1$.

uncharged black hole in anti-de Sitter space with warp factor

$$f(u) = 1 - \left(\frac{u}{u_h}\right)^d . \tag{3.10}$$

As promised, from now on we restrict to the $d = 5$ case because of the observation made in ref. [61] that the zero mode inducing the phase transition is known analytically.

3.3 Critical Behavior

Close to the phase transition, the superconducting black hole solutions can be given as a power series in three small parameters: the order parameter $\epsilon \equiv g^2\langle j_1^x\rangle/2$ and chemical-potential-like objects we call superfluid velocities, $A_x^3(u{=}0) \equiv \mathcal{A}_x^3 = v_\parallel$ and $A_y^3(u{=}0) \equiv \mathcal{A}_y^3 = v_\perp$. Velocity is a bit of a misnomer here as the objects $v_\parallel$ and $v_\perp$, like the chemical potential μ, have mass dimension one. The name is motivated by their canonical conjugacy to the currents j_3^x and j_3^y, and by the discussion in section 1.1.6 that showed that boosting the superfluid to velocity $\vec{v}$ is equivalent to coupling it to an external constant gauge field.

From the AdS/CFT dictionary, the currents $\langle j_a^\mu\rangle$ and external field strength $\mathcal{A}_\mu^a$ in the field theory can be determined from the small u expansion of the bulk gauge field A_μ^a:

$$A_\mu^a = \mathcal{A}_\mu^a + \frac{1}{2}g^2\langle j_\mu^a\rangle u^2 + \ldots . \tag{3.11}$$

3.3.1 The Background

We begin with small steps and construct the solution in the limit $v_\perp = v_\parallel = 0$. In the probe approximation, the equations of motion for the gauge field take the form

$$\mathcal{D}_t A_t^3 = \frac{(A_x^1)^2}{f} A_t^3 \qquad \text{and} \qquad \mathcal{D}_x A_x^1 = -\frac{(A_t^3)^2}{f^2} A_x^1 \,, \tag{3.12}$$

where we have defined the linear second order differential operators

$$\mathcal{D}_t \equiv \partial_u^2 - \frac{1}{u}\partial_u \qquad \text{and} \qquad \mathcal{D}_x = \mathcal{D}_y \equiv \partial_u^2 + \left(\frac{f'}{f} - \frac{1}{u}\right)\partial_u \,. \tag{3.13}$$

To keep the equations simple in what follows, we choose to put the horizon of the black hole at $u_h = 1$. To restore units, dimensionful quantities such as the chemical potential μ, frequencies ω, and wave-vectors k should be replaced with the dimensionless combinations μu_h, ωu_h, and $k u_h$, respectively.

As pointed out by ref. [61], when $A_t^3 = 4(1-u^2)$ there is an analytic solution to the second equation of (3.12) that is regular at the horizon, of the form

$$A_x^1 = \epsilon \frac{u^2}{(1+u^2)^2} \,. \tag{3.14}$$

From eq. (3.11), the meaning of ϵ in the dual field theory is, up to normalization, that of an expectation value for the non-abelian current $\langle j_x^1 \rangle = 2\epsilon/g^2$. The existence of the solution (3.14) indicates that the superfluid phase transition occurs when $\mu = 4$.[6] Given this zero mode, we look for a general solution to eqs. (3.12) as a series expansion

[6] There are in fact a countable set of such zero modes with $\mu = 4k$ where k is a positive integer, which also have analytical expressions. As the higher zero modes have higher free energy, they should not affect the phase diagram of the system.

in ϵ:

$$\begin{aligned} A_x^1 &= \epsilon \frac{u^2}{(1+u^2)^2} + \epsilon^3 w_1 + \epsilon^5 w_2 + \mathcal{O}(\epsilon^7)\,, && (3.15) \\ A_t^3 &= 4(1-u^2) + \epsilon^2 \phi_1 + \epsilon^4 \phi_2 + \mathcal{O}(\epsilon^6)\,. && (3.16) \end{aligned}$$

The solution describes the system for $\mu \gtrsim 4$. Our strategy will be to fix the expectation value of $\langle j_x^1 \rangle = 2\epsilon/g^2$ but to allow the chemical potential to be corrected order by order: $\mu = 4 + \epsilon^2 \delta\mu_1 + \epsilon^4 \delta\mu_2 + \ldots$. Thus, in solving the differential equations, we require the boundary condition that the $\mathcal{O}(u^2)$ term in w_i vanish while $\phi_i(0)$ is allowed to be nonzero.

The differential equation governing ϕ_1 is

$$\mathcal{D}_t \phi_1 = \frac{4u^4}{(1+u^2)^5}\,, \tag{3.17}$$

which has the solution

$$\phi_1 = (1-u^2)\delta\mu_1 + \frac{1}{96}\left(5u^2 - \frac{8u^2(1+3u^2+u^4)}{(1+u^2)^3}\right)\,. \tag{3.18}$$

We applied the boundary condition that ϕ_1 vanish at the horizon. Also, $\delta\mu_1$ corresponds to a shift of the chemical potential by $\epsilon^2\delta\mu_1$. The value of $\delta\mu_1$ is constrained by the solution for w_1, as we now see. The differential equation for w_1 is

$$w_1'' - \frac{1+3u^4}{u(1-u^4)} w_1' + \frac{16}{(1+u^2)^2} w_1 = -\frac{8u^2}{(1-u^2)(1+u^2)^4}\delta\phi_1\,. \tag{3.19}$$

We require the boundary conditions that w_1 be regular at the horizon and vanish at the boundary ($u = 0$). These conditions leave us with the solution

$$w_1 = \frac{cu^2}{(1+u^2)^2} + \frac{u^4(39u^6 - 331u^4 - 819u^2 - 369)}{20{,}160(1+u^2)^5} + \frac{13u^2 \ln(1+u^2)}{1680(1+u^2)^2}\,, \tag{3.20}$$

and the constraint

$$\delta\mu_1 = \frac{71}{6720}\,. \tag{3.21}$$

The term in w_1 proportional to c is just the zero mode, and, consistent with our strategy, we set $c = 0$.

For the free energy calculation we perform below, we also need the next order corrections, ϕ_2 and w_2. The expressions are too cumbersome to repeat here. The structure and boundary conditions are analogous to the case of ϕ_1 and w_1 considered above.

The near boundary expansion of our solution takes the form

$$\begin{aligned}
A^1_x &= \epsilon u^2 + \mathcal{O}(u^4)\,, && (3.22)\\
A^3_t &= \left(4 + \frac{71\epsilon^2}{6720} + \delta\mu_2\epsilon^4 + \mathcal{O}(\epsilon^6)\right) && (3.23)\\
&\quad - \left(4 + \frac{281\epsilon^2}{6720} - \left(\frac{1343 - 1365\ln 2}{2{,}822{,}400} - \delta\mu_2\right)\epsilon^4 + \mathcal{O}(\epsilon^6)\right) u^2 + \mathcal{O}(u^4)\,,
\end{aligned}$$

where

$$\delta\mu_2 = \frac{13(-4{,}015{,}679 + 5{,}147{,}520\ln 2)}{75{,}866{,}112{,}000}\,. \tag{3.24}$$

These expansions match well with numerical solutions that we found close to the transition temperature.

3.3.2 Superfluid flow

In this section, we generalize the background above to allow for the possibility of a superfluid flow. In terms of the bulk solution, this generalization requires turning on a constant value of $A^3_y(u{=}0) = v_\perp$ and $A^3_x(u{=}0) = v_\parallel$ at the boundary corresponding to a non-zero superfluid velocity $(v_\parallel, v_\perp, 0)$. The differential equations describing this background are a modification of eqs. (3.12):

$$u^2 f \mathcal{D}_\lambda A^1_\lambda = g^{\mu\nu}\left(A^3_\mu A^3_\nu A^1_\lambda - A^1_\mu A^3_\nu A^3_\lambda\right)\,, \tag{3.25}$$

$$u^2 f \mathcal{D}_\lambda A^3_\lambda = g^{\mu\nu}\left(A^1_\mu A^1_\nu A^3_\lambda - A^3_\mu A^1_\nu A^1_\lambda\right)\,, \tag{3.26}$$

$$A^1_t \partial_u A^3_t - A^3_t \partial_u A^1_t = f A^1_x \partial_u A^3_x - f A^3_x \partial_u A^1_x + f A^1_y \partial_u A^3_y - f A^3_y \partial_u A^1_y\,, \tag{3.27}$$

where we set $A^2_\mu = A^a_z = A^a_u = 0$. The repeated covariant λ indices on the left hand side are not to be summed over. As before, we solve this system in a small ϵ expansion, but we also add another small expansion parameter $\delta \sim v_\perp \sim v_\parallel$. There is a non-uniformity in the limit $v_\perp \to 0$ and $v_\parallel \to 0$, and we find two branches of solutions for small values of the superfluid velocity. In the case where $v_\perp > v_\parallel$, we find

$$A^1_t = \mathcal{O}(\epsilon^2)\,, \tag{3.28}$$

$$A^1_x = \epsilon\frac{u^2}{(1+u^2)^2} - \epsilon(v_\perp^2 + v_\parallel^2)\frac{u^2(u^2+4\ln(1+u^2))}{24(1+u^2)^2} + \dots\,, \tag{3.29}$$

$$A^1_y = -\epsilon\frac{v_\parallel}{v_\perp}\frac{u^2}{(1+u^2)^2} + \epsilon(v_\parallel^2 + v_\perp^2)\frac{v_\parallel}{v_\perp}\frac{u^2(u^2+4\ln(1+u^2))}{24(1+u^2)^2} + \dots\,, \tag{3.30}$$

$$\begin{aligned} A^3_t &= 4(1-u^2) + \frac{1}{3}(v_\parallel^2 + v_\perp^2)(1-u^2) \\ &\quad + \epsilon^2\frac{v_\perp^2 + v_\parallel^2}{v_\perp^2}\frac{(1-u^2)(71+3u^2-627u^4-279u^6)}{6720(1+u^2)^3} + \dots\,, \end{aligned} \tag{3.31}$$

$$A^3_x = v_\parallel - \epsilon^2\frac{v_\parallel}{v_\perp}\frac{v_\parallel^2+v_\perp^2}{v_\perp}\frac{u^2(3+9u^2+4u^4)}{144(1+u^2)^3} + \dots\,, \tag{3.32}$$

$$A^3_y = v_\perp - \epsilon^2\frac{v_\parallel^2+v_\perp^2}{v_\perp}\frac{u^2(3+9u^2+4u^4)}{144(1+u^2)^3} + \dots\,. \tag{3.33}$$

In the case $v_\perp < v_\parallel$, we find

$$A_t^1 = \epsilon \frac{v_\perp^2 + v_\parallel^2}{v_\parallel} \frac{u^2(1-u^2)}{4(1+u^2)} + \ldots , \tag{3.34}$$

$$A_x^1 = \epsilon \frac{u^2}{(1+u^2)^2} + \epsilon (v_\perp^2 + v_\parallel^2) \frac{u^2(u^2 - 2\ln(1+u^2))}{24(1+u^2)^2} + \ldots , \tag{3.35}$$

$$A_y^1 = \epsilon \frac{v_\perp}{v_\parallel} \frac{u^2}{(1+u^2)^2} + \epsilon (v_\perp^2 + v_\parallel^2) \frac{v_\perp}{v_\parallel} \frac{u^2(u^2 - 2\ln(1+u^2))}{24(1+u^2)^2} + \ldots , \tag{3.36}$$

$$\begin{aligned} A_t^3 = {} & 4(1-u^2) + \frac{1}{6}(v_\parallel^2 + v_\perp^2)(1-u^2) \\ & + \epsilon^2 \frac{v_\perp^2 + v_\parallel^2}{v_\parallel^2} \frac{(1-u^2)(71 + 3u^2 - 627u^4 - 279u^6)}{6720(1+u^2)^3} + \ldots , \end{aligned} \tag{3.37}$$

$$A_x^3 = v_\parallel - \epsilon^2 \frac{v_\parallel^2 + v_\perp^2}{v_\parallel} \frac{u^2(3 + 9u^2 - 2u^4)}{288(1+u^2)^3} + \ldots , \tag{3.38}$$

$$A_y^3 = v_\perp - \epsilon^2 \frac{v_\perp}{v_\parallel} \frac{v_\parallel^2 + v_\perp^2}{v_\parallel} \frac{u^2(3 + 9u^2 - 2u^4)}{288(1+u^2)^3} + \ldots . \tag{3.39}$$

These solutions can be used to compute the speed of second sound perpendicular and parallel to the order parameter A_x^1. In a two component fluid, there are typically two propagating collective modes, ordinary and second sound. In our probe approximation, we see only the superfluid component, and the single collective motion available to us we call second sound. From our holographic perspective, ordinary sound would involve fluctuations of the metric so it is suppressed in the limit $\kappa^2/g^2 \to 0$.

The speed of second sound, like that of ordinary sound, can be computed from derivatives of the state variables. From ref. [69], the second sound speed squared in this probe limit should be

$$c_2^2 = -\left. \frac{\partial j/\partial v}{\partial \rho/\partial \mu} \right|_{v=0} . \tag{3.40}$$

From eq. (3.11), the values of the charge current $j = \langle j_i^x \rangle$ and the charge density $\rho = \langle j_3^t \rangle$ can be read off from the order u^2 pieces of A_i^3 and A_t^3, respectively.

Because our system is not rotationally symmetric, the speed of second sound will depend on the direction of propagation. Let $c_\perp$ and $c_\parallel$ be the speeds perpendicular and parallel to the order parameter A_x^1, respectively. The speed $c_\perp$ can be computed from the background solution $v_\perp > v_\parallel$ while $v_\parallel$ can be computed from the solution with $v_\parallel > v_\perp$. In the case $v_\perp > v_\parallel = 0$, we find that

$$A_t^3 = \mu + \frac{1}{71}(840 - 281\mu)u^2 + \dots \,, \tag{3.41}$$

$$A_y^3 = v_\perp - \frac{140}{71} v_\perp (\mu - 4) u^2 + \dots \,. \tag{3.42}$$

and hence, up to higher order corrections in ϵ,

$$c_\perp^2 \approx \frac{71}{13{,}488}\epsilon^2 \approx \frac{140}{281}(\mu - 4)\,. \tag{3.43}$$

(We used the fact that $\mu - 4 \approx 71\epsilon^2/6720$, which can be read off from (3.23).) In the case $v_\parallel > v_\perp = 0$, at leading order A_t^3 remains the same but now we need

$$A_x^3 = v_\parallel - \frac{70}{71} v_\parallel (\mu - 4) u^2 + \dots \,. \tag{3.44}$$

We find that

$$c_\parallel^2 \approx \frac{1}{2}\frac{71}{13{,}488}\epsilon^2 \approx \frac{70}{281}(\mu - 4)\,. \tag{3.45}$$

We confirm these results for $c_\parallel$ and $c_\perp$ in Sections 3.5.2 and 3.5.3 through an analysis of the hydrodynamic poles in the current-current correlation functions. For numerical results valid when ϵ is not necessarily small, see Figure 3.2.

These perturbative solutions in $v_\perp$ and $v_\parallel$ can also be used to analyze the phase diagram of the system near the critical point $\mu_c = 4$. At the critical point, we expect the order parameter to vanish, so $\epsilon = 0$. The value of A_t^3 at $u = 0$ can be reinterpreted as the value of the chemical potential. These two facts give us a relation between the chemical potential and superfluid velocity along the critical line separating the two phases. For superfluid flow parallel to the order parameter, we expect

$$\mu \approx 4 + \frac{1}{6} v_{\parallel}^2 \tag{3.46}$$

while for flow perpendicular to the order parameter, we have instead

$$\mu \approx 4 + \frac{1}{3} v_{\perp}^2 \,. \tag{3.47}$$

3.3.3 The Free Energy

We compute the contribution to the free energy from the gauge field term in the on-shell action:

$$\begin{aligned} S &= -\frac{1}{4g^2} \int d^5x \sqrt{-g} F^a_{AB} F^{aAB} \\ &= \frac{\beta \mathrm{Vol}_3}{2g^2} \int_0^1 \frac{du}{u} \left((\partial_u A_t^3)^2 - f(u)(\partial_u A_x^1)^2 + \frac{1}{f(u)} (A_x^1 A_t^3)^2 \right) \\ &= -\frac{\beta \mathrm{Vol}_3}{2g^2} \left(\int_0^1 \frac{du}{u} f(u)(\partial_u A_x^1)^2 + \frac{1}{u} A_t^3 (\partial_u A_t^3) \Big|_{u=0} \right) . \end{aligned} \tag{3.48}$$

For a background where $A_x^1 = 0$ and

$$A_t^3 = (4 + \delta\mu_1 \epsilon^2 + \delta\mu_2 \epsilon^4)(1 - u^2) \,, \tag{3.49}$$

the on-shell action is

$$S_{\mathrm{vac}} = \frac{\beta \mathrm{Vol}_3}{4g^2} \left(64 + \frac{71}{210} \epsilon^2 + \left(-\frac{51{,}145{,}217}{2{,}370{,}816{,}000} + \frac{4979}{176{,}400} \ln 2 \right) \epsilon^4 + \mathcal{O}(\epsilon^6) \right) . \tag{3.50}$$

Here Vol_3 is the spatial volume of the field theory while $\beta = 1/T$ is the inverse temperature. For the background where $A_x^1 \neq 0$ has condensed, we find in contrast that

$$S_{\mathrm{sf}} = \frac{\beta \mathrm{Vol}_3}{4g^2}\left(64 + \frac{71}{210}\epsilon^2 + \left(-\frac{48{,}014{,}117}{2{,}370{,}816{,}000} + \frac{4979}{176{,}400}\ln 2\right)\epsilon^4 + \mathcal{O}(\epsilon^6)\right). \quad (3.51)$$

The difference in the values of the two on-shell actions is

$$\beta \Delta P = S_{\mathrm{vac}} - S_{\mathrm{sf}} = \frac{\beta \mathrm{Vol}_3}{4g^2}\left(-\frac{71}{53{,}760}\epsilon^4 + \mathcal{O}(\epsilon^6)\right). \quad (3.52)$$

Now ΔP can be interpreted also as a difference in the free energies because the free energy (in the grand canonical ensemble) is minus the value of the on-shell action. That $\Delta P < 0$ implies that the free energy of the superfluid phase is smaller and thus the superfluid is stable.

Moreover, from the fact that the free energy difference scales as ϵ^4, we see that the phase transition is second order. For small ϵ, $\epsilon^4 \sim (\mu - \mu_c)^2$. If we restore dimensions, then μ should be replaced by $\mu u_h = \mu/\pi T$. Thus, $\epsilon^4 \sim (T_c - T)^2$. The derivative of P with respect to temperature is continuous but non-differentiable.

3.4 Formal Remarks about Green's Functions

A field theory with a non-abelian global symmetry, such as $SU(2)$, has by Noether's Theorem, a conserved current j_a^μ which transforms under the adjoint representation of this symmetry group. In this paper, we are interested in Green's functions for this current, in particular the Fourier transformed retarded current-current correlation functions:

$$G_{ab}^{\mu\nu}(p) = i\int d^4x\, e^{-ip\cdot x}\langle [j_a^\mu(x), j_b^\nu(0)]\rangle \theta(t)\,. \quad (3.53)$$

If the symmetry is non-anomalous, then we can weakly gauge it by coupling the current to an external gauge field $\mathcal{A}_\mu^a$. Gauge invariance then implies that the correlation functions obey a series of Ward identities. For the one-point function, the covariant derivative of the current vanishes:

$$0 = \left(\partial_\mu \delta_a^c + f_{ab}{}^c \mathcal{A}_\mu^b\right) \langle j_c^\mu \rangle \,. \tag{3.54}$$

More usefully for the present discussion, there is also a Ward identity for the retarded two-point function. We give here the Fourier transformed version:

$$0 = (ip_\mu \delta_a^c + f_{ab}{}^c \mathcal{A}_\mu^b) G_{cd}^{\mu\nu}(p) + f_{ad}{}^c \langle j_c^\nu \rangle \,. \tag{3.55}$$

For our gravitational system beyond the phase transition, the gravitational bulk values of both A_t^3 and A_x^1 are non-zero. The AdS/CFT dictionary allows us to read $\mathcal{A}_\mu^a$ and $\langle j_a^\mu \rangle$ from the near boundary expansion of A_μ^a using the relation (3.11). For the system under consideration here, of the components of the external gauge field only $\mathcal{A}_t^3 = \mu$ is non-zero in the field theory. We have two non-vanishing components of the current, $\langle j_1^x \rangle$ and $\langle j_3^t \rangle$.

Below, we compute the Green's functions that describe the response of the system to an external gauge field in the third isospin direction: $G_{a3}^{\mu\nu}$. The relevant Ward identities componentwise are

$$\begin{aligned} 0 &= ip_\mu G_{3a}^{\mu\nu} - \langle j_1^x \rangle \delta_{a2} \delta^{\nu x} \,, && (3.56) \\ 0 &= ip_\mu G_{23}^{\mu\nu} + \mu G_{13}^{t\nu} + \langle j_1^x \rangle \delta^{x\nu} \,, && (3.57) \\ 0 &= ip_\mu G_{13}^{\mu\nu} - \mu G_{23}^{t\nu} \,. && (3.58) \end{aligned}$$

Our Green's functions below obey this set of Ward identities.

Another important observation for the Green's functions under consideration is the symmetry under swapping the indices. We observe that

$$G_{ab}^{\mu\nu}(p) = (-1)^{\phi(a,b)} G_{ba}^{\nu\mu}(p) \,, \tag{3.59}$$

where $\phi(a,b)$ is equal to -1 if either $a=2$ or $b=2$, but not both, and 1 otherwise. This symmetry follows from the discrete symmetries of the system. Given that our currents are even under PT, i.e. parity and time reversal, if PT were a symmetry of the state, we would expect the Green's functions to be symmetric under an index swap. Our state is not symmetric under PT, but it is symmetric under PT times a $\mathbb{Z}_2$ operation on the $\mathfrak{su}(2)$ Lie algebra, $\sigma_1 \to -\sigma_1$ and $\sigma_3 \to -\sigma_3$.

3.4.1 Computation of two-point functions

To compute the current-current correlators (3.53) in the probe approximation, we perturb the background gauge field by sending

$$A^a_A \to A^a_A + \delta A^a_A \,. \tag{3.60}$$

Consequently, the corresponding field strength F^a_{AB} changes to $F^a_{AB} + f^a_{AB}$, with f^a_{AB} given by

$$f^a_{AB} = \partial_A \delta A_B - \partial_B \delta A_A + \epsilon^{abc} \delta A^b_A A^c_B + \epsilon^{abc} A^b_A \delta A^c_B \,. \tag{3.61}$$

From (3.1), one can see that the quadratic action for δA^a_A is

$$S_2 = -\frac{1}{4g^2} \int d^5x \, \sqrt{-g} f^a_{AB} f^{ABa} \,, \tag{3.62}$$

which gives the linearized equations of motion

$$\nabla^A f^a_{AB} + \epsilon^{abc} A^{Ab} f^c_{AB} + \epsilon^{abc} \delta A^{Ab} F^c_{AB} = 0 \,. \tag{3.63}$$

The quadratic action (3.62) is in fact not well defined because the integrand diverges as $\ln u$ at small u as we will see. We will regulate this divergence using holographic renormalization [5].

For definiteness, we will only analyze the case where the background gauge field doesn't depend on t or $\vec{x}$ and where its radial components A_u^a vanish. We choose a similar gauge for the perturbations by requiring $\delta A_u^a = 0$. Equations (3.63) can be solved approximately in the limit of small u. An appropriate series expansion in this limit is

$$\delta A_\mu^a(t, \vec{x}, u) = \alpha_\mu^a(t, \vec{x}) + \tilde{\alpha}_\mu^a(t, \vec{x}) u^2 \ln u + \beta_\mu^a(t, \vec{x}) u^2 + \dots \, . \tag{3.64}$$

for some vector-valued functions $\alpha(t, \vec{x})$, $\tilde{\alpha}(t, \vec{x})$, $\beta(t, \vec{x})$, etc. The values of α and β are the only ones that can be specified independently; all the other functions appearing in this expansion, namely $\tilde{\alpha}$ and higher order corrections, can be expressed in terms of α and β.

Plugging (3.64) into (3.63) and looking at the term with the lowest power of u in the equation with $B = \nu$, one finds a relation between $\tilde{\alpha}_\nu^a$ and α_ν^a:

$$\tilde{\alpha}_\nu^a = -\frac{1}{2} \left[\partial^\mu f_{\mu\nu}^a + \epsilon^{abc} A^{\mu b} f_{\mu\nu}^c + \epsilon^{abc} \delta A^{\mu b} F_{\mu\nu}^c \right] \Big|_{u=0} . \tag{3.65}$$

Upon integration by parts in (3.62), the unregularized on-shell quadratic action can be written as

$$S_2^{\text{on-shell}} = \frac{1}{2g^2} \int d^4x \, \frac{1}{u} (\delta A^{\nu a})(\partial_u \delta A_\nu^a) \Big|_{u=1/\Lambda} . \tag{3.66}$$

The divergence that arises as one takes $\Lambda \to \infty$ comes from the $\partial_u \delta A_\nu^a$ term whose most divergent piece goes like $u \ln u$ at small u. This divergence can be regulated by adding the counterterm

$$S_{\text{ct}} = -\frac{\ln \Lambda}{2g^2} \int d^4x \, \delta A^{\nu a} \left[\partial^\mu f_{\mu\nu}^a + \epsilon^{abc} A^{\mu b} f_{\mu\nu}^c + \epsilon^{abc} \delta A^{\mu b} F_{\mu\nu}^c \right] \Big|_{u=1/\Lambda} . \tag{3.67}$$

Note that this counterterm depends only on the values of the gauge field on the surface $u = 1/\Lambda$ and on its derivatives along this surface, as required by holographic renormalization.

As a side note, a simpler formula for $\tilde{\alpha}^a_\nu$ can be found if only A^3_t and α^3_ν approach non-zero values at the boundary of AdS. In this case, only $\tilde{\alpha}^3_\nu$ is non-zero and is given by

$$\tilde{\alpha}^3_\nu = -\frac{1}{2}\left(\partial_\mu \partial^\mu \alpha^a_\nu - \partial^\mu \partial_\nu \alpha^a_\mu\right) . \tag{3.68}$$

Assuming that $\alpha^a_\nu(t,\vec{x}) = \alpha^a_\nu e^{-i(\omega t - \vec{p}\cdot\vec{x})}$ then

$$\tilde{\alpha}^3_\nu = \frac{1}{2}(\vec{p}^2 - \omega^2)\alpha^3_\nu - \frac{1}{2}p_\nu\left(\omega\alpha^3_t + p_x\alpha^3_x + p_y\alpha^3_y + p_z\alpha^3_z\right) . \tag{3.69}$$

To compute the Fourier transformed two-point function, we first Fourier transform the regulated on-shell action

$$S_2^{\text{on-shell}} = \frac{1}{g^2}\int \frac{d^4p}{(2\pi)^4}\,\alpha^\mu_a(-p)(\beta^a_\mu(p) + c\tilde{\alpha}^a_\mu(p))\,, \tag{3.70}$$

where c is an arbitrary constant introduced by the regularization procedure. Although such an action is not a generating functional for the retarded Green's function, using the procedure outlined by Son and Starinets [52], we can identify the retarded Green's function as[7]

$$G^{a\nu}_{\mu b}(p) = \frac{2}{g^2}\frac{\partial\left[\beta^a_\mu(p) + c\tilde{\alpha}^a_\mu(p)\right]}{\partial\alpha^b_\nu(p)} . \tag{3.71}$$

The linear response of a system to a perturbation $\alpha^b_\nu(p)$ is then a current density of the form

$$\langle j^a_\mu(p)\rangle = G^{a\nu}_{\mu b}(p)\alpha^b_\nu(p)\,. \tag{3.72}$$

For most physical questions, the ambiguity in the choice of c should be irrelevant. More precisely, one can see from (3.65) that schematically $\tilde{\alpha} = \partial\partial\alpha + \partial\alpha + \alpha$, so $G^{a\nu}_{\mu b}(p)$ is ambiguous up to an additive term analytic in p. Its Fourier transform $G^{a\nu}_{\mu b}(x)$ is

[7] For a more precise discussion of how to derive these Green's functions from an action principle and generating functional, see ref. [6]. See also ref. [8] and the discussion in Appendix C of ref. [7].

ambiguous up to an additive term of the from $c_1\delta^4(x) + c_2^\lambda \partial_\lambda \delta^4(x) + c_3^{\lambda\rho}\partial_\lambda\partial_\rho\delta^4(x)$, where c_1, c_2^λ, and $c_3^{\lambda\rho}$ are constants that depend on the particular Green's function we are computing. Since in position space equation (3.72) reads

$$\langle j_a^\mu(x)\rangle = \int d^4x'\, G_{ab}^{\mu\nu}(x-x')\alpha_\nu^b(x')\,, \tag{3.73}$$

it follows that the ambiguity in the choice of c does not affect the result of $\langle j_a^\mu(x)\rangle$ if $\alpha_\nu^b(x) = 0$. In particular, the late-time, large-distance response of the system to localized sources is not affected by this ambiguity. There are many subtleties in these calculations.

3.5 Fluctuations

To calculate the Fourier transformed retarded current-current correlation functions, we need to study fluctuations of the $SU(2)$ gauge fields $A_\mu^a(x)$ in our black hole background.

In the superfluid phase, the expectation value of the order parameter $A_x^1 \neq 0$ breaks rotational symmetry and makes our task richer and more complicated than in the rotationally symmetric case where only $A_t^3 \neq 0$. In the rotationally symmetric case, it would be enough to consider a fluctuation with a time and space dependence of the form $e^{-i\omega t+ikx}$. Given the breaking of rotational symmetry, we should in principle consider a more general dependence where we allow for motion both parallel and transverse to the order parameter: $e^{-i\omega t+ik_x x+ik_y y}$. Because of the complexity of the full result, we shall not present a full accounting of all the Green's functions here. Instead we will content ourselves by studying various informative limits where either $k_x = 0$ or $k_y = 0$.

We make a few other additional simplifying restrictions. Following in the footsteps of refs. [47, 70] where the third isospin direction was interpreted as the $U(1)$ of electricity and magnetism, we will consider Green's functions where at least one of the $SU(2)$ isospin indices is equal to three. In other words, we are interested in the linear response of the system to external electric and magnetic fields.

The last simplifying restriction is to limit our study to the hydrodynamic regime, where the order parameter, the frequency, and the wave-vector are small compared to the temperature. In our dimensionless notation, $\epsilon, k, \omega \ll 1$. It is only in this limit that we have analytic results although it is straightforward to calculate the Green's functions numerically beyond this regime.

We work out the Green's functions in five cases. The first and simplest case, for which we give the most detailed description of the calculation, is for a fluctuation transverse to the order parameter and a wave vector transverse to both the order parameter and the polarization of the fluctuation. We call this fluctuation the pure transverse mode. We next consider fluctuations that correspond to a second sound mode in two different limits, one where the sound is propagating parallel to the order parameter and one where the sound is propagating transverse. These two sets of fluctuations give us independent confirmation of the speeds of second sound computed in Section 3.3.2 from thermodynamics. Finally we consider fluctuations that correspond to a diffusive mode, again in two different limits, one where the diffusion is parallel to the order parameter, one in which the diffusion is transverse. In Section 3.5.6, we discuss conductivities and the London equations.

In what follows, to avoid cumbersome indices, we define new variables for the background values of the gauge field:

$$A_x^1 \equiv W \qquad \text{and} \qquad A_t^3 \equiv \Phi \,. \tag{3.74}$$

3.5.1 Pure transverse mode

The pure transverse mode is described by fluctuations of the field A_y^3 with only z spatial dependence. We decompose the fluctuations into Fourier modes:

$$\delta A_y^3(u,t,z) = a_y(u) e^{-i\omega t + ikz} \,. \tag{3.75}$$

These modes transverse to the order parameter A_x^1 decouple from the other fluctuations of the gauge field and are governed by the differential equation:

$$\mathcal{D}_y a_y = \frac{(k^2+W^2)f - \omega^2}{f^2} a_y \,, \tag{3.76}$$

where $\mathcal{D}_y$ was defined in eq. (3.13).

Near the horizon $u=1$, we find that $a_y \sim (1-u)^{\pm i\omega/4}$ satisfies either ingoing or outgoing plane wave type boundary conditions. Consistent with the presence of an event horizon, it is natural to choose ingoing boundary conditions (the minus sign in the exponent). This choice leads to retarded, as opposed to advanced, Green's functions in the dual field theory [52]. At the boundary $u=0$ of AdS, we would like the freedom to set $a_y(0) = a_{y0}$ to some arbitrary value of our choosing, corresponding to perturbing the dual field theory by a small external field strength. These two boundary conditions along with the differential equation uniquely specify the functional form of a_y.

While an analytic solution to eq. (3.76) does not appear to be available, one can easily solve this equation in the limit of small ω, k, and ϵ. We can write the solution for a_y, valid to order $\epsilon^2 k$, $\epsilon^2\omega$, k^2, and ω^2, in the form

$$a_y = a_{y0} \left(\frac{1-u^2}{1+u^2}\right)^{-i\omega/4} \left(1 + \epsilon^2 a_{y\epsilon} + \epsilon^2 \omega a_{y\omega\epsilon} + k^2 a_{yk} + \omega^2 a_{y\omega} + \ldots\right) \,. \tag{3.77}$$

We find

$$a_{y\epsilon} = -\frac{u^2(3+9u^2+4u^4)}{144(1+u^2)^3}, \tag{3.78}$$

$$a_{y\omega\epsilon} = -\frac{iu^2(12+27u^2+13u^4)}{864(1+u^2)^3}, \tag{3.79}$$

$$a_{yk} = \frac{1}{8}\left(2\ln(u)\ln\left(\frac{1+u^2}{1-u^2}\right)+\mathrm{Li}_2(-u^2)-\mathrm{Li}_2(u^2)\right). \tag{3.80}$$

The expression for $a_{y\omega}$ is too cumbersome to give here. Near the boundary, this solution (3.77) has the expansion

$$a_y = a_{y0} + a_{y0}\left(\frac{i\omega}{2}-\frac{\epsilon^2}{48}-\frac{i\omega\epsilon^2}{72}-\frac{\omega^2\ln 2}{4}+\frac{1}{2}(\omega^2-k^2)\left(\frac{1}{2}-\ln(u)\right)\right)u^2+\dots \tag{3.81}$$

From this near boundary expansion and eq. (3.72), we can calculate the two-point function for the current in the hydrodynamic limit:

$$G_{33}^{yy}(\omega,k) = \frac{2}{g^2}\left(\frac{i\omega}{2}-\frac{\epsilon^2}{48}-\frac{i\omega\epsilon^2}{72}-\frac{\omega^2\ln 2}{4}+(\omega^2-k^2)c\right)+\dots. \tag{3.82}$$

Note that the counter-term ambiguity, proportional to an arbitrary constant c, is of the form predicted in eq. (3.69).

3.5.2 Transverse sound fluctuations

In general, second sound modes are expected to produce poles in the density-density correlation function. We thus need to consider fluctuations in the conjugate field A_t^3. If we consider sound modes moving transverse to the order parameter, we can take the fluctuations to have a y dependence but no x dependence. The self-consistent set of fluctuations to consider that couple to $\delta A_t^3(u,t,y)$ are

$$\begin{aligned}
\delta A_t^3(u,t,y) &= a_t^3(u)e^{-i\omega t+iky}\,,\\
\delta A_y^3(u,t,y) &= a_y^3(u)e^{-i\omega t+iky}\,,\\
\delta A_x^a(u,t,y) &= a_x^a(u)e^{-i\omega t+iky}\,,
\end{aligned} \tag{3.83}$$

where $a = 1, 2$.

The four fluctuations satisfy four second order ordinary differential equations and one first order constraint:

$$\mathcal{D}_x a_x^1 = \left(\frac{-\omega^2-\Phi^2+k^2 f}{f^2}\right) a_x^1 + \frac{2\Phi(i\omega\Phi a_x^2 - W a_t^3)}{f^2}\,, \tag{3.84}$$

$$\mathcal{D}_x a_x^2 = \left(\frac{-\omega^2-\Phi^2+k^2 f}{f^2}\right) a_x^2 - \frac{2i\omega\Phi a_x^1 - iW(\omega a_t^3 + kf a_y^3)}{f^2}\,, \tag{3.85}$$

$$\mathcal{D}_y a_y^3 = \frac{-\omega^2+W^2 f}{f^2}a_y^3 - \frac{k\omega}{f^2}a_t^3 + \frac{ikW}{f}a_x^2\,, \tag{3.86}$$

$$\mathcal{D}_t a_t^3 = -\frac{k^2+W^2}{f}a_t^3 + \frac{\omega k}{f}a_x^3 + \frac{2W\Phi}{f}a_x^1 - \frac{i\omega}{f}a_x^2\,, \tag{3.87}$$

$$0 = \frac{i\omega}{f}\partial_u a_t^3 + ik\partial_u a_y^3 + W\partial_u a_x^2 - (\partial_u W)a_x^2\,, \tag{3.88}$$

where $\mathcal{D}_t$, $\mathcal{D}_x$, and $\mathcal{D}_y$ were defined in eq. (3.13). We checked that the derivative of the constraint equation (3.88) with respect to u is a linear combination of all five differential equations (3.84)–(3.88). Thus if a solution of the first four differential equations satisfies the constraint for some u, it will satisfy the constraint equation at all u.

There are seven integration constants associated with this linear system (3.84)–(3.88). If we look at the horizon of the black hole at $u = 1$, we find seven different kinds of behavior. There exist six solutions that have plane wave behavior for a_x^1, a_x^2, and a_y^3 near the horizon of the form

$$(1-u)^{\pm i\omega/4}\,. \tag{3.89}$$

There is also a pure gauge solution,

$$a_t^3 = -i\omega\,, \qquad a_x^3 = ik\,, \qquad a_x^2 = -W\,. \tag{3.90}$$

As in the pure transverse case, we choose pure ingoing boundary conditions corresponding to $(1-u)^{-i\omega/4}$ behavior. At the boundary $u=0$ of our asymptotically AdS space, we would like to be able to perturb the system with arbitrary boundary values of a_t^3 and a_y^3 but set the "unphysical" components of the gauge field a_x^1 and a_x^2 to zero. These are four constraints and we have only three ingoing solutions. Thus we will also need to make use of the pure gauge solution to enforce our $u=0$ boundary conditions.

We solved the system perturbatively in ω, k, and ϵ. We present the results here in the limit where $\omega \sim k^2 \sim \epsilon^2$. The near boundary expansion ($u=0$) of the solution takes the form

$$\begin{aligned}
a_x^1 &= -\frac{(a_{t0}k + a_{y0}\omega)}{\mathcal{P}} 70k\epsilon\left(48k^2 + 3\epsilon^2 - 248i\omega\right)u^2 + \ldots\,, && (3.91)\\
a_x^2 &= -\frac{a_{t0}k + a_{y0}\omega}{\mathcal{P}}\frac{i\omega\epsilon}{k}\left(21{,}840k^2 + 843\epsilon^2 - 72{,}800i\omega\right)u^2 + a_{y0}\frac{i\epsilon}{k}u^2 + \ldots, && (3.92)\\
a_y^3 &= a_{y0} - \frac{(a_{t0}k + a_{y0}\omega)}{\mathcal{P}}\omega\Big(1120k^4 + 3k^2(117\epsilon^2 - 1120i\omega) && (3.93)\\
&\qquad + \frac{1}{48}(\epsilon^2 - 24i\omega)(843\epsilon^2 - 72{,}800i\omega)\Big)u^2 + \ldots\,, \\
a_t^3 &= a_{t0} + \frac{(a_{t0}k + a_{y0}\omega)}{\mathcal{P}}k\Big(1120k^4 + 3k^2(117\epsilon^2 - 1120i\omega) && (3.94)\\
&\qquad + \frac{1}{48}(\epsilon^2 - 24i\omega)(843\epsilon^2 - 72{,}800i\omega)\Big)u^2 + \ldots\,.
\end{aligned}$$

Note, the expression $(a_{t0}k + a_{y0}\omega)$ is not homogeneous in our scaling limit. We have included the leading corrections proportional to a_{t0} and a_{y0}. There are terms in the expansion proportional to $u^2 \ln u$ but they are subleading in ω, k, and ϵ.

The pole in this limit takes the form

$$\begin{aligned} \mathcal{P} &= -72{,}800 i\omega^3 + \left(43{,}120k^2 + 843\epsilon^2\right)\omega^2 + \frac{7i}{6}(4800k^4 + 553k^2\epsilon^2)\omega + \\ &\quad -\frac{71k^2\epsilon^4}{16} - 141k^4\epsilon^2 - 1120k^6 + \dots \,. \end{aligned} \tag{3.95}$$

Let us study this cubic polynomial in ω in two different limits. First, if $k \ll \epsilon$, we find three poles with the asymptotic form

$$\omega = \pm\sqrt{\frac{71}{13{,}488}}\epsilon k - \frac{147{,}217 i k^2}{947{,}532} + \dots\,, \tag{3.96}$$

$$\omega = -\frac{843 i\epsilon^2}{72{,}800} - \frac{4{,}335{,}443 i k^2}{15{,}397{,}395} + \dots\,. \tag{3.97}$$

The first two poles are propagating modes that we identify with second sound. Indeed, the speed of second sound agrees with the earlier result (3.43) from Section 3.3.2. The position of the third pole in this limit is determined mostly by the size of the order parameter ϵ and so we associate it with the zero mode that causes the phase transition from the superfluid phase back to the normal phase.

In the opposite limit, $k \gg \epsilon$, where the order parameter is small, the behavior should be close to that of the normal fluid. In this limit, we find

$$\omega = \left(\frac{\pm 11 - 3i}{65}\right)k^2 + \left(\frac{\pm 260{,}803 - 131{,}519 i}{26{,}644{,}800}\right)\epsilon^2 + \dots \tag{3.98}$$

$$\omega = -\frac{ik^2}{2} - \frac{5i\epsilon^2}{2928} + \dots\,. \tag{3.99}$$

The first two poles are associated with the zero modes that cause the phase transition from the normal phase to the superfluid phase and were discussed in ref. [47] while the third pole is associated to the diffusive mode of our conserved charge density. Indeed, the location of this diffusive pole is determined by the dynamics of the normal phase and was calculated, without the order ϵ^2 correction, long ago in ref. [71]. As we vary ϵ and k the number of poles cannot change. The two zero mode poles evolve into the sound poles of the previous limit while the diffusive pole becomes the zero mode pole of the previous limit.

From these small u expansions, we can read off the eight Green's functions G^{xt}_{13}, G^{xy}_{13}, G^{xt}_{23}, G^{xy}_{23}, G^{yy}_{33}, G^{yt}_{33}, G^{ty}_{33}, and G^{tt}_{33}. From the discrete symmetries (3.59), we can also read off four more Green's functions with the indices swapped. Note the prefactor $a_{t0}k + a_{y0}\omega$ in the small u expansion. This structure is necessary to satisfy the Ward identities (3.56).

As a further check, we consider a particular static limit of the density-density correlation function. From eqs. (3.72) and (3.94), we can read off the Green's function,

$$G^{tt}_{33} = -\frac{2}{g^2}\frac{k^2}{\mathcal{P}}\left(1120k^4 + 3k^2(117\epsilon^2 - 1120i\omega) + \frac{1}{48}(\epsilon^2 - 24i\omega)(843\epsilon^2 - 72{,}800i\omega)\right). \tag{3.100}$$

We are interested in the long wave-length limit of this Green's function:

$$\lim_{k\to 0} G^{tt}_{33}(0,k) = \frac{2}{g^2}\frac{281}{71}. \tag{3.101}$$

This long wave-length limit is equal to a thermodynamic susceptibility,

$$\lim_{k\to 0} G^{tt}_{33}(0,k) = \frac{\partial^2 P}{\partial\mu^2} = \frac{\partial\rho}{\partial\mu}. \tag{3.102}$$

Given this relation, we see that eq. (3.41) agrees with eq. (3.101).

3.5.3 Longitudinal sound fluctuations

Longitudinal sound modes correspond to the case where the fluctuations in A_t^3 depend only on x. A self-consistent set of perturbations in this case is given by

$$\begin{aligned}\delta A_t^a(u,t,x) &= a_t^a(u)e^{-i\omega t+ikx}\,,\\ \delta A_x^b(u,t,x) &= a_x^b(u)e^{-i\omega t+ikx}\,,\end{aligned}\tag{3.103}$$

where $a, b = 1, 2, 3$. These fields satisfy the following six second order equations and three constraints:

$$\mathcal{D}_t a_t^1 = \frac{1}{f}\left(-W\Phi a_x^3 - ik\Phi a_x^2 + k\omega a_x^1 + k^2 a_t^1\right)\,,\tag{3.104}$$

$$\mathcal{D}_t a_t^2 = \frac{1}{f}\left(2ikWa_t^3 + \left(k^2+W^2\right)a_t^2 + iW\omega a_x^3 + k\omega a_x^2 + ik\Phi a_x^1\right)\,,\tag{3.105}$$

$$\mathcal{D}_t a_t^3 = \frac{1}{f}\left(\left(k^2+W^2\right)a_t^3 - 2ikWa_t^2 + k\omega a_x^3 - iW\omega a_x^2 + 2W\Phi a_x^1\right)\,,\tag{3.106}$$

$$\mathcal{D}_x a_x^1 = \frac{1}{f^2}\left(-2W\Phi a_t^3 + ik\Phi a_t^2 - k\omega a_t^1 + 2i\Phi\omega a_x^2 - \left(\Phi^2+\omega^2\right)a_x^1\right)\,,\tag{3.107}$$

$$\mathcal{D}_x a_x^2 = \frac{1}{f^2}\left(-iW\omega a_t^3 - k\omega a_t^2 - ik\Phi a_t^1 - \left(\Phi^2+\omega^2\right)a_x^2 - 2i\Phi\omega a_x^1\right)\,,\tag{3.108}$$

$$\mathcal{D}_x a_x^3 = \frac{1}{f^2}\left(-k\omega a_t^3 + iW\omega a_t^2 + W\Phi a_t^1 - \omega^2 a_x^3\right)\,,\tag{3.109}$$

$$0 = -\Phi' a_t^2 + i\omega\partial_u a_t^1 + \Phi\partial_u a_t^2 + ifk\partial_u a_x^1\,,\tag{3.110}$$

$$0 = fW' a_x^3 + \Phi' a_t^1 - \Phi\partial_u a_t^1 + i\omega\partial_u a_t^2 + ifk\partial_u a_x^2 - fW\partial_u a_x^3\,,\tag{3.111}$$

$$0 = -fW' a_x^2 + i\omega\partial_u a_t^3 + fW\partial_u a_x^2 + ifk\partial_u a_x^3\,,\tag{3.112}$$

with $\mathcal{D}_t$ and $\mathcal{D}_x$ as defined in (3.13). Again, the three constraint equations are consistent with the second order equations in the sense that if they hold at some u, they hold at all u.

The system (3.104)–(3.112) has nine integration constants. The nine possible behaviors at the horizon are of two types: six plane wave solutions for a_x^1, a_x^2, and a_x^3

that behave as

$$(1-u)^{\pm i\omega/4} \tag{3.113}$$

close to $u=1$, and three pure gauge solutions given by

$$\begin{aligned}
&a_t^1 = -i\omega\alpha^1 - \Phi\alpha^2\,, \qquad a_t^2 = -i\omega\alpha^2 + \Phi\alpha^1\,, \qquad a_t^3 = -i\omega\alpha^3\,,\\
&a_x^1 = ik\alpha^1\,, \qquad a_x^2 = ik\alpha^2 - W\alpha^3\,, \qquad a_x^3 = W\alpha^2 + ik\alpha^3\,,
\end{aligned} \tag{3.114}$$

where α^a are arbitrary constants. As in the previous sections, we require no outgoing modes at the horizon, which amounts to specifying three of the nine integration constants. The other six integration constants are specified in terms of the values of the fields at $u=0$. In order to examine fluctuations in a_t^3 and a_x^3, we set their boundary values to a_{t0} and a_{x0}, respectively, and the boundary values of the other four fields to zero.

Solving the system (3.104)–(3.112) perturbatively in ω, k, and ϵ under the scaling assumption $\omega \sim k^2 \sim \epsilon^2$, we find that the boundary behavior of the fluctuations is

$$a_t^1 = \frac{a_{t0}k + a_{x0}\omega}{\mathcal{P}}\frac{\epsilon\omega}{4}(20{,}160k^2 + 843\epsilon^2 - 72{,}800i\omega)u^2 + \dots\,, \tag{3.115}$$

$$a_t^2 = \frac{a_{t0}k + a_{x0}\omega}{\mathcal{P}}\frac{35k^2\epsilon}{4}(48ik^2 + 3i\epsilon^2 + 320\omega)u^2 + \dots\,, \tag{3.116}$$

$$a_x^1 = -\frac{a_{t0}k + a_{x0}\omega}{\mathcal{P}}35k\epsilon(48k^2 + 3\epsilon^2 - 320i\omega)u^2 + \dots\,, \tag{3.117}$$

$$a_x^2 = -\frac{a_{t0}k + a_{x0}\omega}{\mathcal{P}}\frac{i\epsilon\omega}{k}\big(20{,}160k^2 + 843\epsilon^2 - 72{,}800i\omega\big)u^2 + a_{x0}\frac{i\epsilon}{k}u^2 + \dots \tag{3.118}$$

$$\begin{aligned}
a_t^3 = {} & a_{t0} + \frac{a_{t0}k + a_{x0}\omega}{\mathcal{P}}\frac{k}{96}\big(26{,}880k^4 + 843\epsilon^4 + 192k^2(79\epsilon^2 - 840i\omega) + \\
& -113{,}264i\epsilon^2\omega - 3{,}494{,}400\omega^2\big)u^2 + \dots\,,
\end{aligned} \tag{3.119}$$

$$\begin{aligned}
a_x^3 = {} & a_{x0} - \frac{a_{t0}k + a_{x0}\omega}{\mathcal{P}}\frac{\omega}{96}\big(26{,}880k^4 + 843\epsilon^4 + 192k^2(79\epsilon^2 - 840i\omega) \\
& -113{,}264i\epsilon^2\omega - 3{,}494{,}400\omega^2\big)u^2 + \dots\,.
\end{aligned} \tag{3.120}$$

The pole here is again a cubic polynomial in ω:

$$\begin{aligned}\mathcal{P} &= -72{,}800 i\omega^3 + (39{,}760k^2 + 843\epsilon^2)\omega^2 + \frac{5}{6} i(2688k^4 + 451k^2\epsilon^2)\omega \\ &\quad -\frac{71}{32}k^2\epsilon^4 - 53k^4\epsilon^2 - 280k^6 \,. \end{aligned} \tag{3.121}$$

We consider the roots of the polynomial first in the limit $k \ll \epsilon$:

$$\omega = \pm\sqrt{\frac{1}{2}\frac{71}{13{,}488}}\epsilon k - \frac{103{,}535}{947{,}532} ik^2 + \dots \,, \tag{3.122}$$

$$\omega = -\frac{843}{72{,}800} i\epsilon^2 - \frac{5{,}044{,}459}{15{,}397{,}395} ik^2 + \dots \,. \tag{3.123}$$

The first pair of poles correspond to second sound propagating in the direction parallel to the order parameter with a speed consistent with our earlier result (3.45). The third pole is related to the zero mode that causes a phase transition from the superfluid to the normal phase. Next we consider the limit $k \gg \epsilon$:

$$\omega = \frac{\pm 11 - 3i}{130}k^2 + \frac{\pm 192{,}553 - 95{,}119i}{26{,}644{,}800}\epsilon^2 \,, \tag{3.124}$$

$$\omega = -\frac{ik^2}{2} - \frac{13i\epsilon^2}{2928} + \dots \,. \tag{3.125}$$

The two sound poles have evolved into the zero mode poles, while the zero mode pole has evolved into a diffusive pole.

From the small u expansion, we can read off a large number of Green's functions which we shall not bother to list. Similar to the transverse sound case considered above, the prefactor $(a_{t0}k + a_{x0}\omega)$ in the expansion means that the Ward identities (3.56) will be satisfied. However, there is more structure here. Note that $ika_x^2 = 4a_t^1 - a_{x0}\epsilon u^2$ and $ika_x^1 = -4a_t^2$. In our hydrodynamic limit at leading order in ω and k, these two equations are the Ward identities (3.57) and (3.58), respectively.

Before moving on, we note that

$$\lim_{k\to 0} G^{tt}_{33}(0,k) = \frac{2}{g^2}\frac{281}{71}\,, \tag{3.126}$$

which agrees with eq. (3.101), but k here is parallel rather than transverse to the order parameter.

3.5.4 Transverse diffusive mode

In addition to the sound mode found above, in the limit $k \ll \epsilon$, we expect to find a diffusive mode in the current-current correlator. We begin with the slightly simpler case of a mode polarized transverse to the order parameter but propagating parallel to it, and follow in the next section with a mode polarized longitudinal to the order parameter but propagating transversely. Thus first we look for fluctuations in $\delta A^a_y(u,t,x)$ and any other modes that couple to it. A self-consistent set of fluctuations to consider is

$$\delta A^a_y(u,t,x) = a^a_y(u) e^{-i\omega t + ikx}\,, \tag{3.127}$$

where $a = 1, 2, 3$.

This set of fluctuating modes gives rise to the three differential equations at linear order:

$$\begin{aligned}
\mathcal{D}_x a^3_y &= \frac{(k^2+W^2)f-\omega^2}{f^2} a^3_y - \frac{2iW}{f} a^2_y\,, && (3.128)\\
\mathcal{D}_x a^2_y &= \frac{(k^2+W^2)f-\omega^2-\Phi^2}{f^2} a^2_y + \frac{2iW}{f} a^3_y - \frac{2i\omega\Phi}{f^2} a^1_y\,, && (3.129)\\
\mathcal{D}_x a^1_y &= \frac{k^2 f-\omega^2-\Phi^2}{f^2} a^1_y + \frac{2i\omega\Phi}{f^2} a^2_y\,. && (3.130)
\end{aligned}$$

As before, we solve this set of equations perturbatively in the limit $\omega \sim k^2 \sim \epsilon^2$. The small u expansion of the solutions, from which we may read off the Green's functions, takes the form:

$$
\begin{aligned}
a_y^1 &= \frac{a_{y0}}{\mathcal{P}} 22k\epsilon\omega u^2 + \dots \,, && (3.131)\\
a_y^2 &= \frac{a_{y0}}{\mathcal{P}} 2k\epsilon(2ik^2 + 3\omega)u^2 + \dots \,, && (3.132)\\
a_y^3 &= a_{y0} - \frac{a_{y0}}{\mathcal{P}}\frac{1}{3360}\Big(-1680k^6 + 12k^4(29\epsilon^2 + 700i\omega) - 6k^2(\epsilon^4 + 28i\epsilon^2\omega - 10{,}780\omega^2) + \\
&\quad +\omega(9i\epsilon^4 + 4766\epsilon^2\omega - 109{,}200i\omega^2)\Big)u^2 + \frac{1}{2}a_{y0}k^2u^2\ln u + \dots \,. && (3.133)
\end{aligned}
$$

The poles at leading order in this perturbative expansion come from a quadratic polynomial in ω:

$$
\mathcal{P} = 65\omega^2 + \frac{3i(140k^2 + 3\epsilon^2)}{70}\omega - \frac{(70k^2 + 3\epsilon^2)k^2}{35} \,. \qquad (3.134)
$$

As before, we consider the roots of this polynomial in two limits. First we consider $k \ll \epsilon$, in which case we find

$$
\begin{aligned}
\omega &= -\frac{9i}{4550}\epsilon^2 + \frac{112i}{195}k^2 + \dots \,, && (3.135)\\
\omega &= -\frac{2ik^2}{3} + \dots \,. && (3.136)
\end{aligned}
$$

The first pole is associated with the zero mode that causes the phase transition from the superfluid phase to the normal phase while the second pole comes from a diffusive mode of the charge density.

Next, we consider the limit $k \gg \epsilon$ where we recover the zero modes of the normal phase,

$$
\omega = \frac{\pm 11 - 3i}{65}k^2 + \frac{\pm 33 - 9i}{9100}\epsilon^2 + \dots \,. \qquad (3.137)
$$

At leading order, the location of the pole is the same as that of eq. (3.98). However, the subleading order ϵ^2 corrections are different.

3.5.5 Longitudinal diffusive mode

We continue the discussion by looking at modes polarized longitudinal to the order parameter but propagating transversely. We consider fluctuations $\delta A_x^3(u,t,y)$ and all others coupled to it:

$$\begin{aligned}
\delta A_x^3(u,t,y) &= a_x^3(u)e^{-i\omega t+iky}\,, \\
\delta A_t^a(u,t,y) &= a_t^a(u)e^{-i\omega t+iky}\,, \\
\delta A_y^b(u,t,y) &= a_y^b(u)e^{-i\omega t+iky}\,,
\end{aligned} \tag{3.138}$$

where $a,b=1,2$. This set of fluctuations obeys the five second order equations and two first order constraints:

$$\mathcal{D}_t a_t^1 = \frac{1}{f}\left(-W\Phi a_x^3 - ik\Phi a_y^2 + k^2 a_t^1 + k\omega a_y^1\right)\,, \tag{3.139}$$

$$\mathcal{D}_t a_t^2 = \frac{1}{f}\left(iW\omega a_x^3 + \left(k^2+W^2\right)a_t^2 + k\omega a_y^2 + ik\Phi a_y^1\right)\,, \tag{3.140}$$

$$\mathcal{D}_x a_y^1 = \frac{1}{f^2}\left(ik\Phi a_t^2 + 2i\Phi\omega a_y^2 - k\omega a_t - \left(\Phi^2+\omega^2\right)a_y\right)\,, \tag{3.141}$$

$$\mathcal{D}_x a_y^2 = \frac{1}{f^2}\left(-ifkWa_x^3 - k\omega a_t^2 + \left(fW^2-\Phi^2-\omega^2\right)a_y^2 - ik\Phi a_t - 2i\Phi\omega a_y\right)\,, \tag{3.142}$$

$$\mathcal{D}_x a_x^3 = \frac{1}{f^2}\left(\left(fk^2-\omega^2\right)a_x^3 + iW\omega a_t^2 + ifkWa_y^2 + W\Phi a_t^1\right)\,, \tag{3.143}$$

$$0 = -\Phi' a_t^2 + i\omega\partial_u a_t^1 + \Phi\partial_u a_t^2 + ifk\partial_u a_y^1\,, \tag{3.144}$$

$$0 = fW' a_x^3 + \Phi' a_t^1 - \Phi\partial_u a_t^1 + i\omega\partial_u a_t^2 - fW\partial_u a_x^3 + ifk\partial_u a_y^2\,. \tag{3.145}$$

At the horizon, there are two pure gauge solutions, three ingoing solutions, and three outgoing solutions. We discard the outgoing solutions and use the remaining degrees of freedom to choose the boundary values of the five fluctuations. In particular, we set the boundary values of all the fluctuations to zero save for a_x^3, which we set to

a_{x0}. The near boundary expansion of the solution takes the form:

$$\begin{aligned}
a_t^1 &= \frac{a_{x0}}{\mathcal{P}}\frac{3k^2\epsilon}{4}(k^2-3i\omega)u^2 + a_{x0}\frac{\epsilon}{4}u^2 + \ldots\,, && (3.146)\\
a_t^2 &= \frac{a_{x0}}{\mathcal{P}}\frac{33ik^2\omega\epsilon}{4}u^2 + \ldots\,, && (3.147)\\
a_y^1 &= -\frac{a_{x0}}{\mathcal{P}}33k\epsilon\omega\, u^2 + \ldots\,, && (3.148)\\
a_y^2 &= -\frac{a_{x0}}{\mathcal{P}}3ik\epsilon(k^2-3i\omega)u^2 + \ldots\,, && (3.149)\\
a_x^3 &= a_{x0} + \frac{a_{x0}}{\mathcal{P}}\frac{1}{3360}\Big(840k^6 - 16k^4(13\epsilon^2+420i\omega) - \omega(\epsilon^2-48i\omega)(9i\epsilon^2+4550\omega) \\
&\quad +3k^2(\epsilon^4+125i\epsilon^2\omega-39{,}760\omega^2)\Big)u^2 + \frac{1}{2}a_{x0}k^2u^2\ln u + \ldots\,. && (3.150)
\end{aligned}$$

The pole, similar to the case considered previously, is a quadratic polynomial in ω:

$$\mathcal{P} = 130\omega^2 + \left(6ik^2 + \frac{9i\epsilon^2}{35}\right)\omega - \frac{3}{35}k^2\epsilon^2 - k^4\,. \qquad (3.151)$$

In the limit $k \ll \epsilon$ we find a zero mode and a diffusive mode:

$$\begin{aligned}
\omega &= -\frac{9i}{4550}\epsilon^2 + \frac{56i}{195}k^2 + \ldots\,, && (3.152)\\
\omega &= -\frac{ik^2}{3} + \ldots\,. && (3.153)
\end{aligned}$$

In the opposite limit, we find two zero modes:

$$\omega = \frac{\pm 11 - 3i}{130}k^2 + \frac{\pm 33 - 9i}{9100}\epsilon^2 + \ldots\,. \qquad (3.154)$$

The structure of the small u expansion of the gauge fields is again related to the Ward identities. We see that $ika_y^2 = 4a_t^1 - a_{x0}\epsilon u^2$ and $ika_y^1 = -4a_t^2$, which are restatements of the Ward identities (3.57) and (3.58), respectively.

3.5.6 Conductivity and London Equations

In this section, we begin by studying the response of the system to a homogeneous, time dependent electric field, $\delta A_j^3 \sim e^{-i\omega t}$, and end with a discussion of the London equations. A homogeneous electric field should produce a current in the system via Ohm's Law. To investigate the conductivity in this long wavelength limit, we set $k = 0$ for the two-point functions computed above.

The case of an electric field orthogonal to the order parameter is simple; a current and nothing more is produced. From the pure transverse mode and eq. (3.82), we have

$$G_{33}^{yy}(\omega) = \frac{2}{g^2}\left(-\frac{\epsilon^2}{48} + i\left(\frac{1}{2} - \frac{\epsilon^2}{48}\right)\omega + c\,\omega^2\right) + \dots \,. \tag{3.155}$$

Reassuringly, this result agrees with the $k \to 0$ limit of the Green's functions associated to transverse sound propagation and the transverse diffusive mode.

For an electric field parallel to the order parameter, the physics is richer. We find a current in the x direction but also oscillating (or precessing) charge densities associated with the one and two isospin directions:

$$a_x^3 = a_{x0} + a_{x0}\left(-\frac{\epsilon^2}{96} + i\left(\frac{1}{2} + \frac{\epsilon^2}{288}\right)\omega\right)u^2 + \dots \,, \tag{3.156}$$

$$a_t^1 = a_{x0}\frac{\epsilon}{4}u^2 + \dots \,, \tag{3.157}$$

$$a_t^2 = -a_{x0}\frac{i\epsilon\omega}{16}u^2 + \dots \,. \tag{3.158}$$

This near boundary expansion agrees with the $k \to 0$ limit of the expansions for longitudinal sound and diffusion considered above. The associated Green's functions are

$$G^{xx}_{33}(\omega) = \frac{2}{g^2}\left(-\frac{\epsilon^2}{96} + i\left(\frac{1}{2} + \frac{\epsilon^2}{288}\right)\omega\right) + \dots\,, \tag{3.159}$$

$$G^{tx}_{13}(\omega) = -\frac{2}{g^2}\frac{\epsilon}{4} + \dots\,, \tag{3.160}$$

$$G^{tx}_{23}(\omega) = \frac{2}{g^2}\frac{i\epsilon\omega}{16} + \dots\,. \tag{3.161}$$

Identifying the electric field $E_j = i\omega\,\delta A_j$ and recalling Ohm's Law, the conductivities are related via eq. (3.72) to the retarded Green's functions,

$$\sigma_{xx}(\omega) = \frac{G^{xx}_{33}(\omega)}{i\omega} \qquad \text{and} \qquad \sigma_{yy}(\omega) = \frac{G^{yy}_{33}(\omega)}{i\omega}\,. \tag{3.162}$$

The terms proportional to ϵ^2 in G^{xx}_{33} and G^{yy}_{33} thus produce a pole in the imaginary part of the respective conductivities. As discussed in refs. [12, 13], by the Kramers-Kronig relations (or by properly regularizing the pole) there must be a delta function in the real part of the conductivity, indicating the material loses all resistance to DC currents and suggesting the phase transition is to a superconducting state. While in refs. [12, 13], the pole was seen only numerically, here we can calculate the strength of the pole analytically close the phase transition. Its residue is given by

$$\mathrm{Res}_{\omega=0}\sigma_{xx} = \frac{2}{g^2}\frac{i\epsilon^2}{96} + \dots \qquad \mathrm{Res}_{\omega=0}\sigma_{yy} = \frac{2}{g^2}\frac{i\epsilon^2}{48} + \dots\,. \tag{3.163}$$

In Figure 3.1 we show a comparison between numerical computations of the residues of the poles at $\omega = 0$ in σ_{xx} and σ_{yy}, along with the analytic approximation (3.163) close to $T = T_c$.

In the Drude model for an ideal metal, the conductivity takes the form $\sigma = i\rho/m\omega$ where ρ is the charge density and m is the mass of the charge carrier. In the superconductivity literature (see for example [72]), the pole in the imaginary part of the conductivity is thus often related to a superfluid density. Because our system is not rotationally symmetric, the density to mass ratio defined in this way will depend on the orientation of the superfluid velocity with respect to the order parameter. The proper way to interpret this situation is probably that a suitably defined effective mass of the superfluid depends on the direction of propagation.

An important observation is that in our system, the $\omega \to 0$ and $k \to 0$ limits of the Green's functions commute. The residue of the pole in the conductivity is related to the long wavelength limit of the current-current correlation function in the following way:

$$i\,\mathrm{Res}_{\omega=0}\sigma_{jj} = \lim_{\omega\to 0}\lim_{k\to 0} G^{jj}_{33}(\omega, k)\,. \tag{3.164}$$

The limit in the opposite order is related to a thermodynamic susceptibility:

$$\lim_{k_y\to 0} G^{xx}_{33}(0, k_y) = \frac{\partial^2 P}{\partial v_\parallel^2} \qquad \text{and} \qquad \lim_{k_x\to 0} G^{yy}_{33}(0, k_x) = \frac{\partial^2 P}{\partial v_\perp^2}\,, \tag{3.165}$$

where $v_\parallel$ and $v_\perp$ are superfluid velocities.[8] It follows from eqs. (3.42) and (3.44) that

$$\frac{\partial^2 P}{\partial v_\parallel^2} = \frac{\partial j_\parallel}{\partial v_\parallel} = -\frac{2}{g^2}\frac{\epsilon^2}{96} \qquad \text{while} \qquad \frac{\partial^2 P}{\partial v_\perp^2} = \frac{\partial j_\perp}{\partial v_\perp} = -\frac{2}{g^2}\frac{\epsilon^2}{48}\,. \tag{3.166}$$

When combined with eq. (3.162), these results confirm eq. (3.163).

As emphasized in this context in ref. [13], that the limits commute implies the system really does become a superconductor below T_c. Given that the limits commute, the system obeys a London type equation for small k and ω:

$$\langle j^3_x\rangle \approx -\frac{2}{g^2}\frac{\epsilon^2}{96}\mathcal{A}^3_x \qquad \text{and} \qquad \langle j^3_y\rangle \approx -\frac{2}{g^2}\frac{\epsilon^2}{48}\mathcal{A}^3_y\,. \tag{3.167}$$

If we now imagine the $U(1)$ subgroup generated by $T^3 \in \mathfrak{su}(2)$ is weakly gauged,

[8]Note that to produce a perturbing magnetic field, we require a k that is transverse to the polarization of the current-current correlation function. A perturbation of the form $\delta A^3_x \sim e^{ikx}$ or $\delta A^3_y \sim e^{iky}$ is gauge equivalent to zero and does not produce a response from the system. The Green's function vanishes in this limit.

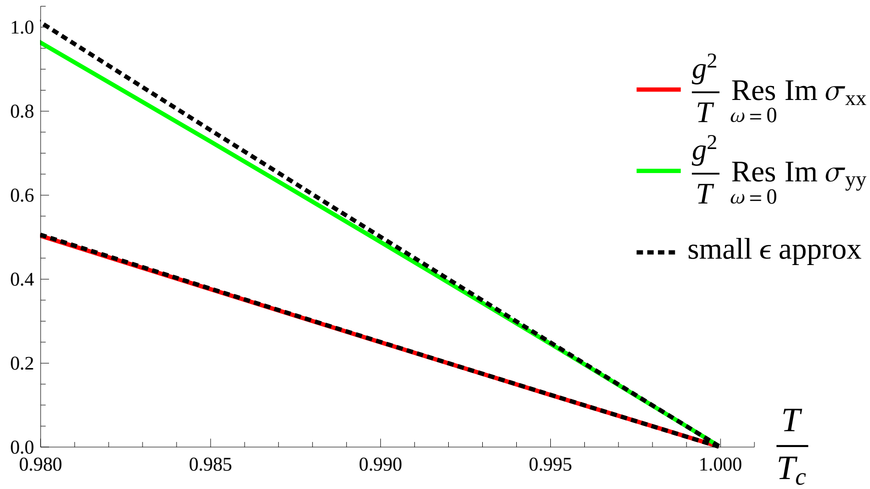

Figure 3.1: Plots of numerical results for $\frac{g^2}{T}\text{Res}_{\omega=0}\text{Im}\sigma_{xx}$ and $\frac{g^2}{T}\text{Res}_{\omega=0}\text{Im}\sigma_{xx}$ as functions of temperature (solid lines), as well as analytical approximations at small ϵ (dotted lines) given by equations (3.163).

these London equations imply not only infinite DC conductivity but also a Meissner effect with London penetration depths that scale as $\lambda_\perp \sim \lambda_\parallel \sim 1/\epsilon \sim (T_c - T)^{-1/2}$.

3.6 Discussion

One of the nicest features of our results is their analytic nature. We were able to confirm a number of previous numeric observations [12, 47, 51, 69, 73, 74] of this superfluid phase transition. In particular, we saw explicitly that the phase transition was second order; the difference in free energy between the phases scaled as $(T_c - T)^2$ below the phase transition. We saw the order parameter grew as $\epsilon \sim (T_c - T)^{1/2}$ below T_c and thus has a mean field critical exponent. We calculated the speed of second sound near the phase transition and observed that it vanished linearly with the reduced temperature $c_\perp \sim c_\parallel \sim (T_c - T)$. We looked at the pole at $\omega = 0$ in the imaginary part of the conductivity and saw the same scaling, $\sigma \sim i(T_c - T)/\omega$, that had been observed numerically in a related model [12] and confirmed that the London penetration depth scales as $\lambda \sim 1/(T_c - T)^{1/2}$. This laundry list of scalings

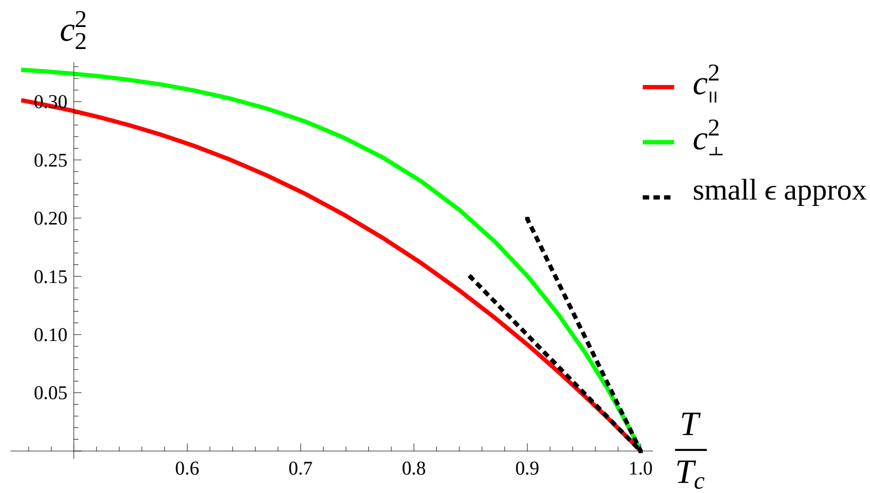

Figure 3.2: The squared speeds of second sound $c_{\parallel}^2$ and $c_{\perp}^2$ as functions of the reduced temperature T/T_c (solid lines) as well as analytical approximations given by eqs. (3.43) and (3.45) close to T_c (dotted lines).

(or critical exponents) is the same observed in the mean-field Landau-Ginzburg model of a superconductor.

Close to T_c, eqs. (3.43) and (3.45) indicate that $c_{\perp}^2 = 2c_{\parallel}^2$, so one may wonder whether such a formula is valid away from T_c as well. Numerical evaluations show that this is not the case: see Figure 3.2. At small temperatures, $c_{\perp}^2$ approaches 1/3. Our numerical evaluations are not sufficiently reliable at small temperatures to see whether $c_{\parallel}^2$ has the same limit.[9]

There are some other results in this paper that are worth emphasizing. For superfluid velocities that are not too large, we were able to determine analytically the critical line in the temperature-superfluid velocity plane separating the normal phase from the superfluid phase. We calculated a large number of current-current correlation functions in the hydrodynamic limit and verified that they satisfied the non-abelian Ward identities. We also investigated how the hydrodynamic poles in

[9] In the case of a scalar order parameter and a phase transition that does not break rotational symmetry, we expect the speed of second sound to approach $(d-1)^{-1/2}$ as $T \to 0$. This limit follows from eq. (3.40) and two observations: 1) At $T = 0$ the Lorentz symmetry breaking due to the temperature disappears and the pressure can depend on μ and v only as $P(\mu^2 - v^2)$. 2) By dimensional analysis, when $T = v = 0$, $P \sim \mu^d$. We would like to thank Amos Yarom for discussion on this point.

these correlation functions move around as a function of k and ϵ.

Optimistically, we hope that someday this system will be more than a toy model. The introduction described possible similarities of this system to helium-3 and p-wave superconductors. Here we add a speculation about a possible connection with QCD. The $SU(2)$ global symmetry of our model could be thought of as the residual approximate isospin symmetry of QCD at low energies and our chemical potential an isospin chemical potential. The phase structure of QCD at non-zero isospin chemical potential has been discussed by ref. [75]. Alas, there is no persistent current in the stable phases they discuss.

There are also some questions left unanswered for the future. While we focused on current-current correlation functions in the third isospin direction in this paper, it would be good to study the full set of Green's functions more carefully. Because of the characteristic magnetic properties of superconductors, it would be interesting to investigate the dependence of the correlation functions on an external magnetic field.

Another interesting direction to pursue is the connection between this work and the membrane paradigm [76] where the horizon of a black hole, rather than the boundary of an asymptotically anti-de Sitter space, is thought of as a fluid. Related to this direction is the observation of ref. [51] that the fraction of the total charge density outside the black hole horizon scales as $T_c - T$ close to T_c, suggesting that this quantity might be related to the superfluid density. We would like to know if this analogy can be made more precise.

Chapter 4

Multi-Matrix Models and Tri-Sasaki Einstein Spaces

This chapter is a lightly edited version of ref. [46], which was written in collaboration with Chris Herzog, Igor Klebanov, and Tibi Tesileanu.

4.1 Introduction

The AdS/CFT correspondence [1–3] provides many predictions about the dynamics of strongly interacting field theories in various numbers of dimensions. We have seen in section 1.2.6 that for the case of three dimensions, the number of low-energy degrees of freedom on N coincident M2-branes is expected to scale as $N^{3/2}$ at large N. This scaling appeared in two quantities: the thermal free energy and the free energy of the Euclidean theory on S^3. Only the second quantity is protected by supersymmetry, and, because of that, there is hope that one would be able to compute it in the field theory even at strong coupling. Indeed, using supersymmetry, this quantity was computed in [23] in the case of ABJM theory at large N and fixed 't Hooft coupling N/k. In the strong coupling limit $N/k \gg 1$, the authors of [23] observed that $F \sim k^{1/2} N^{3/2}$, and that the coefficient matched the prediction from 11-d supergravity given in eq. (1.70).

The paper [23] was in turn based on [44] where the method of localization developed in [45] for four-dimensional theories on S^4 was shown to reduce the path integral of supersymmetric Chern-Simons matter theories on S^3 to finite-dimensional matrix integrals. The idea behind localization is relatively simple: suppose one can find a Q-exact operator $\{Q, \mathcal{V}\}$ whose bosonic part is positive definite, Q being one of the supercharges. Because $\{Q, \mathcal{V}\}$ is Q-exact, the quantity

$$Z = \int [D\phi] \exp\left[-S - t\{Q, \mathcal{V}\}\right] \tag{4.1}$$

is independent of t, so it can be computed for example in the limit of large t. In this limit the path integral localizes on configurations where $\{Q, \mathcal{V}\}$ is smallest, which is $\{Q, \mathcal{V}\} = 0$ because the Q-exact operator was also chosen to have a positive-definite bosonic part. To evaluate Z, one solves the equations $\{Q, \mathcal{V}\} = 0$ and approximates

$$Z = e^{-S_{\text{classical}}} Z_{\text{one-loop}} , \tag{4.2}$$

where $S_{\text{classical}}$ is the classical action S evaluated on the solutions of $\{Q, \mathcal{V}\} = 0$, and $Z_{\text{one-loop}}$ is the one-loop determinant of fluctuations around this classical configuration. Because one can take t to be arbitrarily large, the expression (4.2) is actually exact. Note that the same localization trick goes through if one further inserts into the path integral any operator that is annihilated by Q.

As explained in [44], the path integral of any $\mathcal{N} \geq 2$ Chern-Simons matter theory where the operator scaling dimensions are canonical localizes on configurations where the scalars σ in the $\mathcal{N} = 2$ vector multiplets take constant values and all other fields vanish. The path integral therefore becomes a multi-matrix integral over the σ matrices. Using the $U(N)$ symmetry, one can further reduce this multi-matrix integral to an integral over the eigenvalues of σ. Of course, the calculation performed in [44] does not start with precisely the actions given in section 1.2 because those actions describe superconformal theories on $\mathbb{R}^{2,1}$. To map these theories conformally to S^3 one has to add at the very least conformal mass terms for the scalar fields. The field content of the theories on S^3 and $\mathbb{R}^{2,1}$ is, however, unchanged, so it makes sense to talk about the scalars σ in the S^3 theories as well, even though they were originally defined in the theories on $\mathbb{R}^{2,1}$.

In the ABJM case studied in [23], the exact solution of the matrix model was related by analytic continuation to a solution [77] of another matrix model describing topological Chern-Simons theory on $S^3/\mathbb{Z}_2$. In general, the solution of these matrix models requires some complicated mathematics. One introduces a "resolvent," which is a complex-valued function from whose branch cuts one can extract the eigenvalue configuration that gives the most important contribution to the matrix integral. With the use of holomorphy, one can then restrict the form of the resolvent to a function with just a few parameters that can be found by performing contour integrals. In solving the ABJM matrix model at arbitrary 't Hooft coupling N/k, the authors of [23] noticed that the resolvent of this model could be related by analytic continuation to the resolvent of the $S^3/\mathbb{Z}_2$ model, which had been previously found in [77]. A generalization of the ABJM matrix model to the case where the Chern-Simons levels do not add up to zero was considered in [78].

The goal in this chapter is to build on the progress achieved in [23,44,79] in several ways. Section 4.2 starts by revisiting the matrix integral for the ABJM theory on S^3, and it uncovers the details of the eigenvalue distribution. The matrix eigenvalues are located along the branch cuts of the resolvent of [23, 77]. While the endpoints of the cuts can be read off directly from the resolvent, the cuts themselves are not simply parallel to the real axis, in contrast with the matrix model of [77]. In order to gain intuition for the location of the eigenvalues, we develop a numerical method for finite N and k. This method allows us to access values of N and k that are large enough for the result to be a good approximation to the limit studied in [23,79]. Furthermore, we will focus on the limit where N is sent to infinity at fixed k where the ABJM model is expected to be dual to the $AdS_4 \times S^7/\mathbb{Z}_k$ background of M-theory. In this strong coupling limit, which is not of the 't Hooft type, it can be shown analytically that the structure of the solution simplifies considerably. An ansatz where the real parts of the eigenvalues scale with $\sqrt{N}$ allows us to calculate the free energy analytically. Unlike in [23], the method presented here does not rely on resolvents or mirror symmetry. We confirm that the free energy scales as $N^{3/2}$ with the coefficient found in [23] and also given in eq. (1.70).

Section 4.3.1 contains an extension of the analytic approach described above to the necklace quiver gauge theories with p $U(N)$ gauge groups introduced in section 1.2.5. Recall that these theories can be engineered using a type IIB brane construction involving N D3-branes and p $(1, q_a)$-branes. These theories are dual to M-theory backgrounds of the form $AdS_4 \times Y$, where the spaces Y have three Killing spinors, corresponding to the fact that the field theory has $\mathcal{N} = 3$ supersymmetry. The spaces Y are the bases of hyper-Kähler cones [39–41] and are called tri-Sasakian. They are also Einstein, and we take the Einstein metric on them to be normalized so that $R_{mn} = 6g_{mn}$. The p-matrix models for the gauge theories dual to $AdS_4 \times Y$ may be read off from [44]. In the large N limit we calculate the eigenvalue densities for these matrix models and show that they are piecewise linear. This remarkably simple conclusion allows us to evaluate the coefficient of the $N^{3/2}$ scaling of the free energy as a function of the levels k_a and compare it with the calculation on the gravity side of the AdS/CFT correspondence [22, 23]. For an arbitrary compact space Y we saw in section 1.2.6 that the gravitational free energy was given by eq. (1.69) in terms of Vol(Y). For $p = 3$ the tri-Sasaki Einstein spaces Y are the Eschenburg spaces [80] whose volumes were determined explicitly in [81]. Our 3-matrix model free energy is in perfect agreement with this volume formula.

Furthermore, we carry out calculations of the p-matrix model free energy and use them to conjecture an explicit general formula for the volumes via the AdS/CFT correspondence. For a general p-node quiver with CS levels $k_a = q_{a+1} - q_a$, with $1 \le a \le p$ and $q_{p+1} = q_1$, we conjecture in section 4.4 that

$$\frac{\mathrm{Vol}(Y)}{\mathrm{Vol}(S^7)} = \frac{\sum_{(V,E)\in\mathcal{T}} \prod_{(a,b)\in E} |q_a - q_b|}{\prod_{a=1}^{p} \left[\sum_{b=1}^{p} |q_a - q_b|\right]}\,, \tag{4.3}$$

where the sum in the numerator is over the set $\mathcal{T}$ of all trees (acyclic connected graphs) with p nodes. Such a tree (V,E) consists of the vertices $V = \{1, 2, \dots, p\}$ and $|E| = p-1$ edges. The volumes of the corresponding tri-Sasaki Einstein spaces Y had previously been studied by Yee, who expressed them through an integral formula (eq. (3.49) of [82]). In the cases we have checked, our formula (4.3) is consistent with that of [82]. Equation (4.3) is invariant under permutations of the q_a, supporting the conjectured Seiberg duality for Chern-Simons theories with at least $\mathcal{N} = 2$ supersymmetry [83–85], which may be motivated by interchanging different types of 5-branes in the type IIB brane constructions of these models.

4.2 ABJM Matrix Model

4.2.1 Matrix Model Setup

As discussed in section 1.2, ABJM theory has only two gauge groups, and therefore only two scalars σ and $\tilde{\sigma}$ whose eigenvalues we denote by λ_i and $\tilde{\lambda}_i$, with $1 \le i \le N$. Written as an integral over these eigenvalues, the 2-matrix integral is [44]:

$$Z = \frac{1}{(N!)^2} \int \left(\prod_{i=1}^{N} \frac{d\lambda_i\, d\tilde{\lambda}_i}{(2\pi)^2}\right) \frac{\prod_{i<j} \left(2\sinh\frac{\lambda_i-\lambda_j}{2}\right)^2 \left(2\sinh\frac{\tilde{\lambda}_i-\tilde{\lambda}_j}{2}\right)^2}{\prod_{i,j}\left(2\cosh\frac{\lambda_i-\tilde{\lambda}_j}{2}\right)^2} \exp\left(\frac{ik}{4\pi}\sum_i (\lambda_i^2 - \tilde{\lambda}_i^2)\right)\,, \tag{4.4}$$

where k is the Chern-Simons level, and the precise normalization was chosen as in [23]. The integration contour should be taken to be the real axis in each integration variable. When the number N of eigenvalues is large, the integral in eq. (4.4) can be approximated in the saddle-point limit by $Z = e^{-F}$, where the "free energy" F is an extremum of

$$
\begin{aligned}
F(\lambda_i, \tilde{\lambda}_i) = -i\frac{k}{4\pi}\sum_j (\lambda_j^2 - \tilde{\lambda}_j^2) - \sum_{i<j} \log\left[\left(2\sinh\frac{\lambda_i - \lambda_j}{2}\right)^2 \left(2\sinh\frac{\tilde{\lambda}_i - \tilde{\lambda}_j}{2}\right)^2\right] \\
+ 2\sum_{i,j}\log\left(2\cosh\frac{\lambda_i - \tilde{\lambda}_j}{2}\right) + 2\log N! + 2N\log(2\pi)
\end{aligned}
\tag{4.5}
$$

with respect to λ_i and $\tilde{\lambda}_i$. Varying (4.5) with respect to λ_j and $\tilde{\lambda}_j$ we obtain the saddle-point equations:

$$
\begin{aligned}
-\frac{\partial F}{\partial \lambda_i} &= \frac{ik}{2\pi}\lambda_i - \sum_{j\neq i}\coth\frac{\lambda_j - \lambda_i}{2} + \sum_j \tanh\frac{\tilde{\lambda}_j - \lambda_i}{2} = 0\,, \\
-\frac{\partial F}{\partial \tilde{\lambda}_i} &= -\frac{ik}{2\pi}\tilde{\lambda}_i - \sum_{j\neq i}\coth\frac{\tilde{\lambda}_j - \tilde{\lambda}_i}{2} + \sum_j \tanh\frac{\lambda_j - \tilde{\lambda}_i}{2} = 0\,.
\end{aligned}
\tag{4.6}
$$

The goal of this section is to compute the leading contribution to F in such a large N expansion while holding k fixed.

4.2.2 A Numerical Solution

To gain intuition, one can start by solving the saddle-point equations (4.6) numerically for any values of N and k. One of the simplest ways to do so is to view equations (4.6) as describing the equilibrium configuration of $2N$ point particles whose 2-d coordinates are given by the complex numbers λ_j and $\tilde{\lambda}_j$ and that interact with the forces given by eq. (4.6). This equilibrium configuration can be found by introducing a time dimension and writing down equations of motion for $\lambda_j(t)$ and $\tilde{\lambda}_j(t)$ whose solution approaches the equilibrium configuration (4.6) at late times in the same way as the solution to the heat equation approaches a solution to the Laplace equation at late times. The equations of motion for the eigenvalues are

$$\tau_\lambda \frac{d\lambda_i}{dt} = -\frac{\partial F}{\partial \lambda_i}\,, \qquad \tau_{\tilde{\lambda}} \frac{d\tilde{\lambda}_i}{dt} = -\frac{\partial F}{\partial \tilde{\lambda}_i}\,, \tag{4.7}$$

where τ_λ and $\tau_{\tilde{\lambda}}$ are complex numbers that need to be chosen in such a way that the saddle point we wish to find is an attractive fixed point as $t \to \infty$.

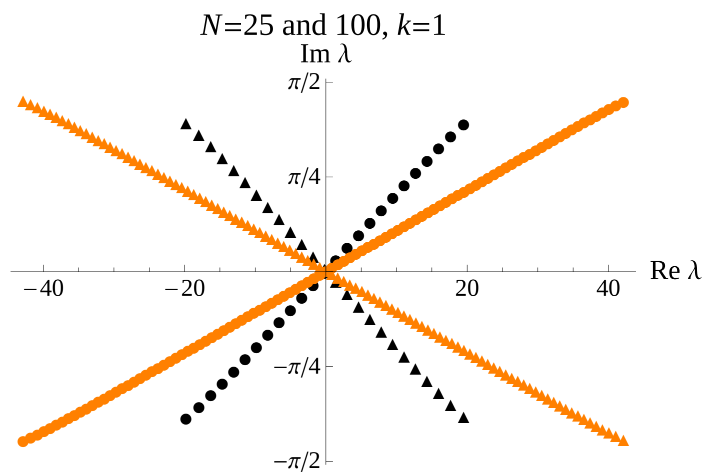

Figure 4.1: Numerical saddle points for the ABJM matrix model. The eigenvalues for $N = 25$ are plotted in black and those for $N = 100$ are plotted in orange. The plot has been obtained with $\tau_\lambda = \tau_{\tilde{\lambda}} = 1$. As mentioned in the text, the real parts of the eigenvalues grow with $\sqrt{N}$.

In figure 4.1 we show typical eigenvalue distributions that can be found using the method we just explained. There are several features of the saddle-point configurations that are worth emphasizing:

- The eigenvalues λ_j and $\tilde{\lambda}_j$ that solve (4.6) are not real.

  That the eigenvalue distributions do not lie on the real axis might be a bit puzzling given that λ_i and $\tilde{\lambda}_i$ are supposed to be eigenvalues of Hermitian matrices. However, it is well known that in general, when using the saddle-point approximation, the main contribution to an oscillatory integral may come from saddles that are not on the original integration contour but through which the integration contour can be made to pass. We will assume that the integration contour that should be chosen in writing down the integral in eq. (4.4) can be deformed so that saddle points like those in figure 4.1 are the only important ones.

- The eigenvalue distributions are invariant under $\lambda_i \to -\lambda_i$ and $\tilde{\lambda}_i \to -\tilde{\lambda}_i$.

  Indeed, the saddle-point equations (4.6) are invariant under these transformations, so it is reasonable to expect that there should be solutions that are also invariant.

- In the equilibrium configuration the two types of eigenvalues are complex conjugates of each other: $\tilde{\lambda}_j = \bar{\lambda}_j$.

  Indeed, it is not hard to see that upon setting $\tilde{\lambda}_j = \bar{\lambda}_j$ the two equations in (4.6) become equivalent, so it is consistent to look for solutions that have this property.

- As one increases N at fixed k, the imaginary part of the eigenvalues stays bounded between $-\pi/2$ and $\pi/2$, while the real part grows with N. We will show shortly that, for the saddle points we find, the real part grows as $N^{1/2}$ as $N \to \infty$.

4.2.3 Large N Analytical Approximation

Let us now find analytically the solution to the saddle-point equations (4.6) in the large N limit. As explained above, we can assume $\tilde{\lambda}_j = \bar{\lambda}_j$ and write[1]

$$\lambda_j = N^\alpha x_j + i y_j \,, \qquad \tilde{\lambda}_j = N^\alpha x_j - i y_j \,, \tag{4.8}$$

[1] After completing this work, we became aware that ref. [86] employs a similar ansatz.

where we introduced a factor of N^α multiplying the real part because we want x_j and y_j to be of order $\mathcal{O}(N^0)$ and become dense in the large N limit. The constant α is so far arbitrary but will be determined later.

In passing to the continuum limit, we define the functions $x, y : [0,1] \to \mathbb{R}$ so that

$$x_j = x(j/N)\,, \qquad y_j = y(j/N)\,. \tag{4.9}$$

Let us assume we order the eigenvalues in such a way that x is a strictly increasing function on $[0,1]$. Introducing the density of the real part of the eigenvalues

$$\rho(x) = \frac{ds}{dx}\,, \tag{4.10}$$

one can approximate (4.5) as

$$F = \frac{k}{\pi} N^{1+\alpha} \int dx\, x\rho(x) y(x) + N^{2-\alpha} \int dx\, \rho(x)^2 f(2y(x)) + \cdots\,, \tag{4.11}$$

where the function f is

$$f(t) = \pi^2 - \left(\arg e^{it}\right)^2\,. \tag{4.12}$$

In other words, f is a periodic function with period π given by

$$f(t) = \pi^2 - t^2 \qquad \text{when} \quad -\pi \le t \le \pi\,. \tag{4.13}$$

We postpone the derivation of eq. (4.11) until the next chapter where this equation is derived in a more general setting.

It may be a little puzzling that while the discrete expression for the free energy in eq. (4.5) is nonlocal, in the sense that there are long-range forces between the eigenvalues, its large N limit (4.11) is manifestly local. One can understand this major simplification from examining, for instance, the first saddle-point equation in (4.6). The force felt by λ_i due to interactions with far-away eigenvalues λ_j and $\tilde{\lambda}_j$ is

$$-\coth\frac{\lambda_j-\lambda_i}{2}+\tanh\frac{\tilde{\lambda}_j-\lambda_i}{2}\approx -\operatorname{sgn}(\operatorname{Re}\lambda_j-\operatorname{Re}\lambda_i)+\operatorname{sgn}(\operatorname{Re}\tilde{\lambda}_j-\operatorname{Re}\lambda_i)\,, \tag{4.14}$$

the corrections to this formula being exponentially suppressed in $\operatorname{Re}\lambda_j-\operatorname{Re}\lambda_i$ and $\operatorname{Re}\tilde{\lambda}_j-\operatorname{Re}\lambda_i$. In other words, the nonlocal part of the interaction force between eigenvalues is given just by the right-hand side of eq. (4.14). The nonlocal part of the force vanishes when $\operatorname{Re}\lambda_j=\operatorname{Re}\tilde{\lambda}_j$, so in assuming that the two eigenvalue distributions are complex conjugates of each other, we effectively arranged for an exact cancellation of nonlocal effects. All that is left are short-range forces, which in the large N limit are described by the local action (4.11).

One can view F as a functional of $\rho(x)$ and $y(x)$ and look for its saddle points in the set

$$\mathcal{C}=\left\{(\rho,y):\int dx\,\rho(x)=1;\rho(x)\geq 0\text{ almost everywhere}\right\}. \tag{4.15}$$

These constraints mean that ρ is a normalized density. Motivated by the numerical analysis we performed, we assume that ρ and y describe a connected distribution of eigenvalues contained in a bounded region of the complex plane.

Let us assume a saddle point for F exists. As $N\to\infty$, we need the two terms in (4.11) to be of the same order in N in order to have nontrivial solutions, so from now on we will set

$$\alpha=\frac{1}{2}\,. \tag{4.16}$$

The real part of the eigenvalues therefore grows as $N^{1/2}$, and to leading order, the free energy behaves as $N^{3/2}$ at large N. In writing (4.11) we omitted the last two terms from eq. (4.5). They do not depend on ρ or y and hence do not affect the saddle-point equations. They are also lower order in N given the choice of α.

To find a saddle point for F, one can add a Lagrange multiplier μ to (4.11) and extremize

$$\tilde{F} = N^{3/2}\left[\frac{k}{\pi}\int dx\, x\rho(x)y(x) + \int dx\, \rho(x)^2 f(2y(x)) - \frac{\mu}{2\pi}\left(\int dx\, \rho(x) - 1\right)\right] \tag{4.17}$$

instead of (4.11). As long as $\rho(x) > 0$, the saddle-point eigenvalue distribution satisfies the equations

$$\begin{aligned} 4\pi\rho(x)f(2y(x)) &= \mu - 2kxy(x)\,, \\ 2\pi\rho(x)f'(2y(x)) &= -kx\,. \end{aligned} \tag{4.18}$$

Plugging (4.13) into (4.18) one obtains

$$\rho(x) = \frac{\mu}{4\pi^3}\,, \qquad y(x) = \frac{\pi^2 kx}{2\mu}\,, \tag{4.19}$$

as long as $-\pi/2 \le y(x) \le \pi/2$. If ρ is supported on $[-x_*, x_*]$ for some $x_* > 0$ that we will determine shortly, we can calculate μ from the normalization of the density $\rho(x)$:

$$\int_{-x_*}^{x_*} dx\, \rho(x) = 1 \qquad \Longrightarrow \qquad \mu = \frac{2\pi^3}{x_*}\,. \tag{4.20}$$

Plugging this formula back into (4.11), we obtain the free energy in terms of x_*:[2]

$$F = \frac{N^{3/2}(12\pi^4 + k^2 x_*^4)}{24\pi^2 x_*} + o(N^{3/2})\,. \tag{4.21}$$

This expression is extremized when

$$x_* = \pi\sqrt{\frac{2}{k}}\,, \qquad y(x_*) = \frac{\pi}{2}\,. \tag{4.22}$$

Luckily, the answer $y(x_*) = \pi/2$ is consistent with our assumption that $-\pi/2 \leq y(x) \leq \pi/2$ without which eq. (4.19) would not be correct. It can be checked that the assumption $y(x_*) > \pi/2$ implies a contradiction. The extremum of F obtained from eqs. (4.21) and (4.22) is

$$F = \frac{\pi\sqrt{2}}{3} k^{1/2} N^{3/2} + o(N^{3/2})\,. \tag{4.23}$$

Quite nicely, this result agrees with the supergravity prediction from eq. (1.70).

In the large N limit the eigenvalues therefore condense on two line segments, and on these two line segments they have uniform density. In figure 4.2 we compare the analytical result for the density with the numerical one.

We would like to compare the location of our eigenvalue distributions with the results of [23]. Noting a similarity between the ABJM matrix model and the $S^3/\mathbb{Z}_2$ matrix model solved in [77], Drukker, Marino, and Putrov [23] wrote down a resolvent for the ABJM model. This resolvent has cuts in the λ plane corresponding to the locations of the eigenvalues. In particular, it has a cut where the λ_i eigenvalues are located and a second cut where the $\tilde{\lambda}_i$ eigenvalues are located but shifted by πi. More

[2] We distinguish between the little-o and big-O notations: $f(N) = o(N^\alpha)$ as $N \to \infty$ means that $\lim_{N\to\infty} f(N)/N^\alpha = 0$, while $f(N) = \mathcal{O}(N^\alpha)$ as $N \to \infty$ means that $|f(N)| \leq AN^\alpha$ for some constant A and all large enough N. In other words $o(N^\alpha)$ stands for terms that grow slower than N^α at large N, while $\mathcal{O}(N^\alpha)$ stands for terms that grow at most as fast as N^α.

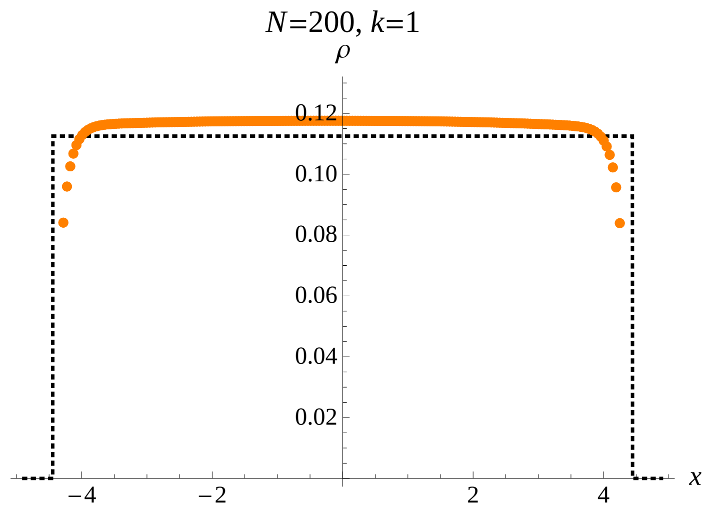

Figure 4.2: Comparison between analytical prediction and numerical results for the density of eigenvalues ρ defined in eq. (4.10). The dotted black line represents the analytical calculation, and the numerical result is shown in orange dots.

specifically, the resolvent has the form

$$\omega(\lambda) = 2\log\left(\frac{1}{2}\left[\sqrt{(e^{\lambda}+b)(e^{\lambda}+1/b)} - \sqrt{(e^{\lambda}-a)(e^{\lambda}-1/a)}\right]\right), \tag{4.24}$$

where $a + 1/a + b + 1/b = 4$ and at strong coupling,

$$a + \frac{1}{a} - b - \frac{1}{b} = 2i\exp\left(\pi\sqrt{\frac{2N}{k} - \frac{1}{12}}\right) + \dots \tag{4.25}$$

The ellipses denote terms exponentially suppressed in N/k relative to the leading term. Solving the equations for a and b, we find that the branch points in the λ plane are at

$$\pm\log a = \pi\sqrt{\frac{2N}{k} - \frac{1}{12}} + \frac{i\pi}{2}, \tag{4.26}$$

$$\pm\log b = -\pi\sqrt{\frac{2N}{k} - \frac{1}{12}} + \frac{i\pi}{2}. \tag{4.27}$$

These expressions are in agreement with (4.22) in the large N limit.

Let us also try to compare our findings with the exact results found for the supersymmetric Wilson loops in ABJM theory [23, 79]. The expectation values of 1/6 and

1/2 supersymmetric Wilson loops are proportional, respectively, to the expectation values of $\sum_{i=1}^{N} e^{\lambda_i}$ and $\sum_{i=1}^{N} \left[e^{\lambda_i} + e^{\tilde{\lambda}_i}\right]$ in the matrix model [23, 44, 79, 87]. In our approach, these quantities become

$$\langle W_{\Box}^{1/6} \rangle = \frac{2\pi i N}{k} \int_{-x_*}^{x_*} e^{\lambda(x)} \rho(x) dx \,, \tag{4.28}$$

$$\langle W_{\Box}^{1/2} \rangle = \frac{2\pi i N}{k} \int_{-x_*}^{x_*} \left(e^{\lambda(x)} + e^{\tilde{\lambda}(x)}\right) \rho(x) dx \,. \tag{4.29}$$

If we evaluate (4.28) and (4.29) using the saddle point we have found, we get

$$\langle W_{\Box}^{1/6} \rangle \approx -\sqrt{\frac{N}{2k}} e^{\pi \sqrt{2N/k}} \,, \tag{4.30}$$

$$\langle W_{\Box}^{1/2} \rangle \approx \frac{i}{2} e^{\pi \sqrt{2N/k}} \,. \tag{4.31}$$

The exponents in these formulae agree with the results in [23, 79, 87].

We should keep in mind that the ABJM model has a type IIA string interpretation only in the limit where $N/k \gg 1$, $N/k^5 \ll 1$. These conditions apply only in the limit where both N and k are taken to infinity. Our approximations are only applicable in the M-theory limit where N is taken to infinity at fixed k. Thus our Wilson loops have a dual interpretation as wrapped M2-branes in M-theory rather than as strings in type IIA string theory.

4.3 Necklace Quiver Gauge Theories

4.3.1 Multi-Matrix Models

As reviewed earlier, the results of [44] show the partition function for the necklace quivers in figure 1.2 localizes on configurations where the scalars σ_a in the $\mathcal{N} = 2$ vector multiplets are constant Hermitian matrices. Denoting by $\lambda_{a,i}$, $1 \leq i \leq N$, the eigenvalues of σ_a, the partition function takes the form of the matrix integral

$$Z = \frac{1}{(N!)^p} \int \left(\prod_{a,i} \frac{d\lambda_{a,i}}{2\pi} \right) \prod_{a=1}^{p} \left(\frac{\prod_{i<j} \left(2 \sinh \frac{\lambda_{a,i} - \lambda_{a,j}}{2}\right)^2}{\prod_{i,j} 2 \cosh \frac{\lambda_{a,i} - \lambda_{a+1,j}}{2}} \exp\left[\frac{i}{4\pi} \sum_i k_a \lambda_{a,i}^2 \right] \right) . \tag{4.32}$$

The normalization of the partition function was chosen so that it agrees with the ABJM result from eq. (4.4) in the case $p = 2$. As in the ABJM case, the integration contour should be taken to be the real axis in each integration variable. The saddle-point equations following from (4.32) are

$$\frac{ik_a}{\pi} \lambda_{a,i} - 2 \sum_{j \neq i} \coth \frac{\lambda_{a,j} - \lambda_{a,i}}{2} + \sum_j \tanh \frac{\lambda_{a+1,j} - \lambda_{a,i}}{2} + \sum_j \tanh \frac{\lambda_{a-1,j} - \lambda_{a,i}}{2} = 0 . \tag{4.33}$$

These equations can be solved numerically using the method described in section 4.2.2: By replacing the right-hand side of these equations by $\tau_a d\lambda_{a,j}/dt$, we obtain a system of first order differential equations whose solution converges at late times t to a solution of eq. (4.33) provided that the constants τ_a are chosen appropriately. We will now show how to obtain an approximate analytical solution valid in the limit where N is taken to be large and k is held fixed.

Based on our intuition from the ABJM model, let us assume that in this case too the real part of the eigenvalues behaves as $N^{1/2}$ at large N while the imaginary part is of order one. So if one writes

$$\lambda_{a,j} = N^{1/2} x_{a,j} + i y_{a,j} , \tag{4.34}$$

then the quantities $x_{a,j}$ and $y_{a,j}$ become dense in the large N limit. Under this assumption, we will be able to solve the saddle-point equations to leading order in

N in a self-consistent way. We can pass to the continuum limit by considering the normalized densities $\rho_a(x)$ of the $x_{a,j}$ together with the continuous functions $y_a(x)$ that describe the imaginary parts of the eigenvalues as functions of x. Let us first make a rough approximation to the saddle-point equations (4.33). When N is large, we have

$$\coth \frac{\lambda_{a,j} - \lambda_{a,i}}{2} \approx \operatorname{sgn}\left(x_{a,j} - x_{a,i}\right) , \qquad \tanh \frac{\lambda_{a,j} - \lambda_{a\pm 1,i}}{2} \approx \operatorname{sgn}\left(x_{a,j} - x_{a\pm 1,i}\right) . \tag{4.35}$$

To leading order in N, the saddle-point equations then become

$$\int dx' \left[2\rho_a(x') - \rho_{a+1}(x') - \rho_{a-1}(x')\right] \operatorname{sgn}(x' - x) = 0 \, . \tag{4.36}$$

Differentiating with respect to x, we immediately conclude that all ρ_a must be equal to one another to leading order in N, so we can write $\rho_a(x) \equiv \rho(x)$ for some density function $\rho(x)$ that is normalized as

$$\int dx \, \rho(x) = 1 \, . \tag{4.37}$$

With the simplifying assumption that the densities ρ_a are equal, one can go back to the integral (4.32) and calculate the free energy functional $F[\rho, y_a]$ to leading order in N:

$$\begin{aligned} F[\rho, y_a] = \frac{N^{3/2}}{2\pi} \int dx \, x\rho(x) \sum_{a=1}^{p} k_a y_a(x) \\ + \frac{N^{3/2}}{2} \int dx \, \rho(x)^2 \sum_{a=1}^{p} f(y_{a+1}(x) - y_a(x)) + o(N^{3/2}) \, , \end{aligned} \tag{4.38}$$

where f is the same function that was defined in (4.12). The derivation of this equation will again be postponed until the next chapter where it will be done in more generality. We wish to evaluate the integral (4.32) in the saddle-point approximation where it equals $Z = e^{-F}$, the free energy F being an appropriate critical point of $F[\rho, y_a]$. Let us assume that the eigenvalue distribution corresponding to this saddle point is connected, symmetric about $x = y = 0$, and bounded.

In looking for the eigenvalue distribution that extremizes (4.38) to order $\mathcal{O}(N^{3/2})$, an important observation is that, in fact, one cannot find this distribution, because to this order in N $F[\rho, y_a]$ has a flat direction given by $y_a(x) \to y_a(x) + \delta y(x)$ for any function $\delta y(x)$. The second term in eq. (4.38) is clearly invariant under this transformation, and the first term is also invariant because $\sum_{a=1}^{p} k_a = 0$. The existence of this flat direction is not a problem at all if one just wants to compute the free energy F to leading order in N. If one's goal is instead to find the eigenvalue distributions for the saddle point, subleading corrections to (4.38) that presumably lift this flat direction must be taken into account. We will content ourselves with calculating the free energy to order $\mathcal{O}(N^{3/2})$, and will leave a careful analysis of how the flat direction gets lifted for future work.

Before we examine the extremization of the free energy functional (4.38) in more detail, let us make a few comments and present a result that follows already from the discussion above. Suppose we manage to find a saddle point of F by extremizing (4.38) for a quiver Chern-Simons gauge theory that in the large N limit and at strong 't Hooft coupling is dual to an $AdS_4 \times Y$ M-theory background. Let us assume that this saddle point gives the most important contribution to the partition function. What can we learn? From (4.38) one may infer that the free energy grows as $N^{3/2}$ at large N as expected from supergravity, so our computation provides a gauge theory explanation of this $N^{3/2}$ behavior. Moreover, one can compare the free energy we obtain with the exact M-theory result (1.69). Via this formula we will compare successfully our matrix model results with the expressions for the volumes of tri-Sasaki Einstein space available in the literature [81, 82].

4.3.2 A Class of Orbifold Chern-Simons Theories

The vacuum moduli space of the nonchiral quivers with alternating Chern-Simons levels $(k, -k, k, -k, \ldots)$ and $N = 1$ is the orbifold $\mathbb{C}^4/\left(\mathbb{Z}_{p/2} \times \mathbb{Z}_{kp/2}\right)$ [38]. There is an induced orbifold action on the unit 7-sphere in $\mathbb{C}^4$, and thus the internal space Y is $S^7/\left(\mathbb{Z}_{p/2} \times \mathbb{Z}_{kp/2}\right)$. Consequently, we expect

$$\mathrm{Vol}(Y) = \frac{4\,\mathrm{Vol}(S^7)}{kp^2} = \frac{4\pi^4}{3kp^2}\,, \tag{4.39}$$

where in the second equality we used the round 7-sphere volume $\mathrm{Vol}(S^7) = \pi^4/3$.

This formula can be reproduced very easily from the matrix model computation. The saddle-point equations (4.33) are solved by setting $\lambda_{2a,i} = \lambda_i$ and $\lambda_{2a+1,i} = \tilde{\lambda}_i$, λ_i and $\tilde{\lambda}_i$ being the eigenvalues for the saddle point of the ABJM matrix model discussed in detail in section 4.2. The free energy of the p-node quiver with CS levels $(k, -k, k, -k, \ldots)$ is therefore $p/2$ times the free energy in the ABJM model, and thus

$$F = \frac{p}{2} F_{\mathrm{ABJM}} = \frac{\pi\sqrt{2}}{6} p k^{1/2} N^{3/2} + o(N^{3/2})\,. \tag{4.40}$$

Using eq. (1.69), one immediately reproduces the volume of the S^7 orbifold in eq. (4.39).

4.3.3 Warm-up: A Four-Node Quiver

Another case we can easily solve using the approximation scheme developed above is that of the four-node quiver with CS levels $k_a = (k, k, -k, -k)$ (see figure 4.3). The two $\mathbb{Z}_2$ symmetries of the quiver, one acting by interchanging nodes $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$ and the other by interchanging nodes $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$, allow us to set

$$\lambda_{1,j} = \lambda_{2,j} = \lambda_j\,, \qquad \lambda_{3,j} = \lambda_{4,j} = \tilde{\lambda}_j\,. \tag{4.41}$$

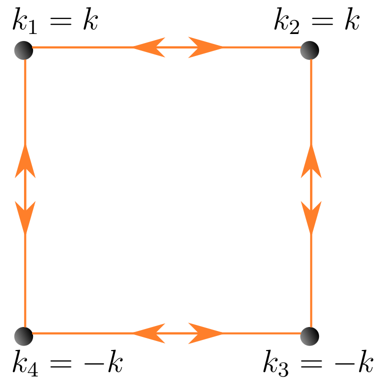

Figure 4.3: Four-node quiver diagram obtained as a particular case of the general quivers presented in figure 1.2.

Moreover, in the saddle-point equations (4.33) it is consistent to set $\tilde{\lambda}_j = \bar{\lambda}_j$ as in the ABJM case, which reduces our task to finding a single eigenvalue distribution λ_i. In passing to the continuum limit, we should therefore set

$$y_1 = y_2 = -y_3 = -y_4 = y\,. \tag{4.42}$$

The free energy functional (4.38) then becomes

$$F[\rho, y] = \frac{2kN^{3/2}}{\pi} \int dx\, x\rho(x)y(x) + N^{3/2} \int dx\, \rho(x)^2 \left[\pi^2 + f(2y(x))\right] + o(N^{3/2})\,. \tag{4.43}$$

In the paragraph following eq. (4.38) we discussed how for arbitrary p-node quivers we would not be able to solve for the y_a themselves, but only for differences of consecutive y_a, because the leading large N contribution to the free energy functional is invariant under the shifts $y_a \to y_a + \delta y$ for any function δy. In the case of the $(k, k, -k, -k)$ quiver we will, however, be able to determine the location of the eigenvalues exactly, because the ansatz (4.42) breaks this shift symmetry.

In order to find the saddle points of (4.43) in the set (4.15), we should add a Lagrange multiplier μ to enforce the normalization condition for ρ and extremize the functional

$$\tilde{F}[\rho, y] = F - \frac{N^{3/2}}{2\pi}\mu\left(\int dx\,\rho(x) - 1\right) . \tag{4.44}$$

Let us assume the eigenvalue distribution is symmetric around $x = y = 0$ and ranges between $[-x_*, x_*]$. Let us focus on the region where $x \geq 0$. Solving the equations of motion we obtain

$$\rho(x) = \frac{\mu}{8\pi^3} , \qquad y(x) = \frac{2k\pi^2 x}{\mu} , \qquad \text{if} \quad |y(x)| \leq \frac{\pi}{2} . \tag{4.45}$$

Since $\rho(x) > 0$ in this region, we have $\mu > 0$ and $y(x) \geq 0$. Assuming $y(x_*) < \pi/2$, we can find μ in terms of x_* from the normalization condition for ρ, and then express F in terms of x_* and extremize it. The extremization yields $x_* = 2^{1/4}\pi/\sqrt{k}$ and $y(x_*) = \pi/\sqrt{2} > \pi/2$, which suggests that the assumption $y(x_*) < \pi/2$ might be wrong. One could imagine that $y(x_*) > \pi/2$, but solving the saddle-point equations in the region where $y > \pi/2$ would yield $\rho(x) < 0$.

The correct answer is $y(x_*) = \pi/2$, and in fact $y(x) = \pi/2$ on some interval $[x_{\pi/2}, x_*]$ with $0 < x_{\pi/2} < x_*$. On this interval,

$$\rho(x) = \frac{\mu - 2k\pi x}{4\pi^3} , \qquad y(x) = \frac{\pi}{2} , \tag{4.46}$$

where in obtaining these equations we only varied (4.44) with respect to ρ. The quantity $x_{\pi/2}$ can be obtained from setting $y(x_{\pi/2}) = \pi/2$ in (4.45):

$$x_{\pi/2} = \frac{\mu}{4\pi k} . \tag{4.47}$$

One can now find μ by imposing the normalization condition for ρ, and then express the free energy F in terms of x_* and extremize with respect to x_*. The result is that

$$x_* = 2x_{\pi/2} = 2\pi\sqrt{\frac{2}{3k}}\,, \qquad \mu = 4\pi^2\sqrt{\frac{2k}{3}}\,. \tag{4.48}$$

The density of eigenvalues is constant on $[-x_{\pi/2}, x_{\pi/2}]$ and then drops linearly to

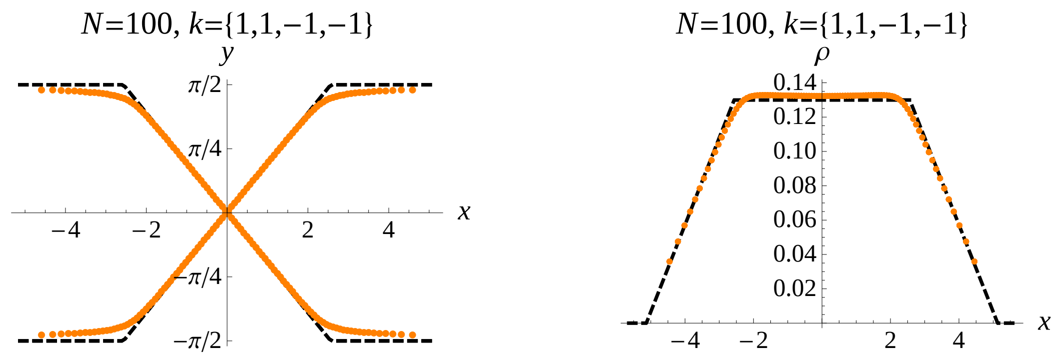

Figure 4.4: Comparison between numerics and analytical prediction for the four-node quiver with $k = \{1, 1, -1, -1\}$. The dotted black lines represent the large N analytical prediction, and the orange dots represent numerical results.

zero on $[-x_*, -x_{\pi/2}]$ and $[x_{\pi/2}, x_*]$. See figure 4.4 for a comparison of this analytical prediction with a numerical solution of the saddle-point equations.

The free energy for this model can be computed from (4.43):

$$F = \sqrt{\frac{32}{27}}\pi k^{1/2} N^{3/2} + o(N^{3/2})\,. \tag{4.49}$$

Using (1.69), we infer that the gravity dual of the Chern-Simons quiver gauge theory with CS levels $(k, k, -k, -k)$ is $AdS_4 \times Y$ where the volume of the compact space Y is

$$\mathrm{Vol}(Y) = \frac{\pi^4}{16k}\,. \tag{4.50}$$

Satisfyingly, this result is in agreement with the calculation of the corresponding integral representation given in [82] for $k = 1$, which we will review in section 4.4.

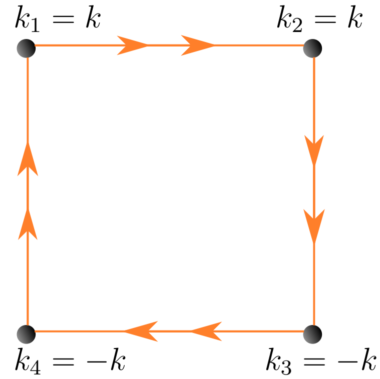

Figure 4.5: The Chern-Simons quiver gauge theory dual to $AdS_4 \times Q^{2,2,2}/\mathbb{Z}_k$ as proposed in [88, 89].

Let us also note that this volume is the same as that of a $\mathbb{Z}_k$ orbifold of the Sasaki-Einstein space $Q^{2,2,2}$, which in turn is a $\mathbb{Z}_2$ orbifold of the coset space $SU(2) \times SU(2) \times SU(2)/\left(U(1) \times U(1)\right)$. If we denote the generators of the three $SU(2)$ factors by $\vec{J}_A$, $\vec{J}_B$, and $\vec{J}_C$, then the two $U(1)$ groups we are modding out by are generated by $J_{A3} + J_{B3}$ and $J_{A3} + J_{C3}$. $Q^{2,2,2}$ admits a toric Sasaki-Einstein metric, and a proposal for the Chern-Simons quiver gauge theory dual to $AdS_4 \times Q^{2,2,2}/\mathbb{Z}_k$ was made in [88, 89]. This proposal is quite similar to the $(k, k, -k, -k)$ nonchiral quiver in figure 4.3, except it is chiral—see figure 4.5. Because of the chiral nature of the quiver, the corresponding matrix model that follows from [44] is somewhat different. Its analysis is beyond the scope of this thesis.

4.3.4 Extremization of the Free Energy Functional and Symmetries

Since the free energy functional (4.38) depends only on differences between consecutive y_a, we find it convenient to introduce the notation $\delta y_a = y_{a-1} - y_a$ and to write

$k_a = q_{a+1} - q_a$ as in eq. (1.65). Equation (4.38) becomes

$$F[\rho, \delta y_a] = \frac{N^{3/2}}{2\pi} \int dx\, x\rho(x) \sum_{a=1}^{p} q_a \delta y_a(x) + \frac{N^{3/2}}{2} \int dx\, \rho(x)^2 \sum_{a=1}^{p} f(\delta y_a(x)) + o(N^{3/2})\,. \tag{4.51}$$

This expression should be extremized over the set

$$\mathcal{C} = \left\{ (\rho, \delta y_a) : \int dx\, \rho(x) = 1; \rho(x) \geq 0, \sum_{a=1}^{p} \delta y_a(x) = 0 \text{ almost everywhere} \right\}\,. \tag{4.52}$$

Since $\sum_{a=1}^{p} \delta y_a = 0$, one could either use this constraint to solve for one of the δy_a and extremize (4.51) only with respect to the remaining ones, or, as we will do, one could introduce a Lagrange multiplier $\nu(x)$ that enforces the constraint and treat all δy_a on equal footing. Because of the normalization constraint (4.37) we also need a Lagrange multiplier μ. We therefore will extremize

$$\tilde{F}[\rho, \delta y_a] = F[\rho, \delta y_a] - \frac{N^{3/2}}{2\pi}\mu \left(\int dx\, \rho(x) - 1 \right) - \frac{N^{3/2}}{2\pi} \int dx\, \rho(x)\nu(x) \sum_{a=1}^{p} \delta y_a(x) \tag{4.53}$$

instead of (4.51). Suppose a saddle point exists. As long as $\rho(x) > 0$, the saddle-point eigenvalue distribution should satisfy the equations

$$\sum_{a=1}^{p} \left[2\pi f(\delta y_a(x))\rho(x) + (q_a x - \nu(x))\ \delta y_a(x) \right] = \mu\,, \tag{4.54a}$$

$$\pi f'(\delta y_a(x))\rho(x) + q_a x = \nu(x)\,. \tag{4.54b}$$

The extremization problem has the following discrete symmetries:

- The free energy functional (4.51) has a $\mathbb{Z}_2$ symmetry under which q_a and δy_a all change sign, so in the large N limit the partition function and the free energy are also invariant under sending $q_a \to -q_a$ for all a. This symmetry acts as $k_a \to -k_a$ and is therefore a parity transformation.

- Equation (4.51) is invariant under an overall shift of all the q_a. This symmetry was to be expected given that, after all, the original integral (4.32) depends only on k_a, which are differences of consecutive q_a.

- Interestingly, the free energy functional we are extremizing is invariant under permutations of the q_a and δy_a, so the partition function and the free energy will also be invariant under permutations of the q_a. Up to order $\mathcal{O}(N^0)$ shifts in the ranks of the gauge groups, which should be dropped in the large N limit we are taking, such permutations correspond to Seiberg dualities in the $\mathcal{N} = 2$ Chern-Simons gauge theories [83–85].

Some of the symmetries discussed above correspond to the action of the dihedral group D_p on the CS levels k_a. Our formalism shows that to leading order in N the free energy is in fact invariant under a larger symmetry group that acts on the q_a and that includes the dihedral group.

4.3.5 Three-Node Quivers

Let us now compute the free energy for arbitrary three-node quivers with CS levels (k_1, k_2, k_3) satisfying $k_1 + k_2 + k_3 = 0$. Since the k_a sum to zero, two of them must have the same sign and be smaller in absolute value than the third. Let us begin by studying the particular case where $k_2 \geq k_1 \geq 0$ and $k_3 < 0$. For simplicity, we choose $\sum_{a=1}^{3} q_a = 0$, which implies

$$q_1 = -\frac{2k_1 + k_2}{3}\,, \qquad q_2 = \frac{k_1 - k_2}{3}\,, \qquad q_3 = \frac{k_1 + 2k_2}{3}\,, \tag{4.55}$$

and we have $q_3 > 0 > q_2 \geq q_1$ and $|q_3| > |q_1| \geq |q_2|$. The solution to eqs. (4.54) is symmetric about $x = \delta y_a = 0$, and when $x \geq 0$ it breaks into three regions:

$$0 \leq x \leq \frac{\mu}{3\pi q_3} : \tag{4.56a}$$
$$\delta y_a = \frac{3\pi^2 x q_a}{\mu} , \qquad \rho = \frac{\mu}{6\pi^3} ,$$

$$\frac{\mu}{3\pi q_3} \leq x \leq -\frac{\mu}{3\pi q_1} : \tag{4.56b}$$
$$\delta y_1 = \frac{(q_1 - q_2)x}{4\pi\rho} - \frac{\pi}{2} , \qquad \delta y_2 = \frac{(q_2 - q_1)x}{4\pi\rho} - \frac{\pi}{2} , \qquad \delta y_3 = \pi ,$$
$$\rho = \frac{2\mu - 3\pi q_3 x}{6\pi^3} ,$$

$$\tag{4.56c}$$

$$-\frac{\mu}{3\pi q_1} \leq x \leq \frac{\mu}{\pi(q_3 - q_1)} : \tag{4.56d}$$
$$\delta y_1 = -\pi , \qquad \delta y_2 = 0 , \qquad \delta y_3 = \pi ,$$
$$\rho = \frac{\mu + (q_1 - q_3)\pi x}{2\pi^3} .$$

The first region ends when one of the three differences δy_a reaches $\pm\pi$. The relations

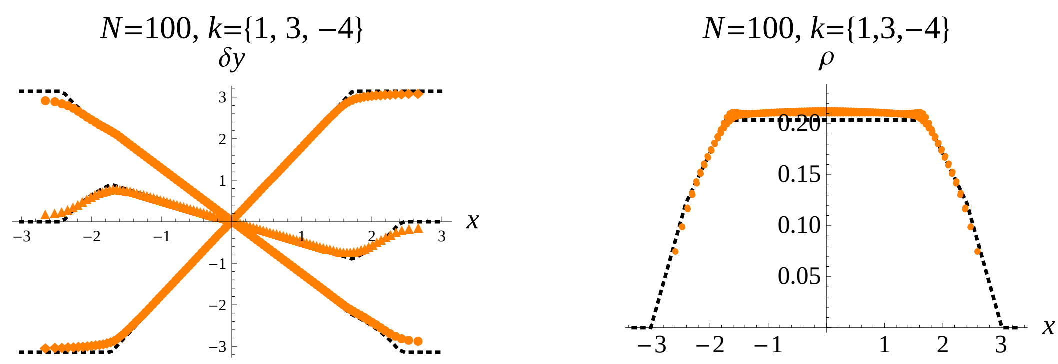

Figure 4.6: Comparison between numerics and analytical results for a three-node quiver. The dotted black lines represent the analytical large N approximation, while the orange dots represent numerical results.

between the q_a imply that at the end of the first region $\delta y_3 = \pi$, while $|\delta y_1| =$

$\pi\,|q_1|\,/\,|q_3| < \pi$ and $|\delta y_2| = \pi\,|q_2|\,/\,|q_3| < \pi$. Throughout the second region $\delta y_3 = \pi$. The second region ends when δy_1 or δy_2 reaches $\pm\pi$. When $q_1 = q_2$, the third region is absent. When $q_1 < q_2 < 0$, in this region δy_2 is monotonically increasing and δy_1 is monotonically decreasing, and since $\sum \delta y_a = 0$ it must be that δy_1 reaches $-\pi$ next. In the third region the δy_a are all constant and the density ρ decreases linearly to zero. See figure 4.6 for a particular example.

The normalization condition on ρ yields

$$\mu = \pi^2\sqrt{\frac{18q_1(q_1+q_2)(2q_1+q_2)}{q_2^2 - 5q_1^2 - 5q_1q_2}} = \pi^2\sqrt{\frac{2(k_1+k_2)(k_2-k_3)(k_1-k_3)}{(k_1k_2 - k_1k_3 - k_2k_3)}}\,. \tag{4.57}$$

Performing the integral (4.38), one obtains

$$F = \frac{N^{3/2}\mu}{3\pi} = \frac{N^{3/2}\pi\sqrt{2}}{3}\sqrt{\frac{(k_1+k_2)(k_2-k_3)(k_1-k_3)}{k_1k_2 - k_1k_3 - k_2k_3}}\,. \tag{4.58}$$

Given the free energy in the case $k_3 < 0 < k_1 \leq k_2$, it is actually possible to compute the free energy for any three-node quivers. Indeed, since in the case where there are only three nodes a permutation of the k_a can be thought of as a relabeling of the nodes, the free energy must be invariant under all such permutations. In addition, the free energy must be invariant under sending $k_a \to -k_a$ according to the second discrete symmetry discussed at the end of section 4.3.4. Combining these two properties, one can find the free energy of an arbitrary quiver with CS levels k_a by constructing the new CS levels $\tilde{k}_1 = \min(|k_1|, |k_2|, |k_3|)$, $\tilde{k}_3 = -\max(|k_1|, |k_2|, |k_3|)$, and $\tilde{k}_2 = -\tilde{k}_1 - \tilde{k}_3$ that satisfy $\tilde{k}_3 < 0 < \tilde{k}_1 \leq \tilde{k}_2$ and for which eq. (4.58) holds. The unique extension of (4.58) that gives the correct answer for arbitrary CS levels is

$$F = \frac{N^{3/2}\pi\sqrt{2}}{3}\sqrt{\frac{(|k_1|+|k_2|)(|k_2|+|k_3|)(|k_1|+|k_3|)}{|k_1|\,|k_2| + |k_1|\,|k_3| + |k_2|\,|k_3|}}\,. \tag{4.59}$$

Quite remarkably, this formula, whose derivation is based solely on gauge theory arguments, agrees with the supergravity prediction: Using (1.69), one can reproduce the volume of a $\mathbb{Z}_{\gcd\{k_1,k_2,k_3\}}$ orbifold of a compact Eschenburg space. The Eschenburg space is specified by three relatively prime integers t_a, and its volume is [81]

$$\frac{\mathrm{Vol}(S(t_1,t_2,t_3))}{\mathrm{Vol}(S^7)} = \frac{t_1t_2 + t_1t_3 + t_2t_3}{(t_1+t_2)(t_2+t_3)(t_1+t_3)} \,. \tag{4.60}$$

In terms of the k_a, the positive integers t_a are $t_a = |k_a| \,/\, \gcd\{k_1,k_2,k_3\}$ [31], so

$$\frac{\mathrm{Vol}(Y)}{\mathrm{Vol}(S^7)} = \frac{1}{\gcd\{k_1,k_2,k_3\}}\frac{\mathrm{Vol}(S(t_1,t_2,t_3))}{\mathrm{Vol}(S^7)} = \frac{|k_1|\,|k_2| + |k_1|\,|k_3| + |k_2|\,|k_3|}{(|k_1|+|k_2|)(|k_2|+|k_3|)(|k_1|+|k_3|)} \,, \tag{4.61}$$

in agreement with (1.69) and (4.59).

4.3.6 General Four-Node Quivers

We can also compute the leading large N contribution to the free energy for arbitrary four-node quivers. Let us first examine the case where $q_4 \geq q_2 \geq q_1 \geq q_3$ and $|q_4|$ is the largest among the q_a. It is convenient to require $\sum_{a=1}^4 q_a = 0$ since many of the intermediate formulae simplify under this assumption. Then we have $q_4 > 0 \geq q_1 \geq q_3$ and $|q_4| \geq |q_3| \geq |q_1| \geq |q_2|$. As in the three-node case, the solution to eqs. (4.54) is symmetric about $x = \delta y_a = 0$, and when $x \geq 0$ it breaks into three regions:

$$
\begin{aligned}
&0 \leq x \leq \frac{\mu}{4\pi q_4}: && (4.62a)\\
&\qquad \delta y_a = \frac{4\pi^2 x q_a}{\mu}, \qquad \rho = \frac{\mu}{8\pi^3},\\
&\frac{\mu}{4\pi q_4} \leq x \leq -\frac{\mu}{4\pi q_3}: && (4.62b)\\
&\qquad \delta y_1 = \frac{(3q_1+q_4)x}{6\pi\rho} - \frac{\pi}{3}, \qquad \delta y_2 = \frac{(3q_2+q_4)x}{6\pi\rho} - \frac{\pi}{3},\\
&\qquad \delta y_3 = \frac{(3q_3+q_4)x}{6\pi\rho} - \frac{\pi}{3}, \qquad \delta y_4 = \pi, \qquad \rho = \frac{3\mu - 4\pi q_4 x}{16\pi^3},\\
&-\frac{\mu}{4\pi q_3} \leq x \leq \frac{\mu}{2\pi(q_2+q_4)}: && (4.62c)\\
&\qquad \delta y_1 = \frac{(q_1-q_2)x}{4\pi\rho}, \qquad \delta y_2 = \frac{(q_2-q_1)x}{4\pi\rho},\\
&\qquad \delta y_3 = -\pi, \qquad \delta y_4 = \pi, \qquad \rho = \frac{\mu + (q_3-q_4)\pi x}{4\pi^3}.
\end{aligned}
$$

The first region ends where δy_4 reaches π. At this endpoint $|\delta y_a| = \pi\,|q_a|\,/\,|q_4| \leq \pi$

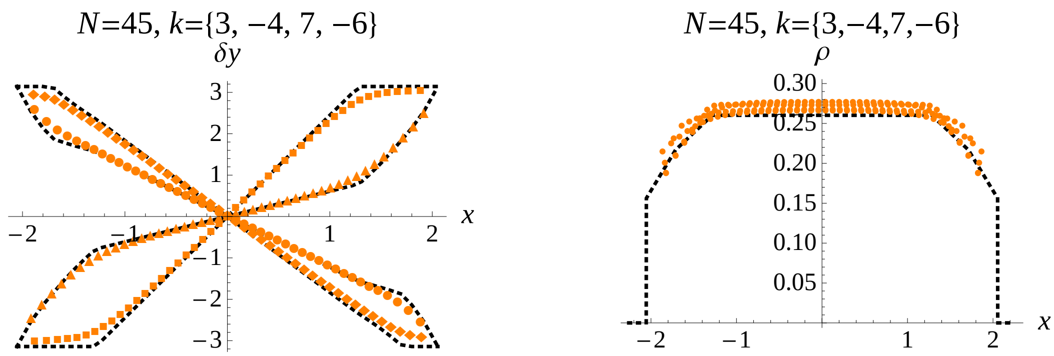

Figure 4.7: Comparison between numerics and analytical results for a four-node quiver. The dotted black lines represent the analytical large N approximation, while the orange dots represent numerical results.

for $a = 1, 2, 3$. The second region ends where $\delta y_3 = -\pi$. At this endpoint $\delta y_1 = \pi(q_1 - q_2)/(q_1 + q_2 - 2q_3)$, and since $q_1 \geq q_3$ and $q_2 \geq q_3$, by the triangle inequality it follows that $|q_1 - q_2| \leq q_1 + q_2 - 2q_3$, so $|\delta y_1| \leq \pi$. Similarly, $|\delta y_2| \leq \pi$ also. Lastly, if $q_2 = q_1$, the third region does not exist. When $q_2 > q_1$ and $q_1 < 0$, δy_1 is monotonically decreasing and δy_2 is monotonically increasing in the third region, and this region ends where $\delta y_1 = -\pi$ and $\delta y_2 = \pi$. See figure 4.7 for an example.

From $\int dx\, \rho(x) = 1$, one can find that μ is given by

$$\frac{8\pi^2}{\mu} = \sqrt{\frac{1}{q_3} - \frac{1}{q_4} + \frac{4(q_2+q_3)}{(q_2+q_4)^2} + \frac{12}{q_2+q_4}}\,. \tag{4.63}$$

The free energy is

$$F = \frac{N^{3/2}\mu}{3\pi} = \frac{8\pi N^{3/2}}{3}\left(\frac{1}{q_3} - \frac{1}{q_4} + \frac{4(q_2+q_3)}{(q_2+q_4)^2} + \frac{12}{q_2+q_4}\right)^{-1/2}\,. \tag{4.64}$$

Given eq. (4.64), one can use the symmetries we discussed at the end of section 4.3.4 to compute the free energy of a quiver gauge theory with arbitrary q_a. Indeed, one can define $\tilde{q}_a$ to be a permutation of the four numbers $q_a - \frac{1}{4}\sum_{b=1}^4 q_b$ that gives $|\tilde{q}_4| \geq |\tilde{q}_3| \geq |\tilde{q}_1| \geq |\tilde{q}_2|$. If $\tilde{q}_4$ is negative, one should flip the sign of all $\tilde{q}_a$, so we can assume $\tilde{q}_4 > 0$. By construction, the $\tilde{q}_a$ sum to zero, so the second and third largest in absolute value, namely $\tilde{q}_3$ and $\tilde{q}_1$, are negative. Therefore, the $\tilde{q}_a$ satisfy all the assumptions under which eq. (4.64) was derived, and since the free energy does not change in going from q_a to $\tilde{q}_a$, one can plug the $\tilde{q}_a$ into eq. (4.64) to find the free energy of an arbitrary four-node quiver theory. The unique extension of (4.64) to arbitrary q_a can also be written as

$$F = \frac{N^{3/2}\pi\sqrt{2}}{3}\sqrt{\frac{\prod_{a=1}^4\left(\sum_{b=1}^4 |q_{ab}|\right)}{\sum_{(a,b)\neq(c,d)\neq(e,f)} |q_{ab}|\,|q_{cd}|\,|q_{ef}| - \sum_{(a,b,c)} |q_{ab}|\,|q_{bc}|\,|q_{ca}|}}\,, \tag{4.65}$$

where q_{ab} denotes $q_a - q_b$, and in the denominator the first sum is over distinct unordered pairs of numbers from 1 to 4 while the second sum is over unordered triplets. Using eq. (1.69), we obtain a prediction for the volume of the compact space

Y:

$$\frac{\mathrm{Vol}(Y)}{\mathrm{Vol}(S^7)} = \frac{\sum_{(a,b)\neq(c,d)\neq(e,f)} |q_{ab}|\,|q_{cd}|\,|q_{ef}| - \sum_{(a,b,c)} |q_{ab}|\,|q_{bc}|\,|q_{ca}|}{\prod_{a=1}^4 \left(\sum_{b=1}^4 |q_{ab}|\right)}\,. \tag{4.66}$$

4.4 A General Formula and its Tests

Equations (4.65) and (4.66) suggest a generalization to arbitrary p-node quivers. Note first that the numerator of eq. (4.66) is a sum over all possible graphs with 4 nodes and 3 edges from which we subtract the sum over all cyclic graphs with 4 nodes and 3 edges, yielding a sum over all possible trees.

We conjecture that for a p-node quiver, the volume of the tri-Sasaki Einstein space Y (normalized so that $R_{mn} = 6g_{mn}$) is given by

$$\frac{\mathrm{Vol}(Y)}{\mathrm{Vol}(S^7)} = \frac{\sum_{(V,E)\in\mathcal{T}} \prod_{(a,b)\in E} |q_a - q_b|}{\prod_{a=1}^p \left[\sum_{b=1}^p |q_a - q_b|\right]}\,, \tag{4.67}$$

where $\mathcal{T}$ is the set of all trees (acyclic connected graphs) with nodes $V = \{1, 2, \ldots, p\}$ and edges

$$E = \{(a_1, b_1), (a_2, b_2), \ldots, (a_{p-1}, b_{p-1})\}\,. \tag{4.68}$$

A standard result in graph theory states that trees with p nodes have $p-1$ edges.

The conjecture in eq. (4.67) is consistent with the results from two-, three-, and four-node quivers, and we also checked it for five- and six-node quivers. This formula is invariant under all the symmetries discussed at the end of section 4.3.4. In particular, a quite nontrivial check of our approach is that this formula is invariant under the Seiberg dualities described in [83–85]. The connection we observe between large N matrix integrals and sums over the tree graphs is reminiscent of the connection between matrix models for 2-d quantum gravity and the Kontsevich matrix model which generates ribbon graphs [90].

An integral representation of volumes of tri-Sasaki Einstein spaces was given by Yee [82]. In general, our spaces Y are $\mathbb{Z}_k$ orbifolds of those considered in [82], where $k = \gcd\{k_a\}$. To simplify the following discussion, let us focus on the $k = 1$ case. In this case [82],

$$\mathrm{Vol}(Y) = \frac{2^{p-2}\pi^4}{3\,\mathrm{Vol}\left(U(1)^{p-2}\right)} \int \prod_{j=1}^{p-2} d\phi^j \prod_{a=1}^{p} \frac{1}{1 + \left(\sum_{j=1}^{p-2} Q_a^j \phi^j\right)^2} \,. \tag{4.69}$$

Here, $\mathrm{Vol}\left(U(1)^{p-2}\right)$ is the volume of a unit cell in the $(p-2)$-dimensional lattice defined by the identifications $\xi_j \sim \xi_j + \eta_j$, where η_j satisfy $\sum_{j=1}^{p-2} Q_a^j \eta_j \in 2\pi\mathbb{Z}$ for all $a = 1, \ldots, p$. The Q_a^j span the kernel of

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ q_1 & q_2 & q_3 & \cdots & q_p \end{pmatrix} \,. \tag{4.70}$$

(The Q_a^j are taken to be relatively prime here.) In the $U(1)^p$ Chern-Simons gauge theory, the Q_a^j are the charges of the bifundamental fields under the unbroken $U(1)^{p-2}$ symmetry [31]. We can take a spanning set of $\vec{Q}^j$ to be, for a fixed j, $Q_1^j = q_2 - q_j$, $Q_2^j = q_j - q_1$, and $Q_j^j = q_1 - q_2$ with all other $Q_a^j = 0$. For this choice of Q_a^j, the volume of $U(1)^{p-2}$ is

$$\mathrm{Vol}\left(U(1)^{p-2}\right) = \frac{(2\pi)^{p-2}}{|q_1 - q_2|^{p-3}} \,. \tag{4.71}$$

Note that $\mathrm{Vol}(Y)$ is invariant under permutation of the q_a. Although we have not carried out the integral in general, we can investigate specific cases with ease. For example, for the choice $\vec{q} = (3, 2, 1, 2)$, corresponding to the $\vec{k} = (1, 1, -1, -1)$ quiver, both our formulae (4.67) and (4.69) give $\mathrm{Vol}(Y) = \pi^4/16$. A more nontrivial choice is $\vec{q} = (3, 2, 1, 5)$ for which both formulae yield $139\pi^4/4725$. By evaluating (4.69) numerically, we were able to check agreement with (4.67) in a number of randomly selected cases for $p = 4, 5$, and 6.

The volume formula (4.67) is invariant under a shift $q_a \to q_a + 1$. In the type IIB brane construction, which involves a sequence of $(1, q_a)$ 5-branes, this symmetry corresponds to the T transformation of the $SL(2, \mathbb{Z})$ S-duality group. We could use the $SL(2, \mathbb{Z})$ symmetry to generalize the free energy to theories whose brane constructions involve general $u_a = (p_a, q_a)$ 5-branes. This generalization is accomplished by replacing the differences $|q_a - q_b|$ in the volume formula with $|u_a \wedge u_b| = |p_a q_b - p_b q_a|$. For special cases where some of the p_a vanish, this formula describes theories with fields in the fundamental representation. For example, for the ABJM model with N_f flavors, corresponding to $u_1 = (1, k), u_2 = (1, 0), u_3 = (0, N_f)$, our formula predicts

$$\frac{\mathrm{Vol}(Y)}{\mathrm{Vol}(S^7)} = \frac{2k + N_f}{2(k + N_f)^2} \,. \tag{4.72}$$

This equation agrees with the explicit matrix model calculation [91] and with the volumes of Eschenburg spaces $S(N_f, N_f, k)$ [81].

4.5 Discussion

In this chapter we have studied p-matrix models describing certain $U(N)^p$ Chern-Simons quiver gauge theories with $\mathcal{N} = 3$ supersymmetry. In the large N limit these theories are dual to eleven-dimensional supergravity on $AdS_4 \times Y$, where Y is a tri-Sasaki Einstein space. By finding an analytical large N limit of the matrix integrals, we were able to check the supergravity prediction that the logarithm of the partition function of the gauge theories on S^3 should grow as $N^{3/2}$. In $AdS_4 \times Y$ the coefficient of proportionality depends on the volume of the compact spaces Y, so we could compare our gauge theory results with the volumes computed earlier using geometric techniques [81, 82]. These successful comparisons constitute new detailed tests of the

AdS_4/CFT_3 dualities. In eq. (4.67) we conjectured an explicit combinatorial volume formula for arbitrary p. It should be possible to derive this formula in an independent way using algebraic geometry techniques similar to those in [92].

Quite generally, the main difficulty in solving matrix models is that the interactions between the eigenvalues are long-ranged, and the saddle-point approximation yields integral equations in the continuum limit. Remarkably, in solving the models described in this paper, one can set up an approximation scheme where the eigenvalue distributions can be found by solving algebraic equations. The limit in which the saddle-point equations simplify is the limit of "large cuts" where the eigenvalues grow as an appropriate positive power of N. Perhaps the key insight in solving these matrix models was that the long-range forces between the eigenvalues can be made to vanish by choosing the distribution of the real parts of the eigenvalues to be the same for each set of eigenvalues. The remaining interaction forces between the eigenvalues are short-ranged, and that is the reason why in the right variables the saddle-point equations were local and algebraic in the large N limit.

While we worked in the limit where N is sent to infinity and the Chern-Simons levels k_a are kept fixed, it is of obvious further interest to relax these assumptions and study $1/N$ corrections. In doing so, a subtle issue that needs a better understanding is the imaginary part of the free energy. At first sight, the imaginary part in the ABJM model is of order $\mathcal{O}(N)$. On the other hand, one could argue that this imaginary part is only defined modulo 2π because a shift of the free energy by an integer multiple of $2\pi i$ leaves the partition function unchanged.

Another interesting generalization of our results is to solve the matrix model in the scaling limit where the Chern-Simons levels are sent to infinity, with N/k_a kept finite. One could calculate the free energy as a function of the 't Hooft-like couplings N/k_a and check that, as predicted by the AdS/CFT correspondence, it should interpolate between an N^2 behavior at small N/k_a dictated by perturbation theory and the

$k^{1/2}N^{3/2}$ behavior at large N/k_a that we found. For $p = 2$ this check was performed in [23] by computing the resolvent of the matrix model using the techniques developed in [77]. We believe a similar check should also be possible for the $\mathcal{N} = 3$ theories studied in this paper, using perhaps similar techniques. Such an approach should also provide access to the ABJ-like cases where the ranks of the p gauge groups are not equal.

Finally, it would be interesting to investigate whether the large N matrix integrals we have calculated play a role in four-dimensional gauge theories, for example, in the 4-d "parent theories" [24, 36] of the 3-d Chern-Simons models we have studied.

Chapter 5

Towards the F-Theorem: $\mathcal{N} = 2$ Field Theories on the Three-Sphere

This chapter is an edited version of ref. [15] written in collaboration with Daniel Jafferis, Igor Klebanov, and Ben Safdi.

5.1 Introduction

Among the earliest tests of the AdS_5/CFT_4 correspondence [1–3] were comparisons of the Weyl anomaly coefficients a and c. On the gravity side these coefficients were calculated in [93] and were found to be equal; their values match the corresponding results in a variety of large N superconformal 4-d gauge theories.

Thanks to the important progress during the past several years, there now also exists a large set of precisely formulated AdS_4/CFT_3 conjectures. Examples with $\mathcal{N} \geq 3$ supersymmetry were reviewed briefly in section 1.2 of Chapter 1. Many similar duality conjectures with $\mathcal{N} = 2$ supersymmetry are also available. While various successful tests of some of these AdS_4/CFT_3 conjectures have been made, it is interesting to ask whether there exists an analog in this dimensionality of the Weyl anomaly matching. At first this question seems silly: of course, there are no anomalies in 3-d field theories. Nevertheless, it has recently been realized [14, 23, 44, 46] that the free energy of the Euclidean field theory on S^3 plays a special role and may be analogous to the anomaly a-coefficient in 4 dimensions.[1] As we have seen in the previous chapter, explicit calculations in unitary 3-d CFTs give *positive* values for F (see also [14, 23, 91]), in contrast with the thermal free energy on $\mathbb{R}^2 \times S^1$, which is negative. More generally, F is positive in all gauge theories with gravity duals—see eq. (1.39).

Recall from Chapter 4 that for field theories with extended supersymmetry, the free energy on S^3 can be calculated using the method of localization that reduces it to certain matrix integrals. We have seen how for theories with M-theory duals and $\mathcal{N} \geq 3$ supersymmetry there is agreement between the matrix model computation and the answer (1.70) predicted by 11-d supergravity. In this chapter we extend these results to theories with $\mathcal{N} = 2$ supersymmetry. For such theories the modification of the localization procedure that takes into account anomalous dimensions was derived in [14,94]. We will solve a variety of corresponding large N matrix models and provide many new tests of AdS_4/CFT_3 conjectures.

These solvable $\mathcal{N} = 2$ theories give rise to some new phenomena that could not be seen in models with higher supersymmetry. In $\mathcal{N} = 2$ theories the constraints of conformal invariance are in general not sufficient to fix all the R-charges of gauge-invariant operators. In such cases it was proposed [14] that the remaining freedom in the R-charges should be fixed by extremizing the free energy on S^3. We apply this idea to various large N models and show that the R-charges determined this way are in agreement with the AdS/CFT correspondence. In fact, in all cases we find that the R-charges locally *maximize* F. This is analogous to the well-known statement that R-charges in four-dimensional $\mathcal{N} = 1$ theories locally maximize the anomaly coefficient a [95].

[1] Similarly, in a 4-d CFT the anomaly a-coefficient may be extracted from the free energy on the four-sphere after removing the power-law divergences and differentiating with respect to $\ln R$.

We also study some pairs of fixed points connected by RG flow and find that F decreases along the flow, just like a does in 4 dimensions (there is growing evidence for the a-theorem in 4-d that states that a decreases along RG trajectories and is stationary at RG fixed points [96]). We also find that, just like a, the free energy F stays constant under exactly marginal deformations. It is therefore tempting to conjecture that there exists a similar F-theorem in 3-d, stating that the free energy on the three-sphere decreases along RG trajectories and is stationary at RG fixed points.

The rest of this chapter is organized as follows. In section 5.2 we review the rules by which one can construct the matrix model associated with a particular $\mathcal{N} = 2$ quiver. These rules are then derived in section 5.3, which can be skipped on a first reading. We show that in gauge theories where the bifundamentals are non-chiral, the total number of fundamentals equals the total number of anti-fundamentals, and the Chern-Simons levels sum to zero, the free energy scales as $N^{3/2}$. In section 5.4 we discuss an infinite class of the necklace quiver gauge theories with $\mathcal{N} = 2$ supersymmetry where the $\mathcal{N} = 3$ theories introduced in section 1.2.5 are deformed by adding a cubic superpotential for the adjoints [97]. In sections 5.5 and 5.7 we display examples of flavored quivers whose quantum corrected moduli space of vacua was constructed in [98,99] and perform F-maximization to find the R-symmetry in the IR. In section 5.6 we discuss deformations of ABJM theory and RG flows. We end with a discussion in section 5.8.

5.2 Matrix models for $\mathcal{N} = 2$ quiver gauge theories

Generalizing the localization argument of [44], it was shown in [14, 94] that the S^3 partition function of $\mathcal{N} = 2$ Chern-Simons-matter theories is also given by a matrix integral over the Cartan subalgebra of the gauge groups. The integrand involves both gaussians determined by the Chern-Simons levels as well as factors appearing from one-loop determinants. The latter depend on the curvature couplings on the sphere, parameterized by trial R-charges Δ:

$$F(\Delta) = -\ln \int \left(\prod_{\text{Cartan}} \frac{d\sigma}{2\pi} \right) \exp\left[\frac{i}{4\pi} \operatorname{tr}_k \sigma^2 - \operatorname{tr}_m \sigma \right] \det{}_{\text{Ad}} \left(2 \sinh \frac{\sigma}{2} \right) \times \prod_{\substack{\text{chirals} \\ \text{in rep } R_i}} \det{}_{R_i} \left(e^{\ell\left(1-\Delta_i + i\frac{\sigma}{2\pi}\right)} \right) , \tag{5.1}$$

where the function

$$\ell(z) = -z \ln\left(1 - e^{2\pi i z}\right) + \frac{i}{2} \left(\pi z^2 + \frac{1}{\pi} \operatorname{Li}_2\left(e^{2\pi i z}\right) \right) - \frac{i\pi}{12} \tag{5.2}$$

satisfies the differential equation $d\ell/dz = -\pi z \cot(\pi z)$ with $\ell(0) = 0$. The integration variables σ are the scalars in the vector multiplets. Since these scalars transform in the adjoint representation of the gauge group, the integration contour for each eigenvalue of σ should be taken to be the real axis. The trace tr_k is normalized so that for each gauge group a it equals the Chern-Simons level k_a times the trace in the fundamental representation. We will explain the term $\operatorname{tr}_m \sigma$ at the end of the next paragraph.

Some of the important ingredients of the $U(N)^p$ CS gauge theories we study are the topological conserved currents $j_{\text{top},a} = * \operatorname{tr} F_a$ and monopole operators $T_{\vec{q}}$ that create q_a units of $\operatorname{tr} F_a$ flux through a two-sphere surrounding the insertion point. In general, the R-symmetry can mix with these topological global symmetries, and the monopole operators $T_{\vec{q}}$ acquire R-charges $R[T_{\vec{q}}] = \gamma_{\vec{q}} + \sum_a \Delta_m^{(a)} q_a$, where $\gamma_{\vec{q}}$ is an anomalous dimension invariant under sending $\vec{q} \to -\vec{q}$, and the $\Delta_m^{(a)}$ are what we call bare monopole R-charges. The anomalous dimensions $\gamma_{\vec{q}}$ can be computed exactly at one-loop in perturbation theory from the matter R-charges, as in refs. [98, 99] based on the work of [100]. Of special interest will be the "diagonal" monopole operators

T corresponding to $\vec{q} = (1,1,1,\ldots)$ and $\tilde{T}$ corresponding to $\vec{q} = (-1,-1,-1,\ldots)$, because they play a crucial role in the construction of the quantum-corrected moduli space of vacua in these theories [98, 99]. Their R-charges satisfy the relation

$$R[T] - R[\tilde{T}] = 2\Delta_m\,, \qquad \Delta_m \equiv \sum_a \Delta_m^{(a)}\,. \tag{5.3}$$

The modification of the couplings to curvature associated to mixing the R-charge with one of the topological charges is precisely the complexification of the FI parameter, appropriately supersymmetrized on S^3. This modification results in the appearance of $\operatorname{tr}_m \sigma$ in the matrix integral (5.1). Note that the trace tr_m is normalized so that for each gauge group it equals the trace in the fundamental representation times the bare monopole R-charge $\Delta_m^{(a)}$; when this charge vanishes the $\operatorname{tr}_m \sigma$ term may be removed from (5.1).

One may worry already that the bare R-charges of the diagonal monopole operators are not gauge-invariant observables because the Chern-Simons coupling makes the monopole operators not gauge-invariant. As we will explain in more detail in section 5.2.3, with an appropriate choice of gauge group one can construct gauge-invariant operators out of T or out of $\tilde{T}$, and from the R-charges of these gauge-invariant operators one can calculate Δ_m. (In passing, note that the same concern can be raised about the R-charges of the bifundamental fields, and the same resolution holds.) In theories with charge conjugation symmetry the R-charge of T should equal that of $\tilde{T}$, which implies $\Delta_m = 0$. Indeed, F-maximization in non-chiral theories is consistent with this observation.

Since the R-symmetry can mix with any other abelian global symmetry, it would be interesting to ask how many such global symmetries there are for a given quiver. We will be interested in quivers with gauge group $U(N)^p$ as well as quivers with gauge group $SU(N)^p \times U(1)$, where the second factor is the diagonal $U(1)$ in $U(N)^p$. If there is no superpotential, we can show that the number of abelian flavor symmetries equals the number of matter representations regardless of which choice of gauge group. Indeed, if all the gauge groups are $SU(N)$, then for each matter field there is a $U(1)$ global symmetry that acts by multiplying that field by a phase. Replacing some of the $SU(N)$ gauge groups by $U(N)$ gauges some of these $U(1)$ symmetries. However, for each new $U(1)$ gauge symmetry there is now an additional topological conserved current $j_{\text{top}} = *F$ in addition to the old conserved current j_{matter}. For Chern-Simons level k the $U(1)$ gauge field now couples to $j_{\text{matter}} + kj_{\text{top}}$. Going from $SU(N)$ to $U(N)$ gauge theory therefore introduces a new topological $U(1)$ symmetry and gauges a linear combination of this $U(1)$ and the diagonal $U(1)$ in $U(N)$. Thus, the total number of global symmetries does not change and stays equal to the number of matter representations for any of choice of gauge group. A non-trivial superpotential will generically break some of these flavor symmetries.

5.2.1 The forces on the eigenvalues

Suppose we have a quiver with nodes $1, 2, \ldots, p$ with $U(N)$ gauge groups and CS levels k_a. Let's denote the eigenvalues corresponding to the ath node by $\lambda_i^{(a)}$, with $i = 1, 2, \ldots, N$. In the saddle point approximation the force acting on $\lambda_i^{(a)}$ can be split into several pieces:

$$F_i^{(a)} = F_{i,\text{ext}}^{(a)} + F_{i,\text{self}}^{(a)} + \sum_b F_{i,\text{inter}}^{(a,b)} + \sum_b F_{i,\text{inter}}^{(b,a)} \,. \tag{5.4}$$

The first term is the external force

$$F_{i,\text{ext}}^{(a)} = \frac{ik_a}{2\pi}\lambda_i^{(a)} - \Delta_m^{(a)} \,, \tag{5.5}$$

where $\Delta_m^{(a)}$ is the corresponding bare monopole R-charge. The second term is due to interactions with eigenvalues belonging to the same node:

$$F_{i,\text{self}}^{(a)} = \sum_{j\neq i} \coth \frac{\lambda_i^{(a)} - \lambda_j^{(a)}}{2} \, . \tag{5.6}$$

Finally, the last two terms in eq. (5.4) correspond to contributions of bifundamental fields (a,b) that transform in the fundamental representation of node a and the anti-fundamental representation of node b. We have

$$F_{i,\text{inter}}^{(a,b)} = \sum_j \left[\frac{\Delta_{(a,b)} - 1}{2} - i\frac{\lambda_i^{(a)} - \lambda_j^{(b)}}{4\pi}\right] \coth\left[\frac{\lambda_i^{(a)} - \lambda_j^{(b)}}{2} - i\pi\left(1 - \Delta_{(a,b)}\right)\right] , \tag{5.7}$$

$$F_{i,\text{inter}}^{(b,a)} = \sum_j \left[\frac{\Delta_{(b,a)} - 1}{2} + i\frac{\lambda_i^{(a)} - \lambda_j^{(b)}}{4\pi}\right] \coth\left[\frac{\lambda_i^{(a)} - \lambda_j^{(b)}}{2} + i\pi\left(1 - \Delta_{(b,a)}\right)\right] . \tag{5.8}$$

We can split the interaction forces between the eigenvalues into long-range forces and short-range forces. We define the long-range forces to be those forces obtained by replacing $\coth(u)$ with its large u approximation, $\operatorname{sgn}\operatorname{Re}(u)$. Since $\operatorname{sgn}\operatorname{Re}(\alpha u) = \operatorname{sgn}\operatorname{Re}(u)$ if $\alpha > 0$, we have

$$\begin{aligned}
F_{i,\text{self}}^{(a)} &\approx \hat{F}_{i,\text{self}}^{(a)} = \sum_{j\neq i} \operatorname{sgn}\operatorname{Re}\left(\lambda_i^{(a)} - \lambda_j^{(a)}\right) , \\
F_{i,\text{inter}}^{(a,b)} &\approx \hat{F}_{i,\text{inter}}^{(a,b)} = \sum_j \left[\frac{\Delta_{(a,b)} - 1}{2} - i\frac{\lambda_i^{(a)} - \lambda_j^{(b)}}{4\pi}\right] \operatorname{sgn}\operatorname{Re}\left(\lambda_i^{(a)} - \lambda_j^{(b)}\right) , \\
F_{i,\text{inter}}^{(b,a)} &\approx \hat{F}_{i,\text{inter}}^{(b,a)} = \sum_j \left[\frac{\Delta_{(b,a)} - 1}{2} + i\frac{\lambda_i^{(a)} - \lambda_j^{(b)}}{4\pi}\right] \operatorname{sgn}\operatorname{Re}\left(\lambda_i^{(a)} - \lambda_j^{(b)}\right) .
\end{aligned} \tag{5.9}$$

5.2.2 General rules for matrix models with no long-range forces

We want to study quiver gauge theories with free energies that scale as $N^{3/2}$ in the large N limit, because these theories are thought to have M-theory duals. One way of achieving this is for the real part of the eigenvalues to scale as $N^{1/2}$ and the imaginary parts to stay order N^0 in the large N limit (see section 5.3 for more details). A necessary condition for this scaling is that the long range forces must vanish at the saddle point of the matrix integral.

A large class of such theories are quiver gauge theories with non-chiral bifundamental superfields, meaning that for each $\mathcal{N} = 2$ chiral superfield $X_{(a,b)}$ transforming in $(\mathbf{N}, \overline{\mathbf{N}})$ of the gauge groups $U(N)_a \times U(N)_b$ there exists another chiral superfield $X_{(b,a)}$ transforming in $(\overline{\mathbf{N}}, \mathbf{N})$. The two fields $X_{(a,b)}$ and $X_{(b,a)}$ need not be related by supersymmetry, and thus their R-charges $\Delta_{(a,b)}$ and $\Delta_{(b,a)}$ need not be equal. In addition to these bifundamental fields we will also allow for equal numbers of fundamental and anti-fundamental fields.[2] The kinetic terms for the vector multiplets could be either Chern-Simons with level k_a or Yang-Mills. Additionally we require

$$\sum_a k_a = 0\,. \tag{5.10}$$

For such theories the condition that the long-range forces (5.9) vanish is equivalent to

$$\sum \Delta_{(a,b)} + \sum \Delta_{(b,a)} = n_{(a)} - 2 \tag{5.11}$$

for each node a, where the sum is taken over all the bifundamental fields transforming non-trivially under $U(N)_a$, and n_a denotes the number of such fields (adjoint fields are supposed to be counted twice: once as part of the first sum and once as part of the second sum).

With these assumptions, it is consistent to assume that, as in Chapter 4, in the

[2] Equal in total number; the number of fundamental and anti-fundamental fields charged under a given gauge group are allowed to differ.

large N limit the eigenvalues $\lambda_i^{(a)}$ behave as

$$\lambda_i^{(a)} = N^{1/2} x_i + i y_{a,i} + o(N^0). \tag{5.12}$$

As we take N to infinity, we can replace x_i and $y_{a,i}$ by continuous functions $x(s)$ and $y_a(s)$ such that $x_i = x(i/N)$ and $y_{a,i} = y_a(i/N)$. In the following discussion, it will be useful to consider the density

$$\rho(x) = \frac{ds}{dx} \tag{5.13}$$

and express the imaginary parts of the eigenvalues as functions $y_a(x)$.

That the long-range forces (5.9) vanish implies that the free energy functional is local. Here are the rules for constructing the free energy functional for any $\mathcal{N} = 2$ quiver theory that satisfies the conditions described above:

1. For each gauge group a with CS level k_a and bare monopole R-charge $\Delta_m^{(a)}$ one should add the term

$$\frac{k_a}{2\pi} N^{3/2} \int dx\, \rho(x) x y_a(x) + \Delta_m^{(a)} N^{3/2} \int dx\, \rho(x) x\,. \tag{5.14}$$

2. For a pair of bifundamental fields, one of R-charge $\Delta_{(a,b)}$ transforming in the $(\mathbf{N}, \overline{\mathbf{N}})$ of $U(N)_a \times U(N)_b$ and one of R-charge $\Delta_{(b,a)}$ transforming in the $(\overline{\mathbf{N}}, \mathbf{N})$ of $U(N)_a \times U(N)_b$, one should add

$$-N^{3/2} \frac{2 - \Delta^+_{(a,b)}}{2} \int dx\, \rho(x)^2 \left[\left(y_a - y_b + \pi \Delta^-_{(a,b)} \right)^2 - \frac{1}{3} \pi^2 \Delta^+_{(a,b)} \left(4 - \Delta^+_{(a,b)} \right) \right] , \tag{5.15}$$

where $\Delta^{\pm}_{(a,b)} \equiv \Delta_{(a,b)} \pm \Delta_{(b,a)}$ satisfies $\Delta^{+}_{(a,b)} < 2$, and $y_a - y_b$ is in the range

$$\left| y_a - y_b + \pi \Delta^{-}_{(a,b)} \right| \leq \pi \Delta^{+}_{(a,b)} \,. \tag{5.16}$$

Outside this range the formula (5.15) is no longer valid, and in fact for arbitrary $y_a - y_b$ the integrand is a non-smooth function. The boundaries of the range (5.16) are points where the integrand should be considered to be non-differentiable. In practice, this means that the equations obtained from varying the free energy functional with respect to $y_a - y_b$ need not hold whenever $\left| y_a - y_b + \pi \Delta^{-}_{(a,b)} \right| = \pm \pi \Delta^{+}_{(a,b)}$.

3. For an adjoint field of R-charge $\Delta_{(a,a)}$, one should add

$$\frac{2\pi^2}{3} N^{3/2} \Delta_{(a,a)} \left(1 - \Delta_{(a,a)}\right) \left(2 - \Delta_{(a,a)}\right) \int dx\, \rho(x)^2 \,. \tag{5.17}$$

4. For a field X_a with R-charge Δ_a transforming in the fundamental of $U(N)_a$, one should add

$$N^{3/2} \int dx\, \rho(x)\, |x| \left(\frac{1 - \Delta_a}{2} - \frac{1}{4\pi} y_a(x) \right) \,, \tag{5.18}$$

while for an anti-fundamental field of R-charge $\tilde{\Delta}_a$ one should add

$$N^{3/2} \int dx\, \rho(x)\, |x| \left(\frac{1 - \tilde{\Delta}_a}{2} + \frac{1}{4\pi} y_a(x) \right) \,. \tag{5.19}$$

These rules will be derived in Section 5.3.

5.2.3 Flat directions and $U(N)$ vs. $SU(N)$

In a theory with p $U(N)$ gauge groups, the matrix integral (5.1), seen as a function of the R-charges of the matter fields as well as the bare monopole R-charges $\Delta_m^{(a)}$, has the following symmetries parameterized by p real numbers $\delta^{(a)}$:

$$\begin{aligned}
\text{eigenvalues } \lambda_i^{(a)} \text{ for } a\text{th gauge group:} &\qquad \lambda_i^{(a)} \to \lambda_i^{(a)} - 2\pi i \delta^{(a)} \,, \\
U(N)_a \times U(N)_b \text{ bifundamental of R-charge } \Delta_{(a,b)}\text{:} &\qquad \Delta_{(a,b)} \to \Delta_{(a,b)} + \delta^{(a)} - \delta^{(b)} \,, \\
U(N)_a \text{ fundamental of R-charge } \Delta_a\text{:} &\qquad \Delta_a \to \Delta_a + \delta^{(a)} \,, \\
U(N)_a \text{ anti-fundamental of R-charge } \tilde{\Delta}_a\text{:} &\qquad \tilde{\Delta}_a \to \tilde{\Delta}_a - \delta^{(a)} \,, \\
\text{bare monopole R-charge } \Delta_m^{(a)} \text{ for } a\text{th gauge group:} &\qquad \Delta_m^{(a)} \to \Delta_m^{(a)} + k_a \delta^{(a)} \,.
\end{aligned} \tag{5.20}$$

The transformations (5.20) leave the matrix integral (5.1) invariant (up to a phase) because they are equivalent to a change of variables where the integration contour for each set of eigenvalues is shifted by a constant amount.

A consequence of this symmetry is that at finite N the free energy $F(\Delta)$ (namely the extremum of the free energy functional for fixed R-charges Δ) has p flat directions parameterized by $\delta^{(a)}$. In the $U(N)^p$ theory, these flat directions are to be expected because, for example, the bifundamental fields in the theory are not gauge-invariant operators. The free energy should depend only on the R-charges of gauge-invariant operators. One then has two options: work with the $U(N)^p$ theory where by maximizing F one can only determine the R-charges of composite gauge-invariant operators (for example, $\Delta_{(a,b)} + \Delta_{(b,a)}$ would be a well-defined number since it is the R-charge of the gauge-invariant operator $\operatorname{tr} X_{(a,b)} X_{(b,a)}$), or ungauge some of the diagonal $U(1)_a$'s in the $U(N)_a$ gauge groups. If the $U(N)_a$ gauge group is replaced by $SU(N)_a$ then the corresponding eigenvalues $\lambda_i^{(a)}$ should satisfy the tracelessness condition

$$\sum_i \lambda_i^{(a)} = 0 \,. \tag{5.21}$$

This condition fixes $\delta^{(a)}$ and removes a flat direction from F.

In the large N limit the free energy will generically have more flat directions than at finite N. For example, at large N there are an additional n_f flat directions coming from the flavors for the following reason. In the theories we consider each fundamental field is paired with an anti-fundamental field. Let the R-charge of one of the fundamental fields be Δ_f and the R-charge of the corresponding anti-fundamental field be $\tilde{\Delta}_f$. At finite N the sum $\Delta_f + \tilde{\Delta}_f$ is fixed by the marginality of the superpotential, leaving one free R-charge. However the finite N free energy will typically be a non-trivial function of both Δ_f and $\tilde{\Delta}_f$. At large N, on the other hand, the free energy really only depends on the sum $\Delta_f + \tilde{\Delta}_f$, as can be seen from eqs. (5.18) and (5.19). This gives us an additional n_f "accidental" flat directions at large N.

Looking at equation (5.14), one can see that at large N the free energy only depends on the sum

$$\Delta_m = \sum_{a=1}^{p} \Delta_m^{(a)}. \tag{5.22}$$

Naively one would think this gives us $p-1$ additional flat directions corresponding to shifts in the individual $\Delta_m^{(a)}$, which leave the sum in equation (5.22) invariant. However, in theories where $\sum_a k_a = 0$, which are all the theories presented in this chapter, the story is slightly more subtle. In these theories we actually only gain $p-2$ additional flat directions in the large N limit. This is because at order $\mathcal{O}(N^{3/2})$ the symmetry corresponding to $\delta^{(a)} = \delta$ is equivalent to one of the "new" flat directions, which correspond to symmetries of the sum (5.22). To summarize, at large N we have a total of $2(p-1)+n_f$ flat directions of the free energy. However, only p of these flat directions correspond to gauge symmetries. The other $p-2+n_f$ flat directions are only there at infinite N.

One could choose to eliminate some of the flat directions in the free energy by changing the gauge groups from $U(N)$ to $SU(N)$. Since the diagonal monopole operators T and $\tilde{T}$ are essential in obtaining the quantum-corrected moduli space in these theories, we would like to keep the diagonal $U(1)$ in $U(N)^p$ as a gauge symmetry. So, let's choose to eliminate all the flat directions in the free energy coming from the abelian gauge symmetries, except for the flat direction corresponding to this diagonal $U(1)$. The R-charges of the (bi)fundamental fields are then gauge invariant quantities, as we will explain below. The residual abelian gauge symmetry gives us $p-1$ gauge invariant combinations of the p bare monopole R-charges $\Delta_m^{(a)}$. However, at large N we will only be able to compute the sums Δ_m and $\Delta_f+\tilde{\Delta}_f$ because of the accidental flat directions.

In going from $U(N)^p$ to $SU(N)^p \times U(1)$ we should regard

$$A_+ = \sum_{a=1}^{p} \operatorname{tr} A_a \tag{5.23}$$

as a dynamical gauge field, while the other gauge fields $A^{(b)} = \sum_a \alpha_a^b \operatorname{tr} A_a$, where α_a^b is a basis of solutions to $\sum_{a=1}^p \alpha_a^b = 0$, should be treated as background fields that we set to zero. The ungauging procedure [101] can be done rigorously by adding $p-1$ vector multiplets whose vector components are B_b, $b=1,2,\ldots,p-1$, and that couple to the topological currents $*F^{(b)} = *\sum_{a=1}^p \alpha_a^b \operatorname{tr} F_a$ through

$$\delta S = \sum_{b=1}^{p-1} \int B_b \wedge F^{(b)}\,, \tag{5.24}$$

with an appropriate supersymmetric completion. Making the fields B_b dynamical, the integration over them in the path integral essentially ungauges $A^{(b)}$. For related discussions, see [102–105].

To summarize so far, in the $U(N)^p$ gauge theory the large N free energy $F(\Delta)$ generically has $2p - 2 + n_f$ flat directions. However, only p of these symmetries correspond to gauge symmetries with the rest being accidental flat direction appearing only at large N. If we want to remove $p - 1$ of the flat directions corresponding to gauge symmetries, we should consider the $SU(N)^p \times U(1)$ gauge theory, where the $U(1)$ gauge field is A_+. In this theory one can construct the baryonic operator

$$\mathcal{B}\left(X_{(a,b)}\right) = \epsilon_{i_1 \cdots i_N} \epsilon^{j_1 \cdots j_N} \left(X_{(a,b)}\right)^{i_1}_{j_1} \cdots \left(X_{(a,b)}\right)^{i_N}_{j_N} , \tag{5.25}$$

which is a gauge-invariant chiral primary with R-charge $N\Delta_{(a,b)}$. In other words, the operator $X_{(a,b)}$ can be assigned a unique R-charge $\Delta_{(a,b)}$ because the baryon $\mathcal{B}\left(X_{(a,b)}\right)$ has the well-defined R-charge $N\Delta_{(a,b)}$. Minimizing $F(\Delta)$ in this theory one can then determine the R-charges of the bifundamenal fields.

Ungauging the $p - 1$ off-diagonal $U(1)$ gauge fields makes it possible to define gauge-invariant baryonic operators at the expense of removing from the chiral ring the off-diagonal monopole operators that generate non-zero numbers of $F^{(b)}$ flux units that exist in the $U(N)^p$ theory. This ungauging doesn't remove, however, the diagonal monopole operators T and $\tilde{T}$, because these operators generate equal numbers of $\operatorname{tr} F_a$ flux units and thus no $F^{(b)}$ flux units. Moreover, the bare monopole R-charges Δ_m of T and $-\Delta_m$ of $\tilde{T}$ are well-defined quantities because one can construct a baryonic-like operator out of T or $\tilde{T}$.

From an AdS/CFT point of view, ungauging $U(1)$ symmetries in the boundary theory is equivalent to changing boundary conditions in the bulk for the bulk gauge fields dual to those $U(1)$ symmetries. In M-theory, the boundary conditions corresponding to the $U(N)^p$ gauge theory allow the existence of M2-branes wrapping topologically non-trivial two-cycles, but disallow the existence of the magnetic dual objects, which would be the M5-branes wrapping the dual five-cycles. The boundary conditions for the $SU(N)^p \times U(1)$ gauge theory allow the existence of M5-branes but disallow M2-branes wrapped on topologically non-trivial cycles. Since these wrapped M2-branes are dual to off-diagonal monopole operators and the M5-branes wrapping topologically non-trivial cycles are dual to baryonic operators, the general picture on the gravity side is consistent with the field theory analysis. See [104, 105] for a more detailed discussion.

In addition to M5-branes wrapping topologically non-trivial cycles that are allowed only in the $SU(N)^p \times U(1)$ gauge theory, on the gravity side one can also consider giant gravitons, which are BPS configurations of M5-branes wrapping topologically trivial five-cycles and rotating within the 7-d space Y [106]. On the field theory side, these objects are thought to be dual to determinants of operators that transform in the adjoint representation of one of the gauge groups (such as determinants of products of bifundamental fields). These determinant operators are gauge invariant in both the $U(N)^p$ and $SU(N)^p \times U(1)$ gauge theory.

In general, the relation between the volume of a five-cycle wrapped by an M5-brane and the dimension of the corresponding gauge theory operator is [107]

$$\Delta = \frac{\pi N}{6} \frac{\mathrm{Vol}(\Sigma_5)}{\mathrm{Vol}(Y)}\,, \tag{5.26}$$

regardless of whether the five-cycle the brane is wrapping is topologically trivial or not. We will make extensive use of this formula, as it provides a way of extracting the expected R-charge of the bifundamental fields (or of certain products of bifundamental fields) from the gravity side. Indeed, after performing F-maximization, we check not only that the extremum of F matches the supergravity prediction (1.69) computed using the volume of Y, but also that the dimensions of the operators dual to wrapped M5-branes agree with eq. (5.26), which involves the volumes of the various five-cycles computed from the gravity side.

5.3 Derivation of matrix model rules

In this section we provide a derivation of the rules we gave in section 5.2.2 for finding the continuum limit of the free energy in eq. (5.1). This section is rather technical and can be skipped on a first reading. We assume that at large N the eigenvalues scale as

$$\lambda_i^{(a)} = N^\alpha x_i + i y_{a,i} + o(1) \tag{5.27}$$

for some number $\alpha \in (0,1)$. We will eventually be interested in setting $\alpha = 1/2$. In writing eq. (5.27) we implicitly assume that as we take N to infinity, the x_i and $y_{a,i}$ become dense, so in the continuum limit we can express y_a as a continuous function $y_a(x)$. It is convenient to define the density

$$\rho(x) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i)\,, \tag{5.28}$$

which as we take $N \to \infty$ also becomes a continuous function of x normalized so that $\int dx\, \rho(x) = 1$.

5.3.1 First rule

For each gauge group a with CS level k_a and bare monopole R-charge $\Delta_m^{(a)}$, the discrete contribution to F is

$$\begin{aligned} F_1 &= \sum_{i=1}^{N} \left[-\frac{ik_a}{4\pi} (\lambda_{a,i})^2 + \Delta_m^{(a)} \lambda_{a,i} \right] \\ &= \sum_{i=1}^{N} \left[-\frac{ik_a}{4\pi} (N^\alpha x_i + i y_{a,i})^2 + \Delta_m^{(a)} (N^\alpha x_i + i y_{a,i}) \right] + o(N^{1+\alpha})\,, \end{aligned} \tag{5.29}$$

where in the second line we used (5.27). Expanding in N we obtain

$$F_1 = -\frac{ik_a}{4\pi}N^{2\alpha}\sum_{i=1}^{N}x_i^2 + N^{\alpha}\sum_{i=1}^{N}\left[\frac{k_a}{2\pi}x_i y_{a,i} + \Delta_m^{(a)}x_i\right] + o(N^{1+\alpha})\,. \tag{5.30}$$

The first term in this sum vanishes when we sum over a because we assume $\sum_a k_a = 0$. In taking the continuum limit we therefore only need to keep the second term and replace $\sum_i(\cdots)$ by $N\int dx\,\rho(x)\,(\cdots)$. We get

$$F_1 = N^{\alpha+1}\int dx\,\rho(x)\left[\frac{k_a}{2\pi}x y_a(x) + \Delta_m^{(a)}x\right] + o(N^{1+\alpha})\,, \tag{5.31}$$

reproducing eq. (5.14) when $\alpha = 1/2$.

5.3.2 Second and third rules

The interaction terms between the eigenvalues contain two types of terms: one coming from the one-loop determinant of the fields in the vector multiplets

$$F_{2,\text{vector}} = -\frac{1}{2}\sum_{a=1}^{p}\sum_{i,j=1}^{N}\ln\left(4\sinh^2\frac{\lambda_i^{(a)}-\lambda_j^{(a)}}{2}\right) \tag{5.32}$$

for each gauge group a, and one coming from the one-loop determinants of the matter fields

$$F_{2,\text{matter}} = -\sum_{\substack{\text{bifundamentals}\\(a,b)}}\sum_{i,j=1}^{N}\ell\left(1-\Delta_{(a,b)} + i\frac{\lambda_i^{(a)}-\lambda_j^{(b)}}{2\pi}\right)\,, \tag{5.33}$$

where the function $\ell(z)$ was defined in eq. (5.2). Since for each gauge group we require the relation (5.11) to hold, we can rewrite the contribution $F_{2,\text{vector}}$ as

$$F_{2,\text{vector}} = -\frac{1}{4} \sum_{\substack{\text{bifundamentals} \\ (a,b)}} \left(1 - \Delta_{(a,b)}\right) \sum_{i,j=1}^{N} \ln \left(16 \sinh^2 \frac{\lambda_i^{(a)} - \lambda_j^{(a)}}{2} \sinh^2 \frac{\lambda_i^{(b)} - \lambda_j^{(b)}}{2}\right) . \tag{5.34}$$

Combining (5.34) and (5.33) one can write the interaction term in the free energy as a sum over the bifundamental fields.

In order to calculate $F_2 = F_{2,\text{matter}} + F_{2,\text{vector}}$ as $N \to \infty$ we find it easier to first calculate the derivatives of F_2 with respect to y_a. We have

$$\begin{aligned}
\frac{\partial F_2}{\partial y_{a,i}} &= \sum_{j=1}^{N} \Bigg[-\frac{i}{2} \left(2 - \Delta_{(a,b)} - \Delta_{(b,a)}\right) \coth \frac{N^\alpha (x_i - x_j) + i(y_{a,i} - y_{a,j})}{2} \\
&- \frac{1}{4\pi} \cot \left[\pi \Delta_{(a,b)} + \frac{i N^\alpha (x_j - x_i) + y_{a,i} - y_{b,j}}{2}\right] \left(2\pi(\Delta_{(a,b)} - 1) + i N^\alpha (x_j - x_i) + y_{a,i} - y_{b,j}\right) \\
&+ \frac{1}{4\pi} \cot \left[\pi \Delta_{(b,a)} - \frac{i N^\alpha (x_j - x_i) + y_{a,i} - y_{b,j}}{2}\right] \left(2\pi(\Delta_{(b,a)} - 1) - i N^\alpha (x_j - x_i) + y_{a,i} - y_{b,j}\right) \Bigg] ,
\end{aligned} \tag{5.35}$$

where in the first term we should not let $j = i$. In the continuum limit this expression becomes

$$\begin{aligned}
\frac{\delta F_2}{\delta y_a(x)} &\approx N^2 \rho(x)^2 \sum_{\substack{\text{bifundamentals} \\ (a,b) \text{ and } (b,a)}} \text{P.V.} \int dx' \Bigg[\frac{i}{2} \left(2 - \Delta_{(a,b)} - \Delta_{(b,a)}\right) \coth \frac{\lambda_a(x) - \lambda_a(x')}{2} \\
&- \frac{1}{4\pi} \cot \left[\pi \Delta_{(a,b)} + \frac{i(\lambda_b(x') - \lambda_a(x))}{2}\right] \left(2\pi(\Delta_{(a,b)} - 1) + i(\lambda_b(x') - \lambda_a(x))\right) \\
&+ \frac{1}{4\pi} \cot \left[\pi \Delta_{(b,a)} - \frac{i(\lambda_b(x') - \lambda_a(x))}{2}\right] \left(2\pi(\Delta_{(b,a)} - 1) - i(\lambda_b(x') - \lambda_a(x))\right) \Bigg] ,
\end{aligned} \tag{5.36}$$

where P.V. denotes principal value integration and by $\lambda_a(x)$ we mean $N^\alpha x + i y_a(x)$. In the sum over pairs of bifundamental fields, adjoint fields should be counted once and should come with an explicit factor of $1/2$. Changing variables from x' to $\xi = N^\alpha(x'-x)$ and taking $N \to \infty$, the integral in (5.36) becomes

$$\frac{\delta F_2}{\delta y_a(x)} \approx N^{2-\alpha}\rho(x)^2 \sum_{\substack{\text{bifundamentals}\\ (a,b) \text{ and } (b,a)}} \int_{-\infty}^{\infty} d\xi$$
$$\left[-\frac{1}{4\pi}\cot\left[\pi\Delta_{(a,b)} + \frac{i\xi + y_a(x) - y_b(x)}{2}\right]\left(2\pi(\Delta_{(a,b)}-1) + i\xi + y_a(x) - y_b(x)\right)\right.$$
$$\left. + \frac{1}{4\pi}\cot\left[\pi\Delta_{(b,a)} - \frac{i\xi + y_a(x) - y_b(x)}{2}\right]\left(2\pi(\Delta_{(b,a)}-1) - i\xi - y_a(x) + y_b(x)\right)\right]. \tag{5.37}$$

This integral converges and can be evaluated to

$$\frac{\delta F_2}{\delta y_a(x)} \approx N^{2-\alpha}\rho(x)^2 \sum_{\substack{\text{bifundamentals}\\ (a,b) \text{ and } (b,a)}} \left[-f_{ab}(x)\left(\frac{f_{ab}(x)}{4\pi} + 1 - \Delta_{(a,b)} - \frac{y_a(x)-y_b(x)}{2\pi}\right)\right.$$
$$\left. + f_{ba}(x)\left(\frac{f_{ba}(x)}{4\pi} + 1 - \Delta_{(b,a)} - \frac{y_b(x)-y_a(x)}{2\pi}\right)\right], \tag{5.38}$$

where we have defined

$$f_{ab}(x) \equiv -i \ln e^{i\left[y_a(x) - y_b(x) + 2\pi(\Delta_{(a,b)} - 1/2)\right]}. \tag{5.39}$$

Integrating this expression with respect to $y_a(x)$ one obtains an expression for F up to y_a-independent terms. The y_a-independent terms can be found by approximating F_2 itself when $y_a = 0$ in the same way that we approximated $\delta F/\delta y_a(x)$ above. The final answer is

$$F_2 = -\frac{N^{2-\alpha}}{12\pi} \sum_{\substack{\text{bifundamentals}\\ (a,b)}} \int dx\, \rho(x)^2 \left[\pi^2 - f_{ab}^2\right]\left[2f_{ab} + 3\left(y_a - y_b + 2\pi(\Delta_{(a,b)} - 1)\right)\right]. \tag{5.40}$$

In the region where $y_a(x) - y_b(x) + 2\pi(\Delta_{(a,b)} - 1/2) \in (-\pi, 3\pi)$, as will be the case most of the time, we have

$$f_{ab}(x) = y_a(x) - y_b(x) + 2\pi\left(\Delta_{(a,b)} - \frac{1}{2}\right) , \tag{5.41}$$

and (5.40) becomes

$$F_2 = -N^{2-\alpha} \sum_{\substack{\text{bifundamentals} \\ (a,b) \text{ and } (b,a)}} \frac{2 - \Delta^+_{(a,b)}}{2} \int dx\, \rho(x)^2 \left[\left(y_a - y_b + \pi\Delta^-_{(a,b)}\right)^2 - \frac{1}{3}\pi^2\Delta^+_{(a,b)}\left(4 - \Delta^+_{(a,b)}\right)\right] , \tag{5.42}$$

where $\Delta^{\pm}_{(a,b)} \equiv \Delta_{(a,b)} \pm \Delta_{(b,a)}$, reproducing eq. (5.15) for $\alpha = 1/2$. Eq. (5.42) is valid in the range

$$\left|y_a - y_b + \pi\Delta^-_{(a,b)}\right| \leq \pi\Delta^+_{(a,b)} . \tag{5.43}$$

In order to reproduce eq. (5.17) for a field transforming in the adjoint of the ath gauge group, we take $y_a = y_b$, $\Delta^+_{(a,a)} = 2\Delta_{(a,a)}$, and $\Delta^-_{(a,a)} = 0$ in one of the terms of (5.42), which we then multiply by a factor of 1/2 as explained above.

5.3.3 Fourth rule

The contribution from the fundamental and anti-fundamental fields is

$$F_3 = -\sum_{\substack{\text{fundamental} \\ a}} \sum_i^N \ell\left(1 - \Delta_a + i\frac{\lambda_i^{(a)}}{2\pi}\right) - \sum_{\substack{\text{anti-fundamental} \\ a}} \sum_i^N \ell\left(1 - \tilde{\Delta}_a - i\frac{\lambda_i^{(a)}}{2\pi}\right) , \tag{5.44}$$

where we denoted the dimension of the fundamentals and anti-fundamentals by Δ and $\tilde{\Delta}$, respectively, to avoid confusion. In the continuum limit, replacing $\sum_i$ by

$N\int dx\,\rho(x)$ as usual and using the scaling ansatz (5.27) we get

$$F_3 = \frac{i(n_f - n_a)}{8\pi} N^{1+2\alpha} \int dx\,\rho x^2 + N^{1+\alpha} \sum_{\substack{\text{fundamental}\\ a}} \int dx\,\rho(x)\,|x| \left(\frac{1-\Delta_a}{2} - \frac{1}{4\pi} y_a(x) \right)$$
$$+ N^{1+\alpha} \sum_{\substack{\text{anti-fundamental}\\ a}} \int dx\,\rho(x)\,|x| \left(\frac{1-\tilde{\Delta}_a}{2} + \frac{1}{4\pi} y_a(x) \right), \tag{5.45}$$

where n_f is the total number of fundamentals and n_a is the total number of anti-fundamentals. When $n_f = n_a$ and $\alpha = 1/2$ one reproduces eqs. (5.18) and (5.19).

5.3.4 Why $\alpha = 1/2$?

When the CS levels sum to zero and the number of fundamentals equals the number of anti-fundamentals, we find $F_1 + F_3 \sim N^{1+\alpha}$ at large N and $F_2 \sim N^{2-\alpha}$. In order to have a non-trivial saddle point we have to balance out these two terms, so $1+\alpha = 2-\alpha$ implying $\alpha = 1/2$. The free energy therefore scales as $N^{3/2}$.

5.4 A class of $\mathcal{N} = 2$ necklace quivers

The first class of quiver gauge theories where we apply the formalism developed in the previous section involves a modification of the necklace $\mathcal{N} = 3$ Chern-Simons theories introduced in section 1.2.5 and studied in Chapter 4. The $\mathcal{N} = 3$ quivers involve p gauge groups with CS levels k_a that satisfy $\sum_{a=1}^{p} k_a = 0$ as well as bifundamental chiral superfields $A_{a,a+1}$ and $B_{a+1,a}$. These theories are natural generalizations of the ABJM model, which corresponds to $p = 2$. They have quartic superpotentials $W \sim \sum_a \frac{1}{k_a} \operatorname{tr}(A_{a,a+1}B_{a+1,a} - B_{a,a-1}A_{a-1,a})$. An equivalent description of these theories

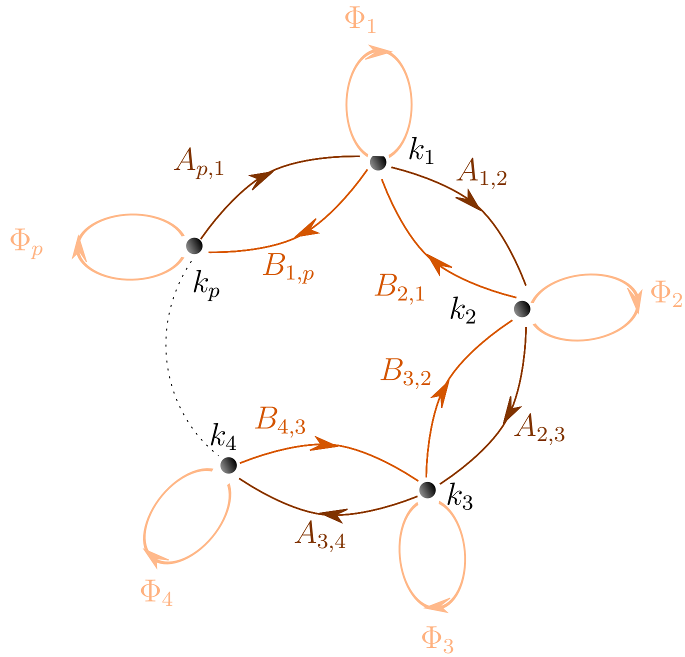

Figure 5.1: A "necklace" quiver diagram for the $\mathcal{N}=3$ Chern-Simons-matter gauge theories with superpotential (5.47) or the $\mathcal{N}=2$ CS-matter gauge theories with superpotential (5.46). We impose the condition that the CS levels k_a should sum to zero.

involves extra adjoint chiral multiplets Φ_a and the superpotential

$$W_{\mathcal{N}=3} \sim \sum_a \mathrm{tr}\left(k_a \Phi_a^2 + \Phi_a (A_{a,a+1}B_{a+1,a} - B_{a,a-1}A_{a-1,a})\right) \tag{5.46}$$

(see figure 5.1). That the two descriptions are equivalent can be seen by simply integrating out the fields Φ_a. If one now changes the superpotential to

$$W_{\mathcal{N}=2} \sim \sum_a \mathrm{tr}\left(\mu_a \Phi_a^3 + \Phi_a (A_{a,a+1}B_{a+1,a} - B_{a,a-1}A_{a-1,a})\right) , \tag{5.47}$$

for some set of parameters μ_a, the resulting theories have only $\mathcal{N}=2$ supersymmetry.[3] If we perturb such an $\mathcal{N}=2$ fixed point by the relevant superpotential deformation $\delta W = \sum_a \mathrm{tr}\,(k_a \Phi_a^2)$ then it should flow to the corresponding $\mathcal{N}=3$ theory.

[3] In the two-node case this model is equivalent to the Martelli-Sparks proposal for the dual of $AdS_4 \times V_{5,2}$ [97].

To keep the discussion as general as possible, let us consider the class of superpotentials

$$W \sim \sum_a \mathrm{tr}\left[\mu_a \Phi_a^{n+1} + \Phi_a(A_{a,a+1}B_{a+1,a} - B_{a,a-1}A_{a-1,a})\right], \tag{5.48}$$

where we assume that all the parameters μ_a are non-vanishing. If $n = 1$ or 2, this theory is dual to $AdS_4 \times Y_n(\vec{k})$. The spaces $Y_n(\vec{k})$ probably have a Sasaki-Einstein metric only when $n \leq 2$, though, because of the Lichnerowicz obstruction of [97,108]. Let us denote by Δ_A and Δ_B the conformal dimensions of the bifundamental fields A_a and B_a, respectively, and by δ the conformal dimensions of the adjoints Φ_a. The condition that the superpotential is marginal implies

$$\delta = 2/(n+1)\,, \qquad \Delta_+ \equiv \Delta_A + \Delta_B = 2n/(n+1)\,. \tag{5.49}$$

Setting the bare monopole R-charge $\Delta_m = 0$,[4] eqs. (5.14)–(5.15) then imply that the free energy functional is

$$\begin{aligned} F_n[\rho, y_a] = &\sum_{a=1}^{p} \frac{k_a}{2\pi} N^{3/2} \int dx\, \rho x y_a + \frac{2\pi^2 p}{3} N^{3/2} \delta(\delta-1)(\delta-2) \int dx\, \rho^2 \\ &- N^{3/2} \frac{2-\Delta_+}{2} \sum_{a=1}^{p} \int dx\, \rho^2 \left[(y_a - y_{a-1} + \pi\Delta_-)^2 - \frac{\pi^2\Delta_+(4-\Delta_+)}{3}\right], \end{aligned} \tag{5.50}$$

with $\Delta_- \equiv \Delta_A - \Delta_B$. Using (5.49), this equation can be simplified to

$$F_n[\rho, y_a] = \sum_{a=1}^{p} \frac{k_a}{2\pi} N^{3/2} \int dx\, \rho x y_a - N^{3/2} \sum_{a=1}^{p} \int dx\, \rho^2 \left[\frac{(y_a - y_{a-1} + \pi\Delta_-)^2}{n+1} - \frac{4\pi^2 n^2}{(n+1)^3}\right]. \tag{5.51}$$

[4] If one includes a non-zero Δ_m in the free energy, F-maximization requires the bare monopole R-charge to vanish.

As discussed after eq. (5.15), this expression holds as long as $|y_a - y_{a-1} + \pi\Delta_-| \leq \pi\Delta_+ = 2\pi n/(n+1)$.

Since the quiver is symmetric under interchanging the A fields with the B fields, we expect that the saddle point has $\Delta_A = \Delta_B$, so $\Delta_- = 0$. In this case, we can absorb the dependence on n into a redefinition of y_a and ρ. By writing

$$y_a = \frac{2n}{n+1}\hat{y}_a\,, \qquad \rho \to \frac{n+1}{2\sqrt{n}}\hat{\rho}\,, \qquad x \to \frac{2\sqrt{n}}{n+1}\hat{x}\,, \tag{5.52}$$

one can easily show that

$$F_n[\rho, y_a] = \frac{4n^{3/2}}{(n+1)^2}F_1[\hat{\rho}, \hat{y}_a]\,. \tag{5.53}$$

Clearly, this relation is also satisfied by the extrema F_n and F_1 of the functionals $F_n[\rho, y_a]$ and $F_1[\hat{\rho}, \hat{y}_a]$, respectively, which given (1.69) implies

$$\mathrm{Vol}(Y_n(\vec{k})) = \frac{(n+1)^4}{16n^3}\,\mathrm{Vol}(Y_1(\vec{k}))\,. \tag{5.54}$$

In particular, we have

$$\mathrm{Vol}(Y_2(\vec{k})) = \frac{81}{128}\,\mathrm{Vol}(Y_1(\vec{k}))\,. \tag{5.55}$$

When $\vec{k} = (1, -1)$ then $Y_1(\vec{k}) = S^7$ with volume $\mathrm{Vol}(S^7) = \pi^4/3$ and $Y_2(\vec{k}) = V_{5,2}$ [97] with volume $\mathrm{Vol}(V_{5,2}) = 27\pi^4/128$ [92], in agreement with eq. (5.55).

We have just shown that for the RG flow between the $\mathcal{N} = 2$ theory (5.47) in the UV deformed by the relevant superpotential deformation $\delta W = \sum_a \mathrm{tr}\,(k_a \Phi_a^2)$ and the $\mathcal{N} = 3$ theory (5.46) in the IR, we have $(F_{\mathrm{IR}}/F_{\mathrm{UV}})^2 = 81/128$. The universal ratio 81/128 is reminiscent of the $a_{\mathrm{IR}}/a_{\mathrm{UV}} = 27/32$ that often arises in $(3+1)$-dimensional RG flows; see [109] for a general argument.

5.5 Flavored gauge theories with one gauge group

The first examples we consider are flavored variations of the 3-d $\mathcal{N} = 8$ Yang-Mills theory, which can be obtained as the dimensional reduction of the $\mathcal{N} = 4$ gauge theory in four dimensions. In $\mathcal{N} = 2$ notation, the 3-d $\mathcal{N} = 8$ vector multiplet consists of an $\mathcal{N} = 2$ vector multiplet with gauge group $U(N)$ or $SU(N)$ as well as three adjoint chiral superfields X_i, $1 \leq i \leq 3$. The superpotential

$$W_0 = \operatorname{tr} X_1 [X_2, X_3] \tag{5.56}$$

ensures that the long-range forces between the eigenvalues vanish, because the requirement that the superpotential is marginal is equivalent to eq. (5.11). The flavoring of this model consists of adding fields q_α and $\tilde{q}_\alpha$ transforming in the anti-fundamental and fundamental representations of the gauge group, respectively, coupled to the adjoints X_i through the superpotential coupling

$$\sum_\alpha q_\alpha \mathcal{O}_\alpha(X_i) \tilde{q}_\alpha \,. \tag{5.57}$$

Here, $\mathcal{O}_\alpha(X_i)$ are polynomials in the X_i with no constant term, which, as operators, also transform in the adjoint representation of the gauge group. It was conjectured in [98,99] that the $U(1)$ quantum corrected moduli space in this case can be described as the embedded codimension one surface

$$T\tilde{T} = \prod_\alpha \mathcal{O}_\alpha(X_i) \tag{5.58}$$

in $\mathbb{C}^5$, where the monopole operators T and $\tilde{T}$ as well as the three fields X_i should be regarded as the five complex coordinates in $\mathbb{C}^5$. This moduli space is a Calabi-Yau space with a conical singularity at $T = \tilde{T} = X_i = 0$. The field theory we just described is then conjectured to be the theory on M2-branes placed at the tip of the Calabi-Yau cone (5.58) and is therefore dual to $AdS_4 \times Y$, where Y is the Sasaki-Einstein base of the cone (5.58).

5.5.1 An infinite family of AdS_4/CFT$_3$ duals

Let's first couple the basic model with superpotential (5.56) to three sets of pairs of chiral superfields $\left(q_j^{(i)}, \tilde{q}_j^{(i)}\right)$, where $i = 1, 2, 3$ and $j = 1, 2, \ldots, n_i$ for some integers $n_i \geq 0$ with at least one of the n_i being strictly positive. The quiver diagram for this theory is shown in figure 5.2. The superpotential of the flavored theory is

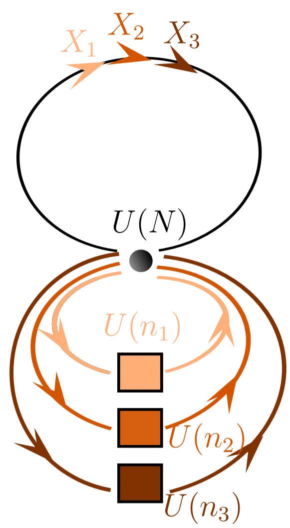

Figure 5.2: The quiver diagram for the flavored theories corresponding to the superpotential in equation (5.59).

$$W \sim W_0 + \mathrm{tr}\left[\sum_{j=1}^{n_1} q_j^{(1)} X_1 \tilde{q}_j^{(1)} + \sum_{j=1}^{n_2} q_j^{(2)} X_2 \tilde{q}_j^{(2)} + \sum_{j=1}^{n_3} q_j^{(3)} X_3 \tilde{q}_j^{(3)}\right] . \tag{5.59}$$

These theories were considered in detail in [98] where it was shown that for each such theory the quantum corrected moduli space of vacua is a toric Calabi-Yau cone. This cone can be parameterized by the complex coordinates X_i as well as the monopole operators T and $\tilde{T}$ subject to the constraint

$$T\tilde{T} = X_1^{n_1} X_2^{n_2} X_3^{n_3} \,. \tag{5.60}$$

The fact that the superpotential should have R-charge $R[W] = 2$ as well as the constraint (5.60) imposes a number of constraints on the R-charges of the various fields:

$$\sum_{i=1}^{3} R[X_i] = 2 \,, \qquad R[T] + R[\tilde{T}] = \sum_{i=1}^{3} n_i R[X_i] \,, \qquad R[q_j^{(i)}] + R[\tilde{q}_j^{(i)}] + R[X_i] = 2 \,. \tag{5.61}$$

With these assumptions, the rules of section 5.2.2 imply that the free energy functional is[5]

$$F[\rho] = 2\pi^2 N^{3/2} \Delta_1 \Delta_2 \Delta_3 \int dx\, \rho^2 + \frac{N^{3/2}}{2} \left(\sum_{i=1}^{3} n_i \Delta_i \right) \int dx\, \rho\, |x| + N^{3/2} \Delta_m \int dx\, \rho x \,, \tag{5.62}$$

where we denoted $R[X_i] = \Delta_i$. We also have $R[T] - R[\tilde{T}] = 2\Delta_m$ (see eq. (5.3)).

The eigenvalue density $\rho(x)$ that maximizes F is supported on $[x_-, x_+]$ with $x_- < 0 < x_+$:

$$\rho = \begin{cases} \dfrac{\left(\sum_{i=1}^{3} n_i \Delta_i\right) - 2\Delta_m}{8\pi^2 \Delta_1 \Delta_2 \Delta_3} (x - x_-) & \text{if } x < 0 \,, \\[2ex] \dfrac{\left(\sum_{i=1}^{3} n_i \Delta_i\right) + 2\Delta_m}{8\pi^2 \Delta_1 \Delta_2 \Delta_3} (x_+ - x) & \text{if } x \geq 0 \,, \end{cases} \tag{5.63}$$

[5]In these non-chiral theories, F-maximization will give $\Delta_m = 0$ due to charge conjugation symmetry, but we will nevertheless keep Δ_m explicitly in the intermediate steps.

where the endpoints of the distribution are such that ρ is continuous at $x = 0$,

$$x_\pm \equiv \pm\sqrt{\frac{8\pi^2 \Delta_1 \Delta_2 \Delta_3 \left[\left(\sum_{i=1}^3 n_i \Delta_i\right) \mp 2\Delta_m\right]}{\left(\sum_{i=1}^3 n_i \Delta_i\right) \left[\left(\sum_{i=1}^3 n_i \Delta_i\right) \pm 2\Delta_m\right]}}\,. \tag{5.64}$$

Plugging these expressions into eq. (5.62), we find that the extremum of $F[\rho]$ at given Δ_i and Δ_m is given by

$$F = \frac{2\sqrt{2}\pi N^{3/2}}{3}\sqrt{\Delta_1 \Delta_2 \Delta_3 \left[\left(\sum_{i=1}^3 n_i \Delta_i\right) - \frac{4\Delta_m^2}{\left(\sum_{i=1}^3 n_i \Delta_i\right)}\right]}\,. \tag{5.65}$$

In order to find Δ_i and Δ_m, one just has to maximize F under the constraint that $\sum_{i=1}^3 \Delta_i = 2$. The maximization problem clearly implies that $\Delta_m = 0$, so

$$F = \frac{2\sqrt{2}\pi N^{3/2}}{3}\sqrt{\Delta_1 \Delta_2 \Delta_3 \left(\sum_{i=1}^3 n_i \Delta_i\right)}\,. \tag{5.66}$$

Finding Δ_i requires solving a system of algebraic equations with no simple closed-form solutions. However, in section 5.5.2 we will examine a variety of special cases where closed-form solutions are available.

It can be shown using toric geometry techniques that the extremum of the free energy (5.65) matches with the gravity prediction based on the volume of the internal space Y and eq. (1.69). We will not reproduce the details of that computation here; the interested reader is referred to section 7 of [15].

5.5.2 Particular cases

$\mathbb{C}^2 \times (\mathbb{C}^2/\mathbb{Z}_{n_1})$

It is instructive to examine particular cases of our general formula (5.66). The first particular case we study is $n_2 = n_3 = 0$ with n_1 arbitrary. The moduli space (5.60) is in this case $\mathbb{C}^2 \times (\mathbb{C}^2/\mathbb{Z}_{n_1})$, where the $\mathbb{Z}_{n_1}$ is generated by $(z_3, z_4) \sim$

$\left(z_3 e^{2\pi i/n_1}, z_4 e^{-2\pi i/n_1}\right)$. This theory should therefore be dual to $AdS_4 \times S^7/\mathbb{Z}_{n_1}$, where the $\mathbb{Z}_{n_1}$ action on S^7 is that induced by the corresponding $\mathbb{Z}_{n_1}$ action on $\mathbb{C}^4$ [110]. Eq. (5.66) is extremized for $\Delta_1 = 1$ and $\Delta_2 = \Delta_3 = 1/2$, which, when combined with (1.69) gives

$$\mathrm{Vol}(Y) = \frac{\pi^4}{3n_1}\,. \tag{5.67}$$

Since the volume of the round seven-sphere is $\mathrm{Vol}(S^7) = \pi^4/3$, this formula is consistent with the expectation that the internal space Y is a $\mathbb{Z}_{n_1}$ orbifold of S^7. Indeed, it was argued in [110] that there is a supersymmetry enhancement to maximal $\mathcal{N} = 8$ supersymmetry when $n_1 = 1$.

$\mathbf{CY}_3 \times \mathbb{C}$ theories

Consider $n_3 = 0$ with arbitrary n_1 and n_2. The equation describing the moduli space reduces to $T\tilde{T} = X_1^{n_1} X_2^{n_2}$, which describes a toric CY_3 cone times $\mathbb{C}$, the complex coordinate in $\mathbb{C}$ being X_3. Since the CY_3 is singular at $X_1 = X_2 = 0$, the space $\mathrm{CY}_3 \times \mathbb{C}$ has non-isolated singularities and so does the base of this cone, the Sasaki-Einstein space Y. These non-isolated singularities might be a reason to worry to what extent AdS/CFT results are applicable in this case, as additional states in M-theory might appear from these singularities. As we will explain, the matrix model computation of the free energy matches the M-theory expectation (1.69) in spite of these potential problems. The free energy (5.66) is extremized by

$$\begin{aligned}
\Delta_1 &= \frac{n_1 - 2n_2 + \sqrt{n_1^2 + n_2^2 - n_1 n_2}}{2(n_1 - n_2)}\,, \qquad \Delta_3 = \frac{1}{2}\,, \\
\Delta_2 &= \frac{n_2 - 2n_1 + \sqrt{n_1^2 + n_2^2 - n_1 n_2}}{2(n_2 - n_1)}\,,
\end{aligned} \tag{5.68}$$

giving

$$F_{n_1,n_2} = \frac{\pi N^{3/2}}{3\sqrt{2}\,|n_1 - n_2|}\Bigg[\left(n_1 + n_2 + \sqrt{n_1^2 + n_2^2 - n_1 n_2}\right)$$
$$\times\left(n_1 - 2n_2 + \sqrt{n_1^2 + n_2^2 - n_1 n_2}\right)\left(-2n_1 + n_2 + \sqrt{n_1^2 + n_2^2 - n_1 n_2}\right)\Bigg]^{1/2} . \tag{5.69}$$

Note that the field X_3 corresponding to the $\mathbb{C}$ factor in $\mathrm{CY}_3 \times \mathbb{C}$ has the canonical R-charge $\Delta_3 = 1/2$.

When $n_1 = n_2 = 1$ the Calabi-Yau three-fold is the well-known conifold $\mathcal{C}$. In this case $\Delta_1 = \Delta_2 = 3/4$, and from eqs. (5.69) and (1.69) one obtains $\mathrm{Vol}(Y) = 16\pi^4/81$. This value can be confirmed from a direct calculation of the volume using the metric on $\mathbb{C} \times \mathcal{C}$ or from toric geometry. See [15] for details.

The D_3 theory

Another fairly simple particular case is $n_1 = n_2 = n_3 = 1$. The associated CY_4 is described by the equation $T\tilde{T} = X_1 X_2 X_3$ and is therefore a complete intersection. While the volume of the Sasaki-Einstein base Y can of course be obtained as a particular case from the toric geometry computation in section 7 of [15], there is actually a simpler way of computing this volume using the results of [92]. Indeed, eq. (16) of that paper with $n = 4$, $d = 6$, $\vec{w} = (3,3,2,2,2,2)$ (so $w = 72$ and $|w| = 12$) gives $\mathrm{Vol}(Y) = 9\pi^4/64$. From the matrix model, the extremum of the free energy (5.66) can be found to be

$$F = \frac{8\pi}{9}\sqrt{\frac{2}{3}}N^{3/2} , \tag{5.70}$$

in agreement with the value we found for $\mathrm{Vol}(Y)$.

5.5.3 Universal RG flows

The theories discussed in section 5.5.2 dual to $\mathrm{CY}_3 \times \mathbb{C}$ have two obvious relevant superpotential deformations: $\mathrm{tr}(X_3)^2$ of R-charge 1 and $\mathrm{tr}(X_3)^3$ of R-charge 3/2. Adding either of these operators to the superpotential causes an RG flow to a new IR fixed point. The RG flows obtained this way are universal in the sense that, as we will now show, the ratio of the IR and UV free energies is independent of the details of the three-fold CY_3. We will only compute this ratio for the toric CY_3 examples of section 5.5.2, but we believe that the same ratio can be obtained for non-toric examples.

To give a unified treatment of the $\mathrm{tr}(X_3)^2$ and $\mathrm{tr}(X_3)^3$ deformations, let's examine the theory obtained by adding $\mathrm{tr}(X_3)^p$ to the superpotentials (5.59) with $n_3 = 0$ but otherwise arbitrary n_1 and n_2. This extra term in the superpotential fixes the R-charge of X_3 to be $\Delta_3 = 2/p$. Fixing Δ_3 to this value and writing for example $\Delta_2 = 2 - \Delta_1 - \Delta_3$ one can find the R-charges of the new IR fixed point by maximizing (5.66). A simple computation shows that the IR R-charges are related to the UV R-charges through

$$\Delta_1^{\mathrm{IR}} = \frac{4(p-1)}{3p}\Delta_1^{\mathrm{UV}}\,, \qquad \Delta_2^{\mathrm{IR}} = \frac{4(p-1)}{3p}\Delta_2^{\mathrm{UV}}\,, \qquad \Delta_3^{\mathrm{IR}} = \frac{2}{p}\,, \tag{5.71}$$

where Δ_1^{UV} and Δ_2^{UV} have the values given in (5.68). Consequently, the IR free energy is also related to the UV free energy in a way independent of which CY_3 space one may want to consider:

$$F^{\mathrm{IR}} = \frac{16(p-1)^{3/2}}{3\sqrt{3}p^2}F^{\mathrm{UV}}\,. \tag{5.72}$$

In particular, for $p = 2$ one obtains $F^{\mathrm{IR}}/F^{\mathrm{UV}} = 4/(3\sqrt{3})$ and for $p = 3$ one obtains $F^{\mathrm{IR}}/F^{\mathrm{UV}} = 32\sqrt{2}/(27\sqrt{3})$.

One obvious question to ask is: what are the gravity duals to these RG flows? For $p = 2$, we believe this holographic RG flow was constructed in [111] (for $p = 3$, we are not aware of a similar holographic construction). Let's examine the holographic RG flow of [111] in more detail. This flow was originally found in 4-d $\mathcal{N} = 8$ gauged supergravity as a flow between two extrema of the gauged supergravity potential—the maximally supersymmetric one and the $U(1)_R \times SU(3)$-symmetric one found in [112]. An uplift of this flow to 11-d supergravity was constructed in [111] where in the UV the geometry aysmptotes to $AdS_4 \times S^7$, and in the IR it asymptotes to a warped product between AdS_4 and a stretched and squashed seven-sphere. It was noticed in [111] that the uplift of the 4-d flow to eleven dimensions was not unique in the sense that an $S^5 \subset S^7$ in the UV geometry could be replaced by the base of any CY_3 cone which is a regular Sasaki-Einstein manifold. Such a generalization of the holographic RG flow [111] should be dual to the flow induced by the superpotential perturbation $\operatorname{tr}(X_3)^2$ in all the gauge theories dual to $\text{CY}_3 \times \mathbb{C}$.

We can compare the field theory prediction (5.72) with the gravity computation. From a four-dimensional perspective, the free energy on S^3 is given by eq. (1.39) in terms of the radius L of AdS_4 and the effective 4-d Newton constant G_4. In the holographic RG-flow of [111], the 4-d Newton constant is kept fixed, so the ratio of free energies is

$$\frac{F_{\text{IR}}}{F_{\text{UV}}} = \left(\frac{L_{\text{IR}}}{L_{\text{UV}}}\right)^2 = \frac{4}{3\sqrt{3}}\,, \tag{5.73}$$

where in the last equation we used $L_{\text{UV}}/L_{\text{IR}} = 3^{3/4}/2$ [111]. Indeed, this expression is in agreement with eq. (5.72).

Two comments are in order. First, when $\text{CY}_3 = \mathbb{C}^3$ supergravity predicts that the IR theory has emergent $U(1)_R \times SU(3)$ symmetry. We now explain why this is a consistent possibility in the field theory. In the field theory, at the IR fixed point one can just integrate out $X_3 \sim [X_1, X_2]$ and obtain the effective superpotential $\mathrm{tr}\,([X_1, X_2]^2 + qX_1\tilde{q})$. The monopole operators have the OPE $T\tilde{T} \sim X_1$, which implies $R[X_1] = R[T] + R[\tilde{T}]$. From eq. (5.71) we see that $\Delta_1^{\mathrm{UV}} = 1$ and $\Delta_2^{\mathrm{UV}} = 1/2$ in the UV (see section 5.5.2) implies $\Delta_1^{\mathrm{IR}} = 2/3$ and $\Delta_2^{\mathrm{IR}} = 1/3$ in IR. The fact that $\Delta_m = 0$ tells us $R[T] - R[\tilde{T}] = 0$, and combining the above observations we conclude $R[T] = R[\tilde{T}] = 1/3$. This leads us to conjecture that T, $\tilde{T}$, and X_2 form a triplet of $SU(3)$, making the expected symmetry enhancement to $U(1)_R \times SU(3)$ a consistent possibility. We thus propose that this gauge theory is dual to Warner's $U(1)_R \times SU(3)$ invariant fixed point of gauged supergravity [112]. Another proposed gauge theory dual is a certain mass-deformed version of ABJM theory [30]; we will solve the corresponding matrix model in section 5.6.

The second comment starts with the observation that the IR free energy of the mass-deformed $\mathbb{C} \times \mathcal{C}$ theory, $\mathcal{C}$ being the conifold, is the same as that of the undeformed $\mathbb{C}^4$ theory. There is a field theory argument that explains this match: The $\mathbb{C}^4$ theory (whose superpotential is $W \sim \mathrm{tr}(X_1[X_2, X_3] + qX_1\tilde{q})$) has a marginal direction where one adds X_1^2 to the superpotential. Integrating out X_1 one obtains $W \sim \mathrm{tr}([X_2, X_3]^2 + q[X_2, X_3]\tilde{q})$. This theory is related by another marginal deformation to $W \sim \mathrm{tr}([X_2, X_3]^2 + qX_2X_3\tilde{q})$, which in turn can be obtained by integrating out X_1 from $W \sim \mathrm{tr}(X_1[X_2, X_3] + qX_2X_3\tilde{q} + mX_1^2)$. The theory with the latter superpotential has the same free energy as the mass-deformed $\mathbb{C} \times \mathcal{C}$ theory.

5.5.4 A non-toric example: The cone over $V_{5,2}/\mathbb{Z}_n$

It was proposed in [99] that the theory dual to the $AdS_4 \times V_{5,2}/\mathbb{Z}_n$ M-theory background is a Yang-Mills $U(N)$ gauge theory with three adjoint fields X_i and $2n$ fields q_j and $\tilde{q}_j$ transforming in $\mathbf{N}$ and $\overline{\mathbf{N}}$ of $U(N)$, respectively, and superpotential

$$W \sim \mathrm{tr}\left[X_1[X_2, X_3] + \sum_{j=1}^{n} q_j(X_1^2 + X_2^2 + X_3^2)\tilde{q}_j\right] . \tag{5.74}$$

The fact that the superpotential has R-charge $R[W]=2$ implies that X_i has R-charge 2/3 and q_j and $\tilde{q}_j$ have R-charge 1/3.

The free energy functional is in this case

$$F[\rho]=\frac{16\pi^2}{27}N^{3/2}\int dx\,\rho^2+\frac{2n}{3}N^{3/2}\int dx\,\rho\,|x|+N^{3/2}\int dx\,\rho x\Delta_m\,. \tag{5.75}$$

Extremizing with respect to ρ under the constraint that ρ is a density, one obtains

$$F=\frac{8\pi\sqrt{n}N^{3/2}}{27}\sqrt{4-\frac{9\Delta_m^2}{n^2}}\,. \tag{5.76}$$

Maximizing this expression with respect to Δ_m gives $\Delta_m=0$ and

$$F=\frac{16\pi\sqrt{n}N^{3/2}}{27}\,. \tag{5.77}$$

Combining this expression with the M-theory expectation (1.69), one obtains

$$\mathrm{Vol}(Y)=\frac{27\pi^4}{128\,n}\,, \tag{5.78}$$

in agreement with the expectation that the space Y is a $\mathbb{Z}_n$ orbifold of $V_{5,2}$.

5.6 Deforming the ABJM theory

In this section we will study some deformations of the ABJM theory that lead to RG flow. Before we do that though, we look at the ABJM theory and assign arbitrary R-charges to the bifundamental fields A_i and B_i that are consistent with the fact that the superpotential

$$W_0\sim\mathrm{tr}\left[\epsilon^{ij}\epsilon^{kl}A_iB_kA_jB_l\right] \tag{5.79}$$

has R-charge two. In other words, denoting $R[A_i] = \Delta_{A_i}$ and $R[B_i] = \Delta_{B_i}$, the constraint the R-charges satisfy is

$$\Delta_{A_1} + \Delta_{A_2} + \Delta_{B_1} + \Delta_{B_2} = 2\,. \tag{5.80}$$

Of course, assigning arbitrary R-charges Δ_{A_i} and Δ_{B_i} breaks SUSY from $\mathcal{N} = 6$ to $\mathcal{N} = 2$. The quiver diagram for this theory is shown in figure 5.3

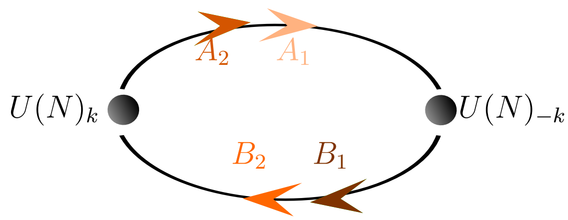

Figure 5.3: The quiver diagram for the ABJM theory at CS level k.

Using the general rules from section 5.2.2, the matrix model free energy functional is

$$\begin{aligned} F[\rho, \delta y] = \frac{k}{2\pi} N^{3/2} \int dx\, \rho x \delta y - N^{3/2} \int dx\, \rho^2 \bigg[(\delta y)^2 + 2\pi \delta y (\Delta_{A_1}\Delta_{A_2} - \Delta_{B_1}\Delta_{B_2}) \\ - 2\pi^2 \Big(\Delta_{A_1}\Delta_{A_2}(\Delta_{B_1} + \Delta_{B_2}) + \Delta_{B_1}\Delta_{B_2}(\Delta_{A_1} + \Delta_{A_2}) \Big) \bigg] + N^{3/2} \int dx\, \rho x \Delta_m\,, \end{aligned} \tag{5.81}$$

where $\delta y \equiv y_1 - y_2$ and $\Delta_m = \Delta_{m1} + \Delta_{m2}$ is the sum of the bare monopole R-charges $\Delta_m^{(1)}$ and $\Delta_m^{(2)}$ for the two gauge groups. In order to find a saddle point of the path integral on S^3, this free energy functional should be extremized as usual under the constraint that ρ is a density, namely $\int dx\, \rho = 1$ and $\rho \geq 0$ almost everywhere. So we should introduce a Lagrange multiplier μ and extremize

$$\tilde{F}[\rho, \delta y] = F - \mu \frac{N^{3/2}}{2\pi} \left(\int dx\, \rho - 1 \right) \tag{5.82}$$

instead of (5.81). Assuming without loss of generality that $\Delta_{A_1} > \Delta_{A_2}$ and $\Delta_{B_1} > \Delta_{B_2}$, one can write the eigenvalue distribution that extremizes (5.82) as a piecewise smooth function:

$$-\frac{\mu}{2\pi(k\Delta_{A_2}-\Delta_m)} < x < -\frac{\mu}{2\pi(k\Delta_{A_1}-\Delta_m)} \quad :$$

$$\rho = \frac{\mu + 2\pi x(k\Delta_{A_2}-\Delta_m)}{8\pi^3(\Delta_{A_2}+\Delta_{B_2})(\Delta_{A_2}+\Delta_{B_1})(\Delta_{A_1}-\Delta_{A_2})}\,, \qquad \delta y = -2\pi\Delta_{A_2}\,, \tag{5.83a}$$

$$-\frac{\mu}{2\pi(k\Delta_{A_1}-\Delta_m)} < x < \frac{\mu}{2\pi(k\Delta_{B_1}-\Delta_m)} \quad :$$

$$\begin{aligned} \rho &= \frac{\mu + \pi x\left[k(\Delta_{A_1}\Delta_{A_2}-\Delta_{B_1}\Delta_{B_2})-2\Delta_m\right]}{4\pi^3(\Delta_{A_1}+\Delta_{B_1})(\Delta_{A_1}+\Delta_{B_2})(\Delta_{A_2}+\Delta_{B_1})(\Delta_{A_2}+\Delta_{B_2})}\,, \\ \delta y &= \frac{2k\pi^2 x\left[\Delta_{A_1}\Delta_{A_2}(\Delta_{B_1}+\Delta_{B_2})+\Delta_{B_1}\Delta_{B_2}(\Delta_{A_1}+\Delta_{A_2})\right]}{\mu + \pi x\left[k(\Delta_{A_1}\Delta_{A_2}-\Delta_{B_1}\Delta_{B_2})-2\Delta_m\right]}\,, \\ &\quad + \frac{\pi(\Delta_{A_1}\Delta_{A_2}-\Delta_{B_1}\Delta_{B_2})(2\pi x\Delta_m-\mu)}{\mu + \pi x\left[k(\Delta_{A_1}\Delta_{A_2}-\Delta_{B_1}\Delta_{B_2})-2\Delta_m\right]}\,, \end{aligned} \tag{5.83b}$$

$$\frac{\mu}{2\pi(k\Delta_{B_1}+\Delta_m)} < x < \frac{\mu}{2\pi(k\Delta_{B_2}+\Delta_m)} \quad :$$

$$\rho = \frac{\mu - 2\pi x(k\Delta_{B_2}+\Delta_m)}{8\pi^3(\Delta_{A_1}+\Delta_{B_2})(\Delta_{A_2}+\Delta_{B_2})(\Delta_{B_1}-\Delta_{B_2})}\,, \qquad \delta y = 2\pi\Delta_{B_2}\,, \tag{5.83c}$$

where

$$\mu^2 = \frac{32\pi^4}{k^3}(k\Delta_{A_1}-\Delta_m)(k\Delta_{A_2}-\Delta_m)(k\Delta_{B_1}+\Delta_m)(k\Delta_{B_2}+\Delta_m)\,. \tag{5.84}$$

By plugging this solution into (5.81) one obtains

$$F=\frac{N^{3/2}\mu}{3\pi}=\frac{N^{3/2}4\sqrt{2}\pi}{3k^{3/2}}\sqrt{(k\Delta_{A_1}-\Delta_m)(k\Delta_{A_2}-\Delta_m)(k\Delta_{B_1}+\Delta_m)(k\Delta_{B_2}+\Delta_m)}\,. \tag{5.85}$$

As expected from the discussion in section 5.2.3, for the $U(N)\times U(N)$ gauge theory the free energy has one flat direction under which $\Delta_{A_i}\to\Delta_{A_i}+\hat{\delta}$, $\Delta_{B_i}\to\Delta_{B_i}-\hat{\delta}$, and $\Delta_m\to\Delta_m+k\hat{\delta}$, corresponding in the notation of section 5.2.3 to $\delta^{(1)}=-\delta^{(2)}=\hat{\delta}/2$. This flat direction is due to the fact that the bifundamental fields as well as the diagonal monopole operators T and $\tilde{T}$ are charged under the $U(1)$ gauge symmetry corresponding to the gauge field $\operatorname{tr}(A_{1\mu}-A_{2\mu})$, so it is not meaningful to assign them individual R-charges. Under this gauge symmetry, the operators A_1 and A_2 have charge 1, B_1 and B_2 have charge -1, and the monopole operators T and $\tilde{T}$ have charges k and $-k$, respectively. The gauge-invariant operators include for example $\operatorname{tr}\tilde{T}(A_i)^k$ and $\operatorname{tr}T(B_i)^k$ with R-charges $k\Delta_{A_i}-\Delta_m$ and $k\Delta_{B_i}+\Delta_m$, and these are indeed the combinations that appear in the expression for F in eq. (5.85).

Regarding $\Delta_m=0$ as a gauge choice, we can maximize (5.85) under the constraint (5.80) that the R-charges of the A_i and B_i fields sum up to two. The maximum is at $\Delta_{A_1}=\Delta_{A_2}=\Delta_{B_1}=\Delta_{B_2}=1/2$, which are the correct R-charges for the $\mathcal{N}=6$ ABJM theory. The value of F at the maximum is

$$F=\frac{\sqrt{2k}\pi N^{3/2}}{3}\,, \tag{5.86}$$

which, when combined with eq. (1.69), implies $\operatorname{Vol}(Y)=\pi^4/(3k)$, in agreement with the fact that ABJM theory is dual to $AdS_4\times S^7/\mathbb{Z}_k$, the volume of S^7 being $\pi^4/3$.

A superpotential deformation of the schematic form $\operatorname{tr}(\tilde{T}A_1)^2$ when $k=1$ or $\operatorname{tr}\tilde{T}A_1^2$ when $k=2$ causes an RG flow to a new IR fixed point where the field A_1 can be integrated out. It was proposed in [30, 113] (see also [114]) that the holographic dual of this RG flow was constructed in [111], first as a flow in 4-d $\mathcal{N} = 8$ gauged supergravity from the maximally symmetric point to the $U(1)_R \times SU(3)$-invariant extremum [112] of the gauged supergravity potential, and then uplifted to M-theory as a flow from $AdS_4 \times S^7$ to a warped product between AdS_4 and a stretched and squashed seven-sphere. (See also section 5.5.3 for another gauge theory realization of the same holographic RG flow.)

Working in the gauge $\Delta_m = 0$, the superpotential deformation mentioned above imposes in the IR the constraint $\Delta_{A_1} = 1$, so

$$F = \frac{4\sqrt{2k}\pi N^{3/2}}{3} \sqrt{\Delta_{A_2}\Delta_{B_1}\Delta_{B_2}}\,. \tag{5.87}$$

This expression should be maximized under the constraint (5.80) that $\Delta_{A_2} + \Delta_{B_1} + \Delta_{B_2} = 1$. By the standard inequality between the geometric and arithmetic mean, the product of three numbers whose sum is kept fixed is maximized when all the numbers are equal, so F has a maximum when $\Delta_{A_2} = \Delta_{B_1} = \Delta_{B_2} = 1/3$. In the IR we therefore have

$$F_{\mathrm{IR}} = \frac{4\sqrt{2k}\pi N^{3/2}}{9\sqrt{3}} = \frac{4}{3\sqrt{3}} F_{\mathrm{UV}}\,, \tag{5.88}$$

where F_{UV} is the free energy of the ABJM theory in eq. (5.86). As already discussed in section 5.5.3, the ratio of F_{IR} to F_{UV} given above is what one expects from the dual holographic RG flow of [111].

5.7 Flavoring the ABJM quiver

In this section we will analyze $\mathcal{N} = 2$ theories that come from adding flavors to the $U(N) \times U(N)$ $\mathcal{N} = 6$ ABJM theory [29] at level k. In general, we could add four pairs of bifundamental fields $(q_j^{(i)}, \tilde{q}_j^{(i)})$ with $i = 1, 2$ and $j = 1, 2, \ldots, n_{ai}$ and $(Q_j^{(i)}, \tilde{Q}_j^{(i)})$

with $i = 1, 2$ and $j = 1, 2, \ldots, n_{bi}$, and we could couple these fields to the ABJM theory (5.79) through the superpotential coupling

$$\delta W \sim \operatorname{tr}\left[\sum_{j=1}^{n_{a1}} q_j^{(1)} A_1 \tilde{q}_j^{(1)} + \sum_{j=1}^{n_{a2}} q_j^{(2)} A_2 \tilde{q}_j^{(2)} + \sum_{j=1}^{n_{b1}} Q_j^{(1)} B_1 \tilde{Q}_j^{(1)} + \sum_{j=1}^{n_{b2}} Q_j^{(2)} B_2 \tilde{Q}_j^{(2)}\right] . \tag{5.89}$$

As far as the matrix model goes, these extra fields corresponds to adding

$$\delta F[\rho, \delta y] = \frac{N^{3/2}}{2} \int dx\, \rho\, |x| \left[\sum_{i=1}^{2} (n_{ai}\Delta_{A_i} + n_{bi}\Delta_{B_i}) + \frac{\delta y}{2\pi}(n_{a1} + n_{a2} - n_{b1} - n_{b2})\right] \tag{5.90}$$

to the free energy functional for ABJM theory in eq. (5.81). The quiver diagram for this theory is given in figure 5.4. It is straightforward to do the extremization of the

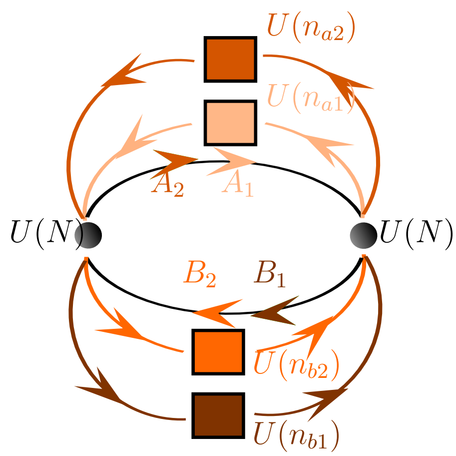

Figure 5.4: The quiver diagram for the flavored theories corresponding to the superpotential in equations (5.79) and (5.89).

free energy functional for arbitrary n_{ai} and n_{bi}, but the resulting formulae are fairly long, so we will just examine a few particular cases.

5.7.1 An infinite class of flavored theories

The first particular case we examine is in some sense a generalization of the flavored quivers we studied in section 5.5.1. Like in the models in that section, there are no Chern-Simons terms and the number of arrows going out of any given node equals the number of arrows going in:

$$n_{a1} + n_{a2} = n_{b1} + n_{b2}\,, \qquad k = 0\,. \tag{5.91}$$

The $U(1)$ quantum corrected moduli space of these theories is given by the relation $T\tilde{T} = A_1^{n_{a1}} A_2^{n_{a2}} B_1^{n_{b1}} B_2^{n_{b2}}$ in $\mathbb{C}^6$ together with a Kähler quotient acting with charges $(0,0,1,1,-1,-1)$ on $(T, \tilde{T}, A_1, A_2, B_1, B_2)$ [98,99]. The free energy is

$$F = \frac{2\pi N^{3/2}}{3} \sqrt{\prod_{i,j=1}^{2} \left(\Delta_{A_i} + \Delta_{B_j}\right) \left[\sum_{i=1}^{2} (n_{ai}\Delta_{A_i} + n_{bi}\Delta_{B_i}) - \frac{4\Delta_m^2}{\sum_{i=1}^{2} (n_{ai}\Delta_{A_i} + n_{bi}\Delta_{B_i})}\right]}\,. \tag{5.92}$$

In order to find the R-charges in the IR, this expression should be locally maximized under the constraint (5.80). Clearly, the maximization over Δ_m yields simply $\Delta_m = 0$, so there is no asymmetry between the R-charges of the monopole operators T and $\tilde{T}$, and the free energy as a function of Δ_{Ai} and Δ_{Bi} reduces to

$$F = \frac{2\pi N^{3/2}}{3} \sqrt{\prod_{i,j=1}^{2} \left(\Delta_{A_i} + \Delta_{B_j}\right) \sum_{i=1}^{2} (n_{ai}\Delta_{A_i} + n_{bi}\Delta_{B_i})}\,. \tag{5.93}$$

In the $U(N)\times U(N)$ gauge theory, the free energy (5.93) is invariant under $\Delta_{A_i} \to \Delta_{A_i} + \hat{\delta}$ and $\Delta_{B_i} \to \Delta_{B_i} - \hat{\delta}$, corresponding to $\delta^{(1)} = -\delta^{(2)} = \hat{\delta}/2$ in the notation of section 5.2.3. As discussed in section 5.2.3, to remove this flat direction one can ungauge the gauge symmetry that rotates A_i and B_i by opposite phases and consider instead a gauge theory with $SU(N)\times SU(N)\times U(1)$ gauge group, where the remaining

$U(1)$ comes from the diagonal $U(1)$ in $U(N) \times U(N)$. The difference between the $SU(N) \times SU(N) \times U(1)$ gauge theory and the $U(N) \times U(N)$ one is that in the former there is an extra constraint

$$\int dx\, \rho \delta y = 0\,. \tag{5.94}$$

Imposing this constraint removes the flat direction mentioned above. An explicit calculation for the saddle point of the theories we are examining in this section gives that eq. (5.94) is equivalent to

$$\Delta_{A_1}\Delta_{A_2} - \Delta_{B_1}\Delta_{B_2} = 0\,. \tag{5.95}$$

In the $SU(N)\times SU(N)\times U(1)$ gauge theory one can therefore determine the R-charges of the bifundamental fields uniquely by maximizing (5.93) under the constraints (5.80) and (5.95).

There are two particular cases where the quantum corrected moduli space can be expressed as a complete intersection and one can apply the methods of [92] to compute the volume of the 7-d Sasaki-Einstein space Y . The first case is $n_{a1} = n_{b1} = 1$ and $n_{a2} = n_{b2} = 0$, where the cone over Y can be described by the equation $z_1 z_2 = z_3 z_4 z_5$ in $\mathbb{C}^5$ [98]. In fact, we encountered this space in section 5.5.2 where we found that the volume was $\mathrm{Vol}(Y) = 9\pi^4/64$. One can indeed reproduce this volume by minimizing (5.93) explicitly and using eq. (1.69).

Another particular case is $n_{a1} = n_{a2} = n_{b1} = n_{b2} = 0$, where the Calabi-Yau cone over Y is the "cubic conifold" described as a complete intersection by the equations $z_1 z_2 = z_3 z_4 = z_5 z_6$ in $\mathbb{C}^6$. Eq. (16) of [92] with $n = 4$, $d = 4$, $\vec{w} = (1, 1, 1, 1, 1, 1)$ (so $w = 1$ and $|w| = 6$) gives $\mathrm{Vol}(Y) = \pi^4/12$. Indeed, extremizing (5.93) and using (1.69) one can reproduce the volume of Y in this case too.

5.7.2 M2-branes probing $\mathbb{C} \times \mathcal{C}$

A quite non-trivial example where the bare monopole R-charge Δ_m plays a crucial role is the case $n_{a1} = 1$ and $n_{a2} = n_{b1} = n_{b2} = 0$ at CS level $k = 1/2$. The CS level is a half-integer because in the IR there is an extra 1/2 shift in the CS level coming from integrating out the fermions in the chiral multiplets $q^{(1)}$ and $\tilde{q}^{(1)}$, which are massive at generic points on the moduli space. The $U(1)$ quantum corrected moduli space is $\mathbb{C}$ times the conifold $\mathcal{C}$.

The $U(N) \times U(N)$ theory has a flat direction given by $\Delta_{A_i} \to \Delta_{A_i} + \hat{\delta}$, $\Delta_{B_i} \to \Delta_{B_i} + \hat{\delta}$, and $\Delta_m \to \Delta_m + \hat{\delta}/2$, so the free energy should only be a function of

$$\hat{\Delta}_{A_i} \equiv \Delta_{A_i} - 2\Delta_m \,, \qquad \hat{\Delta}_{B_i} \equiv \Delta_{B_i} + 2\Delta_m \,. \tag{5.96}$$

Indeed, an explicit extremization of the free energy functional gives

$$F = \frac{2\sqrt{2}\pi N^{3/2}}{3} \sqrt{\frac{\hat{\Delta}_{A_1}\left(\hat{\Delta}_{A_2} + \hat{\Delta}_{B_1}\right)\left(\hat{\Delta}_{A_2} + \hat{\Delta}_{B_2}\right)\left(\hat{\Delta}_{A_1} + 2\hat{\Delta}_{B_1}\right)\left(\hat{\Delta}_{A_2} + 2\hat{\Delta}_{B_2}\right)}{4 - \hat{\Delta}_{A_1}}} \,, \tag{5.97}$$

where $\hat{\Delta}_{A_i}$ and $\hat{\Delta}_{B_i}$ satisfy the constraint $\hat{\Delta}_{A_1} + \hat{\Delta}_{A_2} + \hat{\Delta}_{B_1} + \hat{\Delta}_{B_2} = 2$ coming from eq. (5.80). This expression is maximized for

$$\hat{\Delta}_{A_1} = 1 \,, \qquad \hat{\Delta}_{A_2} = \frac{1}{2} \,, \qquad \hat{\Delta}_{B_1} = \hat{\Delta}_{B_2} = \frac{1}{4} \,, \tag{5.98}$$

yielding

$$F = \frac{\sqrt{3}\pi N^{3/2}}{2\sqrt{2}} \,. \tag{5.99}$$

From (1.69) one obtains that the Sasaki-Einstein base Y of $\mathbb{C} \times \mathcal{C}$ has volume $\mathrm{Vol}(Y) =$

$16\pi^4/81$, as can be checked by an explicit computation using the metric or using toric geometry techniques.

In the $SU(N) \times SU(N) \times U(1)$ theory the flat direction in F is no longer there because one imposes as an additional constraint that $\int dx\, \rho \delta y = 0$. From an explicit computation of the saddle point, one finds that this constraint reduces to

$$\Delta_m = \frac{2\hat{\Delta}_{B_1}\hat{\Delta}_{B_2} - \hat{\Delta}_{A_1}\hat{\Delta}_{A_2}}{2\left(4 - \hat{\Delta}_{A_1}\right)} . \tag{5.100}$$

Using (5.98) one obtains

$$\Delta_m = -\frac{1}{16} , \qquad \Delta_{A_1} = \frac{7}{8} , \qquad \Delta_{A_2} = \Delta_{B_1} = \Delta_{B_2} = \frac{3}{8} . \tag{5.101}$$

5.7.3 Dual to $AdS_4 \times Q^{1,1,1}/\mathbb{Z}_n$

Another example is the theory that was proposed in [98, 99] as a dual of $AdS_4 \times Q^{1,1,1}/\mathbb{Z}_n$. This theory has $n_{a1} = n_{a2} = n$, $n_{b1} = n_{b2} = 0$, and vanishing CS levels $k = 0$. Obtaining an expression for the free energy as a function of arbitrary R-charges Δ_{A_i} and Δ_{B_i} is fairly involved, so using the symmetries of the quiver let's just focus on the subspace where

$$\Delta_{A_i} = \Delta , \qquad \Delta_{B_i} = 1 - \Delta , \tag{5.102}$$

in agreement with the constraint (5.80), and allow an arbitrary bare monopole R-charge Δ_m. The extremization of the free energy functional gives

$$F = \frac{4\pi N^{3/2}}{3\sqrt{n}} \frac{|n^2 - \Delta_m^2|}{\sqrt{3n^2 - \Delta_m^2}} \tag{5.103}$$

as well as

$$\int dx\, \rho \delta y = \pi \left[\frac{4n^2}{3n^2 - \Delta_m^2} - 2\Delta \right] . \tag{5.104}$$

Notice that in this case the free energy F is independent of Δ, because the fact that $k = 0$ implies that the flat direction discussed in section 5.2.3 corresponds to $\Delta \to \Delta + \hat{\delta}$ (where $\hat{\delta} = \delta^{(1)} = -\delta^{(2)}$) leaving Δ_m invariant. Maximizing (5.103) with respect to Δ_m one obtains $\Delta_m = 0$ and

$$F = \frac{4\pi \sqrt{n} N^{3/2}}{3\sqrt{3}} , \tag{5.105}$$

in agreement with the fact that the volume of $Y = Q^{1,1,1}/\mathbb{Z}_n$ is $\mathrm{Vol}(Y) = \pi^4/(8n)$.

As before, for the $U(N) \times U(N)$ theory it doesn't make sense to assign any meaning to Δ because one cannot construct a gauge-invariant operator just from the A_i fields, for example. In the $SU(N) \times SU(N) \times U(1)$ theory, on the other hand, the condition $\int dx\, \rho \delta y = 0$ combined with (5.104) and $\Delta_m = 0$ implies $\Delta = 2/3$. It follows that the baryonic operators constructed out the B_i, such as $\mathcal{B}(B_1)$, have dimensions $N/3$ in agreement with the dimension of wrapped M5-branes [107].

5.8 Discussion

This chapter contained calculations of the three-sphere free energy F for a variety of $\mathcal{N} = 2$ superconformal gauge theories with large numbers of colors. The localization of the free energy for such theories, which allows for varying the R-charges of the fields, was carried out in [14, 94], and we used their results to write down and solve a variety of large N matrix models with the method introduced in [46]. The subsequent maximization of F over the space of trial R-charges consistent with the marginality of the superpotential fixes them and the value of F. The results we find are in complete agreement with the conjectured dual $AdS_4 \times Y$ M-theory backgrounds. We have also studied various RG flows and have found that F decreases in all of them. F is also constant along exactly marginal directions. Thus, F seems to be a good candidate to serve as a 3-d analogue of the a-coefficient in the 4-d Weyl anomaly. This has led us to propose the F-theorem in three dimensions, analogous to the a-theorem in 4-d.

The reader will note that none of the models solved in this paper include chiral bifundamental fields. Instead, we have relied on models with non-chiral bifundamentals, such as the ABJM model, which may be coupled to a rather general set of fundamental fields, either chiral or non-chiral. Constructions of this kind were used in [98, 99] to conjecture gauge theories dual to a variety of $\mathcal{N} = 2$ M-theory backgrounds, including such well-known solutions as $AdS_4 \times Q^{1,1,1}$ and $AdS_4 \times V_{5,2}$. These novel conjectures rely heavily on non-perturbative effects associated with monopole operators: in fact, in these theories the monopole operators play a geometrical role on an equal footing with the fields in the lagrangian. Our work, as well as the superconformal index calculation for the flavored $AdS_4 \times Q^{1,1,1}$ model [115], provides rather intricate tests of these conjectures.

The earlier and perhaps better known conjectures for the gauge theories dual to $AdS_4 \times M^{1,1,1}$, $AdS_4 \times Q^{1,1,1}$ and $AdS_4 \times Q^{2,2,2}$ [32, 85, 88, 89, 116, 117] have instead involved chiral bifundamental fields. The rules derived in [14, 94] seem to apply to these models as well, and we have attempted to study these matrix models both numerically and analytically. Unfortunately, the essential phenomenon in the matrix models exhibiting the $N^{3/2}$ scaling of the free energy, namely the cancellation of long range forces between the eigenvalues, cannot be achieved in the theories with chiral bifundamentals. As a result, the range of the eigenvalues grows as N, rather than $\sqrt{N}$, and the free energy scales as N^2. The latter behavior is in obvious contradiction with the M-theory result (1.69). As N increases, the eigenvalue distribution does not become dense; instead, the gaps do not shrink as N is increased. This leads to an entirely different structure from what we have observed in the various matrix models that do produce the desired $N^{3/2}$ scaling of the free energy. The question whether the matrix models with chiral bifundamentals can be "repaired" is an interesting one and we hope it will be investigated further.

More generally, we find it exciting that the F-theorem for the three-sphere free energy might hold. Such a theorem should be applicable to all 3-d theories, either supersymmetric or not. Further tests of these ideas, as well as attempts at a general field theoretic proof, would be very useful at this stage.

Chapter 6

Conclusions

The first half of this thesis explored connections between black hole physics and superconductivity. The spontaneous breaking of a gauge symmetry by a black hole relied on the fact that AdS space acts like a box in preventing particles with charge of the same sign as the black hole horizon from escaping out to infinity. The only option for these particles is to condense somewhere outside of the black hole horizon, leading to the spontaneous breaking of a global symmetry in the dual field theory interpretation. The symmetry broken state is a superfluid. In Chapters 2 and 3 we computed various transport coefficients related to the phase transition to the superfluid state, and in Chapter 3 we demonstrated analytically that in AdS_5 this phase transition is of second order.

One desirable follow-up to the first half of this thesis would be to understand the physics of the superconducting black holes, or more generally of the gravitational systems used in modeling phenomena encountered in condensed matter physics, in more controlled examples. Over the past few years there have been attempts at embedding the more phenomenological models given by effective actions in AdS into string theory and M-theory. For example: the Abelian Higgs model was embedded into type IIB supergravity in [118] and into 11-d supergravity in [119, 120] (see also

[121]); the p-wave model was embedded using D-branes in [122, 123]; there have been attempts at looking for Fermi surfaces in string and M-theory examples [124–126]; and constructions exhibiting quantum critical behavior [104, 127]. One area that is still unexplored, perhaps because of an insufficient understanding of the more formal aspects of the gauge/gravity duality in that case, relates to higher-spin gauge theories [128–131], which are conjectured to be dual to non-supersymmetric field theories like the critical $O(N)$ model [132] (see [133, 134] for tests of this conjecture). Theories based on the critical $O(N)$ model might in some sense be closer to the models studied in the condensed matter literature than the string theory constructions mentioned above, mainly because they are not supersymmetric.

The more general topic of using AdS/CFT to extract transport coefficients has been a recurring theme in many papers over the past ten years, starting with [71]. The computation of some of these transport coefficients is similar to the computations of absorption coefficients by black branes from more than ten years ago—see, for example, [135–137]. A better understanding of the relation between these two computations would be a worthwhile goal.

The second half of this thesis contained computations of the free energy on S^3 of many $\mathcal{N} \geq 2$ field theories with M-theory dual, with the aim of testing the various AdS_4/CFT_3 that have been proposed in recent years. Quite remarkably, these computations also reproduce the $N^{3/2}$ scaling of the number of degrees of freedom on coincident M2-branes. Even though the $N^{3/2}$ scaling at large N comes out of the field theory computation, this thesis leaves unanswered the question of whether there is any intuition behind this scaling. The recent papers [138, 139] relate conjecturally the saddle-point eigenvalue distributions to numbers of operators in the field theory, suggesting that all eigenvalues whose real parts are between x and $x + dx$ describe properties of just a certain group of chiral operators. A proof of this result could lead to a constructive proof of AdS/CFT.

More generally, much of the recent effort in using localization to compute field theory quantities protected by supersymmetry has been focused on field theories in three and four dimensions only. An obvious follow-up is to extend these computations to theories in higher numbers of dimensions. Perhaps one can learn new things about theories on Dp-branes. Perhaps one can observe the N^3 scaling of the number of degrees of freedom on N coincident M5-branes.

The free energy F of the Euclidean theory on S^3 turned out to be a good measure of the number of degrees of freedom for supersymmetric theories, as we have seen in Chapter 5 in a few examples that this quantity decreases along RG flow (see also [16, 17]). This property was called "the F-theorem." Recently, it was noticed in [18] that the F-theorem also holds for very simple non-supersymmetric RG flows. The possibility of the F-theorem being true for all three-dimensional field theories is tantalizing, as this result would provide strong constraints on possible RG flows. The relation between F and other quantities, such as the equivalence noticed in [21] between F and the entanglement entropy, is worth exploring in more detail.

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