Higher-order derivative gravity and Black Holes

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Abstract

Modifications of Einstein's theory of gravitation have been extensively considered over the last decade, in connection to both cosmology and quantum gravity. Higher-curvature and higher-derivative gravity theories constitute the main examples of such modifications. These theories exhibit, in general, more degrees of freedom than those found in standard General Relativity. In this work we review via both formal arguments the most relevant methods to unveil the gravitational degrees of freedom of a given model, discussing the merits and pitfalls of the various approaches. We also want to shed a light on black holes application of higher order gravities. Since black holes are the most fundamental objects in a theory of gravity, and they provide powerful probes for studying some of the more subtle global aspects of the theory. In the end, we summarize the whole discussions and provide with some insight into the future directions for developments of the higher-order derivative gravity and black holes.

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Chapter 1

Introduction

One of the main issues of General Relativity is an ultraviolet (UV) problem. It emerges in cosmological or black-hole type singularities for classical level. There were many attempts to resolve these divergensies over the last decades. Some of them are: (i) Replacement of spacetime metric with more fundamental approaches (AdS/CFT correspondence) and as a result it can be resolved within this framework; (ii) Modifying the matter stress energy tensor sourcing the gravity. There is two options, first is exotic forms of matter sources such as the Casimir [38] or BCS gap energy [6]; second is modification of the way ordinary matter incorporated to the stress energy tensor at the high energy regime [18]; (iii) Modification of Einstein's equations at the ultra-violet scale but changing space-time continuum description excluding Planckian (or close to Planckian) curvatures. That leads to avoiding of singular Black hole and Big Bang solutions.

In [17] the last statement is considered in detailed. The paper is based on a covariant torsion-free metric theory of gravity containing an infinite set of covariant derivatives. This idea is similar to string theory constractions such as open string field theory [78].

More generally, the systematic study of higher-order gravities provides a deeper understanding of Einstein gravity itself, since it helps unveil what features of the theory are generic, and which ones are specific. Modifying gravity usually is a non-trivial task.

First of all, because of constraints in GR that arise from different tests. These tests are applicable for infrared scale of physics but the theory of gravity which are valid for UV scale will avoid the boundaries of experimental tests. The conditions needed to realize GR for low energy scale are implemented by computing Newtonian potentials [17]. Also it will be helpful for explanation of dark energy via modified gravity theories.

Secondly, the problem with appearing of ghost states in any covariantly modified gravity with higher derivatives. For instance, in [65] it has been shown that fourth order theories of gravity are renormolisable but as side effect it includes the ghost states. The price that has been paid is non unitarity of quantum theory. The presence of ghost states usually signal us about the classical instabilities that the theory possess. Thus it becomes imperative that we avoid them.

Thirdly, one needs to avoid the tachyonic instabilities since it causes superluminal propagation which plagues several dark energy motivated modified gravity theories [52, 26, 29]. What is rather convenient to do is to look at behavior of small fluctuations around a relatively small set of physically important classical backgrounds, such as the Minkowski space-time, Freedman-Lemaitre-Robertson Walker (FLRW) cosmological backgrounds, and spherically symmetric metrics describing spatial regions with astrophysical densities. The most relevant background where a lot of work have been made to understand perturbations is Minkowski spacetime. And obvious question can be asked is whether any consistent gravitational theory satisfies UV completeness.

It has been showed that ghost-free infinite-derivative theories of gravity provides hope for resolving the classical black hole possibly by becoming asymptotically free at short distances. Black holes are the most fundamental objects in a theory of gravity, and they provide powerful probes for studying some of the more subtle global aspects of the theory. It is therefore of considerable interest to investigate the structure of black-hole solutions in theories of gravity with higher-order curvature terms. In this project, we report on some investigations of the static, spherically-symmetric black-hole solutions in four-dimensional Einstein-Hilbert gravity with additional quadratic curvature terms.

The dissertation is organized as follows:

In section 2, we shall sum up some results about the gravitational d.o.f.'s and their various representations. Also we shall highlight the crucial role of the boundary terms, and explore the opportunities of a simple diagnostic tool based on surface counter-terms [15];

In section 3, we shall convey detailed analysis of the linearisation procedure and of its main advantages and dangers as per the extraction of number and nature of the gravitational d.o.f.'s [17]. We shall dive into the momentum representation and its subtleties and focus on the notion of a propagator in Extended Theories of Gravity (ETG) — as long as such concept makes sense — and on its application to higher-order theories;

In section 4, we shall consider auxiliary fields method and expansion around a maximally symmetric spacetime, outlining its pros and cons and discussing the limits of this technique. Some aptly crafted cases will prove how delicate and intricate is the choice of a suitable set of alternative variables, and how the latter affects the dynamical structure;

In section 5, we shall consider static black-hole solutions in the example of Einstein gravity with additional quadratic curvature terms [46]. We then demonstrate the existence of further black hole solutions over and above the Schwarzschild solution. By using the general and quickly convergent parametrization in terms of the continued fractions, we show how numerical solution can be represented in the analytical form, which is accurate not only near the event horizon or far from black hole, but in the whole space [43]. We also discuss some of their thermodynamic properties and show that they obey the first law of thermodynamics.

Chapter 2

Underlying aspects of higher order theories of gravity

Higher–curvature and higher-derivative gravity theories are main examples of modifications of Einstein's theory of gravitation. Since these theories produce more degrees of freedom than in General Relativity its have been extensively studied last decades. It is still remaining a nontrivial task to count, identify and extract the representation of such dynamical variables. As a consequence we can witness some number of classifications of these theories according to the methodology of investigating them. In this chapter we will try to make review of bunch of work that has been done in the past years. The theory of General Relativity recently celebrated one century birthday. During this period of time it has experienced tempestuous development in theoretical and experimental battlefields [1, 2].

It still remains "perfectly unhappy" in many intersections one of which is the micro-world and macroscopic picture [32, 55]. The words of Newton will be appropriate here as never before "whilst the great ocean of truth lay all undiscovered before " [19, Chap.27].

The main goal is quite simple: to find the best description for gravity, by framing the "correct" representation for degrees of freedom (d.o.f.'s) and their dynamics, and satisfying to the observational and experimental constraints. The degrees of freedom here are fundamental notion in description of dynamical systems. The way it can be achieved is the broad variety of theories which are laying in the basis of more general or extended theories of gravity. The extension is rooted in specific geometric interpretation that associated to the center elements of gravitational action and field equations. Very often these duties are falling on the shoulders of the tensor $g_{\mu\nu}$ in Einstein model [49, 74, 77].

Even though geometric structure of manifold sometimes experience evolution the signature and topology remains fixed, but there is still some exceptions [54]. Despite this fundamental ground ETG offers huge variety of theories that based on any sort of variation on the given theme [28]. It is metric and non metric theories, together with metric affine, affine and purely affine proposals. In terms of gravitational degrees of freedom of any nature it is scalar, vectorial, tensorial, spinorial etc. All of these d.o.f.'s are confront to the usual graviton and field equations of arbitraly high order can be easily obtained. Even full non-local models are extensively studied [10]. Even such fundamental notions as equivalence principles might be violated in gravitational and matter sector. The main candidates are theories developed for higher and lower dimensions which were advanced [56], motivated by AdS/CFT correspondence [42].

Gravitational d.o.f.'s is typical element of all ETG along with metric field $g_{\mu\nu}$ where encoded all information about objects. Depends on the specific choice of added building blocks and their coupling to metric we can come up with scalar-tensor, vector-tensor theories, scalar-vector-tensor or multi-scalar-tensor theories [50, 61]. Another way to incorporate new variables is Palatini method of variation. The main concept is enlarging the geometric structure of manifold. It might be implemented by partition of metric and affine structure where correspondent connection coefficients are starting to generate new dynamical variables. These approach is taking as basis for all the subset of affine, purely affine and metric-affine theories [37, 63, 72]. There is another interesting possibility caused by varying torsion [9, 67] or by non-metricity [71] and up to the farthest outcomes [5, 53].

And finally there is another option of ETG that can be built by focusing on dynamics of metric and modifying of gravitational action. This sub class of ETG incorporate the higher-curvature theories and higher-derivative theories. The first one depend on the form of the associated action, the latter depend on the structure of resulting field equations. Although these two expressions are interchangeable and often overlap, it is important to get a crucial meaning of the difference: higher curvature theories correspond to second order of differential equations as in General Relativity - Lanczos-Lovelock models [56, 57] – whereas higher-derivative action will always ended up with higher-order field equations. What is important in ETG is the correct number of gravitational d.o.f.'s and their actual dynamics, the rest is a matter of representations and mathematical rearrangements of variables into relevant interpreted geometric variables. A crucial step towards unveiling the hidden network of relations within the particular theory is extracting and comparing the actual dynamics of the ETG. To achieve this goal there is different approaches and techniques that have pro and cons. Those peculiarities might result in being heavily background-dependent or result in macroscopic modifications of the actions and field equations where comparison of paradigms might become impossible. Further we will shortly review the available protocols and point out what might be best way to deal with this crucial issue in gravitational theory.

2.1 Some relevant technicalities

The main principle of Einstein's GR is that it is purely metric theory [77] which is following out of the variation of Hilbert's action [49, 74]. From this statement it is cleat that all gravitational phenomena come from the dynamics of metric tensor $g_{\mu\nu}$ which is acting on 4–dimensional manifold M.

$$S_{E.H.} = \frac{1}{2\kappa}\int_M R\sqrt{-g}d^4x \tag{2.1}$$

where κ is a coupling constant and R - the scalar curvature.

As we vary the above action with respect to $g_{\mu\nu}$ (or $g^{\mu\nu}$) the field equations are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{2.2}$$

with Λ a fundamental constant and $G_{\mu\nu}$ the symmetric divergence free Einstein tensor. Eq. (2.2) as the system of second-order quasi-linear PDE's allows to define the gravitational configurations of spacetime.

The above formulation of GR does not express the actual number of gravitational degrees of freedom nor it fully explain the emergence of the field equations from action (2.1). The issue is rooted in the lack of correct boundary terms but after fixing it the formulation become quite robust.

The core of higher-order ETG is compound of the same premise behind Einstein's model and arbitraly complex contributions from the curvature tensor. For the general case Einstein-Hilbert's action is then contributed for [20].

$$S = \frac{1}{2\kappa}\int_M R\sqrt{-g}d^4x f(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \nabla_{\alpha_1}R_{\mu\nu\rho\sigma}, ..., \nabla_{(\alpha_1 ...}\nabla_{\alpha_m)}R_{\mu\nu\rho\sigma}) \tag{2.3}$$

with f an defined as a scalar function of the Riemann tensor, its covariant derivatives, and the metric tensor, and κ redefined coupling constant. The change in the action affects the shape of the field equations but the dynamical variables are still incorporated in $g_{\mu\nu}$. We should point out that the function f is chosen such as to be analytic in its arguments, so that it admits a Taylor series expansion in the fileds $g_{\mu\nu}$ or $R_{\alpha\beta\gamma\sigma}$ allowing one to estimate some physical effects at different orders. Quadratic corrections are quite common in the literature, whereas anything beyond order-2 is often studied in connection with simpler actions, as in R_n-theories, to keep the calculations manageable [15].

Since an action with $f(g_{\mu\nu}, \Box R^{\alpha}_{\beta\gamma\delta})$ generates fourth order field equations it can be noticed that with the higher curvature contributions we ended up with field equations of order higher than two. Despite of existence of some exceptions this way of thinking is fairly general result. One of the exceptions is the quadratic Gauss-Bonnet theory that gives back second order only field equations at the same time being higher-curvature ETG [56].

There is might arise the obvious question why we should add higher contributions from the curvature. This idea is based on various grounds starting from observational needs to purely theoretical motivations. One of the reasons is to obtain renormilizable theory for semiclassical treatments of gravity [24]. Weinberg, in turn, noticed that one can treat dynamics of gravity as some sort of non-local interactions thereby producing an "effective" standpoint. Hence, it might be incorporated into ladder of truncated higher-order expansions [76]. Finally, there are some purely formal reasons to go beyond the Hamiltonian action as simple one. It might turn out that it corresponds to lowest step in a ladder of increasing complexity therefore it deserves to be properly investigated.

The first example of such theories are the $f(g_{\mu\nu}, R)$ theories [8, 29, 62]. For this case curvature appear as the fully traced scalar R and it is noteworthy structure for cosmology physics.

Then we have $f(g_{\mu\nu}, R_{\mu\nu})$, where we get the Ricci tensor as a way of the introduction of semiclassical corrections of quantum nature [7, 64].

Finally, one can introduce the Riemann tensor and obtain general form of $f(g_{\mu\nu}, R^{\alpha}_{\beta\gamma\delta})$ [30]. Along with this, another prominent sub-class where only the traceless part of the curvature enters the Lagrangian density is Weyl's conformal gravity. Also, higher order ETG can be expanded by introducing covariant derivative of curvature $f(g_{\mu\nu}, R^{\alpha}_{\beta\gamma\delta}, \nabla_{\sigma} R^{\alpha}_{\beta\gamma\delta})$.

As higher order ETG has been developed we face with logic question: how can we interpret the convoluted behaviour of dynamics of such model? Is there any protocols that allow us to transform higher-curvature actions (and higher-derivative field equations) into more familiar dynamics of second order evolution? The answer for this question is positive. It is possible to remap higher order ETG into second-order theories for the metric and other dynamical objects, such as scalars, vectors, tensors, spinors. However this possibilities cause more questions concerning the true meaning of the expression "purely metric". To date, there is no exact answers to these sort of questions. What is known for sure is that we possess a crowded toolbox of scattered and often ad hoc recipes helping us to determine the actual gravitational d.o.f. of particular theory. But there is still pitfalls and subtleties behind any corner.

2.2 Gravitational degrees of freedom

First, let us introduce some brief introduction the notion to the matter of gravitational degree of freedom. Degree of freedom here is any dynamical variable or parameter involved in the actual description of a physical system. The degrees of freedom in classical mechanics fully describe the evolution of physical system by dynamical entities once all the constraints have been considered.

Since it allows us to get understanding of some aspects of nature it is useful to classify them by the properties of certain symmetry transformation. As is well known, Lorentzian manifolds are locally described by Minkowski spacetime then it is natural to sort the d.o.f. according to the symmetries of flat spacetime or according to the irreducible representations of the Poincare group. It can be implemented by two ways: (i) expanding the theory around flat spacetime; (ii) covariantizing a certain set of conditions that are suitable to fields of spin s in flat spacetime (covariant Fierz-Pauli condition for spin-2 fields [39]). But it is also possible to consider non-flat backgrounds then we will get another classification of d.o.f. according to different group symmetries. In cosmology it performs as a classification of helicity for spatial rotations.

In the field theory, first we need to find the number of dynamical variables and then we have to make a formal character according to a given representation of the theory that consist of specific sets of fields and/or geometric quantities. For the gravity, situation with degrees of freedom is a bit different and it is critical to properly understand the logic behind the theory.

2.3 How degrees of freedom occurs in gravity

The further discussion will be largely taken from [15]. Let us consider the field equation (2.2) with $\Lambda = 0$, then we get $G_{\mu\nu}$ which stands for a system of second-order, quasilinear, hyperbolic partial differential equations governing the dynamics of the symmetric, rank-2 tensor $g_{\mu\nu}$. From this, we can assume that all the gravitational degrees of freedom are encoded in $g_{\mu\nu}$.

We can represent the metric tensor by a $4x4$ matrix in an arbitrary coordinate system. Because of the fundamental symmetry property of metric $g_{\mu\nu} = g_{\nu\mu}$ we can reduce the number of independent components from full sixteen available in a generic rank-2 4D tensor to ten. Another consequence is that we can take into account the background independence of the field equations and of the underlying Lagrangian because the points on the manifold M can be relabelled without affecting actual dynamics. Finally, we can conclude that out of ten gravitational variables we are dealing only with *six* real free parameters that are responsible for actual dynamics and last *four* are unphysical (expression of the freedom in redefining the coordinate system).

Along with metric decomposition we have to analyze the role of field equations. Let us pick an apt foliation of the spacetime manifold into spatial leaves evolving along the streamlines of an affine parameter (a “time” variable). Then project the field equations on this stack of 3-spaces dynamically evolving (this is equivalent to selecting a special set of coordinates). More detailed way of representation can be seen within ADM formalism [15]. In [27] it has been shown that four out of ten field equations are constraints equation thus they do not contribute to the actual dynamics of gravity. Hence, this fact has significant impact on number counting of the degrees of freedom which gets reduced to the final figure of two.

By introducing a weak-field approximation the metric $g_{\mu\nu}$ can be decomposed into the sum of a background flat par $\eta_{\mu\nu}$, and a small perturbation $\varepsilon h_{\mu\nu}$ where ε-infinitesimal parameter accounting for the order in a Taylor-series expansion of the field. Since the background is Minkowskian we can introduce global symmetry represented by Poincare invariance lying beneath the laws of Special Relativity.

As we mentioned above the gravitational field embodied by $\varepsilon h_{\mu\nu}$ produce two degrees of freedom, at the same time the perturbation tensor $\varepsilon h_{\mu\nu}$ can be classified according to the irreducible spinor representations of the Poincare group acting on the flat background. It turns out that we can treat two gravitational degrees of freedom as a spin-2 object living on Minkowski spacetime. As a result one can conclude that according to GR the gravitational phenomena can be mediated by a spin-2 boson. If we take into account the long range character of gravitational interactions spin-2 boson must have vanishing mass. Such agent is commonly known as the *graviton*.

What is important here is that GR has only two gravitational d.o.f.’s which can be treated as the components of a massless spin-2 graviton living on Minkowski or (anti-) de Sitter spacetime. The last statement about (anti-) de Sitter spacetime is concerning of the existence of another universe where spacetime is described by de Sitter-Fantappie-Arcidiacano groups [4, 16].

2.4 The role of boundary terms in identifying of the d.o.f.’s

There are two requirements that needs to satisfy in order to obtain Einstein’s field equations from the variation of the Hilbert action Lagrangian: (i) the manifold M over which integrations in Eq. (2.1) are performed has to have a compact topology; (ii) the variations of $\delta^{\rho}_{\sigma\tau}$ must vanish on the boundary ∂M together with the variations $\delta g_{\mu\nu}$. It is very common that for more general treatment of gravitational phenomena these assumptions do not hold then it might be fixed by adding extra terms if one wants to recover the field equations (2.2).

As an example let us make variation of gravitational action (2.1) without any restrictive assumptions for the boundary ∂M and assume that it is non-compact. After some manipulations we have,

$$\begin{aligned}\delta S\left[g_{\mu\nu}\right] = &\int_M G_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g}d^4x \\ &+ \int_M (\delta^\alpha_\gamma \nabla_\sigma g^{\sigma\beta} - \nabla_\gamma g^{\alpha\beta})\delta\Gamma^\gamma_{\alpha\beta} + \nabla_\gamma(g^{\alpha\beta}\delta\Gamma^\gamma_{\alpha\beta} - g^{\gamma\beta}\delta\Gamma^\alpha_{\alpha\beta})\sqrt{-g}d^4x\end{aligned} \tag{2.4}$$

In order to recover Einstein's equations the last integration has to be vanished. For the first term we have the compatibility condition for the metric $\nabla_\alpha g_{\mu\nu} = 0$ and then it drops out. To get rid of the last term we have to admit the following condition on the boundary or for the case of collapsing the topology of M on a compact object $\delta\Gamma^\rho_{\sigma\tau} = 0$. However such strong assumptions are too restrictive. If we look closer one can notice that compact topology is a quite peculiar configuration and there is no obvious reason to hire these restrictions a priori over any other possible arrangement. The same is true for the requiring of vanishing of the first derivatives of $\delta\Gamma^\rho_{\sigma\tau} \sim \delta\partial_\rho g_{\sigma\tau}$, because it restricts too much the allowed set of field configurations and should be avoided.

The annoying consequences of the last issue can be demonstrated on the example of moving from GR to any higher-order theory of gravity. Let us consider a scheme where the field equations have derivatives of the metric of order r, with $r > 2$. The problem here arising when we want to remove the additional derivative pieces in the action. If we follow the instructions given above concerning vanishing on ∂M of the variations of all the derivatives $\partial^{(k)}_\lambda g_{\mu\nu}$ with $k = \{1, 2, ..., r-1\}$ it will affect the solutions of the field equations. The metric $g_{\mu\nu}$ has to obey an additional set of derivative constraints introduced for making variational problem correct. Since the constraints are set long before the field equations were obtained then the link to the actual dynamics is lost. As a result, the space of possible solutions gets reduced without any intervention of the field equations. So it is quite reasonable decision to eliminate the second assumption and accept only the first requirement of $\delta g_{\mu\nu} = 0$ on the boundary.

There is subtlety that was delivered in [15] "when looking for the actual different solutions of the field equations, the set of initial data specified e.g. on a Cauchy surface must fix the values of the field and its derivatives up to order $r-1$ for the initial-value problem to be meaningful. At this stage, however, we are not dealing with single solutions of the field equations, but rather with the space of all possible solutions, as a whole. While the single-field configurations for a specified matter-energy distribution had rather be pinned down by the initial data, the space of admissible configurations emerging from the action is instead expected to be as large as possible, not to rule out any legitimate candidate."

In GR the solution were found by introducing the so called Gibbons–Hawking–York counter-term [35]. It was noticed that the remaining terms in (2.4) can be obtained from the variation of themselves of twice the trace of the extrinsic curvature K of the sub-manifold ∂M. Thus the Einstein-Hilbert action can be elaborated by the additional surface integral

$$S_{G.H.Y.} = 2\oint_{\partial M} K\sqrt{\gamma}d^3x \tag{2.5}$$

where γ the determinant of the induced three-metric γ_{ab}.

And ultimately we get the full action given by

$$S_{grav} = \frac{1}{2\kappa}\int_M R\sqrt{-g}d^4x - \frac{1}{\kappa}\oint_{\partial M} K\sqrt{\gamma}d^3x, \tag{2.6}$$

which satisfies all the requirements about delivery of Einstein's equations and nothing else, in any possible topological arrangement for the manifold M and with the sole requirement of the vanishing of the $\delta g_{\mu\nu}$ on ∂M.

2.5 The problem with boundary terms in ETG's

The issues with boundary terms that were mentioned above always arise as one considers a higher-order theory of gravity. There are three main aspects that has to be taken into account:

1. Existence of the boundary terms
2. Their property cross out all the uncompensated variations in the higher derivatives of the metric
3. Their use as a diagnostic tool for the defining the nature and number of the actual d.o.f.'s for the particular model

The first two statements mostly overlap each other. There is no exact evidence neither in a positive form nor in a negative about application of boundary terms. Boundary terms cannot always compensate the variations in the action. In that case additional terms must be added by hand [1].

[1]According to [15] a few "lucky" cases exist, though. For instance, Lanczos–Lovelock gravity (the class of n-dimensional generalizations of Gauss–Bonnet theory) is such that all the uncompensated terms can be accounted for by variations of surface terms generalising the Gibbons–Hawking–York counter-terms. While this can be seen more as a mathematical consequence of Chern–Simmons theorem, it physical significance might deserve a deeper analysis

Apart from making the variational problem well-posed it is now can shed the light on the hidden features of the actual theories. Let us consider an $f(R)$-theory where $f(R) = R^2$,

$$S_{grav} = \frac{1}{2\kappa}\int_M f(R)\sqrt{-g}d^4x \tag{2.7}$$

It is possible to use the Gibbons–Hawking–York surface term

$$S_{surf} = \frac{1}{\kappa}\oint_{\partial M} f''(R)K\sqrt{\gamma}d^3x \tag{2.8}$$

If we now sum (2.7) and (2.8) and vary them with respect to g_{ab}, we will end up with the similar to Einstein-Hilbert case cancellations but one term will left proportional to

$$f'(R)\delta R, \tag{2.9}$$

which cannot be compensated by any further boundary term. The only possibility to recover the Einstein's field equations is to set $\delta R = 0$ on the boundary ∂M, together with $\delta g_{\mu\nu}$.

What we get here is the vanishing on the boundary of the variation of all the actual d.o.f. of the theory as it take place in the GR case. But, then the presence of R in the action give an assumption about another degree of freedom (at the bare minimum, a scalar field), hidden somewhere in the free componets of $g_{\mu\nu}$. That scenario works for all the $f(R)$ theories which can be remmaped into scalar tensor models with $f'(R)$ playing the role of the field ϕ in a Brans-Dicke theory [29, 62]. That gives an idea that the proper form of the boundary terms can lead to possible reformulations of some higher-order theories of gravity in terms of other which dynamically tantamount to second-order theories with manifest d.o.f.'s besides the metric.

Unfortunately, the result holding for $f(R)$-theories is almost unique, in the sense that the vanishing of uncompensated terms in the boundary terms does not lead, in general, to any further immediate identification of the additional d.o.f.'s, nor it allows for any easy identification of the geometric nature of the supplementary dynamical variables.

So, we should not to overestimate the relevance of the diagnostic power of this "tool". Admittedly, it is true that, by looking at the variations of the boundary terms, it is possible to notice some telltale that a theory under examination is not as "purely metric" as promised by its action functional.

The gist here is that taking care of the boundary terms in an ETG is a necessary, preliminary step, which results in a well-posed variational formulation of the model. In a few cases (a very tiny subset, in fact), by simply looking at the boundary terms, it is possible to notice that something is hidden beneath a seemingly "purely metric" formulation, and the theory might be recast in terms of additional, non-metric (in the sense of "non-gravitonic") degrees of freedom. At the same time, as the complexity of the starting actions grows, it makes less and less sense to rely on the boundary term analysis to thoroughly grasp the "true" nature of the ETG itself.

Chapter 3

Linearization techniques for higher curvature theory of gravity

One of the techniques that we want to cover is based on the computation of the propagator for a higher-curvature theory of gravity. The main idea behind this approach is splitting of the metric in the sum of a background metric $g^{(0)}_{\mu\nu}$ and a perturbation $\varepsilon h_{\mu\nu}$. The background configuration is a maximally symmetric (MS) flat Riemann spacetime.

There are a bunch of research devoted to this method of extracting the number and type of gravitational d.o.f.'s. The most comprehensive literature for recent contributions are [3, 13, 17, 21]. The pioneer work in higher-order theory of gravity namely quadratic corrections to the Einstein-Hilbert action are performed by Stelle [65, 66].

In this chapter we will show how starting from a generic higher curvature theory the contributions at order ε^2 contain at most quadratic invariants in the Riemann tensor and its derivatives [17]. Then, by obtaining an explicit form for the propagator we represent the issue of gauge invariance. Finally we will make some additional comments concerning the pitfalls and subtleties in using non flat backgrounds.

3.1 Linearization procedure and quadratic gravity

Let us investigate the most general form of higher curvature theory of gravity. In that case "purely metric" theory of gravity imposes invariance under general coordinate transformations. The action must be represented as a scalar function of the Riemann tensor and its covariant derivatives.

Then the metric $g_{\mu\nu}$ can be split into a sum of its background value and fluctuating/perturbation term

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \varepsilon h_{\mu\nu} \tag{3.1}$$

Let us take $g^{(0)}_{\mu\nu}$ to be the Minkowski metric $\eta_{\mu\nu}$. Since we want to consider the quadratic contributions of order ε^2 it can be shown that one does not need to consider the most general action containing the metric tensor, the Riemann tensor and its covariant derivatives given by Eq. (2.3). However it is just enough to examine the following expression [17]

$$S = \frac{1}{\kappa}\int_M \sqrt{-g}d^4x\left(\frac{R}{2} + RF_1(\Box)R + R_{\mu\nu}F_2(\Box)R^{\mu\nu\rho\sigma}\right) \tag{3.2}$$

The expression in (3.2) is a generalization of the theory suggested in [65, 66]. The coefficients of the higher-curvature terms are functions of the d'Alembertian operator and the quadratic term in a Riemann tensor are not rejected.

Given the presence of the $F_3(\Box)$-operator, the Gauss–Bonnet combination $\mathscr{G}$ cannot be deployed to express the quadratic term in the Riemann tensor as a combination of the other two quadratic invariants, R^2 and $R^{\mu\nu}R_{\mu\nu}$.

It is easy to notice that two Riemann tensors are already of order ε^2 ($R^{(0)\alpha}_{\beta}\gamma\delta = 0$) then the covariant derivatives at the same order ε^2. As a result we can obtain the simplified action and the equations of motion for the perturbation field become the same order

$$\begin{aligned} &a(\Box)\Box h_{\mu\nu} + 2b(\Box)\partial_\sigma\partial_{(\mu}h^\sigma_{\nu)} + c(\Box)(\eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \partial_\mu\partial_\nu h) + \\ &+ d(\Box)\eta_{\mu\nu}h + f(\Box)\Box^{-1}\partial_\sigma\partial_\rho\partial_\mu\partial_\nu h^{\rho\sigma} = -2\kappa\tau_{\mu\nu} \end{aligned} \tag{3.3}$$

where $\tau_{\mu\nu}$ is the stress-energy-momentum tensor for matter, and the new symbols were introduced

$$a(\Box) = 1 + 2F_2(\Box)\Box + 8F_3(\Box)\Box, \tag{3.4}$$

$$b(\Box) = -1 - 2F_2(\Box)\Box - 8F_3(\Box)\Box, \tag{3.5}$$

$$c(\Box) = 1 - 8F_1(\Box)\Box - 2F_2(\Box)\Box, \tag{3.6}$$

$$d(\Box) = -1 + 8F_1(\Box)\Box + 2F_2(\Box)\Box, \tag{3.7}$$

$$f(\Box) = 8F_1(\Box)\Box + 4F_2(\Box)\Box + 8F_3(\Box)\Box. \tag{3.8}$$

Initially these coefficients were found by Van Nieuwenhuizen [70] and afterwards Biswas and Talaganis generalized them as functions [17]. In order to recover the GR the functions $F_i(\Box)$ must be analytic in the IR limit one needs to hold following conditions $\lim_{k^2\to 0} F_i(-k^2) \propto -k^2$ and $a(0) = -b(0) = c(0) = -d(0) = 1$, $f(0) = 0$.

3.2 The propagator and gauge fixings

In order to compute the propagator in momentum space we have to introduce a complete set of projectors $\{P^2, P^1, P^0_s, P^0_w\}$ for any symmetric rank-2 tensor. This needs to be done to invert the kinetic operator in (3.3)

$$P^2 = \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\rho}) - \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma}, \tag{3.9}$$

$$P^1 = \frac{1}{2}(\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\nu\sigma}\omega_{\mu\rho}), \tag{3.10}$$

$$P^0_s = \frac{1}{3}(\theta_{\mu\nu}\theta_{\rho\sigma}), \tag{3.11}$$

$$P^0_\omega = \omega_{\mu\nu}\omega_{\rho\sigma}, \tag{3.12}$$

where $\theta_{\mu\nu}$ and $\omega_{\mu\nu}$ are the transverse and longitudinal projectors in the momentum space,

$$\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}, \qquad \omega_{\mu\nu} = \frac{k_\mu k_\nu}{k^2} \tag{3.13}$$

We need to define mapping quantities between spaces with the same spin. For this reason "transfer operators" might be used [59]

$$P^0_{s\omega} = \frac{1}{\sqrt{3}}\theta_{\mu\nu}\omega_{\rho\sigma}, \qquad P^0_{\omega s} = \frac{1}{\sqrt{3}}\omega_{\mu\nu}\theta_{\rho\sigma}. \tag{3.14}$$

By using the combination $O = \sum_{i=1}^{6} c_i P^i$ for the projectors P_i the equations of motion can be rewritten as

$$\sum_{i=1}^{6} c_i P^i h_{\mu\nu} = \kappa(P^2 + P^1 + P^0_s + P^0_\omega)\tau_{\mu\nu}. \tag{3.15}$$

In order to get the final form of the propagator we can use the orthogonality relationship for P_i. But first we need to obtain the explicit forms of the coefficients in Eq. (3.15).

$$ak^2P^2 = \kappa P^2 h \Rightarrow P^2 h = \kappa(\frac{P^2}{ak^2})\tau, \tag{3.16}$$

$$(a+b)k^2P^1 h = \kappa P^1 \tau, \tag{3.17}$$

$$(a+3d)k^2P_s^0 + (c+d)k^2\sqrt{3}P_{s\omega}^0 h = \kappa P_s^0 \tau, \tag{3.18}$$

$$(a+2b+2c+d+f)k^2P_\omega^0 h + (c+d)k^2\sqrt{3}P_{\omega s}^0 h = \kappa P_\omega^0, \tau \tag{3.19}$$

where a,b,c,d,f are the functions of k^2 as we moved into momentum space.

The propagator for the spin-2 can be found straightforward

$$\Pi^{(2)} = \frac{P^2}{ak^2}. \tag{3.20}$$

So, now if we recall that we are dealing with a gauge theory and using the Bianchi identities we can see that some of the coefficients in front of the left-hand sides of eqs. (3.17), ()3.18) and (3.19) vanish. After some mathematical manipulations we can obtain [17].

$$(a+b)\Box h^{\mu\nu}_{\nu,\mu} + (c+d)\Box\partial_\nu h + (b+c+f)\Box h^{\alpha\beta}_{,\alpha\beta\nu} = 0 \tag{3.21}$$

where the right hand side vanishes because of the conservation of $\tau_{\mu\nu}$ and the coefficients in front of each term are also zero as it can be seen from Eq. (3.4).

The Bianchi identities are a byproduct of diffeomorphism invariance therefore the left-hand side of Eq. (3.19) and the mixing term in Eq. (3.18) are singular. This fact means that Eq. (3.17) and Eq. (3.19) cannot be inverted directly. But this procedure can be applied for the spin-2 part and the spin-0s part, and then by using the fact that $d = -c$ we can write

$$\Pi^{(0s)} = \frac{P_s^0}{(a-3c)k^2}, \tag{3.22}$$

The propagator in the sub-space $(2 \bigotimes 0s)$ for tensor product is

$$\Pi = \frac{P^2}{ak^2} + \frac{P_s^0}{(a-3c)k^2}. \tag{3.23}$$

One can rewrite this equation as the sum of GR propagator and some additional terms. The GR propagator is given by

$$\Pi_{GR} = \frac{P^2}{k^2} - \frac{P^0_s}{2k^2}, \tag{3.24}$$

where the longitudinal components of the graviton propagator is compensated by the scalar component. This part of the higher order propagator is gauge independent. But for extraction of other parts the gauge fixing term is needed. For an accurate treatment of the propagator of quadratic gravity (plus the standard Einstein–Hilbert term) on a flat background including the gauge fixing terms, see [3]. Also, as pointed out in [21], the explicit form of the propagator depends in general on the definition of the fluctuating term $\varepsilon h_{\mu\nu}$.

The complete expression of propagator for a higher-curvature ETG has the following form

$$\Pi = \Pi_{GR} + \frac{1-a(-k^2)}{a(-k^2)k^2}P^2 + \frac{1+\frac{a(-k^2)-3c(-k^2)}{2}}{[a(-k^2)-3c(-k^2)]\,k^2}P^0_s. \tag{3.25}$$

By taking into account that $a(0) = c(0) = 1$ one can point out that the infrared limit of (3.25) corresponds to the bare GR-case, i.e. $\lim_{k^2\to 0}\Pi \to \Pi_{GR}$. Thus, the gauge-invariant part of the propagator of a generic higher-curvature ETG on a flat background contains the usual massless spin-2 part (the graviton), and a certain number of additional degrees of freedom given by the zeros of the functions $a(-k^2)$ and $c(-k^2)$. The gauge invariance guarantees that it is

$$a(\Box) = -b(\Box), c(\Box) = -d(\Box), f(\Box) = a(\Box) - c(\Box), \tag{3.26}$$

and therefore just two arbitrary functions survive, to host the gravitational d.o.f.'s.

3.3 Types of theory of gravity according to the propagator content

Now we can constrain the additional propagating d.o.f.'s by imposing some general conditions on Eq. (3.25). In particular, if $a(-k^2) - 1 = 0$ or what is the same $F_2 + 4F_3 = 0$, we can get only graviton without spin-2 particle. For $1 + \frac{a-3c}{2} = 0$ or $3F_1 + F_2 + F_3 = 0$ there will be no additional scalar d.o.f.'s.

In field theories context we can classify higher order theories of gravity by the type of propagator.

- $F_1(-k^2) = \alpha$, $F_2(-k^2) = 0$, with α a constant. This is the case of $f(R)$-theories. Since $a(-k^2) = 1$ and $c(-k^2) = 1 + 8\alpha k^2$ the propagator is

$$\Pi = \Pi_{GR} + \frac{1}{2}\frac{P_s^0}{k^2 + 1/12\alpha}. \tag{3.27}$$

  A new scalar d.o.f emerges, and it has non-tachyonic character (the square of the mass is positive) as long as $\alpha > 0$

- $F_i = const \neq$ and $F_1 = F_3 = -F_2/4$. The resulting action is proportional to the Gauss-Bonnet combination $\mathscr{G}$. It is $a(-k^2) = 1$ and $c(-k^2) = 1$, which ensures that such ETG has no additional degrees of freedom, and its propagator is the as that of GR.

- $F_i = const \neq 0$ and $F_1 = F_3 = -F_2/4$. The resulting action is proportional to the Gauss-Bonnet combination $\mathscr{G}$. It is $a(-k^2) = 1$ and $c(-k^2) = 1$ that guaranteed that such higher order theory of gravity has no additional degrees of freedom and the propagator is identical to GR.

- $F_1(-k^2) = \alpha$, $F_2(-k^2) = \beta$,$F_3(-k^2) = \gamma$. This theory was considered in [65, 66]. This theory includes the most general correction up to quadratic curvature invariants without explicit dependence on differential operators. The square of the Riemann tensor can be changed for R^2 and the Ricci tensor squared after introducing the Gauss-Bonnet combination and a redefinition of the coefficients α, β

$$\Pi = \Pi_{GR} - \frac{P^2}{k^2 - m_0^2} + \frac{P_0^2}{2\left[k^2 + m_2^2\right]}, \tag{3.28}$$

  where $m_0^2 = (2\beta + 8\gamma)^{-1}$ and $m_2^2 = (4\beta + 12\alpha + 4\gamma)^{-1}$. The propagator thus produce a new scalar term and a second spin-2 state. But if we fix the coefficients in such way that the mass of the massive spin-2 is positive, the propagator will anyway have an overall minus sign, which is telltale of the presence of a *ghost* state i.e., a state with negative energy.

- $a(-k^2) = 1 - (k^2/m^2)^2$, $c = 1 - \frac{1}{3}(k^2/m^2)$, whence $8F_1(-k^2) + 2F_2(-k^2) = -\frac{1}{3m^2}$ and $F_2(-k^2) + 8F_3(-k^2) = \frac{1}{m^2}$. With this choice, one obtains the Einstein-Hilbert action plus a term proportional to the Weyl tensor squared, $C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}$. Then the propagator is

$$\Pi = \Pi_{GR} - \frac{P^2}{k^2 + m^2}, \tag{3.29}$$

and the propagator of the massive spin-2 comes with an overall minus sign, therefore it is a ghost state.

- $a(-k^2) = c(-k^2)$. For this particular choice, that corresponds to the condition $2(F_1(-k^2) + F_3(-k^2)) + F_2(-k^2) = 0$, the propagator becomes

$$\Pi = \frac{1}{a(-k^2)}\Pi_{GR}. \tag{3.30}$$

In this case the propagator does not produce any additional pole due to the fact that function $a(-k^2)$ has no zeros. In other words, the theory has no presence of additional states with respect to GR. But it is possible to change situation with large k^2 values or with ultraviolet behavior of the propagator. One of the way to fix the problem is to employ that $a(-k^2)$ is a non-local function [17].

Since the parameters F_1, F_2 and F_3 are functions of the d'Alembertian operator, one can consider, in addition to the previous examples, other models with improved ultraviolet behaviour without the issue of ghost states, such as *non-local theories*.

The question that might arise is what situation will be with studying perturbation around dS/AdS spacetime specifically when ghost states might appear. Before proceeding to this issue let us remind what is dS/AdS spacetime.

The case is the following: if a ghost propagates on flat spacetime, then it can be considered as a feature of the full theory at the non-linear level. The opposite in not true. If there are no ghost states on flat spacetime, this does not mean that they will not emerge on some other type of background metric. There are some research about presence of ghost states and light scalars in higher-curvature theory of gravity, for example [52].

Now it can be concluded that such a modification of GR will in general present a massive ghost-like spin-2 state or so-called the Weyl *poltergeist* and additional scalar d.o.f. that can also be a ghost [17].

The theory goes along with the resluts that have been yield in [65, 66] for the case when the functions $F_i(\Box)$ are constant. However, if the functions depend on k^2, then the theory can reveal a richer structure. For example, the d.o.f.'s depending on the zeros of the functions $a(-k^2)$ and $c(-k^2)$ might be more (or fewer). Moreover, it is very likely that some of such d.o.f.'s can again be ghost field.

The tool we used to identify the propagating d.o.f.'s of higher curvature theory of gravity is very powerful. But it must be applied carefully due to the fact that some characteristics of the fully non-linear model might be lost when considering the quadratic expansion in (3.2). For instance, one of the simplest cases is $F(R) = R + \chi R^3$ [39] - where some pieces of the original action might have a vanishing quadratic term in the flat limit and disappear at the leading order. This model propagates the two helicity states of the graviton plus a scalar field. It turns out that, in the flat limit , the mass of the scalar field becomes infinite, and thus the corresponding propagator vanishes. From this can be concluded that the study of the linearised theory is in general not sufficient to establish unambiguously which are the propagating degrees of freedom. Only a full non-linear analysis can answer the question in a definite way.

Chapter 4

The methods of higher theory of gravity

4.1 Auxiliary fields method

Let us consider the technique that allow us to get second-order field equations by rearrangement the given theory in dynamical form consisting of standard GR terms and additional non-metric variables. This approach is another way to extract information about extra degree of freedoms that has been used in many context [11, 14, 39] and etc.

The action for the metric tensor $g_{\mu\nu}$

$$S = \frac{1}{2\kappa}\int_M \sqrt{-g}d^4xf(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \nabla_{\alpha_1}R_{\mu\nu\rho\sigma}, \dots, \nabla_{(\alpha_1}\dots\nabla_{\alpha_m)}R_{\mu\nu\rho\sigma}) \tag{4.1}$$

The resulting field equation in most cases is commonly in higher order than ordinary field equation for particles and fields. To reconstruct this equation in comprehensive way it is needed to use particular additional techniques. We will consider firstly for the case of fourth order equation then we will turn to higher order equations.

4.1.1 Fourth-order gravity

By simplifying action in (2.3)

$$S = \frac{1}{2\kappa}\int_M \sqrt{-g}d^4xf(g_{\alpha\beta}, R_{\mu\nu\rho\sigma}) \tag{4.2}$$

then equations of motion after variation of action contain fourth-order derivatives of the metric tensor

$$R^{\mu}{}_{\alpha\rho\sigma}\frac{\partial f}{\partial R_{\nu\alpha\rho\sigma}} - 2\nabla_\rho\nabla_\sigma\frac{\partial f}{\partial R_{\rho(\mu\nu)\sigma}} - \frac{1}{2}fg^{\mu\nu} = 0 \tag{4.3}$$

For the case of auxiliary-fields method action might be rewritten in the following way

$$S = \frac{1}{2\kappa}\int_M \sqrt{-g}d^4x\left[f(\rho_{\mu\nu\rho\sigma}) + \frac{\partial f}{\partial \rho_{\mu\nu\rho\sigma}}(R_{\mu\nu\rho\sigma} - \rho_{\mu\nu\rho\sigma})\right], \tag{4.4}$$

The field $\rho_{\mu\nu\rho\sigma}$ does not connect with the metric tensor but possess all the symmetries of the Riemann tensor. The gravitational part can elaborated by adding matter fields that depends only on the metric or $S_{matter} = S_{matter}(g_{\mu\nu})$. By varying action (4.4) with respect to $g_{\mu\nu}$ and $\rho_{\mu\nu\rho\sigma}$ one can obtain the following equations of motions

$$E_{\mu\nu} = T_{\mu\nu}, \tag{4.5}$$

$$\frac{\partial^2 f}{\partial \rho_{\mu\nu\rho\sigma}\partial \rho_{\alpha\beta\gamma\delta}}(R_{\alpha\beta\gamma\delta} - \rho_{\alpha\beta\gamma\delta}) = 0 \tag{4.6}$$

where $E_{\mu\nu}$ is a "generalized Einstein tensor" and $T^{\mu\nu}$ is the stress-energy-momentum tensor of matter. The correspondence between (4.2) and (4.4) is true everywhere except for the values of the field $\rho_{\mu\nu\rho\sigma}$ for which the second derivative is zero. Fields configurations which generated by those "points" will produce a certain number of inequivalent subsets where by applying auxiliary fields it can be unified [30]. In order to work with action (4.4) in canonical form it is needed to perform a field redefinition by introducing new modified metric tensor [39, 62].

$f(R)$-theories

One of the most general example of a higher-derivative theory which has been intensively studied is $f(R)$ gravity [25]. The Einstein-Hilbert action is extended as a more general function of the Ricci scalar. This results in field equations that contain fourth-order derivatives. The action in 4 spacetime dimension has the following form

$$S = \frac{1}{2\kappa}\int_M \sqrt{-g}d^4x f(R) \tag{4.7}$$

By introducing the auxiliary field action can be rewritten as

$$S = \frac{1}{2\kappa}\int_M \sqrt{-g}d^4x\left[f(\psi) + f'(\psi)(R-\psi)\right] \tag{4.8}$$

after variation with respect to auxiliary field

$$f''(\psi)(R-\psi) = 0 \tag{4.9}$$

Eqs. (4.7) and (4.8) are might be equivalent on-shell except for those values of ψ for which $f''(\psi) = 0$. This is sufficient but not necessary condition [62]. Intervals between these "points" define different sectors of the theory. By introducing a new variable defined as $\phi = f'(\psi)$, the action takes the form

$$S = \frac{1}{2\kappa}\int_M \sqrt{-g}d^4x\left[\phi R - V(\phi)\right], \tag{4.10}$$

where

$$V(\phi) = \psi(\phi)\phi - f(\psi(\phi)). \tag{4.11}$$

As we require $f''(R) \neq 0$ the above transformation become invertible and the theory has the precise aspect of a scalar-tensor theory of gravity of the Brans-Dicke type with $\omega = 0$ [61]. As a result "purely metric" action (4.7) has been transformed into a dynamically equivalent model containing the standard GR-contribution as a massless spin-2 graviton and an additional scalar d.o.f. Thus equations of motions are

$$G_{\mu\nu} = \frac{1}{\phi}\left[\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}(\Box\phi - V(\phi)/2)\right], \tag{4.12}$$

$$3\Box\phi + 2V(\phi) - \phi\frac{dV}{d\phi} = 0 \tag{4.13}$$

where $G_{\mu\nu}$ is the standard Einstein tensor. As a result, we obtain the theory of gravity that has been reduced to a theory with only second-order equations of motion. The dynamical content is the same but representation has been shifted from a higher-derivative to a non-minimally coupled second-order structure.

In order to get a canonical kinetic term for the scalar part one can perform a *conformal transformation* on the metric and a new ϕ-redefinition. Such transformations are given by

$$\tilde{g}_{\mu\nu} \equiv \phi g_{\mu\nu}, \tag{4.14}$$

$$\tilde{\phi} \equiv \sqrt{\frac{3}{2\kappa}}log\phi. \tag{4.15}$$

Then the action is

$$S = \int_M \sqrt{-\tilde{g}}d^4x\left[\frac{\tilde{R}}{2\kappa} - \frac{1}{2}\partial_\alpha\tilde{\phi}\partial^\alpha\tilde{\phi} - U(\tilde{\phi})\right] \tag{4.16}$$

The main idea that can be pointed out from $f(R)$ gravity is that the equivalence between the scalar-tensor theory and the original theory of gravity might not be conserved in

Minkowski spacetime. For instance, one can consider the specific case where $f(R) = R + \alpha R^3$ and auxiliary field is $\phi = 1 + \alpha R^2$. Then the invertibility condition for this case is $\alpha R \neq 0$.

It is worth to mention what does "non-minimal" mean. In general relativity one of the main underlying idea is covariance principle. According to this principle all laws of physics should not depend on any particular inertial frame. It can be implemented by converting laws of physics into general covariant form:

$$\eta_{\mu\nu} \to g_{\mu\nu} \qquad\qquad \partial_\mu \to \nabla_\mu \tag{4.17}$$

If we replace the Minkowski metric by a curved spacetime metric, and replace partial derivatives with covariant derivatives. The differentiation for a scalar field equals to its covariant derivative, $f_{;\mu} = f_{,\mu}$. According to this standard procedure, the second term on the right-hand side of (4.16) is obtained from $-\frac{\omega}{\phi}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. In this context, the field ϕ comes to couple to gravity only through $\sqrt{-g}g^{\mu\nu}$. The gravitational coupling obtained by applying this "minimum" rule is called a minimal coupling in analogy with a similar rule for charged fields in electrodynamics. The first term on the right-hand side of (4.16) cannot be obtained by this rule, in flat Minkowski space-time this term simply goes away. This is the origin of the name “nonminimal”.

Quadratic gravity

Quadratic corrections to Einstein general relativity [65]. The action is given by

$$S = \frac{1}{2\kappa}\int_M \sqrt{-g}d^4x \left[R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right] \tag{4.18}$$

By using the definition of the Weyl tensor

$$C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + \frac{R^2}{3}, \tag{4.19}$$

In 4D spacetime the Gauss-Bonnet combination G is a topological invariant, therefore it is possible to add and/or subtract without changing the resulting field equations [31, 12].

$$\begin{aligned} &\frac{\delta}{\delta g^{ab}}\int \sqrt{-g}d^4x \left(R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2\right) \\ &= \frac{\delta}{\delta g^{ab}}\int \sqrt{-g}d^4x \left(C_{abcd}C^{abcd} - 2R_{ab}R^{ab} + \frac{2}{3}R^2\right) = 0 \end{aligned} \tag{4.20}$$

If we drop the term that proportional to the Gauss-Bonnet invariant, then Eq. (4.18) can be transformed

$$S = \frac{1}{2\kappa}\int_M \sqrt{-g}d^4x\left[R + \frac{1}{6m_0^2 - \frac{1}{2m_2^2}C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}}\right] \tag{4.21}$$

where $m_0^{-2} = 6\alpha + 2\beta + 2\gamma$ and $m_2^{-2} = -\beta - 4\gamma$. According to Hindawi [39] it is more convenient to study separately two correction terms keeping the Einstein Hilbert term in the action. He showed that for $m_0 > 0$, the theory has a stable minimum and m_0 is the mass of the perturbations.

The correction term that represented in Weyl tensor corresponds to an additional massive spin-2 field. By using Eq. (4.19), the action (4.21) it can be demonstrated as

$$\begin{aligned} S =& \frac{1}{2\kappa}\int_M \sqrt{-g}d^4x\left[R - \frac{1}{2m_2^2}C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}\right] \\ =& \frac{1}{2\kappa}\int_M \sqrt{-g}d^4x\left[R - \frac{1}{m_2^2}(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2)\right] \\ =& \frac{1}{2\kappa}\int_M \sqrt{-g}d^4x\left[R - G_{\mu\nu}\pi^{\mu\nu} + \frac{1}{4}m_2^2(\pi^{\mu\nu} - \pi^2)\right] \end{aligned} \tag{4.22}$$

where on shell auxiliary field is sutisfied a direct generalization to curved spacetime of the Fierz-Pauli conditions [34]

$$\pi_{\mu\nu} = \frac{2}{m_2^2}\left(R_{\mu\nu} - \frac{1}{6}g_{\mu\nu}R\right). \tag{4.23}$$

By the formal substitutions of that is used in [34] one can obtain

$$\begin{cases} \partial^\mu \phi_{\mu\nu} = 0 \\ \eta^{\mu\nu}\phi_{\mu\nu=0} \end{cases} \Longrightarrow \begin{cases} \nabla^\mu \phi_{\mu\nu} = 0 \\ g^{\mu\nu}\phi_{\mu\nu=0} \end{cases}$$

There are other beneficial outcomes of auxiliary fields method. One of the possibility is to find the mass of the spin-2 field and thereby show that it might be treated as a ghost. But if $\beta = 4\gamma$ then Ricci and Riemann tensors can be removed by using the Gauss-Bonnet combination. As a consequence the mass of the spin-2 field will go to infinity. Such conclusions are true for quadratic action as well. To sum up, it turns out that the massive spin-2 is a ghost, but the graviton and scalar degrees of freedom are not ghosts.

Before proceeding to next section let us consider some issues related of using auxiliary method. This method allowed us to identify the number and nature of degrees of freedom in a non-perturbative fashion. The action in the form of curvature invariants

$$S = \frac{1}{2\kappa}\int_M \sqrt{-g}d^4xf(X_i), \tag{4.24}$$

where the X_i, $i = 1,\dots,n$ are different constractions of the Riemann and the metric tensors. One can introduce auxilary field to define the presence of possible additional d.o.f.'s. If we write an equivalent action with scalar fields Φ_j which non-minimally coupled to the curvature scalars X_i and the potentials for the Φ_j then the resulting action will be still quite complicated [26]. This method of simplifying and extracting of d.o.f.'s needs additional manipulations since it does not always reduce the order of the equations of motion. Therefore it is just provide us with an alternative description of the dynamics of the model.

Since adding more curvature tensors and their contractions does not alter the order of derivatives of $g_{\mu\nu}$ appearing in the action these combinations will still produce equations of motion whose highest order is the fourth. To move forward it is then necessary to take into account an explicit dependence of the action on differential operators acting on the Riemann tensor [20].

4.1.2 Higher than fourth-order gravity

In this section we will discuss how to introduce auxiliary fields for a general diffeo-invariant action as (2.3), which includes differential operators acting on the Riemann tensor. We start by noticing that, in principle, a term of the form R contains up to fourth derivatives of the metric tensor, hence it ought to have been included in last section. But, the term R is in fact a covariant total divergence and, as we are not considering possible issues with boundary terms, we can safely ignore contributions of this type for the moment.

We can calculate the equations of motion for the auxiliary variables and use them in the action to recover Eq. (2.3). This procedure is again very powerful in principle, but in most of the cases not very helpful. Apart from some simple yet relevant ETG's, such protocol does not help in building a general and effective recipe to isolate and identify the additional d.o.f.'s of the higher-curvature theory (and decide whether they are dynamical or not).

In [8, 75, 39] it was shown that every new instance of the d'Alemebertian operator carries two more time derivatives ∂_0^2. Hence, it might be expected that one additional degree of freedom for each power of the box operator. But as it is already known that every term in the action is a total divergence, it will not contribute to the equations of motion, as it occurs with a pure $\Box^kR$-term (k is order of field equations). A term of the kind $\Box^kR\Box^jR$, instead, yields an object of the form $R\Box^{k+j}R$ (after integration by parts), and that actually contributes to the equations of motion.

Regarding these kinds of corrections, it has been shown in [60, 75] at the level of the equations of motion, and of the action [39], that an extended theory of gravity of the type

$$S = \frac{1}{2\kappa}\int_M \sqrt{-g}d^4x f(R, \Box R, \Box^2 R, \dots, \Box^k R), \tag{4.25}$$

can in general be rewritten as a theory describing a set of scalar fields non-minimally coupled to standard GR. The number of auxiliary non-metric d.o.f.'s can either be $2k+1$ or $2k$, based on the emerged functional dependencies in the translated action[1]. Upon writing the function as $f(\lambda, \lambda_1, ..., \lambda_{k+1})$, if $\frac{\partial f}{\lambda_{k+1}}$ is a function of λ_{k+1}, then we are in the first case and k scalar fields are ghost-like, whereas the remaining $k+1$ are not. If instead $\frac{\partial f}{\lambda_{k+1}}$ is not a function of λ_{k+1}, then it is a function of λ_k, in which case we arrive at $2k$ new scalar fields, of which at least $k-1$ are ghost-like.

With this premise, let us look at the theory for which $L = \sqrt{-g}(R + \gamma R\Box R)$. It falls into the first category outlined above, therefore we expect $2 \cdot k = 2$ additional scalar fields to be present in the theory. The reformulated action is given by Wands [75]

$$S = \frac{1}{2k}\int_M \sqrt{-g}d^4 \left[(1 + \gamma\phi_1 + \gamma\Box\phi_0)R - \gamma\phi_0\phi_1\right] \tag{4.26}$$

ignoring possible couplings with the standard matter fields. The field equations for the scalar fields read

$$\gamma\Box R = \gamma\phi_1, \tag{4.27}$$

$$\gamma R = \gamma\phi_0, \tag{4.28}$$

hence the non-degeneracy condition is $\gamma \neq 0$. If we introduce a new scalar defined as $\Phi = (1 + \gamma\phi_1 + \gamma\Box\phi_0)$, the action can be rewritten as

$$S = \frac{1}{2\kappa}\int_M \sqrt{-g}d^4x \left[\Phi R - \phi_0(\Phi - 1) + \gamma\phi_0\Box\phi_0\right] \tag{4.29}$$

[1] Introduction of Lagrange multipliers and auxiliary fields requires the fulfillment of non-degeneracy conditions to ensure the equivalence with the original higher-curvature model. This procedure generates different sectors as in the case of f(R) theory [39]

The expression above can be further manipulated to generate a canonical kinetic term for the scalar field, resulting in

$$S = \frac{1}{2\kappa}\int_M \sqrt{-g}d^4\left[\frac{\Phi R}{2\kappa} + \frac{1}{2}\psi\Box\psi - \frac{1}{\sqrt{4\kappa\gamma}}\phi(\Phi - 1)\right], \tag{4.30}$$

where $\psi = \phi_0$. This is the action of a Brans–Dicke theory with $\omega = 0$, plus an additional scalar field with an interaction potential. It can be noticed the absence of ghost states, as the condition $k - 1 = 0$ preventing the onset of instabilities is here a built-in feature.

It is worth noticing that the dynamical content of the last two examples can be also extracted linearizing the theory around the Minkowski background using the techniques reviewed in the previous section. In doing so, it is easy to show that the scalar part of the propagator (3.25) possesses two additional poles, hence in agreement with the results obtained at the non-linear level using the auxiliary fields method.

4.2 Expansion around maximally symmetric background

In this section we will move for more refined technique to determine the correct number and representations for gravitational degree of freedom of a given higher-order theory of gravity. One can further improve the situation by mixing together the best features of the two protocols discussed so far, namely the expansion of the action, and the apt reformulation of the Lagrangians in terms of curvature invariants. Such method makes it possible to deal more safely with ETG's of the type

$$S = \frac{1}{2\kappa}\int_M f(g_{\mu\nu}, R_{\alpha\beta\gamma\delta})\sqrt{-g}d^4x \tag{4.31}$$

The method that we are going to discuss [23, 26, 39] consists in expanding the action of a given higher theory of gravity up to second order (in curvature invariants) around a specific type of background solution, as long as such solution is admitted by the theory at the full non-linear level. In this case, the spacetime acting as the "ground level" must be a maximally symmetric (MS) solution. In other words, one characterised by a constant value of the scalar curvature R (MS solutions include Minkowski spacetime as a sub-case, and also de Sitter and anti-de Sitter solutions). The linearization of the equations of motion, around MS spacetimes, have also been used in the literature to classify higher-order theories of the type (4.31) on the basis of their spectrum [22, 69]

The outcome of the procedure is a quadratic ETG, its specific form depending on the choice of the initial action, for which it is easier to determine the dynamical content and its possible representations. In this sense, this “non-linear expansion method” deploys the power of the linearization and the generality of a fully non-linear tool.

4.2.1 Some main results of higher-order theory of gravity

Before moving on the dicussing the technique of the method let us first briefly recollect some relevant results for this partucular class of higher order theory of gravity. The general has the following form

$$S_{quadr} = \frac{1}{2\kappa}\int_M \sqrt{-g}d^4x\left[R + \left(\alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R_{\theta\iota\kappa\lambda}R^{\theta\iota\kappa\lambda}\right)\right] \tag{4.32}$$

where α, β, γ three real constants. From the previous chapters we have already shown that there are at most 8 gravitational degrees of freedom where many such dynamical variables will be ghosts [26, 52]. Another way to look at ghosts, at the quantum level at least, is in terms of loss of predictability. Even though an higher-order theory of gravity can be made renormalizable by adding quadratic combinations of the Ricci and Riemann tensor [65], the unitarity of the dynamical evolution gets lost in general.

- $\alpha = \beta = \gamma = 0$. This is the case of Einstein’s General Relativity with 2 propagating degrees of freedom, encoded into a massless spin-2 graviton;
- $\beta = -4\gamma$. This choice reduces the action to that of $f(R)$-ETG’s. Therefore, 3 gravitational d.o.f.’s are expected, and they can be represented by one massive scalar field and the standard graviton (this in view of the proven equivalence between $f(R)$-gravity and scalar-tensor theories of the Brans–Dicke type);
- $\alpha = -(\beta + \gamma)/3$. In this case the extra d.o.f.’s are 5, all gathered into a massive spin-2 field juxtaposing the graviton [65, 66]. The non-graviton part is Weyl’s poltergeist, a type of ghost field [26, 39, 52];
- α, β, γ unconstrained. This is the most general case, and there will be 8 degrees of freedom in total; it is still possible to rearrange them so as to give the massless graviton(2 d.o.f.’s), the massive spin-2 field and 1 scalar field.

4.2.2 Expansion procedure

Following the steps in [26, 39] one can perform an expansion of Eq. (4.31) up to second order in curvature invariants around a MS solution to study the excitations of the theory around such background. It is worth noticing that, regardless of the particular $f(g_{\mu\nu}, R_{\alpha\beta\gamma\delta})$-theory considered, the resulting “effective” Lagrangian emerging after the expansion will always be of the quadratic type (4.32) — the main difference will be the specific set of values retrieved for the constants α, β, γ. Hence, all the main features of the higher-order theory of gravity can be studied already at the level of second-order corrections.

For the expansion, let us consider the MS solutions $g^{(0)}_{\mu\nu}$ such that the two conditions below occur [26, 39]

$$R_{ab} \equiv R^{(0)}_{\mu\nu} = \frac{R^{(0)}}{4} g_{\mu\nu} \qquad R_{\theta\iota\kappa\lambda} \equiv R^{(0)}_{\theta\iota\kappa\lambda} = \frac{R^{(0)}}{12}(g_{\theta\kappa}g_{\iota\lambda} - g_{\theta\lambda}g_{\iota\kappa}), \tag{4.33}$$

with $R^{(0)}$ the constant value of the scalar curvature for the given background. By expanding the action as

$$\begin{aligned} S_q =& \frac{1}{2\kappa}\int_M \sqrt{-g} d^4x \\ & \left\{ a_{(0)} + a_{(1)}(R - R^{(0)}) + \frac{1}{2}\left[a_{(2,1)}(R - R^{(0)})^2 + a_{(2,3)}(R_{\theta\iota\kappa\lambda} - R^{(0)}_{\theta\iota\kappa\lambda})^2 \right] \right\} \end{aligned} \tag{4.34}$$

where we have introduced the shorthand notations

$$a_{(0)} = f|_{R^{(0)}} \tag{4.35}$$

$$a_{(1)}(R - R^0) = \frac{df}{dR_{\theta\iota\kappa\lambda}}\bigg|_{R^{(0)}} (R_{\theta\iota\kappa\lambda} - R^{(0)}_{\theta\iota\kappa\lambda}), \tag{4.36}$$

$$a_{(2,1)}(R - R^0)^2 + a_{(2,2)}(R_{\mu\nu} - R^{(0)}_{\mu\nu})^2 + a_{(2,3)}(R_{\theta\iota\kappa\lambda} - R^{(0)}_{\theta\iota\kappa\lambda})^2 = \tag{4.37}$$

$$\frac{d^2 f}{dR_{\rho\sigma\tau\upsilon} dR_{\xi\zeta\varsigma\omega}}\bigg|_{R^{(0)}} (R_{\rho\sigma\tau\upsilon} - R^{(0)}_{\rho\sigma\tau\upsilon})(R_{\xi\zeta\varsigma\omega} - R^{(0)}_{\xi\zeta\varsigma\omega}) \tag{4.38}$$

By making use of the constant curvature condition $a_{(0)}R^{(0)} = 2a_{(0)}$ which is the restrictions of the field equations to constant curvature solutions. If we collect the pieces order by order, we get the final form of the action

$$S_q = \frac{\xi}{2\kappa}\int_M \sqrt{-g} d^4x \left[-\frac{R^{(0)}}{2} + R + \frac{1}{6m^2_{(0)}} R^2 - \frac{1}{m^2_{(2)}}\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2 \right) + \zeta G \right], \tag{4.39}$$

where the four ξ, ζ, $m_{(0)}$ and $m_{(2)}$ are

$$\xi = a_{(1)} - \left(a_{(2,1)} + \frac{1}{4}a_{(2,2)} + \frac{1}{6}a_{(2,3)}\right) R^{(0)}, \tag{4.40}$$

$$m_{(0)}^2 = \frac{\xi}{3a_{(2,1)} + a_{(2,2)} + a_{(2,3)}} \tag{4.41}$$

$$m_{(2)}^2 = -\frac{2\xi}{a_{(2,2)} + 4a_{(2,3)}}, \tag{4.42}$$

$$\zeta = \frac{a_{(2,3)}}{2\xi} \tag{4.43}$$

It can be seen from the action that the expanded Lagrangian is nothing but that of a quadratic higher derivative gravity with an effective cosmological constant term and a gravitational coupling rescaled by ξ. One of the significant thing we can conclude from this method is that the actual dynamics of any fourth-order higher-curvature theory of gravity can be effectively framed and identified. Moreover the dynamical variables emerged so far are those of a quadratic higher derivative gravity whose "weighting coefficients" are determined by the specific form of the starting action (the dependence of $\xi, \zeta, m_{(0)}$ and $m_{(2)}$ on the elements of the starting action).

Despite the accurateness of this method for the extraction of d.o.f.'s there are still some structural problems. The expansion around MS solution is undoubtedly a simple and fast method to study higher-order theory of gravity around their vacua, but its imperfections quickly reveal. In [52] it has been already discussed some issues that emerge when examining the case of a relatively minimal $f(R)$-theory defined by $f(R) = R + \chi R^3$, with χ a coupling constant. When expanding this action around Minkowski space (which itself is a MS solution), the scalar degree of freedom does not appear. At the same time, as soon as it was performed the very same expansion around a non-flat MS solution, the scalar d.o.f. will eventually crop up. This last conclusion suggests that the decoupling of the non-metric scalar field is just the occasional effect of the expansion around a single, specific background, with peculiar properties. The situation gets worse as there are other cases where the additional d.o.f.'s cannot be made manifest after an expansion around any MS solution of the theory and this is a serious limitation for the technique.

Chapter 5

Black holes in higher order theory of gravity

Black holes are the most fundamental objects in a theory of gravity, and they provide powerful probes for studying some of the more subtle global aspects of the theory. It is therefore of considerable interest to investigate the structure of black-hole solutions in theories of gravity with higher-order curvature terms. Here, we report on some investigations of the static, spherically-symmetric black-hole solutions in four-dimensional Einstein-Hilbert gravity with added quadratic curvature terms, for which the most general action can be taken to be

$$S = \int d^4x\sqrt{-g}(\gamma R - \alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \beta R^2), \tag{5.1}$$

where α, β and γ are constants and $C_{\mu\nu\rho\sigma}$ is the Weyl tensor.

5.1 Classical gravity with higher derivatives

We shall work use units where we set $\gamma = 1$, and the equations of motion following from [68]. One of the coupling constants can be fixed when choosing the system of units, so $\gamma = 1$. Then, the equations of motion take the form of

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - 4\alpha B_{\mu\nu} + 2\beta R(R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}) \\ +2\beta(g_{\mu\nu}\Box R - \nabla_\mu\nabla_\nu R) = 0, \end{aligned} \tag{5.2}$$

where

$$B_{\mu\nu} = (\nabla^\rho \nabla^\sigma + \frac{1}{2} R^{\rho\sigma}) C_{\mu\nu\rho\sigma} \tag{5.3}$$

is the Bach tensor, which is tracefree.

In general, the theory describes a system with a massive spin-2 mode with mass-squared $m_2^2 = 1/(2\alpha)$ and a massive spin-0 mode with mass-squared $m_0^2 = 1/(6\beta)$, in addition to the massless spin-2 graviton. According to [66] these massive modes will be associated with rising and falling Yukawa type behaviour in the metric modes near infinity, of the form $\frac{1}{r}e^{\pm m_2 r}$ and $\frac{1}{r}e^{\pm m_0 r}$. In particular, one can expect that if generic initial data is set at some small distance, the rising exponentials will eventually dominate, leading to singular asymptotic behaviour. In seeking black-hole solutions, the question then arises as to whether the rising exponentials can be avoided for appropriately finely-tuned initial data.

It can easily be seen that any solution of pure Einstein gravity will also be a solution [65], and so in particular the usual Schwarzschild black hole continues to be a solution in the higher-order theory. The question we wish to address, then, is whether there exist any other static black hole solutions.

5.2 Static and spherically symmetric solutions

Static, spherically-symmetric black-hole solutions have been investigated in [51], using generalisations of the Lichnerowicz and Israel theorems. According to this generalisations Lichnerowicz theorem [45], tells us that the only static, asymptotically flat, geodesically complete, vacuum, solution to Einstein's equations is flat space-time. Israel theorem [40] demonstrates that the only static, asymptotically flat, vacuum space-time, which contains past and future event horizons (that intersect on a surface that is topologically S^2 is given by the Schwarzschild metric). Also Israel theorem shows that the Reissner-Nordstrom metric must be the solution if the vacuum is replaced by an electromagnetic field. This 'no-hair' theorem is a striking result in classical GR and extensions of it to more general gravity theories, in particular as one approaches the scales on which GR is expected to be violated, would provide us with insight into the transition from GR to full quantum gravity.

Since we will arrive at somewhat different conclusions, we shall briefly summarise the key elements in [51]. We consider static metrics of the form $ds_4^2 = -\lambda^2 dt^2 + h_{ij} dx^i dx^j$, where λ and h_{ij} are functions only of the three spatial coordinates x. Taking the trace of the field equations (5.2) gives $\beta(\Box - m_0^2)R = 0$. We then multiply this by λR and integrate over the spatial domain from a putative horizon out to infinity. Expressed in terms of the covariant derivative D_i with respect to the spatial 3-metric h_{ij}, this gives

$$\int \sqrt{h} d^3x \left[D^i(\lambda R D_i R) - \lambda (D_i R)^2 - m_0^2 \lambda R^2 \right] = 0 \tag{5.4}$$

Since λ vanishes on the horizon, it follows that if $D_i R$ goes to zero sufficiently rapidly at spatial infinity the total derivative (i.e. surface term) gives no contribution, and the non-positivity of the remaining terms then implies $R = 0$. In other words, as shown in [51], any static black-hole solution of (5.1) must have vanishing Ricci scalar. This leads to a great simplification, and it means that one can, without loss of generality, study the case of pure Einstein-Weyl gravity (i.e. (5.1) with $\beta = 0$), since obviously the quadratic term in R makes no contribution to the old equations for a configuration with $R = 0$. Furthermore, the trace of the old equations (5.2) for Einstein- Weyl gravity immediately implies $R = 0$. In fact, the two differential equations for h and f are both now of only second order in derivatives.

The second stage of the discussion in [51] then involved looking at the remaining content of (2.2), i.e. the non trace part. According to [51], this led to another integral identity that then implied, under certain assumptions, that $R_{\mu\nu} = 0$. If this were correct, then the conclusion would be the usual Schwarzschild solution is the only static black hole solution of the theory described by (5.1). Setting $R = 0$, as already argued above, multiplying (5.2) by $\lambda R_{\mu\nu}$, and then integrating over the spatial region outside the horizon gives

$$\begin{aligned} \int \sqrt{h} d^3x [D^i W_i - \frac{1}{4}\lambda (D_i \bar{R} - 4D^j R_{ij})^2 + 4\lambda (D^j R_{ij})^2 \\ -4\lambda (D_{[i} R_{j]k})^2 + \lambda (D_i R_{jk})^2 - \frac{1}{4}\lambda \bar{R}^2 (m_2^2 + \bar{R}) \\ -\lambda (m_2^2 R^{ij} R_{ij} - 2R^{ij} R_{jk} R_i^k)] = 0, \end{aligned} \tag{5.5}$$

where $W_i = \lambda R^{jk} D_i R_{jk} + \frac{1}{4}\lambda \bar{R} D_i \bar{R} - 2\lambda R^{jk} D_j R_{ik} - \lambda \bar{R} D_j R_i^j$ and $\bar{R}$ is the Ricci scalar of the spatial metric h_{ij}. Although the surface term will give zero, the mix of positive and negative signs in the bulk terms prevents one from obtaining any kind of vanishing theorem for the Ricci tensor of the four-dimensional metric. This raises the intriguing possibility that there might in fact exist static, spherically symmetric black-hole solutions over and above the Schwarzschild solution.

5.3 Schwarzschild and non-Schwarzschild Black Holes

The equations of motion following from (5.1) are too complicated to be able to solve explicitly, even for the case of the static, spherically-symmetric ansatz. The line element of a black hole is given by

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{5.6}$$

In [46], it has been carried out a numerical investigation of the solutions. To do this, first needs to assume that there exists a black-hole horizon at some radius $r = r_0 > 0$, at which the metric functions h and f vanish, and then it was obtained near-horizon Taylor expansions for $h(r)$ and $f(r)$, of the form

$$h(r) = c[(r - r_0) + h_2(r - r_0)^2 + h_s(r - r_0)^3 + \ldots] \tag{5.7}$$

$$f(r) = f_1(r - r_0) + f_2(r - r_0)^2 + f_s(r - r_0)^3 + \ldots \tag{5.8}$$

Substituting into the equations of motion (5.2), with β set to zero for the reasons discussed above, the coefficients h_i and f_i for $i \geqslant 2$ can be solved in terms of the two non-trivial free parameters r_0 and f_1. There is also a "trivial" parameter, corresponding to the freedom to rescale the time coordinate, which was written in the form of an overall scaling of $h(r)$. Thus one can get

$$h_2 = \frac{1 - 2f_1 r_0}{f_1 r_0^2} + \frac{1 - f_1 r_0}{8\alpha f_1^2 r_0}, \qquad f_2 = \frac{1 - 2f_1 r_0}{r_0^2} - \frac{3(1 - f_1 r_0)}{8\alpha f_1 r_0}, \tag{5.9}$$

and so on. (It has been used Taylor expansions to $O((r - r_0)^9)$ in numerical integrations.) The Schwarzschild solution corresponds to $f_1 = 1/r_0$, and so it is convenient to parametrise f_1 as

$$f_1 = \frac{1 + \delta}{r_0} \tag{5.10}$$

with non-vanishing δ characterising the extent to which the near-horizon solution deviates from Schwarzschild.

Considering variation of this non-Schwarzschild parameter away from the $\delta = 0$ Schwarzschild value, it is clear that changing it generally has to do something to the solution at infinity. For a solution assumed to have a horizon and holding $R = 0$, the only thing that can happen initially is that a rising exponential is turned on, i.e. asymptotic fatness is lost at spatial infinity. So, for asymptotically flat solutions with a horizon in the near vicinity of the Schwarzschild solution, the only spherically symmetric static solution generally is the Schwarzschild solution itself.

This conclusion is formalized in [47] by considering infinitesimal variations of a solution away from Schwarzschild and proving a "no-hair" theorem for the linearised equation governing the variation. This can successfully be done for coefficients α that are not too large (i.e. for spin-two masses m_2 that are not too small). One concludes that the Schwarzschild black hole is at least in general isolated as an asymptotically flat solution with a horizon.

Now the question arises: what happens when one moves a finite distance away from Schwarzschild in terms of the non-Schwarzschild parameter? Does the loss of asymptotic flatness persist, or does something else happen, with solutions arising that cannot be treated by a linearised analysis in the deviation from Schwarzschild?

This can be answered numerically [46]. The task becomes one of finding values of $\delta \neq 0$ for which the generic rising exponential behaviour as $r \to 0$ is suppressed. What one finds is that there does indeed exist an asymptotically flat family of non-Schwarzschild black holes which crosses the Schwarzschild family at a special horizon radius r_0. For $\alpha = \frac{1}{2}$, one can find the following phases of black holes in Figure 5.1

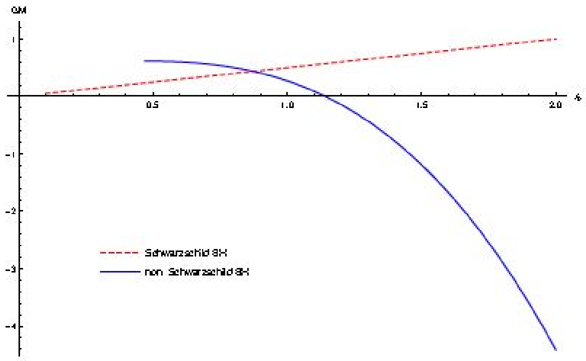

Figure 5.1: Black-hole masses as a function of horizon radius r_0, with a crossing point at $r^{Lich} \simeq 0.876$. The dashed straight-line $r_0 = 2M$ family consists of Schwarzschild black holes and the curved solid-line family consists of non-Schwarzschild black holes.[58]

The crossing point $r_0 \simeq 0.876$ between these two black-hole families is key to a further understanding of the overall structure of the black-hole solutions to the quadratic curvature theory.

The plots of the metric functions f and h for the examples of a positive-mass black hole with $r_0 = 1$, and a negative-mass black hole, with $r_0 = 2$, are shown in Figure 5.2

5.4 Analytical solution for the Non-Schwarzschild black hole

In [43] it was represented the analytical form of the numerical solutions that has been considered in the previous section. It was performed by using the convergent parametrization

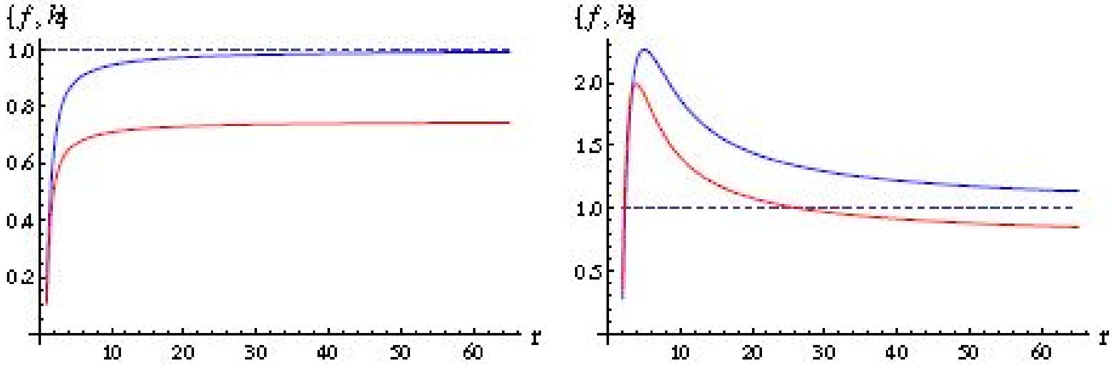

Figure 5.2: The non-Schwarzschild black hole for $r_0 = 1$ (left plot) and $r_0 = 2$ (right plot). In each plot the upper curve is $f(r)$ and the lower curve is $h(r)$. For clarity we have chosen a rescaling of h so that it approaches $3/4$, rather than 1, to avoid an asymptotic overlap of the curves.[46]

in terms of continued fractions. This result is accurate not only near the event horizon or far from black hole, but in the whole space. Using this analytical form it has been possible now to study easily all the further properties of the black hole, such as thermodynamics, Hawking radiation, particle motion, accretion, perturbations, stability, quasinormal spectrum, etc.

It is useful to introduce the dimensionless parameter, which parametrizes the solutions up to the rescaling

$$p = \frac{r_0}{\sqrt{2\alpha}} \tag{5.11}$$

It can be noticed that for all p the Schwarzschild metric is the exact solution of the Einstein-Weyl equations as well, but at some minimal nonzero p_{min}, in addition to the Schwarzschild solution, there appears the non-Schwarzschild branch that was found numerically in [46] which describes the asymptotically flat black hole, whose mass is decreasing, when p grows, and vanishing at some p_{max}. The approximate maximal and minimal values of p are

$$p_min \approx 1054/1203 \approx 0.876, \qquad p_{max} \approx 1.14 \tag{5.12}$$

according to the parametrization procedure the functions A and B were defined through the following relations

$$h(r) \equiv xA(x), \tag{5.13}$$

$$\frac{h(r)}{f(r)} \equiv B(x)^2, \tag{5.14}$$

where x is the dimensionless compact coordinate

$$x \equiv 1 - \frac{r_0}{r}. \tag{5.15}$$

The above functions represented as follows

$$A(x) = 1 - \varepsilon(1-x) + (a_0 - \varepsilon)(1-x)^2 + \bar{A}(x)(1-x)^3 \tag{5.16}$$

$$B(x) = 1 + b_0(1-x) + \bar{B}(x)(1-x)^2 \tag{5.17}$$

where $\bar{A}(x)$ and $\bar{B}(x)$ are introduced in terms of the continued fractions, in order to describe the metric near the event horizon $x = 0$

$$\bar{A}(x) = \cfrac{a_1}{1 + \cfrac{a_2 x}{1 + \cfrac{a_3 x}{1 + \cfrac{a_4 x}{1 + \ldots}}}} \tag{5.18}$$

$$\bar{B}(x) = \cfrac{b_1}{1 + \cfrac{b_2 x}{1 + \cfrac{b_3 x}{1 + \cfrac{b_4 x}{1 + \ldots}}}} \tag{5.19}$$

At the event horizon one has: $\bar{A}(0) = a_1, \bar{B}(0) = a_1$.

After some mathematical manipulations and choosing the accuracy of a fraction 0.1% for the metric functions $f(r)$ and $h(r)$ (see Figure 5.3) one can find the above coefficients and substitute them into (5.18) and obtain the final analytic expressions(represented in [43]) for the metric functions as the forth order continued fraction expansion

The obtained analytical metric represents asymptotically flat black hole which has the same post-Newtonian behaviour as in General Relativity, but is essentially different in the strong field regime. The metric is expressed in terms of event horizon radius r_0 and the dimensionless parameter $p = r_0/\sqrt{2\alpha}$, where α is the coupling constant. The minimal value of $p \approx 0.876$ corresponds to the merger of the Schwarzschild and non-Schwarzschild solutions, while at $p \approx 1.14$ the black-hole mass approaches zero $\varepsilon = -1$.

The obtained analytical approximation for the metric has two evident advantages over the numerical solution. First, it allows one to solve all the above enumerated problems

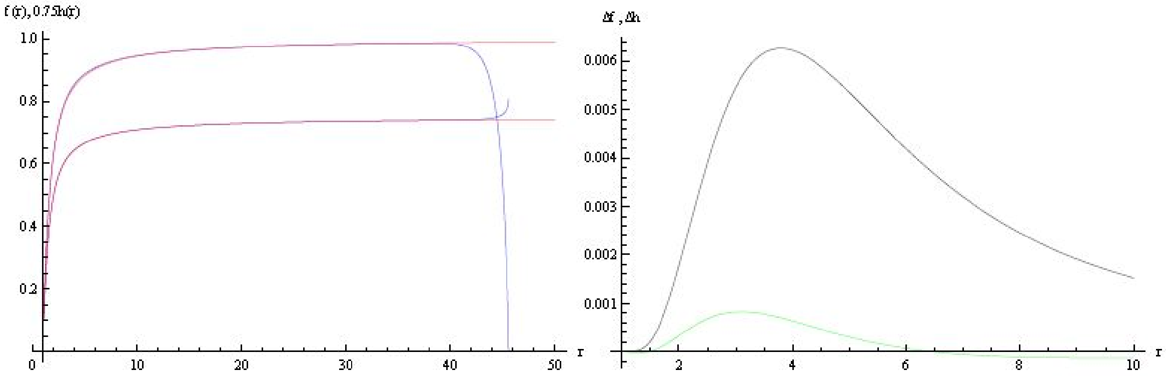

Figure 5.3: Comparison of numerical and analytical approximations for the metric functions: $r_0 = 1, \alpha = 0.5(p = 1)$.Left panel:$f(r)$ (upper) and, rescaled, $h(r)$ (lower). Numerical approximation (blue) fails at sufficiently large distance while our analytical approximation (red) has the correct behavior both near and far from the event horizon. Right panel: the difference between analytical and numerical approximations for $f(r)$ (black, upper) and $h(r)$ (green, lower). The largest difference is around the innermost stable circular orbit of a massive particle and photon circular orbit, where it still remains smaller than 0.1%.[43]

mentioned above in a much more economic and elegant way. Second, the analytical metric allows applications of a greater variety of methods for its analysis.

5.5 Thermodynamic implications for stability

Having established the existence of the non-Schwarzschild black holes, it is instructive to study some of their thermodynamic properties, and to compare these with the properties of the Schwarzschild black holes. In order to do this, it has been collected the numerical results for a sequence of black-hole solutions with r_0 in the range $r_0^{min} \approx 0.876 < r_0 < 1.5$, and then filtered the data to appropriate polynomials. Since we have buiseness with higher-derivative theory, the entropy is not simply given by one quarter of the area of the event horizon, and instead one needs to use the formula derived by Wald [73, 41].

This has been evaluated for the ansatz (5.6) in quadratic curvature gravities in [33], and applied to the case with $\beta = 0$ and $\gamma = 1$ in (5.1) this gives $S = \pi r_0^2 + 4\pi\alpha(1 - f_1 r_0) = \pi r_0^2 - 4\pi\alpha\delta$. There is a freedom to add a constant multiple of the Gauss-Bonnet invariant to the Lagrangian, which shifts the entropy by a parameter-independent constant without affecting the equations of motion. This fact has been used to ensure the entropy of the Schwarzschild black hole vanishes when the mass vanishes. Then one might find that the mass and the temperature of these non-Schwarzschild black holes, as a function of the entropy, take the form

$$M \approx 0.168 + 0.131S - 0.00749S^2 - 0.000139S^3 + \dots, \tag{5.20}$$

$$T \approx 0.131 - 0.0151S - 0.000428S^2 + \dots. \tag{5.21}$$

It can be seen that $\partial M/\partial S \approx 0.131 - 0.0150S - 0.000417S^2$, which is very close to the expression for the temperature. Thus the non-Schwarzschild black holes are seen to obey the first law of thermodynamics $dM = TdS$ to quite a high precision. It can be noticed that the expressions for M and T as a function of S for the Schwarzschild black holes are very different in form, with $M = (S/4\pi)^{1/2}$ and $T = \frac{1}{4}(\pi S)^{-1/2}$ (Figure 5.4)

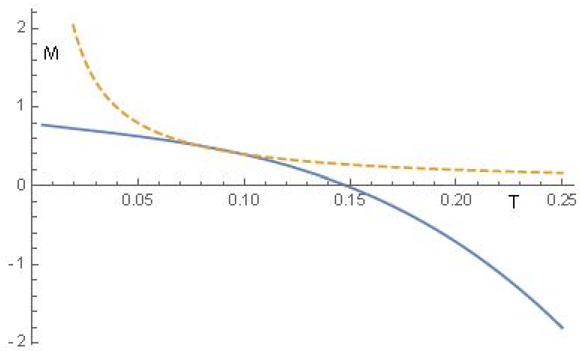

Figure 5.4: Mass M versus temperature T relations for Schwarzschild (dashed line) and non-Schwarzschild (solid line) black holes.[46]

It is interesting to note that the entropy of the non-Schwarzschild black hole of a given mass is always less than the entropy of the Schwarzschild black hole of the same mass. The two entropies approach each other asymptotically as r_0 approaches r_0^{min}. This can be seen in the Figure 5.5.

From the slope of M(T) it can be seen that the specific heat $C = dM/dT$ is negative for both black holes, and more negative for the non-Schwarzschild black hole at a given temperature (Figure 5.6)

From Figure 5.6, one can observe that for higher temperatures, which for Schwarzschild black holes correspond to small masses M and small radii r_0 and which for non-Schwarzschild black holes correspond to smaller (and eventually negative) masses but larger radii, the non-Schwarzschild black holes have a more negative specific heat than the Schwarzschild black holes. Accordingly, since Schwarzschild black holes are known to be subject to classical Gregory-Laflamme instabilities in this portion of their family trajectory, the suggestion is that the non-Schwarzschild black holes are more unstable in this "hot" portion of their family trajectory than the Schwarzschild black holes.

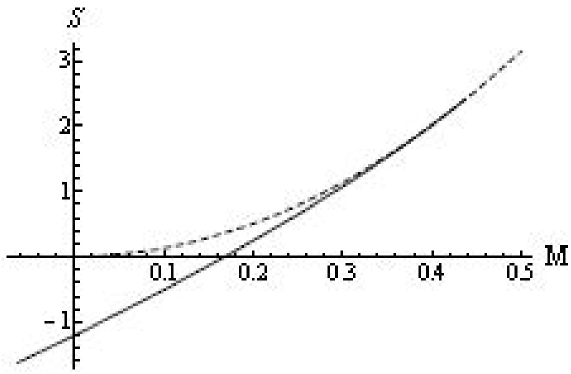

Figure 5.5: The entropy as a function of mass, for the Schwarzschild (dashed line) and non-Schwarzschild (solid line) black holes.[46]

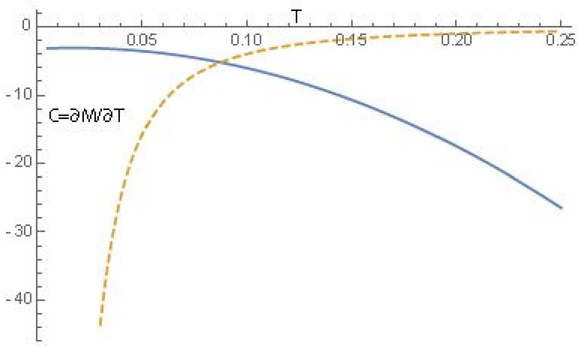

Figure 5.6: Specific heat C versus temperature T relations for Schwarzschild (dashed line) and non-Schwarzschild (solid line) black-holes families.[48]

Conversely, for lower temperatures, which for Schwarzschild black holes correspond to large masses M and large radii r_0 and for non-Schwarzschild black holes correspond to larger (now all positive) masses but smaller radii, the non-Schwarzschild black holes have a less negative specific heat than the Schwarzschild black holes. Accordingly, since Schwarzschild black holes are known to be classically immune to Gregory-Laamme instabilities [1] in this portion of their family trajectory, the suggestion is that the non-Schwarzschild black holes are more stable than the in this "cold" portion of their family trajectory than the Schwarzschild black holes.

The same inferences may be drawn by considering a graph of the free energy $F = M - TS$ versus temperature T relations of the two black-hole families, as shown in Figure 5.7, where again one sees a crossing of the two family curves

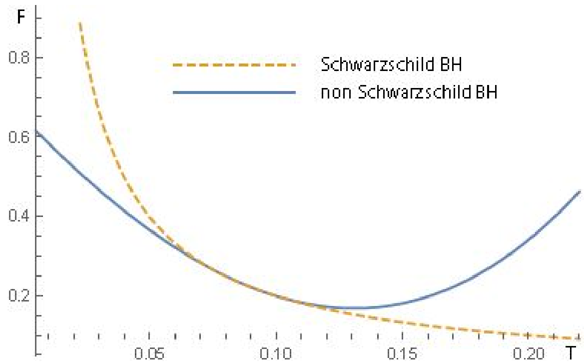

Figure 5.7: Free energy $F = M - TS$ versus temperature T relations for Schwarzschild (dashed line) and non-Schwarzschild (solid line) black holes families.[46]

Thus, a coherent suggestion emerges for the phase structure of dynamical stability and instability ranges of the Schwarzschild and non-Schwarzschild black hole families, as shown in Figure 5.8. This suggestion arises from two interrelated observations. The first is the existence of threshold unstable modes in linear perturbations away from the Schwarzschild and the non-Schwarzschild black hole families at the Lichnerowicz crossing point, as shown in the previous sections. The second is the pattern of relative susceptibilities to thermodynamic instability as revealed by study of the specific heats or the free energies of the solution families.

[1] In their seminal papers in 1993 and 1994, Gregory and Laflamme showed that certain branes and Higher-dimensional Einstein gravity black string solutions in theories of gravity in higher dimensions ($D \geq 5$) are found to exhibit an instability to small perturbations. The end point of this instability, particularly whether it leads to a phase transition forming a black hole. This has been studied to higher dimensions and a critical dimension has been found to exist below which the end state of instability is a black hole phase, i.e., for $(5 \leq D \leq 13)$. Above the critical dimension the instability drives to a non-uniform black ring phase [36, 44]

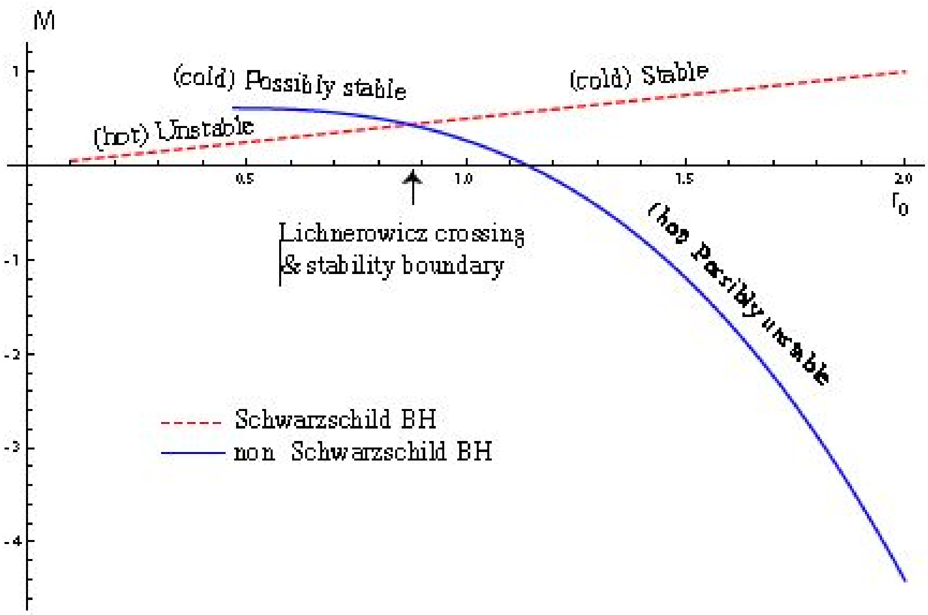

Figure 5.8: Classical stability ranges for Schwarzschild (dashed line) and suggested classical stability ranges for non-Schwarzschild (solid line) black holes.[48]

Chapter 6

Conclusions

In this work we have considered the issue of uncovering the actual degrees of freedom characterising a theory of gravity and relation to the field content. The most crucial aspect that actual degrees of freedom is intrinsic property whereas the filed content that embodies them is a conventional choice and this is not always unique. In fact it has been shown that some higher-curvature theory of gravity can be recast as GR plus a variable amount of extra non-minimally coupled fields.

In the case of quadratic theory of gravity this reformulation is possible and leads to the appearance of unstable extra degrees of freedom taking the form of ghost field.

At the same time from what we have reviewed here it should be noted that extracting and isolating the actual d.o.f.'s is not a trivial task: most of the simpler methods do provide inconclusive results, a few noticeable exceptions teach very little about the underlying structural difficulties, and the advanced techniques are plagued by pitfalls and computational fatigue.

For the sake of convenience we briefly sum up the main conclusion:

(a) Linear expansion around some suitable maximally symmetric background e.g. Minkowski flat spacetime is effective tool to uncover the extra d.o.f.'s. However a short-sighted application of the recipe can expose only a fraction of extra d.o.f.'s because of high symmetry of the background. It freezes some fields preventing their emergence at the linear level. If one goes way from the highly symmetric backgrounds would get the fields carrying the extra dynamical variables with highly non-negligible consequences.

(b) In auxiliary method one needs to introduce new fields and reshuffle the dynamical variables in order to reduce the differential order of higher order theory of gravity to familiar number two. For the case of models quadratic in the curvature eight dynamical d.o.f.'s emerge and then encoded into suitable number of spin-0 and spin-2 particles. The spin-2 objects can be massless or massive. As a result such theories can cast as GR plus a massive graviton and s scalar field. But these method become unmanageable as one goes beyond fourth-order theories and considers general $f(g_{\mu\nu}, R_{\mu\nu\rho\sigma})$ Lagrangians.

(c) In the framework of another technique one needs to expand the action of extended theory of gravity up to second order in curvature invariants around maximally symmetric spacetime solutions. As the outcome the study of extended theories of gravity provides results which holds for a vast class of models up to second order corrections. This protocols is different from linearizing the metric around a fixed background, but it ends up exhibiting the same old problem. The expansion around some MS spacetime in view of the high degree of symmetry in the background is unable to fully describe the actual d.o.f.'s of a theory.

(d) In this dissertation the black hole was used as laboratory to probe some of the consequences of the interpreting the action as complete classical action. From the analysis one can conclude that there is a second branch of static, spherically symmetric black holes over and above the Schwarzschild solutions. There are not Ricci flat, although they do have vanishing Ricci scalar. In a regime where α is small, which one might hope would correspond to a small correction to Einstein gravity, the second branch of black holes will be tiny and will actually have very large curvature near the horizon thus tending to invalidate the requirement that the curvature squared should be small. The fact that their mass can be negative, violating the usual positive-mass theorem of standard Einstein gravity, indicates that the ghost-like nature of the quadratically-corrected action is becoming dominant in this regime, one might view the contribution of the ghost-like massive spin-2 modes.

Also, just before the closing remarks, it is perhaps worth inserting a number of other interesting problems associated with the obtained metric that have physical significance [43]

- Perturbations and analysis of stability of the non-Schwarzschild black hole;
- Quasinormal modes of gravitational and test fields in its vicinity. As higher curvature corrections frequently lead to a new branch of non-perturbative (in coupling constant) modes, it is interesting to check whether this phenomena takes place for the considered here quadratic gravity;
- Analysis of massless and massive particles motion, binding energy, innermost stable circular orbits;
- Analysis of the accretion disks and the corresponding radiation in the electromagnetic spectra;
- Consideration of tidal and external magnetic fields in the vicinity of a black hole, etc.
- Hawking radiation in the semiclassical and beyond semiclassical regimes;

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