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# LIE TRANSFORMATIONS AND TRANSPORT EQUATIONS FOR COMBINED-FUNCTION DIPOLES

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This paper gives the Lie transformation of order 3 for a dipole including quadrupole and sextupole components. From these the first and second order TRANSPORT coefficients are derived for a set of canonical variables. The paper shows which changes have to be made to the conventional TRANS-PORT formalism in order to use canonical variables. The transformations for drifts, homogeneous field dipoles, quadrupoles, and sextupoles can be found by putting the irrelevant coefficients to zero. When doing so, the results agree with the results found in earlier papers. The third-order Lie transformations for fringing fields in the hard-edge approximation are included.

#### 1. INTRODUCTION

Lie transformations have been recommended<sup>3,4</sup> for tracking charged particles. The transformation coefficients are listed in Ref. 3 for various magnetic elements. To complement the catalog, this paper gives the Lie coefficients of order 3 for a dipole including quadrupole and sextupole components. From these the first and second order TRANSPORT coefficients<sup>1,2</sup> are derived for a set of canonical variables. The paper shows which changes have to be made to the conventional TRANSPORT formalism in order to use canonical variables. The transformations for drifts, homogeneous field dipoles, quadrupoles, and sextupoles can be found by putting the irrelevant coefficients to zero. When doing so, the results agree with the results found in earlier papers.<sup>1,2</sup>

Section 2 derives the fourth-order Hamiltonian for a combined-function dipole. Section 3 lists the definitions used. Section 4 presents the Lie transformation to order 3 for the body of a combined function dipole. Section 5 displays the corresponding TRANSPORT coefficients, and Section 6 compares these results to the original TRANSPORT.<sup>1</sup> Finally, Sections 7 and 8 present the TRANSPORT coefficients and the third-order Lie transformations for entrance and exit fringe fields in the hard-edge approximation.

# 2. THE HAMILTONIAN TO ORDER 4 FOR A COMBINED FUNCTION DIPOLE

This paper uses the conventions of the MAD program<sup>5</sup> with the following canonical variables:

x The local horizontal axis pointing to the left,

- $p_x$  The horizontal canonical momentum, divided by the reference momentum  $p_0$ ,
- y The local vertical axis pointing up,
- $p_{y}$  The vertical canonical momentum, divided by the reference momentum  $p_{0}$ ,
- $\tau$  The velocity of light times the negative time difference:  $\tau = -c\delta t$ ,
- δ The positive energy difference, divided by the reference momentum times the velocity of light:  $\delta = \delta E/(cp_0)$ ,
- s The arc length along the reference orbit.

Except for the signs adopted here for  $\tau$  and  $\delta$ , Refs. 3 and 4 use the same variables. In the limit of fully relativistic particles (v = c, E = pc), the variables  $\tau$  and  $\delta$  used here agree with those used in TRANSPORT.<sup>1,2</sup> This means that  $\tau$  becomes the negative path length difference, while  $\delta$  becomes the fractional momentum error. The reference momentum  $p_0$  must be a constant in order to keep the system canonical.

The MAD program uses the following Taylor expansion for the field on the midplane (y = 0)

$$B_{\nu}(x, 0) = B_0 + B_1 x / 1! + B_2 x^2 / 2! + B_3 x^3 / 3! + \cdots$$

Note the factorials in the denominators which are not present in TRANSPORT. The meanings of the field coefficients are

- $B_0$  The dipole field, with a positive value in the positive y direction, i.e. a positive field bends a positively charged particle to the right.
- $B_1$  The quadrupole strength =  $\partial B_y/\partial x$ , with a positive value corresponding to horizontal focusing of a positively charged particle.
- $B_2$  The sextupole strength =  $\partial^2 B_{\rm v} / \partial x^2$ .
- $B_3$  The octupole strength =  $\partial^3 B_{\rm v} / \partial x^3$ .

Using this expansion and the curvature h of the reference orbit, the longitudinal component of the vector potential to order 4 is

$$A_{s} = -B_{0}[x - hx^{2}/2 + h^{2}x^{3}/2 - h^{3}x^{4}/2 + \cdots]$$
  
-  $B_{1}[(x^{2} - y^{2})/2 - hx^{3}/6 + h^{2}(4x^{4} - y^{4})/24 + \cdots]$   
-  $B_{2}[(x^{3} - 3xy^{2})/6 - h(x^{4} - y^{4})/24 + \cdots]$   
-  $B_{3}[(x^{4} - 6x^{2}y^{2} + y^{4})/24 + \cdots] + \cdots.$ 

Taking the curl of  $A_s$  in curvilinear coordinates the field components can be computed as

$$B_{x}(x, y) = B_{1}[y + h^{2}y^{3}/6 + \cdots]$$
  
+  $B_{2}[xy - hy^{3}/6 + \cdots]$   
+  $B_{3}[(3x^{2}y - y^{3})/6 + \cdots] + \cdots$   
$$B_{y}(x, y) = B_{0}$$
  
+  $B_{1}[x - hy^{2}/2 + h^{2}xy^{2}/2 + \cdots]$   
+  $B_{2}[(x^{2} - y^{2})/2 - hxy^{2}/2 + \cdots]$   
+  $B_{3}[(x^{3} - 3xy^{2})/6 + \cdots] + \cdots$ 

It can be easily verified that the curl and divergence of the magnetic field are both zero to order 2. If we now introduce the particle charge e and the relativistic constants  $\beta = v/c$ , the relative velocity of the reference particle, and  $\gamma = E/mc^2$ , the relative mass of the reference particle, the Hamiltonian for a curved reference system can be written<sup>4</sup> as

$$H = + (1+\delta) - (1+hx)(e/p_0)A_s - (1+hx)[(1+2\delta/\beta+\delta^2) - (p_x^2+p_y^2)]^{1/2}.$$

Note the sign change for  $2\delta/\beta$  with respect to Ref. 4. In order to find the Lie transformation, we expand the Hamiltonian as a power series and separate the terms of equal orders:

$$H = H_2 + H_3 + H_4$$

Defining the multipole coefficients as

$$K_n = eB_n/p_0 = B_n/B\rho = (1/B\rho)(\partial^n B_v/\partial x^n),$$

where  $B\rho$  is the magnetic rigidity and  $h = K_0$ , the resulting homogeneous polynomials are

$$\begin{split} H_2 &= + (hK_0 + K_1)x^2/2 - K_1y^2/2 \\ &+ [p_x^2 + p_y^2 + (\delta/\beta\gamma)^2]/2 - hx\delta/\beta \\ H_3 &= + (K_2 + 2hK_1)x^3/6 - (K_2 + hK_1)xy^2/2 \\ &+ (hx - \delta/\beta)[p_x^2 + p_y^2 + (\delta/\beta\gamma)^2]/2 \\ H_4 &= + (K_3 + 3hK_2)x^4/24 - (K_3 + 2hK_2)x^2y^2/6 + (K_3 + hK_2 - h^2K_1)y^4/24 \\ &- (hx - \delta/\beta)[p_x^2 + p_y^2 + (\delta/\beta\gamma)^2]\delta/2\beta \\ &+ [p_x^2 + p_y^2 + (\delta/\beta\gamma)^2]^2/8 \end{split}$$

### 3. DEFINITIONS

In order to simplify numerical evaluations, the transformation equations are given in terms of a few simple integrals. Using the magnet length L, we define the focusing functions as

$$c(k, L) = \cos(kL) = \cosh(ikL)$$
  

$$s(k, L) = \int_0^L c(k, s) \, ds = \sin(kL)/k = \sinh(ikL)/ik$$
  

$$d(k, L) = \int_0^L s(k, s) \, ds = [1 - c(k, L)]/k^2$$
  

$$f(k, L) = \int_0^L d(k, s) \, ds = [L - s(k, L)]/k^2$$

For the horizontal plane, we also use the definitions

$$k_x^2 = h^2 + K_1$$

$$c_x = c(k_x, L)$$

$$s_x = s(k_x, L)$$

$$d_x = d(k_x, L),$$

and for the vertical plane

$$k_y^2 = -K_1$$
  

$$c_y = c(k_y, L)$$
  

$$s_y = s(k_y, L).$$

Finally, we use the integrals

$$J_{1} = \int_{0}^{L} d_{x} ds = (L - s_{x})/k_{x}^{2}$$

$$J_{2} = \int_{0}^{L} d_{x}^{2} ds = (3L - 4s_{x} + s_{x}c_{x})/(2k_{x}^{4})$$

$$J_{3} = \int_{0}^{L} d_{x}^{3} ds = (15L - 22s_{x} + 9s_{x}c_{x} - 2s_{x}c_{x}^{2})/(6k_{x}^{6})$$

$$J_{c} = [c(2k_{y}, L) - c(k_{x}, L)]/(k_{x}^{2} - 4k_{y}^{2})$$

$$J_{s} = \int_{0}^{L} J_{c} ds = [s(2k_{y}, L) - s(k_{x}, L)]/(k_{x}^{2} - 4k_{y}^{2})$$

$$J_{d} = \int_{0}^{L} J_{s} ds = [d(2k_{y}, L) - d(k_{x}, L)]/(k_{x}^{2} - 4k_{y}^{2})$$

$$J_{f} = \int_{0}^{L} J_{d} ds = [f(2k_{y}, L) - f(k_{x}, L)]/(k_{x}^{2} - 4k_{y}^{2})$$

For  $k_x = 0$ ,  $k_y = 0$ , or  $(k_x^2 - 4k_y^2) = 0$  some of these integrals are indeterminate. Formulas to evaluate these integrals are given in Appendix A.

# 4. THE LIE TRANSFORMATION FOR A COMBINED-FUNCTION DIPOLE

In the following we refer to the phase-space vector

$$V = \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ \tau \\ \delta \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix}$$

The linear transformation is given by

$$v_i^{(2)} = \sum_{k=1}^6 R_{ik} v_k^{(1)}.$$

Using this notation, the transfer matrix R for the body of the dipole is

$$R = \begin{pmatrix} c_{x} & s_{x} & 0 & 0 & 0 & +(h/\beta)d_{x} \\ -k_{x}^{2}s_{x} & c_{x} & 0 & 0 & 0 & +(h/\beta)s_{x} \\ 0 & 0 & c_{y} & s_{y} & 0 & 0 \\ 0 & 0 & -k_{y}^{2}s_{y} & c_{y} & 0 & 0 \\ -(h/\beta)s_{x} & -(h/\beta)d_{x} & 0 & 0 & 1 & -(h/\beta)^{2}J_{1} + L/\beta^{2}\gamma^{2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The third-order Lie Polynomial  $F_3$  is found according to the method given in Ref. 3. First we transform the Hamiltonian using

$$V(s) = R(s)V^{(1)}$$

The transfer matrix R is written as a function of the arc length s. After this transformation, the polynomial  $H_2$  vanishes and the Hamiltonian becomes

$$H = H_3[R^{-1}(s)V] + H_4[R^{-1}(s)V].$$

In Ref. 3, it is shown that the third-order Lie Polynomial is

$$F_3 = -\int_0^L H_3[R^{-1}(s)V] \, ds.$$

We rewrite  $F_3$  in the form

$$F_3 = \sum_{i=1}^{6} \sum_{j=1}^{6} \sum_{k=1}^{6} F_{ijk} v_i v_j v_k / 3!.$$

The coefficients of  $F_3$  form a fully symmetric array. This implies that the coefficients must be completed by the symmetry condition

$$F_{ijk} = F_{ikj} = F_{jik} = F_{jki} = F_{kji} = F_{kij}.$$

The values of the  $F_{ijk}$  are the partial derivatives of order 3 of the  $F_3$  polynomial. This is different from Ref. 3 where the  $F_{ijk}$  are defined as the coefficients of the polynomial. In this paper they are thus larger by a factor 3! than in Ref. 3 if all three indices are equal. They are larger by a factor 2! if two indices are equal, and they are the same if all indices are different. The non-zero coefficients of  $F_3$  are listed below.

$$F_{111} = -(1/3)(K_2 + 2hK_1)s_x(2 + c_x^2) - hk_x^4s_x^3$$
  

$$F_{112} = +(1/3)(K_2 + 2hK_1)(d_x + s_x^2c_x) - hk_x^2s_x^2c_x$$
  

$$F_{122} = -(1/3)(K_2 + 2hK_1)s_x^3 - hs_xc_x^2$$
  

$$F_{222} = +(1/3)(K_2 + 2hK_1)d_x^2(2 + c_x) + h(d_x + s_x^2c_x)$$

$$\begin{split} F_{116} &= -(h/6\beta)(K_2 + 2hK_1)(3J_1 - 3s_xd_x + 2s_x^3) + (h^2/\beta)k_x^2s_x^3 \\ &+ (1/2\beta)K_1(L - s_xc_x) \\ F_{126} &= +(h/6\beta)(K_2 + 2hK_1)d_x^2(1 + 2c_x) + (h^2/\beta)s_x^2c_x \\ &+ (1/2\beta)K_1s_x^2 \\ F_{226} &= -(h/3\beta)(K_2 + 2hK_1)(s_xd_x^2 + J_2) - (h^2/2\beta)[J_1 + s_xd_x(1 + 2c_x)] \\ &+ (1/2\beta)(L + s_xc_x) \\ F_{166} &= -(h^2/3\beta^2)(K_2 + 2hK_1)(s_xd_x^2 - 2J_2) - (h^3/\beta^2)s_x^3 \\ &- (h/\beta^2)K_1(J_1 + s_xd_x) - (h/\beta^2\gamma^2)s_x \\ F_{266} &= +(h^2/3\beta^2)(K_2 + 2hK_1)d_x^3 + (h^3/\beta^2)s_x^2d_x \\ &- (h/\beta^2)s_x^2 + (h/\beta^2\gamma^2)d_x \\ F_{666} &= -(h^3/\beta^3)(K_2 + 2hK_1)J_3 - (h^4/\beta^3)(s_xd_x^2 + J_2) \\ &+ (3h^2/2\beta^3)(J_1 + s_xd_x) + (3/\beta^3\gamma^2)(L - h^2J_1) \\ F_{133} &= +2K_1K_2(k_x^2s_xJ_d + c_xJ_s) + (K_2 + hK_1)s_x \\ F_{134} &= -K_2(k_x^2s_xJ_d + c_xJ_s) - hs_x \\ F_{233} &= +2K_1K_2(c_xJ_d - s_xJ_s) - (K_2 + hK_1)d_x \\ F_{234} &= -K_2(c_xJ_a - s_xJ_c) \\ F_{244} &= +2K_2(c_xJ_a - s_xJ_s) + hd_x \\ F_{336} &= +2(h/\beta)K_1K_2(J_f + d_xJ_s - s_xJ_d) + (h/\beta)(K_2 + hK_1)J_1 \\ &- (1/2\beta)K_1(L - s_yc_y) \\ F_{346} &= -(h/\beta)K_2(J_d + d_xJ_c - s_xJ_s) \\ &- (1/2\beta)K_1s_y^2 \\ F_{446} &= +2(h/\beta)K_2(J_f + d_xJ_s - s_xJ_d) - (h^2/\beta)J_1 \\ &+ (1/2\beta)(L + s_yc_y) \\ \end{split}$$

# 5. THE TRANSPORT COEFFICIENTS FOR A COMBINED-FUNCTION DIPOLE

From the third-order Lie transformation given above, the second-order TRANS-PORT coefficients can be derived using the cascade of transformations<sup>3,4</sup>

$$V^{(2)} = RV^{(1)}$$
  
 $V^{(3)} = [F_3, V^{(2)}].$ 

This leads to the expressions for the TRANSPORT coefficients

$$T_{1ik} = -(1/2) \sum_{m=1}^{6} \sum_{n=1}^{6} F_{2mn} R_{mi} R_{nk}$$

$$T_{2ik} = + (1/2) \sum_{m=1}^{6} \sum_{n=1}^{6} F_{1mn} R_{mi} R_{nk}$$
$$T_{3ik} = - (1/2) \sum_{m=1}^{6} \sum_{n=1}^{6} F_{4mn} R_{mi} R_{nk}$$
$$T_{4ik} = + (1/2) \sum_{m=1}^{6} \sum_{n=1}^{6} F_{3mn} R_{mi} R_{nk}$$
$$T_{5ik} = - (1/2) \sum_{m=1}^{6} \sum_{n=1}^{6} F_{6mn} R_{mi} R_{nk}$$
$$T_{6ik} = + (1/2) \sum_{m=1}^{6} \sum_{n=1}^{6} F_{5mn} R_{mi} R_{nk}.$$

In order to simplify notation, the elements of T have been defined here as a symmetric rectangular array. The elements of the T matrix whose second and third index are different have thus half the value of those listed in Ref. 1. Using the MAD conventions, the second-order equations read

$$v_i^{(2)} = \sum_{j=1}^6 R_{ij} v_j^{(1)} + \sum_{j=1}^6 \sum_{k=1}^6 T_{ijk} v_j^{(1)} v_k^{(1)},$$

The T array is now square rather than triangular. The non-zero coefficients are listed below. We refer to section 3 above for definitions.

$$\begin{split} T_{111} &= -(1/6)(K_2 + 2hK_1)(s_x^2 + d_x) - (h/2)k_x^2 s_x^2 \\ T_{112} &= -(1/6)(K_2 + 2hK_1)s_x d_x + (h/2)s_x c_x \\ T_{122} &= -(1/6)(K_2 + 2hK_1)d_x^2 + (h/2)c_x d_x \\ T_{116} &= -(h/12\beta)(K_2 + 2hK_1)(3s_x J_1 - d_x^2) + (h^2/2\beta)s_x^2 \\ &\quad + (1/4\beta)K_1 Ls_x \\ T_{126} &= -(h/12\beta)(K_2 + 2hK_1)(s_x d_x^2 - 2c_x J_2) + (h^2/4\beta)(s_x d_x + c_x J_1) \\ &\quad - (1/4\beta)(s_x + Lc_x) \\ T_{166} &= -(h^2/6\beta^2)(K_2 + 2hK_1)(d_x^3 - 2s_x J_2) + (h^3/2\beta^2)s_x J_1 \\ &\quad - (h/2\beta^2)Ls_x - (h/2\beta^2\gamma^2)d_x \\ T_{133} &= +K_1K_2J_d + (1/2)(K_2 + hK_1)d_x \\ T_{134} &= +(1/2)K_2J_s \\ T_{144} &= +K_2J_d - (h/2)d_x \end{split}$$

$$T_{211} = -(1/6)(K_2 + 2hK_1)s_x(1 + 2c_x)$$
  

$$T_{212} = -(1/6)(K_2 + 2hK_1)d_x(1 + 2c_x)$$
  

$$T_{222} = -(1/3)(K_2 + 2hK_1)s_xd_x - (h/2)s_x$$
  

$$T_{216} = -(h/12\beta)(K_2 + 2hK_1)(3c_xJ_1 + s_xd_x)$$
  

$$-(1/4\beta)K_1(s_x - Lc_x)$$

$$\begin{split} T_{226} &= -(h/12\beta)(K_2 + 2hK_1)(3s_xJ_1 + d_x^2) \\ &+ (1/4\beta)K_1Ls_x \\ T_{266} &= -(h^2/6\beta^2)(K_2 + 2hK_1)(s_xd_x^2 - 2c_xJ_2) \\ &- (h/2\beta^2)K_1(c_xJ_1 - s_xd_x) - (h/2\beta^2\gamma^2)s_x \\ T_{233} &= +K_1K_2J_s + (1/2)(K_2 + hK_1)s_x \\ T_{234} &= +(1/2)K_2J_c \\ T_{244} &= +K_2J_s - (h/2)s_x \end{split}$$

$$\begin{split} T_{313} &= + (1/2)K_2(c_yJ_c - 2K_1s_yJ_s) + (h/2)K_1s_xs_y \\ T_{314} &= + (1/2)K_2(s_yJ_c - 2c_yJ_s) + (h/2)s_xc_y \\ T_{323} &= + (1/2)K_2(c_yJ_s - 2K_1s_yJ_d) + (h/2)K_1d_xs_y \\ T_{324} &= + (1/2)K_2(s_yJ_s - 2c_yJ_d) + (h/2)d_xc_y \\ T_{336} &= + (h/2\beta)K_2(c_yJ_d - 2K_1s_yJ_f) + (h^2/2\beta)K_1J_1s_y \\ &- (1/4\beta)K_1Ls_y \\ T_{346} &= + (h/2\beta)K_2(s_yJ_d - 2c_yJ_f) + (h^2/2\beta)J_1c_y \\ &- (1/4\beta)(s_y + Lc_y) \\ \hline T_{413} &= + (1/2)K_1K_2(2c_yJ_s - s_yJ_c) + (1/2)(K_2 + hK_1)s_xs_y \\ T_{423} &= + (1/2)K_1K_2(2c_yJ_d - s_yJ_s) + (1/2)(K_2 + hK_1)d_xc_y \\ T_{424} &= + (1/2)K_2(2K_1s_yJ_d - c_yJ_s) + (1/2)(K_2 + hK_1)d_xs_y \\ T_{436} &= + (h/2\beta)K_1K_2(2c_yJ_f - s_yJ_d) + (h/2\beta)(K_2 + hK_1)J_1c_y \\ &+ (1/4\beta)K_1(s_y - Lc_y) \\ \hline T_{446} &= + (h/2\beta)K_2(2K_1s_yJ_f - c_yJ_d) + (h/2\beta)(K_2 + hK_1)J_1s_y \\ &- (1/4\beta)K_1Ls_y \\ \end{split}$$

$$\begin{split} T_{511} &= +(h/12\beta)(K_2 + 2hK_1)(s_xd_x + 3J_1) \\ &-(1/4\beta)K_1(L - s_xc_x) \\ T_{512} &= +(h/12\beta)(K_2 + 2hK_1)d_x^2 \\ &+(1/4\beta)K_1s_x^2 \\ T_{522} &= +(h/6\beta)(K_2 + 2hK_1)J_2 - (1/2\beta)s_x \\ &-(1/4\beta)K_1(J_1 - s_xd_x) \\ T_{516} &= +(h^2/12\beta^2)(K_2 + 2hK_1)(3d_xJ_1 - 4J_2) \\ &+(h/4\beta^2)K_1J_1(1 + c_x) + (h/2\beta^2\gamma^2)s_x \\ T_{526} &= +(h^2/12\beta^2)(K_2 + 2hK_1)(d_x^3 - 2s_xJ_2) \\ &+(h/4\beta^2)K_1s_xJ_1 + (h/2\beta^2\gamma^2)d_x \end{split}$$

$$\begin{split} T_{566} &= + (h^3/6\beta^3)(K_2 + 2hK_1)(3J_3 - 2d_xJ_2) \\ &+ (h^2/6\beta^3)K_1[s_xd_x^2 - J_2(1 + 2c_x)] + (3/2\beta^3\gamma^2)(h^2J_1 - L) \\ T_{533} &= -(h/\beta)K_1K_2J_f - (h/2\beta)(K_2 + hK_1)J_1 \\ &+ (1/4\beta)K_1(L - c_ys_y) \\ T_{534} &= -(h/2\beta)K_2J_d \\ &- (1/4\beta)K_1s_y^2 \\ T_{544} &= -(h/\beta)K_2J_f + (h^2/2\beta)J_1 \\ &- (1/4\beta)(L + c_ys_y) \end{split}$$

#### 6. COMPARISON WITH TRANSPORT

From the Hamiltonian given in section 2, we find the canonical equations of motion

$$dx/ds = p_x(1 + hx - \delta/\beta) \tag{1}$$

$$dy/ds = p_{v}(1 + hx - \delta/\beta)$$
<sup>(2)</sup>

$$dp_{x}/ds = -(K_{1} + hK_{0})x + h\delta/\beta$$
  
-(K\_{2} + 2hK\_{1})x^{2}/2 + (K\_{2} + hK\_{1})y^{2}/2 - h[p\_{x}^{2} + p\_{y}^{2} + (\delta/\beta\gamma)^{2}]/2  
$$dp_{y}/ds = +K_{1}y + (K_{2} + hK_{1})xy.$$

By differentiation and substitution, we find the second-order equations

$$d^{2}x/ds^{2} = -(K_{1} + hK_{0})x + h\delta/\beta$$
  
-(K<sub>2</sub>+4hK<sub>1</sub>+2h<sup>2</sup>K<sub>0</sub>)x<sup>2</sup>/2+(K<sub>2</sub>+hK<sub>1</sub>)y<sup>2</sup>/2  
+(h<sup>2</sup>+hK<sub>0</sub>+K<sub>1</sub>)x\delta/\beta+(h/2)(p\_{x}^{2}-p\_{y}^{2}) - h\delta^{2}/\beta^{2} - h\delta^{2}/2\beta^{2}\gamma^{2}(3)  
$$d^{2}y/ds^{2} = +K_{1}y + (K_{2} + 2hK_{1})xy - K_{1}y\delta/\beta + hp_{x}p_{y}.$$
(4)

Putting  $\beta = 1$ ,  $1/\gamma = 0$  and making the replacements

$$K_0 = h$$
$$K_1 = -h^2 n$$
$$K_2 = 2h^3 \beta,$$

Eqs. (3) and (4) become identical to those given in Ref. 1.

The TRANSPORT coefficients listed in this paper may also be found directly as follows:

1. Find the initial conditions for the derivatives at the entrance using Eqs. (1) and (2):

$$dx/ds = p_x(1 + hx - \delta/\beta)$$
$$dy/ds = p_y(1 + hx - \delta/\beta)$$

2. Solve the second-order equations of motion (3) and (4) to step through the dipole using these initial conditions.

3. Convert the derivatives at the exit back to canonical momenta by inverting Eqs. (1) and (2) to second order:

$$p_x = (1 - hx + \delta/\beta) \, dx/ds$$
$$p_y = (1 - hx + \delta/\beta) \, dy/ds$$

This procedure reproduces the T matrix elements found above, but by different means. The particular choice of canonical variables in this paper adds the  $\delta/\beta$  terms in Eqs. (1) and (2) as compared to Ref. 1. These terms, together with the relativistic limit, account for the entire difference of the results contained in this paper and those in Ref. 1.

# 7. TRANSPORT COEFFICIENTS FOR HARD-EDGE FRINGING FIELDS

Following the TRANSPORT conventions, we define the fringe field angles  $\psi_1$  and  $\psi_2$  as positive if they cause the outward normal of the pole face to be rotated away from the centre of curvature, i.e. if the fringing field causes an additional horizontally defocusing quadrupole. The pole face curvatures  $\eta_1$  and  $\eta_2$  are positive if the pole face is convex, i.e. if there is an additional negative sextupole effect in the fringing field. The first-order transfer matrix for an entrance or exit fringing field is then in the hard-edge approximation<sup>1</sup>

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ +h \tan \psi & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -h \tan \psi & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The second-order coefficients must be modified with respect to Ref. 1 using Eqs. (1) and (2) of Section 6 when using canonical variables. This causes the coefficients  $T_{216}$  and  $T_{436}$  to vanish. The non-zero coefficients for the entrance fringing field are

$$T_{111} = -(h/2) \tan^2 \psi_1$$

$$T_{133} = +(h/2) \sec^2 \psi_1$$

$$T_{211} = +(h/2) \eta_1 \sec^3 \psi + K_1 \tan \psi_1$$

$$T_{212} = +(h/2) \tan^2 \psi_1$$

$$T_{233} = -(h/2) \eta_1 \sec^3 \psi_1 - K_1 \tan \psi_1 + (h^2/2) \tan \psi_1 (1 + \sec^2 \psi_1)$$

$$T_{234} = -(h/2) \tan^2 \psi_1$$

$$T_{313} = +(h/2) \tan^2 \psi_1$$

$$T_{413} = -(h/2) \eta_1 \sec^3 \psi_1 - K_1 \tan \psi_1$$

$$T_{414} = -(h/2) \tan^2 \psi_1$$

$$T_{423} = -(h/2) \sec^2 \psi_1.$$

For the exit fringing field, the non-zero second-order coefficients are

$$T_{111} = +(h/2) \tan^2 \psi_2$$

$$T_{133} = -(h/2) \sec^2 \psi_2$$

$$T_{211} = +(h/2) \eta_2 \sec^3 \psi_2 + K_1 \tan \psi_2 - (h^2/2) \tan^3 \psi_2$$

$$T_{212} = -(h/2) \tan^2 \psi_2$$

$$T_{233} = -(h/2) \eta_2 \sec^3 \psi_2 - K_1 \tan \psi_2 - (h^2/2) \tan^3 \psi_2$$

$$T_{234} = +(h/2) \tan^2 \psi_2$$

$$T_{313} = -(h/2) \tan^2 \psi_2$$

$$T_{413} = -(h/2) \eta_2 \sec^3 \psi_2 - K_1 \tan \psi_2 + (h^2/2) \tan \psi_2 \sec^2 \psi_2$$

$$T_{414} = +(h/2) \tan^2 \psi_2$$

$$T_{413} = +(h/2) \tan^2 \psi_2$$

$$T_{413} = +(h/2) \tan^2 \psi_2$$

Thus in both cases there is an effective sextupole with the strength

$$S = -h\eta \sec^3 \psi - 2K_1 \tan \psi$$

The terms of the T array without  $\eta$  or  $K_1$  depend on the edge angle  $\psi$  and the curvature h only.

# 8. LIE TRANSFORMATIONS FOR HARD-EDGE FRINGING FIELDS

The Lie transformations for fringing fields are found by inversion of the formulas given in section 5. The linear transfer matrix is the same as in section 7. The third order Lie polynomial for the entrance fringing field has the non-zero coefficients

$$F_{111} = + (h/6)\eta_1 \sec^3 \psi_1 + (K_1/3) \tan \psi_1 - (h^2/3) \tan^3 \psi_1$$
  

$$F_{112} = + (h/6) \tan^2 \psi_1$$
  

$$F_{133} = - (h/6)\eta_1 \sec^3 \psi_1 - (K_1/3) \tan \psi_1 + (h^2/6) \tan \psi_1$$
  

$$F_{134} = - (h/6) \tan^2 \psi_1$$
  

$$F_{233} = - (h/6) \sec^2 \psi_1.$$

For the exit fringing field, the polynomial has the non-zero coefficients

$$F_{111} = + (h/6)\eta_2 \sec^3 \psi_2 + (K_1/3) \tan \psi_2 + (h^2/6) \tan^3 \psi_2$$
  

$$F_{112} = - (h/6) \tan^2 \psi_2$$
  

$$F_{133} = - (h/6)\eta_2 \sec^3 \psi_2 + (K_1/3) \tan \psi_2 + (h^2/6) \tan^3 \psi_2$$
  

$$F_{134} = + (h/6) \tan^2 \psi_2$$
  

$$F_{233} = + (h/6) \sec^2 \psi_2$$

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# APPENDIX A

#### **Evaluation of the Key Integrals**

The formulas below guarantee a relative error smaller than  $10^{-8}$  for all integrals, when evaluated to 12 significant digits.

Consider the integrals  $J_1$ ,  $J_2$ , and  $J_3$ . For  $(k_x L)^2 < 10^{-2}$  use the first three terms of the three Taylor series

$$J_{1} = L^{3} \sum_{n=0}^{\infty} (-k_{x}^{2}L^{2})^{n} / (2n+3)!$$

$$J_{2} = 2L^{5} \sum_{n=0}^{\infty} (4^{n+1} - 1)(-k_{x}^{2}L^{2})^{n} / (2n+5)!$$

$$J_{3} = (L^{7}/4) \sum_{n=0}^{\infty} [9^{n+3} - 6(4^{n+3}) + 15](-k_{x}^{2}L^{2})^{n} / (2n+7)!$$

For  $(k_x L)^2 > 10^{-2}$  use the formulas

$$J_{1} = (L - s_{x})/k_{x}^{2}$$
  
=  $[L - s(k_{x}, L)]/k_{x}^{2}$   
$$J_{2} = (3L - 4s_{x} + s_{x}c_{x})/(2k_{x}^{4})$$
  
=  $[3L - 4s(k_{x}, L) + s(2k_{x}, L)]/(2k_{x}^{4})$   
$$J_{3} = (15L - 22s_{x} + 9s_{x}c_{x} - 2s_{x}c_{x}^{2})/(6k_{x}^{6})$$
  
=  $[10L - 15s(k_{x}, L) + 6s(2k_{x}, L) - s(3k_{x}, L)]/(4k_{x}^{6})$ 

Now consider the four integrals  $J_c$ ,  $J_s$ ,  $J_d$ , and  $J_f$ . For max  $(k_x^2, 4k_y^2) < 10^{-2}$  use the first three terms of the four Taylor series

 $\sim$ 

$$J_{c} = \sum_{n=0}^{\infty} C_{n} L^{2n+2} / (2n+2)!$$
$$J_{s} = \sum_{n=0}^{\infty} C_{n} L^{2n+3} / (2n+3)!$$

$$J_d = \sum_{n=0}^{\infty} C_n L^{2n+4} / (2n+4)!$$
$$J_f = \sum_{n=0}^{\infty} C_n L^{2n+5} / (2n+5)!$$

with the coefficients

$$C_n = (-1)^n \sum_{m=0}^n (k_x^2)^{n-m} (4k_y^2)^m$$

Note that  $k_x^2$  and  $k_y^2$  cannot be negative at the same time. For  $k_x^2 < 0$  or  $k_y^2 < 0$  use

$$J_{c} = [c(2k_{y}, L) - c(k_{x}, L)]/(k_{x}^{2} - 4k_{y}^{2})$$
  

$$J_{s} = [s(2k_{y}, L) - s(k_{x}, L)]/(k_{x}^{2} - 4k_{y}^{2})$$
  

$$J_{d} = [d(2k_{y}, L) - d(k_{x}, L)]/(k_{x}^{2} - 4k_{y}^{2})$$
  

$$J_{f} = [f(2k_{y}, L) - f(k_{x}, L)]/(k_{x}^{2} - 4k_{y}^{2})$$

For  $k_x^2 > 4k_y^2 > 0$  use

$$J_{c} = s[(k_{x} + 2k_{y})/2, L]s[(k_{x} - 2k_{y})/2, L]/2$$
  

$$J_{s} = \{s[(k_{x} + 2k_{y})/2, L]c[(k_{x} - 2k_{y})/2, L] - c[(k_{x} + 2k_{y})/2, L]s[(k_{x} - 2k_{y})/2, L]\}/4k_{x}k_{y}$$
  

$$J_{d} = [d(2k_{y}, L) - J_{c}]/k_{x}^{2}$$
  

$$J_{f} = [f(2k_{y}, L) - J_{s}]/k_{x}^{2}$$

For  $4k_y^2 > k_x^2 > 0$  use

$$J_{c} = s[(k_{x} + 2k_{y})/2, L]s[(k_{x} - 2k_{y})/2, L]/2$$

$$J_{s} = \{s[(k_{x} + 2k_{y})/2, L]c[(k_{x} - 2k_{y})/2, L]]$$

$$- c[(k_{x} + 2k_{y})/2, L]s[(k_{x} - 2k_{y})/2, L]\}/4k_{x}k_{y}$$

$$J_{d} = [d(k_{x}, L) - J_{c}]/4k_{y}^{2}$$

$$J_{f} = [f(k_{x}, L) - J_{s}]/4k_{y}^{2}$$