THE ERGODICITY OF THE MOTION OF A PARTICLE IN A POTENTIAL FIELD

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1. Introduction

The ergodicity of classical dynamical systems which appear actually in the statistical mechanics was discussed by Ya. G. Sinai [11]. He announced the ergodicity of the dynamical system of particles with central potentials of a special type which are enclosed in a rectangular box. However no proofs have been published yet, except the simplest one-particle model which is called a Sinai billiard system. The ergodicity of the system and the K-property are shown in [13], [8], and the Bernoulli property is shown in [5]. The purpose of this report is to show the ergodicity and the Bernoulli property of the motion of a particle in a potential field. The proofs for a special case of a compound central potential field are separately presented in [9], [10].

2. Observation

Let T be a two dimensional torus and let $U_\iota(q)$, $\iota = 1,2,\cdots,I$, be several potential functions with finite ranges $\overline{Q}_\iota$'s defined on T. Suppose that the potential ranges do not overlap and that the boundary ∂Q_ι of the range $\overline{Q}_\iota$ is a closed curve of C^3-class. Assume that every $U_\iota(q)$ is continuous in the torus T and is continuously differentiable in $\overline{Q}_\iota$.

Observe the motion of a particle with mass m and energy E in the potential field. Denote by $\{S_t\}$ the flow induced from the dynamical system ; that is, for each (q, p) the point $S_t(q, p)$ means the state of the particle at time t whose initial state is (q, p), where $q = (q^{(1)}, q^{(2)}$ and $p = (p^{(1)}, p^{(2)})$ are the position and the momentum. As usual the flow $\{S_t\}$ can be restricted to the energy surface M_E, $M_E \equiv \{(q, p) ; (p^{(1)})^2+(p^{(2)})^2 = 2m(E-U(q)), \quad q \in Q_E\}$, where $Q_E \equiv \{q ; U(q) \le E\}$. Moreover the measure

$$d\mu_E \equiv \text{const. } d\omega dq^{(1)}dq^{(2)}$$

on M_E is invariant under $\{S_t\}$, where $(p^{(1)}, p^{(2)}) = (\kappa\cos\omega, \kappa\sin\omega)$ with $\kappa^2 = 2m(E-U(q))$. Let π be the natural projection from M_E to the configuration space Q_E ; $\pi(q, p) = q$. Put $Q \equiv T - \bigcup_{\iota=1}^{I} \overline{Q}_\iota$ and $M_0 \equiv \pi^{-1}(Q)$. It is easily seen that almost every orbit of the particle crosses the boundary $\partial Q = \bigcup_\iota \partial Q_\iota$ of Q. A point q of the boundary can be parametrized by (ι, r), where ι is the number of the curve ∂Q_ι which contains q, and r is the arclength between the point q and a fixed origin of ∂Q_ι measured along the curve ∂Q_ι clockwise. Further, a point (q, p) in the set $\pi^{-1}(\partial Q)$ can be parametrized by (ι, r, φ), where φ is the angle between p and the inward normal of ∂Q_ι at q. Put

$$M \equiv \{(\iota, r, \varphi) \in \pi^{-1}(\partial Q) ; \tfrac{1}{2}\pi \le \varphi \le \tfrac{3}{2}\pi\},$$

namely M is the set of all incident vectors at ∂Q. Then a transformation T_* of M is defined by that $T_*^{-1}(\iota, r, \varphi) = (\iota_1, r_1, \varphi_1)$ shows the next incident position and angle after starting from (ι, r, φ). Then it is seen that T_* preserves the measure ν of M defined by

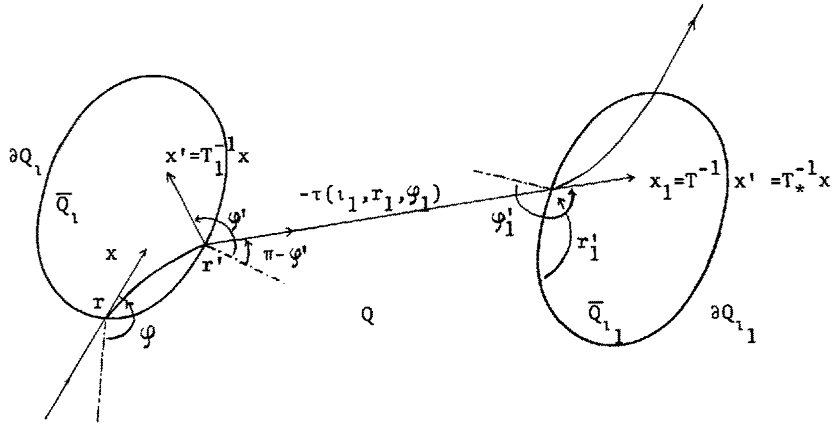
$$d\nu = -\nu_0 \cos\varphi d\varphi dr d\iota$$

with normalized constant ν_0, where $d\iota$ means the unit mass on each ι.

Since it is true that $\{S_t\}$ is ergodic if and only if T_* is ergodic, it is important to investigate the transformation T_*. Then the transformation T_* can be resolved into the product

$$T_* = T_1 T,$$

where T_1 is the transformation such that for $(\iota, r', \varphi') \equiv T_1^{-1}(\iota, r, \varphi)$, the point $(\iota, r', \pi-\varphi')$ shows the position and the momentum at the instant when the particle goes out from $\overline{Q}_\iota$, and T is *the Sinai billiard transformation* which maps $T_*^{-1}(\iota, r, \varphi) = (\iota_1, r_1, \varphi_1)$ to (ι, r', φ').



Generally, let us call a transformation T_* *a perturbed billiard transformation* if T_* is expressed in the form

$$T_* = T_1 T$$

with the Sinai billiard transformation T defined for Q and with a C^2-diffeomorphism T_1 of M which preserves the measure ν and each set $\pi^{-1}(\partial Q_\iota)$, $\iota = 1, 2, \cdots, I$.

Obviously, the transformation T_* induced from $\{S_t\}$ is a perturbed billiard transformation. If the perturbation of T by T_1 is small, then the ergodic theoretical properties of T should be preserved under the perturbation.

3. Theorems

In this section, assume the following assumptions (H-1) ∿ (H-3) :

(H-1) Every $\overline{Q}_\iota$ is a strictly convex domain with the boundary ∂Q_ι of C^3-class.

(H-2) T_1 is a C^2-diffeomorphism of M which preserves the measure ν and satisfies

$$T_1(\iota, r, \tfrac{1}{2}\pi) = (\iota, r, \tfrac{1}{2}\pi) \quad \text{and} \quad T_1(\iota, r, \tfrac{3}{2}\pi) = (\iota, r, \tfrac{3}{2}\pi).$$

Denote by $k(\iota, r)$ the curvature of Q_ι at r and by $-\tau(\iota_1, r_1, \varphi_1)$ the distance between (ι_1, r_1) and (ι, r') measured along the orbit with $(\iota, r', \varphi') = T(\iota_1, r_1, \varphi_1)$. Put $A \equiv \frac{\partial r_1}{\partial r}$, $B \equiv \frac{\partial r_1}{\partial \varphi}$, $C \equiv \frac{\partial \varphi_1}{\partial r}$ and $D \equiv \frac{\partial \varphi_1}{\partial \varphi}$.

(H-3) There exist positive constants η, L_{max}, $L^+_{min}(\iota)$ and $L^-_{min}(\iota)$, $1 \le i \le I$, such that for (ι, r, φ) with $(\iota, r', \varphi') = T_1^{-1}(\iota, r, \varphi)$ and $(\iota_1, r_1, \varphi_1) = T_*^{-1}(\iota, r, \varphi)$ and for $s \ge L^+_{min}(\iota)$ (resp. $s \ge L^-_{min}(\iota_1)$) the following inequalities hold

$$-Cs-D \ge 1+\eta \qquad \left(\text{resp. } -\frac{\cos\varphi_1}{\cos\varphi}(Cs+A) \ge 1+\eta\right),$$

$$L_{max} \ge \frac{As+B}{Cs+D} \ge L^+_{min}(\iota_1) \qquad \left(\text{resp. } L_{max} \ge \frac{Ds+B}{Cs+A} \ge L^-_{min}(\iota)\right),$$

$$\frac{\cos\varphi_1}{\cos\varphi}(A+\tfrac{1}{s}B)(Cs+D) \ge 1+\eta \qquad \left(\text{resp. } \frac{\cos\varphi_1}{\cos\varphi}(D+\tfrac{1}{s}B)(Cs+A) \ge 1+\eta\right).$$

The assumption (H-3) may look rather complicated. However the assumption is satisfied if the perturbation by T_1 is small. For example, the following proposition gives sufficient conditions under which the assumption (H-3) is satisfied. For a function f, put $[f]^+ \equiv \max\{f, 0\}$.

Proposition 1.

(i) If there exists a positive constant η such that

$$-D \ge 1+\eta, \quad -\frac{\cos\varphi_1}{\cos\varphi} A \ge 1+\eta$$

and if $-B \ge 0$, $-C \ge 0$, then the assumption (H-3) is fulfilled.

(ii) If

$$\frac{1}{k_{min}}\left[-\frac{\partial\varphi'}{\partial r}\right]^+ + \left(1 + \frac{1}{k_{min}|\tau|_{min}}\right)\left[-\frac{\partial\varphi'}{\partial\varphi} + 1\right]^+ < 1,$$

$$k_{max}\left(1 + \frac{2}{k_{min}|\tau|_{min}}\right)\left[\frac{\partial\varphi}{\partial r'}\right]^+ + \left(1 + \frac{1}{k_{min}|\tau|_{min}}\right)\left[-\frac{\partial\varphi}{\partial\varphi'} + 1\right]^+ < 1,$$

then the assumption (H-3) is fulfilled, where $k_{min} \equiv \min_{\iota,r} k(\iota, r)$,

$k_{max} \equiv \max_{\iota,r} k(\iota, r)$ and $|\tau|_{min} \equiv \min_{\iota,r,\varphi} |\tau(\iota, r, \varphi)|$.

Theorem 2. Under the assumptions (H-1), (H-2) and (H-3), the transformation $T_* = T_1 T$ is Bernoullian, and hence it is ergodic and a K-system.

Theorem 3. Let $\{S_t\}$ be the flow given in §1 which is governed by potential functions $\{U_\iota\}$. If $\{U_\iota\}$ and the energy E of the particle admit the assumptions (H-1), (H-2) and (H-3), then $\{S_t\}$ is ergodic. Moreover, if $\{S_t\}$ has no point spectrum, then $\{S_t\}$ is a K-system and a Bernoulli flow.

If U_ι's are central potentials, then one can state the sufficient condition for the ergodicity in terms of $\{U_\iota\}$. A central potential $V(s)$ is called *bell-shaped* if

(BS-1) $V(s)$ is continuous for $s > 0$ and $V(s) = 0$ for $s > R$ with some R,

(BS-2) $V(s)$ belongs to C^2-class in $(0, R)$ and there exist left derivatives $V'(R-0)$ and $V''(R-0)$,

(BS-3) $-sV'(s)$ is monotone decreasing and $V'(R-0) < 0$.

Theorem 4. If every U_ι is bell-shaped and if the energy E satisfies the condition

$$0 < E < \frac{1}{4} \min_{\iota}\{- \frac{R_\iota |\tau|_{min}}{R_\iota + |\tau|_{min}} U'(R_\iota - 0)\},$$

then $\{S_t\}$ is ergodic. Moreover if $\{S_t\}$ has no point spectrum, then $\{S_t\}$ is a K-system and a Bernoulli flow.

REFERENCES

[1] Бунимович, Л. А. : Об ергодических свойствах некотрых бильярдов. Функц. анализ. 8(1974), 73-74.

[2] Бунимович, Л. А. : Центральная предельная теорема для одного класса бильярдов. Теор. вероят. и ее примен. 19 (1974), 63-83.

[3] Бунимович, Л. А. : О бильярдах, близких к рассеивающим. Матем. сб. 94(1974), 49-73.

[4] Bunimovich, L.A. = Sinai, Ya.G. : On a fundamental theorem in the theory of dispersing billiards, Math. USSR-Sb. 90 (1973), 407-423.

[5] Galavotti, G.=Ornstein, D.S. : Billiards and Bernoulli schemes (preprint).

[6] Kubo, I. : Ergodicity of the dynamical system of a particle on a domain with irregular walls, Proc. 2nd Japan-USSR Symp. Prob. Theory. Lecture Notes in Math. 330, Springer (1973), 287-295.

[7] Kubo, I. : The motion of a particle in a central field, 数理解析研究所講究録 204, RIMS (1974), 100-109.

[8] Kubo, I. : Billiard systems, Lecture Notes (preparing).

[9] Kubo, I. : Perturbed billiard systems I. The ergodicity of the motion of a particle in a compound central field. (preprint).

[10] Kubo, I. = Murata, H. : Perturbed billiard systems II. The Bernoulli property. (preparing).

[11] Sinai, Ya. G. : On the foundations of the ergodic hypothesis for a dynamical system of statistical mechanics, Sov. Mat. Dokl. 4 No.6 (1963), 1818-1822.

[12] Sinai, Ya. G. : Classical dynamical systems with countably multiple Lebesgue spectra II, A.M.S. Transl. (2) 68 (1968), 34-68.

[13] Sinai, Ya. G. : Dynamical systems with elastic reflections, Russian Math. Surveys 25 (1970), 137-189.