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Hidden Grassmann Structure in the XXZ Model IV: CFT Limit

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Abstract The Grassmann structure of the critical XXZ spin chain is studied in the limit to conformal field theory. A new description of Virasoro Verma modules is proposed in terms of Zamolodchikov's integrals of motion and two families of fermionic creation operators. The exact relation to the usual Virasoro description is found up to level 6.

1 Introduction

In the present paper we continue the series of works (1; 2; 3) on the XXZ model. In (3) we considered it in the presence of the Matsubara direction, or equivalently the six vertex model on a cylinder. We computed the normalised partition function with a defect localised between two horizontal lines, which corresponds to an insertion of a quasi-local operator:

$$Z^\kappa \left\{ q^{2\alpha S(0)} \vartheta \right\} = \frac{\text{Tr}_S \text{Tr}_M \left(T_{S,M} q^{2\kappa S + 2\alpha S(0)} \vartheta \right)}{\text{Tr}_S \text{Tr}_M \left(T_{S,M} q^{2\kappa S + 2\alpha S(0)} \right)}. \quad (1.1)$$

Here $q = e^{\pi i v}$ is related to the coupling parameter (see (2.2) below), and $T_{S,M}$ stands for the monodromy matrix on the two tensor products of evaluation representations of $U_q(\widehat{\mathfrak{sl}}_2)$: one for the horizontal (or 'space') direction S , and another for the vertical (or Matsubara) direction M . For more details see Sect. 2 below, in particular *fig. 1*. It was important in (3) to incorporate inhomogeneities in the

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Matsubara chain. This allows, for example, to consider the temperature expectation values in the spirit of (4; 5), by adjusting inhomogeneities and taking the limit to the infinite chain in the Matsubara direction.

The clue to our calculation was the introduction of operators $\mathbf{t}^*(\zeta), \mathbf{b}^*(\zeta), \mathbf{c}^*(\zeta)$ (2) which, by acting on the primary field $q^{2\alpha S(0)}$, create the space of quasi-local operators on the horizontal chain. More precisely, quasi-local operators are created by Taylor coefficients of $\mathbf{t}^*(\zeta), \mathbf{b}^*(\zeta), \mathbf{c}^*(\zeta)$ at the point $\zeta^2 = 1$. In this paper we change the definition of $\mathbf{b}^*(\zeta), \mathbf{c}^*(\zeta)$ from those of (2; 3) by applying a certain Bogolubov transformation. Compared with the original ones, they have better asymptotic properties. We shall explain this in Sect. 2.

There is an obvious similarity with conformal field theory (CFT), where the descendants are created from the primary field by the action of the Virasoro algebra. Our aim in this paper is to examine the scaling limit of our construction in the critical regime, and to establish its precise relation with CFT. We shall consider the case of a homogeneous Matsubara chain.

The functional (1.1) is non-trivial only for operators \mathcal{O} of spin zero. As it turns out, for the study of the scaling limit, it is quite useful to relax this restriction. We shall first introduce the following generalisation of (1.1) which is of interest on its own right. For $s > 0$ we define

$$Z^{\kappa, s} \left\{ q^{2\alpha S(0)} \mathcal{O} \right\} = \frac{\text{Tr}_S \text{Tr}_M \left(Y_M^{(-s)} T_{S, M} q^{2\kappa S} \mathbf{b}_{\infty, s-1}^* \cdots \mathbf{b}_{\infty, 0}^* \left(q^{2\alpha S(0)} \mathcal{O} \right) \right)}{\text{Tr}_S \text{Tr}_M \left(Y_M^{(-s)} T_{S, M} q^{2\kappa S} \mathbf{b}_{\infty, s-1}^* \cdots \mathbf{b}_{\infty, 0}^* \left(q^{2\alpha S(0)} \right) \right)},$$

where the $\mathbf{b}_{\infty, j}^*$'s denote the coefficients of the singular part of $\mathbf{b}^*(\zeta)$ at $\zeta^2 = 0$. When $s < 0$, a similar definition is in force using the expansion coefficients of $\mathbf{c}^*(\zeta)$. As long as $Y_M^{(-s)}$ is taken generically, this definition is independent of its choice (see Sect. 2 for more details). In general the dependence on $Y_M^{(-s)}$ enters, but only in a ‘‘topological’’ way.

Needless to say, what we are dealing with is a lattice analogue of the screening operators à la Feigin-Fuchs-Dotsenko-Fateev (6; 7). It is interesting to see that quasi-local operators and screening operators both arise from the same operators $\mathbf{b}^*(\zeta), \mathbf{c}^*(\zeta)$, as expansions either around $\zeta^2 = 1$ or $\zeta^2 = 0$.

After these modifications the main formula of (3) remains valid. It reads

$$\begin{aligned} & Z^{\kappa, s} \left\{ \mathbf{t}^*(\zeta_1^0) \cdots \mathbf{t}^*(\zeta_p^0) \mathbf{b}^*(\zeta_1^+) \cdots \mathbf{b}^*(\zeta_r^+) \mathbf{c}^*(\zeta_r^-) \cdots \mathbf{c}^*(\zeta_1^-) \left(q^{2\alpha S(0)} \right) \right\} \\ &= \prod_{i=1}^p 2\rho(\zeta_i^0 | \kappa, \kappa + \alpha, s) \times \det \left(\omega(\zeta_i^+, \zeta_j^- | \kappa, \alpha, s) \right)_{i, j=1, \dots, r}, \end{aligned} \quad (1.2)$$

where the functions $\rho(\zeta_i^0 | \kappa, \kappa + \alpha, s)$ and $\omega(\zeta_i^+, \zeta_j^- | \kappa, \alpha, s)$ are defined by the data of the Matsubara direction. We refer to this formula as the determinant formula. Due to the Bogolubov transformation of \mathbf{b}^* and \mathbf{c}^* the function $\omega(\zeta, \xi | \kappa, \alpha, 0)$ is slightly different from $\omega(\zeta, \xi | \kappa, \alpha)$ used in (3; 8).

Now let us turn to the scaling limit. We have two twisted transfer matrices $T_M(\zeta, \kappa + \alpha)$ and $T_M(\zeta, \kappa)$ in the Matsubara direction. As the number of sites \mathbf{n}

becomes large, their Bethe roots tend to distribute densely on \mathbb{R}_+ . Fixing $R > 0$ and introducing the step of the lattice a , we consider the limit

$$\mathbf{n} \rightarrow \infty, \quad a \rightarrow 0, \quad \mathbf{n}a = 2\pi R \text{ fixed.} \quad (1.3)$$

At the same time we rescale the spectral parameter as

$$\zeta = (Ca)^{\nu} \lambda, \quad \lambda \text{ fixed,} \quad (1.4)$$

so that the Bethe roots close to 0 stay finite in terms of the variable λ . Here C is a constant chosen for fine tuning (see Sect. 8, (8.3)). In this limit the twisted transfer matrices turn into the transfer matrices of chiral CFT on the cylinder $Cyl = \mathbb{C}/2\pi iR\mathbb{Z}$, introduced and studied by Bazhanov, Lukyanov and Zamolodchikov (9; 10). We wish to mention here that the present work owes a great deal to these remarkable papers without which it would have been impossible. Details about the scaling limit can be found in Sect. 8 below. The relevant CFT has central charge

$$c = 1 - \frac{6\nu^2}{1-\nu}.$$

We shall parametrise the conformal dimension as

$$\Delta_{\alpha} = \frac{\nu^2}{4(1-\nu)} ((\alpha-1)^2 - 1),$$

and write the action of the Virasoro algebra on a local field $\phi(y)$ as $\mathbf{L}_n(\phi)(y)$.

In the following we set $\bar{a} = Ca$, and let \lim_{scaling} indicate the scaling limit (1.3), (1.4). The functions entering the determinant formula (1.2) also have finite limits,

$$\begin{aligned} \rho^{\text{sc}}(\lambda | \kappa, \kappa') &= \lim_{\text{scaling}} \rho(\lambda \bar{a}^{\nu} | \kappa, \kappa + \alpha, s), \\ 4 \omega^{\text{sc}}(\lambda, \mu | \kappa, \kappa', \alpha) &= \lim_{\text{scaling}} \omega(\lambda \bar{a}^{\nu}, \mu \bar{a}^{\nu} | \kappa, \alpha, s), \end{aligned}$$

where

$$\kappa' = \kappa + \alpha + 2 \frac{1-\nu}{\nu} s. \quad (1.5)$$

So all these partition functions (1.2) have finite limits. They should have some definite meaning in the context of CFT. We contend that they are the three point functions of the descendants of the chiral primary field $\phi_{\alpha}(0)$, computed in the presence of two other primary fields (or their descendants) inserted at the two ends of the cylinder.

More specifically, we conjecture that the following picture holds true. First, the creation operators tend to a limit,

$$2\tau^*(\lambda) = \lim_{\text{scaling}} \mathbf{t}^*(\lambda \bar{a}^{\nu}), \quad 2\beta^*(\lambda) = \lim_{\text{scaling}} \mathbf{b}^*(\lambda \bar{a}^{\nu}), \quad 2\gamma^*(\lambda) = \lim_{\text{scaling}} \mathbf{c}^*(\lambda \bar{a}^{\nu}).$$

As $\lambda \rightarrow \infty$, these operators have asymptotic expansions of the form

$$\begin{aligned} \log(\tau^*(\lambda)) &\simeq \sum_{j=1}^{\infty} \tau_{2j-1}^* \lambda^{-\frac{2j-1}{\nu}}, \\ \frac{1}{\sqrt{\tau^*(\lambda)}} \beta^*(\lambda) &\simeq \sum_{j=1}^{\infty} \beta_{2j-1}^* \lambda^{-\frac{2j-1}{\nu}}, \quad \frac{1}{\sqrt{\tau^*(\lambda)}} \gamma^*(\lambda) \simeq \sum_{j=1}^{\infty} \gamma_{2j-1}^* \lambda^{-\frac{2j-1}{\nu}}. \end{aligned} \quad (1.6)$$

In the limit, the quasi-local operator $q^{2\alpha S(0)}$ becomes the product of two chiral primary fields $\phi_\alpha(0) \otimes \bar{\phi}_{-\alpha}(0)$. The operators τ_{2j-1}^* and the quadratic combinations $\beta_{2j-1}^* \gamma_{2k-1}^*$ act only on the left component $\phi_\alpha(0)$, and create the entire Verma module spanned by the Virasoro descendants

$$\mathbf{l}_{-m_1} \cdots \mathbf{l}_{-m_s}(\phi_\alpha)(0).$$

Furthermore, if $Y_{\mathbf{M}}^{(-s)}$ is chosen to be generic then

$$\begin{aligned} \lim_{\text{scaling}} Z_R^{\kappa, s} \left\{ \mathbf{t}^*(\zeta_1^0) \cdots \mathbf{t}^*(\zeta_p^0) \mathbf{b}^*(\zeta_1^+) \cdots \mathbf{b}^*(\zeta_r^+) \mathbf{c}^*(\zeta_r^-) \cdots \mathbf{c}^*(\zeta_1^-) \left(q^{2\alpha S(0)} \right) \right\} \\ = 2^{p+2r} Z_R^{\kappa, \kappa'} \left\{ \tau^*(\lambda_1^0) \cdots \tau^*(\lambda_p^0) \beta^*(\lambda_1^+) \cdots \beta^*(\lambda_r^+) \gamma^*(\lambda_r^-) \cdots \gamma^*(\lambda_1^-) (\phi_\alpha(0)) \right\}. \end{aligned} \quad (1.7)$$

In the right hand side, the symbol $Z_R^{\kappa, \kappa'} \{X(0)\}$ stands for the three point function normalised as $Z_R^{\kappa, \kappa'} \{\phi_\alpha(0)\} = 1$, with $X(0)$ inserted at $x=0$ and the primary fields $\phi_{\kappa+1}, \phi_{-\kappa'+1}$ being inserted at $x=\infty$ and $x=-\infty$, respectively (see (6.8) below). Non-generic choice of $Y_{\mathbf{M}}^{(-s)}$ corresponds to replacing the primary fields at $x=\pm\infty$ by their descendants. In this paper we discuss only the case of generic $Y_{\mathbf{M}}^{(-s)}$.

The coefficients in (1.6) are homogeneous operators in the sense that

$$[\mathbf{l}_0, \tau_{2j-1}^*] = (2j-1) \tau_{2j-1}^*, \quad [\mathbf{l}_0, \beta_{2i-1}^* \gamma_{2j-1}^*] = (2i+2j-2) \beta_{2i-1}^* \gamma_{2j-1}^*.$$

Hence, for each degree, the descendants created by \mathbf{l}_{-k} 's, and those created by τ_{2j-1}^* 's, β_{2j-1}^* 's and γ_{2j-1}^* 's, must be finite linear combinations of each other. The main goal of this paper is to show, for low degrees, that this is indeed the case, and that the coefficients can be found explicitly.

To determine the coefficients of the linear combination, we compare the values of $Z_R^{\kappa, \kappa'}$. For the Virasoro descendants, they can be easily computed by the conformal Ward-Takahashi identities. For the descendants by τ_{2j-1}^* and others, we need the coefficients of the asymptotic expansion of the functions $\rho_R(\lambda|\kappa, \kappa')$ and $\omega_R(\lambda, \mu|\kappa, \kappa', \alpha)$. In Sect. 10 we develop a systematic method for computing them. We note that in both cases the results are polynomials in the conformal dimensions $\Delta_{\kappa+1}, \Delta_{\kappa'+1}$. We may regard them as independent variables and compare the coefficients, since s in (1.5) can take any integer values. This was one of reasons for us to introduce the screening operators.

We consider first τ_{2j-1}^* . CFT allows an integrable structure based on Zamolodchikov's integrals of motion \mathbf{i}_{2m-1} (11). With the above procedure we are led to a result which should not be surprising,

$$\tau_{2m-1}^* = C_m \cdot \mathbf{i}_{2m-1},$$

where C_m are some ν -dependent constants which can be found in (10).

We then consider the action of β_{2j-1}^* 's and γ_{2j-1}^* 's. Because of a technical difficulty we have not been able to compute the asymptotics of $\omega_R(\lambda, \mu|\kappa, \kappa', \alpha)$ for $\kappa \neq \kappa'$. Here we restrict to the case $\kappa = \kappa'$. Since $Z_R^{\kappa, \kappa}(\mathbf{i}_{2m-1}(X)) = 0$ for any

X , restricting to $\kappa = \kappa'$ means that we consider the quotient space of the Verma module modulo the action of the integrals of motion. We assume that the vectors

$$\mathbf{i}_{2k_1-1} \cdots \mathbf{i}_{2k_r-1} \mathbf{I}_{-2m_1} \cdots \mathbf{I}_{-2m_s}(\phi_\alpha(0))$$

span the Verma module, so the quotient space is created by the \mathbf{I}_{-2m} 's. With primary fields as asymptotical states, we can compare up to the level 6. It should be added, however, that up to this level the system of equations is overdetermined. So the very possibility of finding a solution is the strongest support of our fermionic picture.

We give one example on the level 4:

$$\beta_1^* \gamma_3^*(\phi_\alpha(0)) = \frac{1}{2} D_1(\alpha) D_3(2-\alpha) \left(\mathbf{I}_{-2}^2 + \frac{2c-32-4d_\alpha}{9} \mathbf{I}_{-4} \right) (\phi_\alpha(0)),$$

where

$$d_\alpha = \frac{1}{6} \sqrt{(25-c)(24\Delta_\alpha + 1 - c)},$$

$$D_{2n-1}(\alpha) = \sqrt{\frac{i}{v}} \Gamma(v)^{-\frac{2n-1}{v}} (1-v)^{\frac{2n-1}{2}} \cdot \frac{1}{(n-1)!} \frac{\Gamma\left(\frac{\alpha}{2} + \frac{1}{2v}(2n-1)\right)}{\Gamma\left(\frac{\alpha}{2} + \frac{(1-v)}{2v}(2n-1)\right)}.$$

In the right-hand side, we have a particular combination of the Virasoro descendants. This equation says that its three-point function remains of the same determinant form before and after integrable perturbation.

The text is organised as follows. In Sect. 2 we review the results of (3) and describe the Bogolubov transformation mentioned above. In Sect. 3 we define the functions ρ and ω . In Sect. 4 we define screening operators on the lattice, and describe a generalisation of the previous results. In Sect. 5 we start discussing the scaling limit of the XXZ chain, examining the behaviour of the Bethe roots in the Matsubara direction as the length of the chain becomes infinite. Sections 6 and 7 are a review of the CFT integrals of motion on the cylinder, and the series of works of Bazhanov, Lukyanov and Zamolodchikov (BLZ). We explain in Sect. 8 how the Matsubara transfer matrix turns into that of BLZ in the continuous limit. In Sect. 9 we discuss the CFT interpretation of the scaling limit in the space direction. In Sect. 10 we study the asymptotics of Thermodynamic Bethe Ansatz (TBA) function \mathfrak{a} for CFT. In Sect. 11 we find the asymptotical expansion of ω for $\kappa = \kappa'$. In Sect. 12 we compare descendants created by \mathbf{I}_{-2m} with those created by β_{2j-1}^* 's and γ_{2j-1}^* and give some concluding remarks. In the Appendix we present general properties of asymptotics of ω which apply to the case $\kappa \neq \kappa'$.

2 Review of Previous Results

Let us start with a brief review of the papers (1; 2; 3). Consider the XXZ spin chain in the infinite volume. The space of states of the model is

$$\mathfrak{H}_S = \bigotimes_{j=-\infty}^{\infty} \mathbb{C}^2,$$

and the Hamiltonian is given by

$$H = \frac{1}{2} \sum_{k=-\infty}^{\infty} (\sigma_k^1 \sigma_{k+1}^1 + \sigma_k^2 \sigma_{k+1}^2 + \Delta \sigma_k^3 \sigma_{k+1}^3), \quad \Delta = \frac{1}{2}(q + q^{-1}). \quad (2.1)$$

We consider the critical XXZ model in the following range of the coupling:

$$q = e^{\pi i \nu}, \quad 1/2 < \nu < 1. \quad (2.2)$$

Together with \mathfrak{H}_S we consider the Matsubara space \mathfrak{H}_M . In (3) the most general case was treated: namely, \mathfrak{H}_M was the tensor product of spaces of different dimensions, and to every site \mathbf{m} an independent inhomogeneity parameter $\tau_{\mathbf{m}}$ was attached. In the present paper we shall restrict ourselves to the case

Fig. 1 Partition function on the cylinder. The functional Z^K is a ratio of two partition functions on the cylinder. On each crossing of a row and a column, one associates the Boltzmann weights of the six vortex model. On a particular row there are also twist fields $q^{\kappa\sigma_j^3}$ (marked by *crosses*) and $q^{(\sigma+\kappa)\sigma_j^3}$ (marked by *circles*). The numerator of Z^K corresponds to a lattice with defects representing an insertion of a local operator

$$\mathfrak{H}_{\mathbf{M}} = \bigotimes_{j=1}^{\mathbf{n}} \mathbb{C}^2.$$

We consider the monodromy matrix $T_{S,\mathbf{M}}$. Mathematically this is nothing but the image of the universal R matrix of $U_q(\widehat{\mathfrak{sl}}_2)$ on the tensor product of evaluation representations corresponding to \mathfrak{H}_S and $\mathfrak{H}_{\mathbf{M}}$. It has been said that we shall consider a homogeneous Matsubara chain only. In the notation of (3), this corresponds to setting $\tau_{\mathbf{m}} = q^{\frac{1}{2}}$ for all \mathbf{m} . We shall absorb this into redefinition of the L -operator comparing to (3). Let us write the definition explicitly:

$$T_{S,\mathbf{M}} = \overset{\curvearrowright}{\prod}_{j=-\infty}^{\infty} T_{j,\mathbf{M}},$$

where

$$T_{j,\mathbf{M}} \equiv T_{j,\mathbf{M}}(1), \quad T_{j,\mathbf{M}}(\zeta) = \overset{\curvearrowright}{\prod}_{\mathbf{m}=1}^{\mathbf{n}} L_{j,\mathbf{m}}(\zeta),$$

with

$$L_{j,\mathbf{m}}(\zeta) = q^{-\frac{1}{2}\sigma_j^3\sigma_{\mathbf{m}}^3} - \zeta^2 q^{\frac{1}{2}\sigma_j^3\sigma_{\mathbf{m}}^3} - \zeta(q - q^{-1})(\sigma_j^+ \sigma_{\mathbf{m}}^- + \sigma_j^- \sigma_{\mathbf{m}}^+).$$

A local operator \mathcal{O} on \mathfrak{H}_S is by definition an operator which acts nontrivially only on a finite number of the tensor components \mathbb{C}^2 of \mathfrak{H}_S . More generally we consider quasi-local operators with tail α , which are operators of the form

$$q^{2\alpha S(0)} \mathcal{O}, \quad S(k) = \frac{1}{2} \sum_{j=-\infty}^k \sigma_j^3,$$

with \mathcal{O} being local. In this notation $S = S(\infty)$ is the total spin. In (3) we computed the expectation values defined by

$$Z^K \left\{ q^{2\alpha S(0)} \mathcal{O} \right\} = \frac{\mathrm{Tr}_S \mathrm{Tr}_{\mathbf{M}} \left(T_{S,\mathbf{M}} q^{2\kappa S + 2\alpha S(0)} \mathcal{O} \right)}{\mathrm{Tr}_S \mathrm{Tr}_{\mathbf{M}} \left(T_{S,\mathbf{M}} q^{2\kappa S + 2\alpha S(0)} \right)}. \quad (2.3)$$

This is a linear functional on the space $\mathscr{W}_{\alpha,0}$ of spinless quasi-local operators with tail α .

It is helpful to think of the functional Z^K as a ratio of partition functions of the six vertex model on the cylinder. For example, the numerator of (2.3) can be presented graphically as follows:

On this picture the summation is performed over all edges except the broken ones in the middle. The arrows on the broken edges are fixed, representing the local operator \mathcal{O} . The one dimensional sublattice going in the infinite space direction will be called the space chain, while the compact one dimensional sublattice in the Matsubara direction will be referred to as the Matsubara chain.

We computed the expectation values (2.3) using the fermionic description of $\mathcal{W}_{\alpha,0}$ found in (2). Let us briefly recall it. Consider the space

$$\mathscr{W}^{(\alpha)} = \bigoplus_{s=-\infty}^{\infty} \mathscr{W}_{\alpha-s,s},$$

where $\mathscr{W}_{\alpha-s,s}$ denotes the space of quasi-local operators of spin s with tail $\alpha - s$.

We have the creation operators $\mathbf{t}^*(\zeta), \mathbf{b}^*(\zeta), \mathbf{c}^*(\zeta)$ and the annihilation operators $\mathbf{b}(\zeta), \mathbf{c}(\zeta)$ which act on $\mathscr{W}^{(\alpha)}$. To be precise the operators $\mathbf{t}^*(\zeta), \mathbf{b}^*(\zeta), \mathbf{c}^*(\zeta)$ were defined in (2) as formal power series in $\zeta^2 - 1$, the quasi-local operators in $\mathscr{W}^{(\alpha)}$ are created by coefficients of these series. However, when the series are substituted into Z^κ the result allows analytical continuation. So, in the present paper we shall adopt another point of view which is similar to that of CFT. Namely, we shall consider the operators $\mathbf{t}^*(\zeta), \mathbf{b}^*(\zeta), \mathbf{c}^*(\zeta)$ as analytical functions. Then the relation to real quasi-local operators is achieved by considering $\mathbf{t}^*(\zeta), \mathbf{b}^*(\zeta), \mathbf{c}^*(\zeta)$ around the point $\zeta^2 = 1$.

The creation-annihilation operators have the block structures

$$\begin{aligned} \mathbf{t}^*(\zeta) &: \mathscr{W}_{\alpha-s,s} \rightarrow \mathscr{W}_{\alpha-s,s}, \\ \mathbf{b}^*(\zeta), \mathbf{c}(\zeta) &: \mathscr{W}_{\alpha-s+1,s-1} \rightarrow \mathscr{W}_{\alpha-s,s}, \\ \mathbf{c}^*(\zeta), \mathbf{b}(\zeta) &: \mathscr{W}_{\alpha-s-1,s+1} \rightarrow \mathscr{W}_{\alpha-s,s}. \end{aligned} \quad (2.4)$$

The operator $\tau = \mathbf{t}_1^*/2$ plays a special role: it is the right shift by one site along the space chain.

We have

$$\begin{aligned} \mathbf{b}(\zeta)(q^{2\alpha S(0)}) &= 0, & \mathbf{c}(\zeta)(q^{2\alpha S(0)}) &= 0, \\ [\mathbf{c}(\xi), \mathbf{c}^*(\zeta)]_+ &= \psi(\xi/\zeta, \alpha), & [\mathbf{b}(\xi), \mathbf{b}^*(\zeta)]_+ &= -\psi(\zeta/\xi, \alpha), \end{aligned}$$

where

$$\psi(\zeta, \alpha) = \zeta^\alpha \frac{\zeta^2 + 1}{2(\zeta^2 - 1)}. \quad (2.5)$$

The operators in the space $\mathscr{W}_{\alpha,0}$ are created from the primary field $q^{2\alpha S(0)}$ by action of \mathbf{t}^* 's and of equal number of \mathbf{b}^* 's and \mathbf{c}^* 's. The completeness (12) says that the entire space $\mathscr{W}_{\alpha,0}$ is generated by coefficients of the creation operators considered as series in $\zeta^2 - 1$. Certainly, this description is reminiscent of CFT.

The main result of (3) is the following relations which allow for recursive computations of the expectation values:

$$Z^\kappa \{\mathbf{t}^*(\zeta)(X)\} = 2\rho(\zeta|\kappa, \kappa + \alpha) Z^\kappa \{X\}, \quad (2.6)$$

$$Z^\kappa \{\mathbf{b}^*(\zeta)(X)\} = \frac{1}{2\pi i} \oint_\Gamma \omega(\zeta, \xi|\kappa, \alpha) Z^\kappa \{\mathbf{c}(\xi)(X)\} \frac{d\xi^2}{\xi^2}, \quad (2.7)$$

$$Z^\kappa \{\mathbf{c}^*(\zeta)(X)\} = -\frac{1}{2\pi i} \oint_\Gamma \omega(\xi, \zeta|\kappa, \alpha) Z^\kappa \{\mathbf{b}(\xi)(X)\} \frac{d\xi^2}{\xi^2}, \quad (2.8)$$

where Γ goes around all the singularities of the integrand except $\xi^2 = \zeta^2$. We think no further explanation is needed nowadays when the method of CFT is a part of common knowledge.

The functions $\rho(\zeta|\kappa, \kappa + \alpha)$ and $\omega(\zeta, \xi|\kappa, \alpha)$ will be defined in Sect. 3. We changed the notation for the former from (3). The present notation agrees better with the explicit formula given below. The set of equations (2.6), (2.7), (2.8) implies a determinant representation for the expectation values (3), we shall discuss this later.

Let us describe the modification of $\mathbf{b}^*(\zeta), \mathbf{c}^*(\zeta)$ by a Bogolubov transformation which was mentioned in the Introduction. Denoting the operators used in (2; 3) by $\mathbf{b}_{\text{rat}}^*(\zeta), \mathbf{c}_{\text{rat}}^*(\zeta)$, define

$$\begin{aligned}\mathbf{b}^*(\zeta) &= \mathbf{b}_{\text{rat}}^*(\zeta) + \frac{1}{2\pi i} \int_{\Gamma} D_{\zeta} D_{\xi} \Delta_{\zeta}^{-1} \psi(\zeta/\xi, \alpha) \cdot \mathbf{c}(\xi) \frac{d\xi^2}{\xi^2}, \\ \mathbf{c}^*(\zeta) &= \mathbf{c}_{\text{rat}}^*(\zeta) - \frac{1}{2\pi i} \int_{\Gamma} D_{\zeta} D_{\xi} \Delta_{\zeta}^{-1} \psi(\xi/\zeta, \alpha) \cdot \mathbf{b}(\xi) \frac{d\xi^2}{\xi^2},\end{aligned}\quad (2.9)$$

where D_{ζ} is the following finite difference operator of the second order:

$$D_{\zeta} f(\zeta) = f(\zeta q) + f(\zeta q^{-1}) - \mathbf{t}^*(\zeta) f(\zeta).$$

The function $\Delta_{\zeta}^{-1} \psi(\zeta, \alpha)$ is transcendental. For

$$\zeta^2 > 0, \quad -\frac{1}{v} < \text{Re } \alpha < 0$$

we define it as

$$\Delta_{\zeta}^{-1} \psi(\zeta, \alpha) = -VP \int_0^{\infty} \frac{1}{2v \left(1 + (\zeta/\eta)^{\frac{1}{v}}\right)} \psi(\eta, \alpha) \frac{d\eta^2}{2\pi i \eta^2}, \quad (2.10)$$

where the principal value is taken with regards to the pole at $\eta^2 = 1$. In general we define it by analytic continuation with respect to both α and ζ^2 , obtaining a meromorphic function of $\log \zeta$. It is bounded at $\log \zeta \rightarrow \pm\infty$, and its singularities closest to the real axis are the simple poles at $\log \zeta = \pm\pi i v$ with residues of opposite signs.

The function $D_{\zeta} D_{\xi} \Delta_{\zeta}^{-1} \psi(\zeta/\xi, \alpha)$ is regular at $\zeta = \xi$, so the Taylor series for $\mathbf{b}^*(\zeta), \mathbf{c}^*(\zeta)$ at $\zeta^2 = 1$ are well-defined. The function ω changes following the change of $\mathbf{b}^*, \mathbf{c}^*$:

$$\omega(\zeta, \xi|\kappa, \alpha) = \omega_{\text{rat}}(\zeta, \xi|\kappa, \alpha) + \overline{D}_{\zeta} \overline{D}_{\xi} \Delta_{\zeta}^{-1} \psi(\zeta/\xi, \alpha), \quad (2.11)$$

where

$$\overline{D}_{\zeta} f(\zeta) = f(\zeta q) + f(\zeta q^{-1}) - 2\rho(\zeta|\kappa, \kappa + \alpha) f(\zeta). \quad (2.12)$$

Let us mention one marvellous property of our modified operators $\mathbf{b}^*, \mathbf{c}^*$.

The main subject of our original study (1; 2) was the following normalised matrix element

$$Z_{\infty}\{q^{2\alpha S(0)} \mathcal{O}\} = \frac{\langle \text{vac} | q^{2\alpha S(0)} \mathcal{O} | \text{vac} \rangle}{\langle \text{vac} | q^{2\alpha S(0)} | \text{vac} \rangle}, \quad (2.13)$$

where $|\text{vac}\rangle$ denotes the ground state of the XXZ Hamiltonian in the infinite volume. Equivalently, the numerator of (2.13) is the partition function of the six vertex model on the plane with a defect localised between two horizontal lines.

It is easy to see using the formulae from (1; 2) that, if we create the quasi-local fields by the operators $\mathbf{t}^*(\zeta), \mathbf{b}^*(\zeta), \mathbf{c}^*(\zeta)$, then Z_∞ vanishes on all of them with the sole exception of the descendants created by $\tau = \mathbf{t}_1^*/2$. For the latter we have

$$Z_\infty\{\tau^m(q^{2\alpha S(0)}\mathcal{O})\} = 1, \quad m \in \mathbb{Z}.$$

We know from the algebraic construction (2) that the Taylor coefficients of the part $\mathbf{b}_{\text{rat}}^*(\zeta), \mathbf{c}_{\text{rat}}^*(\zeta)$ in (2.9) produce only rational functions of q and q^α . Hence all transcendental pieces of the expectation values (2.9) come from rewriting a given quasi-local operator in the fermionic basis and picking Taylor coefficients of the function $\bar{D}_\zeta \bar{D}_\xi \Delta_\zeta^{-1} \psi(\zeta/\xi, \alpha)$.

The property of $\mathbf{t}^*(\zeta)$ (except \mathbf{t}_1^*) and $\mathbf{b}^*(\zeta), \mathbf{c}^*(\zeta)$ mentioned above is analogous to that of the Virasoro generators in CFT: on the Riemann sphere, all normalised one point functions of the descendants vanish due to the conformal invariance. Strictly speaking, the one point function of the primary field also vanishes, so a word of clarification is necessary. Take a massive model with a mass scale m and consider the conformal limit $m \rightarrow 0$. While the normalised one point function of the primary field stays equal to 1, those of the descendants vanish in the limit, because for dimensional reasons they contain additional powers of m .

At this point one may wonder why we did not do the Bogolubov transformation killing completely the function ω in (2.7), (2.8). The answer is that it is impossible to rewrite the left hand sides of (2.7), (2.8) because the function ω cannot be written as a function of ρ .

3 Functions ρ and ω

Introduce the twisted Matsubara transfer matrix:

$$T_{\mathbf{M}}(\zeta, \kappa) = \text{Tr}_j \left(T_{j, \mathbf{M}}(\zeta) q^{\kappa \sigma_j^3} \right). \quad (3.1)$$

Let $|\kappa\rangle$ be the eigenvector of $T(1, \kappa)$ whose eigenvalue is maximal in the absolute value. Similarly let $\langle \kappa + \alpha |$ be the eigenvector of $T(1, \kappa + \alpha)$ whose eigenvalue is maximal in the absolute value. It is well-known that these eigenvectors have spin 0. We call them the maximal eigenvectors, and assume that they are not orthogonal. We denote the eigenvalues of $T_{\mathbf{M}}(\zeta, \kappa)$ (resp. $T_{\mathbf{M}}(\zeta, \kappa + \alpha)$) on $|\kappa\rangle$ (resp. $\langle \kappa + \alpha |$) by $T(\zeta, \kappa)$ (resp. $T(\zeta, \kappa + \alpha)$).

Then $\rho(\zeta|\kappa, \kappa + \alpha)$ is defined by $T(\zeta, \kappa)$ and $T(\zeta, \kappa + \alpha)$. We have

$$\rho(\zeta|\kappa, \kappa + \alpha) = \frac{T(\zeta, \kappa + \alpha)}{T(\zeta, \kappa)}. \quad (3.2)$$

The function $\omega(\zeta, \xi|\kappa, \alpha)$ is more complicated. In (3) it was shown that it is completely determined by two requirements the singular part and the normalisation condition. Then they were explicitly solved in terms of q -deformed Abelian integrals. In the present paper it is convenient to use an alternative, TBA-like, description used in (8). Let us explain this simplifying a little the notation of (8).

Together with the transfer matrix $T_{\mathbf{M}}(\zeta, \kappa)$ we consider in (3) Baxter's Q -operators $Q_{\mathbf{M}}^{\pm}(\zeta, \kappa)$. In this paper we use only one of them: $Q_{\mathbf{M}}^{-}(\zeta, \kappa)$, denoting it just as $Q_{\mathbf{M}}(\zeta, \kappa)$. It is defined by the trace over the highest weight representation of the q -oscillator algebra with generators $\mathbf{a}, \mathbf{a}^*, D$ (see (2) for notation),

$$Q_{\mathbf{M}}(\zeta, \kappa) = \zeta^{-\kappa + S_{\mathbf{M}}} (1 - q^{-2(\kappa - S_{\mathbf{M}})}) \text{Tr}^{-} \left(T_{Osc, \mathbf{M}}^{-}(\zeta) q^{-2\kappa D_A} \right), \quad (3.3)$$

where

$$T_{Osc, \mathbf{M}}^{-}(\zeta) = \overset{\curvearrowright}{\prod_{\mathbf{m}=1}^{\mathbf{n}}} L_{Osc, \mathbf{m}}^{-}(\zeta),$$

$$L_{Osc, \mathbf{m}}^{-}(\zeta) = \left(I - \zeta^2 q^{2D+1} \sigma_{\mathbf{m}}^{-} \sigma_{\mathbf{m}}^{+} - \zeta q^{-\frac{1}{2}} (\mathbf{a} \sigma_{\mathbf{m}}^{-} + \mathbf{a}^* \sigma_{\mathbf{m}}^{+}) \right) q^{D\sigma_{\mathbf{m}}^3}.$$

Its eigenvalue on the eigenvector discussed above will be denoted by $Q(\zeta, \kappa)$. The main role in TBA is played by the function:

$$\mathfrak{a}(\zeta, \kappa) = \frac{d(\zeta)Q(\zeta q, \kappa)}{a(\zeta)Q(\zeta q^{-1}, \kappa)}, \quad (3.4)$$

where

$$a(\zeta) = (1 - q\zeta^2)^{\mathbf{n}}, \quad d(\zeta) = (1 - q^{-1}\zeta^2)^{\mathbf{n}}.$$

It follows from the Baxter equation that the solutions to the equation

$$\mathfrak{a}(\zeta, \kappa) = -1,$$

are the zeros of either $Q(\zeta, \kappa)$ or $T(\zeta, \kappa)$. The function $\mathfrak{a}(\zeta, \kappa)$ satisfies the non-linear integral equation (13; 4; 14):

$$\log \mathfrak{a}(\zeta, \kappa) = -2\pi i \nu \kappa + \log \left(\frac{d(\zeta)}{a(\zeta)} \right) - \int_{\gamma} K(\zeta/\xi) \log(1 + \mathfrak{a}(\xi, \kappa)) \frac{d\xi^2}{\xi^2}, \quad (3.5)$$

where the cycle γ goes around the zeros of $Q(\zeta, \kappa)$ (Bethe roots) in the *clockwise* direction, as opposed to all other contours.

We shall need a slightly more general kernel than $K(\zeta/\xi)$, so, let us define them together. First, we introduce operations:

$$\begin{aligned} \Delta_{\zeta} f(\zeta) &= f(\zeta q) - f(\zeta q^{-1}), \\ \delta_{\zeta}^{-} f(\zeta) &= f(\zeta q) - \rho(\zeta | \kappa, \kappa + \alpha) f(\zeta). \end{aligned} \quad (3.6)$$

Then

$$K(\zeta, \alpha) = \frac{1}{2\pi i} \Delta_{\zeta} \Psi(\zeta, \alpha), \quad K(\zeta) = K(\zeta, 0). \quad (3.7)$$

We shall use the the following notation:

$$f \star g = \int_{\gamma} f(\eta) g(\eta) dm(\eta),$$

where the measure is given by

$$dm(\eta) = \frac{d\eta^2}{\eta^2 \rho(\eta|\kappa, \kappa + \alpha)(1 + \mathfrak{a}(\eta, \kappa))}. \quad (3.8)$$

Now we introduce the resolvent of a certain integral operator

$$R_{\text{dress}} - R_{\text{dress}} \star K_\alpha = K_\alpha, \quad (3.9)$$

where K_α stands for the integral operator with the kernel $K(\zeta/\xi, \alpha)$. Introducing further the two kernels

$$f_{\text{left}}(\zeta, \xi) = \frac{1}{2\pi i} \delta_\zeta^- \psi(\zeta/\xi, \alpha), \quad f_{\text{right}}(\zeta, \xi) = \delta_\xi^- \psi(\zeta/\xi, \alpha), \quad (3.10)$$

we are ready to write the definition of (8) cleaning it from irrelevant auxiliary objects and taking into account the modification (2.11):

$$\begin{aligned} \frac{1}{4} \omega(\zeta, \xi | \kappa, \alpha) &= (f_{\text{left}} \star f_{\text{right}} + f_{\text{left}} \star R_{\text{dress}} \star f_{\text{right}})(\zeta, \xi) \\ &\quad + \delta_\zeta^- \delta_\xi^- \Delta_\zeta^{-1} \psi(\zeta/\xi, \alpha). \end{aligned} \quad (3.11)$$

4 Introducing Screening Operators on the Lattice

Now we want to describe an important generalisation of results of (3). We have three constants: α which defines the tail of the operator and $\kappa, \kappa + \alpha$ which defines the twist of Matsubara transfer matrices at $+\infty$ and $-\infty$. As it has been said in the Introduction we need more freedom. Let us explain how an additional parameter $s \in \mathbb{Z}$ can be introduced into Z^κ . We shall see later that in the scaling limit introduction of this parameter leads to emancipation of $\kappa + \alpha$ from κ and α .

Let us consider the trace

$$\text{Tr}_S \text{Tr}_M \left(Y_M^{(-s)} T_{S, M} q^{2\kappa S + 2(\alpha - s)S(0)} \mathcal{O}^{(s)} \right), \quad (4.1)$$

where $Y_M^{(-s)}$ carries spin $-s$. For definiteness we suppose $s > 0$. It follows from the ice condition that in this situation the operator $\mathcal{O}^{(s)}$ must have spin s .

We assume that among the eigenvectors of the transfer matrices $T_M(\zeta, \kappa)$ and $T_M(\zeta, \kappa + \alpha - s)$ there are maximal ones $|\kappa\rangle, |\kappa + \alpha - s, s\rangle$ with eigenvalues $T(\zeta, \kappa)$ and $T(\zeta, \kappa + \alpha - s, s)$ which are defined by the requirement that $T(1, \kappa) \cdot T(1, \kappa + \alpha - s, s)$ is of maximal absolute value among all the pairs of eigenvectors. We assume further the generality condition:

$$\langle \kappa | Y_M^{(-s)} | \kappa + \alpha - s, s \rangle \neq 0. \quad (4.2)$$

Obviously the difference between spins of $|\kappa + \alpha - s, s\rangle$ and $|\kappa\rangle$ must be equal to s . We make the technical assumption that a spin of $|\kappa\rangle$ remains equal to zero.

The natural idea is to create the operators $q^{2(\alpha - s)S(0)} \mathcal{O}^{(s)}$ by $\mathbf{b}^*, \mathbf{c}^*, \mathbf{t}^*$ having an excess of operators \mathbf{b}^* :

$$\begin{aligned} q^{2(\alpha - s)S(0)} \mathcal{O}^{(s)} &= \mathbf{b}^*(\xi_1) \cdots \mathbf{b}^*(\xi_s) \\ &\quad \times \mathbf{b}^*(\zeta_m^+) \cdots \mathbf{b}^*(\zeta_1^+) \mathbf{c}^*(\zeta_1^-) \cdots \mathbf{c}^*(\zeta_m^-) \mathbf{t}^*(\zeta_1^0) \cdots \mathbf{t}^*(\zeta_n^0) \left(q^{2\alpha S(0)} \right). \end{aligned}$$

However, the formulae (2.7) are not applicable in this case. Let us explain why it is so.

The method of (3) requires to start the consideration by the operator $\mathbf{b}^*(\xi_1)$ which is the closest to $T_{S,M}q^{2\kappa S}$. The operator $\xi_1^{-\alpha}\mathbf{b}_{\text{rat}}^*(\xi_1)(X)$ is a meromorphic function of ξ_1^2 with singularities at the points $(\zeta_j^-)^2$. It satisfies $\mathbf{n} + 1$ normalisation conditions (3). The additional term in (2.9) is of the form satisfying (2.7) from the very beginning. These singularities and normalisation conditions are studied algebraically, they do not change comparing to (3) where the spin of X was equal to -1 . However, the behaviour at zero changes: in the present case the spin of X equals $s - 1$, and according to (2) $\xi_1^{-\alpha}\mathbf{b}^*(\xi_1)(X)$ does not vanish at zero. Generally,

$$\zeta^{-\alpha}\mathbf{b}^*(\zeta)(X) = \sum_{j=0}^{s-1} \zeta^{-2j}\mathbf{b}_{\infty,j}^*(X) + \zeta^{-\alpha}\mathbf{b}_{\text{reg}}^*(\zeta)(X), \quad \text{spin}(X) = s - 1, \quad (4.3)$$

where $\zeta^{-\alpha}\mathbf{b}_{\text{reg}}^*(\zeta)(X)$ vanishes at 0. But in (3) the fact that $\zeta^{-\alpha}\mathbf{b}^*(\zeta)(X)$ vanishes at 0 for $\text{spin}(X) = -1$ was important when deriving (2.7). As a result in the present case we do not have enough conditions to define ω .

Let us turn this problem into advantage. The operators $\mathbf{b}_{\infty,j}^*, j = 0, \dots, s - 1$ constitute a finite Grassmann algebra. So, we just consider $\xi_j \rightarrow 0$ and replace $\mathbf{b}^*(\xi_1) \dots \mathbf{b}^*(\xi_s)$ by $\mathbf{b}_{\infty,s-1}^* \dots \mathbf{b}_{\infty,0}^*$. Now move one of the remaining operators \mathbf{b}^* , namely, $\mathbf{b}^*(\zeta_1^+)$ to the left. Obviously, $(\zeta_1^+)^{-\alpha}\mathbf{b}^*(\zeta_1^+)$ vanishes as $(\zeta_1^+)^2 \rightarrow 0$ because the singular part disappears due to multiplication by $\mathbf{b}_{\infty,s-1}^* \dots \mathbf{b}_{\infty,0}^*$. Effectively, this operator reduces to $\mathbf{b}_{\text{reg}}^*(\zeta_1^+)$.

It is not hard to see that we can introduce $\omega(\zeta, \xi)$ and obtain (2.6), (2.7), (2.8) for the functional on $\mathscr{W}_{\alpha,0}$:

$$Z^{\kappa,s} \left\{ q^{2\alpha S(0)} \mathcal{O} \right\} = \frac{\text{Tr}_S \text{Tr}_M \left(Y_M^{(-s)} T_{S,M} q^{2\kappa S} \mathbf{b}_{\infty,s-1}^* \dots \mathbf{b}_{\infty,0}^* \left(q^{2\alpha S(0)} \mathcal{O} \right) \right)}{\text{Tr}_S \text{Tr}_M \left(Y_M^{(-s)} T_{S,M} q^{2\kappa S} \mathbf{b}_{\infty,s-1}^* \dots \mathbf{b}_{\infty,0}^* \left(q^{2\alpha S(0)} \right) \right)}. \quad (4.4)$$

The function ρ changes in the most natural way to

$$\rho(\zeta | \kappa, \kappa + \alpha, s) = \frac{T(\zeta, \kappa + \alpha - s, s)}{T(\zeta, \kappa)}. \quad (4.5)$$

The function $\omega(\zeta, \xi | \kappa, \alpha, s)$ is defined by (3.11), replacing $\rho(\zeta | \kappa, \kappa + \alpha)$ by $\rho(\zeta | \kappa, \kappa + \alpha, s)$ but keeping the same $\mathfrak{a}(\zeta, \kappa)$.

Let us discuss one important property of the Bethe vector of spin s . The basic objects in the theory are the transfer matrix $T_M(\zeta, \kappa + \alpha - s)$ and Baxter's Q -operator $Q_M(\zeta, \kappa + \alpha - s)$. Their eigenvalues on the vector $|\kappa + \alpha - s, s\rangle$ have the following analytical properties: $T(\zeta, \kappa + \alpha - s, s)$ is a polynomial of ζ^2 of degree \mathbf{n} ,

$$Q(\zeta, \kappa + \alpha - s, s) = \zeta^{-\alpha - \kappa + 2s} A(\zeta, \kappa + \alpha - s, s),$$

where $A(\zeta, \kappa + \alpha - s, s)$ is a polynomial in ζ^2 of degree $\mathbf{n}/2 - s$. In terms of T and A the Baxter equation reads

$$\begin{aligned} T(\zeta, \kappa + \alpha - s, s) A(\zeta, \kappa + \alpha - s, s) \\ = q^{-\kappa'} d(\zeta) A(\zeta q, \kappa + \alpha - s, s) + q^{\kappa'} a(\zeta) A(\zeta q^{-1}, \kappa + \alpha - s, s), \end{aligned} \quad (4.6)$$

with

$$\kappa' = \kappa + \alpha + 2 \frac{1-\nu}{\nu} s. \quad (4.7)$$

Formally, κ' is defined modulo $2\mathbb{Z}/\nu$. However, the eigenvalues are multi-valued functions of κ' and we have to be careful about the choice of the branch. The choice in (4.7) is consistent with the semi-classical limit $\nu \rightarrow 1$. We shall return to this point when we discuss the scaling limit.

Notice that if we write the Baxter equation for twist κ' and spin 0:

$$T(\zeta, \kappa') A(\zeta, \kappa') = q^{-\kappa'} d(\zeta) A(\zeta q, \kappa') + q^{\kappa'} a(\zeta) A(\zeta q^{-1}, \kappa'), \quad (4.8)$$

it looks exactly the same as (4.6) if we identify $T(\zeta, \kappa')$ with $T(\zeta, \kappa + \alpha - s, s)$ and $A(\zeta, \kappa')$ with $A(\zeta, \kappa + \alpha - s, s)$. There is, however an important difference: the polynomial $A(\zeta, \kappa')$ in (4.8) is of degree $\mathbf{n}/2$ while the polynomial $A(\zeta, \kappa + \alpha - s, s)$ in (4.6) is of degree $\mathbf{n}/2 - s$. Still, this similarity will be very important for us later when we discuss the scaling limit.

Similarly to the previous discussion we can consider an operator $Y_{\mathbf{M}}^{(s)}$ which carries positive spin s . Then the operator $\mathbf{c}^*(\zeta)$ will have nontrivial behaviour at $\zeta^2 = 0$:

$$\zeta^\alpha \mathbf{c}^*(\zeta)(X) = \sum_{j=0}^{s-1} \zeta^{-2j} \mathbf{c}_{\infty, j}^*(X) + \zeta^\alpha \mathbf{c}_{\text{reg}}^*(\zeta)(X), \quad \text{spin}(X) = -s + 1, \quad (4.9)$$

and we can repeat the entire procedure using $\mathbf{c}_{\infty, j}^*$ instead of $\mathbf{b}_{\infty, j}^*$. So, s in (4.7) can take any integer value.

Obviously, what we are doing here is nothing else but introducing screening operators on the lattice. This is important for relating to the CFT. The screening operators $\mathbf{b}_{\infty, j}^*$ anticommute among themselves, the same is true for $\mathbf{c}_{\infty, j}^*$. Naively, one could say that $\mathbf{b}_{\infty, j}^*$ anticommute with $\mathbf{c}_{\infty, i}^*$, but this does not make sense because the product of these operators do not act nontrivially on any subspace of $\mathscr{W}^{(\alpha)}$.

Remark 41 Let us mention one more, less dramatic, generalisation of the results of (3). Clearly the functional $Z^{\kappa, s}$ is independent of the choice of $Y_{\mathbf{M}}^{(s)}$ provided the condition (4.2) is satisfied. However different choices are also possible. For instance one can take any eigenvector $|A\rangle$ and eigencovector $\langle B|$ of the Matsubara transfer matrices and consider the projector $|A\rangle\langle B|$. The main formula is applicable in this more general setting.

5 Scaling Limit

We take α, κ to be real, and restrict our consideration to the region

$$|\kappa| < 1, \quad |\kappa'| < 1. \quad (5.1)$$

Then we continue analytically.

The crucial data for our construction are the eigenvectors $|\kappa + \alpha - s, s\rangle, |\kappa\rangle$. Consider the second of them. The corresponding eigenvalue $T(\zeta, \kappa)$ corresponds to the maximal eigenvector $|\kappa\rangle$ of spin zero. Denote by $Q(\zeta, \kappa)$ the eigenvalue of the Q -operator on $|\kappa\rangle$. These eigenvalues have the form

$$Q(\zeta, \kappa) = \zeta^{-\kappa} \prod_{p=1}^{\mathbf{n}/2} (1 - \zeta^2 / \xi_p^2),$$

$$T(\zeta, \kappa) = (q^\kappa + q^{-\kappa}) \prod_{p=1}^{\mathbf{n}} (1 - \zeta^2 / \theta_p^2).$$

In the domain $|\kappa| < 1$, the vector $|\kappa\rangle$ is uniquely characterised by the two requirements for the roots: $\xi_k^2, \theta_j^2 \in \mathbb{R}$, and $\xi_k^2 > 0 > \theta_j^2$ for all k, j .

Let us study the behaviour of the Bethe roots ξ_p as $\mathbf{n} \rightarrow \infty$. Suppose they are numbered in the order $\xi_1^2 < \xi_2^2 < \xi_3^2 < \dots$. In the limit $\mathbf{n} \rightarrow \infty$ they are subject to the Lieb distribution (15): for $1 \ll m$ we have

$$\frac{\xi_{m+1}}{\xi_m} - 1 = \frac{\pi v}{\mathbf{n}} \left(\xi_m^{\frac{1}{v}} + \xi_m^{-\frac{1}{v}} \right) + O\left(\frac{1}{\mathbf{n}^2}\right). \quad (5.2)$$

We are interested in the Bethe roots which are not very far from $\zeta^2 = 0$. In other words we assume $1 \ll m \ll \mathbf{n}$. Since ξ_m is small, we can drop the term $\xi_m^{\frac{1}{v}}$ in the (5.2), obtaining

$$\xi_m \simeq \left(\pi \frac{m}{\mathbf{n}} \right)^v. \quad (5.3)$$

Similar power law is obeyed by θ_m .

So far we have concentrated on the ground states in Matsubara direction. But according to Remark 41 the main formulae can be generalised to arbitrary Bethe states. There are low-lying excited states which satisfy (5.3), and to which the same analysis as for the ground states apply. Readers who are familiar with the papers by Bazhanov, Lukyanov and Zamolodchikov (9; 10; 16) would immediately say that in the limit we obtain the CFT transfer matrices treated by them. We shall come to this relation later in Sect. 7.

The formulae (2.6), (2.7) and (2.8) imply the explicit expression

$$Z^{\kappa, s} \left\{ \mathbf{t}^*(\zeta_1^0) \dots \mathbf{t}^*(\zeta_p^0) \mathbf{b}^*(\zeta_1^+) \dots \mathbf{b}^*(\zeta_r^+) \mathbf{c}^*(\zeta_r^-) \dots \mathbf{c}^*(\zeta_1^-) \left(q^{2\alpha S(0)} \right) \right\}$$

$$= \prod_{i=1}^p 2\rho(\zeta_i^0 | \kappa, \kappa + \alpha, s) \times \det \left(\omega(\zeta_i^+, \zeta_j^- | \kappa, \alpha, s) \right)_{i, j=1, \dots, r}. \quad (5.4)$$

We consider the scaling limit in the Matsubara direction,

$$\mathbf{n} \rightarrow \infty, \quad a \rightarrow 0, \quad \mathbf{n}a = 2\pi R \text{ fixed.} \quad (5.5)$$

Let us introduce the following strange-looking notation:

$$\bar{a} = Ca, \quad (5.6)$$

where C is some ν -dependent constant which will be needed for fine tuning comparing the scaling limit with CFT.

The following limits exist for finite λ :

$$\begin{aligned} T^{\text{sc}}(\lambda, \kappa) &= \lim_{\mathbf{n} \rightarrow \infty, a \rightarrow 0, 2\pi R = \mathbf{n}a} T(\lambda \bar{a}^\nu, \kappa), \\ Q^{\text{sc}}(\lambda, \kappa) &= \lim_{\mathbf{n} \rightarrow \infty, a \rightarrow 0, 2\pi R = \mathbf{n}a} \bar{a}^{\nu\kappa} Q(\lambda \bar{a}^\nu, \kappa). \end{aligned} \quad (5.7)$$

The eigenvalues of $T^{\text{sc}}(\lambda, \kappa), Q^{\text{sc}}(\lambda, \kappa)$ are given by convergent infinite products due to (5.3). In particular, it is easy to see from (5.3) that the following asymptotics hold:

$$\log Q^{\text{sc}}(\lambda, \kappa)_{\lambda^2 \rightarrow -\infty} \sim 2\pi R \cdot \frac{C}{\sin \frac{\pi}{2\nu}} (-\lambda^2)^{\frac{1}{2\nu}}. \quad (5.8)$$

Certainly, the limits exist for eigenvalues corresponding to any eigenvectors satisfying (5.3).

Now we turn to the operators $T_{\mathbf{M}}(\zeta, \kappa + \alpha - s), Q_{\mathbf{M}}(\zeta, \kappa + \alpha - s)$ for which the eigenvectors of spin s are considered. The definitions of $T^{\text{sc}}(\lambda, \kappa + \alpha - s, s), Q^{\text{sc}}(\lambda, \kappa + \alpha - s, s)$ are the same as before. The important statement is that

$$T^{\text{sc}}(\lambda, \kappa + \alpha - s, s) = T^{\text{sc}}(\lambda, \kappa'), \quad Q^{\text{sc}}(\lambda, \kappa + \alpha - s, s) = Q^{\text{sc}}(\lambda, \kappa'),$$

where κ' is given by (4.7). The right hand sides are understood as ‘‘correct’’ analytical continuations from the spin 0 sector. We have to explain two points: first, what is the reason that in the scaling limit the eigenvalues in the spin s sector equals analytical continuations of those in the spin 0 sector; second, what we mean by ‘‘correct’’ analytic continuations. The first point is simple: recall the discussion concerning the similarity of Eqs. (4.6), (4.8). The only difference between them was the number of Bethe roots, but this number is infinite in the scaling limit, so, the difference disappears. On the other hand the eigenvalue $T^{\text{sc}}(\lambda, \kappa)$ is a multi-valued function of κ , so we have to explain the choice of its branch. At this point we refer to the semi-classical domain: ν close to 1. We take a good branch at this domain, and then continue analytically. Notice that $T(0, \kappa) = 2 \cos(\pi\nu\kappa)$. We require that introducing s does not deviate us far from this value. This was the reason for choosing definition (3.1) because with this definition we have

$$T(0, \kappa - s, s) = 2 \cos(\pi\nu(\kappa + 2\frac{1-\nu}{\nu}s)),$$

which stays close to $2 \cos(\pi\nu\kappa)$ for all s if ν is close to 1.

From now on we shall often consider κ and κ' as arbitrary numbers implying the possibility of analytical continuation from values (4.7).

Using $Q^{\text{sc}}(\lambda, \kappa), Q^{\text{sc}}(\lambda, \kappa')$ we obtain finite limits

$$\rho^{\text{sc}}(\lambda|\kappa, \kappa') = \lim_{\mathbf{n} \rightarrow \infty, a \rightarrow 0, 2\pi R = \mathbf{n}a} \rho(\lambda \bar{a}^V|\kappa, \alpha, s), \quad (5.9)$$

$$4 \omega^{\text{sc}}(\lambda, \mu|\kappa, \kappa', \alpha) = \lim_{\mathbf{n} \rightarrow \infty, a \rightarrow 0, 2\pi R = \mathbf{n}a} \omega(\lambda \bar{a}^V, \mu \bar{a}^V|\kappa, \alpha, s), \quad (5.10)$$

where κ' is given by (4.7), and then for $\rho^{\text{sc}}(\lambda|\kappa, \kappa'), \omega^{\text{sc}}(\lambda, \mu|\kappa, \kappa', \alpha)$ the analytical continuation with respect to κ' is used.

According to Remark 41 (5.4) remains valid for any Bethe states in the Matsubara direction. It has been already said that to Bethe states close to the ground states the scaling procedure applies. We shall argue later on that these vectors span the Verma module of chiral CFT. So, we would like to use the right hand side of (5.4) in order to consider the scaling limit in the space direction

$$ja = x \text{ finite.}$$

In our setting it amounts to considering the operators:

$$2\tau^*(\lambda) = \lim_{a \rightarrow 0} \mathbf{t}^*(\lambda \bar{a}^V), \quad (5.11)$$

$$2\beta^*(\lambda) = \lim_{a \rightarrow 0} \mathbf{b}^*(\lambda \bar{a}^V), \quad 2\gamma^*(\lambda) = \lim_{a \rightarrow 0} \mathbf{c}^*(\lambda \bar{a}^V),$$

and

$$\Phi_\alpha(0) = \lim_{a \rightarrow 0} q^{2\alpha S(0)},$$

so that (5.4) gives in the scaling limit

$$\begin{aligned} & Z_R^{\kappa, \kappa'} \{ \tau^*(\lambda_1^0) \cdots \tau^*(\lambda_p^0) \beta^*(\lambda_1^+) \cdots \beta^*(\lambda_r^+) \gamma^*(\lambda_r^-) \cdots \gamma^*(\lambda_1^-) (\Phi_\alpha(0)) \} \\ &= \prod_{i=1}^p \rho^{\text{sc}}(\lambda_i^0|\kappa, \kappa') \times \det \left(\omega^{\text{sc}}(\lambda_i^+, \lambda_j^-|\kappa, \kappa', \alpha) \right)_{i,j=1, \dots, r}. \end{aligned} \quad (5.12)$$

We understand the formula (5.4) as giving the expectation values of certain non-local operators making contact with quasi-local ones near $\zeta^2 = 1$. After introducing a this point moves to $\lambda^2 = \bar{a}^{-2V}$, and in the scaling limit it goes further to $\lambda^2 = +\infty$. It should not be a surprise that this limit is described by CFT. To make this statement precise, we have to establish certain asymptotic properties of $\rho^{\text{sc}}(\lambda|\kappa, \kappa'), \omega^{\text{sc}}(\lambda, \mu|\kappa, \kappa', \alpha)$ for $\lambda^2, \mu^2 \rightarrow +\infty$. But first we shall need some information about the integrable structure of CFT.

6 CFT on a Cylinder and Three Point Functions

In this section we introduce our notation concerning CFT, and collect a few facts which will be relevant to the subsequent sections.

Consider chiral CFT on a cylinder $Cyl = \mathbb{C}/2\pi iR\mathbb{Z}$ with circumference $2\pi R$, the points x and $x + 2\pi iR$ being identified. Along with the local coordinate x , we shall also use the global coordinate

$$z = e^{-\frac{x}{R}}.$$

The two points $x = -\infty, \infty$ on the boundary of Cyl correspond respectively to the points $z = \infty, 0$ on the Riemann sphere.

Let

$$T(x) = \sum_{n=-\infty}^{\infty} l_n x^{-n-2}$$

be the energy-momentum tensor in the coordinate x , where the l_n 's satisfy the commutation relations of the Virasoro algebra with the central charge

$$c = 1 - \frac{6v^2}{1-v}. \quad (6.1)$$

The energy-momentum tensor in the coordinate z ,

$$\tilde{T}(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2},$$

is related to $T(x)$ via the transformation rule $\tilde{T}(z)(dz)^2 = (T(x) - (c/12)\{z;x\})(dx)^2$. Here $\{z;x\}$ denotes the Schwarzian derivative. In turn, $T(x)$ is written as

$$T(x) = \frac{1}{R^2} \left(\sum_{n=-\infty}^{\infty} L_n e^{\frac{nx}{R}} - \frac{c}{24} \right).$$

The Virasoro algebra acts on a local field $O(y)$ by the contour integral

$$(\mathbf{l}_n O)(y) = \int_{C_y} \frac{dx}{2\pi i} (x-y)^{n+1} T(x) O(y), \quad (6.2)$$

where C_y encircles the point y anticlockwise.

From now on, we fix a primary field $\phi_\Delta(y)$ with the scaling dimension Δ :

$$(\mathbf{l}_0 \phi_\Delta)(y) = \Delta \phi_\Delta(y), \quad (\mathbf{l}_n \phi_\Delta)(y) = 0 \quad (n > 0),$$

and study the expectation values

$$\langle T(x_k) \cdots T(x_1) \phi_\Delta(y) \rangle_{\Delta_+, \Delta_-}. \quad (6.3)$$

The suffix indicates that we consider (6.3) in the presence of two other primary fields inserted at $x = \pm\infty$. More precisely, we impose the boundary conditions

$$\lim_{x \rightarrow \pm\infty} T(x) = \frac{1}{R^2} \left(\Delta_\pm - \frac{c}{24} \right) \quad (6.4)$$

inside the expectation values (6.3), where Δ_\pm are the conformal dimensions of the inserted primary fields. For readers who prefer the language of representation theory, we are considering a highest weight vector $|\Delta_+\rangle$ at $z = 0$ satisfying $L_n |\Delta_+\rangle = \delta_{n,0} \Delta_+ |\Delta_+\rangle$ ($n \geq 0$), and a co-vector $\langle \Delta_-|$ at $z = \infty$ satisfying $\langle \Delta_-| L_n = \delta_{n,0} \Delta_- \langle \Delta_-|$ ($n \leq 0$).

The singular part of (6.3) is known from OPEs. In order to write them in the coordinate x , it is useful to introduce the function

$$\chi(x) = \frac{1}{2} \coth\left(\frac{x}{2R}\right) = \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} \left(\frac{x}{R}\right)^{2n-1}.$$

Here $B_0 = 1, B_2 = 1/6, B_4 = -1/30, \dots$ are the Bernoulli numbers. The main OPEs then read, as $x \rightarrow y$,

$$T(x)T(y) = -\frac{c}{12R}\chi'''(x-y) - \frac{2T(y)}{R}\chi'(x-y) + \frac{T'(y)}{R}\chi(x-y) + \mathcal{O}(1),$$

(6.5)

$$T(x)\phi_\Delta(y) = -\frac{\Delta\phi_\Delta(y)}{R}\chi'(x-y) + \frac{\phi'_\Delta(y)}{R}\chi(x-y) + O(1), \quad (6.6)$$

where the prime stands for the derivative.

The OPEs (6.5), (6.6), combined with (6.4) and the behaviour

$$\chi(x) = \pm \frac{1}{2} + O(e^{\mp x/R}) \quad (x \rightarrow \pm\infty),$$

allow us to write the conformal Ward-Takahashi identity which determines (6.3) recursively:

$$\begin{aligned} & \langle T(x_k) \cdots T(x_1)\phi_\Delta(y) \rangle_{\Delta_+, \Delta_-} \\ &= -\frac{c}{12R} \sum_{j=2}^k \chi'''(x_1 - x_j) \langle T(x_k) \cdots \overset{j}{\cdot} \cdots T(x_2)\phi_\Delta(y) \rangle_{\Delta_+, \Delta_-} \\ &+ \left\{ \sum_{j=2}^k \left(-\frac{2}{R}\chi'(x_1 - x_j) + \frac{1}{R}(\chi(x_1 - x_j) - \chi(x_1 - y)) \frac{\partial}{\partial x_j} \right) - \frac{\Delta}{R}\chi'(x_1 - y) \right. \\ &\left. + (\Delta_+ - \Delta_-) \frac{1}{R^2}\chi(x_1 - y) + \frac{1}{2R^2}(\Delta_+ + \Delta_-) - \frac{c}{24R^2} \right\} \langle T(x_k) \cdots T(x_2)\phi_\Delta(y) \rangle_{\Delta_+, \Delta_-}. \end{aligned}$$

From these one can extract, for example,

$$\frac{\langle (\mathbf{1}_{-n}\phi_\Delta)(y) \rangle_{\Delta_+, \Delta_-}}{\langle \phi_\Delta(y) \rangle_{\Delta_+, \Delta_-}} = \begin{cases} \frac{\delta_{n,2}}{2R^2}(\Delta_+ + \Delta_- - \frac{c}{12}) - \frac{B_n\Delta}{n(n-2)!R^n} & (n : \text{even}); \\ \frac{(\Delta_+ - \Delta_-)B_{n-1}}{(n-1)!R^n} & (n : \text{odd}). \end{cases}$$

In general, the normalised three point function of any particular descendant

$$\frac{\langle (\mathbf{1}_{-n_k} \cdots \mathbf{1}_{-n_1}\phi_\Delta)(y) \rangle_{\Delta_+, \Delta_-}}{\langle \phi_\Delta(y) \rangle_{\Delta_+, \Delta_-}} \quad (6.7)$$

is determined as a polynomial in $\Delta, \Delta_+, \Delta_-$.

In writing these formulas, we have tacitly assumed that $\langle \phi_\Delta(y) \rangle_{\Delta_+, \Delta_-}$ is non-trivial. Actually, in the theory with $c < 1$, for a given generic value of Δ there is a discrete but an infinite collection of such Δ_+, Δ_- . In view of the polynomial dependence mentioned above, one can regard (6.7) as a linear functional, defined for arbitrary $\Delta, \Delta_+, \Delta_-$, on the Verma module consisting of the descendants of $\phi_\Delta(y)$. Henceforth we shall adopt this point of view. In later sections we shall use the parametrisation $\Delta = \Delta_\alpha, \Delta_+ = \Delta_{\kappa+1}, \Delta_- = \Delta_{-\kappa'+1}$, and write this functional as

$$Z_R^{\kappa, \kappa'} \{X(y)\} = \frac{\langle X(y) \rangle_{\Delta_{\kappa+1}, \Delta_{-\kappa'+1}}}{\langle \phi_{\Delta_\alpha}(y) \rangle_{\Delta_{\kappa+1}, \Delta_{-\kappa'+1}}}, \quad (6.8)$$

where $X(y)$ is a descendant of $\phi_{\Delta_\alpha}(y)$.

In (11), A. Zamolodchikov introduced the local integrals of motion which survive the $\phi_{1,3}$ -perturbation of CFT. They act on local operators as

$$(\mathbf{i}_{2n-1}O)(y) = \int_{C_y} \frac{dx}{2\pi i} h_{2n}(x)O(y) \quad (n \geq 1). \quad (6.9)$$

The densities $h_{2n}(y)$ are certain descendants of the identity operator I . The simplest examples are

$$h_2(x) = (\mathbf{I}_{-2}I)(x) = T(x), \quad h_4(x) = (\mathbf{I}_{-2}T)(x), \quad (6.10)$$

for which we have

$$(\mathbf{i}_1O)(y) = (\mathbf{I}_{-1}O)(y), \quad (\mathbf{i}_3O)(y) = 2 \sum_{n=0}^{\infty} (\mathbf{I}_{-n-2}\mathbf{I}_{n-1}O)(y).$$

In general, for a descendant $h(x)$ of the identity operator, the three point function

$$\int_{C_y} \frac{dx}{2\pi i} \langle h(x)O(y) \rangle_{\Delta_+, \Delta_-}$$

is reducible to that of $O(y)$. Indeed, it can be rewritten as

$$- \int_{C(u)} \frac{dx}{2\pi i} \langle h(x)O(y) \rangle_{\Delta_+, \Delta_-} + \int_{C(v)} \frac{dx}{2\pi i} \langle O(y)h(x) \rangle_{\Delta_+, \Delta_-},$$

where $C(u)$ ($u \in \mathbb{R}$) is a circle with the real part u , starting from $u - \pi Ri$ and ending at $u + \pi Ri$. Choosing $u \ll \operatorname{Re} y \ll v$ and using the boundary conditions at $x \rightarrow \pm\infty$, one can show that each of them reduce to a constant.

From the above remark it follows that

$$\langle \mathbf{i}_{2n-1}(O(y)) \rangle_{\Delta_+, \Delta_-} = (I_{2n-1}^+ - I_{2n-1}^-) \cdot \langle O(y) \rangle_{\Delta_+, \Delta_-}, \quad (6.11)$$

where I_{2n-1}^{\pm} denote the vacuum eigenvalues of the local integrals of motion on the Verma module with conformal dimension Δ_{\pm} . Their explicit formulas for small n can be found in (9) (see also Sect. 10, (10.18)–(10.20) below). Notice that, in the special case $\Delta_+ = \Delta_-$, the three point function vanishes on the image of the local integrals.

As mentioned in the Introduction, we accept the conjectural statement that the Verma module is spanned by the elements

$$\mathbf{i}_{2k_1-1} \cdots \mathbf{i}_{2k_p-1} \mathbf{I}_{-2m_1} \cdots \mathbf{I}_{-2m_q}(\phi_{\alpha}(0)).$$

Formula (6.11) tells that for the computation of the linear functional (6.7) it suffices to consider the descendants by the even Virasoro generators $\{\mathbf{I}_{-2n}\}_{n \geq 1}$.

Before closing this section, let us comment on a point which could be a source of confusion. The local integrals of motion arise in two different ways. At the boundary of the cylinder, they appear as the operators I_{2n-1} constructed from the modes L_n of the energy-momentum tensor in the coordinate $e^{-x/R}$. These operators preserve the subspace of the Verma module of a given degree and can be diagonalised. In the classical limit, the eigenvalues of I_{2n-1} correspond to the values of the integrals of motion on quasi-periodic solutions to the KdV hierarchy. In contrast, the action of the integrals of motion on local fields \mathbf{i}_{2n-1} are constructed from the modes \mathbf{I}_n in the coordinate x . Unlike in the first case, they do not commute with \mathbf{I}_0 (for example, the first integral of motion is \mathbf{I}_{-1}). They act as a creation part of the Heisenberg algebra. In the classical limit, they correspond to the action of the Hamiltonian vector fields generated by the local integrals of motion. So it does not make sense to talk about their diagonalisation.

7 Brief Review of BLZ

In a series of papers (9; 10; 16) Bazhanov, Lukyanov and Zamolodchikov (BLZ) studied the integrable structure of the chiral CFT on a circle. We shall recall these results briefly, since they are quite relevant to us.

It is convenient to write the energy-momentum tensor in terms of the chiral boson

$$\varphi(x) = iQ + P\frac{x}{R} + \sum_{n \neq 0} \frac{a_n}{n} e^{\frac{nx}{R}}.$$

The operators P, Q are canonically conjugate, and the a_n 's satisfy the Heisenberg algebra

$$[P, Q] = \frac{i(1-\nu)}{2}, \quad [a_m, a_n] = \frac{m(1-\nu)}{2} \delta_{m+n,0}.$$

The energy-momentum tensor is expressed as

$$(1-\nu)T(x) = : \varphi'(x)^2 : + \nu \varphi''(x) - \frac{1-\nu}{24R^2}.$$

The chiral vertex operator

$$\phi_\alpha(x) = e^{-\frac{\nu^2 \alpha^2}{4(1-\nu)} \frac{x}{R}} : e^{\frac{\nu}{1-\nu} \alpha \varphi(x)} : \quad (7.1)$$

is a primary field of scaling dimension

$$\Delta_\alpha = \frac{\alpha(\alpha-2)\nu^2}{4(1-\nu)}. \quad (7.2)$$

We note that the parameter β^2 used in (10) is identified as

$$\beta^2 = 1 - \nu,$$

hence their $q = e^{\pi i \beta^2}$ is our $-q^{-1}$. We have also changed the sign of P, a_n and the normal ordering convention in (10) to

$$: e^{\lambda \varphi(x)} : := e^{\lambda \sum_{n < 0} \frac{a_n}{n} e^{nx/R}} e^{i\lambda Q} e^{\lambda P x/R} e^{\lambda \sum_{n > 0} \frac{a_n}{n} e^{nx/R}},$$

which results in the appearance of a scalar factor $e^{-\frac{\nu^2 \alpha^2}{4(1-\nu)} \frac{x}{R}}$ in (7.1).

The main object studied by BLZ is the universal monodromy matrix in CFT. It is an element of $U_q(\mathfrak{b}^+) \otimes A_{\mathbf{H}}$, where $U_q(\mathfrak{b}^+)$ is the Borel subalgebra of $U_q(\widehat{\mathfrak{sl}}_2)$ generated by e_0, e_1, \hbar_1 , and $A_{\mathbf{H}}$ is the algebra generated by P, Q, a_n (the suffix \mathbf{H} stands for Heisenberg). Set

$$\mathcal{H}_{\mathbf{H}}(x) = e_0 \otimes V_+(x) + e_1 \otimes V_-(x), \quad \mathcal{H}_{\mathbf{H}} = -\frac{2P}{1-\nu}, \quad (7.3)$$

$$V_{\pm}(x) = e^{-(1-\nu)\frac{x}{R}} : e^{\pm 2\varphi(x)} :, \quad (7.4)$$

so that $[\mathcal{H}_{\mathbf{H}}, V_{\pm}(x)] = \pm 2V_{\pm}(x)$. The universal monodromy matrix is defined to be

$$\mathcal{T}_{\mathbf{H}}(\lambda) = \mathcal{P} \exp \left(\lambda \int_0^{2\pi R} \mathcal{K}_{\mathbf{H}}(-iy) dy \right) q^{-\frac{1}{2}h_1 \otimes \mathcal{H}_{\mathbf{H}}}. \quad (7.5)$$

Here $\mathcal{P} \exp$ stands for the path ordered exponential. Formula (7.5) is understood as a power series in λ . The integrals in each term converge in the domain $1/2 < \nu < 1$. Otherwise divergences occur and a regularisation is needed.

We have considered two maps: $U_q(\mathfrak{b}_+) \rightarrow \text{End}(V_a)$ with two-dimensional V_a , and $U_q(\mathfrak{b}_+) \rightarrow \text{Osc}_A$ with Osc_A being the q -oscillator algebra (see (2) for the notation). The images of $\mathcal{T}_{\mathbf{H}}(\lambda)$ under these maps are denoted by $\mathcal{T}_{a,\mathbf{H}}(\lambda)$ and $\mathcal{T}_{A,\mathbf{H}}(\lambda)$. Then following (10) we define

$$\begin{aligned} T_{\mathbf{H}}^{\text{CFT}}(\lambda) &= \text{Tr}_a \left(\mathcal{T}_{a,\mathbf{H}}(\lambda) e^{-2\pi i(\sigma_a^3 \otimes P)} \right), \\ Q_{\mathbf{H}}^{\text{CFT}}(\lambda) &= \lambda^{\frac{2P}{\nu}} (1 - e^{2\pi i P}) \text{Tr}_A \left(\mathcal{T}_{A,\mathbf{H}}(\lambda) e^{2\pi i(D_A \otimes P)} \right). \end{aligned} \quad (7.6)$$

There is a slight difference with (10) due to different notation for the q -oscillator algebra.

These operators satisfy the Baxter equation

$$T_{\mathbf{H}}^{\text{CFT}}(\lambda) Q_{\mathbf{H}}^{\text{CFT}}(\lambda) = Q_{\mathbf{H}}^{\text{CFT}}(\lambda q^{-1}) + Q_{\mathbf{H}}^{\text{CFT}}(\lambda q). \quad (7.7)$$

An important property of these transfer matrices is that they commute with the local integrals of motion. The first local integral of motion is nothing but $L_0 - \frac{c}{24}$, which commutes with the transfer matrices as mentioned above. Hence each of their eigenstates on a Verma module belongs to the subspace of a definite degree. In particular, the highest weight vector of the Verma module (primary field) is an eigenvector.

Actually, the local integrals of motion are all encoded in the transfer matrix $T_{\mathbf{H}}^{\text{CFT}}(\lambda)$. The latter is known (9) to be an entire function of λ^2 . One of the main statements of (9) is that it has the following asymptotics for $\lambda^2 \rightarrow \infty, \lambda^2 \notin \mathbb{R}_{<0}$:

$$\log(T_{\mathbf{H}}^{\text{CFT}}(\lambda)) \sim RC_0 \lambda^{\frac{1}{\nu}} + \sum_{n=1}^{\infty} C_n \lambda^{-\frac{2n-1}{\nu}} I_{2n-1}, \quad (7.8)$$

where C_n are known constants which can be found in (10); they can be also extracted from Sect. 10 of the present paper. For the moment the only point relevant to us is the fact that

$$C_1 < 0.$$

This means that for sufficiently large λ^2 the highest weight vector is the eigenvector of $T_{\mathbf{H}}^{\text{CFT}}(\lambda)$ with the maximal absolute value. The eigenvalue of L_0 on the highest weight vector equals

$$\frac{1}{1-\nu} \left(P^2 - \frac{\nu^2}{4} \right). \quad (7.9)$$

One may wonder why the asymptotic expansion (7.8) is given as a series in the fractional power $\lambda^{\frac{1}{\nu}}$. As explained in (9), the reason is that $\lambda^{\frac{1}{\nu}}$ is a dimensional quantity having the dimension of the inverse length. Indeed, consider the L -operator (7.3). It must have the dimension of the inverse length in order that the exponential in (7.5) be dimensionless. But the operators $V_{\pm}(x)$ carry the anomalous dimension $1 - \nu$. So, clearly the dimension of λ equals ν .

We shall be interested in the Bethe roots, which are the zeros $\lambda^2 = \lambda_n^2$ of $Q_{\mathbf{H}}^{\text{CFT}}(\lambda)$. They behave as

$$\lambda_n^2 = O(n^{2\nu}), \quad n \rightarrow \infty.$$

Of equal significance are the zeros $\lambda^2 = \mu_n^2$ of $T_{\mathbf{H}}^{\text{CFT}}(\lambda)$. The eigenvalue corresponding to the primary field has the characteristic property (10) that all $\lambda_i^2 > 0 > \mu_j^2$ for all i, j .

8 Conformal Limit in the Matsubara Direction

Let us return to the XXZ model. Using the notation introduced in Sect. 5, we write the Baxter equation

$$T_{\mathbf{M}}(\lambda \bar{a}^{\nu}, \kappa) Q_{\mathbf{M}}(\lambda \bar{a}^{\nu}, \kappa) = a(\lambda \bar{a}^{\nu}) Q_{\mathbf{M}}(\lambda \bar{a}^{\nu} q^{-1}, \kappa) + d(\lambda \bar{a}^{\nu}) Q_{\mathbf{M}}(\lambda \bar{a}^{\nu} q, \kappa) \quad (8.1)$$

where

$$a(\lambda \bar{a}^{\nu}) = (1 - q \bar{a}^{2\nu} \lambda^2)^{\mathbf{n}}, \quad d(\lambda \bar{a}^{\nu}) = (1 - q^{-1} \bar{a}^{2\nu} \lambda^2)^{\mathbf{n}}.$$

We are interested in the maximal eigenvector $|\kappa\rangle$ of $T_{\mathbf{M}}(\lambda \bar{a}^{\nu}, \kappa)$.

We want to consider the limit $\mathbf{n} \rightarrow \infty, \bar{a} \rightarrow 0$, while keeping $\mathbf{n} \bar{a} = 2\pi R \cdot C$ and λ fixed. In this limit, for $1/2 < \nu < 1$,

$$a(\lambda) \rightarrow 1, \quad d(\lambda) \rightarrow 1,$$

so, if we identify

$$\nu \kappa = -2P, \quad (8.2)$$

the Baxter equation (8.1) turns into (7.7).

Now we want to fix the constant C in order to make the equivalence between Q^{sc} and Q^{CFT} exact. We had the asymptotics (5.8). On the other hand, it is known (10) that

$$\log Q^{\text{CFT}}(\lambda, \kappa) \sim R \cdot \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1-\nu}{2\nu}\right) \Gamma\left(1 - \frac{1}{2\nu}\right) \Gamma(\nu)^{\frac{1}{\nu}} (-\lambda^2)^{\frac{1}{2\nu}}.$$

So, comparing we see that the agreement is exact if

$$C = \frac{\Gamma\left(\frac{1-\nu}{2\nu}\right)}{2\sqrt{\pi} \Gamma\left(\frac{1}{2\nu}\right)} \Gamma(\nu)^{\frac{1}{\nu}}. \quad (8.3)$$

So, we come to

$$Q^{\text{sc}}(\lambda, \kappa) = Q^{\text{CFT}}(\lambda) \Big|_{P=-\frac{\nu\kappa}{2}}. \quad (8.4)$$

Let us argue that the vector $|\kappa\rangle$ goes to the primary field with the dimension

$$\Delta_{\kappa+1} = \frac{\nu^2}{4(1-\nu)}(\kappa^2 - 1).$$

On the lattice the maximal eigenvector $|\kappa\rangle$ is defined by the requirement that the eigenvalue $T(1, \kappa)$ is of maximal absolute value. In the scaling limit this corresponds to the requirement that $T_{\mathbf{H}}^{\text{CFT}}(\lambda, \kappa)$ is maximal for λ^2 large and positive. But the asymptotic behaviour of the BLZ transfer matrix is given by (7.8), so in the domain

$$-\frac{\pi\nu}{2} < \arg \lambda < \frac{\pi\nu}{2} \quad (8.5)$$

the maximal eigenvalue corresponds to the primary field. Comparing with (7.2) and (7.9), we find that in the picture of Sect. 6 we have

$$\Delta_+ = \Delta_{\kappa+1}.$$

Hence the boundary conditions at $+\infty$ is described by $\phi_{\kappa+1}(+\infty)$. Similarly, considering the left Matsubara transfer matrix we find that its ground state corresponds to the scaling dimension

$$\Delta_- = \Delta_{-\kappa'+1}. \quad (8.6)$$

In other words, in the picture of Sect. 6 we have the left boundary condition described by the primary field $\phi_{-\kappa'+1}(-\infty)$.

The argument above is not rigorous, because it involves two limits which are *a priori* non-commutative. Nevertheless we believe it makes sense because of integrability, which stipulates that $|\kappa\rangle$ is an eigenvector of $T_{\mathbf{M}}(\lambda\bar{a}^\nu, \kappa)$ for all values of λ . As a supporting argument, we quote from (17) knowledge that the eigenvalue $T(\zeta, \kappa)$ of the lattice transfer matrix for $\tau = q^{1/2}$ has maximal absolute value in the range $|\zeta| = 1$, $-\pi\nu/2 < \arg \zeta < \pi\nu/2$. This agrees exactly with (8.5).

We call the previous reasoning a macroscopic one. For completeness let us give a less formal, microscopic derivation providing at the same time a constant which is important for physics. Consider the Matsubara transfer matrix $T_{\mathbf{M}}(\lambda\bar{a}^\nu, \kappa)$, which is given explicitly by

$$T_{\mathbf{M}}(\lambda\bar{a}^\nu, \kappa) = \text{Tr}_j \left(L_{j,\mathbf{n}}(\lambda\bar{a}^\nu) \cdots L_{j,1}(\lambda\bar{a}^\nu) q^{\kappa\sigma_j^3} \right), \quad (8.7)$$

where

$$L_{j,\mathbf{m}}(\lambda\bar{a}^\nu) = \begin{pmatrix} q^{-\frac{1}{2}\sigma_{\mathbf{m}}^3} - \bar{a}^{2\nu}\lambda^2 q^{\frac{1}{2}\sigma_{\mathbf{m}}^3} & -(q - q^{-1})\lambda\bar{a}^\nu \sigma_{\mathbf{m}}^- \\ -(q - q^{-1})\lambda\bar{a}^\nu \sigma_{\mathbf{m}}^+ & q^{\frac{1}{2}\sigma_{\mathbf{m}}^3} - \bar{a}^{2\nu}\lambda^2 q^{-\frac{1}{2}\sigma_{\mathbf{m}}^3} \end{pmatrix}_j.$$

Let us make the gauge transformation

$$L_{j,\mathbf{m}}(\lambda\bar{a}^\nu) = q^{-\frac{1}{2}\sigma_j^3 \sum_{i=1}^{\mathbf{m}} \sigma_i^3} \widehat{L}_{j,\mathbf{m}}(\lambda\bar{a}^\nu) q^{\frac{1}{2}\sigma_j^3 \sum_{i=1}^{\mathbf{m}-1} \sigma_i^3}, \quad (8.8)$$

where

$$\widehat{L}_{j,\mathbf{m}}(\lambda \bar{a}^\nu) = \begin{pmatrix} 1 - \bar{a}^{2\nu} \lambda^2 q^{\sigma_{\mathbf{m}}^3} & -(q - q^{-1}) \lambda \bar{a}^\nu q^{-\frac{1}{2} + \sum_{i=1}^{\mathbf{m}-1} \sigma_i^3} \sigma_{\mathbf{m}}^- \\ -(q - q^{-1}) \lambda \bar{a}^\nu q^{-\frac{1}{2} - \sum_{i=1}^{\mathbf{m}-1} \sigma_i^3} \sigma_{\mathbf{m}}^+ & 1 - \bar{a}^{2\nu} \lambda^2 q^{-\sigma_{\mathbf{m}}^3} \end{pmatrix}_j.$$

Now we recall known formulae concerning the continuous limit of the XXZ chain (18). A very accurate account of this matter is given in Lukyanov's paper (19). Notice, however, that the Hamiltonian in (19) differs from ours by a similarity transformation with the operator $U = \prod_{\mathbf{m}: \text{odd}} \sigma_{\mathbf{m}}^3$. Having this in mind we rewrite the main order formulae for $\mathbf{n} \rightarrow \infty, y = \mathbf{m}a$ (formulae (2.19) in (19)) as follows:

$$\sigma_{\mathbf{m}}^3 \rightarrow \frac{a}{i\pi(1-\nu)} \partial_y (\varphi(-iy) - \bar{\varphi}(-iy)), \quad (8.9)$$

$$\sigma_{\mathbf{m}}^\pm \rightarrow (-1)^{\mathbf{m}} \sqrt{F/2} a^{\frac{1}{2}(1-\nu)} : e^{\pm(\varphi(-iy) + \bar{\varphi}(-iy))} : .$$

Here $\varphi(x), \bar{\varphi}(x)$ are two chiral bosons with the same normalisation as in Sect. 7. The fractional power of a in the second formula is needed in order to compensate the anomalous dimension of $: e^{\pm(\varphi(-iy) + \bar{\varphi}(-iy))} :$, and F is related to the one point function of the latter (19). From these formulae we see that

$$W_m^\pm = q^{\mp \sum_{i=1}^{\mathbf{m}-1} \sigma_i^3} \sigma_{\mathbf{m}}^\pm \rightarrow \sqrt{Z} a^{1-\nu} V_\pm(-iy), \quad (8.10)$$

where $V_\pm(x)$ are the chiral vertex operators (7.4). The power of a changed due to a normal reordering, while the constant Z is obviously related to the asymptotical behaviour at $m \rightarrow \infty$ of the following two-point function for XXZ model:

$$\langle W_m^+ W_0^- \rangle = \frac{Z}{m^{2(1-\nu)}}.$$

We do not know a direct way to fix this constant, however, our construction allows an indirect one. Indeed, in order to have complete agreement with CFT on this microscopic level we need that

$$\widehat{L}_{j,\mathbf{m}}(\lambda \bar{a}^\nu) = 1 + a\lambda \mathcal{H}_{j,\mathbf{H}}(-iy) + O(a^{2\nu}), \quad (8.11)$$

which would imply

$$\widehat{L}_{j,\mathbf{n}}(\lambda \bar{a}^\nu) \cdots \widehat{L}_{j,1}(\lambda \bar{a}^\nu) \rightarrow \mathcal{P} \exp \left(\lambda \int_0^{2\pi R} \mathcal{H}_{j,\mathbf{H}}(-iy) dy \right), \quad (8.12)$$

where we have set

$$\mathcal{H}_{j,\mathbf{H}}(x) = iq^{-\frac{1}{2}} \left(\sigma_j^+ V_-(x) + \sigma_j^- V_+(x) \right).$$

So the microscopic picture agrees with the macroscopic one if

$$Z = \frac{1}{4 \sin^2(\pi\nu) C^{2\nu}}.$$

Altogether we obtain from (8.7),

$$T_{\mathbf{M}}(\lambda \bar{a}^\nu, \kappa) \rightarrow \text{Tr}_j \left[e^{\pi i \nu \kappa} \mathcal{P} \exp \left(\lambda \int_0^{2\pi R} \mathcal{H}_{j,\mathbf{H}}(-iy) dy \right) \right], \quad (8.13)$$

giving rise to the BLZ transfer matrix (7.6) with the identification (8.2). In particular, the second chirality decouples.

9 Conformal Limit in the Space Direction

Let us return to the formula

$$\begin{aligned} & Z_R^{\kappa, \kappa'} \{ \tau^*(\lambda_1^0) \cdots \tau^*(\lambda_p^0) \beta^*(\lambda_1^+) \cdots \beta^*(\lambda_r^+) \gamma^*(\lambda_r^-) \cdots \gamma^*(\lambda_1^-) (\Phi_\alpha(0)) \} \\ &= \prod_{i=1}^p \rho^{\text{sc}}(\lambda_i^0 | \kappa, \kappa') \det \left(\omega^{\text{sc}}(\lambda_i^+, \lambda_j^- | \kappa, \kappa', \alpha) \right)_{i,j=1, \dots, r}. \end{aligned} \quad (9.1)$$

We have seen that the functions in the right-hand side are defined through the eigenvalues of the BLZ transfer matrix on the primary fields $\phi_{\kappa+1}, \phi_{-\kappa'+1}$. Thus the right hand side of (9.1) is defined. We want to interpret the left-hand side of this equation. Our arguments are far from being mathematically rigorous, so, we formulate our statement as a conjecture.

Conjecture *Asymptotics of (9.1) for $\lambda_i^\pm, \lambda_i^0 \rightarrow \infty$ describes the expectation values of descendants for chiral CFT with $c = 1 - 6 \frac{v^2}{1-v}$ of the primary field $\phi_\alpha(0)$ inserted on the cylinder with the asymptotic conditions described by $\phi_{-\kappa'+1}$ and $\phi_{\kappa+1}$.*

Recall that we start with κ, κ', α satisfying (4.7), and then continue analytically. The three point function of the operators $\phi_{-\kappa'+1}(-\infty), \phi_\alpha(0), \phi_{\kappa+1}(\infty)$ does not vanish because (4.7) can be rewritten as

$$(-\kappa' + 1) + \alpha + (\kappa + 1) = 2 - 2 \frac{1-v}{v} s,$$

which coincides in our normalisation with the Dotsenko-Fateev condition (7) for one type of screening operators condition. We do not know if the second set of screening operators can be defined starting from the lattice model.

In the present section we shall first present qualitative arguments in favour of this conjecture, and then explain how it can be verified quantitatively. The actual verification will be done in Sects. 11, 12.

Consider the primary field $q^{2\alpha S(0)}$ on the lattice. Making the scaling limit in the horizontal direction on the cylinder in the same way as it was done in the vertical one we conclude that this operator turns into $\Phi_\alpha(0) = \phi_\alpha(0) \otimes \bar{\phi}_\alpha(0)$. Typical operators in the space $\mathcal{W}_{\alpha-s, s}$ are of the form $q^{2(\alpha-s)S(0)} \sigma_{k_1}^+ \cdots \sigma_{k_s}^+ \mathcal{O}(s \geq 0)$ or $q^{2(\alpha-s)S(0)} \sigma_{k_1}^- \cdots \sigma_{k_{-s}}^- \mathcal{O}(s < 0)$, where \mathcal{O} is spinless. Then the same bosonisation formulae as (8.9)

$$q^{2\alpha S(0)} \sim e^{\frac{v}{1-v} \alpha (\varphi(0) - \bar{\varphi}(0))}, \quad q^{-2S(k-1)} \sigma_k^+ \sim e^{2\varphi(x)}$$

imply that

$$\text{Scaling limit}(\mathcal{W}_{\alpha-s, s}) \subset \mathcal{V}_{\alpha+2 \frac{1-v}{v} s} \otimes \overline{\mathcal{V}}_{-\alpha}.$$

So, the operators $\tau^*(\lambda), \beta^*(\lambda), \gamma^*(\lambda)$ do not change the Verma module for the second chirality. This is one reason to assume that they do not act on it at all. Let us give the first evidence for this claim.

Consider the operator $\tau^*(\lambda)$. It originates from its counterpart on the lattice, $\mathbf{t}^*(\zeta)$. We know that close to $\zeta^2 = 1$ the operator $\mathbf{t}^*(\zeta)$ describes the adjoint action

of XXZ local integrals of motion. So, we expect the same in the CFT. Using the BLZ formulae (7.8) we see that

$$\log \rho^{\text{sc}}(\lambda | \kappa, \kappa') \simeq \sum_{n=1}^{\infty} \lambda^{-\frac{2n-1}{\nu}} C_n (I_{2n-1}(\kappa) - I_{2n-1}(\kappa')), \quad (9.2)$$

which implies together with (6.11) that

$$\tau^*(\lambda) \simeq \exp \left(\sum_{n=1}^{\infty} \lambda^{-\frac{2n-1}{\nu}} C_n \mathbf{i}_{2n-1} \right).$$

So, as it has been expected, the local operators are extracted from the action of $\tau^*(\lambda)$ in the asymptotics at $\lambda \rightarrow \infty$ as coefficients of fractional degrees $\lambda^{\frac{1}{\nu}}$ which have the dimension of inverse length. Certainly, this exercise is quite tautological, and we would not write this paper if this were the only thing we can do. But it demonstrates the chiral nature of our operators.

In the Appendix we prove the general statement that $\omega^{\text{sc}}(\lambda, \mu | \kappa, \kappa', \alpha)$ has the following asymptotics for $\lambda^2, \mu^2 \rightarrow +\infty$:

$$\omega^{\text{sc}}(\lambda, \mu | \kappa, \kappa', \alpha) \simeq \sqrt{\rho^{\text{sc}}(\lambda | \kappa, \kappa')} \sqrt{\rho^{\text{sc}}(\mu | \kappa, \kappa')} \sum_{i,j=1}^{\infty} \lambda^{-\frac{2i-1}{\nu}} \mu^{-\frac{2j-1}{\nu}} \omega_{i,j}(\kappa, \kappa', \alpha). \quad (9.3)$$

The proof will be given for the primary fields $\phi_{-\kappa'-1}, \phi_{\kappa+1}$ as asymptotical conditions, but it can be generalised to arbitrary descendants. So, in the weak sense the following operators are defined:

$$\log(\tau^*(\lambda)) \simeq \sum_{j=1}^{\infty} \tau_{2j-1}^* \lambda^{-\frac{2j-1}{\nu}}, \quad (9.4)$$

$$\frac{1}{\sqrt{\tau^*(\lambda)}} \beta^*(\lambda) \simeq \sum_{j=1}^{\infty} \beta_{2j-1}^* \lambda^{-\frac{2j-1}{\nu}}, \quad (9.5)$$

$$\frac{1}{\sqrt{\tau^*(\lambda)}} \gamma^*(\lambda) \simeq \sum_{j=1}^{\infty} \gamma_{2j-1}^* \lambda^{-\frac{2j-1}{\nu}}, \quad (9.6)$$

which act between different Verma modules:

$$\begin{aligned} \tau_{2j-1}^* &: \mathcal{V}_{\alpha+2\frac{1-\nu}{\nu}s} \rightarrow \mathcal{V}_{\alpha+2\frac{1-\nu}{\nu}s}, \\ \beta_{2j-1}^* &: \mathcal{V}_{\alpha+2\frac{1-\nu}{\nu}(s-1)} \rightarrow \mathcal{V}_{\alpha+2\frac{1-\nu}{\nu}s}, \\ \gamma_{2j-1}^* &: \mathcal{V}_{\alpha+2\frac{1-\nu}{\nu}(s+1)} \rightarrow \mathcal{V}_{\alpha+2\frac{1-\nu}{\nu}s}. \end{aligned}$$

Consider the Verma module \mathcal{V}_{α} . We obtain different elements of this module by operators τ_{2j-1}^* acting on the primary field ϕ_{α} , and by the same number of operators β_{2j-1}^* and γ_{2j-1}^* acting further. Due to the completeness in the lattice case (12), in this way we obtain linearly independent vectors from \mathcal{V}_{α} . Counting the characters we see that the entire Verma module is created in this way. Indeed,

from a combinatorial point of view, τ^* is one odd boson, and β^*, γ^* are Gross-Neveu fermions which in the uncharged sector produce one even boson.

Now we shall proceed to computation of the coefficients $\omega_{i,j}(\kappa, \kappa', \alpha)$ and comparing them with the three-point functions of CFT. Since the operator $\tau^*(\lambda)$ is already settled we shall consider the case $\kappa = \kappa'$ which means to ignore the image of the actions by the local integrals of motion \mathbf{i}_{2n-1} in the Verma module \mathcal{V}_α . Acting on the primary field $\phi_\alpha(0)$, the even generators of the Virasoro algebra, L_{-2k} , create the quotient space of the Verma module by these descendants.

10 Asymptotics of $\log a^{\text{sc}}$

In this section we study the asymptotic behaviour of the function

$$a^{\text{sc}}(\lambda, \kappa) = \frac{Q^{\text{sc}}(\lambda q, \kappa)}{Q^{\text{sc}}(\lambda q^{-1}, \kappa)}$$

as $\lambda^2 \rightarrow \infty$. Following closely the analysis developed in (10), we give a recursive algorithm for determining the coefficients of the asymptotic expansion.

It is known (see (10), (3.17) and (3.23)) that for large κ the smallest Bethe root behaves as $\lambda_1^2 \sim c(\nu)\kappa^{2\nu}$, where

$$\begin{aligned} c(\nu) &= \Gamma(\nu)^{-2} e^{\delta} \left(\frac{\nu}{2R} \right)^{2\nu}, \\ \delta &= -\nu \log \nu - (1 - \nu) \log(1 - \nu). \end{aligned} \quad (10.1)$$

The main technical idea in (10) is to consider the limit

$$\lambda^2, \kappa \rightarrow \infty, \quad \text{keeping } t = c(\nu)^{-1} \frac{\lambda^2}{\kappa^{2\nu}} \text{ fixed.}$$

Henceforth we change the variable from λ to t and write

$$F(t, \kappa) = \log a^{\text{sc}}(\lambda, \kappa).$$

This function is to be determined from the non-linear integral equation. In order to write the equation, it is convenient to redefine some functions in terms of the variables t, u . We use

$$K(t) = \frac{1}{2\pi i} \cdot \frac{1}{2} \left(\frac{tq^2 + 1}{tq^2 - 1} - \frac{tq^{-2} + 1}{tq^{-2} - 1} \right).$$

We also use $R(t, u)$ to represent the following resolvent kernel.

$$R(t, u) - \int_1^\infty \frac{dv}{v} R(t, v) K(v/u) = K(t/u), \quad (t, u > 1).$$

An explicit formula for $R(t, u)$ will be given below.

The non-linear integral equation for $F(t, \kappa)(t > 1)$ reads

$$F(t, \kappa) - \int_1^\infty K(t/u)F(u, \kappa) \frac{du}{u} = -2\pi i \nu \kappa - \left(\int_1^{e^{i\varepsilon \cdot \infty}} K(t/u) \log(1 + e^{F(u, \kappa)}) \frac{du}{u} - \int_1^{e^{-i\varepsilon \cdot \infty}} K(t/u) \log(1 + e^{-F(u, \kappa)}) \frac{du}{u} \right), \quad (10.2)$$

where ε is a small positive number. From Appendix A, we see that

$$F(t, \kappa) = -F_+(\arg t, \kappa) |t|^{\frac{1}{2\nu}} + O(|t|^{-\frac{1}{2\nu}}), \quad 0 < \arg t < \pi, \\ F(t, \kappa) = F_-(\arg t, \kappa) |t|^{\frac{1}{2\nu}} + O(|t|^{-\frac{1}{2\nu}}), \quad -\pi < \arg t < 0,$$

where F_\pm is positive. We seek for the solution of (10.2) in an asymptotic series in κ^{-1} ,

$$F(t, \kappa) \simeq \sum_{n=0}^{\infty} \kappa^{-2n+1} F_n(t).$$

Consider first the leading coefficient $F_0(t)$. For $t > 1$, from (9.2),

$$((I - K)F_0)(t) = -2\pi i \nu \quad (10.3)$$

follows, where K denotes the integral operator on the interval $[1, \infty)$,

$$Kf(t) = \int_1^\infty K(t/u)f(u) \frac{du}{u}.$$

Equation (10.3) can be solved by the standard Wiener-Hopf technique.

Quite generally, for a function $f(t)$ let

$$\hat{f}(k) = \int_0^\infty f(t) t^{-ik} \frac{dt}{t}, \quad f(t) = \int_{-\infty}^\infty \hat{f}(k) t^{ik} \frac{dk}{2\pi}$$

denote the Mellin transform and its inverse transform. For the solution of (10.3) we shall need

$$\hat{K}(k) = \frac{\sinh(2\nu - 1)\pi k}{\sinh \pi k},$$

along with the Riemann-Hilbert factorisation

$$1 - \hat{K}(k) = S(k)^{-1} S(-k)^{-1}, \\ S(k) = \frac{\Gamma(1 + (1 - \nu)ik) \Gamma(1/2 + i\nu k)}{\Gamma(1 + ik) \sqrt{2\pi(1 - \nu)}} e^{i\delta k},$$

where δ is defined in (10.1).

The function $S(k)$ is holomorphic on the lower half plane $\text{Im } k < 1/2\nu$, and for $k \rightarrow \infty$ behaves as $S(k) = 1 + O(k^{-1})$. If we demand that

$$F_0(t) = \text{const. } t^{\frac{1}{2\nu}} + O\left(t^{-\frac{1}{2\nu}}\right) \quad (t \rightarrow \infty),$$

then (10.3) admits a unique solution given by

$$F_0(t) = \int_{\mathbb{R} - \frac{i}{2\nu} - i0} dl t^{il} S(l) \frac{-if}{l(l + \frac{i}{2\nu})}, \quad (t > 1), \quad (10.4)$$

where

$$f = \frac{1}{2\sqrt{2(1-\nu)}}.$$

The right-hand side of (9.4) gives a continuous function $\tilde{F}_0(t)$ on the half line $(0, \infty)$ such that $\tilde{F}_0(t) = 0$ for $0 < t \leq 1$. However, the function $F_0(t)$ for $t > 1$ can be analytically continued to the sector $|\arg t| < 2(1-\nu)\pi$, rewriting Eq. (9.3):

$$F_0(t) = -2\pi i\nu + (KF_0)(t), \quad (10.5)$$

$$(KF_0)(t) = \int_{\mathbb{R} - \frac{i}{2\nu} - i0} dl t^{il} S(l) \hat{K}(l) \frac{-if}{l(l + \frac{i}{2\nu})}. \quad (10.6)$$

It is also possible to check directly the consistency of the formulas (10.4) and (10.5). The difference of two integrals (10.4) and (10.6) has the only pole in the upper half plane $\text{Im } l + \frac{1}{2\nu} > 0$ at $l = 0$, where we pick up the residue $-2\pi i\nu$.

Now we turn to the higher order terms. Similarly as above, the Wiener-Hopf method allows us to find $R(t, u)$ for $t > 1$,

$$R(t, u) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dl}{2\pi} \frac{dm}{2\pi} t^{il} u^{im} S(l) S(m) \hat{K}(m) \frac{-i}{l+m-i0}.$$

The analytic continuation is given by

$$R(t, u) = K(t/u) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dl}{2\pi} \frac{dm}{2\pi} t^{il} u^{im} S(l) S(m) \hat{K}(l) \hat{K}(m) \frac{-i}{l+m-i0}.$$

From this

$$R(t, u) = \int_{-\infty}^{\infty} \frac{dl}{2\pi} t^{il} S(l) \hat{K}(l) \hat{R}(l, u)$$

follows, where

$$\begin{aligned} \hat{R}(l, u) &= \int_{-\infty}^{\infty} \frac{dm}{2\pi} u^{im} S(m) \frac{-i}{l+m-i0} \\ &= u^{-il} S(l)^{-1} + \int \frac{dm}{2\pi i} \frac{1}{l+m-i0} u^{im} S(m) \hat{K}(m). \end{aligned} \quad (10.7)$$

The last line shows that $\hat{R}(l, e^x)$ is analytic near $x = 0$. Equation (10.2) can be converted into

$$\begin{aligned} F(t, \kappa) &= \kappa F_0(t) \\ &- \left(\int_1^{e^{i\epsilon \cdot \infty}} R(t, u) \log(1 + e^{F(u, \kappa)}) \frac{du}{u} - \int_1^{e^{-i\epsilon \cdot \infty}} R(t, u) \log(1 + e^{-F(u, \kappa)}) \frac{du}{u} \right). \end{aligned} \quad (10.8)$$

Motivated by the formula (10.5), let us set

$$F(t, \kappa) = \kappa F_0(t) + \int_{-\infty}^{\infty} dl t^{il} S(l) \hat{K}(l) (\Psi(l, \kappa) - \kappa \Psi_0(l)), \quad (10.9)$$

where $\Psi(l, \kappa)$ has an asymptotic expansion,

$$\Psi(l, \kappa) \simeq \sum_{n=0}^{\infty} \kappa^{-2n+1} \Psi_n(l), \quad \Psi_0(l) = \frac{-if}{l(l + \frac{i}{2v})}. \quad (10.10)$$

We show below that each coefficient $\Psi_n(l)$ ($n \geq 1$) can be determined as a polynomial in l by a purely algebraic procedure.

With a change of integration variable, (10.8) is brought further into the form

$$\begin{aligned} \Psi(l, \kappa) - \kappa \Psi_0(l) = & -\frac{i}{f\kappa} \left\{ \int_0^{-i\infty+\varepsilon} \frac{dx}{2\pi} \hat{R}(l, e^{ix/f\kappa}) \log(1 + e^{F(e^{ix/f\kappa}, \kappa)}) \right. \\ & \left. + \int_0^{i\infty+\varepsilon} \frac{dx}{2\pi} \hat{R}(l, e^{-ix/f\kappa}) \log(1 + e^{-F(e^{-ix/f\kappa}, \kappa)}) \right\} \end{aligned} \quad (10.11)$$

Since $\log(1 + e^{\pm F(u, \kappa)})$ decays exponentially for $\pm \text{Im} u > 0$, the asymptotics of the right hand side of (10.11) is completely determined from the behaviour of the integrand at $x = 0$.

In order to develop a systematic expansion, let us first make a general remark. Consider a Fourier integral

$$G(x) = \int_{-\infty}^{\infty} e^{ikx} g(k) dk.$$

We assume that $g(k)$ is the boundary value of a holomorphic function on the lower half plane $\text{Im} k < 0$, satisfying the asymptotic expansion

$$g(k) \simeq \sum_{n=-n_0}^{\infty} g_n (ik)^{-n} \quad (k \rightarrow \infty, \text{Im} k < 0).$$

Then integration by parts shows that for any $N > 0$ we have

$$G(x) = \sum_{n=-n_0}^0 g_n 2\pi \delta^{(n)}(x) + 2\pi \sum_{n=1}^N \frac{g_n}{(n-1)!} x_+^{n-1} + R_N,$$

where $R_N = O(x^N)$ as $x \rightarrow 0$. Suppose further that $G(x)$ can be prolonged analytically around $x = 0$. In this situation its Taylor expansion can be computed from the right hand side, discarding the delta function terms. The result is summarised in a compact form

$$\int_{-\infty}^{\infty} e^{ikx} g(k) dk = 2\pi i \text{res}_k [e^{ikx} g(k)],$$

where $\text{res}_k [\dots]$ signifies the coefficient of k^{-1} in the expansion at $k = \infty$.

The above consideration applies to (10.7), and we obtain the Taylor expansion at $x = 0$,

$$\hat{R}(l, e^{ix/f\kappa}) = \text{res}_h \left[\frac{e^{-hx/f\kappa}}{l+h} S(h) \right]. \quad (10.12)$$

For the factor $\log(1 + e^{F(u, \kappa)})$, we proceed as follows. Set

$$F(e^{ix/f\kappa}, \kappa) = -2\pi(x - \bar{F}(x, \kappa)).$$

Similarly as above, the Taylor expansion of $\bar{F}(x, \kappa)$ at $x = 0$ is calculated as

$$\bar{F}(x, \kappa) = x + \text{res}_h \left[e^{-hx/f\kappa} S(h) i\Psi(h, \kappa) \right]. \quad (10.13)$$

Actually the term x is cancelled by a term coming from $\kappa\Psi_0(h)$, so that $\bar{F}(x, \kappa) = O(\kappa^{-1})$. We rewrite the corresponding piece of the integrand as

$$\log(1 + e^{\pm F(e^{\pm ix/f\kappa}, \kappa)}) = \sum_{n=0}^{\infty} \frac{\bar{F}(\pm x, \kappa)^n}{n!} \left(\mp \frac{\partial}{\partial x} \right)^n \log(1 + e^{-2\pi x}). \quad (10.14)$$

Substituting (10.14), (10.12) into (10.11), we arrive at

$$\begin{aligned} & i\Psi(l, \kappa) - i\kappa\Psi_0(l) \\ & \simeq \frac{2}{f\kappa} \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^{\infty} \frac{dx}{2\pi} \left\{ \text{res}_h \left[\frac{e^{-hx/f\kappa}}{l+h} S(h) \right] \bar{F}(x, \kappa)^n \left(-\frac{\partial}{\partial x} \right)^n \right\}_{\text{even}} \log(1 + e^{-2\pi x}). \end{aligned} \quad (10.15)$$

Here $\{\dots\}_{\text{even}}$ (resp. $\{\dots\}_{\text{odd}}$) means the even (resp. odd) part in x . To evaluate the integral in (10.15) we need only to develop the integrand into a Taylor series and apply the formula

$$\int_0^{\infty} \frac{dx}{2\pi} x^m \left(-\frac{\partial}{\partial x} \right)^n \log(1 + e^{-2\pi x}) = m!(1 - 2^{-m-1+n}) \frac{\zeta(m-n+2)}{(2\pi)^{m-n+2}}.$$

In summary, the asymptotic expansion (10.10) can be calculated order by order in κ^{-2} , from the set of equations (10.15) and (10.13).

The first few terms of the expansion read

$$\begin{aligned} i\Psi(l, \kappa) &= \frac{1}{l(l + \frac{i}{2v})} f\kappa + \frac{1}{24} \frac{1}{f\kappa} \\ &+ \frac{7}{2^6 \cdot 90} \left(l - \frac{i}{2v} \right) \left(l - i \frac{2v^2 - 6v + 6}{7v(1-v)} \right) \frac{1}{(f\kappa)^3} + \dots \end{aligned}$$

In general, the coefficients have the structure

$$\Psi_n(l) = \prod_{j=1}^{n-1} \left(l - \frac{i(2j-1)}{2v} \right) \times (\text{Polynomial in } l \text{ of degree } n-1). \quad (10.16)$$

From the knowledge of $\log a^{\text{sc}}(\lambda, \kappa)$, it is straightforward to extract the asymptotic expansion of $\log T^{\text{sc}}(\lambda, \kappa)$ (10):

$$\begin{aligned} \log T^{\text{sc}}(\lambda, \kappa) &\simeq \pi i \nu \kappa + \frac{\sqrt{1-\nu}}{\sqrt{2\pi}} \int_{\mathbb{R} - \frac{i}{2\nu} - i0} dl \frac{\Gamma(1-il)\Gamma(\frac{1}{2}+i\nu l)}{\Gamma(1-i(1-\nu)l)} \Psi(l, \kappa) \left(\frac{e^{\delta - \pi i \nu} \lambda^2}{\kappa^{2\nu} c(\nu)} \right)^{il} \\ &\simeq \sum_{n=0}^{\infty} C_n I_{2n-1}(\kappa) \lambda^{-\frac{2n-1}{\nu}}, \end{aligned}$$

where

$$\begin{aligned} C_n &= -\frac{\sqrt{\pi}}{\nu} \frac{1}{n!} \frac{\Gamma(\frac{2n-1}{2\nu})}{\Gamma(1 + \frac{1-\nu}{2\nu}(2n-1))} (1-\nu)^n \Gamma(\nu)^{-\frac{2n-1}{\nu}}, \\ I_{2n-1}(\kappa) &= -i\Psi\left(\frac{i(2n-1)}{2\nu}, \kappa\right) \times n(2n-1)(2\nu^2)^{n-1} (f\kappa)^{2n-1} R^{-2n+1}, \end{aligned} \quad (10.17)$$

and we set by definition $I_{-1} = R$. Notice that $C_0 > 0$ while $C_n < 0$ for $n \geq 1$.

The factors in (10.16) ensure that at these special values of l the asymptotic series (10.10) truncates. This has to be the case, because according to (10), $I_{2n-1}(\kappa)$ ($n \geq 1$) are the vacuum eigenvalues of the integrals of motion which are polynomials in c and $\Delta_{\kappa+1}$.

For instance

$$I_1(\kappa) = \frac{1}{R} \left(\Delta_{\kappa+1} - \frac{c}{24} \right), \quad (10.18)$$

$$I_3(\kappa) = \frac{1}{R} I_1(\kappa)^2 - \frac{1}{6R^2} I_1(\kappa) + \frac{c}{1440R^3}, \quad (10.19)$$

$$I_5(\kappa) = \frac{1}{R} I_3(\kappa) I_1(\kappa) - \frac{1}{3R^2} I_3(\kappa) + \frac{c+5}{360R^4} I_1(\kappa) - \frac{c(5c+28)}{181440R^5}. \quad (10.20)$$

We have verified upto $n = 4$ that (10.17) matches perfectly the formulas for I_{2n-1} given in (9).

11 Asymptotics of ω for $\kappa = \kappa'$

In this section, we restrict our consideration to the case $\kappa = \kappa'$, so that $\rho^{\text{sc}}(\lambda | \kappa, \kappa') = 1$. Our goal is to give an algorithm for deriving the asymptotic expansion of the function $\omega^{\text{sc}}(\lambda, \mu | \kappa, \kappa, \alpha)$.

We start from the representation

$$\omega^{\text{sc}}(\lambda, \mu | \kappa, \kappa, \alpha) = (f_{\text{left}} \star f_{\text{right}} + f_{\text{left}} \star R_{\text{dress}} \star f_{\text{right}})(\lambda, \mu) + \omega_0(\lambda, \mu | \alpha), \quad (11.1)$$

where

$$\begin{aligned} f_{\text{left}}(\lambda, \mu, \alpha) &= \frac{1}{2\pi i} \delta_{\lambda}^{-} \psi(\lambda/\mu, \alpha), \quad f_{\text{right}}(\lambda, \mu, \alpha) = \delta_{\mu}^{-} \psi(\lambda/\mu, \alpha), \\ \omega_0(\lambda, \mu | \alpha) &= \delta_{\lambda}^{-} \delta_{\mu}^{-} \Delta_{\lambda}^{-1} \psi(\lambda/\mu, \alpha), \end{aligned}$$

and R_{dress} denotes the resolvent for the integral equation

$$R_{\text{dress}} - R_{\text{dress}} \star K_{\alpha} = K_{\alpha}. \quad (11.2)$$

Here we have set

$$f \star g = \left(\int_{\sigma^2}^{e^{i\epsilon}\infty} - \int_{\sigma^2}^{e^{-i\epsilon}\infty} \right) f(\lambda)g(\lambda)dm(\lambda),$$

$$dm(\lambda) = \frac{d\lambda^2}{\lambda^2(1 + \mathfrak{a}^{\text{sc}}(\lambda, \kappa))},$$

and σ^2 is a point lying between the smallest Bethe root and the largest zero of $T^{\text{sc}}(\lambda, \kappa)$. Strictly speaking, \star and $dm(\lambda)$ have slightly different meaning than those used in Sect. 3. Since we use them here only locally, there should not be fear of confusion.

From now until (11.5) below, we shall work with the variables

$$t = c(\nu)^{-1}\lambda^2/\kappa^{2\nu}, \quad u = c(\nu)^{-1}\mu^2/\kappa^{2\nu},$$

and write

$$K(t, \alpha) = \frac{1}{2\pi i} \cdot \frac{1}{2} \left((tq^2)^{\alpha/2} \frac{tq^2 + 1}{tq^2 - 1} - (tq^{-2})^{\alpha/2} \frac{tq^{-2} + 1}{tq^{-2} - 1} \right).$$

Again we start with the leading order approximation as $\kappa \rightarrow \infty$, where (11.2) becomes

$$R(t, u, \alpha) - \int_1^{\infty} \frac{dv}{v} R(v, u, \alpha) K(t/v, \alpha) = K(t/u, \alpha).$$

This can be solved in the same manner as before, using

$$\hat{K}(k, \alpha) = \frac{\sinh \pi \left((2\nu - 1)k - \frac{i\alpha}{2} \right)}{\sinh \pi \left(k + \frac{i\alpha}{2} \right)}.$$

The only point worth noting is that in writing the Riemann-Hilbert factorisation

$$1 - \hat{K}(k, \alpha) = S(k, \alpha)^{-1} S(-k, 2 - \alpha)^{-1},$$

$$S(k, \alpha) = \frac{\Gamma \left(1 + (1 - \nu)ik - \frac{\alpha}{2} \right) \Gamma \left(\frac{1}{2} + i\nu k \right)}{\Gamma \left(1 + ik - \frac{\alpha}{2} \right) \sqrt{2\pi} (1 - \nu)^{(1-\alpha)/2}} e^{i\delta k},$$

we are naturally led to assume that

$$0 < \alpha < 2.$$

So the naïve symmetry $(-k, -\alpha) \rightarrow (k, \alpha)$ of $\hat{K}(k, \alpha)$ is replaced by the symmetry $(k, \alpha) \rightarrow (-k, 2 - \alpha)$. With our normalisation of α , the reflection $\alpha \rightarrow 2 - \alpha$ is the usual one for CFT with $c < 1$. For the resolvent kernel we obtain the representation

$$R(t, u, \alpha) = K(t/u, \alpha)$$

$$+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dl}{2\pi} \frac{dm}{2\pi} t^{il} u^{im} S(l, \alpha) S(m, 2 - \alpha) \hat{K}(l, \alpha) \hat{K}(m, 2 - \alpha) \frac{-i}{l + m - i0}.$$

The ‘dressed’ resolvent kernel $R_{\text{dress}}(t, u)$ satisfies

$$\begin{aligned} R_{\text{dress}}(t, u) - R(t, u, \alpha) &= - \int_1^{e^{i\epsilon_\infty}} \frac{1}{1 + e^{-F(v, \kappa)}} R(t, v, \alpha) R_{\text{dress}}(v, u) \\ &\times \frac{dv}{v} - \int_1^{e^{-i\epsilon_\infty}} \frac{1}{1 + e^{F(v, \kappa)}} R(t, v, \alpha) R_{\text{dress}}(v, u) \frac{dv}{v}. \end{aligned}$$

Setting

$$\begin{aligned} R_{\text{dress}}(t, u) &= K(t/u, \alpha) \\ &+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dl}{2\pi} \frac{dm}{2\pi} t^{il} u^{im} S(l, \alpha) S(m, 2 - \alpha) \hat{K}(l, \alpha) \hat{K}(m, 2 - \alpha) \Theta(l, m | \kappa, \alpha), \end{aligned} \quad (11.3)$$

$$\Theta(l, m | \kappa, \alpha) \simeq \sum_{n=0}^{\infty} \Theta_n(l, m | \alpha) \kappa^{-2n}, \quad \Theta_0(l, m) = \frac{-i}{l+m}, \quad (11.4)$$

and repeating the analysis of the previous section, we arrive at the linear recursion relation for the $\Theta_n(l, m | \alpha)$:

$$\begin{aligned} \Theta(l, m | \kappa, \alpha) - \Theta_0(l, m) &\simeq - \frac{2}{f\kappa} \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^{\infty} dx \left\{ \text{res}_{l'} \left[\frac{e^{-l'x/f\kappa}}{l+l'} S(l', 2 - \alpha) \right] \right. \\ &\times \left. \text{res}_{m'} \left[e^{-m'x/f\kappa} S(m', \alpha) \Theta(m', m | \kappa, \alpha) \right] \bar{F}(x, \kappa)^n \left(-\frac{\partial}{\partial x} \right)^n \right\}_{\text{odd}} \times \frac{1}{1 + e^{2\pi x}}. \end{aligned}$$

The coefficients of the series (11.4) can be calculated by Taylor expanding the integrand and applying

$$\int_0^{\infty} dx x^m \left(-\frac{\partial}{\partial x} \right)^n \frac{1}{1 + e^{2\pi x}} = m! (1 - 2^{-m+n}) \frac{\zeta(m-n+1)}{(2\pi)^{m-n+1}}.$$

The first non-trivial term reads

$$i\Theta(l, m | \kappa, \alpha) \simeq \frac{1}{l+m} + \frac{i}{24v} \frac{1}{(f\kappa)^2} \left(-iv(l+m) - \frac{1}{2} + \Delta_\alpha \right) + O\left(\frac{1}{\kappa^4}\right).$$

Returning to the original variables λ and μ , the formula for $\omega^{\text{sc}}(\lambda, \mu | \kappa, \kappa, \alpha)$ can be obtained from (11.1). We have

$$\begin{aligned} \omega^{\text{sc}}(\lambda, \mu | \kappa, \kappa, \alpha) &\simeq \frac{1}{2\pi i} \iint dldm \tilde{S}(l, \alpha) \tilde{S}(m, 2 - \alpha) \Theta(l + i0, m | \kappa, \alpha) \\ &\times \left(\frac{e^{\delta + \pi i v} \lambda^2}{\kappa^{2v} c(v)} \right)^{il} \left(\frac{e^{\delta + \pi i v} \mu^2}{\kappa^{2v} c(v)} \right)^{im}, \quad (11.5) \\ \tilde{S}(k, \alpha) &= \frac{\Gamma(-ik + \frac{\alpha}{2}) \Gamma(\frac{1}{2} + ivk)}{\Gamma(-i(1-v)k + \frac{\alpha}{2}) \sqrt{2\pi} (1-v)^{(1-\alpha)/2}}, \end{aligned}$$

where $l + i0$ is important only in $\Theta_0(l, m)$. Notice that originally in $R(t, u, \alpha)$ we had rather $l - i0$. The change appeared due to the addition of $\omega_0(\lambda, \mu)$ which

explains the importance of this term. Picking the residues at the poles in the upper half plane, its asymptotics as $t, u \rightarrow \infty$ can be calculated:

$$\begin{aligned} \omega^{\text{sc}}(\lambda, \mu | \kappa, \alpha) &\simeq - \sum_{r,s=1}^{\infty} \frac{1}{r+s-1} D_{2r-1}(\alpha) D_{2s-1}(2-\alpha) \\ &\quad \times \lambda^{-\frac{2r-1}{v}} \mu^{-\frac{2s-1}{v}} \Omega_{2r-1,2s-1}(\kappa, \alpha), \end{aligned} \quad (11.6)$$

where

$$D_{2n-1}(\alpha) = \sqrt{\frac{i}{v}} \Gamma(v)^{-\frac{2n-1}{v}} (1-v)^{\frac{2n-1}{2}} \cdot \frac{1}{(n-1)!} \frac{\Gamma\left(\frac{\alpha}{2} + \frac{1}{2v}(2n-1)\right)}{\Gamma\left(\frac{\alpha}{2} + \frac{(1-v)}{2v}(2n-1)\right)}, \quad (11.7)$$

and

$$\Omega_{2r-1,2s-1}(\kappa, \alpha) = -\Theta\left(\frac{i(2r-1)}{2v}, \frac{i(2s-1)}{2v} \middle| \kappa, \alpha\right) \times \frac{r+s-1}{v} \left(\frac{\sqrt{2} f \kappa v}{R}\right)^{2r+2s-2}.$$

The counterpart of the factorisation (10.16) for $\Psi_n(l)$ is the vanishing property

$$\Theta_n\left(\frac{i(2r-1)}{2v}, \frac{i(2s-1)}{2v} \middle| \alpha\right) = 0 \quad (n \geq r+s).$$

This ensures that the coefficients $\Omega_{2r-1,2s-1}(\kappa, \alpha)$ are polynomials in $\Delta_{\kappa+1}, \alpha$ and c . For instance,

$$\begin{aligned} \Omega_{1,1}(\kappa, \alpha) &= \frac{1}{R} I_1(\kappa) - \frac{\Delta_\alpha}{12R^2}, \\ \Omega_{1,3}(\kappa, \alpha) &= \frac{1}{R} I_3(\kappa) - \frac{\Delta_\alpha}{6R^3} I_1(\kappa) + \frac{\Delta_\alpha^2}{144R^4} + \frac{c+5}{1080R^4} \Delta_\alpha \mp \frac{\Delta_\alpha}{360R^4} d_\alpha, \\ \Omega_{1,5}(\kappa, \alpha) &= \frac{1}{R} I_5(\kappa) - \frac{\Delta_\alpha}{4R^3} I_3(\kappa) + \left(\frac{\Delta_\alpha^2}{48R^5} + \frac{c+11}{360R^5} \Delta_\alpha\right) I_1(\kappa) \\ &\quad - \frac{\Delta_\alpha^3}{1728R^6} - \frac{13(c+35)}{90720R^6} \Delta_\alpha^2 - \frac{2c^2+21c+70}{60480R^6} \Delta_\alpha \\ &\quad \mp \left(\frac{\Delta_\alpha}{120R^5} I_1(\kappa) - \frac{1}{1440R^6} \Delta_\alpha^2 - \frac{c+7}{7560R^6} \Delta_\alpha\right) d_\alpha, \\ \Omega_{3,3}(\kappa, \alpha) &= \frac{1}{R} I_5(\kappa) - \frac{\Delta_\alpha}{4R^3} I_3(\kappa) + \left(\frac{\Delta_\alpha^2}{48R^5} + \frac{c+2}{360R^5} \Delta_\alpha + \frac{c+2}{1440R^5}\right) I_1(\kappa) \\ &\quad - \frac{1}{1728R^6} \Delta_\alpha^3 - \frac{5c-14}{18144R^6} \Delta_\alpha^2 - \frac{10c^2+37c+70}{362880R^6} \Delta_\alpha - \frac{1/2c^2+c}{36288R^6}. \end{aligned}$$

Here $I_{2n-1}(\kappa)$ are given in (10.18)–(10.20), and

$$d_\alpha = \frac{v(v-2)}{v-1} (\alpha-1) = \frac{1}{6} \sqrt{(25-c)(24\Delta_\alpha+1-c)}. \quad (11.8)$$

These structures exhibit a remarkable consistency with our fermionic picture.

12 Final Results and Conclusions

Now we clearly see the structure of our fermions in the CFT limit. They naturally split into two parts:

$$\beta_{2m-1}^* = D_{2m-1}(\alpha)\beta_{2m-1}^{\text{CFT}*}, \quad \gamma_{2m-1}^* = D_{2m-1}(2-\alpha)\gamma_{2m-1}^{\text{CFT}*}, \quad (12.1)$$

the multipliers $D_{2m-1}(\alpha), D_{2m-1}(2-\alpha)$ absorb all the transcendental dependence on α , the operators $\beta_{2m-1}^{\text{CFT}*}, \gamma_{2m-1}^{\text{CFT}*}$ are purely CFT-objects.

The fermions act between different Verma modules. In order to stay in one Verma module it is convenient to introduce the bilinear combinations of fermions

$$\begin{aligned} \phi_{2m-1,2n-1}^{\text{even}} &= (m+n-1)\frac{1}{2}\left(\beta_{2m-1}^{\text{CFT}*}\gamma_{2n-1}^{\text{CFT}*} + \beta_{2n-1}^{\text{CFT}*}\gamma_{2m-1}^{\text{CFT}*}\right), \\ \phi_{2m-1,2n-1}^{\text{odd}} &= d_\alpha^{-1}(m+n-1)\frac{1}{2}\left(\beta_{2n-1}^{\text{CFT}*}\gamma_{2m-1}^{\text{CFT}*} - \beta_{2m-1}^{\text{CFT}*}\gamma_{2n-1}^{\text{CFT}*}\right). \end{aligned}$$

The Verma module has a basis consisting of the vectors

$$\mathbf{i}_{2k_1-1} \cdots \mathbf{i}_{2k_p-1} \mathbf{l}_{-2l_1}, \cdots, \mathbf{l}_{-2l_q}(\phi_\alpha). \quad (12.2)$$

Conjecturally the same space is also created by the action of the \mathbf{i}_{2k-1} 's and fermions:

$$\mathbf{i}_{2k_1-1} \cdots \mathbf{i}_{2k_p-1} \phi_{2m_1-1,2n_1-1}^{\text{even}} \cdots \phi_{2m_r-1,2n_r-1}^{\text{even}} \phi_{2\bar{m}_1-1,2\bar{n}_1-1}^{\text{odd}} \phi_{2\bar{m}_s-1,2\bar{n}_s-1}^{\text{odd}}(\phi_\alpha). \quad (12.3)$$

For small degrees, the transition coefficients between (12.2) and (12.3), modulo descendants of the \mathbf{i}_{2k-1} , can be determined by taking the expectation values with $\kappa = \kappa'$ and equating like powers of κ . Abbreviating ϕ_α and writing Δ_α as Δ , we find

$$\begin{aligned} \phi_{1,1}^{\text{even}} &\equiv \mathbf{l}_{-2}, \\ \phi_{1,3}^{\text{even}} &\equiv \mathbf{l}_{-2}^2 + \frac{2c-32}{9} \mathbf{l}_{-4}, \\ \phi_{1,3}^{\text{odd}} &\equiv \frac{2}{3} \mathbf{l}_{-4}, \\ \phi_{1,5}^{\text{even}} &\equiv \mathbf{l}_{-2}^3 + \frac{c+2-20\Delta+2c\Delta}{3(\Delta+2)} \mathbf{l}_{-4}\mathbf{l}_{-2} \\ &\quad + \frac{-5600\Delta+428c\Delta-6c^2\Delta+2352\Delta^2-300c\Delta^2+12c^2\Delta^2+896\Delta^3-32c\Delta^3}{60\Delta(\Delta+2)} \mathbf{l}_{-6}, \\ \phi_{1,5}^{\text{odd}} &\equiv \frac{2\Delta}{\Delta+2} \mathbf{l}_{-4}\mathbf{l}_{-2} + \frac{56-52\Delta-2c+4c\Delta}{5(\Delta+2)} \mathbf{l}_{-6}, \\ \phi_{3,3}^{\text{even}} &\equiv \mathbf{l}_{-2}^3 + \frac{6+3c-76\Delta+4c\Delta}{6(\Delta+2)} \mathbf{l}_{-2}\mathbf{l}_{-4} \\ &\quad + \frac{-6544\Delta+498c\Delta-5c^2\Delta+2152\Delta^2-314c\Delta^2+10c^2\Delta^2-448\Delta^3+16c\Delta^3}{60\Delta(\Delta+2)} \mathbf{l}_{-6}. \end{aligned} \quad (12.4)$$

At the next degree, there are 5 Virasoro descendants $\mathbf{I}_{-2}^4, \mathbf{I}_{-4}\mathbf{I}_{-2}^2, \mathbf{I}_{-4}^2, \mathbf{I}_{-6}\mathbf{I}_{-2}, \mathbf{I}_{-8}$, which are polynomials in $\Delta_{\kappa+1}$ of degree 4, 2, 1, 1, 0, respectively. With the data at hand, obtained from the primary field $\phi_{\kappa+1}$, there remains one parameter undetermined. This can be fixed considering the first descendent $L_{-1}\phi_{\kappa+1}$ which we hope to do in the future. Nevertheless we have checked that the determinant,

$$\beta_1^{\text{CFT}^*} \beta_3^{\text{CFT}^*} \gamma_3^{\text{CFT}^*} \gamma_1^{\text{CFT}^*},$$

after subtracting a suitable multiple of \mathbf{I}_{-2}^4 , has the correct degree 2 in $\Delta_{\kappa+1}$. We regard it as further supporting evidence in favour of the fermionic structure.

Let us pass to conclusions.

We believe that the fermionic description will provide new results for the theory of integrable models. For example, there is an obvious similarity between our fermions and those introduced in (20). With the formulae (12.4) at hand, it should be possible to upgrade the qualitative description of form factors of descendants in (20) to a quantitative level. We hope to explain this in future works. Here, however, we would like to emphasise that, even for CFT, the fermionic description must give something completely new. Let us explain that.

Consider the functional $Z_R^{\kappa, \kappa'}$ with $\kappa = \kappa'$. It describes the three point function for descendants of ϕ_α and two primary fields $\phi_{-\kappa+1}, \phi_{\kappa+1}$ of equal dimension $\Delta_{-\kappa+1} = \Delta_{\kappa+1}$. It was said several times that the construction generalises if we replace the asymptotic states described by $\phi_{\kappa+1}$ by any other eigenstate of the integrals of motion I_{2n-1} . The only change is that the function ω is to be computed for the new asymptotic condition. It is assumed (9; 10) that the joint spectrum of I_{2n-1} is simple, so, in this way we compute all the three-point functions for a descendant of ϕ_α and descendants of $\phi_{-\kappa+1}, \phi_{\kappa+1}$, provided the latter are eigenstates of the integrals of motion. Notice that the descendant of $\phi_{\kappa+1}$ can be very deep in the Verma module. In that case the usual CFT computation is rather hard to perform. Let us be more precise appealing to the classical limit.

In the classical limit $\nu \rightarrow 1$, the eigenstates of I_{2n-1} are in correspondence with the periodic solutions of the KdV equation. Let us give some explanation about this point.

Consider the classical KdV hierarchy with the second Poisson structure:

$$\{u(y_1), u(y_2)\} = 2(u(y_1) + u(y_2))\delta'(y_1 - y_2) + \delta'''(y_1 - y_2).$$

The integrability of the KdV equation is due to existence of the auxiliary linear problem:

$$(\partial_y^2 + u(y))\psi(y, \lambda) = \lambda^2 \psi(y, \lambda).$$

We consider the periodic case $u(y + 2\pi R) = u(y)$. In this case one defines the monodromy matrix $M(\lambda)$ for the auxiliary linear problem in a standard way. Then the local integrals of motion in involution are found in the asymptotical expansion of $T^{\text{cl}}(\lambda) = \text{Tr}M(\alpha)$ for $\lambda^2 \rightarrow +\infty$:

$$\log(T^{\text{cl}}(\lambda)) \simeq 2\pi R\lambda + \sum_{n=1}^{\infty} C^{\text{cl}}_{I_{2n-1}^{\text{cl}}} \lambda^{-(2n-1)},$$

where

$$C_n^{\text{cl}} = -\sqrt{\pi} \frac{\Gamma\left(\frac{2n-1}{2}\right)}{n!}.$$

The integrals of motion I_{2n-1}^{cl} are well-known functionals of $u(y)$. Here we use for them the normalisation of $f(9)$. So, the classical limit is

$$T(-iy) \rightarrow \frac{1}{1-\nu} u(y).$$

It brings the Virasoro commutation relations to the second Poisson structure of the KdV hierarchy, and ensures the finite limits

$$(1-\nu)^n I_{2n-1} \rightarrow I_{2n-1}^{\text{cl}}.$$

It is well known that periodic solutions of KdV are in correspondence with hyper-elliptic Riemann surfaces which are a two-fold covering of the Riemann sphere of λ^2 . In particular, the solution corresponding after the quantisation to the primary field $\phi_{\kappa+1}$ corresponds to the Riemann surface of genus 0:

$$\mu^2 = \lambda^2 - \kappa^2.$$

From the point of view of classical theory, this is a completely trivial case which describes a constant solution of KdV. This case becomes non-trivial after the quantisation, because KdV is a theory with infinitely many degrees of freedom, and quantising the simplest classical solution one has to take into account infinitely many zero oscillations (see (21) for a relevant discussion). Still it is rather unpleasant to be able to quantise only trivial classical solutions. The consideration of the usual, low-lying descendants of $\phi_{\kappa+1}$ does not change the situation seriously: they describe excitations for the same classical solution. What are really interesting solutions in the classical case? They correspond to other Riemann surfaces. The simplest one is described by the elliptic curve:

$$\mu^2 = (\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2)(\lambda^2 - \lambda_3^2).$$

At the quantum level, this solution corresponds to the following distribution of the Bethe roots over the real axis in the plane of λ^2 . Going from $\lambda^2 = -\infty$ we first have no Bethe roots. Then there is a large interval where the Bethe roots are dense. Then there is a large interval without Bethe roots, wherein we find one or several zeros of $T^{\text{sc}}(\lambda, \kappa)$. Then starting from a certain point and up to $\lambda^2 = \infty$, the Bethe roots are again dense.

For a reader who is not familiar with periodic solutions of KdV, it is useful to think about this solution as a periodic analogue of one-soliton solution which we really obtain in the limit $R \rightarrow \infty$. Everybody would agree that quantising only the trivial solutions when there are solitons around is a waste of such possibility.

Our fermionic construction gives a possibility to treat this kind of asymptotic states. Moreover, we suppose that the function ω has a clear algebra-geometric meaning in the classical limit. We hope to return to all that in one of our future publications.

A General Results on the Asymptotics of $\omega^{\text{sc}}(\lambda, \mu | \kappa, \kappa', \alpha)$

In this section we derive the asymptotic behaviour (9.3) of $\omega^{\text{sc}}(\lambda, \mu | \kappa, \kappa', \alpha)$ when $\lambda^2, \mu^2 \rightarrow \infty$. The main point of the argument is that in a certain domain, which we call the A-domain, the expansion (9.2) of $\log \rho^{\text{sc}}(\lambda | \kappa, \kappa')$ holds for both λ and λq^{-1} , and by the cancellation due to

$$\lambda^{\frac{1}{v}} = -(\lambda q^{-1})^{\frac{1}{v}},$$

we have $\rho^{\text{sc}}(\lambda | \kappa, \kappa') \rho^{\text{sc}}(\lambda q^{-1} | \kappa, \kappa') \simeq 1$. We shall suppress the arguments κ, κ' and α in $\rho^{\text{sc}}(\lambda | \kappa, \kappa')$ and $\omega^{\text{sc}}(\lambda, \mu | \kappa, \kappa', \alpha)$. We set

$$\rho^{\text{sc}}(\lambda) = \frac{T^{\text{sc}}(\lambda, \kappa')}{T^{\text{sc}}(\lambda, \kappa)}, \quad \alpha^{\text{sc}}(\lambda) = \frac{Q^{\text{sc}}(\lambda q, \kappa)}{Q^{\text{sc}}(\lambda q^{-1}, \kappa)}.$$

For $\omega^{\text{sc}}(\lambda, \mu)$, after simple computations we get:

$$\omega^{\text{sc}}(\lambda, \mu) = (f_{\text{left}} \star f_{\text{right}} + f_{\text{left}} \star R_{\text{dress}} \star f_{\text{right}})(\lambda, \mu) + \delta_{\lambda}^{-} \delta_{\mu}^{-} \Delta_{\lambda}^{-1} \psi(\lambda/\mu, \alpha). \quad (\text{A.1})$$

The symbol \star stands for integration over the contour γ going clockwise around the zeros of $Q^{\text{sc}}(\lambda, \kappa)$ with the measure

$$dm(\theta) = \frac{d\theta^2}{\theta^2 \rho^{\text{sc}}(\theta) (1 + \alpha^{\text{sc}}(\theta))}.$$

Here again $\star, dm(\lambda)$ and R_{dress} are slightly different than those used in Sect. 3 or Sect. 11, but this should not cause any confusion.

The measure $dm(\lambda)$ has simple poles at the zeros of $Q^{\text{sc}}(\lambda, \kappa)$ and $T^{\text{sc}}(\lambda, \kappa')$. For simplicity of presentation, we assume κ and κ' are close enough so that any zero of $T^{\text{sc}}(\lambda, \kappa')$ is smaller than any zero of $Q^{\text{sc}}(\lambda, \kappa)$.

In this section if we say $f(\lambda) \simeq g(\lambda)$ on some half line of $\arg(\lambda^2)$, it means $f(\lambda) = g(\lambda) + O(|\lambda|^{-N})$ for all N there.

From (10) one concludes that

$$\begin{aligned} \log \alpha^{\text{sc}}(\lambda) &= -F_+(\arg(\lambda^2)) |\lambda|^{\frac{1}{v}} + O(|\lambda|^{-\frac{1}{v}}), \quad 0 < \arg(\lambda^2) < \pi, \\ \log \alpha^{\text{sc}}(\lambda) &= F_-(\arg(\lambda^2)) |\lambda|^{\frac{1}{v}} + O(|\lambda|^{-\frac{1}{v}}), \quad -\pi < \arg(\lambda^2) < 0, \end{aligned}$$

where $F_{\pm}(\arg(\lambda^2))$ are some functions taking positive values in corresponding domains. Hence $\alpha^{\text{sc}}(\lambda)$ decays rapidly in the upper half plane and grows rapidly in the lower half plane.

Following (10) we write in the integral over the upper bank in (A.1) using

$$\frac{1}{1 + \alpha^{\text{sc}}(\eta)} = 1 - \frac{1}{1 + \bar{\alpha}^{\text{sc}}(\eta)}, \quad \bar{\alpha}^{\text{sc}}(\eta) = \frac{1}{\alpha^{\text{sc}}(\eta)},$$

in order to separate the rapidly decreasing part. To formalise the story we introduce the notation

$$f \star g = f \circ g - f * g,$$

where

$$f \circ g = \int_{\sigma^2}^{e^{i0}\infty} f(\lambda) g(\lambda) \frac{d\lambda^2}{\lambda^2 \rho^{\text{sc}}(\lambda)}, \quad f * g = \int_{\sigma^2}^{\infty} f(\lambda) g(\lambda) d\tilde{m}(\lambda).$$

Here σ^2 is an arbitrary point lying between the largest zero of $T(\alpha, \kappa')$ and the smallest zero of $Q^{\text{sc}}(\lambda, \kappa)$, and the modified measure is

$$d\tilde{m}(\lambda) = \frac{d\lambda^2}{\lambda^2 \rho^{\text{sc}}(\lambda)} \left(\frac{1}{1 + \bar{\alpha}^{\text{sc}}(\lambda e^{i0})} + \frac{1}{1 + \bar{\alpha}^{\text{sc}}(\lambda e^{-i0})} \right).$$

Introduce the resolvent R by the equation

$$R - K_\alpha \circ R = K_\alpha,$$

and two “dressed” kernels:

$$F_{\text{left}} = f_{\text{left}} + f_{\text{left}} \circ R, \quad F_{\text{right}} = f_{\text{right}} + R \circ f_{\text{right}}, \quad (\text{A.2})$$

where the functions $f_{\text{left}}(\lambda, \eta)$, $f_{\text{right}}(\eta, \lambda)$ are defined in (3.10). They are singular at $\eta^2 = \lambda^2$. According to our general prescription we understand real λ^2 in them as $\lambda^2 e^{-i0}$, and then continue analytically. The equation for the resolvent takes the form

$$R_{\text{dress}} + R * R_{\text{dress}} = R * R_{\text{dress}} + R_{\text{dress}} = R,$$

and the definition of ω can be rewritten as

$$\begin{aligned} \omega^{\text{sc}}(\lambda, \mu) &= \omega^{(1)}(\lambda, \mu) + \omega^{(2)}(\lambda, \mu), \\ \omega^{(1)}(\lambda, \mu) &= (-F_{\text{left}} * F_{\text{right}} + F_{\text{left}} * R_{\text{dress}} * F_{\text{right}})(\lambda, \mu), \\ \omega^{(2)}(\lambda, \mu) &= (f_{\text{left}} \circ F_{\text{right}})(\lambda, \mu) + \delta_\lambda^- \delta_\mu^- \Delta_\lambda^{-1} \psi(\lambda/\mu, \alpha). \end{aligned}$$

Now we are ready to study the asymptotic behaviour. We shall consider λ^2 and μ^2 in the A-domain defined as follows: $\pi(2\nu - 1) < \arg(\lambda^2), \arg(\mu^2) < \pi$. We prove the correct asymptotic behaviour there, then assume that it is valid for all $-\pi < \arg(\lambda^2), \arg(\mu^2) < \pi$. The latter assumption is not even necessary for our goals, but we do not see why it should not be true having in mind that the only infinite series of poles of $\omega(\lambda, \mu)$ are the zeros of $T^{\text{sc}}(\lambda, \kappa)$ which accumulate to $\lambda^2 = -\infty$.

The importance of the A-domain is due to the fact that in it

$$\rho(\lambda)\rho(\lambda q^{-1}) \simeq 1. \quad (\text{A.3})$$

Introduce the operation

$$\delta_\lambda^+ f(\lambda) = f(\lambda) + \rho(\lambda)f(\lambda q^{-1}). \quad (\text{A.4})$$

Using the definitions it is not hard to show that for λ^2, μ^2 in the A-domain

$$\delta_\lambda^+ F_{\text{left}}(\lambda, \eta) \simeq 0, \quad \delta_\mu^+ F_{\text{right}}(\eta, \mu) \simeq 0.$$

These equations imply

$$\begin{aligned} F_{\text{left}}(\lambda, \eta) &\simeq \sqrt{\rho(\lambda)} \sum_{k=1}^{\infty} \lambda^{-\frac{2k-1}{\nu}} F_{\text{left}, k}(\eta), \\ F_{\text{right}}(\eta, \mu) &\simeq \sqrt{\rho(\mu)} \sum_{k=1}^{\infty} \mu^{-\frac{2k-1}{\nu}} F_{\text{right}, k}(\eta). \end{aligned} \quad (\text{A.5})$$

It is easy to argue that $F_{\text{left}, k}(\eta), F_{\text{right}, k}(\eta)$ grow for $\eta \rightarrow \infty$ as powers of η . This is enough to ensure that the “connected part” $\omega^{(1)}(\lambda, \mu)$ has the desired asymptotics: we just substitute the asymptotics (A.5) into the formula for $\omega^{(1)}(\lambda, \mu)$ and observe that all the integrals converge because of exponential in η decay of $d\tilde{m}(\eta)$. With the “disconnected part” $\omega^{(2)}(\lambda, \mu)$ the situation is far more delicate, studying it we shall understand the importance of the term $\delta_\lambda^- \delta_\mu^- \Delta_\lambda^{-1} \psi(\lambda/\mu, \alpha)$ in $\omega(\lambda, \mu)$.

Let us evaluate the last term in $\omega^{(2)}(\lambda, \mu)$ considering $\lambda^2, \mu^2 > \sigma^2$. Using the definition (2.10) it is easy to see that

$$\delta_\mu^- \Delta_\lambda^{-1} \psi(\lambda/\mu) = - \int_0^{e^{i0\infty}} \frac{1}{2\nu \left(1 + (\lambda/\eta)^{\frac{1}{\nu}}\right)} f_{\text{right}}(\eta, \mu) \frac{d\eta^2}{2\pi i \eta^2} + \delta_\mu^+ \frac{1}{4\nu \left(1 - (\lambda/\mu)^{\frac{1}{\nu}}\right)}.$$

The function $F_{\text{right}}(\eta, \mu)$ allows analytical continuation with respect to η , so, we shall use it for all $\eta^2 \in \mathbb{R}_+$. Substitute $f_{\text{right}} = F_{\text{right}} - K\alpha \circ F_{\text{right}}$ and compute the integral

$$\int_0^\infty \frac{K_\alpha(\theta, \eta)}{2\nu(1 + (\lambda/\theta)^{\frac{1}{\nu}})} \frac{d\theta^2}{\theta^2} = -\psi(\lambda/\eta, \alpha) - \frac{1}{2\nu(1 - (\lambda/\eta)^{\frac{1}{\nu}})}.$$

We get after some straightforward computations

$$\omega^{(2)}(\lambda, \mu) = \omega^{(3)}(\lambda, \mu) + \omega^{(4)}(\lambda, \mu),$$

where

$$\omega^{(3)}(\lambda, \mu) = -\int_0^{\sigma^2} \delta_\lambda^- \frac{1}{2\nu(1 + (\lambda/\eta)^{\frac{1}{\nu}})} \cdot F_{\text{right}}(\eta, \mu) \frac{d\eta^2}{2\pi i \eta^2}, \quad (\text{A.6})$$

$$\omega^{(4)}(\lambda, \mu) = -VP \int_{\sigma^2}^\infty \delta_\lambda^- \delta_\eta^+ \frac{1}{2\nu(1 - (\lambda/\eta)^{\frac{1}{\nu}})} \cdot F_{\text{right}}(\eta, \mu) \frac{d\eta^2}{2\pi i \eta^2 \rho(\eta)}, \quad (\text{A.7})$$

where the principal value refers to the pole at $\eta^2 = \mu^2$.

For $\omega^{(3)}(\lambda, \mu)$ the asymptotics of the kind (9.3) follows immediately from (A.5) and

$$\delta_\lambda^+ \delta_\lambda^- \frac{1}{2\nu(1 + (\lambda/\eta)^{\frac{1}{\nu}})} \simeq 0.$$

Consider $\omega^{(4)}(\lambda, \mu)$. We check the equations

$$\delta_\lambda^+ \omega^{(4)}(\lambda, \mu) \simeq 0, \quad \delta_\mu^+ \omega^{(4)}(\lambda, \mu) \simeq 0. \quad (\text{A.8})$$

The first of them follows immediately from two facts. First,

$$\delta_\lambda^+ \delta_\lambda^- \delta_\eta^+ \frac{1}{2\nu(1 - (\lambda/\eta)^{\frac{1}{\nu}})} \simeq 0.$$

Second, writing explicitly

$$\delta_\lambda^- \delta_\eta^+ \frac{1}{2\nu(1 - (\lambda/\eta)^{\frac{1}{\nu}})} = \frac{\rho(\eta) - \rho(\lambda)}{2\nu(1 - (\lambda/\eta)^{\frac{1}{\nu}})} + \frac{1 - \rho(\lambda)\rho(\eta)}{2\nu(1 + (\lambda/\eta)^{\frac{1}{\nu}})}, \quad (\text{A.9})$$

and recalling the asymptotic expansion for $\rho(\lambda), \rho(\eta)$, we see that asymptotically for $\lambda^2 \rightarrow +\infty, \eta^2 \rightarrow +\infty$ the singularities in (A.9) disappear. Altogether we have for the asymptotics in both arguments,

$$\delta_\lambda^- \delta_\eta^+ \frac{1}{2\nu(1 - (\lambda/\eta)^{\frac{1}{\nu}})} \simeq \sqrt{\rho(\lambda)} \sum_{m,n=1}^\infty \lambda^{-\frac{2m-1}{\nu}} \eta^{-\frac{n}{\nu}} C_{m,n}. \quad (\text{A.10})$$

To prove the second equation in (A.8) it is not sufficient to use (A.5) because $F_{\text{right}}(\eta, \mu)$ has simple poles at $\eta^2 = \mu^2$ and $\eta^2 = \mu^2 q^2$ which contribute to the analytic continuation $\mu \rightarrow \mu q^{-1}$. However, it is easy to see that the corresponding contributions cancel.

Using the first of Eqs. (A.8) we get

$$\omega^{(4)}(\lambda, \mu) \simeq \sqrt{\rho(\lambda)} \sum_{m=1}^\infty \lambda^{-\frac{2m-1}{\nu}} \omega_m^{(4)}(\mu),$$

where due to (A.10) the functions $\omega_m^{(4)}(\mu)$ are given by convergent integrals. These functions satisfy $\delta_\mu^+ \omega_m^{(4)}(\mu) \simeq 0$, and do not grow for $\mu^2 \rightarrow +\infty$. Hence $\omega^{(4)}(\lambda, \mu)$ has the asymptotics of the kind (9.3).

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