"Moduli space of self-dual connections in dimension greater than four for abelian Gauge groups"

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Abstract

In 1954, C. Yang and R. Mills created a Gauge Theory for strong interaction of Elementary Particles. More generally, they proved that it is possible to define a Gauge Theory with an arbitrary compact Lie group as Gauge group. Within this context, it is interesting to find critical values of a functional defined on the space of connections: the Yang-Mills functional. If the based manifold is four dimensional, there exists a natural notion of (anti-)self-dual 2-form, which gives a natural notion of (anti-)self-dual connection. Such connections give critical values of the Yang-Mills functional. Moreover, the Gauge group acts on the set of (anti-)self-dual connections. The set of (anti-)self-dual connections modulo the Gauge group is called the Moduli space of (anti-)self-dual connections. It is interesting for physicists because it provides critical values of the Yang-Mills functional and for mathematicians because it is an invariant of the based manifold. In dimension greater than four...

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Moduli space of self-dual connections in dimension greater than four for abelian Gauge groups

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Introduction

§1. State of the art

In 1954, Chen N. Yang and Robert L. Mills published their groundbreaking paper *Conservation of Isotopic Spin and Isotopic Gauge Invariance* [YM54] in the Physical Review that would revolutionize the Physics. Generalizing the well-known Gauge Theory for Electromagnetism, they built a Gauge Theory for the strong interaction of Elementary Particles. In this context, the Gauge group is $SU(2)$ instead of $U(1)$. In fact, they even discovered that this theory can be generalized with any arbitrary compact Lie group as Gauge group.

G. 't Hooft has summarized this idea in the preface of ['H05] published 50 years later:

> Gauge Theory has [...] grown into a pivotal concept in the Theory of Elementary Particles, and it is expected to play an equally essential role in even more basic theoretical constructions that are speculated upon today, with the aim of providing an all-embracing picture of the universal Laws of Physics. (['H05, Preface])

One of the main ideas of C.N. Yang and R.L. Mills was to examine invariants under *local*, instead of global, symmetries of the strong interaction in Elementary Particle Physics. Mathematical objects that model local symmetries are the *principal bundles* with the Gauge groups as Lie groups.

Another main idea of C.N. Yang and R.L. Mills was to add a quadratic term, a commutator, to the field strength to get a Gauge invariant field strength. This commutator is in fact hidden in Electromagnetic Gauge Theory because of the commutativity of the Gauge group. C.N. Yang has expressed this idea in ['H05]:

> It was only in 1953-1954 when Bob Mills and I revisited the problem and tried adding quadratic terms to the field

> strength $F_{\mu\nu}$ that an elegant theory emerged. For Mills and me it was many years later that we realized the quadratic terms were in fact *natural* from the mathematical viewpoint. (['H05, Chapter 1])

He means that a field strength with its added commutator corresponds mathematically to the *curvature of a connection on the principal bundle.*

If the reader would like to learn more about the physical viewpoint of Yang-Mills theory, a must would be their initial article [YM54]. Moreover, among all the books and articles covering this subject, we point out *50 years of Yang-Mills Theory* edited by G. 't Hooft in 2005 ['H05]. In our text, we are looking at some mathematical viewpoints.

Let P be a principal bundle with compact Lie group G over a compact Riemannian manifold M. Let us denote by E_{Ad} the associated vector bundle corresponding to the adjoint representation of G (definition in Subsection 1.5.1). The curvature F of a connection 1-form α on P (Definitions 1.5.2 and 1.5.4) can be seen as an element of $\Lambda^2(M)\otimes\Gamma^\infty(E_{Ad})$ (Definition 1.5.10). The *Yang-Mills functional* is defined on the set of connection 1-forms on P by

$$\mathcal{YM}(\alpha) = \int_M |F|^2\epsilon, \tag{1}$$

where the norm on $\Lambda^2(M)\otimes\Gamma^\infty(E_{Ad})$ is defined with respect to the Riemannian metric and a given Ad_G-invariant scalar product on $\mathfrak{g} := Lie(G)$ (which exists because G is compact) and ϵ is a given volume form on M.

The Euler-Lagrange equation of the Yang-Mills action is

$$D^*F = 0 \tag{2}$$

for $D : \Lambda^k(P)\otimes\mathfrak{g} \to \Lambda^{k+1}(P)\otimes\mathfrak{g}$ the exterior covariant differentiation ([KN63, Section II.5]) and D^* its adjoint.

A curvature F of a connection 1-form α on P which is a solution of this Euler-Lagrange equation is called a *Yang-Mills field*. In general, searching Yang-Mills fields is not easy.

With respect to the Hodge-star operator $*$ (definition in Section 2.1),

$$D^* = - * D * .$$

So the curvature F of a connection 1-form α is a Yang-Mills field if and only if

$$D * F = 0.$$

If M is 4-dimensional, it is easy to characterize some Yang-Mills fields. To build them, let us remark that $*^2 = 1$ on $\Lambda^2(M)$ so $\Lambda^2(M)$ splits into two parts: the eigenspace of eigenvalue 1, written $\Lambda^2_+(M)$, and the eigenspace of eigenvalue -1, written $\Lambda^2_-(M)$. Elements of $\Lambda^2_+(M)$ (respectively $\Lambda^2_-(M)$) are called *self-dual* 2*-forms* (respectively *anti-self-dual* 2*-forms*) (Definition 2.1.1).

A connection 1-form is called *(anti-)self-dual* if the 2-form part of its curvature is (anti-)self-dual. The curvature F of any self-dual or anti-self-dual connection 1-form is automatically a Yang-Mills field because the Bianchi identity says that $DF = 0$. Moreover, self-dual connections minimize the Yang-mills functional (see [Tau82, Section 1] and [GP87, Section 1]).

An (anti-)self dual connection is mapped to an (anti-)self-dual connection by the (global) Gauge group (i.e. the group of vertical automorphisms of the principal bundle). Hence mathematicians and physicists are interested in the space of (anti-)self-dual connections on P modulo Gauge equivalence. It is called the *moduli space of (anti-)self-dual connections on P* and denoted by $\mathcal{M}$.

Topological and differential structures of this moduli space have been important studies for both mathematicians and physicists during the seventies and eighties. Famous scientists have worked on this project. We can for example list A.A. Belavin, A.M. Polyakov, A.S. Schwartz and Y.S. Tyupkin in 1975 [BPST75], M.F. Atiyah, N.J. Hitchin, V.G. Drinfel′d and Y.I Manin in 1978 [AHDM78], M.F. Atiyah, N.J. Hitchin and I.M. Singer in 1978 [AHS78], T.H. Parker in 1982 [Par82], C.H. Taubes in 1982 [Tau82], M.F. Atiyah and R. Bott in 1983 [AB83], S.K. Donaldson in 1983 [Don83], M. Itoh in 1983 [Ito83], D.S. Freed and K.K. Uhlenbeck 1984 [FU84]...

But physicists and mathematicians are interested in the search of Yang-Mills fields in every dimension. Hence they have been quickly interested in the search of possible generalized definitions of (anti-)self-dual connections and moduli spaces in any dimension. It was first of all initiated by the physicists E. Corrigan, C. Devchand, D. Fairlie and J. Nuyts in 1983 [CDFN83] and then by the mathematicians S.K. Donaldson and R.P. Thomas in 1998 [DT98]. In particular, they have worked on spaces of dimension greater than 4. Papers about Yang-Mills theory, (anti-)self-duality and moduli spaces in dimension greater than four have emerged in subsequent years: G. Tian in 2000 [Tia00], S.K. Donaldson and E. Segal in 2011 [DS11], A. Haydys in 2012 [Hay12], Y. Tanaka in 2012 [Tan12], S. Wang in 2015 [Wan15], V. Muñoz and C.S. Shahbazi in 2017 [MS17]...

In particular, they have found different suitable ways to extend the notion of (anti-)self-duality in higher dimension. One of those is particularly interesting for us: if Ω is a closed $(n-4)$-form on M (where $dim(M) = n$), we can say that a 2-form μ is (anti-)self-dual if

$$*\mu = \pm\mu \wedge \Omega.$$

This definition of (anti-)self-duality is used, among others, by G. Tian in [Tia00]. As in the four dimensional case, curvatures of (anti-)self-dual connections are automatically Yang-Mills fields. Indeed, if α is (anti-)self-dual, then

$$\begin{aligned} D * F &= \pm D(F \wedge \Omega) \\ &= \pm\left((DF) \wedge \Omega + F \wedge d\Omega\right) \\ &= 0 \end{aligned}$$

because Ω is closed and thanks to the Bianchi identity. Moreover, if the norm of Ω is less than 1, anti-self-dual connections minimize the Yang-Mills functional. This fact is claimed in [Tia00, Section 1].

§2. Our contribution

We present in this text the author's contribution in this framework. We are working in a context which provides the form Ω for free: almost Kähler manifolds (Definition 2.2.3). If (M, g, J, ω) is an almost Kähler manifold of real dimension $2n$, then

$$\frac{\omega^{\wedge(n-2)}}{(n-1)!}$$

is a closed $(2n-4)$-form. Hence, we can consider the following definition: a 2-form μ on M is (anti-)self-dual (Definition 2.2.4) if

$$*\mu = \pm\mu \wedge \frac{\omega^{\wedge(n-2)}}{(n-1)!}.$$

Lemma 2.1.8 shows that taking $\frac{\omega^{\wedge(n-2)}}{(n-2)!}$ as closed $(2n-4)$-form would be consistent too. We will give a word about our choice of normalization in the conclusion. Moreover, the same lemma proves that there is no anti-self-dual 2-form for our choice of definition. So we are interested in self-dual 2-forms for the $(2n-4)$-form $\frac{\omega^{\wedge(n-2)}}{(n-1)!}$.

The definitions of self-dual connections, Gauge group and moduli space of self-dual connections are naturally adapted (Definitions 3.1.1, 3.1.3 and 3.1.5). As explain above, curvatures of such self-dual connections are Yang-Mills fields. However, we do not know if self-dual connections minimize the Yang-Mills functional. The goal of our work is to identify suitable hypotheses under which we are able to characterize the moduli space of self-dual connections for our choice of definition and to build a Lie group structure on it.

First of all, with proper hypotheses, we characterize the moduli space $\mathcal{M}$ of self-dual connections. The more restrictive hypothesis asks that the Gauge group is *abelian*. If the Gauge group is abelian, we write it Z instead of G to avoid confusions. Moreover, M has to be compact, connected and of real dimension $2n > 4$. If $\pi : P \to M$ is a Z-principal bundle, we prove that either $\mathcal{M}$ is empty, or $\mathcal{M}$ is in bijection with

$$H^1(M, \mathfrak{z})/K^Z,$$

where $H^1(M, \mathfrak{z})$ is the de Rham cohomology of M valued in $\mathfrak{z} := Lie(Z)$ and $K^Z := \{[\overline{\varphi}^{-1}d\overline{\varphi}] | \overline{\varphi} \in \mathcal{C}^\infty(M, Z)\}$ (Theorem 3.2.3).

Secondly, if we add a connectedness hypothesis on the Gauge group and if $\mathcal{M}$ is non empty, then we prove that there exists a manifold structure on $\mathcal{M}$ which turns it into an abelian Lie group (Theorem 3.2.9).

For now, physicists are of course mainly interested in non-abelian Gauge group. We hope that our theorems will be generalized in the future. We will give a word about it in the conclusion.

Here is the outline of the thesis. In order to prove both theorems, we remind in Chapter 1 well known notions of algebraic topology and differential geometry. It deals among others with de Rham cohomology, symplectic vector spaces and manifolds, line integrals, Lie groups, the path lifting property, fiber bundles and symmetric spaces. It is not a complete text about discussed subjects. It simply gives definitions and theorems needed for the understanding of what follows thereafter. Each topic is given with suitable references where the reader can find more details.

Chapter 2 deals with the generalized definition of self-duality of 2-forms for spaces of dimension greater than 4. The first section is devoted to the case of a vector space and the second one to the case of a manifold.

The first section begins with some recalls about the Hodge-star operator and the notion of (anti-)self-duality in dimension 4 (Definition 2.1.1). Then we give the chosen definition of self-duality for spaces of dimension greater than 4. Vector spaces suitable for our generalized definition are the Kähler vector spaces (Definition 2.1.2). On those spaces, we consider the following definition (Definition 2.1.3):

Definition A. *Let* (V, ω, J, g) *be a Kähler vector space of real dimension* $2n \geq 4$ *endowed with the orientation given by the canonical volume form of the underlying symplectic vector space* (V, ω) *(see Section 1.1). A* 2*-form* μ *on* V *is called* generalized self-dual *(or simply* self-dual*) if*

$$*\mu = \mu \wedge \frac{\omega^{\wedge(n-2)}}{(n-1)!}.$$

Eventually, and it is where it gets interesting, we prove a characterization of the space of self-dual 2-forms (Proposition 2.2.5):

Proposition B. *Let* (V, ω, J, g) *be a Kähler vector space of real dimension* $2n > 4$ *endowed with the orientation given by the canonical volume form of the underlying symplectic vector space* (V, ω) *(see Section 1.1). Then a* 2*-form* μ *on* V *is self-dual if and only if there exists* $c \in \mathbb{R}$ *such that* $\mu = c\omega$.

This proposition holds only if the dimension of V is strictly greater than 4. It is a beautiful result, easy to use in the following.

The proof is based on Lemma 2.1.6. The latter asserts that there exists an orthonormal Darboux basis on each Kähler vector space (i.e. an orthonormal basis for the metric which is also a Darboux basis for the symplectic form). With respect to an orthonormal Darboux basis, we compute explicitly the Hodge star operator and determine a useful decomposition of $\Lambda^2(V^*)$ (Lemmas 2.1.7 and 2.1.8) which eventually allows to prove the characterization.

The structure of the second section is a carbon copy of the structure of the first one, extended to the case of almost Kähler manifolds. The definition of the self-duality is clear (Definition 2.2.4):

Definition C. *Let* (M, g, J, ω) *be an almost Kähler manifold of real dimension* $2n \geq 4$ *endowed with the orientation given by the canonical volume form of the underlying symplectic manifold* (M, ω) *(see Section 1.1). A* 2*-form* μ *on* M *is called* generalized self-dual *(or simply* self-dual*) if*

$$*\mu = \mu \wedge \frac{\omega^{\wedge(n-2)}}{(n-1)!}.$$

The characterization of the self-dual 2-forms comes directly from the vector space case (Theorem 2.1.9).

Theorem D. *Let (M, g, J, ω) be an almost Kähler manifold of real dimension $2n > 4$ endowed with the orientation given by the canonical volume form of the underlying symplectic manifold (M, ω) (see Section 1.1). Then a 2-form μ is self-dual if and only if there exists $c \in \mathcal{C}^\infty(M, \mathbb{R})$ such that*

$$\mu = c\omega.$$

In Chapter 3, we study the moduli space of self-dual connections. This chapter is split into two sections. While the first one gives general definitions of self-dual connection, Gauge group and moduli space, the second one looks at moduli space of self-dual connections in a particular context: the case of an abelian Gauge group. it Z in stead of G.

To define what is a self-dual connection, by abuse, we can say that its curvature has to be a self-dual 2-form on M. As explained above, the curvature of a connection can be seen as an element of $\Lambda^2(M) \otimes \Gamma^\infty(E_{Ad})$. Precisely, a connection is called self-dual if the 2-form part of its curvature is self-dual. In the beginning of Section 3.1, we give this definition (Definition 3.1.1) and two examples (Example 3.1.2): the first one is constructed from the Heisenberg group, the second one is constructed on some Hermitian symmetric spaces.

The second part of Section 3.1 is devoted to the definition of the moduli space of self-dual connections (Definition 3.1.5). For this, we recall what a Gauge transformation is (Definition 3.1.3) and we show that self-duality is preserved by those transformations (Proposition 3.1.4). Moreover, we prove that the Gauge group is isomorphic to $\mathcal{C}^\infty(P, G)^G$ - the group of G-equivariant functions where G acts on itself by conjugation (with group structure induced by the group structure of G).

If the Gauge group is abelian, some simplifications appear. The first part of Section 3.2 explains these simplifications. E_{Ad} is the trivial bundle so the curvature of a connection can be seen as a 2-form valued in $\mathfrak{z} := Lie(Z)$. In the same way, the difference of two connections can be seen as an element of $\Lambda^1(M) \otimes \Gamma^\infty(E_{Ad})$, hence as a 1-form on M valued in $\mathfrak{z}$. The curvature of a connection is its differential plus a bracket term. So in the abelian case, it is simply the differential of the connection. Eventually, the Gauge group is isomorphic to $\mathcal{C}^\infty(M, Z)$ because the conjugation on Z is trivial.

In the second part of Section 3.2, we prove the following theorem (Theorem 3.2.3):

Theorem E. *Let Z be an abelian compact Lie group, (M, g, J, ω) be a compact connected almost Kähler manifold of real dimension $2n > 4$ and $\pi : P \to M$ be a Z-principal bundle. Then, either the moduli space of* *self-dual connections $\mathcal{M}$ is empty, or $\mathcal{M}$ is in bijection with*

$$H^1(M,\mathfrak{z})/K^Z,$$

where $K^Z := \{[\overline{\varphi}^{-1}d\overline{\varphi}] | \overline{\varphi} \in \mathcal{C}^\infty(M,Z)\}$*.*

If $\mathcal{M}$ is non empty, there exists a self-dual connection α_0. For every other self-dual connection α, $\alpha - \alpha_0$ can be seen as a 1-form on M valued in $\mathfrak{z}$. Lemma 3.2.1 proves that the map

$$\mathcal{M} \to H^1(M,\mathfrak{z})/K^{S^1} : [\alpha] \mapsto [\alpha - \alpha_0]$$

is well-defined, injective and surjective.

The end of Section 3.2 gives an interesting structure on the moduli space of self-dual connections, if one more hypothesis holds: the connectedness of the Gauge group (i.e. the Gauge group is a k-torus for $k \in \mathbb{N}_0$). It gives the following theorem (Theorem 3.2.9):

Theorem F. *Let Z be a k-torus for $k \in \mathbb{N}_0$, (M,g,J,ω) be a compact connected almost Kähler manifold of real dimension $2n > 4$ and $\pi : P \to M$ be a Z-principal bundle. Then, either $\mathcal{M}$ is empty, or there exists a manifold structure on $\mathcal{M}$ which turns $\mathcal{M}$ into an abelian Lie group.*

$H^1(M,\mathfrak{z})$ is a real finite dimensional vector space so, in particular, a Lie group. By Section 1.3, we have simply to prove that K^{S^1} is closed in $H^1(M,\mathfrak{z})$. Lemma 3.2.7 gives a characterization of K^{S^1} which allows to prove it easily. Eventually, we give some examples of moduli spaces of self-dual connections (Example 3.2.10).

A last chapter concludes our work and presents open questions. In our definition of self-duality in dimension greater than 4, we chose $\frac{1}{(n-1)!}$ as coefficient. We could have chosen $\frac{1}{(n-2)!}$ instead. Moreover, hypothesis that we used are restrictive (in particular, the fact that the Gauge group has to be abelian). The main open questions are: what would happen for the coefficient $\frac{1}{(n-2)!}$ and in a more general context ?

In the Appendix, the reader will find some words about the Loos and the Grassmann connections. Before the beginning of our work about moduli space of self-dual connections, we looked at the equality of these two well-known definitions of connections. Although it is not linked to moduli spaces, it seems to us that it is a nice result, so we decided to include it as an appendix.

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Notations

The following notations will be commonly used throughout this text:

- $2n$: the dimension of the symplectic spaces,
- $\hat{\bullet}$: omission of the factor $\bullet$,
- Ad: the adjoint representation of G on $\mathfrak{g}$,
- α: a connection 1-form,
- $\widetilde{\alpha_2 - \alpha_1}$: see Definition 1.5.10,
- $\overline{\alpha_2 - \alpha_1}$: see Section 3.2,
- F: the curvature of a connection 1-form α,
- $\widetilde{F}$: see Definition 1.5.10,
- $\overline{F}$: see Section 3.2,
- $*$: the Hodge-star operator,
- B^τ, for $\tau \in \mathbb{R}$: see Lemma 2.1.8,
- $C_g : G \to G$: the conjugation by $g \in G$, for G a Lie group,
- $\mathcal{C}^\infty(P, V)^G$: the vector space of smooth G-equivariant functions, where G is a Lie group acting on a vector space V on the left and P is the total space of a G-principal bundle,
- d: the exterior differentiation of forms,
- E_{Ad}: the vector bundle associated to a G-principal bundle for the adjoint representation,
- $^\flat$: the flat operator, i.e. on (V, ω) a symplectic vector space, $^\flat : V \to V^*$ is defined by $v^\flat = \omega(v, .)$ for every $v \in V$,

- g: an inner product on a vector space or a Riemannian metric on a manifold,
- G: a Lie group in general ; the transvection group in the symmetric space framework,
- $\mathfrak{g}$, (respectively $\mathfrak{k}$ and $\mathfrak{z}$): the Lie algebra $Lie(G)$ (respectively $Lie(K)$ and $Lie(Z)$) of the Lie group G (respectively K and Z),
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: the decomposition of $\mathfrak{g}$ with respect to σ in the symmetric space framework,
- $\Gamma^\infty(E)$: the vector space of smooth sections of a vector bundle E,
- $T_pP = Q_p \oplus G_p$: the decomposition of T_pP in horizontal and vertical vectors, where P is the total space of a principal bundle endowed with a connection 1-form,
- $h : T_pP \to Q_p$: the projection on the horizontal part of T_pP, where P is the total space of a principal bundle endowed with a connection 1-form,
- $H^k(M, V)$: the de Rham cohomology of k-forms on a manifold M valued in a real finite vector space V,
- J: a complex structure,
- K: the isotropy group in the symmetric space framework,
- $K^Z := \{[\overline{\varphi}^{-1} d\overline{\varphi}] | \overline{\varphi} \in \mathcal{C}^\infty(M, Z)\}$, where M is a manifold and Z a Lie group,
- $\Lambda^k(V)$: the vector space of skew-symmetric k-forms on a real finite vector space V,
- $\Lambda^*(V)$: the graded algebra of skew-symmetric forms on a real finite vector space V,
- $\Lambda^k(M)$: the vector space of k-differential forms on a manifold M,
- $\Lambda^*(M)$: the graded algebra of differential forms on a manifold M,
- $\Lambda^2_+(M)$: the vector space of self-dual 2-forms on an oriented 4-dimensional Riemannian manifold M,
- $\Lambda^2_-(M)$: the vector space of anti-self-dual 2-forms on an oriented 4-dimensional Riemannian manifold M,

- $\mathcal{M}$: the moduli space of self-dual connections,
- ∇^G: the Grassmann connection,
- ω: a symplectic form,
- P: the total space of a principal bundle,
- φ: a smooth section on an associated vector bundle or a Gauge transformation on a principal bundle,
- $\hat{\varphi}$ or $(\varphi)\hat{}$: the equivariant function corresponding to a smooth section φ on an associated vector bundle,
- $\tilde{\varphi}$: the equivariant function corresponding to a Gauge transformation φ on a principal bundle or the lifting of the path $\overline{\varphi} \circ \gamma$ in the proof of Proposition 3.2.7,
- $\overline{\varphi}$: the smooth function corresponding to a Gauge transformation φ on a principal bundle if the Gauge group is abelian (see Section 3.2),
- $f^*\mu$: the pullback of the form μ by the (piecewise) smooth map f,
- R_g: right product by $g \in G$ on a Lie group G,
- s: a local smooth section of a principal bundle,
- σ: the natural automorphism of the transvection group in the symmetric space framework,
- T^k: the k-torus for $k \in \mathbb{N}_0$,
- V^*: the vector space dual to a vector space V,
- $\wedge$: the wedge product on forms,
- X^*: a fundamental vector field,
- $\mathcal{YM}$: the Yang-Mills functional,
- Z: an abelian compact Lie group,
- $\mathfrak{z}(\mathfrak{k})$: the center of the Lie algebra $\mathfrak{k}$.

Chapter 1

Useful mathematical background

The first chapter recalls some mathematical background useful for the understanding of this text. In Section 1, we deal with differential geometry. We begin by fixing the notion of manifold which is not universal. Then we recall basic facts about the de Rham cohomology and symplectic vector spaces and manifolds.

In Section 2, we recall the notion of line integral and its basic properties. Line integral is used in the proof of Proposition 3.2.7. This proposition characterizes a subgroup, that we denote by K^{S^1} and which is fundamental in the understanding of the moduli space of self-dual connections in our context (Theorems 3.2.3 and 3.2.9).

In Section 3, we give the definitions of Lie groups and Lie subgroups and three proposition and theorems about it. These properties are central in the proof of Theorem 3.2.9 which states that the moduli space of self-dual connections in our context is an abelian Lie group.

Section 4 is a short section about algebraic topology. It recalls the path lifting property useful for the proof of Proposition 3.2.7 too.

Section 5 is the longest and probably the most important section of this first chapter. It deals with fiber bundles which are the mathematical foundations of the Yang-Mills theory. It is divided into two subsections. The first one recalls some definitions. First of all, it defines the notions of principal bundles and connections and curvatures on principal bundles. These notions are of fundamental importance for the definition of the moduli space of self-dual connections (Definition 3.1.5). Secondly, it recalls the notion of associated vector bundle. It is useful directly in the second subsection where we prove a technical proposition about connections and curvatures. This proposition allows us to consider Definition 1.5.10, which will be used in Chapter 3.

Eventually, Section 6 recalls some definitions and basic facts about symmetric spaces. Symmetric spaces give examples of applications of our theorems in Chapter 3.

This first chapter is certainly not an exhaustive text about those basic notions. It has two goals: first of all fixing the main definitions and notations that we use and secondly pointing out definitions and theorems which are central for the understanding of our text. In each section, we give references for more details about these subjects.

1.1. Basic notions of differential geometry

The notion of manifold is not universal in differential geometry. Hence, we begin by fixing our definition of manifold. Then we recall what is the de Rham cohomology. Eventually, we give some words about symplectic vector spaces and symplectic manifolds. For more details about basic differential geometry, we refer to [KN63], [Hel62] and [War83].

In our text, a *smooth n-manifold* (or simply a *n-manifold* or a *manifold*) is an Hausdorff second countable space with a differentiable structure of class $\mathcal{C}^\infty$ of dimension n. Let us remark that this notion of manifold is quite different in [KN63] and [Hel62] because they do not ask second countability.

We denote by $\Lambda^k(V)$ the vector space of *skew-symmetric k-forms* on a vector space V and

$$\Lambda^*(V) := \cup_{k\in\mathbb{N}}\Lambda^k(V)$$

the graded algebra of skew-symmetric forms on V. In the same way, we denote by $\Lambda^k(M)$ the vector space of *k-differential forms* (or simply k-forms) on a manifold M and

$$\Lambda^*(M) := \cup_{k\in\mathbb{N}}\Lambda^k(M)$$

the graded algebra of forms on M. The *wedge product* $\wedge$ is defined on forms on vector spaces and on manifolds as follows ([KN63, Section I.1]): for $k \in \mathbb{N}_0$ and $\mu_1, ..., \mu_k$ k 1-forms on V (respectively M) and $X_1, ..., X_k$ k vectors of V (respectively vector fields on M), $\mu_1 \wedge ... \wedge \mu_k$ is the k-form defined by

$$\mu_1 \wedge ... \wedge \mu_k(X_1, ..., X_k) := \frac{1}{k!} \det\left(\mu_i(X_j)\right)_{1\leq i,j\leq k}.$$

Exterior differentiation on forms on M can be characterized by ([KN63, Section I.1]):

- $d : \Lambda^k(M) \to \Lambda^{k+1}(M)$ is an $\mathbb{R}$-linear mapping,
- if $f \in \mathcal{C}^\infty(M)$, df is the total differential of f,

- for $\mu \in \Lambda^k(M)$ and $\nu \in \Lambda^l(M)$,

$$d(\mu \wedge \nu) = d\mu \wedge \nu + (-1)^k \mu \wedge d\nu,$$

- $d^2 = 0$,

  for every $k, l \in \mathbb{N}$.

In particular, we point out the following ([KN63, Section I.3]): for α a 1-form on M and X, Y vector fields on M,

$$d\alpha(X,Y) = \frac{1}{2}\left(X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y])\right).$$

Some authors drop the factor $\frac{1}{k!}$ in their definition of the wedge product. It influences the definition of the exterior differential too.

As $d^2 = 0$, $(\oplus_{k\in\mathbb{N}}\Lambda^k(M), d)$ forms a *differential complex* and we can consider the related cohomology, called the *de Rham cohomology*,

$$H^*(M,\mathbb{R}) = \sum_{k\in\mathbb{N}} H^k(M,\mathbb{R})$$

where

$$H^k(M,\mathbb{R}) = Ker(d) \cap \Lambda^k(M)/Im(d) \cap \Lambda^k(M).$$

Elements of $Ker(d) \cap \Lambda^k(M)$ are called *closed forms of degree k* and elements of $Im(d) \cap \Lambda^k(M)$ *exact forms of degree k* ([BT82, Section I.1]).

In the compact case, we have the following:

Lemma 1.1.1. [BT82, Theorem 5.1 and Proposition 5.3.1] *If M is a compact manifold, its de Rham cohomology is finite dimensional.*

More details about forms, wedge product and differentiation can be found in [KN63].

Let us now turn to symplectic vector spaces and manifolds. A real vector space V endowed with a 2-form ω is called a *symplectic vector space* if ω is skew-symmetric and non-degenerate. Such vector space has to be of even dimension. A symplectic vector space of dimension $2n$ admits a natural orientation given by its *canonical volume form*

$$\Omega := \frac{1}{n!}\omega \wedge ... \wedge \omega \ (n\text{-times}).$$

With respect to a *Darboux basis* (i.e. a basis $\{e_i, f_j\}_{1\le i,j\le n}$ such that $\omega = \sum_i \xi_i \wedge \eta_i$ for $\xi_i := e_i^\flat$ and $\eta_i := f_i^\flat$),

$$\Omega = \xi_1 \wedge \eta_1 \wedge \xi_2 \wedge \eta_2 \wedge ... \wedge \xi_n \wedge \eta_n.$$

(M, ω) is called a *symplectic manifold* if M is a real manifold and ω a non-degenerate closed 2-form on M. As in the symplectic vector space case, a symplectic manifold has even dimension. Moreover, a symplectic manifold of dimension $2n$ admits a natural orientation given by its *canonical volume form*

$$\Omega := \frac{1}{n!}\omega \wedge ... \wedge \omega \ (n\text{-times}).$$

In the compact case, we have the following:

Lemma 1.1.2. [Wal11, Proposition 1.1.3(ii)] *If (M, ω) is a compact symplectic manifold, then ω is a non exact form.*

1.2. Line integrals

In Proposition 3.2.7 appears a line integral. It allows us to characterize a subgroup, denoted by K^{S^1}, which appears in the understanding of the moduli space of self-dual connections in our context (Theorems 3.2.3 and 3.2.9).

This notion of line integral is explained in details in [Lee03] in Chapter 11. Here, we recall this definition, first of all for 1-forms defined on $\mathbb{R}$ and secondly for 1-forms defined on a manifold. Eventually, we give some useful properties of it.

First of all, let us define the line integral of a 1-form defined on an interval of $\mathbb{R}$.

Definition 1.2.1. *Let $[a, b] \subseteq]c, d[$ be intervals of $\mathbb{R}$ and β be a 1-form on $]c, d[$. If t denotes the standard coordinate in $\mathbb{R}$, β can be written*

$$\beta = f(t)dt$$

for $f :]c, d[\to \mathbb{R}$ a smooth function. The line integral of β along $[a, b]$ *is defined by*

$$\int_{[a,b]} \beta := \int_a^b f(t)dt.$$

Secondly, let us look at the corresponding definition for a 1-form on a smooth manifold. This line integral is defined along what is called a piecewise smooth curve segment. We explain what it is and then give the definition of line integral in this context.

A *piecewise smooth curve segment* on a smooth manifold M is a continuous curve $\gamma : [a, b] \to M$ with the property that there exists a finite subdivision $a = a_0 < a_1 < ... < a_k = b$ of $[a, b]$ such that, for each $j \in \{0, ..., k-1\}$, $\gamma_{[a_j,a_{j+1}]}$ is smooth (i.e. it has a smooth extension to an open set containing $[a_j, a_{j+1}]$).

Definition 1.2.2. *Let us suppose that M is a smooth manifold and β a 1-form on M. If $\gamma : [a, b] \to M$ is a piecewise smooth curve segment with underlying decomposition $a = a_0 < a_1 < ... < a_k = b$, the* line integral of β along γ *is defined by*

$$\int_\gamma \beta := \sum_{j=0}^{k-1} \int_{[a_j,a_{j+1}]} \gamma^* \beta.$$

Eventually, we state the following proposition, which is proved in [Lee03]:

Proposition 1.2.3 ([Lee03], Proposition 11.34 and Proposition 11.37)**.** *Let M be a smooth manifold. If $\gamma : [a, b] \to M$ is a piecewise smooth curve segment and β_1 and β_2 are 1-forms on M, then*

(i) for any $c_1, c_2 \in \mathbb{R}$,

$$\int_\gamma (c_1\beta_1 + c_2\beta_2) = c_1 \int_\gamma \beta_1 + c_2 \int_\gamma \beta_2,$$

(ii) for $c \in]a, b[$, if we denote $\gamma_1 := \gamma|_{[a,c]}$ and $\gamma_2 := \gamma|_{[c,b]}$, then

$$\int_\gamma \beta_1 = \int_{\gamma_1} \beta_1 + \int_{\gamma_2} \beta_1.$$

(iii) $\int_{\gamma^{-1}} \beta_1 = -\int_\gamma \beta_1$, where γ^{-1} is the path γ covered in the reverse side.

1.3. Lie groups

Lie groups are fundamental in Yang-Mills theory. Indeed, one of the ideas of C.N. Yang and R.L Mills was to take a general compact Lie group as Gauge group, instead of $U(1)$ or $SU(2)$. General theory about Lie groups and Lie algebras is developed in details in a lot of books. We refer for example to [War83]. The goal of this section is quite different. It points out properties about Lie groups and Lie subgroups useful in the proof that the moduli space of self-dual connections in our context is a Lie group (Theorem 3.2.9). We still begin by giving the definition of Lie groups and Lie subgroups.

A *Lie group* is a manifold with a group structure such that the maps $G \times G \to G : (g, h) \mapsto gh$ and $G \to G : g \mapsto g^{-1}$ are smooth maps. K is a *Lie subgroup* of a Lie group G if K is a Lie group, a submanifold of G and an abstract subgroup. We denote by $\mathfrak{g}$ (respectively $\mathfrak{k}$, $\mathfrak{z}$) the Lie algebra of a Lie group G (respectively K, Z).

The following proposition and theorems about Lie groups are essential in Chapter 3:

Proposition 1.3.1. [War83, Section 3.3] *The product $G \times H$ of two Lie groups is itself a Lie group with the product manifold structure and the direct product group structure ; that is, for every (g_1, h_1) and $(g_2, h_2) \in G \times H$, $(g_1, h_1).(g_2, h_2) := (g_1 g_2, h_1 h_2)$.*

Theorem 1.3.2. [War83, Section 3.42] *Let G be a Lie group, and let K be a closed abstract subgroup of G. Then K has a unique manifold structure which makes K into a Lie subgroup of G.*

Theorem 1.3.3. [War83, Section 3.64] *Let G be a Lie group and let K be a closed normal subgroup of G. Then the homogeneous manifold G/K with its natural group structure is a Lie group.*

1.4. Path lifting property

We need simply to recall one notion of algebraic topology: the *path lifiting property* [Hat02, Section 1.3]. As already mention above, it will be useful in the proof of the characterization of K^{S^1} (Proposition 3.2.7), a subgroup appearing in the identification of the moduli space of self-dual connections in our context (Theorem 3.2.3).

If $p : \tilde{M} \to M$ is a covering space (i.e. there exists an open cover $\{U_i\}$ of M such that for each i, $p^{-1}(U_i)$ is a disjoint union of open sets of $\tilde{M}$ which are all homeomorphic to U_i), then for each continuous path $f : I \to M$ (I is an interval of $\mathbb{R}$ containing a) and each $\tilde{x_0} \in \tilde{M}$ such that $p(\tilde{x_0}) = f(a) \in M$, there exists a unique path $\tilde{f} : I \to \tilde{M}$ such that $\tilde{f}(a) = \tilde{x_0}$ and $p \circ \tilde{f} = f$.

1.5. Fiber bundles

The notion of fiber bundle is central in Yang-Mills theory. The most important fiber bundles used in our text are the principal bundles. We will recall this definition and the notions of connections, curvatures and associated vector bundles. Moreover, we will give some of their properties. Definitions and more details about fiber bundles can be found for example in [KN63].

This section is divided into two subsections. The first one recalls some definitions and basic properties about principal bundles, connections, curvatures and associated vector bundles. The second one identifies the difference of two connections on a principal bundle and the curvature of a connection on a principal bundle with forms on the based manifold. These properties are fundamental for the identification of the moduli space of self-dual connections in our context (Theorem 3.2.3).

1.5.1. Definitions and basic facts

In this subsection, first of all, we define the notion of principal bundle. Secondly, we define the notion of connection on a principal bundle, the underlying notions of horizontal lifts and horizontal sections and the notion of curvature of a connection. Eventually, we turn to the definition of associated vector bundles. This last point will be useful directly in the following subsection.

Let us begin with the notion of principal bundle.

Definition 1.5.1. [KN63, Section I.5] *Let M be a manifold and G a Lie group. A* principal bundle over M with group G (or a G-principal bundle) *consists of a manifold P and an action of G on P satisfying the following conditions:*

(i) G acts freely on P on the right: $(p, g) \in P \times G \mapsto pg = R_g p \in P$;

(ii) M is the quotient space of P by the equivalence relation induced by G, $M = P/G$, and the canonical projection $\pi : P \to M$ is smooth;

(iii) P is locally trivial, that is, every point x of M has a neighborhood U such that $\pi^{-1}(U)$ is isomorphic with $U \times G$ in the sense that there is a diffeomorphism $\psi : \pi^{-1}(U) \to U \times G$ such that $\psi(p) = (\pi(p), \varphi(p))$ where φ is a mapping of $\pi^{-1}(U)$ into G satisfying $\varphi(pg) = (\varphi(p)g)$ for all $p \in \pi^{-1}(U)$ and $g \in G$.

Now, we can define the notion of connections on a principal bundle and the underlying notions of horizontal lifts and horizontal sections.

Definition 1.5.2. [KN63, Section II.1] *A* connection on a G-principal bundle $P \to M$ *is a 1-form α on P valued in the Lie algebra $\mathfrak{g}$ such that* *(i) for every* $X \in \mathfrak{g}$ *and every* $p \in P$, $\alpha_p(X^*_p) = X$, *for* X^* *the fundamental vector field corresponding to* X, *i.e.* $X^*_p := \frac{d}{dt}\Big|_0 p\exp(tX)$,

(ii) for every $g \in G$, $R^*_g\alpha = Ad_{g^{-1}}\alpha$.

On a G-principal bundle $P \to M$ endowed with a connection 1-form α, a vector $v \in T_pP$ is said to be *horizontal* if $\alpha_p(v) = 0$. The subspace of horizontal vectors at p is denoted by Q_p. If G_p denotes the subspace of vectors tangent to the fiber at p (i.e. vectors $v \in T_pP$ such that $\pi_{*p}v = 0$), we have a decomposition of T_pP for each $p \in P$:

$$T_pP = G_p \oplus Q_p.$$

We denote by $h : T_pP \to Q_p$ the projection on the horizontal part at $p \in P$.

For every $X \in \Gamma^\infty(TM)$, there exists a unique horizontal vector field on P which projects on X. It is called the *horizontal lift* of X and it is denoted by $\overline{X} \in \Gamma^\infty(TP)$. A local section s of P in a neighborhood of a point $x \in M$ is called *horizontal at* x if its differential ds_x at x maps T_xM to $Q_{s(x)}$ [Mor07, Definition 5.6]. The following proposition about horizontal sections is proved in [Mor07]:

Proposition 1.5.3. [Mor07, Lemma 5.7] *Let* $P \to M$ *be a* G*-principal bundle on a manifold* M *endowed with a connection* 1*-form. Then for every* $x \in M$ *and* $p \in P$ *such that* $\pi(p) = x$, *there exist local sections of* P *horizontal at* x.

We can turn to the notion of curvature of a connection 1-form.

Definition 1.5.4. [KN63, Section II.5] *On a* G*-principal bundle* $P \to M$, *we can define the curvature* F *of a connection* 1*-form* α *as follows:*

$$F := (d\alpha)(h(.), h(.)). \tag{3}$$

In this text, we denote by F (respectively F_0, F_1, F_2) the curvature of a connection 1-form α (respectively $\alpha_0, \alpha_1, \alpha_2$).

Before ending this subsection, let us deal with the notion of associated vector bundle. For G a Lie group and M a manifold, we consider a G-principal bundle $\pi : P \to M$ and a representation ρ of G on an $\mathbb{R}$-vector space V of dimension k. We define an equivalence relation on $P \times V$ by $(p, v) \sim (pg, \rho(g^{-1})v)$ for every $g \in G, p \in P, v \in V$ and obtain a vector bundle

$$E_\rho := P \times V/\sim\, \to M : [(p, v)] \mapsto \pi(p),$$

called *associated vector bundle*. We denote by $\mathcal{C}^\infty(P, V)^G$ the vector space of G-equivariant functions from P to V, i.e. the vector space of smooth functions $f : P \to V$ such that $f(pg) = \rho(g^{-1})f(p)$ for every $p \in P$ and $g \in G$, where the vector space structure comes naturally from the vector space structure of V. There exists an isomorphism between the vector space of smooth sections on E_ρ and $\mathcal{C}^\infty(P,V)^G$ given by $\varphi(\pi(p)) = [(p, \hat{\varphi}(p)]$ for each $p \in P$, for φ a smooth section on E_ρ and $\hat{\varphi}$ the corresponding G-equivariant function. In our text, we denote always by $\hat{\varphi}$ or $(\varphi)\hat{}$ the G-equivariant function corresponding to a smooth section φ.

More details about connections and curvatures can be found in [KN63].

1.5.2. Useful properties

In this subsection, we write the difference of two connection 1-forms and the curvature of a connection 1-form on a principal bundle as forms on the based manifold valued in a given vector bundle. First of all, we give some lemmas. They will drive us to the proof of Proposition 1.5.9, which asserts that these forms are well-defined.

The proofs of the following lemmas can be found in [KN63, Section II.5].

Lemma 1.5.5. *On a G-principal bundle $P \to M$, the curvature F of a connection 1-form α can be computed as*

$$F(X,Y) = d\alpha(X,Y) + \frac{1}{2}[\alpha(X), \alpha(Y)]$$

for X and Y smooth vector fields on P.

Lemma 1.5.6. *On a G-principal bundle $P \to M$, the curvature F of a connection 1-form α can be computed as*

$$F(X,Y) = -\frac{1}{2}\alpha([X,Y])$$

for X and Y horizontal *smooth vector fields on P.*

Moreover, we need two other lemmas that we will prove here.

Lemma 1.5.7. *Let $P \to M$ be a G-principal bundle endowed with a connection 1-form α. Then for every $g \in G$, R_{g*} and h commute.*

Proof. The lemma follows from this remark: for every $p \in P$, $v \in T_pP$ and $g \in G$,

(i) $\pi_* R_{g*} v = \pi_* v$, then v is vertical if and only if $R_{g*}v$ is vertical,

(ii) $\alpha(R_{g*}v) = Ad_{g^{-1}}\alpha(v)$ then v is horizontal if and only if $R_{g*}v$ is horizontal.

□

Lemma 1.5.8. *Let $P \to M$ be a G-principal bundle endowed with a connection 1-form α. Then*

$$F(R_{g*}v, R_{g*}w) = Ad_{g^{-1}}F(v, w)$$

for every v and $w \in T_pP$ and $g \in G$.

Proof. Let X and Y be two vector fields such that $X_p = v$ and $Y_p = w$. Then using Equation (3) and Lemmas 1.5.6 and 1.5.7, we see that

$$\begin{aligned}
F(R_{g*}X, R_{g*}Y) &= F\left(h(R_{g*}X), h(R_{g*}Y)\right) \\
&= -\frac{1}{2}\alpha\left([h(R_{g*}X), h(R_{g*}Y)]\right) \\
&= -\frac{1}{2}\alpha\left([R_{g*}h(X), R_{g*}h(Y)]\right) \\
&= -\frac{1}{2}\alpha\left(R_{g*}[h(X), h(Y)]\right) \\
&= Ad_{g^{-1}}\left(-\frac{1}{2}\alpha([h(X), h(Y)])\right) \\
&= Ad_{g^{-1}}F(X, Y)
\end{aligned}$$

So

$$F(R_{g*}v, R_{g*}w) = Ad_{g^{-1}}F(v, w).$$

□

The following proposition will allow the Definition 1.5.10.

Proposition 1.5.9. *Let $\pi : P \to M$ be a G-principal bundle. Then*

(i) if α_1 and α_2 are connection 1-forms on P and $X \in \Gamma^\infty(TM)$, the function

$$f_X : P \to \mathfrak{g} : p \mapsto (\alpha_2 - \alpha_1)(s_* X_{\pi(p)}),$$

where $s : U \subseteq M \to P$ is a local section defined on $U \ni \pi(p)$ such that $s(\pi(p)) = p$, is independent of the choice of s and equivariant (i.e. $f_X(pg) = Ad_{g-1}f_X(p)$ for every $p \in P$ and $g \in G$), *(ii) if α is a connection* 1*-form on P and $X, Y \in \Gamma^\infty(TM)$, the function*

$$g_{X,Y} : P \to \mathfrak{g} : p \mapsto F(s_* X_{\pi(p)}, s_* Y_{\pi(p)}),$$

where $s : U \subseteq M \to P$ is a local section defined on $U \ni \pi(p)$ such that $s(\pi(p)) = p$, is independent of the choice of s and equivariant (i.e. $g_{X,Y}(pg) = Ad_{g-1}(g_{X,Y}(p))$ for every $p \in P$ and $g \in G$).

Proof. (i) First of all, let us show that f_X is independent of the choice of local section.

If $s' : U' \subseteq M \to P$ is another local section of P such that $s'(\pi(p)) = p$, then there exists a smooth function

$$g : U \cap U' \to G$$

such that $s'(y) = s(y).g(y)$ for all $y \in U \cap U'$ and $g(\pi(p)) = 1$.

Hence, if we consider $\gamma : I \subseteq \mathbb{R} \to M$ such that $\gamma(0) = \pi(p)$ and $\frac{d}{dt}\Big|_0 \gamma(t) = X_{\pi(p)}$,

$$\begin{aligned} s'_* X_{\pi(p)} &= \frac{d}{dt}\Big|_0 s'(\gamma(t)) \\ &= \frac{d}{dt}\Big|_0 s(\gamma(t))g(\gamma(t)) \\ &= s_*(X_{\pi(p)}) + \Big(dg(X_{\pi(p)})\Big)^*_p \end{aligned}$$

and using the definition of a connection 1-form, we find

$$\begin{aligned} & (\alpha_2 - \alpha_1)(s'_* X_{\pi(p)}) \\ = \ & (\alpha_2 - \alpha_1)\left(s_*(X_{\pi(p)}) + \Big(dg(X_{\pi(p)})\Big)^*_p\right) \\ = \ & (\alpha_2 - \alpha_1)(s_* X_{\pi(p)}). \end{aligned}$$

This shows that f_X is well-defined.

Now, let us show that f_X is equivariant.

For every $g \in G$, $s' := s.g$ is a local section of P such that $s'(\pi(pg)) = pg$. Then, for $\gamma : I \subseteq \mathbb{R} \to M$ such that $\gamma(0) = \pi(pg)$ and $\frac{d}{dt}\Big|_0 \gamma(t) = X_{\pi(pg)}$,

$$\begin{aligned} s'_* X_{\pi(pg)} &= \frac{d}{dt}\Big|_0 s'(\gamma(t)) \\ &= \frac{d}{dt}\Big|_0 s(\gamma(t)).g \\ &= R_{g*} s_* X_{\pi(p)}. \end{aligned}$$

Using the definition of a connection 1-form, we find

$$\begin{aligned} f_X(pg) &= (\alpha_2 - \alpha_1)(s'_* X_{\pi(pg)}) \\ &= (\alpha_2 - \alpha_1)(R_{g*} s_* X_{\pi(pg)}) \\ &= Ad_{g^{-1}}(\alpha_2 - \alpha_1)(s_* X_{\pi(p)}) \\ &= Ad_{g^{-1}} f_X(p). \end{aligned}$$

(ii) First of all, let us show that $g_{X,Y}$ is independent of the choice of local section.

If $s' : U' \subseteq M \to P$ is another local section of P such that $s'(\pi(p)) = p$, then there exists a smooth function

$$g : U \cap U' \to G$$

such that $s'(y) = s(y).g(y)$ for all $y \in U \cap U'$ and $g(\pi(p)) = 1$.

Hence, with the same kind of computation as in item (i), we show that

$$s'_* X_{\pi(p)} = s_*(X_{\pi(p)}) + \left(dg(X_{\pi(p)})\right)^*_p$$

and

$$s'_* Y_{\pi(p)} = s_*(Y_{\pi(p)}) + \left(dg(Y_{\pi(p)})\right)^*_p$$

and using Equation (3), we find

$$\begin{aligned} & F(s'_* X_{\pi(p)}, s'_* Y_{\pi(p)}) \\ = \; & F\left(s_*(X_{\pi(p)}) + \left(dg(X_{\pi(p)})\right)^*_p, s_*(Y_{\pi(p)}) + \left(dg(Y_{\pi(p)})\right)^*_p\right) \\ = \; & F\left(h\left(s_*(X_{\pi(p)}) + \left(dg(X_{\pi(p)})\right)^*_p\right),\right. \\ & \left. h\left(s_*(Y_{\pi(p)}) + \left(dg(Y_{\pi(p)})\right)^*_p\right)\right) \\ = \; & F\left(h(s_*(X_{\pi(p)})), h(s_*(Y_{\pi(p)}))\right) \\ = \; & F\left(s_*(X_{\pi(p)}), s_*(Y_{\pi(p)})\right). \end{aligned}$$

This shows that $g_{X,Y}$ is well-defined.

Now, let us show that $g_{X,Y}$ is equivariant.

For every $g \in G$, $s' := s.g$ is a local section of P such that $s'(\pi(pg)) = pg$.

With the same kind of computation as in item (i),

$$s'_* X_{\pi(pg)} = R_{g*} s_* X_{\pi(p)}.$$

and

$$s'_* Y_{\pi(pg)} = R_{g*} s_* Y_{\pi(p)}.$$

Then, using Lemma 1.5.8, we find

$$\begin{aligned} g_{X,Y}(pg) &= F(s'_* X_{\pi(pg)}, s'_* Y_{\pi(pg)}) \\ &= F(R_{g*} s_* X_{\pi(p)}, R_{g*} s_* Y_{\pi(p)}) \\ &= Ad_g^{-1} F(s_* X_{\pi(p)}, s_* Y_{\pi(p)}) \\ &= Ad_{g^{-1}}(g_{X,Y}(p)). \end{aligned}$$

□

We are now ready to write the difference of two connection 1-forms and the curvature of a connection 1-form on a principal bundle $P \to M$ as forms on M valued in a given vector bundle. This fact is claimed in [AHS78, Sections 2 and 6].

Definition 1.5.10. *Let $\pi : P \to M$ be a G-principal bundle. For $\alpha, \alpha_1, \alpha_2$ connection 1-forms on P, let us consider $\widetilde{\alpha_2 - \alpha_1} \in \Lambda^1(M) \otimes \Gamma^\infty(E_{Ad})$, defined by*

$$\left(\widetilde{\alpha_2 - \alpha_1}(X)\right)\hat{} := f_X$$

for all $X \in \Gamma^\infty(TM)$, and $\widetilde{F} \in \Lambda^2(M) \otimes \Gamma^\infty(E_{Ad})$, defined by

$$\left(\widetilde{F}(X, Y)\right)\hat{} := g_{X,Y}$$

for all $X, Y \in \Gamma^\infty(TM)$.

Thanks to Proposition 1.5.9, $\widetilde{\alpha_2 - \alpha_1}$ and $\widetilde{F}$ are well-defined.

1.6. Symmetric spaces

Symmetric spaces give examples of applications of our theorems of Chapter 3. We present here the main definition and basic facts about symmetric spaces. After the definition, we show that a connected symmetric space can be seen as a homogeneous space. Then, we look at a decomposition of the Lie algebra of its transvection group (the group of compositions of even numbers of symmetries). Eventually, we present a natural principal bundle and a natural connection 1-form defined on it. For more details, we refer to [Loo69].

First of all, let us give the definition of symmetric spaces.

Definition 1.6.1. [Loo69, Chapter II] *A* symmetric space *is a manifold M with a smooth multiplication $\mu : M \times M \to M$, written as $\mu(x, y) = s_x y$, and with the following properties: for every $x, y \in M$,*

(i) $s_x x = x$,

(ii) $s_x^2 = Id$,

(iii) $s_x s_y s_x = s_{s_x y}$,

(iv) *x is an isolated fixed point of s_x, i.e. there exists a neighbourhood U of x such that $s_x z = z$ implies $z = x$ for all $z \in U$.*

Secondly, let us show that a connected symmetric space can be seen as a homogeneous manifold. A group G is canonically attached to a symmetric space: the *transvection group*. The elements of this group are the compositions of even numbers of symmetries

$$G := < s_x \circ s_y | x, y \in M > .$$

If M is connected, then G is a finite dimensional Lie group acting transitively on M (see e.g. Proposition 1.4.9 of [Vog11]).

For the rest of the section, we suppose that M is connected and we fix a point o in M. The subgroup

$$K := \{g \in G | g.o = o\}$$

is a closed subgroup of G (see e.g. [War83, Section 3.61]). It is called the *isotropy group at o*.

In Theorem 3.58 of [War83], F.K. Warner proves that G/K admits a unique manifold structure such that the projection $\pi : G \to G/K$ is smooth and there exist local smooth sections of G/K in G, (i.e. for every $gK \in G/K$, there exists an open set U of G/K containing gK and a smooth map $s : U \to G$ such that $\pi \circ s = id$). F.K. Warner proves in the Theorem 3.62 of [War83] that the map

$$G/K \to M : gK \mapsto g.o$$

is a diffeomorphism for this natural manifold structure.

Thirdly, we define a natural automorphism of G which allows a decomposition of its Lie algebra. Let us consider the automorphism

$$\sigma : G \to G : g \mapsto s_o g s_o.$$

We remark that then $s_{g_0K} gK = g_0\sigma(g_0^{-1}g)K$ for every $g_0K, gK \in M \simeq G/K$. $\sigma^2 = Id_G$ implies $\sigma_{*e}^2 = Id_{\mathfrak{g}}$. So the only possible eigenvalues of σ_{*e} are 1 and -1. As $G_0^\sigma \subseteq K \subseteq G^\sigma$ where $G^\sigma := \{g \in G | \sigma(g) = g\}$ and G_0^σ is its identity component, $\mathfrak{k}$ is the space of eigenvectors of eigenvalue 1. Let us write $\mathfrak{p}$ the space of eigenvectors of eigenvalue -1. Then there exists a canonical decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

Eventually, we observe that there exist a natural principal bundle and a natural connection 1-form on each connected symmetric space. Thanks to the foregoing, it is easy to verify that the projection

$$G \to G/K$$

defines naturally a K-principal bundle.

A natural connection 1-form can be defined on it:

$$\alpha : TG \to \mathfrak{k} : X_g \mapsto pr_{\mathfrak{k}}(L_{g^{-1}*}(X_g)).$$

It is called the *Loos connection.*

To understand the link between the more famous Loos connection on the tangent space of a symmetric space (the only one which is s_x-invariant for every $x \in M$) and this Loos connection 1-form, we refer to the Appendix.

Chapter 2

From self-duality of 2-forms in dimension four to a generalized definition

This text is devoted to the research of critical points of the Yang-Mills functional (see Equation (1) in the introduction). In dimension 4, (anti-)self-dual connections provide such critical points. The notion of (anti-)self-dual connections comes directly from the natural notion of (anti-)self-dual 2-forms. Since in dimension 4, the Hodge-star operator $*$ maps a 2-form to a 2-form, a 2-form μ is called (anti-)self-dual simply if $*\mu = \pm\mu$ (Definition 2.1.1).

To find critical points of the Yang-Mills action on compact manifolds of dimension greater than 4, we would like to consider a notion of self-duality of connections, so of 2-forms, on these spaces. In this chapter, we define and study a generalized notion of self-duality of 2-forms. The first section works on some vector spaces and the second one on some manifolds.

Our definition makes sense on Kähler vector spaces (respectively almost Kähler manifolds). We denote by $2n$ the real dimension of the space and ω the underlying symplectic form. Since on $2n$-dimensional oriented inner product spaces and on $2n$-dimensional oriented Riemannian manifolds the Hodge-star operator $*$ maps a 2-form to a $(2n-2)$-form, we say that a 2-form μ is self-dual if (Definition 2.1.3)

$$*\mu = \mu \wedge \frac{\omega^{\wedge(n-2)}}{(n-1)!}.$$

This definition is a particular case of the well-known generalized definition presented among others in [Tia00, Lemma 1.2.1 and Remark 1].

The first section of this chapter recalls first of all the notion of Hodge-star operator and of (anti-)self-duality on vector spaces of dimension 4. Secondly, it generalizes the definition of self-duality for dimension greater than 4 on the case of vector spaces. The $\frac{1}{(n-1)!}$-factor chosen in the definition leads to a characterization of the space of self-dual 2-forms if $2n > 4$ (Proposition 2.1.9): a 2-form μ is self-dual if and only if there exists $c \in \mathbb{R}$ such that $\mu = c\omega$. We will understand in the following why this proposition is not valid in dimension 4. The end of the first section is a sequence of technical lemmas which give the proof of this proposition.

This proof is based on four lemmas. The fundamental one is Lemma 2.1.6. The latter states that there exists an orthonormal Darboux basis (i.e. an orthonormal basis for the metric which is also a Darboux basis for the symplectic form) on each Kähler vector space. Lemma 2.1.7 computes explicitly the Hodge-star operator of 2-forms written with respect to an orthonormal Darboux basis. Lemma 2.1.8 gives a decomposition of $\Lambda^2(M)$ which takes the role of the decomposition $\Lambda^2(M) = \Lambda^2_+(M) \oplus \Lambda^2_-(M)$ in dimension 4. The proof of Proposition 2.1.9 comes directly from this last lemma.

The structure of the second section is a carbon copy of the structure of the first one, in the manifold's case. The characterization in this case comes directly from the vector space case: a 2-form μ is self-dual if and only if there exists $c \in \mathcal{C}^\infty(M, \mathbb{R})$ such that $\mu = c\omega$ (Proposition 2.2.5).

2.1. On vector spaces

First, we recall the definition of the Hodge-star operator on an oriented inner product space and the definition of self-duality and anti-self-duality on their natural framework: real 4-dimensional oriented inner product spaces. Secondly, we define the notion of Kähler vector spaces and explain in details how we generalize the self-duality on such spaces of dimension ≥ 4. Thirdly, four lemmas drive us to the main proposition of this section. It determines the space of generalized self-dual 2-forms on Kähler vector spaces, for real dimension $2n > 4$.

First of all, let us recall the definition of the Hodge-star operator and the notion of (anti-)self-duality in dimension 4. We consider (V, g) an inner product space of dimension n, i.e. an $\mathbb{R}$-vector space of dimension n endowed with g a positive definite symmetric bilinear form (inner product). It gives naturally an inner product on V^* that we can extend to an inner product on $\Lambda^k V^*$ for every $k \in \{1, ..., n\}$ by

$$g(\alpha_1 \wedge ... \wedge \alpha_k, \beta_1 \wedge ... \wedge \beta_k) := \det\left(g(\alpha_i, \beta_j)\right)_{i,j\in\{1,...n\}}$$

for every $\alpha_1 \wedge ... \wedge \alpha_k$ and $\beta_1 \wedge ... \wedge \beta_k \in \Lambda^k V^*$. By abuse of notation we write all these inner products g.

We fix an orientation on V choosing a preferred n-form $\epsilon \in \Lambda^n V^*$. It allows us to give the definition of the *Hodge-star operator* $*$ on $\Lambda^k V$ for

$k \in \{1, ..., n\}$ by

$$*: \Lambda^k V^* \to \Lambda^{n-k} V^* : \mu \mapsto *\mu$$

where $*\mu$ is the only $(n-k)$-form such that

$$\lambda \wedge *\mu = g(\lambda, \mu)\epsilon \tag{4}$$

for every $\lambda \in \Lambda^k V^*$.

$*$ respects the important following property:

$$*^2 = (-1)^{k(n-k)}. \tag{5}$$

The proof that the Hodge-star operator is well defined and the proof of Equation (5) (for general signature of g) can be found in [Dra99].

If we restrict to a vector space of dimension 4,

$$*: \Lambda^2 V^* \to \Lambda^2 V^*.$$

So there exists a natural notion of (anti-)self-duality on 4-dimensional vector spaces:

Definition 2.1.1. *Let (V, g) be an oriented inner product space of real dimension 4. A 2-form μ is called self-dual if*

$$*\mu = \mu$$

and anti-self-dual if

$$*\mu = -\mu.$$

Now, we extend the definition of self-duality on vector spaces of dimension greater than 4. The way that we choose to extend this definition makes sense on Kähler vector spaces. We recall what is a Kähler vector space. Then we give the extended definition.

Definition 2.1.2. [Boa09, Section 4] *A* Kähler vector space of real dimension n (V, ω, J, g) *is an inner product space (V, g) of dimension n endowed with*

- *ω a symplectic form on V, i.e. a skew-symmetric and non-degenerate bilinear form,*
- *$J : V \to V$ a complex structure, i.e. an $\mathbb{R}$-linear endomorphism such that $J^2 = -Id_V$,*

such that

$$\omega(v, Jw) = g(v, w)$$

for each $v, w \in V$.

Definition 2.1.3. *Let (V, ω, J, g) be a Kähler vector space of real dimension $2n \geq 4$ endowed with the orientation given by the canonical volume form of the underlying symplectic vector space (V, ω) (see Section 1.1). A 2-form μ on V is called* generalized self-dual *(or simply* self-dual*) if*

$$*\mu = \mu \wedge \frac{\omega^{\wedge(n-2)}}{(n-1)!}.$$

This definition is a particular case of the definition used in [Tia00, Lemma 1.2.1 and Remark 1].

Remark 2.1.4. *We will see in Lemma 2.1.8 that there does not exist what we would like to call* anti-self-dual 2-form*, i.e. 2-form μ such that*

$$*\mu = -\mu \wedge \frac{\omega^{\wedge(n-2)}}{(n-1)!}.$$

It is why we consider only self-dual *2-forms in the following.*

To characterize the space of self-dual 2-forms on Kähler vector spaces, we need four technical lemmas. The first one deals with the J-invariance of g in a Kähler vector space.

Lemma 2.1.5. *Let (V, ω, J, g) be a Kähler vector space. Then $g(Jv, Jw) = g(v, w)$ for every $v, w \in V$.*

Proof.

$$\begin{aligned} g(Jv, Jw) &= \omega(Jv, -w) && (6)\\ &= \omega(w, Jv) && (7)\\ &= g(w, v) && (8)\\ &= g(v, w) && (9) \end{aligned}$$

□

The second lemma shows that there exists a basis on each Kähler vector space which is orthonormal for the inner product and Darboux for the symplectic form.

Lemma 2.1.6. *Let (V, ω, J, g) be a Kähler vector space of real dimension $2n$. Then there exists $\{e_i, f_j\}_{1 \leq i,j \leq n}$ a basis of V such that*

(i) $\{e_i, f_j\}_{1 \leq i,j \leq n}$ is an orthonormal basis for g,

(ii) $\{e_i, f_j\}_{1\leq i,j\leq n}$ *is a Darboux basis for* ω*, i.e.*

$$\omega = \sum_{i=1}^{n} \xi_i \wedge \eta_i$$

for $\xi_i := e_i^\flat$ *and* $\eta_i := f_i^\flat$*, where* $.^\flat$ *means the flat operator for* ω*.*

Such a basis is called an orthonormal Darboux basis *in our text.*

Proof. Let us write $E_0 := V$ and iterate the following for $1 \leq i \leq n$:

We choose e_i a normalized vector of E_{i-1} and denote $f_i := Je_i$ and $E_i := E_{i-1}^{\perp <e_i,f_i>}$ (the vector subspace of E_{i-1} orthogonal to $< e_i, f_i >$).

By induction, we can easily see that for every $i \in \{1, ..., n\}$, E_i is stable under J thanks to Lemma 2.1.5. Lemma 2.1.5 also tells us that f_i is normalized and orthogonal to e_i.

At the end of the iteration, as $\omega(v, Jw) = g(v, w)$ for every $v, w \in V$, $\{e_i, f_j\}_{1\leq i,j\leq n}$ is an orthonormal Darboux basis of V. □

The following lemma computes the Hodge-star operator of 2-forms written with respect to an orthonormal Darboux basis.

Lemma 2.1.7. *Let* (V, ω, J, g) *be a Kähler vector space of real dimension* $2n \geq 4$*. Let us consider an orthonormal Darboux basis* $\{e_i, f_j\}_{1\leq i,j\leq n}$ *and write* $\xi_i := e_i^\flat$ *and* $\eta_i := f_i^\flat$ *for every* $i \in \{1, ..., n\}$*. We consider the orientation of* V *given by the canonical volume form of the underlying symplectic vector space* (V, ω) *(see Section 1.1). Then*

(i) for $i < j \in \{1, ..., n\}$*,*

$$*(\xi_i \wedge \xi_j) = -\eta_i \wedge \eta_j \wedge \frac{\omega^{\wedge(n-2)}}{(n-2)!},$$

(ii) for $i < j \in \{1, ..., n\}$*,*

$$*(\eta_i \wedge \eta_j) = -\xi_i \wedge \xi_j \wedge \frac{\omega^{\wedge(n-2)}}{(n-2)!},$$

(iii) for $i \neq j \in \{1, ..., n\}$*,*

$$*(\xi_i \wedge \eta_j) = \eta_i \wedge \xi_j \wedge \frac{\omega^{\wedge(n-2)}}{(n-2)!},$$

(iv) the vectors $*(\xi_i \wedge \eta_i)$ *for* $1 \leq i \leq n$ *are linearly independent,*

(v) for every $i \in \{1, ..., n\}$,

$$\xi_i \wedge \eta_i \wedge \frac{\omega^{\wedge(n-2)}}{(n-2)!} = * \sum_{k \neq i} \xi_k \wedge \eta_k.$$

Proof. The proofs are similar for the items (i)-(iii). We remark that the left and the right hand sides of the equalities are equal to

(i) $-\eta_i \wedge \eta_j \wedge \xi_1 \wedge \eta_1 \wedge ... \wedge \hat{\xi_i} \wedge \hat{\eta_i} \wedge ... \wedge \hat{\xi_j} \wedge \hat{\eta_j} \wedge ... \wedge \xi_n \wedge \eta_n$,

(ii) $-\xi_i \wedge \xi_j \wedge \xi_1 \wedge \eta_1 \wedge ... \wedge \hat{\xi_i} \wedge \hat{\eta_i} \wedge ... \wedge \hat{\xi_j} \wedge \hat{\eta_j} \wedge ... \wedge \xi_n \wedge \eta_n$,

(iii) $\eta_i \wedge \xi_j \wedge \xi_1 \wedge \eta_1 \wedge ... \wedge \hat{\xi_i} \wedge \hat{\eta_i} \wedge ... \wedge \hat{\xi_j} \wedge \hat{\eta_j} \wedge ... \wedge \xi_n \wedge \eta_n$,

where $\hat{\bullet}$ stands for omission.

For the left hand side, it is simply verification of the Equation (4). For the right hand side, it is long but straightforward combinatorial computations.

For $i \in \{1, ..., n\}$, we can check that

$$*(\xi_i \wedge \eta_i) = \xi_1 \wedge \eta_1 \wedge ... \wedge \hat{\xi_i} \wedge \hat{\eta_i} \wedge ... \wedge \xi_n \wedge \eta_n.$$

Then clearly, the vectors

$$*(\xi_i \wedge \eta_i)$$

for $1 \leq i \leq n$ are linearly independent.

Moreover, a combinatorial computation shows that

$$\xi_i \wedge \eta_i \wedge \frac{\omega^{\wedge(n-2)}}{(n-2)!} = \sum_{k \neq i} \xi_1 \wedge \eta_1 \wedge ... \wedge \hat{\xi_k} \wedge \hat{\eta_k} \wedge ... \wedge \xi_n \wedge \eta_n. \tag{10}$$

So

$$\xi_i \wedge \eta_i \wedge \frac{\omega^{\wedge(n-2)}}{(n-2)!} \quad = \quad * \sum_{k \neq i} \xi_k \wedge \eta_k. \tag{11}$$

□

The last lemma provides a decomposition of $\Lambda^2 V^*$ if $2n > 4$.

Lemma 2.1.8. *Let* (V, ω, J, g) *be a Kähler vector space of real dimension* $2n > 4$. *We consider the orientation of* V *given by the canonical volume form of the underlying symplectic vector space* (V, ω) *(see Section 1.1). Let us denote*

$$B^\tau := \left\{ \mu \in \Lambda^2 V^* | * \mu = \tau \mu \wedge \omega^{\wedge(n-2)} \right\}$$

for each $\tau \in \mathbb{R}$.

Then,

$$\Lambda^2 V^* = B^{\frac{1}{(n-2)!}} \oplus B^{\frac{-1}{(n-2)!}} \oplus B^{\frac{1}{(n-1)!}}.$$

Moreover, with respect to an orthonormal Darboux basis $\{e_i, f_j\}_{1\leq i,j\leq n}$*,*

$$B^{\frac{1}{(n-2)!}} = \oplus_{i<j} \langle \xi_i \wedge \xi_j - \eta_i \wedge \eta_j, \xi_i \wedge \eta_j + \eta_i \wedge \xi_j \rangle \,,$$

$$\begin{aligned} B^{\frac{-1}{(n-2)!}} &= \oplus_{i<j} \langle \xi_i \wedge \xi_j + \eta_i \wedge \eta_j, \xi_i \wedge \eta_j - \eta_i \wedge \xi_j \rangle \oplus & (12) \\ & \quad \oplus_{i\neq 1} \langle \xi_1 \wedge \eta_1 - \xi_i \wedge \eta_i \rangle & (13) \end{aligned}$$

and

$$B^{\frac{1}{(n-1)!}} = \left\langle \sum_{i=1}^{n} \xi_i \wedge \eta_i \right\rangle = \langle \omega \rangle$$

where $\xi_i := e_i^\flat$ *and* $\eta_i := f_i^\flat$ *for every* $i \in \{1, ..., n\}$.

Proof. Lemma 2.1.7 (i), (ii) and (iii) implies that

$$\oplus_{i<j} \langle \xi_i \wedge \xi_j - \eta_i \wedge \eta_j, \xi_i \wedge \eta_j + \eta_i \wedge \xi_j \rangle \subseteq B^{\frac{1}{(n-2)!}}$$

and

$$\oplus_{i<j} \langle \xi_i \wedge \xi_j + \eta_i \wedge \eta_j, \xi_i \wedge \eta_j - \eta_i \wedge \xi_j \rangle \subseteq B^{\frac{-1}{(n-2)!}}.$$

It remains to work on the vector subspace

$$\oplus_{i\in\{1,...,n\}} \langle \xi_i \wedge \eta_i \rangle \,.$$

Let us fix $\alpha_1, ..., \alpha_n \in \mathbb{R}$ and suppose that

$$\sum_{i=1}^{n} \alpha_i \xi_i \wedge \eta_i \in B^\tau.$$

Thanks to Lemma 2.1.7 (v),

$$\begin{aligned} \sum_{i=1}^{n} \alpha_i * (\xi_i \wedge \eta_i) &= * \sum_{i=1}^{n} \alpha_i \xi_i \wedge \eta_i & (14) \\ &= \tau \sum_{i=1}^{n} \alpha_i \xi_i \wedge \eta_i \wedge \omega^{\wedge(n-2)} & (15) \\ &= \tau \sum_{i=1}^{n} \alpha_i (n-2)! * \sum_{k\neq i} \xi_k \wedge \eta_k & (16) \\ &= \sum_{k=1}^{n} \sum_{i\neq k} \tau (n-2)! \alpha_i * (\xi_k \wedge \eta_k) \,. & (17) \\ & & (18) \end{aligned}$$

By Lemma 2.1.7 (iv),

$$\alpha_k = \sum_{i\neq k} \tau(n-2)!\alpha_i. \tag{19}$$

Let us denote $T := \tau(n-2)!$. Equation (19) means that the vector $(\alpha_1, ..., \alpha_n)$ is a solution of the homogeneous system

$$\begin{pmatrix} 1 & -T & ... & -T & -T \\ -T & 1 & ... & -T & -T \\ ... & ... & ... & ... & ... \\ -T & -T & ... & -T & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix} = 0$$

which is equivalent to the system

$$\begin{pmatrix} 1 & -T & ... & -T & -T \\ 0 & 1-T^2 & ... & -T-T^2 & -T-T^2 \\ ... & ... & ... & ... & ... \\ 0 & -T-T^2 & ... & -T-T^2 & 1-T^2 \end{pmatrix} \begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix} = 0.$$

To solve this system, let us decompose the problem into two cases: $T = -1$ (i.e. $\tau = \frac{-1}{(n-2)!}$) and $T \neq -1$. In the first case, the system is equivalent to the system

$$x_1 + ... + x_n = 0.$$

This shows that

$$\oplus_{i\neq 1} \langle \xi_1 \wedge \eta_1 - \xi_i \wedge \eta_i \rangle \subseteq B^{\frac{-1}{(n-2)!}}.$$

In the second case, the system is equivalent to the systems

$$\begin{pmatrix} 1 & -T & ... & -T & -T \\ 0 & 1-T & ... & -T & -T \\ ... & ... & ... & ... & ... \\ 0 & -T & ... & -T & 1-T \end{pmatrix} \begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix} = 0,$$

$$\begin{pmatrix} 1 & -T & -T & ... & -T & -T \\ 0 & 1 & 0 & ... & 0 & -1 \\ 0 & 0 & 1 & ... & 0 & -1 \\ ... & ... & ... & ... & ... & ... \\ 0 & -T & -T & ... & -T & 1-T \end{pmatrix} \begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix} = 0$$

and eventually

$$\begin{pmatrix} 1 & -T & -T & ... & -T & -T \\ 0 & 1 & 0 & ... & 0 & -1 \\ 0 & 0 & 1 & ... & 0 & -1 \\ ... & ... & ... & ... & ... & ... \\ -1 & 0 & 0 & ... & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix} = 0.$$

Then $(\alpha_1, ..., \alpha_n)$ is a solution of the systems

$$\begin{cases} x_1 - Tx_2 - ... - Tx_n = 0 \\ x_1 = x_2 = ... \qquad = x_n \end{cases}$$

and

$$\begin{cases} (1 - T(n-1))x_1 = 0 \\ x_1 = x_2 = ... \quad = x_n. \end{cases}$$

So if $T \neq \frac{1}{n-1}$ (i.e. $\tau \neq \frac{1}{(n-1)!}$), the unique solution is $(0, ..., 0)$. Otherwise, the solutions are

$$\{(\alpha, ..., \alpha) | \alpha \in \mathbb{R}\}.$$

In particular, this shows that

$$\left\langle \sum_{i=1}^{n} \xi_i \wedge \eta_i \right\rangle \subseteq B^{\frac{1}{(n-1)!}}.$$

The lemma follows by counting dimensions. □

To finish this section, we determine the set of self-dual 2-forms on a Kähler vector space of real dimension $2n > 4$ with the orientation given by the canonical volume form of the underlying symplectic vector space (see Section 1.1).

Proposition 2.1.9. *Let (V, ω, J, g) be a Kähler vector space of real dimension $2n > 4$ endowed with the orientation given by the canonical volume form of the underlying symplectic vector space (V, ω) (see Section 1.1). Then a 2-form μ on V is self-dual if and only if there exists $c \in \mathbb{R}$ such that $\mu = c\omega$.*

Proof. The proof is clear thanks to Lemma 2.1.8. □

Before ending this section, let us insist on an important fact: our work is only valid on vector spaces of real dimension strictly greater than 4. Indeed, the decomposition of $\Lambda^2 V^*$ of the Lemma 2.1.8 is no more valid in dimension 4 because in this case, $n = 2$ and hence $(n-1)! = (n-2)! = 1$. In fact, in dimension 4,

$$\Lambda^2 V^* = B^1 \oplus B^{-1}.$$

So this text is not a generalization but a complement of the job made for dimension 4.

2.2. On manifolds

The structure of this section is a carbon copy of the structure of the previous one but it treats the notion of self-duality of 2-forms on manifolds instead of self-duality of 2-forms on vector spaces. First of all we explain how the definitions of the Hodge-star operator and of (anti-)self-duality in dimension 4 of the previous section are adapted on oriented Riemannian manifolds. Secondly, we recall the notion of almost Kähler manifolds. Manifolds on which, thirdly, we define our generalized notion of self-duality. We end this section with one of the most important theorem of our text which characterizes the space of self-dual 2-forms on almost Kälher manifolds of real dimension strictly greater than 4.

The Hodge-star operator on an oriented Riemannian manifold is simply the Hodge-star operator of the previous section at each point of the manifold.

Definition 2.2.1. [AHS78, Section 1] *On (M, g) an oriented Riemannian manifold of dimension n, the* Hodge-star operator $* : \Lambda^k(M) \to \Lambda^{n-k}(M)$ *is defined for every $x \in M$ by*

$$(*\mu)_x := *(\mu_x)$$

where $*(\mu_x)$ is the Hodge-star operator of μ_x on the oriented inner product space $(T_x M, g_x)$.

The natural definition of (anti-)self-duality on a manifold of real dimension 4 is clear.

Definition 2.2.2. [AHS78, Section 1] *On (M, g) an oriented Riemannian manifold of dimension 4, a 2-form μ is called* self-dual *if*

$$*\mu = \mu$$

and anti-self-dual *if*

$$*\mu = -\mu.$$

To extend this definition to real dimension greater than 4, we copy the previous section. Manifolds corresponding to Kähler vector spaces are the almost Kähler manifolds. Let us recall this definition and give the extended definition of self-duality.

Definition 2.2.3. *An* almost Kähler manifold (M, g, J, ω) *is a Riemannian manifold* (M, g) *with*

- *an almost complex structure* J*, i.e.* $J \in \Gamma^\infty(End(TM))$ *such that* $J^2 = -id_{TM}$*,*
- *a symplectic* 2*-form* ω*, i.e. a closed nondegenerate* 2*-form*

such that

$$\omega(., J.) = g(., .).$$

Definition 2.2.4. *Let* (M, g, J, ω) *be an almost Kähler manifold of real dimension* $2n \geq 4$ *endowed with the orientation given by the canonical volume form of the underlying symplectic manifold* (M, ω) *(see Section 1.1). A* 2*-form* μ *on* M *is called* generalized self-dual *(or simply* self-dual*) if*

$$*\mu = \mu \wedge \frac{\omega^{\wedge(n-2)}}{(n-1)!}.$$

The following theorem comes from Proposition 2.1.9. It will be fundamental for the identification of the moduli space of self-dual connections in our context in Chapter 3.

Theorem 2.2.5. *Let* (M, g, J, ω) *be an almost Kähler manifold of real dimension* $2n > 4$ *endowed with the orientation given by the canonical volume form of the underlying symplectic manifold* (M, ω)*. Then a* 2*-form* μ *is self-dual if and only if there exists* $c \in \mathcal{C}^\infty(M, \mathbb{R})$ *such that*

$$\mu = c\omega.$$

Proof. By Proposition 2.1.9, if μ is self-dual, there exists a function $c : M \to \mathbb{R}$ such that $\mu = c\omega$. If we look at this equality with respect to a Darboux basis $\{dq_i, dp_i\}_{1 \leq i,j \leq n}$, we see that μ is written locally

$$c \sum_i dq_i \wedge dp_i.$$

μ is smooth so the function c has to be locally smooth, so globally. The opposite implication is clear. □

Chapter 3

Generalized moduli space of self-dual connections

The generalized definition of self-duality of 2-forms of previous chapter provides a natural definition of self-duality of connections. As in real dimension 4, self-dual connections on manifolds of real dimension greater than 4 are critical points of the Yang-Mills functional. Moreover the set of self-dual connections on these spaces is stable under the action of the Gauge group. The space of self-dual connections modulo the Gauge group is called the moduli space of self-dual connections. It is interesting for both mathematicians and physicists. This chapter is devoted to its study. It is split into two sections.

The first one gives definitions of self-dual connections, Gauge group and moduli space of self-dual connections. Moreover, it gives two examples of self-dual connections (Examples 3.1.2). One is constructed on a torus bundle over a torus, thanks to the Heisenberg group. The other one is constructed on the natural principal bundle over an Hermitian symmetric space with proper properties.

The second section deals with the study of moduli space of self-dual connections if the Gauge group is abelian. To avoid confusion, if the Gauge group is abelian, we denote it by Z instead of G. In this case, four things simplify:

- the curvature of a connection can be seen as a 2-form on M valued in $\mathfrak{z} := Lie(Z)$,
- the difference of two connection 1-forms can be seen as a 1-form on M valued in $\mathfrak{z}$,
- the curvature of a connection α is simply $d\alpha$,
- the Gauge group is isomorphic to $\mathcal{C}^\infty(M, Z)$.

This section is split into four parts. The first one explains these simplifications. The second one identifies the moduli space of self-dual connections $\mathcal{M}$ when the based manifold is compact connected of real dimension $2n > 4$. It asserts that $\mathcal{M}$ is either empty, or in bijection with $H^1(M, \mathfrak{z})/K^Z$, where K^Z is a subgroup of $H^1(M, \mathfrak{z})$. For this proof, in the non-empty case, we define a natural map from $\mathcal{M}$ to $H^1(M, \mathfrak{z})/K^Z$. The proof that this map is well-defined, injective and surjective comes directly from Lemma 3.2.1.

The third part of this section gives a structure on the moduli space of self-dual connections (if it is non empty) if one hypothesis is added: the Gauge group has to be connected. We prove that there exists an abelian Lie group structure on $\mathcal{M}$. Section 1.3 tells us that the only difficult thing to prove is that K^{S^1} is closed in $H^1(M, Lie(S^1))$. It is quite easily proved thanks to the characterization of K^{S^1} formulated with a line integral (Proposition 3.2.7).

The fourth part of this section studies some examples of moduli space of self-dual connections (Examples 3.2.10).

3.1. Definition

In this section, we define the notion of moduli space of self-dual connections. First of all, we define the self-duality of connections and present some examples of self-dual connections. Secondly, we define the notion of Gauge transformations. Thirdly, we show that a Gauge transformation preserves the self-duality and eventually we give the definition of moduli space of self-dual connections.

We generalize the definition of self-duality of connection of [AHS78, Section 2] to even dimension greater than four using Chapter 2.

Definition 3.1.1. *Let G be a compact Lie group, (M, g, J, ω) be an almost Kähler manifold of real dimension $2n \geq 4$ and $P \to M$ be a G-principal bundle. A connection 1-form α on P is called* self-dual *if $\widetilde{F}$ (defined in Definition 1.5.10) is a self-dual 2-form on M.*

By "$\widetilde{F}$ is self-dual" we mean the following: if $\widetilde{F} = \sum_j \mu_j \otimes s_j$ for $\mu_j \in \Lambda^2(M)$ and $s_j \in \Gamma^\infty(E_{Ad})$, each μ_j has to be a self-dual 2-form on M for the generalized Definition 2.2.4. It is independent of the choice of the representative $\sum_j \mu_j \otimes s_j$ in $\Lambda^2(M) \otimes \Gamma^\infty(E_{Ad})$ because the self-duality of 2-forms is stable under the product by a smooth function on M.

Now, let us look at two examples of self-dual connection 1-forms.

Example 3.1.2. *(i) Our first example of self-dual connection will be constructed on a torus bundle over a torus. R.S. Palais and T.E.*

Stewart explain in [PS61] *that bundles of this form has compact* 2*-step nilmanifold as total space*[1]*. An easy example of* 2*-step nilpotent compact Lie group is*

$$G := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & I_n & b \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b \in \mathbb{K}^n, c \in \mathbb{K} \right\},$$

the Heisenberg group*, for* $\mathbb{K} = \mathbb{R}, \mathbb{C}$ *or* $\mathbb{H}$ *(the real, complex or quaternion numbers). If we denote by* Γ *the subgroup of* G *of matrices with coefficients in* $Z := \mathbb{Z}$ *(respectively* $\mathbb{Z} \oplus i\mathbb{Z}$ *or* $\mathbb{Z} \oplus i\mathbb{Z} \oplus j\mathbb{Z} \oplus k\mathbb{Z}$*) then*

$$Hsbrg^n(\mathbb{K}) := G/\Gamma$$

is a compact 2*-step nilmanifold.*

Let us denote by $T^k \simeq \mathbb{K}/Z$ *the* k*-torus for* $k := dim_{\mathbb{R}}(\mathbb{K})$ *and let us consider the well-defined free right action of* T^k *on* $Hsbrg^n(\mathbb{K})$*:*

$$Hsbrg^n(\mathbb{K}) \times T^k \to Hsbrg^n(\mathbb{K})$$

$$\left(\left[\begin{pmatrix} 1 & a & c \\ 0 & I_n & b \\ 0 & 0 & 1 \end{pmatrix} \right], [x] \right) \mapsto \left[\begin{pmatrix} 1 & a & c+x \\ 0 & I_n & b \\ 0 & 0 & 1 \end{pmatrix} \right].$$

$$\pi : Hsbrg^n(\mathbb{K}) \to T^{2nk} : \left[\begin{pmatrix} 1 & a & c \\ 0 & I_n & b \\ 0 & 0 & 1 \end{pmatrix} \right] \mapsto [(a_1, b_1, ..., a_n, b_n)]$$

endowed with this T^k*-right action defines a* T^k*-principal bundle structure on* $Hsbrg^n(\mathbb{K})$*. We will construct a connection* 1*-form* α_0 *on it, if* $2kn > 4$*.*

For $X_u \in T_u\,(Hsbrg^n(\mathbb{K}))$*, let us consider* $\gamma : I \subseteq \mathbb{R} \to G$ *such that* $[\gamma(0)] = u$ *and* $\frac{d}{dt}|_0[\gamma(t)] = X_u$*. We write* $\gamma(t) = \begin{pmatrix} 1 & a(t) & c(t) \\ 0 & I_n & b(t) \\ 0 & 0 & 1 \end{pmatrix} \in G$ *and define*

$$\alpha_0 : T\,(Hsbrg^n(\mathbb{K})) \to Lie(T^k) \simeq \mathbb{K} : X_u \mapsto c'(0) - a'(0).b(0).$$

[1]Roughly speaking, a compact 2-step nilmanifold is a quotient space N/H where N is a 2-step nilpotent compact Lie group and H a closed subgroup of N (we refer to [Wil82] for precise definitions and details about nilmanifolds).

It's easy to show that α_0 is independent of the chosen curve γ in G. Moreover, it is a connection 1-form. Indeed,

- *If $T \in Lie(T^k)$ and $u = \left[\begin{pmatrix} 1 & a & c \\ 0 & I_n & b \\ 0 & 0 & 1 \end{pmatrix}\right] \in Hsbrg^n(\mathbb{K})$,*

$$\begin{aligned} T_u^* &:= \frac{d}{dt}\Big|_0 \left[\begin{pmatrix} 1 & a & c \\ 0 & I_n & b \\ 0 & 0 & 1 \end{pmatrix}\right][\exp tT] \\ &= \frac{d}{dt}\Big|_0 \left[\begin{pmatrix} 1 & a & c + \sum_k \frac{(tT)^k}{k!} \\ 0 & I_n & b \\ 0 & 0 & 1 \end{pmatrix}\right]. \end{aligned}$$

  So $\alpha_0(T_u^*) = T$.

- *If $[\tau] \in T^k$ and $X_u \in T\left(Hsbrg^n(\mathbb{K})\right)$ for $u = \left[\begin{pmatrix} 1 & a & c \\ 0 & I_n & b \\ 0 & 0 & 1 \end{pmatrix}\right]$, let us denote by $\gamma : I \subseteq \mathbb{R} \to G$ a curve in G such that $[\gamma(0)] = u$ and $\frac{d}{dt}|_0[\gamma(t)] = X_u$ and let us write $\gamma(t) = \begin{pmatrix} 1 & a(t) & c(t) \\ 0 & I_n & b(t) \\ 0 & 0 & 1 \end{pmatrix}$.*

  Then

$$\begin{aligned} R_{[\tau]}^* \alpha_0(X_u) &= \alpha_0\left(R_{[\tau]*} X_u\right) \\ &= \alpha_0\left(\frac{d}{dt}\Big|_0 \left[\begin{pmatrix} 1 & a(t) & c(t) \\ 0 & I_n & b(t) \\ 0 & 0 & 1 \end{pmatrix}\right][\tau]\right) \\ &= \alpha_0\left(\frac{d}{dt}\Big|_0 \left[\begin{pmatrix} 1 & a(t) & c(t) + \tau \\ 0 & I_n & b(t) \\ 0 & 0 & 1 \end{pmatrix}\right]\right) \\ &= c'(0) - a'(0).b(0) \\ &= \alpha(X_u) \\ &= Ad_{[\tau]}^{-1} \alpha(X_u). \end{aligned}$$

There exists a natural structure of Kähler manifold on T^{2nk}. So we can try to show that α_0 is self-dual with respect to the underlying metric and symplectic form.

Let us denote

$$I_{\mathbb{R}} := \left]-\frac{1}{2}, \frac{1}{2}\right[\left(\text{respectively } I_{\mathbb{C}} := \left]-\frac{1}{2}, \frac{1}{2}\right[+ i \left]-\frac{1}{2}, \frac{1}{2}\right[\right.$$

$$\text{or } I_{\mathbb{H}} := \left]-\frac{1}{2},\frac{1}{2}\right[+i\left]-\frac{1}{2},\frac{1}{2}\right[+j\left]-\frac{1}{2},\frac{1}{2}\right[+k\left]-\frac{1}{2},\frac{1}{2}\right[\Bigg).$$

For every $[(a,b)] := [(a_1, b_1, ..., a_n, b_n)] \in T^{2kn}$, *let us consider the open subset of* T^{2kn}

$$U_{[(a,b)]} := \left\{\ [(a_1 + x_1, b_1 + y_1, ..., a_n + x_n, b_n + y_n)] \,|\, x, y \in (I_{\mathbb{K}})^n \right\}$$

and the local section $s : U_{[(a,b)]} \to Hsbrg^n(\mathbb{K})$ *defined by*

$$\begin{aligned} & s\left([(a_1 + x_1, b_1 + y_1, ..., a_n + x_n, b_n + y_n)]\right) \\ = \quad & \left[\begin{pmatrix} 1 & a+x & 0 \\ 0 & I_n & b+y \\ 0 & 0 & 1 \end{pmatrix}\right] \end{aligned}$$

for every $[(a_1 + x_1, b_1 + y_1, ..., a_n + x_n, b_n + y_n)] \in U_{[(a,b)]}$.

Hence

$$s^*\alpha_0 = -\sum_{i=1}^{n}(b_i + y_i)dx_i$$

and

$$s^*F_0 = s^*d\alpha_0 = \sum_{i=1}^{n} dx_i \wedge dy_i = \omega,$$

where ω *denotes the symplectic structure on* T^{2nk}.

If $2nk > 4$, *Theorem 2.2.5 asserts that* α_0 *is a self-dual connection* 1*-form on* $Hsbrg^n(\mathbb{K})$.

(ii) The second example is constructed on some kind of Hermitian symmetric spaces. *The symmetric structure gives canonical principal bundle and connection: the* K*-principal bundle* $G \to G/K$ *and its Loos connection defined in Section 1.6. Naturally, we wonder if the Loos connection is self-dual. In general, the answer is no. But we will show that with restrictive hypotheses, the Loos connection is still self-dual. We refer to* [KN96, Section XI.9] *for the definition of Hermitian symmetric spaces and details about it and to the Section 1.6 for basic facts about symmetric spaces.*

Let (M, g, J, ω) *be an Hermitian symmetric space.* M *is in particular a connected symmetric space so we can consider the corresponding* K*-principal bundle* $G \to G/K$ *and its Loos connection as in Section 1.6. If* G *is simple,* K *a* 1*-dimensional Lie subgroup of* G *and* M *of real dimension strictly greater than* 4, *then the Loos connection* α_0 *on* $G \to G/K$ *is self-dual.*

Indeed, ω *is compatible with the symmetric structure so* s_x*-invariant for every* $x \in M$*. In particular,* ω *is* G*-invariant.*

So $(\pi^*\omega)_e|_{\mathfrak{p}\times\mathfrak{p}}$ *is* $\mathfrak{k}$*-invariant, i.e.*

$$(\pi^*\omega)_e|_{\mathfrak{p}\times\mathfrak{p}}(ad_X Y, Z) = -\,(\pi^*\omega)_e|_{\mathfrak{p}\times\mathfrak{p}}(Y, ad_X Z)$$

for every $Y, Z \in \mathfrak{p}$ *and* $X \in \mathfrak{k}$*.*

G *is simple, so by* [Bie98, Theorem 2.1]*, there exists* $Z \in \mathfrak{z}(\mathfrak{k})$ *such that*

$$(\pi^*\omega)_e|_{\mathfrak{p}\times\mathfrak{p}} = -\beta(Z, [.,.])$$

for β *the Killing form on* $\mathfrak{g}$*.* K *is* 1*-dimensional, hence* $\mathfrak{z}(\mathfrak{k}) = \mathfrak{k}$ *and* $\{Z\}$ *forms a basis of* $\mathfrak{k}$

A vector field is horizontal with respect to α_0 *if and only if it is a left-invariant vector field on* G *corresponding to a vector in* $\mathfrak{p}$*.*

By Lemma 1.5.6, for every $X, Y \in \mathfrak{p}$*,*

$$\begin{aligned}
F_0(\widetilde{X}_g, \widetilde{Y}_g) &= -\frac{1}{2}\alpha_0([\widetilde{X}, \widetilde{Y}]_g) \\
&= -\frac{1}{2}\alpha_0(\widetilde{[X, Y]}_g) \\
&= -\frac{1}{2}pr_{\mathfrak{k}}([X, Y]) \\
&= -\frac{1}{2}\frac{\beta(Z, [X, Y])Z}{\|Z\|^2} \\
&= \frac{1}{2}(\pi^*\omega)_e(X, Y)\frac{Z}{\|Z\|^2} \\
&= (\pi^*\omega)_g(\widetilde{X}_g, \widetilde{Y}_g)\frac{Z}{2\|Z\|^2}
\end{aligned}$$

because $\pi^*\omega$ *is* G*-invariant.*

If we denote by Q_g *the horizontal part of* T_gG *with respect to* α_0*, on* $Q_g \times Q_g$*,*

$$(F_0)_g = (\pi^*\omega)_g \otimes \frac{Z}{2\|Z\|^2}.$$

As both expressions are equal to 0 *on* $G_g \times G_g$*,* $G_g \times Q_g$ *and* $Q_g \times G_g$ *(for* G_g *the vertical part of* T_gG*),*

$$F_0 = (\pi^*\omega) \otimes \frac{Z}{2\|Z\|^2}$$

on TG*.*

For every $g \in G$, every $s : U \subseteq M \to G$ local section such that $\pi(g) \in U$ and $s(\pi(g)) = g$ and every $X, Y \in \Gamma^\infty(TM)$,

$$\begin{aligned}\left(\tilde{F}_0(X,Y)\right)\hat{}\,(g) &= g_{X,Y}(g)\\ &= F_0(s_*X_{\pi(g)}, s_*Y_{\pi(g)})\\ &= (\pi^*\omega)(s_*X_{\pi(g)}, s_*Y_{\pi(g)})\frac{Z}{2\|Z\|^2}\\ &= \omega(X_{\pi(g)}, Y_{\pi(g)})\frac{Z}{2\|Z\|^2}.\end{aligned}$$

Hence Theorem 2.2.5 asserts that α_0 is self-dual.

Let us remark that the conditions to have a self-dual Loos connection are restrictive. We have not found an explicit example of Hermitian symmetric space which respects these hypotheses. Nevertheless, we wrote this theoretical example in our text because we built our general theory from this Hermitian symmetric case.

The definitions of Gauge group and moduli space come directly from the 4-dimensional case. Let us begin with the definition of the Gauge group.

Definition 3.1.3. *For G a compact Lie group, let $P \to M$ be a G-principal bundle over a manifold M. A* Gauge transformation *is a vertical isomorphism of the principal bundle, i.e. a diffeomorphism $\varphi : P \to P$ such that for every $p \in P$ and $g \in G$,*

$$\pi(\varphi(p)) = \pi(p)$$

and

$$\varphi(pg) = \varphi(p)g.$$

We denote by $\mathcal{G}$ the set of Gauge transformations. It forms a group for the composition law. It is called the Gauge group *of P.*

Let us denote by $\mathcal{C}^\infty(P, G)^G$ the group of G-equivariant functions for the action of G on itself by conjugation $C_g : G \to G : g' \mapsto gg'g^{-1}$, i.e. the group of smooth functions $\tilde{\varphi} : P \to G$ such that $\tilde{\varphi}(pg) = C_{g^{-1}}\tilde{\varphi}(p)$, where the group laws come naturally from the group laws of G. Then there exists a isomorphism between $\mathcal{G}$ and $\mathcal{C}^\infty(P, G)^G$ given by

$$\varphi(p) = p\tilde{\varphi}(p)$$

for $\varphi \in \mathcal{G}$ and $\tilde{\varphi}$ the corresponding G-equivariant function. The local triviality of P proves that it is indeed a bijection. In our text, we denote always by $\tilde{\varphi}$ the G-equivariant function corresponding to the Gauge transformation φ.

The following proposition shows that the set of self-dual connection 1-forms is stable by the action of the Gauge group. It is fundamental for the definition of the moduli space.

Proposition 3.1.4. *For G a compact Lie group and (M, g, J, ω) an almost Kähler manifold of real dimension $2n \geq 4$, let $\pi : P \to M$ be a G-principal bundle endowed with a connection 1-form α and φ be a Gauge transformation. Then*

(i) $\varphi^*\alpha$ is a connection 1-form,

(ii) α is self-dual if and only if $\varphi^*\alpha$ is self-dual.

Proof. (i) $\varphi^*\alpha$ is a 1-form on P valued in $\mathfrak{g}$. We have to check the Definition 1.5.2. The first point is clear if we remark that, for $X \in \mathfrak{g}$ and $p \in P$,

$$\varphi_* X_p^* = \left.\frac{d}{dt}\right|_0 \varphi(p \exp(tX)) = \left.\frac{d}{dt}\right|_0 \varphi(p) \exp(tX) = X_{\varphi(p)}^*.$$

Let us compute the second point. For every $X_p \in T_pP$ and $\gamma : I \subseteq \mathbb{R} \to P$ such that $\gamma(0) = p$ and $\left.\frac{d}{dt}\right|_0 \gamma(t) = X_p$ and for every $g \in G$,

$$\begin{aligned} R_g^*\varphi^*\alpha(X_p) &= \alpha(\varphi_* R_{g*} X_p) \\ &= \alpha\left(\left.\frac{d}{dt}\right|_0 \varphi(\gamma(t)g)\right) \\ &= \alpha\left(\left.\frac{d}{dt}\right|_0 \varphi(\gamma(t))g\right) \\ &= Ad_{g^{-1}}\alpha(\varphi_* X_p) \\ &= Ad_{g^{-1}}\varphi^*\alpha(X_p). \end{aligned}$$

(ii) First of all, we remark that, by Proposition 1.5.5, the curvature of $\varphi^*\alpha$ is

$$d\varphi^*\alpha + \frac{1}{2}[\varphi^*\alpha(.), \varphi^*\alpha(.)] = \varphi^* F.$$

Let us look at $\widetilde{\varphi^* F}$ of Definition 1.5.10. For $X, Y \in \Gamma^\infty(TM)$, $p \in P$ and $s : U \subseteq M \to P$ a local section defined on U such that

$\pi(p) \in U$ and $s(\pi(p)) = p$,

$$\begin{aligned}\left(\widetilde{\varphi^* F}(X,Y)\right)\hat{}(p) &= \varphi^* F(s_* X_{\pi(p)}, s_* Y_{\pi(p)}) \\ &= F\left(\varphi_* s_* X_{\pi(p)}, \varphi_* s_* Y_{\pi(p)}\right) \\ &= F\left((\varphi \circ s)_* X_{\pi(p)}, (\varphi \circ s)_* Y_{\pi(p)}\right) \\ &= \left(\tilde{F}(X,Y)\right)\hat{}(\varphi(p))\end{aligned}$$

because $\varphi \circ s : U \subseteq M \to P$ is a local section of P such that $\varphi \circ s(\pi(p)) = \varphi(p)$.

So if $\widetilde{F} = \sum_j \mu_j \otimes s_j$, then $\widetilde{\varphi^* F} = \sum_j \mu_j \otimes s_j^\varphi$ where

$$\widehat{s_j^\varphi} = \widehat{s_j} \circ \varphi.$$

Hence $\varphi^* \alpha$ is self-dual if and only if α is self-dual.

□

Proposition 3.1.4 says that we can define an equivalence relation on the set of self-dual connection 1-forms on P with respect to the Gauge group:

$$\varphi^* \alpha \sim \alpha$$

for every $\varphi \in \mathcal{G}$.

We can generalize the well known notion of moduli space of self-dual connections to almost Kähler manifolds of real dimension greater than four. The classical definition can be found for example in [AHS78, Section 6].

Definition 3.1.5. *For G a compact Lie group and (M, g, J, ω) an almost Kähler manifold of real dimension $2n \geq 4$, let $\pi : P \to M$ be a G-principal bundle. The* moduli space of self-dual connections on P *is the set of self-dual connections on P modulo the equivalence relation given by $\mathcal{G}$, the Gauge group on P.*

We denote by $\mathcal{M}$ the moduli space of self-dual connections on P.

3.2. For abelian Gauge groups

In this section we identify the moduli space of self-dual connections on a compact connected almost Kähler manifold of real dimension strictly greater than four if the Gauge group is abelian. If we consider an abelian Lie group, we write it Z instead of G as usual.

The fact that Z is abelian simplifies most of what we did until now. In this section, first of all, we present all these simplifications. It will lead us to two theorems which characterize the moduli space of self-dual connections in this context. Both of them are presented with some propositions and lemmas. We end this section by some examples of applications of these theorems.

The Lie group is abelian so E_{Ad} is isomorphic to the trivial bundle $M \otimes \mathfrak{z}$ and $\Gamma^\infty(E_{Ad}) \simeq \mathcal{C}^\infty(M, \mathfrak{z})$. By Definition 1.5.10, the difference of two connection 1-forms α_1 and α_2 defines a element $\widetilde{\alpha_2 - \alpha_1} \in \Lambda^1(M) \otimes \Gamma^\infty(E_{Ad})$. So if the Lie group is abelian, it defines naturally a global element $\overline{\alpha_2 - \alpha_1} \in \Lambda^1(M, \mathfrak{z})$. Locally

$$\overline{\alpha_2 - \alpha_1} := s^* (\alpha_2 - \alpha_1)$$

for some $s : U \subseteq M \to P$ local section of P.

In the same way, the curvature F of a connection 1-form α defines naturally a global element $\overline{F} \in \Lambda^2(M, \mathfrak{z})$ which is locally

$$\overline{F} := s^* F$$

for some $s : U \subseteq M \to P$ local section of P.

If we denote

$$\overline{F} =: \sum_j \mu_j \otimes Y_j \in \Lambda^2(M, \mathfrak{z}) = \Lambda^2(M) \otimes \mathfrak{z},$$

α is self-dual if and only if the μ_j's are self-dual 2-forms on M for every j.

We remark moreover that $F = d\alpha$ because the Lie algebra $\mathfrak{z}$ is abelian. Eventually let us remark that $\mathcal{C}^\infty(P, Z)^Z$ is isomorphic to $\mathcal{C}^\infty(M, Z)$ because the action of Z on itself by conjugation is trivial. So $\mathcal{G} \simeq \mathcal{C}^\infty(M, Z)$. For $\varphi \in \mathcal{G}$, we write $\overline{\varphi}$ the corresponding smooth function from M to Z. We have

$$\varphi(p) = p\left(\overline{\varphi} \circ \pi(p)\right)$$

for every $p \in P$.

The following lemma will give us directly the proof of our first theorem, which characterizes $\mathcal{M}$.

Lemma 3.2.1. *For Z an abelian compact Lie group and (M, g, J, ω) a compact connected almost Kähler manifold of real dimension $2n > 4$, let $\pi : P \to M$ be a Z-principal bundle. Then* *(i) if α_1 and α_2 are self-dual connections on P, $\overline{\alpha_2 - \alpha_1}$ is a closed 1-form on M valued in $\mathfrak{z}$,*

(ii) if α_0, α and β are connection 1-forms on P, then there exists $\varphi \in \mathcal{G}$ such that $\beta = \varphi^* \alpha$ if and only if there exists $\overline{\varphi} \in \mathcal{C}^\infty(M, Z)$ such that $\overline{\beta - \alpha_0} = \overline{\alpha - \alpha_0} + \overline{\varphi}^{-1} d\overline{\varphi}$,

(iii) if σ is a closed 1-form on M valued in $\mathfrak{z}$ and α a self-dual connection 1-form on P, then $\pi^*\sigma + \alpha$ is a self-dual connection 1-form on P.

Remark 3.2.2. *By $\overline{\varphi}^{-1} d\overline{\varphi}$ we mean, for $X_x \in T_xM$ and $\gamma : I \subseteq \mathbb{R} \to M$ such that $\gamma(0) = x$ and $\dot{\gamma}(0) = X_x$,*

$$\overline{\varphi}^{-1} d\overline{\varphi}(X_x) := \left.\frac{d}{dt}\right|_0 \left(\overline{\varphi}(x)\right)^{-1} \overline{\varphi}(\gamma(t)) \in \mathfrak{z}.$$

Proof. (i) Let us denote $\overline{F}_1 := \sum_j \mu_j^1 \otimes Y_j^1$ and $\overline{F}_2 := \sum_j \mu_j^2 \otimes Y_j^2$ where $\mu_j^i \in \Lambda^2(M)$ and $Y_j^i \in \mathfrak{z}$ for $i = 1, 2$. α_1 and α_2 are self-dual so by definition μ_j^i are self-dual 2-forms on M for $i = 1, 2$ and for every j. By Theorem 2.2.5, there exist $c_j^i \in \mathcal{C}^\infty(M, \mathbb{R})$ such that $\mu_j^i = c_j^i \omega$. Then

$$\overline{F}_1 = \sum_j c_j^1 \omega \otimes Yj^1 = \omega \otimes Y^1$$

for $Y^1 := \sum_j c_j^1 Y_j^1 \in \mathcal{C}^\infty(M, \mathfrak{z})$ and in the same way

$$\overline{F}_2 = \omega \otimes Y^2$$

for $Y^2 := \sum_j c_j^2 Y_j^2 \in \mathcal{C}^\infty(M, \mathfrak{z})$.
For some $s : U \subseteq M \to P$ local section of P,

$$\begin{aligned} \omega \otimes (Y^2 - Y^1) &= \overline{F_2} - \overline{F_1} \\ &= s^*(F_2 - F_1) \\ &= s^*(d\alpha_2 - d\alpha_1) \\ &= ds^*(\alpha_2 - \alpha_1) \\ &= d\overline{\alpha_2 - \alpha_1} \end{aligned}$$

and so globally

$$\omega \otimes (Y^2 - Y^1) = d\overline{\alpha_2 - \alpha_1}. \tag{20}$$

Hence, to finish the proof, it is enough to show that $Y^2 - Y^1 = 0$. First of all, let us show that Y^1 and Y^2 are constants. For $i = 1$ and 2,

$$dF_i = d^2\alpha_i = 0.$$

Then for $s : U \subseteq M \to P$ some local section of P,

$$d\overline{F_i} = ds^* F_i = s^* dF_i = 0.$$

But

$$d\overline{F_i} = d\omega \otimes Y_i + \omega \otimes dY_i = \omega \otimes dY_i$$

so Y_i is constant on M because M is connected.

Now, we can show that $Y^2 - Y^1 = 0$. We consider $\{X_k\}_{k\in\{1,...,l\}}$ a basis of $\mathfrak{z}$ and we denote

$$Y^2 - Y^1 = \sum_{k=1}^{l} f_k X_k$$

for $f_k \in \mathbb{R}$ and

$$\overline{\alpha_2 - \alpha_1} = \sum_{k=1}^{l} \tau_k \otimes X_k$$

for $\tau_k \in \Lambda^1(M, \mathbb{R})$.

Then for every $k \in \{1, ..., l\}$, $f_k\omega = d\tau_k$. If $f_k > 0$ or $f_k < 0$, $\omega = d\frac{\tau_k}{f_k}$. So ω is an exact symplectic form which contradicts the compactness of M, thanks to Lemma 1.1.2.

So $f_k = 0$ for every $k \in \{1, ..., l\}$ and $Y^2 - Y^1 = 0$. Hence $\overline{\alpha_2 - \alpha_1}$ is a closed 1-form on M by Equation (20).

(ii) For every $X_x \in T_xM$ and $\gamma : I \subseteq \mathbb{R} \to M$ such that $\gamma(0) = x$ and $\frac{d}{dt}\Big|_0 \gamma(t) = X_x$ and for every $s : U \subseteq M \to P$ local section such that $x \in U$, we remark that for every $\varphi \in \mathcal{G}$,

$$\begin{aligned}
\varphi^*\alpha(s_*X_x) &= \alpha(\varphi_* s_*(X_x)) \\
&= \alpha\left(\left.\frac{d}{dt}\right|_0 \varphi\left(s(\gamma(t))\right)\right) \\
&= \alpha\left(\left.\frac{d}{dt}\right|_0 s(\gamma(t))\overline{\varphi}\circ\pi\left(s(\gamma(t))\right)\right) \\
&= \alpha\left(\left.\frac{d}{dt}\right|_0 s(\gamma(t))\overline{\varphi}(\gamma(t))\right) \\
&= \alpha\left(R_{\overline{\varphi}(x)*}s_*X_x + \left(\overline{\varphi}^{-1}d\overline{\varphi}(X_x)\right)^*_{\varphi(s(x))}\right) \\
&= Ad_{\overline{\varphi}(x)^{-1}}\alpha\left(s_*X_x\right) + \overline{\varphi}^{-1}d\overline{\varphi}(X_x) \\
&= \alpha\left(s_*X_x\right) + \overline{\varphi}^{-1}d\overline{\varphi}(X_x).
\end{aligned}$$

So for every $s : U \subseteq M \to P$ local section,

$$s^*(\varphi^*\alpha - \alpha) = \overline{\varphi}^{-1}d\overline{\varphi}. \tag{21}$$

If there exists $\varphi \in \mathcal{G}$ such that $\beta = \varphi^*\alpha$, then by Equation (21),

$$\overline{\beta - \alpha} = \overline{\varphi}^{-1}d\overline{\varphi}$$

for $\overline{\varphi}$ the smooth function from M to Z corresponding to φ.

Conversely, if there exists $\overline{\varphi} \in \mathcal{C}^\infty(M, Z)$ such that

$$\overline{\beta - \alpha} = \overline{\varphi}^{-1}d\overline{\varphi},$$

then for every $s : U \subseteq M \to P$ local section,

$$s^*(\beta - \alpha) = \overline{\varphi}^{-1}d\overline{\varphi},$$

i.e. thanks to Equation (21), $s^*\beta = s^*\alpha + \overline{\varphi}^{-1}d\overline{\varphi} = s^*\varphi^*\alpha$ for φ the Gauge transformation corresponding to $\overline{\varphi}$.

To show that $\beta = \varphi^*\alpha$, we will show that they have the same horizontal vectors. For every $p \in P$, we can apply Proposition 1.5.3 with respect to β at $\pi(p)$. So we can find an horizontal local section s at $\pi(p)$ such that $s(\pi(p)) = p$. If $X_p \in T_pP$ is horizontal with respect to β, $s_*\pi_*X_p = X_p$, because there exists a unique horizontal lift of π_*X_p at p.

Then

$$\begin{aligned} 0 &= \beta(X_p) \\ &= (s^*\beta)(\pi_* X_p) \\ &= (s^*\varphi^*\alpha)(\pi_* X_p) \\ &= \varphi^*\alpha(X_p). \end{aligned}$$

Hence X_p is horizontal with respect to $\varphi^*\alpha$ too. In the same way, an horizontal vector for $\varphi^*\alpha$ is horizontal for β. So β and $\varphi^*\alpha$ are the same connection forms.

(iii) Clearly $\pi^*\sigma + \alpha \in \Lambda^1(P) \otimes \mathfrak{z}$. Moreover, it is a connection 1-form. Indeed, for every $X \in \mathfrak{z}$ and every $p \in P$,

$$\begin{aligned} (\pi^*\sigma + \alpha)(X_p^*) &= \pi^*\sigma(X_p^*) + \alpha(X_p^*) \\ &= 0 + X \\ &= X \end{aligned}$$

and for every $z \in Z$ and $X_p \in T_pP$,

$$\begin{aligned} R_z^*(\pi^*\sigma + \alpha)(X_p) &= (\pi \circ R_z)^*\sigma(X_p) + R_z^*\alpha(X_p) \\ &= \pi^*\sigma(X_p) + Ad_{z^{-1}}\alpha(X_p) \\ &= Ad_{z^{-1}}(\pi^*\sigma + \alpha)(X_p) \end{aligned}$$

because $Ad_{z^{-1}} = Id_{\mathfrak{z}}$. Finally, $d(\pi^*\sigma+\alpha) = \pi^* d\sigma + d\alpha = 0 + F = F$ so $\pi^*\sigma + \alpha$ and α have the same curvature and $\pi^*\sigma + \alpha$ is self-dual.

$\square$

We can now present our theorem which characterizes $\mathcal{M}$:

Theorem 3.2.3. *Let Z be an abelian compact Lie group, (M, g, J, ω) be a compact connected almost Kähler manifold of real dimension $2n > 4$ and $\pi : P \to M$ be a Z-principal bundle. Then, either the moduli space of self-dual connections $\mathcal{M}$ is empty, or $\mathcal{M}$ is in bijection with*

$$H^1(M, \mathfrak{z})/K^Z,$$

where $K^Z := \{[\overline{\varphi}^{-1} d\overline{\varphi}] | \overline{\varphi} \in \mathcal{C}^\infty(M, Z)\}$.

Remark 3.2.4. *If $\mathcal{M}$ is non empty, then it depends only on M and Z. It is independent of the total space of the principal bundle. To identify $\mathcal{M}$, the knowledge of this total space is useful simply to construct a first self-dual connection 1-form and so to show that $\mathcal{M}$ is non empty.*

Before going to the proof of this theorem, let us remark that the expression $H^1(M,\mathfrak{z})/K^Z$ makes sense:

Remark 3.2.5. *$H^1(M,\mathfrak{z})$ is an $\mathbb{R}$-vector space so, in particular, an abelian group. We will show that K^Z is a well-defined subgroup of $H^1(M,\mathfrak{z})$. For this, we prove:*

(i) for every $\overline{\varphi} \in \mathcal{C}^\infty(M,Z)$, $\overline{\varphi}^{-1}d\overline{\varphi}$ is a closed 1-form,

(ii) for every $\overline{\varphi_1},\overline{\varphi_2} \in \mathcal{C}^\infty(M,Z)$, $\overline{\varphi}_1{}^{-1}d\overline{\varphi}_1 + \overline{\varphi}_2{}^{-1}d\overline{\varphi}_2 \in K^Z$,

(iii) if $\overline{\varphi}^{-1}d\overline{\varphi} \sim \alpha$ in $H^1(M,\mathfrak{z})$, then there exists $\overline{\psi} \in \mathcal{C}^\infty(M,Z)$ such that $\overline{\psi}^{-1}d\overline{\psi} = \alpha$.

Hence, (i) and (iii) prove that K^Z is a well-defined subset of $H^1(M,\mathfrak{z})$ and (ii) proves that it is a subgroup.

(i) Using the fact that Z is abelian, we see that for every $\overline{\varphi} \in \mathcal{C}^\infty(M,Z)$,

$$\begin{aligned}\overline{\varphi}d\overline{\varphi}^{-1} + \overline{\varphi}^{-1}d\overline{\varphi} &= d\left(\overline{\varphi}^{-1}\overline{\varphi}\right)\\ &= d(1)\\ &= 0\end{aligned}$$

so

$$d\overline{\varphi}^{-1} = -\overline{\varphi}^{-2}d\overline{\varphi}$$

and

$$\begin{aligned}d\left(\overline{\varphi}^{-1}d\overline{\varphi}\right) &= d\left(\overline{\varphi}^{-1}\right)\wedge d\overline{\varphi} + \overline{\varphi}^{-1}d^2\overline{\varphi}\\ &= -\overline{\varphi}^{-2}d\overline{\varphi}\wedge d\overline{\varphi}\\ &= 0.\end{aligned}$$

(ii) For $X_x \in T_xM$ and $\gamma : I \subseteq \mathbb{R} \to M$ such that $\gamma(0) = x$ and $\dot{\gamma}(0) = X_x$, using the fact that Z is abelian, we find

$$\begin{aligned}&\left(\overline{\varphi_1}.\overline{\varphi}_2\right)^{-1} d\left(\overline{\varphi_1}.\overline{\varphi}_2\right)(X_x)\\ =\ & \left.\frac{d}{dt}\right|_0 \left(\overline{\varphi_1}.\overline{\varphi}_2\right)^{-1}(x)\left(\overline{\varphi_1}.\overline{\varphi}_2\right)(\gamma(t))\\ =\ & \left.\frac{d}{dt}\right|_0 \left(\overline{\varphi_1}(x)\right)^{-1}\overline{\varphi_1}(\gamma(t)).\left(\overline{\varphi}_2(x)\right)^{-1}\overline{\varphi_2}(\gamma(t))\\ =\ & \left.\frac{d}{dt}\right|_0 \left(\overline{\varphi_1}(x)\right)^{-1}\overline{\varphi_1}(\gamma(t)).\left(\overline{\varphi}_2(x)\right)^{-1}\overline{\varphi_2}(\gamma(0))\\ &+ \left.\frac{d}{dt}\right|_0 \left(\overline{\varphi_1}(x)\right)^{-1}\overline{\varphi_1}(\gamma(0)).\left(\overline{\varphi}_2(x)\right)^{-1}\overline{\varphi_2}(\gamma(t))\\ =\ & \overline{\varphi_1}^{-1}d\overline{\varphi_1}(X_x) + \overline{\varphi_2}^{-1}d\overline{\varphi_2}(X_x).\end{aligned}$$

(iii) *If α and $\overline{\varphi}^{-1}d\overline{\varphi}$ are equivalent in $H^1(M,\mathfrak{z})$, then there exists $f \in \mathcal{C}^\infty(M,\mathfrak{z})$ such that $\alpha = \overline{\varphi}^{-1}d\overline{\varphi} + df$. For $X_x \in T_xM$ and $\gamma : I \subseteq \mathbb{R} \to M$ such that $\gamma(0) = x$ and $\dot{\gamma}(0) = X_x$,*

$$\begin{aligned} & (\exp\circ f)^{-1}\, d\,(\exp\circ f)\,(X_x) \\ = \;& \left.\frac{d}{dt}\right|_0 ((\exp\circ f)\,(x))^{-1}\,(\exp\circ f)\,(\gamma(t)) \\ = \;& \left.\frac{d}{dt}\right|_0 \exp(f(\gamma(t)) - f(x)) \\ = \;& df(X_x). \end{aligned}$$

Hence $\alpha = \overline{\varphi}^{-1}d\overline{\varphi} + (\exp\circ f)^{-1}d(\exp\circ f)$. So, thanks to (ii), $\alpha = (\overline{\varphi}\exp\circ f)^{-1}d(\overline{\varphi}\exp\circ f)$.

We can now prove the Theorem 3.2.3.

Proof. If $\mathcal{M} = \emptyset$, there is nothing to prove. So let us fix α_0 a self-dual connection 1-form on P.

We consider the map

$$[\alpha] \in \mathcal{M} \mapsto [\overline{\alpha - \alpha_0}] \in H^1(M,\mathfrak{z})/K^Z.$$

Thanks to Lemma 3.2.1 items (i) and (ii), this map is well-defined and injective. Eventually, by Lemma 3.2.1 item (iii), every $[\sigma] \in H^1(M,\mathfrak{z})/K^Z$ is the image of $[\pi^*\sigma + \alpha_0] \in \mathcal{M}$ so the map is surjective. $\square$

The next part of this section is devoted to the proof of the fact that $H^1(M,\mathfrak{z})/K^Z$ admits a differentiable structure which turns it into an abelian Lie group if we add one hypothesis: the Gauge group has to be connected (so a k-torus for $k \in \mathbb{N}_0$). For this, we need one lemma and one proposition.

Lemma 3.2.6. *Let M be a compact manifold and Z_1 and Z_2 abelian compact Lie groups. Then*

$$H^1(M, Lie(Z_1\times Z_2))/K^{Z_1\times Z_2} \simeq H^1(M,\mathfrak{Z}_1)/K^{Z_1} \times H^1(M,\mathfrak{Z}_2)/K^{Z_2}.$$

Proof. Clearly,

$$H^1(M, Lie(Z_1\times Z_2)) \simeq H^1(M,\mathfrak{z}_1) \oplus H^1(M,\mathfrak{z}_2)$$

because $Lie(Z_1 \times Z_2) \simeq \mathfrak{z}_1 \oplus \mathfrak{z}_2$.

Moreover, for every $\overline{\varphi} = (\overline{\varphi_1}, \overline{\varphi_2}) \in \mathcal{C}^\infty(M, Z_1 \times Z_2)$,

$$[\overline{\varphi}^{-1} d\overline{\varphi}] = [\overline{\varphi_1}^{-1} d\overline{\varphi_1}] + [\overline{\varphi_2}^{-1} d\overline{\varphi_2}] \in H^1(M, \mathfrak{z}_1) \oplus H^1(M, \mathfrak{z}_2)$$

so the isomorphism descends to a well-defined isomorphism to the quotient spaces □

Theorem 4.22 of [Lee03] inspired us the next proposition.

Proposition 3.2.7. *Let M be a connected n-manifold and α a 1-form on M valued in Lie(S^1). Then $\alpha \in K^{S^1}$ if and only if*

$$\int_\gamma \alpha \in i2\pi\mathbb{Z}$$

for every γ closed piecewise smooth curve segment on M.

Remark 3.2.8. *The line integral is defined on 1-forms valued in $\mathbb{R}$ while α in this proposition is a 1-form valued in Lie(S^1). The integral is defined thanks to the identification Lie$(S^1) \simeq \mathbb{R}$ which consists in dropping the i in Lie$(S^1) \simeq i\mathbb{R}$.*

Proof. First of all, let us show that if $\alpha \in K^{S^1}$, then

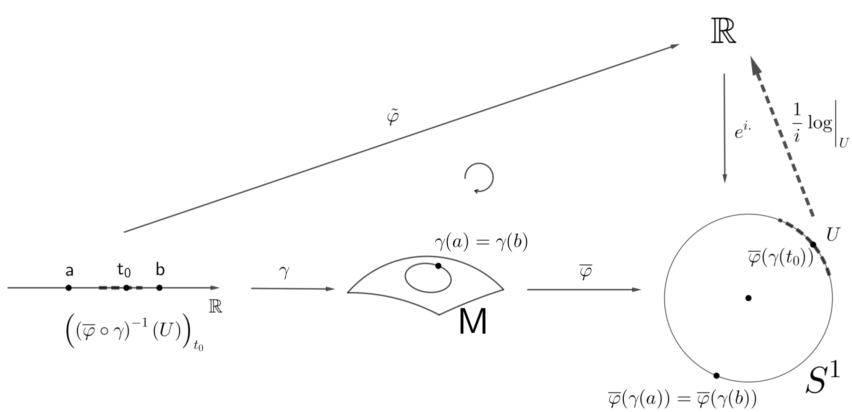
$$\int_\gamma \alpha \in i2\pi\mathbb{Z}$$

for every γ closed piecewise smooth curve segment on M.

We fix $\gamma : [a,b] \subseteq \mathbb{R} \to M$ a closed piecewise smooth curve segment on M and we decompose $[a,b]$ in $a = a_0 < \ldots < a_k = b$ such that $\gamma|_{[a_j, a_{j+1}]}$ is smooth for every $j \in \{0, \ldots, k-1\}$.

We can denote $\alpha = \overline{\varphi}^{-1} d\overline{\varphi}$ for $\overline{\varphi} \in \mathcal{C}^\infty(M, S^1)$. We consider the map

$$\overline{\varphi} \circ \gamma : [a,b] \to S^1.$$



By the lifting property (see Section 1.4) applied to the covering space $\mathbb{R} \to S^1 : \theta \mapsto e^{i\theta}$, there exists a continuous map $\tilde{\varphi} : [a,b] \to \mathbb{R}$ such that

$$e^{i\tilde{\varphi}(t)} = \overline{\varphi}(\gamma(t))$$

for every $t \in [a,b]$.

First, let us show that $\tilde{\varphi}$ is a piecewise smooth curve segment on $[a,b]$ for the same decomposition of $[a,b]$. For each $t_0 \in [a,b]$, let us consider $U \subseteq S^1$ an open set containing $\overline{\varphi}(\gamma(t_0))$ on which we can define a smooth $\log : U \to Lie(S^1)$. There exists $l \in \mathbb{Z}$ such that, for every t in the connected component of $(\overline{\varphi} \circ \gamma)^{-1}(U)$ containing t_0,

$$\log(e^{i\tilde{\varphi}(t)}) = i\tilde{\varphi}(t) + l2\pi i$$

because $\tilde{\varphi}$ is continuous. So $\tilde{\varphi} = -l2\pi + \frac{1}{i}\log \circ \overline{\varphi} \circ \gamma$ on an open set containing t_0.

Secondly, let us remark that

$$e^{i\tilde{\varphi}(a)} = \overline{\varphi}(\gamma(a)) = \overline{\varphi}(\gamma(b)) = e^{i\tilde{\varphi}(b)}$$

so $\tilde{\varphi}(b) - \tilde{\varphi}(a) \in 2\pi\mathbb{Z}$.

Now, we will look at $\int_\gamma \overline{\varphi}^{-1} d\overline{\varphi}$. By definition,

$$\int_\gamma \overline{\varphi}^{-1} d\overline{\varphi} = \sum_{j=0}^{k-1} \int_{[a_j, a_{j+1}]} \left(\gamma|_{[a_j,a_{j+1}]}\right)^* \overline{\varphi}^{-1} d\overline{\varphi}.$$

For every $j \in \{0, ..., k-1\}$ and for every $t_0 \in [a_j, a_{j+1}]$,

$$\begin{aligned}
& \left(\gamma|_{[a_j,a_{j+1}]}\right)^* \left(\overline{\varphi}^{-1} d\overline{\varphi}\right) \left(\frac{\partial}{\partial t}\right)_{t_0} \\
= \; & \left.\frac{d}{dt}\right|_0 \left(\overline{\varphi}(\gamma(t_0))\right)^{-1} \overline{\varphi}\left(\gamma(t_0 + t)\right) \\
= \; & \left.\frac{d}{dt}\right|_0 e^{i(\tilde{\varphi}(t_0+t) - \tilde{\varphi}(t_0))} \\
= \; & i \left.\frac{d}{dt}\right|_0 \tilde{\varphi}(t_0 + t) \\
= \; & i\tilde{\varphi}'(t_0).
\end{aligned}$$

So

$$\left(\gamma|_{[a_j,a_{j+1}]}\right)^* \left(\overline{\varphi}^{-1} d\overline{\varphi}\right) = i\tilde{\varphi}' dt.$$

By the fundamental theorem, we can compute $\int_\gamma \overline{\varphi}^{-1} d\overline{\varphi}$:

$$\begin{aligned}
\int_\gamma \overline{\varphi}^{-1} d\overline{\varphi} &= \sum_{j=0}^{k-1} \int_{[a_j, a_{j+1}]} \left(\gamma|_{[a_j, a_{j+1}]}\right)^* \overline{\varphi}^{-1} d\overline{\varphi} \\
&= \sum_{j=0}^{k-1} \int_{a_j}^{a_{j+1}} i\tilde{\varphi}'(t)dt \\
&= i \sum_{j=0}^{k-1} \left(\tilde{\varphi}(a_{j+1}) - \tilde{\varphi}(a_j)\right) \\
&= i\left(\tilde{\varphi}(b) - \tilde{\varphi}(a)\right) \in i2\pi\mathbb{Z}.
\end{aligned}$$

Now, let us show the reverse side of the proposition, i.e. let us suppose that

$$\int_\gamma \alpha \in i2\pi\mathbb{Z}$$

for every γ closed piecewise smooth curve segment on M and let us show that $[\alpha] \in K^{S^1}$.

We fix $x_0 \in M$ and consider the function

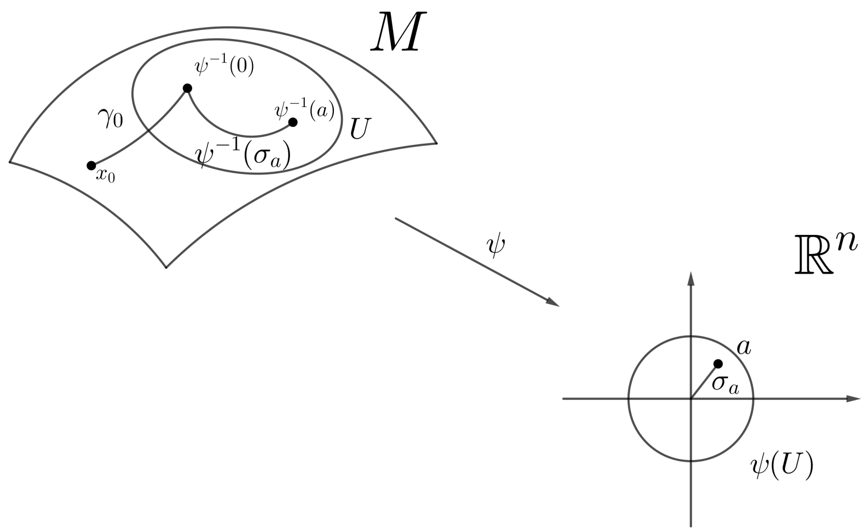
$$\overline{\varphi} : M \to S^1 : x \mapsto e^{\int_\gamma \alpha},$$

where γ is a piecewise smooth curve segment from x_0 to x. We will show that $\overline{\varphi}$ is well-defined, smooth and that $\overline{\varphi}^{-1} d\overline{\varphi} = \alpha$.

First of all, let us remark that for every $x, y \in M$ and every γ_1 and γ_2 piecewise smooth curve segments on M from x to y, by Proposition 1.2.3,

$$\int_{\gamma_1} \alpha - \int_{\gamma_2} \alpha = \int_{\gamma_1 \# \gamma_2^{-1}} \alpha \in i2\pi\mathbb{Z}$$

for $\#$ the concatenation law on paths. So $\overline{\varphi}$ is well-defined.



To prove last points, we will work on $(U \subseteq M, \psi : U \to \mathbb{R}^n)$ a coordinate chart of M such that $\psi(U)$ is an open ball of $\mathbb{R}^n$ centered at 0. We need to consider the following function:

$$f : \psi(U) \to Lie(S^1) : a \mapsto \int_\gamma \left(\psi^{-1}\right)^* \alpha$$

where γ is a piecewise smooth curve segment from 0 to a in $\psi(U)$. By Poincaré lemma, $\left(\psi^{-1}\right)^* \alpha$ is exact on $\psi(U)$. By the Theorem 4.22 of [Lee03] and its proof, f is independent of γ, smooth and $\left(\psi^{-1}\right)^* \alpha = df$.

Now, let us fix γ_0 a piecewise smooth curve segment from x_0 and $\psi^{-1}(0)$. For every $a \in \psi(U)$, let us denote by $\sigma_a : [0,1] \to \psi(U)$ the piecewise smooth curve segment defined by $\sigma_a(t) := ta$. By Proposition 1.2.3, we observe that

$$\begin{aligned}
\overline{\varphi}\left(\psi^{-1}(a)\right) &= e^{\left(\int_{\gamma_0} \alpha + \int_{\psi^{-1}\circ\sigma_a} \alpha\right)} \\
&= e^{\int_{\gamma_0} \alpha} e^{\int_{\psi^{-1}\circ\sigma_a} \alpha} \\
&= e^{\int_{\gamma_0} \alpha} e^{\int_{[0,1]} \left(\psi^{-1}\circ\sigma_a\right)^* \alpha} \\
&= e^{\int_{\gamma_0} \alpha} e^{\int_{\sigma_a} \left(\psi^{-1}\right)^* \alpha} \\
&= e^{\int_{\gamma_0} \alpha} e^{f(a)}.
\end{aligned}$$

It shows first of all that $\overline{\varphi} \circ \psi^{-1}$ is smooth. So $\overline{\varphi}$ is smooth and we can consider $\overline{\varphi}^{-1} d\overline{\varphi}$.

Moreover, for every $a \in \psi(U)$ and $i \in \{1, ..., n\}$,

$$
\begin{aligned}
& \left(\left(\psi^{-1}\right)^* \left(\overline{\varphi}^{-1} d\overline{\varphi}\right)\right)_a \left(\frac{\partial}{\partial x_{i\,a}}\right) \\
= \; & \left.\frac{d}{dt}\right|_0 \left(\overline{\varphi}(\psi^{-1}(a))\right)^{-1} \overline{\varphi}(\psi^{-1}(a + (0, ..., 0, t, 0, ..., 0))) \\
= \; & \left.\frac{d}{dt}\right|_0 e^{-\int_{\gamma_0} \alpha} e^{-f(a)} e^{\int_{\gamma_0} \alpha} e^{f(a+(0,...,0,t,0,...,0))} \\
= \; & \left.\frac{d}{dt}\right|_0 e^{(f(a+(0,...,0,t,0,...,0))-f(a))} \\
= \; & \left.\frac{d}{dt}\right|_0 f(a + (0, ..., 0, t, 0, ..., 0)) \\
= \; & df_a \left(\frac{\partial}{\partial x_{i\,a}}\right)
\end{aligned}
$$

where in $(0, ..., 0, t, 0, ..., 0)$, t is in the i^{th} place.

Then on $\psi(U)$,

$$\left(\psi^{-1}\right)^* \left(\overline{\varphi}^{-1} d\overline{\varphi}\right) = df = \left(\psi^{-1}\right)^* \alpha$$

and so $\overline{\varphi}^{-1} d\overline{\varphi} = \alpha$. □

We are ready to prove our second main theorem:

Theorem 3.2.9. *Let Z be a k-torus for $k \in \mathbb{N}_0$, (M, g, J, ω) be a compact connected almost Kähler manifold of real dimension $2n > 4$ and $\pi : P \to M$ be a Z-principal bundle. Then, either $\mathcal{M}$ is empty, or there exists a manifold structure on $\mathcal{M}$ which turns $\mathcal{M}$ into an abelian Lie group.*

Proof. By Theorem 3.2.3, if $\mathcal{M}$ is non empty, then it is in bijection with $H^1(M, \mathfrak{z})/K^Z$. Let us show that it admits an abelian Lie group structure.

Z is isomorphic to $(S^1)^k$ for some $k \in \mathbb{N}$. Thanks to Proposition 1.3.1 and Lemma 3.2.6, it is enough to show that $H^1(M, Lie(S^1) \simeq \mathbb{R})/K^{S^1}$ admits a structure of abelian Lie group. M is compact so $H^1(M, \mathbb{R})$ is a finite vector space thanks to Lemma 1.1.1 and in particular an abelian Lie group for the trivial differentiable structure on finite dimensional vector spaces.

Thanks to Remark 3.2.5, K^{S^1} is an abstract abelian group. By Theorems 1.3.2 and 1.3.3, there exists a manifold structure on

$$H^1(M, Lie(S^1))/K^{S^1}$$

which turns it into a Lie group if K^{S^1} is closed in $H^1(M, Lie(S^1))$. So it is enough to prove that K^{S^1} is closed in $H^1(M, Lie(S^1))$.

Let $\{[\beta_n]\}_{n\in\mathbb{N}}$ be a sequence in K^{S^1} which converges to $[\beta] \in H^1(M, Lie(S^1))$. Let us show that $[\beta] \in K^{S^1}$. By Proposition 3.2.7, it is enough to show that

$$\int_\gamma \beta \in i2\pi\mathbb{Z}$$

for every γ closed piecewise smooth curve segment on M. We fix any such γ. The same proposition says that

$$\int_\gamma \beta_n \in i2\pi\mathbb{Z}$$

for every $n \in \mathbb{N}$. By Proposition 1.2.3, the map

$$\int_\gamma : H^1(M, Lie(S^1)) \to \mathbb{R} : [\alpha] \mapsto \int_\gamma \alpha$$

is linear, so continuous. So

$$\left\{\int_\gamma \beta_n\right\}_{n\in\mathbb{N}}$$

is a sequence in $i2\pi\mathbb{Z} \subset Lie(S^1)$ which converges to $\int_\gamma \beta$.

As $i2\pi\mathbb{Z}$ is discrete, there exists $N \in \mathbb{N}$ such that, for each $n \geq N$, $\int_\gamma \beta_n = \int_\gamma \beta$. In particular, $\int_\gamma \beta \in i2\pi\mathbb{Z}$. The theorem is proved.

□

We end this section with some examples of moduli spaces of self-dual connections.

Example 3.2.10. *(i) Example 3.1.2 item (i) gives a self-dual connection on the T^k-principal bundle $Hsbrg^n(\mathbb{K}) \to T^{2kn}$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, $k := \dim_{\mathbb{R}} \mathbb{K}$ and $n \in \mathbb{N}$ such that $2nk > 4$.*

$\mathcal{M}$ is then non empty. As T^k is compact and abelian and T^{2kn} is a compact connected Kähler manifold of real dimension $2nk > 4$, Theorem 3.2.3 asserts that

$$\mathcal{M} \simeq H^1(M, Lie(T^k))/K^{T^k}.$$

Moreover T^k is a torus so Theorem 3.2.9 says that $\mathcal{M}$ is an abelian Lie group.

We will show that

$$\mathcal{M} \simeq \left(\mathbb{R}^{2kn}/\mathbb{Z}^{2kn}\right)^k \simeq T^{2k^2n}.$$

As $T^k \simeq S^1 \times ... \times S^1$ *(k-times),*

$$\begin{aligned}\mathcal{M} &\simeq H^1(T^{2kn}, Lie(T^k))/K^{T^k} \\ &= H^1(T^{2kn}, Lie(S^1))/K^{S^1} \times ... \\ &\quad \times H^1(T^{2kn}, Lie(S^1))/K^{S^1} \text{ (k-times)},\end{aligned}$$

by Lemma 3.2.6. So it is enough to show that

$$H^1(T^{2kn}, Lie(S^1))/K^{S^1} \simeq \mathbb{R}^{2kn}/\mathbb{Z}^{2kn}. \tag{22}$$

Using the isomorphism $T^{2kn} \simeq S^1 \times ... \times S^1$ *(2kn-times), for* $j \in \{1, ..., 2kn\}$ *we can consider the local map*

$$\theta_j : T^{2kn} \simeq S^1 \times ... \times S^1 \to Lie(S^1) : (e^{i2\pi\theta_1}, ..., e^{i2\pi\theta_n}) \mapsto i\theta_j.$$

The corresponding 1*-form* $d\theta_j$ *on* T^{2kn} *is a well-defined and*

$$\{[d\theta_j]\}_{j\in\{1,...,2kn\}}$$

is a basis of $H^1(T^{2kn}, Lie(S^1))$*.*

To prove Isomorphism (22)*, we will prove that*

$$\sum_{j=1}^{2kn} c_j[d\theta_j] \in K^{S^1}$$

if and only if $c_j \in \mathbb{Z}$ *for every* $j \in \{1, ..., 2kn\}$*. By Proposition 3.2.7, it is equivalent to prove that*

$$\int_\gamma \sum_{j=1}^{2kn} c_j d\theta_j \in i2\pi\mathbb{Z}$$

for all γ *closed piecewise smooth curve segment on* T^{2kn} *if and only if* $c_j \in \mathbb{Z}$ *for all* $j \in \{1, ..., 2kn\}$ *.*

Let us fix $\gamma : [a, b] \to T^{2kn}$ *a closed piecewise smooth curve segment on* T^{2kn} *and let us consider the covering*

$$\mathbb{R}^{2kn} \to T^{2kn} : (\theta_1, ...\theta_{2kn}) \mapsto (e^{i\theta_1}, ..., e^{i\theta_{2kn}}).$$

Then there exists $\tilde{\gamma} = (\tilde{\gamma}_1, ..., \tilde{\gamma}_{2kn}) : [a, b] \to \mathbb{R}^{2kn}$ *a closed piecewise smooth curve segment on* $\mathbb{R}$ *such that*

$$(e^{i\tilde{\gamma}_1(t)}, ..., e^{i\tilde{\gamma}_{2kn}(t)}) = \gamma(t).$$

We can see that if γ is smooth on some subinterval $[c,d]$ in $[a,b]$, then for every $j \in \{1,...,2kn\}$,

$$\left(\gamma|_{[c,d]}\right)^* d\theta_j = i\,(\tilde{\gamma}_j)'\,dt.$$

Then for every $j \in \{1,...,2kn\}$, by the fundamental theorem,

$$\int_\gamma d\theta_j \quad = \quad i(\tilde{\gamma}_j(b) - \tilde{\gamma}_j(a)) \in i2\pi\mathbb{Z}$$

because $\gamma(a) = \gamma(b)$.

If more details about this computation are needed, we refer to the first part of the proof of Proposition 3.2.7 where we made the same kind of computation with all the details.

It shows first of all that if $c_j \in \mathbb{Z}$ for every $j \in \{1,...,2kn\}$, then

$$\int_\gamma \sum_{j=1}^{2kn} c_j d\theta_j \in i2\pi\mathbb{Z}$$

for every γ closed piecewise smooth curve segment.

For the opposite implication, for $j_0 \in \{1,...,2kn\}$ fixed, let us consider the smooth curve

$$\gamma_{j_0} : [0,2\pi] \to T^{2kn} \simeq (S^1)^{2kn} : t \mapsto (0,...,0,e^{it},0,...,0)$$

where e^{it} stands in the j_0^{th} coordinate. Then, for every $j \in \{1,...,2kn\}$, we can choose $\tilde{\gamma}_j(t) = \delta_{jj_0}t$ for every $t \in [0,2\pi]$ where δ_{jj_0} is the Kronecker delta.

By Equation (23),

$$\begin{aligned}
\int_{\gamma_{j_0}} \sum_{j=1}^{2kn} c_j d\theta_j \quad &= \quad \sum_{j=1}^{2kn} c_j \int_{\gamma_{j_0}} d\theta_j \\
&= \quad \sum_{j=1}^{2kn} c_j i(\tilde{\gamma}_j(2\pi) - \tilde{\gamma}_j(0)) \\
&= \quad \sum_{j=1}^{2kn} c_j i(\delta_{jj_0}2\pi - \delta_{jj_0}0) \\
&= \quad c_{j_0} i2\pi.
\end{aligned}$$

If $\int_{\gamma_{j_0}} \sum_{j=1}^{2kn} c_j d\theta_j \in i2\pi\mathbb{Z}$, then $c_{j_0} \in \mathbb{Z}$.

Hence

$$\int_\gamma \sum_{j=1}^{2kn} c_j d\theta_j \in i2\pi\mathbb{Z}$$

for every γ closed piecewise smooth curve segment on T^{2kn} if and only if $c_j \in \mathbb{Z}$ for all $j \in \{1, ..., 2kn\}$.

As claimed, we can conclude that

$$\mathcal{M} \simeq T^{2k^2n}.$$

(ii) The second example of Examples 3.1.2 gives self-dual connections on some Hermitian symmetric spaces of real dimension strictly greater than 4 (those with underlying Lie group G simple and isotropy group Z of dimension 1). Hence, in this case, $\mathcal{M}$ is non empty. If moreover the Hermitian symmetric space is compact, then $\mathcal{M} \simeq H^1(M, \mathfrak{z})/K^Z$ by Theorem 3.2.3. Moreover, if $Z = S^1$, $\mathcal{M}$ is an abelian Lie group by Theorem 3.2.9.

(iii) For $n > 2$, $\mathbb{CP}^n$ is a connected and simply connected compact Kähler manifold of real dimension $2n$ strictly greater than 4. Theorem 3.2.3 asserts that the moduli space of self-dual connections on a Z-principal bundle over $\mathbb{CP}^n$, for Z an abelian compact Lie group, is either empty or a singleton.

In fact, it asserts that for every connected and simply connected compact almost Kähler manifold of real dimension strictly greater than 4, the moduli space of self-dual connections on a Z-principal bundle, for Z an abelian compact Lie group, is either empty or a singleton.

Conclusion and open questions

The notion of self-duality of 2-forms is natural on spaces of dimension 4. The second chapter of this text generalizes this definition on Kähler vector spaces and on almost Kähler manifolds of real dimension strictly greater than 4: a 2-form μ is called generalized self-dual if

$$*\mu = \mu \wedge \frac{\omega^{\wedge(n-2)}}{(n-1)!}$$

where ω is the underlying symplectic form and $2n$ is the real dimension of the space.

We proved the following (Theorem 2.2.5):

Theorem. *Let (M, g, J, ω) be an almost Kähler manifold of real dimension $2n > 4$ endowed with the orientation given by the canonical volume form of the underlying symplectic manifold (M, ω) (see Section 1.1). Then a 2-form μ is self-dual if and only if there exists $c \in \mathcal{C}^\infty(M, \mathbb{R})$ such that*

$$\mu = c\omega.$$

In the first section of Chapter 3, we extended the definitions of self-dual connections, Gauge transformations and moduli space of self-dual connections from oriented Riemannian manifolds of real dimension 4 to almost Kähler manifolds of real dimension greater than 4 using Chapter 2. We identified the moduli space of self-dual connections on principal bundles over compact connected almost Kähler manifolds of real dimension strictly greater that 4 with abelian compact Gauge groups. Moreover, adding a connectedness hypothesis on the Gauge group, we showed that the moduli space of self-dual connections is an abelian Lie group.

Two natural questions appear directly. First of all, why did we choose the coefficient $\frac{1}{(n-1)!}$ in the definition of self-duality ? This choice seams smart. Indeed, this definition of self-duality is consistent with the one in dimension 4. Moreover, if the manifold is compact, self-dual connections corresponding to this definition of self-dual 2-forms provide critical points of the Yang-Mills functional, as in the 4-dimensional case. But it is not the only smart coefficient: $\frac{1}{(n-2)!}$ is suitable too.

Lemma 2.1.8 tells us that the space of self-dual 2-forms for the coefficient $\frac{1}{(n-2)!}$ is of real dimension $n(n-1)$ and the space of anti-self-dual 2-forms for this coefficient is of real dimension $(n+1)(n-1)$. Clearly, the situation would be more difficult with this coefficient. Nevertheless, it is certainly interesting to study the moduli space of (anti-)self-dual connections for the coefficient $\frac{1}{(n-2)!}$ too.

Secondly, what happens in the non-abelian case ? Again the problem is more complicated. Indeed, four simplifications appear in the abelian case that are no more valid in general (see Section 3.2). Nevertheless, the rest of the text stays right. In particular, Theorem 2.2.5 holds in general. It would be interesting to consider this theorem as starting point for a study of the moduli space of self-dual connections in the non-abelian case.

To conclude, the definition of self-duality that we consider and the hypothesis requested by our work are quite strict. An important open question is trying to extend this result in a more general context !

Nevertheless, with this suitable definition and this suitable hypothesis, we present a nice theory: we identified the moduli space of self-dual connections and we showed that it admits a structure of abelian Lie group.

Appendix

The Loos and the Grassmann connections

On the one hand, there exists a unique connection on the tangent bundle of a connected symmetric space which is s_x-invariant for every $x \in M$. This connection is called the *Loos connection*. For useful informations about symmetric spaces and Loos connections, see Section 1.6. For details about it, we refer to [Loo69].

On the other hand, on each projective $\mathcal{A}$-module of finite type $\mathcal{E}$ (for $\mathcal{A} = \mathcal{C}^\infty(M, \mathbb{R})$, the algebra of smooth functions on a given manifold M), we can define a natural connection called the *Grassmann connection*. It is constructed in the following way: $\mathcal{E}$ is finitely generated so there exists $N \in \mathbb{N}$ and a surjective module morphism $\mu : \mathcal{A}^N \to \mathcal{E}$, where $\mathcal{A}^N = \mathbb{R}^N \otimes \mathcal{A}$. Moreover, $\mathcal{E}$ is projective so there exists a module morphism $\lambda : \mathcal{E} \to \mathcal{A}^N$ such that $\mu \circ \lambda = Id_{\mathcal{E}}$. For $\varphi \in \mathcal{E}$, $\lambda(\varphi)$ is a vector composed of N elements of $\mathcal{A}$. We can take the differential of all these functions. We will denote this expression by $d(\lambda(\varphi))$. For $X \in \Gamma^\infty(TM)$, $d(\lambda(\varphi))(X)$ is again an element of $\mathcal{A}^N$. We can compose this with μ. It gives back an element of $\mathcal{E}$. It is the definition of the Grassmann connection. We can resume this with the following expression:

$$\nabla^G_X \varphi = \mu \left(d \left(\lambda \varphi \right) \left(X \right) \right).$$

We refer to [Lan97] for details about it.

The Serre-Swan theorem (which can be extended to a priori non-compact connected smooth manifolds: see [Nes03]) asserts that, for M a connected smooth manifold and $\mathcal{A} := \mathcal{C}^\infty(M, \mathbb{R})$, an $\mathcal{A}$-module $\mathcal{E}$ is projective of finite type if and only if it is the space of smooth sections of a vector bundle E over M (see [Swa62]). So it is possible to define naturally a Grassmann connection on the tangent bundle of each manifold.

Now, let (M, s) be a connected symmetric space. Serre-Swan theorem drives us to a natural question: *Is it possible to define a Grassmann* *connection equal to the Loos connection ?* i.e. a Grassmann connection s_x-invariant for every $x \in M$?

The answer is *Yes*. This text is devoted to the construction of suitable λ and μ and the proof that the underlying Grassmann connection is s_x-invariant for every $x \in M$.

To construct λ and μ, it is easier to see TM from a different point of view: as a vector bundle associated to the natural K-principal bundle $G \to G/K \simeq M$ defined in Section 1.6, where G is the transvection group of M and K the isotropy group at a fixed point $o \in M$.

We will write $Ad_K^{\mathfrak{p}}$ the representation of K which provides this associated vector bundle. It is defined as

$$Ad_K^{\mathfrak{p}} : K \to Gl(\mathfrak{p}) : k \mapsto Ad_k|_{\mathfrak{p}}$$

for Ad the usual adjoint representation of G. It is well-defined because $Ad_k(\mathfrak{p}) \subseteq \mathfrak{p}$ for every $k \in K$.

The isomorphism of vector bundle is given by

$$\Psi : E_{Ad_K^{\mathfrak{p}}} \to TM : [(g, X)] \mapsto X_g^* := \left.\frac{d}{dt}\right|_0 gexp(tX)K.$$

For $x = g_0K$, let us recall that $s_x gK = g_0\sigma(g_0^{-1}g)K$ for every $gK \in M$. An action of s_x on $\Gamma^\infty(E_{Ad_K^{\mathfrak{p}}})$ for every $x = g_0K \in M$ is naturally defined: for every $\varphi \in \Gamma^\infty(E_{Ad_K^{\mathfrak{p}}})$ and $y = gK \in M$,

$$\begin{aligned}
(s_x.\varphi)(y) &= \Psi^{-1}\left(s_{x*}\Psi(\varphi)\right)_y \\
&= \Psi^{-1}s_{x*}\left(\Psi(\varphi)\right)_{s_x^{-1}(y)} \\
&= \Psi^{-1}s_{x*}\left.\frac{d}{dt}\right|_0 g_0\sigma(g_0^{-1}g)\exp(t\hat{\varphi}(g_0\sigma(g_0^{-1}g))) \\
&= \Psi^{-1}\left.\frac{d}{dt}\right|_0 s_x\left(g_0\sigma(g_0^{-1}g)\exp(t\hat{\varphi}(g_0\sigma(g_0^{-1}g)))\right) \\
&= \Psi^{-1}\left.\frac{d}{dt}\right|_0 \left(g\sigma(\exp(t\hat{\varphi}(g_0\sigma(g_0^{-1}g))))\right) \\
&= \Psi^{-1}\left.\frac{d}{dt}\right|_0 \left(g\exp(t\sigma_*\hat{\varphi}(g_0\sigma(g_0^{-1}g)))\right) \\
&= \Psi^{-1}\left.\frac{d}{dt}\right|_0 \left(g\exp(-t\hat{\varphi}(g_0\sigma(g_0^{-1}g)))\right) \\
&= [(g, -\hat{\varphi}(g_0\sigma(g_0^{-1}g)))].
\end{aligned}$$

Hence

$$(s_x.\varphi)\hat{}\,(g) = -\hat{\varphi}(g_0\sigma(g_0^{-1}g))$$

for every $g \in G$.

For every $x \in M$, the natural notion of s_x-invariance on connections on TM transposes on connections on $E_{Ad_K^{\mathfrak{p}}}$ as follows: a connection ∇ on $E_{Ad_K^{\mathfrak{p}}}$ is s_x-invariant if and only if

$$(s_x.\nabla) = \nabla$$

where

$$(s_x.\nabla)_X \varphi = s_x.\left(\nabla_{s_{x*}^{-1}X} s_x^{-1}.\varphi\right).$$

To construct a s_x-invariant Grassmann connection for every $x \in M$, we have to choose suitable $\lambda : E_{Ad_K^{\mathfrak{p}}} \to \mathcal{A}^N$ and $\mu : \mathcal{A}^N \to E_{Ad_K^{\mathfrak{p}}}$ such that $\mu \circ \lambda = Id$ for some $N \in \mathbb{N}$.

Let us take $N = \dim(G)$. By abuse, we identify a vector of $\mathfrak{g}$ with an element of $\mathbb{R}^N$ using a fixed basis of $\mathfrak{g}$. We define $\mu : \mathcal{A}^N \to E_{Ad_K^{\mathfrak{p}}}$ by

$$\left(\mu\begin{pmatrix} f_1 \\ \dots \\ f_N \end{pmatrix}\right)\hat{}\,(g) := \pi_{\mathfrak{p}}\left(Ad_{g^{-1}}\begin{pmatrix} f_1(gK) \\ \dots \\ f_N(gK) \end{pmatrix}\right)$$

and $\lambda : E_{Ad_K^{\mathfrak{p}}} \to \mathcal{A}^N$ by

$$\lambda\varphi(gK) := Ad_g(\hat{\varphi}(g)).$$

Hence,

$$\begin{aligned}(\mu \circ \lambda(\varphi))\hat{}\,(g) &= \pi_{\mathfrak{p}}\left(Ad_{g^{-1}}Ad_g(\hat{\varphi}(g))\right)\\ &= \hat{\varphi}(g)\end{aligned}$$

so $\mu \circ \lambda = Id$.

The Grassmann connection is then defined as follows: for all $\varphi \in E_{Ad_K^{\mathfrak{p}}}$, $Y \in \Gamma^\infty(TM)$, $g \in G$ and $X \in \mathfrak{p}$ such that $X^*_{gK} = Y_{gK}$, then

$$\begin{aligned}(\nabla^G_Y\varphi)\hat{}\,(g) &= \mu\left(d\left(\lambda\varphi\right)(Y)\right)\hat{}\,(g)\\ &= pr_{\mathfrak{p}}Ad_{g^{-1}}d(\lambda\varphi)(Y_{gK})\\ &= pr_{\mathfrak{p}}Ad_{g^{-1}}d(\lambda\varphi)(X^*_{gK})\\ &= pr_{\mathfrak{p}}Ad_{g^{-1}}\left.\frac{d}{dt}\right|_0(\lambda\varphi)(g\exp tX.K)\\ &= pr_{\mathfrak{p}}Ad_{g^{-1}}\left.\frac{d}{dt}\right|_0 Ad_{g\exp tX}\hat{\varphi}(g\exp tX)\\ &= pr_{\mathfrak{p}}\left.\frac{d}{dt}\right|_0 Ad_{\exp tX}\hat{\varphi}(g\exp tX)\\ &= pr_{\mathfrak{p}}([X,\hat{\varphi}(g)] + d\hat{\varphi}(L_{g*}X))\\ &= d\hat{\varphi}(L_{g*}X)\end{aligned}$$

because $[\mathfrak{p},\mathfrak{p}] \subseteq \mathfrak{k}$ and $d\hat{\varphi} : TG \to \mathfrak{p}$.

It remains to show that this connection is s_x-invariant for every $x \in M$. For this, let us remark that if $Y \in \Gamma^\infty(TM)$, $g \in G$, $X \in \mathfrak{p}$ such that $X^*_{gK} = Y_{gK}$ and $x = g_0K \in M$, then

$$\begin{aligned}
\left(s_{x*}^{-1}Y\right)_{g_0\sigma(g_0^{-1}g)K} &= \left(s_{x*}^{-1}Y\right)_{s_x^{-1}(gK)} \\
&= s_{x*}^{-1}\left(Y\right)_{gK} \\
&= \left.\frac{d}{dt}\right|_0 g_0\sigma(g_0^{-1}g\exp tX).K \\
&= \left.\frac{d}{dt}\right|_0 g_0\sigma(g_0^{-1}g)\exp(-tX).K \\
&= (-X)^*_{g_0\sigma(g_0^{-1}g)K}\,.
\end{aligned}$$

Hence for every $Y \in \Gamma^\infty(TM)$, $g \in G$, $X \in \mathfrak{p}$ such that $X^*_{gK} = Y_{gK}$ and $x = g_0K \in M$,

$$\begin{aligned}
&\left(\left(s_x.\nabla^G\right)_Y \varphi\right)\hat{}\,(g) \\
= \;& \left(s_x.\left(\nabla^G_{s_{x*}^{-1}Y}(s_x^{-1}.\varphi)\right)\right)\hat{}\,(g) \\
= \;& -\left(\nabla^G_{s_{x*}^{-1}Y}(s_x^{-1}.\varphi)\right)\hat{}\,(g_0\sigma(g_0^{-1}g)) \\
= \;& -d(s_x^{-1}.\varphi)\hat{}\left(L_{g_0\sigma(g_0^{-1}g)*}(-X)\right) \\
= \;& \left.\frac{d}{dt}\right|_0 \hat{\varphi}\left(g_0\sigma(g_0^{-1}g_0\sigma(g_0^{-1}g)\exp(-tX))\right) \\
= \;& \left.\frac{d}{dt}\right|_0 \hat{\varphi}\left(g\sigma(\exp -tX)\right) \\
= \;& \left.\frac{d}{dt}\right|_0 \hat{\varphi}\left(g\exp tX\right) \\
= \;& d\hat{\varphi}(L_{g*}X)) \\
= \;& (\nabla^G_Y\varphi)\hat{}\,(g).
\end{aligned}$$

Hence ∇^G is s_x-invariant for every $x \in M$ and equal to the Loos connection.

Remark. *If $X \in \mathfrak{p}$, the vector field $L_{g*}X$ on G is horizontal with respect to the Loos connection 1-form defined in Section 1.6. The explicit expression for the Grassmann connection*

$$(\nabla^G_Y\varphi)\hat{}\,(g) = d\hat{\varphi}(L_{g*}X)$$

(for every $Y \in \Gamma^\infty(TM)$, $\varphi \in \Gamma^\infty(E_{Ad_K^{\mathfrak{p}}})$, $g \in G$ *and* $X \in \mathfrak{p}$ *such that* $X^*_{gK} = Y_{gK}$*) tells us that the Grassmann connection, so the Loos connection, is the associated connection on the associated vector bundle* $E_{Ad_K^{\mathfrak{p}}}$ *of the Loos connection* 1*-form. It explains the link between both definitions of Loos connection.*

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