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A new class of Euler equation on the dual of the $N = 1$ extended Neveu-Schwarz algebra

Yanyan Ge[a)] and Dafeng Zuo[a)]
School of Mathematical Science, University of Science and Technology of China, Hefei 230026, People's Republic of China

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Let $\mathfrak{G}$ be the $N = 1$ extended Neveu-Schwarz algebra and $\mathfrak{G}^*_{\mathbf{reg}}$ its regular dual. In this paper, we will study a super-Euler system with seven parameters (s_1, s_2, c_1, ..., c_5) associated with $\mathfrak{G}^*_{\mathbf{reg}}$. We will show that the super-Euler system is (1) local bi-superbihamiltonian if $\mathbf{s_1} = \frac{1}{4}\mathbf{c_1}$ and $\mathbf{s_2} = \frac{1}{2}\mathbf{c_2}$; (2) supersymmetric if $\mathbf{s_1} = \mathbf{c_1}$ and $\mathbf{s_2} = \mathbf{c_2}$; (3) local bi-superbihamiltonian and supersymmetric if $\mathbf{s_1} = \mathbf{c_1} = \mathbf{0}$ and $\mathbf{s_2} = \mathbf{c_2} = \mathbf{0}$. By choosing different parameters, we could obtain several supersymmetric or bi-superhamiltonian generalizations of some well-known integrable systems including the Ito equation, the 2-component Camassa-Holm equation, the 2-component Hunter-Saxton equation, and, especially, the Whitham-Broer-Kaup dispersive water-wave system. *Published by AIP Publishing.* https://doi.org/10.1063/1.5051755

I. INTRODUCTION

Let $\mathfrak{G}$ be a Lie (super-) algebra and $\mathfrak{G}^*$ (the regular part of) its dual, and let $\langle\ ,\ \rangle^*$ denote a natural pairing between $\mathfrak{G}$ and $\mathfrak{G}^*$. *The Euler equation on* $\mathfrak{G}^*$ *is defined by the following system* (e.g.,, Refs. 7 and 9):

$$\frac{dm}{dt} = -ad^*_{\mathcal{A}^{-1}m}m \tag{1.1}$$

as an evolution of a point $m \in \mathfrak{G}^*$*, where* $\mathcal{A}: \mathfrak{G} \to \mathfrak{G}^*$ *is an invertible self-adjoint operator, called the inertia operator.*

Let $\mathrm{Vect}(\mathbb{S}^1)$ be the Lie algebra of vector fields, denoted by $\mathrm{Vect}(\mathbb{S}^1) = \{f(x,\cdot)\partial \,|\, f(x,\cdot) \in C^\infty(\mathbb{S}^1)\}$, where $\partial = \frac{\partial}{\partial x}$. Its nontrivial central extension is the Virasoro algebra $\mathfrak{vir} = \mathrm{Vect}(\mathbb{S}^1) \oplus \mathbb{R}$ with the Lie bracket

$$[(f\partial, s)), (g\partial, s_1)] = \left((fg_x - f_x g)\partial, \int_{\mathbb{S}^1} f g_{xxx} dx\right).$$

The Euler equation (1.1) on $\mathfrak{vir}^*$ for $\mathcal{A} = c_1 - c_4\partial^2$ for $c_1, c_4 \in \mathbb{R}$ is local bihamiltonian and reads

$$u_t + 2uf_x + u_x f + \varsigma f_{xxx} = 0, \quad \varsigma_t = 0, \tag{1.2}$$

where $u = \mathcal{A}(f)$. More precisely, let $\varsigma \in \mathbb{R}$; the Euler equation (1.2) reduces to

- the Korteweg-de Vries (KdV in brief) equation $f_t + 3ff_x + \varsigma f_{xxx} = 0$ if $c_1 = 1$ and $c_4 = 0$ (Ref. 11);
- the Camassa-Holm (CH in brief) equation $u_t + 2uf_x + u_x f + \varsigma f_{xxx} = 0$ if $c_1 = 1$ and $c_4 = 1$, where $u = f - f_{xx}$ (e.g., Refs. 13 and 14);
- the Hunter-Saxton (HS in brief) equation $f_{xxt} + 2f_x f_{xx} + ff_{xxx} - \varsigma f_{xxx} = 0$ if $c_1 = 0$ and $c_4 = 1$ (Refs. 15 and 16).

Besides the Virasoro algebra, there are many generalizations including the extended Virasoro algebra (Refs. 17–19), the $N \le 3$ superconformal algebra (Refs. 11, 20, 22–25), and the Frobenius-Virasoro algebra (Refs. 26 and 27); please see, e.g., Refs. 8, 9, and 25 and references therein for details. In this paper, we are concerned with an extension of the $N = 1$ Neveu-Schwarz algebra (Ref. 6) and will study local bi-superhamiltonian Euler equations and supersymmetric Euler equations.

[a)]Authors to whom correspondence should be addressed: yanyange@mail.ustc.edu.cn and dfzuo@ustc.edu.cn

For brevity, we use the following notations:

- ϕ, χ, ψ, α, β, and γ are fermionic functions;
- f, g, u, a, b, and v are bosonic functions;
- $\widehat{F} = \left(f\partial, \phi \mathrm{dx}^{-\frac{1}{2}}, a, \alpha \mathrm{dx}^{\frac{1}{2}}, \vec{\sigma}\right)^T, \quad \widehat{G} = \left(g\partial, \chi \mathrm{dx}^{-\frac{1}{2}}, b, \beta \mathrm{dx}^{\frac{1}{2}}), \vec{\tau}\right)^T \in \mathfrak{G}$;
- $\widehat{U} = (u\mathrm{dx}^2, \psi \mathrm{dx}^{\frac{3}{2}}, v\mathrm{dx}, \gamma \mathrm{dx}^{\frac{1}{2}}, \vec{\varsigma})^T \in \mathfrak{G}^*$;
- $V_B = \mathrm{Vect}(\mathbb{S}^1) \oplus \mathrm{C}^\infty(\mathbb{S}^1)$ and $V_F = \mathrm{C}^\infty(\mathbb{S}^1) \oplus \mathrm{C}^\infty(\mathbb{S}^1)$.

Definition 1.1 (Ref. 6). *The $N = 1$ extended Neveu-Schwarz algebra $\mathfrak{G}$ is defined by*

$$\mathfrak{G} = V_B \oplus V_F \oplus \mathbb{R}^3$$

with the commutation relation

$$[\widehat{F}, \widehat{G}] = \begin{pmatrix} (fg_x - f_x g + \frac{1}{2}\phi\chi)\partial \\ (f\chi_x - \frac{1}{2}f_x\chi - g\phi_x + \frac{1}{2}g_x\phi)dx^{-\frac{1}{2}} \\ fb_x - a_x g + \frac{1}{2}\phi\beta + \frac{1}{2}\alpha\chi \\ (f\beta_x + \frac{1}{2}f_x\beta - \frac{1}{2}a_x\chi - g\alpha_x - \frac{1}{2}g_x\alpha + \frac{1}{2}b_x\phi)dx^{\frac{1}{2}} \\ (\omega_1, \omega_2, \omega_3) \end{pmatrix}, \tag{1.3}$$

where $\omega_1 = \int_{\mathbb{S}^1}(fg_{xxx} + \phi\chi_{xx})dx$, $\omega_2 = \int_{\mathbb{S}^1}(f_{xx}b - g_{xx}a - \phi_x\beta + \chi_x\alpha)dx$ and $\omega_3 = \int_{\mathbb{S}^1}(2ab_x + 2\alpha\beta)dx$.

Let $\mathfrak{G}^*_{\mathbf{reg}}$ be the regular part of the dual space $\mathfrak{G}^*$ to $\mathfrak{G}$ under the following pair:

$$\langle \widehat{U}, \widehat{F}\rangle^* = \int_{\mathbb{S}^1}(uf + \psi\phi + va + \gamma\alpha)dx + \vec{\varsigma}\cdot\vec{\sigma}.$$

With the use of the integration of parts and the definition

$$\langle ad^*_{\widehat{F}}(\widehat{U}), \widehat{G}\rangle^* = -\langle \widehat{U}, [\widehat{F}, \widehat{G}]\rangle^*,$$

one has the coadjoint action of $\mathfrak{G}$ on $\mathfrak{G}^*_{\mathbf{reg}}$ as follows (Ref. 25):

$$ad^*_{\widehat{F}}(\widehat{U}) = \begin{pmatrix} (2uf_x + u_x f + \varsigma_1 f_{xxx} + \varsigma_2 a_{xx} + \frac{3}{2}\psi\phi_x + \frac{1}{2}\psi_x\phi \\ +\frac{1}{2}\gamma\alpha_x - \frac{1}{2}\gamma_x\alpha + va_x)\,\mathrm{dx}^2 \\ (-\varsigma_1\phi_{xx} - \varsigma_2\alpha_x - \frac{1}{2}u\phi - \frac{1}{2}v\alpha + \frac{3}{2}f_x\psi + f\psi_x + \frac{1}{2}\gamma a_x)\,\mathrm{dx}^{\frac{3}{2}} \\ ((vf)_x + \frac{1}{2}(\gamma\phi)_x - \varsigma_2 f_{xx} + 2\varsigma_3 a_x)\,\mathrm{dx} \\ (\gamma_x f + \frac{1}{2}\gamma f_x - \frac{1}{2}v\phi + \varsigma_2\phi_x - 2\varsigma_3\alpha)\,\mathrm{dx}^{\frac{1}{2}} \\ 0 \end{pmatrix}.$$

Let $\mathcal{A}: \mathfrak{G} \to \mathfrak{G}^*_{\mathbf{reg}}$ be the arbitrary inertia operator; then, the Euler equation (1.1) for $\widehat{U} = \mathcal{A}(\widehat{F})$ on $\mathfrak{G}^*_{\mathbf{reg}}$ reads

$$\begin{aligned} u_t &= -\varsigma_1 f_{xxx} - \varsigma_2 a_{xx} - 2uf_x - u_x f - va_x - \frac{3}{2}\psi\phi_x - \frac{1}{2}\psi_x\phi - \frac{1}{2}\gamma\alpha_x + \frac{1}{2}\gamma_x\alpha, \\ \psi_t &= \frac{1}{2}u\phi + \frac{1}{2}v\alpha + \varsigma_1\phi_{xx} + \varsigma_2\alpha_x - \frac{3}{2}f_x\psi - f\psi_x - \frac{1}{2}\gamma a_x, \\ v_t &= \varsigma_2 f_{xx} - 2\varsigma_3 a_x - (vf)_x - \frac{1}{2}(\gamma\phi)_x, \\ \gamma_t &= \frac{1}{2}v\phi - \gamma_x f - \frac{1}{2}\gamma f_x - \varsigma_2\phi_x + 2\varsigma_3\alpha, \\ \vec{\varsigma}_t &= 0. \end{aligned} \tag{1.4}$$

In this paper, we will consider an inertia operator $\mathcal{A}: \mathfrak{G} \to \mathfrak{G}^*_{\mathbf{reg}}$ with seven parameters being of the form

$$\mathcal{A} = \begin{pmatrix} c_1 - c_4\partial^2 & 0 & c_2 - c_5\partial & 0 & 0 \\ 0 & c_4\partial - s_1\partial^{-1} & 0 & c_5 - s_2\partial^{-1} & 0 \\ c_2 + c_5\partial & 0 & c_3 & 0 & 0 \\ 0 & -c_5 - s_2\partial^{-1} & 0 & -c_3\partial^{-1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix}, \tag{1.5}$$

where $s_1, s_2, c_i \in \mathbb{R}$ for $i = 1, \ldots, 5$. We will show the following:

Theorem 1.2. *Let $\varsigma_1, \varsigma_2, \varsigma_3$ be constants; the Euler equation (1.1) with the given inertia operator (1.5) is*

- *local bi-superbihamiltonian if* $\mathbf{s_1 = \frac{1}{4}c_1}$ *and* $\mathbf{s_2 = \frac{1}{2}c_2}$*;*
- *supersymmetric if* $\mathbf{s_1 = c_1}$ *and* $\mathbf{s_2 = c_2}$*;*
- *not only local bi-superbihamiltonian but also supersymmetric if* $\mathbf{s_1 = c_1 = 0}$ *and* $\mathbf{s_2 = c_2 = 0}$.

We remark that

- When $s_2 = c_2 = c_5 = 0$, this result has been obtain in Ref. 25. In this case, we have presented several supersymmetric or bi-superhamiltonian generalizations of some well-known integrable systems including the Ito equation, the 2-component CH equation, and the 2-component HS equation.
- When $s_1 = c_1 = c_5 = 0$, the above Euler equation will give a new class of bi-superhamiltonian or supersymmetric systems including the bi-superhamiltonian Whitham-Broer-Kaup (Kuper-WBK) dispersive water-wave system and the $N = 1$ supersymmetric WBK system.

II. BI-SUPERHAMILTONIAN EULER EQUATIONS ON $\mathfrak{G}^*_{\mathbf{reg}}$

In this section, we will study local bi-superhamiltonian Euler equations and show that

Theorem 2.1. *Suppose that*

$$\mathbf{s_1 = \frac{1}{4}c_1, s_2 = \frac{1}{2}c_2}. \tag{2.1}$$

Then the Euler equation (1.4) is local bi-superhamiltonian on $\mathfrak{G}^*_{\mathbf{reg}}$ with a freezing point $\widehat{U}_0 = (\frac{c_1}{2}dx^2, 0, c_2 dx^{\frac{3}{2}}, 0, (-c_4, -c_5, \frac{c_3}{2}))^T \in \mathfrak{G}^*_{\mathbf{reg}}$.

Proof. For brevity, we denote

$$\widetilde{F} = (f, \eta, v, \mu)^T, \quad \widetilde{U} = (u, \psi, v, \gamma)^T, \quad \frac{\delta H}{\delta \widetilde{U}} = (\frac{\delta H}{\delta u}, \frac{\delta H}{\delta \psi}, \frac{\delta H}{\delta v}, \frac{\delta H}{\delta \gamma})^T$$

for any smooth functional $H = H[\widetilde{U}]$.[28]

Setting $\phi = \eta_x$ and $\alpha = \mu_x$, the Euler equation (1.4) with the condition (2.1) reads

$$\begin{aligned} u_t &= -\varsigma_1 f_{xxx} - \varsigma_2 a_{xx} - 2uf_x - u_x f - v a_x - \frac{3}{2}\psi\eta_{xx} - \frac{1}{2}\psi_x\eta_x - \frac{1}{2}\gamma\mu_{xx} + \frac{1}{2}\gamma_x\mu_x, \\ \psi_t &= \frac{1}{2}u\eta_x + \varsigma_1\eta_{xxx} + \varsigma_2\mu_{xx} - \frac{3}{2}f_x\psi - f\psi_x + \frac{1}{2}v\mu_x - \frac{1}{2}\gamma a_x, \\ v_t &= \varsigma_2 f_{xx} - 2\varsigma_3 a_x - (vf)_x - \frac{1}{2}(\gamma\eta_x)_x, \\ \gamma_t &= \frac{1}{2}v\eta_x - \gamma_x f - \frac{1}{2}\gamma f_x - \varsigma_2\eta_{xx} + 2\varsigma_3\mu_x, \end{aligned} \tag{2.2}$$

where $\widetilde{U} = \mathbf{\Lambda}\widetilde{F} = \mathbf{\Lambda}(f, \eta, a, \mu)^T$ and

$$\mathbf{\Lambda} = \begin{pmatrix} c_1 - c_4\partial^2 & 0 & c_2 - c_5\partial & 0 \\ 0 & c_4\partial^2 - \frac{1}{4}c_1 & 0 & c_5\partial - \frac{1}{2}c_2 \\ c_2 + c_5\partial & 0 & c_3 & 0 \\ 0 & -\frac{1}{2}c_2 - c_5\partial & 0 & -c_3 \end{pmatrix}.$$

From the identity $\widetilde{U} = \mathbf{\Lambda}\widetilde{F}$, it follows that

$$\frac{\delta H}{\delta \widetilde{U}} = \mathbf{\Lambda}^{-1}\frac{\delta H}{\delta \widetilde{F}}. \tag{2.3}$$

By the definition of the Euler equation, it is natural to rewrite the system (2.2) as a superhamiltonian system

$$\widetilde{U}_t = \mathcal{J}_2\frac{\delta H_1}{\delta \widetilde{U}}, \quad H_1 = \frac{1}{2}\int_{\mathbb{S}^1}(uf + \psi\phi + va + \gamma\alpha)dx,$$

where $\mathcal{J}_2$ is the induced superhamiltonian operator from the canonical super Lie-Poisson bracket on $\mathfrak{G}^*_{\mathbf{reg}}$ being of the form

$$\mathcal{J}_2 = \begin{pmatrix} -\varsigma_1\partial^3 - u\partial - \partial u & -\psi\partial - \frac{1}{2}\partial\psi & -\varsigma_2\partial^2 - v\partial & -\gamma\partial + \frac{1}{2}\partial\gamma \\ -\partial\psi - \frac{1}{2}\psi\partial & \frac{1}{2}u + \varsigma_1\partial^2 & -\frac{1}{2}\gamma\partial & \frac{1}{2}v + \varsigma_2\partial \\ \varsigma_2\partial^2 - \partial v & -\frac{1}{2}\partial\gamma & -2\varsigma_3\partial & 0 \\ -\partial\gamma + \frac{1}{2}\gamma\partial & \frac{1}{2}v - \varsigma_2\partial & 0 & 2\varsigma_3 \end{pmatrix}.$$

Setting $\mathcal{J}_1 = \mathcal{J}_2|_{\widehat{U}=\widehat{U}_0}$, then $\mathcal{J}_1 = -\mathrm{diag}(\partial, 1, \partial, 1)\mathbf{\Lambda}$. Taking

$$\begin{aligned} H_2 = \int_{\mathbb{S}^1}\Big(&\frac{c_1}{2}f^3 - \frac{c_4}{2}ff_x^2 + c_2af^2 - \frac{c_5}{2}a_xf^2 + \frac{c_3}{2}a^2f \\ &+ \frac{\varsigma_1}{2}ff_{xx} + \varsigma_2a_xf + \varsigma_3a^2 + \frac{\varsigma_1}{2}\eta_x\eta_{xx} + \varsigma_2\mu\eta_{xx} - \varsigma_3\mu\mu_x \\ &- \frac{3c_1}{8}f\eta\eta_x - \frac{c_2}{4}a\eta\eta_x - \frac{c_4}{2}f\eta_x\eta_{xx} - \frac{c_3}{2}f\mu\mu_x \\ &+ \frac{c_2}{4}f\mu_x\eta - \frac{3c_2}{4}f\mu\eta_x + c_5f\mu_x\eta_x - \frac{c_3}{2}a\mu\eta_x\Big)dx, \end{aligned}$$

a direct computation gives

$$\begin{aligned} \frac{\delta H_2}{\delta f} &= \varsigma_1f_{xx} + \varsigma_2a_x + \frac{3c_1}{2}f^2 - c_4ff_{xx} - \frac{c_4}{2}f_x^2 + 2c_2af - c_5a_xf \\ &\quad + \frac{c_3}{2}a^2 - \frac{3c_1}{8}\eta\eta_x - \frac{c_4}{2}\eta_x\eta_{xx} - \frac{3c_2}{4}\mu\eta_x + \frac{c_2}{4}\mu_x\eta + c_5\mu_x\eta_x - \frac{c_3}{2}\mu\mu_x, \\ \frac{\delta H_2}{\delta \eta} &= \frac{3c_4}{2}f_x\eta_{xx} + c_4f\eta_{xxx} + \frac{c_4}{2}f_{xx}\eta_x - \frac{3c_1}{4}f\eta_x - \frac{3c_1}{8}f_x\eta - \frac{3c_2}{4}f_x\mu - c_2f\mu_x \\ &\quad + c_5f_x\mu_x + c_5f\mu_{xx} - \frac{c_2}{2}a\eta_x - \frac{c_2}{4}a_x\eta - \frac{c_3}{2}a_x\mu - \frac{c_3}{2}a\mu_x - \varsigma_1\eta_{xxx} - \varsigma_2\mu_{xx}, \\ \frac{\delta H_2}{\delta a} &= 2\varsigma_3a + c_3af + c_5ff_x + c_2f^2 - \varsigma_2f_x - \frac{c_2}{4}\eta\eta_x - \frac{c_3}{2}\mu\eta_x, \\ \frac{\delta H_2}{\delta \mu} &= -c_2f\eta_x - c_5f_x\eta_x - c_5f\eta_{xx} - \frac{c_2}{4}f_x\eta - c_3f\mu_x - \frac{c_3}{2}f_x\mu - \frac{c_3}{2}a\eta_x + \varsigma_2\eta_{xx} - 2\varsigma_3\mu_x. \end{aligned} \tag{2.4}$$

With the help of (2.3) and (2.4), one has

$$\mathcal{J}_1\frac{\delta H}{\delta \widetilde{U}} = -\mathrm{diag}(\partial, 1, \partial, 1)\mathbf{\Lambda}\frac{\delta H}{\delta \widetilde{U}} = -\mathrm{diag}(\partial, 1, \partial, 1)\frac{\delta H}{\delta \widetilde{F}} = \widetilde{U}_t.$$

Obviously (Refs. 5 and 9), $\mathcal{J}_1$ and $\mathcal{J}_2$ are compatible, and we thus complete the proof of the theorem. □

We remark that if $\mathbf{s_2} = \mathbf{c_2} = \mathbf{c_5} = \mathbf{0}$ and $\mathbf{s_1} = \frac{1}{4}\mathbf{c_1}$, the system (2.2) is exactly the local bi-superhamiltonian Euler equation obtained in Ref. 25 including the Kuper-Ito system and the two-component Kuper-CH system.

Example 2.2. Assume that

$$\mathbf{s_1} = \mathbf{c_1} = \mathbf{c_3} = \mathbf{c_4} = \mathbf{c_5} = \mathbf{0}, \quad \mathbf{c_2} = \mathbf{1}, \quad \mathbf{s_2} = \frac{\mathbf{1}}{\mathbf{2}}.$$

Then the system (2.2) *reads*

$$\begin{aligned} \mathbf{a_t} + \mathbf{2(af)_x} + \varsigma_1 \mathbf{f_{xxx}} + \varsigma_2 \mathbf{a_{xx}} &= \tfrac{3}{4}\mu\eta_{xx} + \tfrac{1}{2}\mu_x\eta_x - \tfrac{1}{4}\mu_{xx}\eta, \\ \mathbf{f_t} + \mathbf{2f\,f_x} - \varsigma_2 \mathbf{f_{xx}} + \mathbf{2}\varsigma_3 \mathbf{a_x} &= \tfrac{1}{4}\eta\eta_{xx}, \\ \boldsymbol{\mu_t} + \mathbf{2}\varsigma_1 \boldsymbol{\eta_{xxx}} + \mathbf{2}\varsigma_2 \boldsymbol{\mu_{xx}} &= -a\eta_x - \tfrac{1}{2}a_x\eta - \tfrac{3}{2}f_x\mu - 2f\mu_x, \\ \boldsymbol{\eta_t} - \mathbf{2}\varsigma_2 \boldsymbol{\eta_{xx}} + \mathbf{4}\varsigma_3 \boldsymbol{\mu_x} &= -2f\eta_x - \tfrac{1}{2}f_x\eta. \end{aligned} \tag{2.5}$$

Especially,

(1) *when* $\zeta_1 = 0$, $\zeta_2 = 1$, *and* $\zeta_3 = 2$, *the system* (2.5) *reduces to the super long-wave equation (Refs.* 10 and 12*);*
(2) *when* $\mu = \eta = 0$, *the system* (2.5) *reduces to the WBK system (Ref.* 4 *for* $\varsigma_3 = 1$*)*

$$\begin{aligned} &\mathbf{a_t} + \mathbf{2(af)_x} + \varsigma_1 \mathbf{f_{xxx}} + \varsigma_2 \mathbf{a_{xx}} = 0, \\ &\mathbf{f_t} + \mathbf{2f\,f_x} - \varsigma_2 \mathbf{f_{xx}} + \mathbf{2}\varsigma_3 \mathbf{a_x} = 0, \end{aligned} \tag{2.6}$$

which has been derived by (up to a scaling of the variable t)

- *Whitham in Ref.* 1 *for* $\varsigma_1 = \varsigma_2 = 0$,
- *Broer in Ref.* 2 *for* $\varsigma_1 = \frac{2}{3}$, $\varsigma_2 = 0$, *and* $\varsigma_3 = 1$, *and*
- *Kaup in Ref.* 3 *for* $\varsigma_2 = 0$, $\varsigma_3 = 1$.

We thus would like to call the system (2.5) *the* **Kuper-WBK system**.

III. SUPERSYMMETRIC EULER EQUATIONS ON $\mathfrak{G}^*_{\mathbf{reg}}$

In this section, we assume that $\mathbf{s_1} = \mathbf{c_1}$, $\mathbf{s_2} = \mathbf{c_2}$ and hope to study (local bi-superhamiltonian and) supersymmetric Euler equations.

Setting $\phi = \eta_x$ and $\alpha = \mu_x$, the Euler equation (1.4) reads

$$\begin{aligned} u_t &= -\varsigma_1 f_{xxx} - \varsigma_2 a_{xx} - 2uf_x - u_x f - va_x - \frac{3}{2}\psi\eta_{xx} - \frac{1}{2}\psi_x\eta_x - \frac{1}{2}\gamma\mu_{xx} + \frac{1}{2}\gamma_x\mu_x, \\ \psi_t &= \frac{1}{2}u\eta_x + \varsigma_1\eta_{xxx} + \varsigma_2\mu_{xx} - \frac{3}{2}f_x\psi - f\psi_x + \frac{1}{2}v\mu_x - \frac{1}{2}\gamma a_x, \\ v_t &= \varsigma_2 f_{xx} - 2\varsigma_3 a_x - (vf)_x - \frac{1}{2}(\gamma\eta_x)_x, \\ \gamma_t &= \frac{1}{2}v\eta_x - \gamma_x f - \frac{1}{2}\gamma f_x - \varsigma_2\eta_{xx} + 2\varsigma_3\mu_x, \end{aligned} \tag{3.1}$$

where $\begin{pmatrix} u \\ \psi \\ v \\ \gamma \end{pmatrix} = \begin{pmatrix} c_1 - c_4\partial^2 & 0 & c_2 - c_5\partial & 0 \\ 0 & c_4\partial^2 - c_1 & 0 & c_5\partial - c_2 \\ c_2 + c_5\partial & 0 & c_3 & 0 \\ 0 & -c_2 - c_5\partial & 0 & -c_3 \end{pmatrix}\begin{pmatrix} f \\ \eta \\ a \\ \mu \end{pmatrix}$.

Let $\mathbb{S}^{1|1}$ be a supercircle with local coordinates x, θ, where x is a local coordinate on $\mathbb{S}^1$ and θ is an odd Grassmann coordinate. Let $\mathcal{D} = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x}$ be a superderivative. Let $C^\infty(\mathbb{S}^{1|1})$ be the set of all smooth superfields on $S^{1|n}$. Taking two odd superfields $\Phi = \eta + \theta f$ and $\Omega = \mu + \theta a \in C^\infty(\mathbb{S}^{1|1})$, then the Euler equation (3.2) could be rewritten as

$$
\begin{aligned}
\Psi_t &= -\varsigma_1\Phi_{xxx} - \varsigma_2\Omega_{xx} - \frac{3}{2}c_1\Big(\Phi(\mathcal{D}\Phi)\Big)_x + c_5\Big((\mathcal{D}\Phi)\Omega_x\Big)_x \\
&\quad - \frac{1}{2}c_2\Big(\Phi(\mathcal{D}\Omega) + 3\Omega(\mathcal{D}\Phi)\Big)_x - \frac{1}{2}c_3\Big(\Omega(\mathcal{D}\Omega)\Big)_x \\
&\quad + c_4\Big((\mathcal{D}\Phi)\Phi_{xxx} + \frac{1}{2}\Phi_x(\mathcal{D}\Phi_{xx}) + \frac{3}{2}(\mathcal{D}\Phi_x)\Phi_{xx}\Big), \\
\Gamma_t &= \varsigma_2\Phi_{xx} - 2\varsigma_3\Omega_x - \frac{1}{2}c_2\Big(\Phi(\mathcal{D}\Phi_x) + 3\Phi_x(\mathcal{D}\Phi)\Big) \\
&\quad - \frac{1}{2}c_3\Big((\mathcal{D}\Omega)\Phi_x + 2\Omega_x(\mathcal{D}\Phi) + \Omega(\mathcal{D}\Phi_x)\Big) - c_5\Big(\Phi_x(\mathcal{D}\Phi)\Big)_x,
\end{aligned}
\tag{3.2}
$$

where $\Psi = c_1\Phi - c_4\Phi_{xx} + c_2\Omega - c_5\Omega_x$ and $\Gamma = c_2\Phi + c_5\Phi_x + c_3\Omega$. We thus have

Theorem 3.1. *When taking $s_1 = c_1$ and $s_2 = c_2$, the Euler equation (1.4) reduces to the system (3.2), which is supersymmetic, i.e., invariant under the supersymmetric transformation*

$$\delta f = \theta\eta_x, \quad \delta\eta = \theta f, \quad \delta a = \theta\mu_x, \quad \delta\mu = \theta a. \tag{3.3}$$

By choosing different parameters, the system (3.2) reduces to (Ref. 25)

- the $N = 1$ supersymmetric two-component KdV equation if $\mathbf{c_1 = c_3 = 1}$ and $\mathbf{c_2 = c_4 = c_5 = 0}$;
- the $N = 1$ supersymmetric two-component CH equation if $\mathbf{c_1 = c_3 = c_4 = 1}$ and $\mathbf{c_2 = c_5 = 0}$;
- the $N = 1$ supersymmetric two-component HS equation if $\mathbf{c_3 = c_4 = 1}$ and $\mathbf{c_1 = c_2 = c_5 = 0}$.

Here we want to present a new example, that, is the $N = 1$ **supersymmetric WBK** system.

Example 3.2. Let $\mathbf{c_2 = 1}$ and $\mathbf{c_1 = c_3 = c_4 = c_5 = 0}$; then the system (3.2) becomes

$$
\begin{aligned}
\Omega_t &= -\varsigma_1\Phi_{xxx} - \varsigma_2\Omega_{xx} - \frac{1}{2}\Big(\Phi(\mathcal{D}\Omega) + 3\Omega(\mathcal{D}\Phi)\Big)_x, \\
\Phi_t &= \varsigma_2\Phi_{xx} - 2\varsigma_3\Omega_x - \frac{1}{2}\Big(\Phi(\mathcal{D}\Phi_x) + 3\Phi_x(\mathcal{D}\Phi)\Big).
\end{aligned}
$$

We would like to call this the $N = 1$ **supersymmetric WBK** *system. Equivalently, in componentwise forms*

$$
\begin{aligned}
&\mathbf{a_t + 2(af)_x + \varsigma_1 f_{xxx} + \varsigma_2 a_{xx}} = \frac{3}{2}\mu\eta_{xx} + \mu_x\eta_x - \frac{1}{2}\mu_{xx}\eta, \\
&\mathbf{f_t + 2f\,f_x - \varsigma_2 f_{xx} + 2\varsigma_3 a_x} = \frac{1}{2}\eta\eta_{xx}, \\
&\boldsymbol{\mu_t + 2\varsigma_1\eta_{xxx} + 2\varsigma_2\mu_{xx}} = -\frac{1}{2}(a\eta)_x - \frac{3}{2}(f\mu)_x, \\
&\boldsymbol{\eta_t - \varsigma_2\eta_{xx} + 2\varsigma_3\mu_x} = -\frac{3}{2}f\eta_x - \frac{1}{2}f_x\eta.
\end{aligned}
\tag{3.4}
$$

With the use of **Theorem 2.1** and **Theorem 3.1**, we thus obtain

Theorem 3.3. *When taking $c_1 = c_2 = 0$, the Euler equation (3.2) reduces to*

$$
\begin{aligned}
c_4\Phi_{xt} + c_5\Omega_{xt} &= \varsigma_1\Phi_{xxx} + \varsigma_2\Omega_{xx} + \frac{1}{2}c_3\Big(\Omega(\mathcal{D}\Omega)\Big)_x - c_5\Big((\mathcal{D}\Phi)\Omega_x\Big)_x \\
&\quad - c_4\Big((\mathcal{D}\Phi)\Phi_{xxx} + \frac{1}{2}\Phi_x(\mathcal{D}\Phi_{xx}) + \frac{3}{2}(\mathcal{D}\Phi_x)\Phi_{xx}\Big), \\
c_5\Phi_{xt} + c_3\Omega_t &= \varsigma_2\Phi_{xx} - 2\varsigma_3\Omega_x - c_5\Big(\Phi_x(\mathcal{D}\Phi)\Big)_x \\
&\quad - \frac{1}{2}c_3\Big((\mathcal{D}\Omega)\Phi_x + 2\Omega_x(\mathcal{D}\Phi) + \Omega(\mathcal{D}\Phi_x)\Big),
\end{aligned}
\tag{3.5}
$$

which is not only supersymmetric but also local bi-superhamiltonian. Notice that in order to assure that the inertia operator is invertible, we should assume $\mathbf{c_3c_4 \neq c_5^2}$.

Example 3.4 (Ref. 25). *Let* $c_3 = -c_4 = 1$ *and* $\mathbf{c_5 = 0}$*; the system* (3.5) *reduces to*

$$-\Phi_{xxt} = \varsigma_1\Phi_{xxx} + \varsigma_2\Omega_{xx} + \frac{1}{2}\Big(\Omega(\mathcal{D}\Omega)\Big)_x$$

$$+ \Big((\mathcal{D}\Phi)\Phi_{xxx} + \frac{1}{2}\Phi_x(\mathcal{D}\Phi_{xx}) + \frac{3}{2}(\mathcal{D}\Phi_x)\Phi_{xx}\Big),$$

$$\Omega_t = \varsigma_2\Phi_{xx} - 2\varsigma_3\Omega_x - \frac{1}{2}\Big((\mathcal{D}\Omega)\Phi_x + 2\Omega_x(\mathcal{D}\Phi) + \Omega(\mathcal{D}\Phi_x)\Big).$$

Especially, (i) setting $\varsigma_1 = \varsigma_2 = \varsigma_3 = 0$ *and* $\eta = \mu = 0$*, we obtain the two-component HS equation (Ref.* 18*)*

$$-f_{xxt} = 2f_xf_{xx} + ff_{xxx} - aa_x, \quad a_t = -(af)_x.$$

(ii) Setting $\varsigma_1 = \varsigma_2 = \varsigma_3 = 0$ *and* $\Omega = 0$*, the above system becomes*

$$-\Phi_{xxt} = (\mathcal{D}\Phi)\Phi_{xxx} + \frac{1}{2}\Phi_x(\mathcal{D}\Phi_{xx}) + \frac{3}{2}(\mathcal{D}\Phi_x)\Phi_{xx},$$

which is the supersymmetric HS equation in Refs. 21 and 23.

IV. CONCLUSIONS

In summary, we have studied Euler equations associated with the $N = 1$ extended Neveu-Schwarz algebra and shown the conditions under which there are superymmetric or local bi-superhamiltonian. As an application, we have proposed two super generalizations of the WBK system: the $N = 1$ supersymmetric WBK system (3.4) and the local bi-superhamiltonian Kuper-WBK system (2.5).

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[28] The variational derivatives $\frac{\delta H}{\delta u}$, $\frac{\delta H}{\delta \psi}$, $\frac{\delta H}{\delta v}$, and $\frac{\delta H}{\delta \gamma}$ are defined by

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} H[\widetilde{U} + \epsilon\delta\widetilde{U}] = \int\left(\delta u\frac{\delta H}{\delta u} + \delta\psi\frac{\delta H}{\delta\psi} + \delta v\frac{\delta H}{\delta v} + \delta\gamma\frac{\delta H}{\delta\gamma}\right)dx.$$

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