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# Hopf maps and integrable systems

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# Contents

# Introduction

The integrable systems play a distinguished role in the theoretical physics [1]. In fact, many of interesting physical theories are small deformations of corresponding integrable system. The term "integrable system" is coming from the Liouville's theorem. It can be formulated as follows[2]:

**Theorem**: Let on an $n$-dimensional symplectic manifold one has $n$ mutually commuting functions $F_i$:

$$F_1, F_2, \ldots, F_n, \quad \{F_\mu, F_\nu\} = 0, \quad \mu, \nu = 1, \ldots, n : \tag{0.1}$$

Let us define the level surface $M_f$ as follows:

$$M_f = \{x : F_\mu = f_\mu, \quad \mu = 1, \ldots n\}. \tag{0.2}$$

It follows from 0.1 that the one-forms $dF_i$ on $M_f$ are linearly independent. Then

1. $M_f$ is a smooth manifold invariant under the phase flow of

function $H = F_1$.

2. If $M_f$ is compact and connected then it is diffeomorphic to the $n$-dimensional torus:

$$T^n = \{(\phi_1, \dots \phi_n) \quad modd \quad 2\pi\} \tag{0.3}$$

3. The phase flow of the function $H$ defines a conditionally periodic motion on $M_f$, i.e. in terms of conditional coordinates the equations of motion have the following form:

$$\frac{d\boldsymbol{\phi}}{dt} = \boldsymbol{\omega}, \quad \boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{f}) \tag{0.4}$$

4. The canonical equations with Hamiltonian $H$ are integrable in quadratures.

This theorem establishes the connection between the integrals of motion and the integrability of the system. In other words, it states, that if the system has $n-1$ mutually commuting integrals of motion then we can pass to variables where the system separates on $n$ independent oscillators. The coordinates $\boldsymbol{\phi}$ together with their conjugate momenta $\mathbf{I}$ are called action angle variables.

Such formulation of Liouville's theorem prompts us to use geometric method for the investigation of integrable systems (see e.g.

[3]).

As a tool for the construction and investigation of some integrable systems we will use the Hopf maps. These maps describe fibrations of spheres over spheres: $S^{2n-1}/S^{n-1} = S^n$, $n = 1, 2, 4, 8$ and are strongly related to so called normed division algebras. (For the review see e.g. [4] and references therein). Explicitly or not they appear almost in all the fields of the modern theoretical physics. For example, the distinct dimensions of the supergravity theories are related to the existence of Hopf maps[9]. They can be used for the construction of supersymmetric systems as well as in the theory of superconductors.

We are interested in Hopf maps due to the existence of the reduction procedure, which allows us to relate $2p$-dimensional oscillator with $p + 1$-dimensional systems with monopoles for $p = 1, 2, 4$.

The reduction procedure seems to be a canonical way for the construction of integrable systems. However, it was described in terms of the Hamiltonian approach only. On the other hand, there are many problems in physics, whose natural language is the Lagrangian formalism(e.g. some problems in superymmetric mechanics). Thus, it would be nice to have an algorithm to perform the reduction procedure without passing to Hamiltonian description of the system. In the second chapter the procedure of the Lagrangian

reduction was developed. In particular, it was considered the reduction procedure related to the first and second Hopf maps. The resulting systems describe particle moving in the field of Dirac and Yang monopoles respectively.

The systems with magnetic monopoles are interesting for their duality property [5], their application in condensed matter theory etc. On the other hand, the oscillator system is much more easier to investigate. It is, perhaps, the most known (super)integrable system. The oscillator system appears almost in all the fields of theoretical physics at least for the description of small oscillations near the point of equilibrium. Thus, the construction of its integrable generalizations(with additional terms or on the curved manifolds etc.) seems to be an important task.

Surprisingly, a multi-center generalization of Higgs oscillator can be constructed from the spherical part of the Calogero model[45] using simple geometry. On the other hand the Calogero model is itself an interesting subject for investigation. It is an example of many-body integrable system. Although it is formally integrable(in sense of the Liouville's theorem), its practical integration is very hard task. The problem is that the motion integrals found using the standard method of Lax pairs have power of momenta higher than 2. This leads to the necessity to solve algebraic equations of

the same order. We hope that the investigation of this model using geometric methods will allow us to round these difficulties.

The work is organized as follows.

In the first Chapter we describe the Hopf fibrations in terms of normed division algebras paying special attention on the first and second maps.

In the Chapter 2 we develop the reduction procedure connected to the first and second Hopf maps and investigate obtained systems.

In the Chapter 3 we construct a (pseudo)spherical analogue of anisotropic inharmonic Higgs oscillator and apply to it the reduction procedure.

In Chapter 4 we consider Calogero models relating it to multi-center Higgs oscillator as well as construct action-angle variables for them.

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# Chapter 1

# Hopf maps

Hopf maps play distinguished role in modern theoretical physics: numerous constructions and models are related with them. Hopf maps are of the special importance in the supersymmetry [7], monopoles [8], and more generally, in supergravity/string theories. Hopf maps are useful in the study of the problems of classical and quantum (supersymmetric) mechanics as well. They are useful for the construction of the mechanical systems (including supersymmetric ones) with monopoles by the reduction method, including supersymmetric mechanics.

The Hopf maps describe a fibration of $2p-1$-dimensional sphere over a $p$-dimensional one with fiber $(p-1)$-sphere($p = 1, 2, 4, 8$). The first($p = 2$) Hopf map was discovered in 1931 by Heinz Hopf [10] and is the first example of a map from a higher-dimensional sphere

to a lower-dimensional one which is not null-homotopic. The distinguished dimensions of Hopf maps are related with the existence of normed division algebras: the first Hopf map is related with complex, the second with quaternionic and the third one with octonionic numbers. The modern theory of octonions and their relations to the Hopf maps can be found in [4] and references therein. The content of this chapter is based on the [38, 80]

## 1.1 General consideration

The Hopf maps (or Hopf fibrations) are the fibrations of the sphere over a sphere,

$$\begin{aligned} S^1/Z_2 &= S^1 \quad \text{(zero Hopf map)}; \\ S^3/S^1 &= S^2 \quad \text{(first Hopf map)}; \\ S^7/S^3 &= S^4 \quad \text{(second Hopf map)} \\ S^{15}/S^7 &= S^8 \quad \text{(third Hopf map)} \end{aligned} \tag{1.1}$$

These fibrations reflect the existence of real ($p = 1$), complex ($p = 2$), quaternionic ($p = 4$) and octonionic ($p = 8$) numbers.

Let us describe them in explicit terms. For this purpose, we

consider the functions $\mathbf{x}(u_\alpha, \bar{u}_\alpha), x_{p+1}(u_\alpha, \bar{u}_\alpha)$

$$\mathbf{x} = 2\bar{\mathbf{u}}_2\mathbf{u}_1, \quad x_{p+1} = \bar{\mathbf{u}}_2\mathbf{u}_2 - \bar{\mathbf{u}}_1\mathbf{u}_1, \tag{1.2}$$

where $\mathbf{u}_1, \mathbf{u}_2$ are complex numbers for the $p = 2$ case (first Hopf map), quaternionic numbers for the $p = 4$ case (second Hopf map) and octonionic numbers for $p = 8$ case(third Hopf map). One can consider them as coordinates of the $2p$-dimensional space $\mathbb{R}^{2p}$ ($p = 2$ for $\mathbf{u}_{1,2}$ complex numbers; $p = 4$ for $\mathbf{u}_{1,2}$ quaternionic numbers; $p = 8$). In all cases $x_{p+1}$ is a real number while $\mathbf{x}$ is, respectively, a complex number ($p = 2$), a quaternionic ($p = 4$) or an octonionic one($p = 8$),

$$\mathbf{x} \equiv x_n + \sum_{k=1,\ldots,n-1} \mathbf{e}_k x_k, \qquad \mathbf{e}_i\mathbf{e}_j = -\delta_{ij} + C_{ijk}\mathbf{e}_k \tag{1.3}$$

where the structure constants $C_{ijk}$ are totally antisymmetric by indices $(ijk)$, so that $\mathbf{e}_k \equiv 0$ for $n = 1$; $\mathbf{e}_k \equiv \mathbf{i}$, $\mathbf{i}^2 = -1$, $c_{ijk} = 0$ for $n = 2$; $\mathbf{e}_k \equiv (\mathbf{i}, \mathbf{j}, \mathbf{k})$, $C_{ijk} = \varepsilon_{ijk}$ for $n = 4$. For $n = 8$ the structure constants $c_{ijk}$ are defined by the relations

$$C_{123} = C_{147} = C_{165} = C_{246} = C_{257} = C_{354} = C_{367} = 1, \tag{1.4}$$

while all other non-vanishing components are determined by the total antisymmetry.

Hence, $(x_{p+1}, \mathbf{x})$ parameterize the $(p+1)$-dimensional space $\mathbb{R}^{p+1}$.

One could immediately check that the following equation holds:

$$r^2 \equiv \mathbf{x}\bar{\mathbf{x}} + x_{p+1}^2 = (\mathbf{u}_1\bar{\mathbf{u}}_1 + \mathbf{u}_2\bar{\mathbf{u}}_2)^2 \equiv R^4. \tag{1.5}$$

Thus, defining the $(2p-1)$-dimensional sphere in $\mathbb{R}^{2p}$ of radius $R$, $\mathbf{u}_\alpha\bar{\mathbf{u}}_\alpha = R^2$, we will get the $p$-dimensional sphere in $\mathbb{R}^{p+1}$ with radius $r = R^2$.

The expressions (1.2) can be easily inverted by the use of

$$\begin{gathered}\mathbf{u}_\alpha = \mathbf{g}r_\alpha, \qquad \text{where} \quad r_1 \equiv r_+ = \frac{\sqrt{\bar{\mathbf{x}}\mathbf{x}}}{\sqrt{2(r+x_{p+1})}}, \\ r_2 = \sqrt{\frac{r+x_{p+1}}{2}}, \quad \bar{\mathbf{g}}\mathbf{g} = 1.\end{gathered} \tag{1.6}$$

It is seen that

$$\mathbf{g}^2 = \mathbf{u}_2(\overline{\mathbf{u}}_2)^{-1}. \tag{1.7}$$

It follows from the last equation in (1.6) that $\mathbf{g}$ parameterizes the $(p-1)$-dimensional sphere of unit radius.

One can notice from this expressions that the transformation

$$\mathbf{g} \mapsto \mathbf{G}\mathbf{g} \tag{1.8}$$

preserves the functions $x_\mu$. Here $\mathbf{G}$ is an arbitrary element of corresponding algebra with unit length: $\mathbf{G}\bar{\mathbf{G}} = 1$ and, therefore, parametrizes $(p-1)$-dimensional sphere. So, also taking in account

the identitiy 1.5 we come to the factorization 1.2.

For the first and second Hopf maps the transformation 1.8 for the quantities $\mathbf{u_{1,2}}$ can be written as follows:

$$\mathbf{u}_\alpha \mapsto \mathbf{G}\mathbf{u}_\alpha, \quad \alpha = 1, 2 \tag{1.9}$$

while for the third Hopf map it looks a little more complicated:

$$\mathbf{u}_\alpha = \frac{(\mathbf{G}\mathbf{u}_2)(\mathbf{u}_2\mathbf{u}_\alpha)}{\bar{\mathbf{u}}_2\bar{\mathbf{u}}_2} \tag{1.10}$$

The complexity, in fact, comes from the fact that the algebra of octonions which corresponds to the third Hopf map is not associative. It is shown in [50] that this transformation cannot be made global. Thus, one can consider only its simplification choosing the parameter $\mathbf{G}$ depending on $\mathbf{u}_{1,2}$.

## 1.2 First and second Hopf maps

As it was mentioned in the Introduction, the reduction procedure related to the third Hopf map does not exist because of the lack of associativity of octonions. Since, we will use the Hopf maps to perform the reduction, we will need more detailed formulae. Let us give the description of first and second Hopf maps in internal terms, using the decomposition $\mathbb{R}^{2p} = \mathbb{R}^1 \times \mathrm{S}^{2p-1}$, $\mathbb{R}^{p+1} = \mathbb{R}^1 \times \mathrm{S}^p$, and

paramerizing $S^p$ by inhomogeneous projective coordinates

$$z = \frac{\bar{\mathbf{u}}_1 \mathbf{u}_2}{\bar{\mathbf{u}}_1 \mathbf{u}_1}, \quad \Rightarrow \quad |\mathbf{u}_1|^2 = \frac{r}{1+\bar{z}z}. \tag{1.11}$$

Hence, we get

$$\mathbf{u}_1 = \frac{\mathbf{g}\sqrt{r}}{\sqrt{1+\bar{z}z}}, \qquad \mathbf{u}_2 = \mathbf{u}_1 z = \frac{\mathbf{g}\sqrt{r}z}{\sqrt{1+\bar{z}z}} \tag{1.12}$$

For $r = \text{const}$ we get the description of $S^{2p-1}$ in terms of the coordinates of the base manifold $S^p$ and of the fiber coordinates $\mathbf{g}$. The internal coordinate $z$ of the sphere $S^p$ is related with the Cartesian coordinates of the ambient space $\mathbb{R}^{p+1}$ (1.2) as follows

$$\mathbf{x} = r\mathbf{h}_+, \quad x_{p+1} = r h_{p+1}, \qquad \mathbf{h}_+ = \frac{2z}{1+\bar{z}z}, \quad h_{p+1} = \frac{1-\bar{z}z}{1+\bar{z}z}. \tag{1.13}$$

For $S^1$ the group element and the corresponding left-invariant one-form can be presented as follows

$$S^1 \ : \quad \mathbf{g} = \mathrm{e}^{\mathbf{i}\varphi}, \quad \overline{\mathbf{g}} d\mathbf{g} = \mathbf{i} d\varphi, \quad \varphi \in [0, 2\pi) \tag{1.14}$$

Hence, the ambient coordinates of the $S^3$ sphere of unit radius are related with the internal coordinates of $S^1$ and $S^2$ by (1.12), where we put $r = 1$ and $\mathbf{g} = \mathrm{e}^{\mathbf{i}\varphi}$.

In quaternionic case we get the following expressions for the

$SU(2)$ group element and its left-invariant form

$$S^3 \ : \quad \mathbf{g} = \mathrm{e}^{\mathbf{i}\gamma}\frac{1+\mathbf{j}z}{\sqrt{1+z\bar{z}}}, \quad \overline{\mathbf{g}}d\mathbf{g} = \Lambda_3\mathbf{i} + \Lambda_+\mathbf{j}, \qquad \Lambda_+ = (\Lambda_2 + \mathbf{i}\Lambda_1), \tag{1.15}$$

where

$$\Lambda_3 = h_3 d\gamma + \frac{\mathbf{i}}{2}\frac{\bar{z}dz - zd\bar{z}}{1+z\bar{z}} \qquad \Lambda_+ = \mathbf{i}\mathbf{h}_+ d\gamma + \frac{d\bar{z}}{1+z\bar{z}} \qquad i,j,k = 1,2,3. \tag{1.16}$$

Here $h_3, \mathbf{h}_\pm$ are the Euclidean coordinates of the ambient space $\mathbb{R}^3$ given by (1.13): simultaneously they play the role of Killing potentials of the Kähler structure on $S^2$.

The vector fields dual to the above one-forms look as follows

$$\mathbf{V}_3 = \frac{\partial}{\partial\gamma} + 2\mathbf{i}\left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial\bar{z}}\right), \quad \mathbf{V}_+ = \overline{\mathbf{V}}_- = \frac{\partial}{\partial\bar{z}} + z^2\frac{\partial}{\partial z} - \mathbf{i}\frac{z}{2}\frac{\partial}{\partial\gamma} \tag{1.17}$$

$$\Lambda_3(\mathbf{V}_3) = \Lambda_\pm(\mathbf{V}_\pm) = 1, \quad \Lambda_\pm(\mathbf{V}_\mp) = \Lambda_\pm(\mathbf{V}_3) = \Lambda_3(\mathbf{V}_\pm) = 0. \tag{1.18}$$

Let us also write down the following expressions

$$-(\bar{\mathbf{g}}d\mathbf{g})^2 = \Lambda_i\Lambda_i = \left(d\gamma - \frac{\mathbf{i}}{2}\frac{\bar{z}dz - zd\bar{z}}{1+z\bar{z}}\right)^2 + \frac{dzd\bar{z}}{(1+z\bar{z})^2}. \tag{1.19}$$

We also need another $SU(2)$ group element parameterizing the sphere $S^3$ and "commuting" with (1.15):

$$\widetilde{\mathbf{g}} = \frac{1+\mathbf{j}z}{\sqrt{1+z\bar{z}}}\mathrm{e}^{-\mathbf{i}\gamma}, \qquad \bar{\mathbf{g}}\bar{\widetilde{\mathbf{g}}}\mathbf{g}\widetilde{\mathbf{g}} = 1. \tag{1.20}$$

The corresponding left-invariant forms are given by the expressions

$$\begin{aligned} \bar{\widetilde{\mathbf{g}}}d\widetilde{\mathbf{g}} = \widetilde{\Lambda}_3\mathbf{i} + \widetilde{\Lambda}_+\mathbf{j}, \qquad \widetilde{\Lambda}_+ = \widetilde{\Lambda}_2 + \mathbf{i}\widetilde{\Lambda}_1, \\ \widetilde{\Lambda}_3 = d\gamma + \tfrac{\mathbf{i}}{2}\tfrac{zd\bar{z}-\bar{z}dz}{1+z\bar{z}} \qquad \widetilde{\Lambda}_+ = \tfrac{\mathrm{e}^{2\mathbf{i}\gamma}d\bar{z}}{1+z\bar{z}}, \end{aligned} \tag{1.21}$$

while the vector fields dual to these forms look as follows:

$$\mathbf{U}_3 = -\frac{\partial}{\partial\gamma}, \quad \mathbf{U}_+ = \overline{\mathbf{U}}_- = \mathrm{e}^{-2\mathbf{i}\gamma}\left((1+z\bar{z})\,\frac{\partial}{\partial\bar{z}} + \frac{\mathbf{i}z}{2}\frac{\partial}{\partial\gamma}\right) \tag{1.22}$$

$$\widetilde{\Lambda}_3(\mathbf{U}_3) = \widetilde{\Lambda}_\pm(\mathbf{U}_\pm) = 1, \quad \widetilde{\Lambda}_\pm(\mathbf{U}_\mp) = \widetilde{\Lambda}_\pm(\mathbf{U}_3) = \widetilde{\Lambda}_3(\mathbf{U}_\pm) = 0. \tag{1.23}$$

From the second expression in (1.21) follows the commutativity of the $\mathbf{V}_a$ and $\mathbf{U}_a$ fields. This pair forms the the $so(4) = so(3)\times so(3)$ algebra of isometries of the $S^3$ sphere.

$$[\mathbf{V}_i, \mathbf{V}_j] = 2\varepsilon_{ijk}\mathbf{V}_k, \qquad [\mathbf{U}_i, \mathbf{U}_j] = 2\varepsilon_{ijk}\mathbf{U}_k, \qquad [\mathbf{V}_i, \mathbf{U}_j] = 0, \tag{1.24}$$

with $i, j, k = 1, 2, 3$.

The commutativity of $\mathbf{V}_i$ and $\mathbf{U}_i$ plays a key role in our further considerations. Notice also that we can pass from the parametrization (1.21) to (1.15) via the $z \to \tilde{z}\mathrm{e}^{-2\mathbf{i}\tilde{\gamma}}$, $\gamma = -\tilde{\gamma}$ transformation.

For our further considerations this is all we need to know from the Hopf maps.

# Chapter 2

# Hopf maps and Reductions

In this chapter we develop reduction procedures in two different mechanical approaches- Lagrangian and Hamiltonian. Although theoretically *in the considered cases* they are equivalent, the practical passing from the first one to the second does not seem to be obvious. On the other hand, the Lagrangian approach allows us to write the supersymmetric extension more easily [38] while the Hamiltoinian one gives us the ability to relate oscillator systems and their generalizations with the (MICZ)Kepler ones [64].The content of this chapter is based on the [64, 38, 78].

## 2.1 Lagrangian Reduction

Let us consider a free particle on the $2p$-dimensional space equipped with the $G$-invariant conformal flat metric. Taking into account the

expressions (1.6) we can represent its Lagrangian as follows

$$\mathcal{L}_{2p} = g(\overline{\mathbf{u}} \cdot \mathbf{u})\dot{\overline{\mathbf{u}}}_\alpha \dot{\mathbf{u}}_\alpha =$$

$$= g(r_\pm, r_2)\left(\dot{r}_+\dot{r}_- + \dot{r}_2^2 - r\left(\overline{\mathbf{g}}\dot{\mathbf{g}}\mathcal{A} + \mathcal{A}r\overline{\mathbf{g}}\dot{\mathbf{g}}\right) - r(\overline{\mathbf{g}}\dot{\mathbf{g}})^2\right) = \tag{2.1}$$

$$= g\left(\dot{r}_+\dot{r}_- + \dot{r}_2^2\right) + gr\dot{\Lambda}_a A_a - gr\dot{\Lambda}_a\dot{\Lambda}_a,$$

Here and in the following $\dot{\Lambda}_a$ are defined in (1.16), with the differentials replaced by the time derivatives, while

$$\mathcal{A} = A_a\mathbf{e}_a \equiv \frac{\dot{r}_+ r_- - r_+\dot{r}_-}{2r} = \frac{\mathbf{x}\dot{\overline{\mathbf{x}}} - \dot{\mathbf{x}}\overline{\mathbf{x}}}{2r(r + x_{p+1})}. \tag{2.2}$$

We have used the identity $r_+r_- + r_2^2 = r$ and the notation $\overline{\mathbf{u}} \cdot \mathbf{u} \equiv \overline{\mathbf{u}}_\alpha \cdot \mathbf{u}_\alpha$. One can see, for the $p = 2$ case (the complex numbers) that $\mathcal{A}$ defines a Dirac monopole potential

$$\mathcal{A} = \imath A_D = \imath\frac{x_1\dot{x}_2 - x_2\dot{x}_1}{r(r + x_3)}. \tag{2.3}$$

In the $p = 4$ case (the quaternionic numbers) $A_a$ defines the potential of the the $SU(2)$ Yang monopole. The explicit formulae for $A_a$ in terms of the real coordinates $r_+ = \rho_4 + \rho_a e_a$ look as follows:

$$A_a = \eta^a_{bc}\rho_b\dot{\rho}_c = \frac{\eta^a_{bc}x_b\dot{x}_c}{r\,(r + x_5)}, \qquad \eta^a_{bc} = \delta_{ab}\delta_{4c} - \delta_{4b}\delta_{ac} + \varepsilon_{abc},$$

where $\eta^a_{bc}$ is the t'Hooft symbol.

The Lagrangian (2.1) is manifestly invariant under the $G-$group action.

In the $p = 2$ case the generator of the $G = U(1)$ group is given by the vector field $\mathbf{V} = \partial/\partial\varphi$; indeed, taking into account (1.14), one can see that, for $p = 2$, $\varphi$ is a cyclic variable in (2.1).

In the $p = 4$ case the generators of the $G = SU(2)$ group are given by the vector fields $\mathbf{U}_a$ (1.22). It is in agreement with the fact that $\mathbf{U}_a$ define the isometries of the eight-dimensional Lagrangian (2.1).

By making use of the Noether constants of motion we can decrease the dimensionality of the system.

In the $p = 2$ case we have a single Noether constant of motion defined by the vector field dual to the left-invariant form $\dot{\Lambda} = \dot{\varphi}$; this is precisely the momentum conjugated to $\varphi$, which appears in the Lagrangian (2.1) as a cyclic variable. Hence, excluding this variable, we shall get, for $p = 2$, a three-dimensional system.

On the other hand, in the $p = 4$ case, thanks to the non-Abelian nature of the $G = SU(2)$ group, only the $\gamma$ variable is a cyclic one, even if $z, \bar{z}$ appear in the Lagrangian (2.1) without time-derivatives too. It is therefore expected that in this second case the reduction procedure would be more complicated. In contrast with the Hamiltonian reduction procedure, the Lagrangian reduction is a less com-

mon, or at least a less developed, procedure which deserves being done with care.

For this reason we shall describe the Lagrangian counterparts of the Hamiltonian reduction procedures separately for both the $p = 2$ and the $p = 4$ cases.

### 2.1.1 The $U(1)$ reduction

Let us consider the reduction of the four-dimensional particle given by the Lagrangian (2.1) to a three-dimensional system. Taking into account the expression (1.14) we can re-write the Lagrangian as follows:

$$\mathcal{L} = g\left(\dot{r}_+\dot{r}_- + \dot{r}_2^2 - 2r\dot{\varphi}\mathcal{A}_D + r\dot{\varphi}^2\right). \tag{2.4}$$

Since $\varphi$ is a cyclic variable, its conjugated momentum is a conserved quantity

$$p_\varphi = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = -2rg\mathcal{A}_D + 2gr\dot{\varphi} \quad \Leftrightarrow \quad \dot{\varphi} = \frac{p_\varphi}{2gr} + \mathcal{A}_D. \tag{2.5}$$

Naively one could expect that the reduction would require fixing the value of the Noether constant and substituting the corresponding expression for $\dot{\varphi}$ in the Lagrangian (2.4). However, acting in this way, we shall get a three-dimensional Lagrangian without a linear term in the velocities, i.e. without a magnetic field (of the Dirac

monopole). This is in obvious contradiction with the result of the Hamiltonian reduction of the four-dimensional system via the $U(1)$ group action!

*The contradiction is due to the incorrectness of the proposed procedure, since we are dealing with a variational principle.*

The correct Lagrangian counterpart of the Hamiltonian reduction procedure should look as follows. At first we have to replace the Lagrangian (2.4) by the following, variationally equivalent, one (obtained by performing the Legendre transformation for $\dot{\varphi}$):

$$\widetilde{\mathcal{L}} = p_\varphi \dot{\varphi} - p_\varphi \mathcal{A}_D - \frac{p_\varphi^2}{rg} - r A_D^2 + g\left(\dot{r}_+ \dot{r}_- + \dot{r}_2^2\right). \qquad (2.6)$$

Indeed, varying the independent variable $p_\varphi$, we shall arrive to the initial Lagrangian.

The isometry of the Lagrangian (2.6), corresponding to the $U(1)$-generator $\mathbf{V} = \frac{\partial}{\partial \varphi}$, is given by the same vector field. It defines the Noether constant of motion

$$I = p_\phi \qquad (2.7)$$

*Upon fixing the value of the Noether constant $p_\varphi = s$, the first term of the new Lagrangian transforms as a full time derivative and can therefore be ignored.*

As a result, we shall get the following three-dimensional Lagrangian

$$\mathcal{L}_3 = g\left(\dot{r}_+\dot{r}_- + \dot{r}_2^2\right) - s\mathcal{A}_D - r\mathcal{A}_D^2 - \frac{s^2}{2rg} = \frac{g}{r}\dot{x}_i\dot{x}_i - s\mathcal{A}_D - \frac{s^2}{2rg}. \quad (2.8)$$

Clearly, it describes the motion of a particle moving in a three-dimensional space equipped by the metric $\widetilde{g}_{ij} = \frac{g}{r}\delta_{ij}$ in presence of a Dirac monopole generating a magnetic field with strength

$$\vec{B} = \frac{s\vec{x}}{\widetilde{g}x^3}. \quad (2.9)$$

A further reduction of the system to two dimensions corresponds to a system with a nonlinear chiral multiplet $(2, 4, 2)$, obtained by fixing the "radius" $r = const$. Since the Dirac monopole potential $\mathcal{A}_D$ does not depend on $r$, we shall get a two-dimensional system moving in the same magnetic field. It applies in particular to a particle on the sphere moving in a constant magnetic field (the Dirac monopole is located at the center of the sphere), i.e. the Landau problem on a sphere.

We are now ready to discuss the analogous reduction associated with the second Hopf map.

### 2.1.2 The $SU(2)$ reduction

In the case of the second Hopf map we have to reduce the Lagrangian (2.1) with $p = 4$ via the action of the $SU(2)$ group expressed by the vector fields (1.22). Due to the non-Abelian nature of the $SU(2)$ group the system will be reduced to a five (or higher)-dimensional one.

For a correct reduction procedure we have to replace the initial Lagrangian by one which is variationally equivalent, extending the initial configuration space with new variables, $\pi$, $\bar{\pi}$, $p_\gamma$, playing the role of conjugate momenta to $z$, $\bar{z}$, $\gamma$. In other words, we will replace the sphere $S^3$ (parameterized by $z$, $\bar{z}$, $\gamma$) by its cotangent bundle $T^*S^3$ parameterized by the coordinate $z$, $\bar{z}$, $\gamma$, $\pi$, $\bar{\pi}$, $p_\gamma$. Let us further define, on $T^*S^3$, the Poisson brackets given by the relations

$$\{\pi, z\} = 1, \qquad \{\bar{\pi}, \bar{z}\} = 1, \qquad \{p_\gamma, \gamma\} = 1. \tag{2.10}$$

We introduce the Hamiltonian generators $P_a$ corresponding to the vector fields (1.17) (replacing the derivatives entering the vector fields $\mathbf{V}_a$ by the corresponding momenta)

$$P_+ = \frac{P_2 - \imath P_1}{2} = \pi + \bar{z}^2\bar{\pi} - \frac{\imath\bar{z}}{2}p_\gamma, \quad P_- = \bar{P}_-, \quad P_3 = \frac{p_\gamma}{2} - \imath\left(z\pi - \bar{z}\bar{\pi}\right). \tag{2.11}$$

In the same way we introduce the Hamiltonian generators $I_a$ corresponding to the vector fields (1.22):

$$I_3 = -p_\gamma, \quad I_+ = \frac{I_2 - \imath I_1}{2} = \frac{\imath p_\gamma z + 2\bar{\pi}\left(1 + z\bar{z}\right)}{2} e^{-2\imath\gamma}, \quad I_- = \overline{I}_+. \tag{2.12}$$

These quantities define, with respect to the Poisson bracket (2.10), the $so(4) = so(3) \times so(3)$ algebra

$$\{P_a, P_b\} = \varepsilon_{abc} P_c, \quad \{I_a, I_b\} = \varepsilon_{abc} I_c, \quad \{I_a, P_b\} = 0. \tag{2.13}$$

The functions $P_a$, $I_a$ obey the following equality, important for our considerations

$$I_a I_a = P_a P_a. \tag{2.14}$$

At this point we replace the initial Lagrangian (2.1) by the following one, which is variationally equivalent

$$\mathcal{L}_{int} = \left(P_+\Lambda_+ + P_-\Lambda_- + P_3\Lambda_3\right) - P_a A_a - \frac{P_a P_a}{gr} -$$

$$-\frac{gr}{4}\left(A_+ A_- + A_3^2\right) + g\left(\dot{r}_+\dot{r}_- + \dot{r}_2^2\right). \tag{2.15}$$

The isometries of this modified Lagrangian corresponding to (1.22) are defined by the vector fields

$$\widetilde{\mathbf{U}}_a \equiv \{I_a, \ \ \}, \tag{2.16}$$

where $I_a$ are given by (2.12) and the Poisson brackets are given by (2.10). The quantities $I_a$ entering (2.16) are the Noether constants of motion of the modified Lagrangian (2.15). This can be easily seen by taking into account the following equality

$$P_+\Lambda_+ + P_-\Lambda_- + P_3\Lambda_3 = p_\gamma\dot{\gamma} + \pi\dot{z} + \bar{\pi}\dot{\bar{z}}. \tag{2.17}$$

We have now to perform the reduction via the action of the $SU(2)$ group given by the vector fields (2.16). For this purpose we have to fix the Noether constants of motion (2.12), setting

$$I_a = s_a = const, \qquad s_a s_a \equiv s^2.$$

Since the constants of motion $I_a$ do not dependent on the $r_\pm, r_5$ coordinates we can perform an orthogonal rotation so that only the third component of this set, $I_3$, assumes a value different from zero. Equating $I_+$ and $I_-$ with zero we obtain:

$$I_3 = p_\gamma = s, \qquad \bar{\pi} = -s\frac{\imath}{2}\frac{z}{1+z\bar{z}}, \quad \pi = s\frac{\imath}{2}\frac{\bar{z}}{1+z\bar{z}}. \tag{2.18}$$

Hence,

$$P_+ = -s\frac{\imath z}{1+z\bar{z}}, \quad P_- = s\frac{\imath\bar{z}}{1+z\bar{z}}. \tag{2.19}$$

Therefore $P_a$ coincide with the Killing potentials of the $S^2$ sphere!

This is by no means an occasional coincidence.

Taking in mind the equality (2.17) we can conclude that the third term entering (2.15) can be ignored because it is a full time derivative. Besides that, taking into account (2.14), we can rewrite the Lagrangian as follows:

$$\mathcal{L}_{red} = g\frac{\dot{x_\mu}\dot{x_\mu}}{r} - \imath s\frac{\bar{z}\dot{z} - z\dot{\bar{z}}}{1+z\bar{z}} - sA_a h_a(z,\bar{z}) - \frac{s^2}{gr}, \qquad \mu = 1,\ldots,5, \tag{2.20}$$

where we have used the identity

$$-\frac{gr}{4}\left(A_+A_- + A_3^2\right) + g\left(\dot{r}_+\dot{r}_- + \dot{r}_2^2\right) = g\frac{\dot{x_\mu}\dot{x_\mu}}{r}.$$

The second term in the above reduced Hamiltonian is the one-form defining the symplectic (and Kähler) structure on $S^2$, in agreement with the previous observation that $P_a$ coincide with the Killing potentials of $S^2$.

It therefore follows that the Noether constants of motion do not allow us to exclude the $z, \bar{z}$ variables. However, their time derivatives appear in the Lagrangian in a linear way only and define the internal degrees of freedom of the five-dimensional isospin particle interacting with a Yang monopole. As a consequence, the dimensionality of the phase space of the reduced system is $2 \cdot 5 + 2 = 12$. Only for the particular case $s = 0$, corresponding to the absence

of the Yang monopole, we obtain a five-dimensional system. This means that *locally* the Lagrangian of the system can be formulated in a six-dimensional space. Such a representation seems, however, useless, in contrast with the one presented here. Notice that a detailed desciption of the dynamics of the isospin particle can be found in [49].

The further reduction of the constructed $(5 + ...)$-dimensional system to a $(4+...)$-dimensional one would be completely similar to the $U(1)$ case: it requires fixing the radial variable $r$. The resulted system describes the isospin particle moving in a four-dimensional space and interacting with the BPST instanton.

In this Section we have considered the Lagrangian reduction procedures, restricting ourselves to $2p$-dimensional systems with *conformally flat metrics* only. From our considerations it is however clear that similar reductions can be performed also for particles moving on other $G$-invariant $2p$-dimensional spaces (not necessarily conformally flat), in presence of a $G$-invariant potential. The modifications do not yield any qualitative difference with the proposed reduction procedures and will be reflected in more complicated forms of the resulting Lagrangians. It deserves to be mentioned that the presence of monopoles in the reduced systems (including those with

additional potential terms) do not change their solvability property, neither at the classical, nor at the quantum level. As an example, for $SO(2p)$-invariant systems, the only change in the spectrum after inclusion of the respective monopole is the change in the validity range of the orbital momentum [50]. In supersymmetric systems, on the other hand, the presence of a monopole can change essentially the supersymmetric properties. This question will be considered in one of the next sections.

## 2.2 Hamiltonian Reduction

Let us show how to implement the reduction procedure in Hamiltonian approach. While the Lagrangian language is more convenient for the supersymmetrization we will see that the Hamiltonian one allows us to establish the connection between reduced system and (MICZ)Kepler one.

### 2.2.1 The $U(1)$ reduction

For the description of the reduction procedure related to the first Hopf map it is more convenient instead of 4 real coordinates consider two complex ones: $z_1, z_2$ and their conjugated momenta $\pi_1, \pi_2$. The Poisson brackets have canonical form

$$\{\pi_\alpha, z^\beta\} = 1, \quad \{\bar{\pi}_\alpha, \bar{z}^\beta\} = 1, \qquad \alpha, \beta = 1, 2\ . \tag{2.21}$$

with all others vanishing.

In this coordinates the Hamiltonian corresponding to 2.1 have the following form:

$$H = \frac{\pi_\mu \bar{\pi}_\mu}{g(z\bar{z})} + U, \tag{2.22}$$

where the dependency of the potential $U$ on the coordinates will be considered later.

The symmetry generator corresponding to the syymetry 1.8 looks as foolows:

$$J = \frac{\imath}{2}(\pi z - \bar{z}\bar{\pi}), \tag{2.23}$$

For the case of the first Hopf map the transformation 1.2 can be written in the following way:

$$\mathbf{x} = z\boldsymbol{\sigma}\bar{z}, \quad \mathbf{p} = \frac{z\boldsymbol{\sigma}\pi + \bar{\pi}\boldsymbol{\sigma}\bar{z}}{2(z\bar{z})}, \tag{2.24}$$

where $\boldsymbol{\sigma}$ are the Pauli matrices.

As a result, the reduced Poisson brackets read

$$\{p_i, x^j\} = \delta_i^j, \quad \{p_i, p_j\} = s\frac{\epsilon_{ijk}x^k}{x^3}, \quad x = |\mathbf{x}| \tag{2.25}$$

where $s$ is value of the generator (2.23): $J = s$.

After substitution 2.24 one finds for the Hamiltonian the following expression:

$$H = \frac{x}{2g}\left(\mathbf{p}^2 + \frac{s^2}{x^2}\right) + U(x). \tag{2.26}$$

It is senn from the Poisson brackets 2.25 that the reduced system contains a Dirac monopole in the origin. Indeed, replacing

$$\mathbf{p} \mapsto \mathbf{p} - s\mathbf{A}_D, \quad \mathbf{A}_D = \frac{(x_2, -x_1, 0)}{x(x + x_3)}, \tag{2.27}$$

one will come to the canonical form of Poisson brackets between $x$ and $p$.

### 2.2.2 The $SU(2)$ reduction

Again we start from the Hamiltonian of 8-dimensional free particle system

$$H = \frac{\mathcal{P}^2}{2g} + U, \tag{2.28}$$

defined on a manifold with conform flat metric $g$ and parametrized by coordinates $u$ and their conjugate momentum $\mathcal{P}$.

For the description of the reduction procedure related to the second Hopf map $S^7/S^3 = S^4$ we first introduce five $8 \times 8$ matrices $\Gamma^\mu$

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\delta^{\mu\nu}\mathbf{1}_8 \tag{2.29}$$

with the following relations:

$$\Gamma^1 = \tau_A \otimes \tau_1 \otimes \tau_A, \quad \Gamma^2 = \tau_A \otimes \tau_2 \otimes \tau_A, \quad \Gamma^3 = \tau_A \otimes \tau_A \otimes \mathbf{1}_2,$$

$$\Gamma^4 = \tau_1 \otimes \mathbf{1}_2 \otimes \mathbf{1}_2, \quad \Gamma^5 = \tau_2 \otimes \mathbf{1}_2 \otimes \mathbf{1}_2, \tag{2.30}$$

where

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau_A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.31}$$

and $\{A, B\}$ denotes the anticommutator. For our purposes we have also to introduce three $8 \times 8$ antisymmetric matrices $\Sigma_a$:

$$\Sigma^1 = \frac{1}{2}\mathbf{1}_2 \otimes \tau_A \otimes \tau_1, \quad \Sigma^2 = \frac{1}{2}\mathbf{1}_2 \otimes \tau_A \otimes \tau_2, \quad \Sigma^3 = \frac{1}{2}\mathbf{1}_2 \otimes \mathbf{1_2} \otimes \tau_{\mathbf{A}}. \tag{2.32}$$

which commute with all matrices $\Gamma_\mu$, anticommute with each other and satisfy the $su(2)$ algebra relations:

$$\left[\Gamma^\mu, \Sigma^i\right] = 0, \quad \left\{\Sigma^i, \Sigma^j\right\} = -2\delta^{ij}\mathbf{1}_8, \quad \left[\Sigma^i, \Sigma^j\right] = \varepsilon_{ijk}\Sigma^k \tag{2.33}$$

We also consider functions $x_\mu$ and $p_\mu$, which are connected with

$u_A$ by the following relations

$$x^\mu = u\Gamma^\mu u, \quad p_\mu = \frac{\mathcal{P}\Gamma_\mu u}{u^2} \tag{2.34}$$

where $u$ is an 8 dimensional column vector with elements $u_A$, $\mathcal{P}$ is the corresponding momentum and $u^2 = u^T u$.

One can can notice that the transformation

$$u \to (\lambda_0 \mathbf{1}_8 + \lambda_i \Sigma_i)\, u, \quad \lambda_0^2 + \sum \lambda_i^2 = 1 \tag{2.35}$$

leaves invariant the $x_\mu$ quantities. Therefore, the fibration 2.34 identifies all points which differ by the transformation 2.35. It can be checked explicitly that

$$x_\mu x_\mu \equiv r^2 = (u_A u_A)^2 \equiv R^4. \tag{2.36}$$

Thus, defining the seven- dimensional sphere in $\mathbb{R}^8$ of radius $R$: $\mathbf{u}_\alpha \bar{\mathbf{u}}_\alpha = R^2$, we get a 4-dimensional sphere in $\mathbb{R}^5$ with radius $r = R^2$, i.e. we obtain the second Hopf map. Taking into account the relation 2.36 and the fact that the second relation in 2.35 defines the $S^3$ sphere, one can conclude that the second Hopf map is a fibration of the sphere $S^7$ over $S^3$:

$$S^7/S^3 = S^4$$

In order to invert expressions let us introduce 3 additional coordinates:

$$z = \frac{u_7 - \mathbf{i}u_8}{u_5 - \mathbf{i}u_6}, \quad \bar{z} = \frac{u_7 + \mathbf{i}u_8}{u_5 + \mathbf{i}u_6}, \quad \gamma = \arctan\frac{u_5}{u_6} \tag{2.37}$$

It is easy to see that the coordinates $z, \bar{z}, \gamma$ parametrize the sphere $x_\mu = const.$

The matrices $\Sigma^i$ define a set of functions on $S^3$ that form the $su(2)$ algebra:

$$\mathbf{I}_i = u_A \Sigma^i_{AB} \mathcal{P}_B. \tag{2.38}$$

In terms of the new coordinates these functions can be written as follows:

$$\mathbf{I}_3 = -\frac{p_\gamma}{2} \qquad \mathbf{I}_2 + \mathbf{i}\mathbf{I}_1 = \mathbf{I}_+ = \frac{\mathrm{e}^{-2\imath\gamma}}{4}\left(2\left(1 + z\bar{z}\right)\bar{\pi} + \imath z p_\gamma\right), \qquad \mathbf{I}_- = \bar{\mathbf{I}}_+ : \tag{2.39}$$

We shall need also another $SU(2)$ group elements parameterizing the sphere $S^3$ and commuting with (2.39):

$$\mathbf{P}_3 = \frac{p_\gamma}{2} + \imath\left(z\pi - \bar{z}\bar{\pi}\right), \qquad \mathbf{P}_+ = \frac{1}{2}\left(\bar{\pi} + z^2\pi + \imath\frac{z}{2}p_\gamma\right), \qquad \mathbf{P}_- = \overline{\mathbf{P}}_+ \tag{2.40}$$

These quantities define, with respect to the Poisson brackets

(2.21), the $so(4) = so(3) \times so(3)$ algebra

$$\{P_i, P_j\} = \varepsilon_{ijk} P_k, \quad \{I_i, I_j\} = \varepsilon_{ijk} I_k, \quad \{I_i, P_j\} = 0. \tag{2.41}$$

The functions $P_i$, $I_i$ obey the following equality, which is important for our considerations

$$I_i I_i = P_i P_i. \tag{2.42}$$

Let us perform now the reduction by the action of the $SU(2)$ group given by the functions (2.39). For this purpose we have to fix constants of motion(2.39):

$$I_i = s_i = const, \qquad s_i s_i \equiv s^2.$$

Since $I_i$ are constants of motion independent on the $x_\mu$ coordinates we can perform an orthogonal rotation, so that only the third component of this set, $I_3$, will be different from zero. Equating $I_+$ and $I_-$ with zero we obtain:

$$-I_3 = \frac{p_\gamma}{2} = s, \qquad \bar{\pi} = -s\imath \frac{z}{1 + z\bar{z}}, \quad \pi = s\imath \frac{\bar{z}}{1 + z\bar{z}}. \tag{2.43}$$

Hence,

$$P_+ = -s\frac{\imath z}{1 + z\bar{z}} \equiv \imath h_+, \quad P_- = s\frac{\imath \bar{z}}{1 + z\bar{z}} \equiv \imath h_-, \quad P_3 = s\frac{1 - z\bar{z}}{1 + z\bar{z}} \equiv h_3 \tag{2.44}$$

Thus $P_i$ become precisely the Killing potentials of the $S^2$ sphere! It is not an occasional coincidence, indeed.

After fixing the values of motion integrals the Poisson brackets between quantities $p_\mu$ read:

$$\{p_\mu, p_\nu\} = s\left(\partial_\mu A^i_\nu h_i - \partial_\nu A^i_\mu h_i - \varepsilon_{ijk} h_i A^j_\mu A^k_\nu\right) \equiv s F^i_{\mu\nu} h_i, \quad (2.45)$$

where

$$A^i_\mu = \frac{\eta^i_{\nu\mu} x^\nu}{r\,(r + x_5)} \quad \text{for} \quad \mu = 1,2,3,4, \quad A^i_5 = 0 \quad (2.46)$$

defines the potential of the $SU(2)$ Yang monopole. Here

$$\eta^i_{\nu\mu} = \delta_{i\nu}\delta_{4\mu} - \delta_{4\nu}\delta_{i\mu} + \varepsilon_{4i\nu\mu} \quad (2.47)$$

is t'Hooft symbol.

After substitution Hamiltonian takes the following form:

$$H = \frac{x}{2g}(p^2 + \frac{s^2}{x^2}) + U. \quad (2.48)$$

As it was expected exactly the corrsponding Hamiltonian for 2.20.

# Chapter 3

# Anisotropic Inharmonic Higgs Oscillator and related systems with monopoles

The oscillator and Kepler systems are the best known examples of mechanical systems with hidden symmetries [1]. Due to the existence of hidden symmetry these systems admit separation of variables in few coordinate systems. Despite of their qualitative difference, they can be related with each other in some cases. Namely, $(p+1)-$dimensional Kepler system can be obtained by the appropriate reduction procedures from the $2p-$dimensional oscillator for $p = 1, 2, 4$ (for the review see, e.g. [11]).

These procedures, which are known as Levi-Civita (or Bohlin) [12], Kustaanheimo-Stiefel [13] and Hurwitz [14] transformations

imply the reduction of the oscillator by the action of $Z_2$, $U(1)$, $SU(2)$ group, respectively, and yield, in general case, the Kepler-like systems with monopoles [15, 17, 19]. The second system (with $U(1)$ (Dirac) monopole) is best known and most important among them. It was invented independently by Zwanziger and by McIntosh and Cisneros [16] and presently is refered as MICZ-Kepler system. There are few deformations of oscillator and Kepler systems, which preserve part of hidden symmetries, e.g., anisotropic oscillator, Kepler system with additional linear potential, two-center Kepler system [1], as well as their "MICZ-extensions" [56]. The Kepler system with linear potential is of special importance due to its relevance to the Stark effect. One can observe that the four-dimensional oscillator with additional anisotropic term

$$U_A = \frac{\Delta\omega^2}{2}\sum_{i=1}^{p=2}(x_i^2 - x_{i+p}^2) \tag{3.1}$$

results in the (MICZ-)Kepler system with potential

$$V_{cos} = \frac{\Delta\omega^2}{4}\cos\theta = \frac{\Delta\omega^2}{4}\frac{x_{p+1}}{|\mathbf{x}|}, \tag{3.2}$$

which is the textbook example of the deformed Kepler system admitting the separation of variables in parabolic coordinates. While (three-dimensional) Kepler system with additional linear potential

(which is also separable in parabolic coordinates) is originated in the (four-dimensional) oscillator system with fourth-order anisotropic potential term

$$U_{nlin} = -2\varepsilon_{el} \sum_{i=1}^{p=2} x_i^4 - x_{i+p}^4. \tag{3.3}$$

The corresponding potentials in other dimensions look similarly.

Oscillator and Kepler systems admit the generalizations on a $d-$dimensional sphere and a two-sheet hyperboloid (pseudosphere). They are defined, respectively, by the following potentials [46, 47]

$$U_{osc} = \frac{\omega^2 R_0^2}{2} \frac{\mathbf{x}^2}{x_0^2}, \quad V_{Kepler} = -\frac{\gamma}{R_0} \frac{x_0}{|\mathbf{x}|}, \tag{3.4}$$

where $\mathbf{x}, x_0$ are the Cartesian coordinates of the ambient (pseudo)Euclidean space $\mathbb{R}^{\mathrm{d}+1}(\mathbb{R}^{\mathrm{d}.1})$: $\epsilon\mathbf{x}^2 + x_0^2 = R_0^2$, $\epsilon = \pm 1$. The $\epsilon = +1$ corresponds to the sphere and $\epsilon = -1$ corresponds to the pseudosphere. These systems also possess nonlinear hidden symmetries providing them with the properties similar to those of conventional oscillator and Kepler systems. Various aspects of these systems were investigated in [53]. Let us notice also mention the Ref. [57], where the integrability of the spherical two-center Kepler system was proved.

Completely similar to the planar case one can relate the oscillator and MICZ-Kepler systems on pseudospheres (two-sheet hyperboloids). In the case of sphere, the relation between these systems is

slightly different: the oscillator on sphere results in the oscillator on hyperboloid [51]. After appropriate "Wick rotation" (compare with [54]) of the MICZ-Kepler system on hyperboloid one can obtain the MICZ-Kepler system on the sphere, constructed in [55].

As far as we know, the integrable (pseudo)spherical analogs of the anisotropic oscillator and of the oscillator with nonlinear potential (3.3) were unknown up to now, as well as the (pseudo)spherical analog of the (MICZ-)Kepler system with linear and $\cos\theta$ potential terms. The construction of these (pseudo)spherical systems is not only of the academic interest. They could be useful for the study of the various physical phenomena in nanostructures, as well as in the early Universe. For example, the spherical generalization of the anisotropic oscillator potential can be used as the confining potential restricting the motion of particles in the asymmetric segments of the thin (pseudo)spherical films. While with the (pseudo)spherical generalization of the linear potential at hands one can study the impact of the curvature of space in the Stark effect.

For the simplicity we will start our consideration from $p = 4$ euclidean system. The generalization to higher dimensions is straightforward [78]. The content of this chapter is based on the [64, 38, 78, 18].

## 3.1 Euclidean system

In complex coordinates $z_1, z_2$ and their corresponding momenta $\pi_1, \pi_2$ the Hamiltonian of isotropic oscillator reads

$$\mathcal{H}_0 = \pi\bar{\pi} + \omega^2 z\bar{z}. \tag{3.5}$$

Its rotational symmetry generators are defined by the expressions

$$J = \tfrac{\imath}{2}(\pi z - \bar{z}\bar{\pi}), \tag{3.6}$$

$$\mathbf{J} = \tfrac{\imath}{2}(\pi\boldsymbol{\sigma} z - \bar{z}\boldsymbol{\sigma}\bar{\pi}), \tag{3.7}$$

$$J_{\alpha\beta} = \tfrac{1}{2}\pi_\alpha\bar{z}^\beta, \quad J_{\bar{\alpha}\bar{\beta}} = \tfrac{1}{2}\bar{\pi}_\alpha z^\beta\ , \tag{3.8}$$

and the hidden symmetry generators read

$$\mathbf{A} = \tfrac{1}{2}(\pi\boldsymbol{\sigma}\bar{\pi} + \omega^2\bar{z}\boldsymbol{\sigma} z), \tag{3.9}$$

$$A_{\alpha\beta} = \tfrac{1}{2}(\pi_\alpha\pi_\beta + \omega^2\bar{z}^\alpha\bar{z}^\beta), \quad A_{\bar{\alpha}\bar{\beta}} = \overline{A}_{\beta\alpha} \tag{3.10}$$

Note, that 2.23 is exactly the operator corresponding to the $U(1)$ symmetry of the first Hopf map. The integrable anisotropic inharmonic deformation of this system looks as follows

$$\mathcal{H}_{aosc} = \mathcal{H}_0 + (\Delta\omega^2 + 2\varepsilon_{el} z\bar{z})z\sigma_3\bar{z}. \tag{3.11}$$

Its constants of motion are given by (2.23), by the third component of (3.7), and by the hidden symmetry generator

$$A = A_3 + \frac{\Delta\omega^2}{2}(z\bar{z}) + \frac{\varepsilon_{el}}{2}\left((z\bar{z})^2 + (z\sigma_3\bar{z})^2\right), \tag{3.12}$$

Clearly, the first term in the additional potential decouples the initial isotropic oscillator in the anisotropic one with the frequencies $\omega_\pm = \sqrt{\omega^2 \pm \Delta\omega^2}$. The second part of the deformation term given by (3.11) has no such simple explanation. After transformation of the initial system in the Kepler-like one it results in the linear potential.

Performing the reduction procedure described in the previous setcion for the resulting system we find:

$$H^{red}_{aosc} = \frac{xp^2}{2} + \omega^2 x + \Delta\omega^2 x_3 + \varepsilon_{el} x x_3 \tag{3.13}$$

Following ([51]), we can now transform the reduced oscillator to a Kepler-like system. For this purpose we should fix the energy surface $\mathcal{H}^{red}_{aosc} = E \equiv \gamma/2$ and multiply by $1/x$, to get

$$\left(\mathcal{H}^{red}_{aosc} - E^{red}_{aosc}\right)\frac{1}{x} = 0 \equiv \mathcal{H}_{MICZS} - E_{MICZS}, \quad E_{MICZS} = -\omega^2 \tag{3.14}$$

$$\mathcal{H}_{MICZ} = \frac{1}{2}\left(\mathbf{p}^2 + \frac{s^2}{x^2}\right) + \frac{\gamma}{x} + \Delta\omega^2\frac{x_3}{x} + \varepsilon_{el}x_3 \tag{3.15}$$

For any motion integral $\mathcal{I}$ we have:

$$\{H_{MICZS}, \mathcal{I}\} = \left\{H^{red}_{aosc}, \mathcal{I}\right\}|_{H^{red}_{aosc}=const} = 0 \tag{3.16}$$

Hence, the new Hamiltonian has the same number of motion integrals and therefore preserves the integrability of the initial system.

It is seen that 3.15 defines the MICZ-Kepler system with the additional $\cos\theta$ potential in the presence of constant electric field pointed along $x_3$-axes. For the completeness, let us write down the constants of motion of the constructed system reducing the constants of motion of the four-dimensional oscillator. The $J_3$ results in the corresponding component of angular momentum,

$$J = \mathbf{n}_3\mathbf{J}, \qquad \mathbf{J} = \mathbf{p}\times\mathbf{x} + s\frac{\mathbf{x}}{x}. \tag{3.17}$$

The reduced generator $A$ looks as follows

$$A = \mathbf{n_3}\mathbf{A} + \frac{\varepsilon_{el}}{2}\left(\mathbf{n}_3\times\mathbf{x}\right)^2 + \Delta\omega^2\frac{\left(\mathbf{n}_3\times\mathbf{x}\right)^2}{x} \tag{3.18}$$

where

$$\mathbf{A} = \mathbf{p}\times\mathbf{J} + \gamma\frac{\mathbf{x}}{x} \tag{3.19}$$

is the Runge-Lenz vector of the unperturbed MICZ-Kepler system.

## 3.2 Anisotropic inharmonic Higgs Oscillator

All the calculations in this section we will make for $p = 4$ case, however, the generalization to the higher dimensions is straightforward.

The Hamiltonian of four-dimensional Higgs oscillator looks as follows

$$\mathcal{H}_0^{\epsilon} = \frac{(1+\epsilon z\bar{z})^2 \pi\bar{\pi}}{2R_0^2} + \frac{2\omega^2 R_0^2 z\bar{z}}{(1-\epsilon z\bar{z})^2}. \tag{3.20}$$

The symmetries of (pseudo)sphere are defined by the generators (2.23)-(3.8), and

$$J_\alpha = (1-\epsilon z\bar{z})\pi_\alpha + \varepsilon(\pi z + \bar{\pi}\bar{z})\bar{z}^\alpha, \qquad J_{\bar{\alpha}} = \bar{J}_\alpha. \tag{3.21}$$

It is clear that the generators (2.23)-(3.8) define the $so(4)$ (rotational) symmetry algebra of the Higgs oscillator, while the generators (3.21) define the translations on (pseudo)the sphere. By their use one can construct the generators of hidden symmetries of the Higgs oscillator,

$$A_{\alpha\beta} = \frac{J_\alpha J_\beta}{2R_0^2} + 2\omega^2 R_0^2 \frac{\bar{z}^\alpha \bar{z}^\beta}{(1-\epsilon z\bar{z})^2}, \quad I_{\bar{\alpha}\bar{\beta}} = \bar{I}_{\alpha\beta} \tag{3.22}$$

and

$$\mathbf{A} = \frac{(J\boldsymbol{\sigma}\bar{J})}{2R_0^2} + 2\omega^2 R_0^2 \frac{(z\boldsymbol{\sigma}\bar{z})}{(1-\epsilon z\bar{z})^2}. \tag{3.23}$$

Let us construct the integrable (pseudo)spherical analog of the anisotropic inharmonic oscillator (3.11). We consider the class of Hamiltonians

$$\mathcal{H}^{\epsilon}_{aosc} = \mathcal{H}^{\epsilon}_{0} + (z\sigma_3\bar{z})\Lambda(z\bar{z}), \tag{3.24}$$

which besides the symmetries defined by the generators $J$ and $J_3$, possess the hidden symmetry defined by the constant of motion

$$A = A_3 + g(z\bar{z}) + (z\widehat{\sigma}_3\bar{z})^2 h(z\bar{z}). \tag{3.25}$$

Here $\Lambda(z\bar{z})$, $g(z\bar{z})$ and $h(z\bar{z})$ are some unknown functions, and $A_3$ is the third component of (3.23).

Surprisingly, from the requirement that $A$ is the constant of motion, we uniquely (up to constant parameters) define the functions $\Lambda, g, h$, i.e. find the integrable anisotropic generalization of Higgs oscillator. Namely, the function $\Lambda$ in (3.24) reads

$$\Lambda \equiv \frac{2R_0^2\Delta\omega^2}{(1+\epsilon z\bar{z})^2} + \frac{8\varepsilon_{el}R_0^4}{(1-(z\bar{z})^2)^2}\frac{(1+(z\bar{z})^2)(z\bar{z})}{(1-\epsilon z\bar{z})^2}, \tag{3.26}$$

and the hidden symmetry generator looks as follows

$$A = A_3 + \frac{2R_0^2\Delta\omega^2 z\bar{z}}{(1+\epsilon z\bar{z})^2} +$$

$$+4\varepsilon_{el}R_0^4\left(\frac{(z\bar{z})^2}{(1-(z\bar{z})^2)^2}+\frac{(z\widehat{\sigma}_3\bar{z})^2}{(1-\epsilon z\bar{z})^4}\right). \tag{3.27}$$

One can easily see that the constructed system results in (3.11) results in the limit $R_0 \to \infty$.

*Hence, we have got the well-defined (pseudo)spherical generalization of (3.11).*

In coordinates $y$ of the ambient space the potential of the constructed system looks much simpler. The potential of (isotropic) Higgs reads

$$U_{Higgs}=\frac{\omega^2 R_0^2}{2}\frac{R_0^2-y_0^2}{y_0^2}, \tag{3.28}$$

while the anisotropy terms is defined by the expression

$$U_{AI}=\left(\frac{\Delta\omega^2}{2}+\epsilon\varepsilon_{el}R_0^2\frac{(R_0^4-y_0^4)}{y_0^4}\right)\mathbf{y}\widehat{\sigma}_3\bar{\mathbf{y}} \tag{3.29}$$

Let us mention that for the above construction the dimensionality of the space is essential. The same reasonings for any number of real coordinates (instead of two complex z)and any orthogonal matrix (instead of Hermitean $\sigma_3$) will lead us to the similar oscillator in corresponding dimension. In particular, for the $n = 8$ one can use $\Gamma_5$ defined in 2.30 as an orthogonal matrix one can find an integrable 8-dimensional anisotropic inharmonic Higgs oscillator which can be reduced following the procedure described in the second chapter.

## 3.3 MICZ-Kepler-like systems on pseudosphere

In this Section performing Kustaanheimo-Stiefel transformation of the constructed system we shall get the pseudospherical analog of the Hamiltonian (3.15). This procedure is completely similar to those of the isotropic Higgs oscillator [51].

At first, we must reduce the system by the Hamiltonian action of the generator (2.23). Choosing the functions (2.24) as the reduced coordinates, and fixing the level surface $J = s$, we shall get the six-dimensional phase space equipped by the Poisson brackets (2.25). Then we fix the energy surface of the oscillator on the (pseudo)sphere, $\mathcal{H}^{\epsilon}_{aosc} = E_{aosc}$, and multiply it by $(1-\epsilon x^2)^2/x^2$. As a result, the energy surface of the reduced system takes the form

$$\mathcal{H}^{-}_{AMICZ} = \mathcal{E}^{-}_{AMICZ}, \tag{3.30}$$

where

$$\mathcal{H}^{-}_{AMICZ} = \frac{(1-x^2)^2}{8r_0^2}(\mathbf{p}^2 + \frac{s^2}{x^2}) - \frac{\gamma}{2r_0}\frac{1+x^2}{x} + \\ + \frac{\Delta\omega^2}{2}\left(\frac{1-\epsilon x}{1+\epsilon x}\right)^2 \frac{x_3}{x} + 2\varepsilon_{el} r_0 \frac{1+x^2}{1-x^2}\frac{x_3}{1-x^2}, \tag{3.31}$$

$$r_0 = R_0^2, \quad \gamma = \frac{E_{aosc}}{2}, \quad \mathcal{E}^{-}_{AMICZ} = -\frac{\omega^2}{2} + \epsilon\frac{E_{aosc}}{2r_0}. \tag{3.32}$$

Interpreting $\mathbf{x}$ as the stereographic coordinates of three-dimensional

pseudosphere

$$\mathbf{y} = r_0 \frac{2\mathbf{x}}{1-x^2}, \quad y_0 = r_0 \frac{1+x^2}{1-x^2}, \tag{3.33}$$

we conclude that (3.31) defines the pseudospherical analog of the MICZ-Kepler system with linear and $\cos\theta$ potential terms (3.15). It is clear that in quantum mechanical consideration the presence linear term will lead to the analogue of Stark effect.

The constants of motion of the anisotropic oscillators, $J_3$ and $A$ yield, respectively, the third component of angular momentum (3.17) and the hidden symmetry generator

$$A = \mathbf{n}_3\mathbf{A} + \frac{r_0\Delta\omega^2}{(1+\epsilon x)^2}\left[\frac{x^2-x_3^2}{x}\right] + 2\varepsilon_{el}r_0^2\frac{x^2-x_3^2}{(1-x^2)^2} \tag{3.34}$$

where

$$\mathbf{A} = \frac{\mathbf{T}\times\mathbf{J}}{2r_0} + \gamma\frac{\mathbf{x}}{x}$$

is the Runge -Lenz vector of the MICZ-Kepler system on pseudosphere, $\mathbf{J}$ is the generator of the rotational momentum defined by the expression (3.17), and

$$\mathbf{T} = \left(1+x^2\right)\mathbf{p} - 2(\mathbf{xp})\,\mathbf{x}. \tag{3.35}$$

is translation generator.

This term also looks simply in Euclidean coordinates of ambient

space:

$$V_{AI} = \frac{\Delta\omega^2}{2}\left(\frac{y_3}{y} + \epsilon y_0 y_3\right) + \varepsilon_{el} y_0 y_3 \tag{3.36}$$

Let us notice, that the term proportional to $\Delta\omega^2$ depends on $\epsilon$, i.e., formally, the anisotropic terms yield different pseudospherical generalizations of potential $\cos\theta$. However, this difference is rather trivial: it is easy to observe, that one potential transforms in other one upon spatial reflection.

Presented Kepler-like system admits the separation of variables in the following generalization of parabolic coordinates (compare with [52]):

$$x_1 + ix_2 = \frac{2\sqrt{\xi\eta}}{r_0 + \frac{\sqrt{\sqrt{(r_0^2+\xi^2)(r_0^2+\eta^2)}+\xi\eta+r_0^2}}{\sqrt{2}}} \mathrm{e}^{\imath\varphi},$$

$$x_3 = \frac{\sqrt{2}\sqrt{\sqrt{(r_0^2+\xi^2)(r_0^2+\eta^2)}-\xi\eta-r_0^2}}{r_0 + \frac{\sqrt{\sqrt{(r_0^2+\xi^2)(r_0^2+\eta^2)}+\xi\eta+r_0^2}}{\sqrt{2}}}. \tag{3.37}$$

In these coordinates the metric reads

$$ds^2 =$$

$$= r_0^2 \frac{\xi+\eta}{4}\left(\frac{d\xi^2}{\xi(r_0^2+\xi^2)} + \frac{d\eta^2}{\eta(r_0^2+\eta^2)}\right) + \xi\eta d\varphi^2. \tag{3.38}$$

Passing to the canonical momenta, one can represent the Hamilto-

nian (3.31) as follows

$$\mathcal{H}^{-}_{MICZ} = \frac{2\xi(r_0^2+\xi^2)}{r_0^2(\xi+\eta)}p_\xi^2 + \frac{2\eta(r_0^2+\eta^2)}{r_0^2(\xi+\eta)}p_\eta^2 + \frac{1}{\xi\eta}\frac{p_\varphi^2}{2} +$$

$$\frac{sp_\varphi + s^2}{r_0(\xi+\eta)}\left(\frac{r_0+\sqrt{r_0^2+\xi^2}}{\xi} + \frac{r_0-\sqrt{r_0^2+\eta^2}}{\eta}\right) +$$

$$+\frac{\Delta\omega^2 r_0}{2}\frac{\xi\sqrt{r_0^2+\xi^2} - \eta\sqrt{r_0^2+\eta^2} + \xi^2 - \eta^2}{\xi+\eta} -$$

$$-\frac{\gamma}{r_0}\frac{\sqrt{r_0^2+\xi^2}+\sqrt{r_0^2+\eta^2}}{\xi+\eta} + \varepsilon_{el}\frac{\xi-\eta}{2} \qquad (3.39)$$

So, the corresponding generating function has to have the additive form- $S = \mathcal{E}_{AMICZ}t + p_\varphi\varphi + S_1(\xi) + S_2(\eta)$ . Replacing $p_\xi$ and $p_\eta$ by $dS_1(\xi)/d\xi$ and $dS_2(\eta)/d\eta$ respectively, we obtain the following ordinary differential equations

$$\frac{2\xi(r_0^2+\xi^2)}{r_0^2}\left(\frac{dS_1(\xi)}{d\xi}\right)^2 + (sp_\varphi + s^2)\frac{r_0+\sqrt{r_0^2+\xi^2}}{r_0\xi}$$

$$+\frac{\Delta\omega^2 r_0}{2}(\xi\sqrt{r_0^2+\xi^2}+\xi^2) -$$

$$-\frac{\gamma}{r_0}\sqrt{r_0^2+\xi^2} + \varepsilon_{el}\xi^2 - \mathcal{E}_{AMICZ}\xi + \frac{p_\varphi^2}{\xi} = \beta \qquad (3.40)$$

$$\frac{2\eta(r_0^2+\eta^2)}{r_0^2}\left(\frac{dS_2(\eta)}{d\eta}\right)^2 + (sp_\varphi + s^2)\frac{r_0-\sqrt{r_0^2+\eta^2}}{r_0\eta} +$$

$$-\frac{\Delta\omega^2 r_0}{2}(\eta\sqrt{r_0^2+\eta^2}+\eta^2) -$$

$$-\frac{\gamma}{r_0}\sqrt{r_0^2+\eta^2}-\varepsilon_{el}\eta^2-\mathcal{E}_{AMICZ}\eta+\frac{p_\varphi^2}{\eta}=-\beta \tag{3.41}$$

From these equations we can immediately find the explicit expression for the generating function. We have separated the variables for the pseudospherical generalization of the Coulomb system with linear and $\cos\theta$ potential.

The above equations looks much simpler in the new coordinates $(\chi,\zeta)$, where $\xi=r_0\sinh\chi$, $\eta=r_0\sinh\zeta$.

$$\left(\frac{dS_1(\chi)}{d\chi}\right)^2=\frac{\mathcal{E}_{AMICZ}}{2}-\frac{\Delta\omega^2 r_0^4}{2}(\cosh\chi+\sinh\chi)+$$

$$+(\frac{\gamma r_0}{2}-s^2-sp_\varphi)\coth\chi-\frac{\varepsilon_{el}r_0^3}{2}\sinh\chi-\frac{p_\varphi^2}{2\sinh^2\chi}+$$

$$+\frac{\beta r_0-s^2-sp_\varphi}{2\sinh\chi}, \tag{3.42}$$

$$\left(\frac{dS_2(\zeta)}{d\zeta}\right)^2=\frac{\mathcal{E}_{AMICZ}}{2}+\frac{\Delta\omega^2 r_0^4}{2}(\cosh\zeta+\sinh\zeta)+$$

$$+(\frac{\gamma r_0}{2}+s^2+sp_\varphi)\coth\zeta+\frac{\varepsilon_{el}r_0^3}{2}\sinh\zeta-\frac{p_\varphi^2}{2\sinh^2\zeta}-$$

$$-\frac{\beta r_0+s^2+sp_\varphi}{2\sinh\zeta}. \tag{3.43}$$

**Remark 2.** In the same manner the $2p-$dimensional anisotropic inharmonic oscillator on (pseudo)sphere can be connected to the $(p+1)-$dimensional Kepler-like systems on pseudosphere also for

the $p = 1, 4$ . For $p = 1$ we should just assume that $z^\alpha$ are *real* coordinates. In this case we should not perform any reduction at the classical level (in quantum case we have to reduce the initial system by the discreet $Z_2$ group action, see [15]). For the $p = 4$ we have to assume, that $z^\alpha$ are *quaternionic* coordinates (equivalently, that $z^\alpha$ are complex coordinates with $\alpha = 1, \dots 4$). In contrast with $p = 1, 2$ cases, we should reduce the initial system by the $SU(2)$ group action [19].

**Remark 3.** The planar (MICZ)-Kepler system with linear potential can be obtained as a limiting case of the two-center (MICZ-) Kepler system, when one of the forced centers is placed at infinity (see, e.g. [1]). The two-center (pseudo)spherical Kepler system is the integrable system as well [57]. However, presented pseudospherical generalization of the (MICZ-)Kepler system with linear potential could not be obtained from the two-center pseudospherical Kepler system: it can be easily checked, that in contrast with pseudospherical Kepler potential, it does not obey the corresponding Laplas equation.

### 3.3.1 Transition to the sphere

To get the spherical counterpart of the Hamiltonian (3.31), let us perform its "Wick rotation" which yields

$$\mathcal{H}^{+} = \mathcal{H}_0^{+} + 2\varepsilon_{el} r_0 \frac{1-x^2}{1+x^2}\frac{x_3}{1+x^2} + \\ + \frac{\Delta\omega^2}{2}\left(\frac{1-i\epsilon x}{1+i\epsilon x}\right)^2 \frac{x_3}{x}, \tag{3.44}$$

where

$$\mathcal{H}_0^{+} = \frac{(1+x^2)^2}{8r_0^2}\left(\mathbf{p}^2 + \frac{s^2}{x^2}\right)^2 - \gamma\frac{1-x^2}{2r_0 x} \tag{3.45}$$

is the Hamiltonian of unperturbed MICZ-Kepler system on the sphere. The hidden symmetry of this system is defined by the expression

$$A = \mathbf{n}_3\mathbf{A} + \Delta\omega^2 \left[\frac{x^2-x_3^2}{(1+i\epsilon x)^2 x}\right] + 2\varepsilon_{el} r_0^2 \frac{x^2-x_3^2}{(1+x^2)^2}. \tag{3.46}$$

where

$$\mathbf{A} = \mathbf{J}\times\mathbf{T} + \gamma\frac{\mathbf{x}}{x} \tag{3.47}$$

is Runge-Lenz vector of the spherical MICZ-Kepler system, with the angular momentum $\mathbf{J}$ given by (3.17) and with the translation generator

$$\mathbf{T} = \left(1-x^2\right)\mathbf{p} + 2(\mathbf{xp})\mathbf{q}. \tag{3.48}$$

One can see, that due to the last term in (3.44) this Hamiltonian is a complex one. Taking its real part we shall get the integrable spherical analog of the MICZ-Kepler system with linear and $\cos\theta$ potentials,

$$\mathcal{H}^{+}_{MICZ} = \mathcal{H}^{+}_{0} + \frac{\Delta\omega^2}{2}\frac{1-6x^2+x^4}{1+x^2}\frac{x_3}{x} + \\ +2\varepsilon_{el}\frac{1-x^2}{1+x^2}\frac{x_3}{1+x^2}. \tag{3.49}$$

The generator of its hidden symmetry is also given by the real part of (3.46)

$$A = \mathbf{n_3 A} + \left[\Delta\omega^2 r_0 \frac{1-x^2}{x} + \frac{\varepsilon_{el}}{2}\right]\frac{x^2-x_3^2}{(1+x^2)^2}. \tag{3.50}$$

In the terms of ambient space $\mathbb{R}^4$ the anisotropy term is defined by the expression (3.29).

**Remark 4.** It is clear from our consideration, that the addition to the constructed system of the potential

$$c_0 \ \mathrm{Im}\ \left(\frac{1-i\epsilon x}{1+i\epsilon x}\right)^2 \frac{x_3}{x} \tag{3.51}$$

will also preserve the integrability. The hidden symmetry generator will be given by the expression

$$A + c_0 \,\mathrm{Im}\frac{\Delta\omega^2}{2(1+i\epsilon x)^2}\left[\frac{x^2 - x_3^2}{x}\right]. \tag{3.52}$$

However, it is easy to see, that this additional potential coinsides with (3.44), i.e. we do not get anything new in this way.

### 3.3.2 Dipole transitions and Stark effect in the charge-dyon system

One can investigate the influence of Dirac monopole on the Coulumb system. For this reason let us consider the dipole transitions in the MICZ-Kepler system interacting with planar monochromatic electromagnetic wave, which are completely similar to the ones in the "dyogen atom" [58].

$$\mathcal{H} = \frac{(\mathbf{p}-\mathbf{A})^2}{2} + \frac{s^2}{2r^2} - \frac{1}{r} \approx \mathcal{H}_{\mathrm{MIC}} - \mathbf{Ap}, \tag{3.53}$$

where $\mathcal{H}_{\mathrm{MIC}}$ is defined by (3.15) and

$$\mathbf{A} = A_0 \mathbf{u}\cos(\omega t - \mathbf{kr}) \,, \quad \nabla\cdot\mathbf{A} = 0 \tag{3.54}$$

is the vector potential of the wave.

Following the procedure described in [61] for the matrix elements

of dipole transitions we find

$$\mathbf{ud}_{n,l,m|n',l',m'} = I(n,l|n',l')\left[\frac{u_x+\imath u_y}{2}\left(\frac{l+1}{2(2l+1)}\sqrt{(l+m)(l^2-s^2)}\delta_{m-1|m'}\delta_{l-1|l'} + \right.\right.$$

$$\left.+\frac{\sqrt{(l+1)(l-m+1)(l-m+2)((l+1)^2-s^2)}}{2\sqrt{l+2}(l+1)(2l+2)}\delta_{m-1|m'}\delta_{l+1|l'} + s\frac{\sqrt{(l-m+1)(l+m)}}{l(l+1)}\delta_{m-1|m'}\delta_{l|l'}\right)$$

$$-\frac{u_x-\imath u_y}{2}\left(\frac{l+2}{2(2l+3)}\sqrt{(l+m+2)((l+1)^2-s^2)}\delta_{m+1|m'}\delta_{l+1|l'}\right.$$

$$\left.-\sqrt{\frac{l}{l+1}}\frac{\sqrt{(l-m-1)(l-m)(l^2-s^2)}}{l(2l-1)}\delta_{m+1|m'}\delta_{l-1|l'} + s\frac{\sqrt{(l+m+1)(l-m)}}{l(l+1)}\delta_{m+1|m'}\delta_{l|l'}\right) +$$

$$u_z\left(\frac{\sqrt{(l+1)(l^2-m^2)(l^2-s^2)}}{\sqrt{l}l(2l+1)}\delta_{l-1|l'} + \right.$$

$$\left.\left.\frac{\sqrt{(l+1)((l+1)^2-m^2)((l+1)^2-s^2)}}{\sqrt{l+2}(l+1)(2l+1)}\delta_{l+1|l'} + s\frac{m}{l(l+1)}\delta_{l|l'}\right)\delta_{m|m'}\right] \tag{3.55}$$

where

$$I(n,l|n',l') = \int_0^\infty c_{nl}c^*_{n'l'}r^{l+l'}\mathrm{e}^{-(\frac{r}{n'}+\frac{r}{n})}F(l-n+1,2l+2,\frac{2r}{n})F(l'-n'+1,2l'+2,\frac{2r}{n})r^3dr, \tag{3.56}$$

and

$$c_{nl} = \frac{2^l}{n^{l+2}(2l+1)!}\sqrt{\frac{(2l+1)(n+l)!}{\pi(n-l-1)!}} \tag{3.57}$$

are the normalization constants of non-perturbed charge-dyon systems. It is seen that the presence of Dirac monopole changes the

selection rules of the system. Namely, in the absence of monopole one has

$$u_z \neq 0 \quad : \quad m = m',\ l' = l - 1 \tag{3.58}$$

$$|u_x + \imath u_y| \neq 0 \quad : \quad m' = m \pm 1\ , l' = l \pm 1\ ; \qquad m' = m \pm 1\ , l' = l \tag{3.59}$$

In the presence of Dirac monopole, when $s \neq 0$ another transitions are also possible [58, 59]

$$u_z \neq 0 \quad : \ m = m',\ l' = l \tag{3.60}$$

$$|u_x + \imath u_y| \neq 0 \quad : \ m' = m \pm 1\ , l' = l \tag{3.61}$$

So, the presence of monopole makes the selection rules less rigorous. Namely, besides (3.59), the transitions preserving the orbital quantum number $l$ become also allowed, (3.61). When the electromagnetic wave has transversal polarization ($u_z = 0$), the transitions preserving the orbital quantum number, and changing the azimuth quantum number become possible. When longitudal mode in the electromagnetic wave appears ($u_z \neq 0$), the transitions, preserving both orbital and azimuth quantum numbers are also admissible.

The other useful example is the Stark effect in the system with presence of Dirac monopole.The Hamiltonian of the MICZ-Kepler system (interpreting as charge-dyon system) in the external con-

stant uniform electric field is of the form:

$$\mathcal{H}_{\text{Stark}} = \mathcal{H}_{MIC} + \mathbf{Er}. \tag{3.62}$$

To find the energy spectrum we shall follow the procedure described in [62]. For this reason let us assume that the electic field is directed along the $x_3$ axes and pass to the parabolic coordinate systems. After the separation of variables in the Schrödinger equation for the Hamiltonian (3.62) we arrive at the system [60]

$$\begin{aligned}
&\frac{d}{d\xi}\left(\xi\frac{d\Phi_1}{d\xi}\right) + \left[\frac{\mathcal{E}}{2}\xi - \frac{|\mathbf{E}|}{4}\xi^2 - \frac{(m+s)^2}{4\xi}\right]\Phi_1 = -\beta_1\Phi_1, \\
&\frac{d}{d\eta}\left(\eta\frac{d\Phi_2}{d\eta}\right) + \left[\frac{\mathcal{E}}{2}\eta + \frac{|\mathbf{E}|}{4}\eta^2 - \frac{(m-s)^2}{4\eta}\right]\Phi_2 = -\beta_2\Phi_2, \quad \beta_1 + \beta_2 = 1
\end{aligned} \tag{3.63}$$

For $s = 0$ these equations coincide with the similar equations for the hydrogen atom in the parabolic coordinates [62]. Hence, similar to that, we can consider the energy $\mathcal{E}$ as a fixed parameter, and $\beta_{1,2}$ as the eigenvalues of corresponding operators. These quantities are defined after solving the above equations, as the functions on $\mathcal{E}$ and $\mathbf{E}$. Then, due to the relation $\beta_1 + \beta_2 = 1$, the energy $\mathcal{E}$ becomes a function on the external field $\mathbf{E}$.

It is seen from the above expressions, that the calculation of the first and second order corrections to the $\beta_{1,2}^{(0)}$ will be completely similar to the ones in the Coulomb problem [62], if one replaces

$|m| \rightarrow |m+s|$ in $\beta_1$, $\Phi_1$, and $|m| \rightarrow |m-s|$ in $\beta_2$, $\Phi_2$. These substitutions yields the following expressions

$$\beta_a^{(1)} = -\frac{(-1)^a|\mathbf{E}|}{4\kappa^2}\left(6n_a^2 + 6n_a|m_a| + m_a^2 + 6n_a + 3|m_a| + 2\right) \quad (3.64)$$

$$\beta_a^{(2)} = -\frac{|\mathbf{E}|^2}{16\kappa^5}\left(|m_a| + 2n_a + 1\right)\left(4m_a^2 + 17(2|m_a|n_a + 2n_a^2 + \right.$$
$$\left. + |m_a| + 2n_a) + 18\right). \quad (3.65)$$

Then we get

$$\beta_1^0 + \beta_2^{(0)} = \kappa n, \quad \beta_1^{(1)} + \beta_2^{(1)} = \frac{3|\mathbf{E}|}{2\kappa^2}A, \quad \beta_1^{(2)} + \beta_2^{(2)} = -\frac{|\mathbf{E}|^2}{16\kappa^5}B, \quad (3.66)$$

where we introduce the notations

$$A \equiv nn_- - \frac{ms}{3}, \quad B \equiv 17n^3 - 3nn_-^2 + 54An_- + 19n - 9n(m^2 + s^2), \quad (3.67)$$

and the quantum numbers

$$n = n_1 + n_2 + \frac{|m+s| + |m-s|}{2} + 1\ , \quad n_- \equiv n_1 - n_2 + \frac{|m+s| - |m-s|}{2}. \quad (3.68)$$

Taking into account, that $\beta_1 + \beta_2 = 1$, we get

$$\kappa n + \frac{3|\mathbf{E}|A}{2\kappa^2} - \frac{|\mathbf{E}|^2 B}{16\kappa^5} = 1 \quad (3.69)$$

Iteratively solving this equation, we get

$$\kappa = \kappa_0 + |\mathbf{E}|\kappa_1 + |\mathbf{E}|^2\kappa_2 \,, \qquad \kappa_0 = \frac{1}{n}, \quad \kappa_1 = -\frac{3An}{2}, \quad \kappa_2 = n^3\left(\frac{Bn}{16} - \frac{9A^2}{2}\right). \tag{3.70}$$

Then, from $E = -\kappa^2/2$ we find the energy of the system

$$\mathcal{E} = -\frac{1}{2n^2} + \frac{3|\mathbf{E}|}{2}\left(nn_- - \frac{ms}{3}\right) - \frac{|\mathbf{E}|^2 n^2}{16}\left(17n^4 - 3(nn_- - 3ms)^2 - \right.$$

$$\left. -9n^2m^2 + 19n^2 - 9n^2s^2 + 21(ms)^2\right) \tag{3.71}$$

One can represent the quantum numbers (3.68) as follows

$$n = \begin{cases} n_1 + n_2 + |s| + 1 & \text{for} \quad |m| \le |s| \\ n_1 + n_2 + |m| + 1 & \text{for} \quad |m| > |s| \end{cases},$$

$$n_- = \begin{cases} n_1 - n_2 + m \text{ sgn } s & \text{for} \quad |m| \le |s| \\ n_1 - n_2 + s \text{ sgn } m & \text{for} \quad |m| > |s| \end{cases}. \tag{3.72}$$

The ground state of the non-perturbed charge-dyon system corresponds to the following values of quantum numbers: $n_1 = n_2 = 0$, $|m| \le |s|$. Hence,

$$n = |s| + 1, \quad n_- = m \text{ sgn } s \,, \quad m = -|s|, -|s| + 1, \ldots, |s| - 1, |s|. \tag{3.73}$$

Substituting these expressions in (3.71), we get

$$\mathcal{E}_0 = -\frac{1}{2(|s|+1)^2} + m \text{ sgn } s \, |\mathbf{E}| \left( |s| + \frac{3}{2} \right) -$$

$$-\frac{|\mathbf{E}|^2(|s|+1)^2}{16} \left[ 17(|s|+1)^4 + (|s|+1)^2(19-9s^2) - 6m^2 \, (|s|+2) \right]. \tag{3.74}$$

It is seen, that the ground state of the non-perturbed charge-dyon system has $(2|s|+1)$-fold degeneracy (by azimuth quantum number $m$), while the linear Stark effect completely removes the degeneracy on $m$. It is proportional to the azimuth quantum number $m$, while its sign depends on the relative sign of monopole number $s$ and $m$ (the linear Stark effect in the "dyogen atom" possesses similar properties [59]). In contrast to linear Stark effect, the quadratic Stark effect of the ground state is independent neither on sign $s$, no on sign $m$.

# Chapter 4

# Calogero model

The Calogero model [20, 21, 22] and its various extensions and generalizations play a distinguished role among other multi-particle integrable systems. They have attracted much attention due to their rich internal structure and numerous applications in many areas of physics (see, e.g., the recent review [23] and references therein).

In the continuum or thermodynamic limit, i.e. for large particle numbers, the Calogero model gives rise to a Yang-Mills theory [24] on a cylinder, while its superconformal extension describes a black hole in the near-horizon limit [25]. In this limit, the system have soliton solutions corresponding to the fundamental excitations [26].

The quantum Calogero model describes free particles with fractional statistics whose type is determined by the interaction strength [27]. Moreover, the variational ground state of the fractional quan-

tum Hall effect (known as the Laughlin state [28]) can be considered as some deformation of the ground state of Calogero model [29]. The trigonometric analogue of the model [30] is related to integrable spin-1/2 chains with long range interactions, which possess a resonating-valence-bond ground state [31]. Recently, the relation to the Benjamin-Ono equation arising in the hydrodynamics of stratified fluids has been established [32].

The Calogero model and its modifications appear also in matrix models [33], $W_\infty$-algebras [34], Yangian quantum groups [35], random matrices [36] and many other areas of physics and mathematics.

The content of this chapter is based on the [45, 72, 73].

## 4.1 Cuboctahedric Higgs oscillator from the rational Calogero model

In this chapter, we will study the classical rational Calogero system without confining potential[72]. It describes one-dimensional particles with inverse-square interaction [20, 21, 22]:

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{i<j} \frac{g}{(x_i - x_j)^2}, \qquad \{p_i, x_j\} = \delta_{ij}. \tag{4.1}$$

One of the important features of the system is its manifest conformal invariance, which was essential for the invention of the model, as well as for its further studies.

In the pioneering paper [20], the three-particle model had been considered first. After excluding the center of mass (with á priory conserving momentum) and taking into account the conformal invariance, the model was reduced to a one-dimensional exactly solvable system on circle considered by Jacobi in the middle of XIX century [37]:

$$\mathcal{I} = \frac{p_\varphi^2}{2} + \frac{9g}{2\cos^2 3\varphi}. \tag{4.2}$$

For more particles, the analysis of the Calogero model becomes more complicated. In particular, the construction of the complete set of the constants of motion assumes the use of the powerful method of Lax pair [22]. This approach allowed to relate the Calogero system to $A_{N-1}$ Lie algebras, as well as to construct its integrable modifications related to other Lie algebras [39]. The Calogero systems can be obtained from the free-particle system by an appropriate reduction procedure known as the projection method [1]. Recently, it has been generalized to the Calogero model extensions corresponding to the root systems [40].

However, the analog of the system (4.2) has not been properly

studied for the case of more than three particles. Such a study would be an interesting problem from few viewpoints.

Already in the pioneering papers [20, 21] it was observed that the spectrum of the Calogero model with additional oscillator potential is similar to the spectrum of free $N$-dimensional oscillator. It was claimed there that a similarity transformation to the free-oscillator system may exist, at least, in the part of Hilbert space. However, this transformation has been written explicitly only three decades later [41]. In Ref. [42], it has been related to the conformal group $SU(1,1)$. This similarity transformation has a very transparent geometric explanation for the two-particle Calogero model (the "conformal mechanics"): it corresponds to the inversion in the Klein model of the Lobachevsky space, which describes the phase space of the system. A natural way to extend this picture to the multi-particle Calogero system is to identify the coordinates of its "radial" part with the coordinates of the Klein model. In other words, one must extract and investigate the angular part of the system. The other stimulation for the study of the angular part of the Calogero model the translation of the discrete symmetries of the one-dimensional multi-particle system to the higher-dimensional one-particle one. This would provide us with á priori integrable higher-dimensional one-particle system with some discrete symmetry.

### 4.1.1 Center-of-mass system

For our purpose we need to introduce $N$-dimensional vectors $b^{ij}$ as follows:

$$b_k^{ij} = \frac{1}{\sqrt{2}} \left( \delta_{i_k} - \delta_{jk} \right). \tag{4.3}$$

They satisfy to the relations

$$\sum_k (b_k^{ij})^2 = 1 \quad and \quad (\mathbf{e}, b^a) = 0, \tag{4.4}$$

where

$$\mathbf{e} = (1, 1, ...., 1),$$

and round brackets define scalar product. After this denotation, we can rewrite the Hamiltonian 4.1 as follows.

$$\mathcal{H}_N = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + \sum_{a=1}^{N(N-1)/2} \frac{g}{2 \left( \sum_{k=1}^{N} b_k^a x_k \right)^2}, \qquad \{p_i, x_j\} = \delta_{ij}, \tag{4.5}$$

where $a \equiv (i, j)$ is $N(N-1)/2$-valued index, which enumerates pairs of interacting particles, $p_i$ are the corresponding momenta. The second relation in 4.4 means that all vectors $b^a$ lie in a hypersurface, which is orthogonal to the vector $\mathbf{e}$. This mean, that the set of vectors is not $N$-dimensional one and can be putted in $N-1$ dimensional space by an orthogonal space rotation using an appropriate matrix $A_{ik}$. Dynamically, this rotation is equivalent to

the transition to the center of mass system. Let us write down the explicit formula for such rotation.

$$A_{km} = \begin{cases} 1/\sqrt{N} & \text{for} \quad m = 1 \\ -1/\sqrt{(N-m+2)(N-m+1)} & \text{for} \quad k \geq m > 1 \\ \sqrt{N-k}/\sqrt{N-k+1} & \text{for} \quad m = k+1 \\ 0 & \text{for} \quad \text{the rest} \end{cases} \tag{4.6}$$

Here and further, we will use this definition for matrix $A_{ik}$, if not specially mentioned. After performing the rotation we have:

$$\mathcal{H}_N = \frac{1}{2}\sum_{i=1}^{N-1} p_i^2 + \sum_{a=1}^{N(N-1)/2} \frac{g}{2\left(\sum_{k=2}^{N} b_k^a y_k\right)^2}, \qquad \{p_i, y_j\} = \delta_{ij}, \tag{4.7}$$

Where $y_k$ are new coordinates and $p_i$ corresponding momenta. It is clear, that the expression (4.7) defines the constant of motion of the Calogero model. It can be considered as the Hamiltonian of some $(N-1)$-dimensional system. From the orthogonality of the matrix $A_{ik}$ we have:

$$\cos\alpha_{ij,i'j'} = \sum_k b_k^{ij} b_k^{i'j'} = \frac{1}{2}\sum_k (\delta_{ik}-\delta_{jk})(\delta_{i'k}-\delta_{j'k}) = \frac{1}{2}(\delta_{ii'}+\delta_{jj'}-\delta_{ij'}-\delta_{i'j}), \tag{4.8}$$

the quantities $\mathbf{b}^a = (b^a_1, \ldots, b^a_{N-1})$ are unit vectors in $(N-1)$-dimensional space, and $\alpha_{ij,i'j'}$ are the angles between them. Hence, the angle between two arbitrary vectors can take the following values only

$$\alpha_{ij,i'j'} = \frac{\pi}{3}, \quad \frac{2\pi}{3}, \quad \frac{\pi}{2}. \tag{4.9}$$

The reduced system can be interpreted as the one-particle system in $(N-1)$-dimensional space. Let us extract the radius $r$ of the obtained $(N-1)$-dimensional system. This could be done, for instance, in $(N-1)$-dimensional spherical coordinates. In these terms, the Hamiltonian of the Calogero model looks as follows

$$\mathcal{H} = \frac{p_r^2}{2} + \frac{\mathcal{I}_{N-2}(p_{\varphi_\alpha}, \varphi_\alpha)}{r^2},$$

with

$$\mathcal{I}_{N-2} = \frac{K_{\text{sph}}}{2} + \sum_a \frac{g}{2\cos^2\theta_a}, \qquad \{p_{\varphi_\alpha}, \varphi_\alpha\} = \delta_{\alpha\beta}, \qquad \alpha, \beta = 2, \ldots, N-1. \tag{4.10}$$

Here $K_{\text{sph}}$ is the standard kinetic term of the particle on $(N-2)$-dimensional sphere with unit radius, and $\theta_a$ is the angle between $\mathbf{b}_a$ and the unit vector directed from the center of the sphere to the particle, $\mathbf{n} = \mathbf{r}/r$. Since $\mathcal{I}_{N-2}$ is independent from $p_r$ and $r$, it commutes with the Hamiltonian $\mathcal{H}_{N-1}$. So, it is a constant of

motion of the Calogero model. Note that this integral is quadratic on the momenta (while in the standard Lax-pair based approach the only constant of motion, which is quadratic on momenta, is the Hamiltonian). It can be considered as the Hamiltonian of the particle moving on the $(N-2)$-dimensional sphere with $N(N-1)/2$ force centers defined by the vectors $\mathbf{b}^a$. Since this system is invariant under reflections $\mathbf{b}^a \to -\mathbf{b}^a$ for any $a$, sometimes it is reasonable to consider the $N(N-1)$ properly located force centers.

In order to clarify the physical meaning of the obtained system, let us rewrite its potential as follows

$$V_{\rm sph} = \sum_a \frac{g}{2\cos^2\theta_a} = \frac{N(N-1)g}{4} + \frac{g}{2}\sum_a \tan^2\theta_a \ .$$

Let us remind that the potential

$$V_{\rm Higgs} = \frac{\omega^2 r_0^2 \tan^2\theta}{2}$$

is well-known potential of the Higgs oscillator. It defines the generalization of the oscillator potential to the sphere with the radius $r_0$, which inherits all hidden symmetries of ordinary oscillator [47].

*Hence, we obtained the integrable $N(N-1)/2$-center $N$-dimensional Higgs oscillator of the frequency $\omega = \sqrt{g}$.* The location of the force centers is quite rigid, and deserves to be considered in more details.

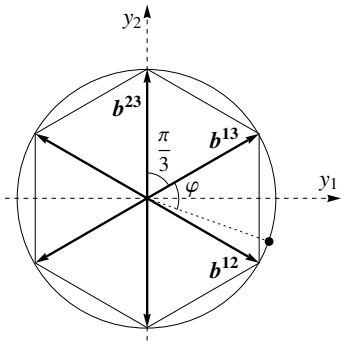


Figure 4.1: The force centers ($\mathbf{b}^{12}$, $\mathbf{b}^{23}$, $\mathbf{b}^{13}$ and their opposites), which form the root system of $su(3)$ and constitute an hexagon. The angle $\varphi$ describes the position of a particle on cycle.

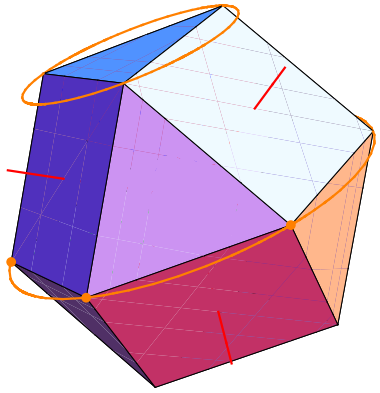

Figure 4.2: The vectors (4.16) together with their opposites form a cuboctahedron and are equivalent to the root system of $su(4)$. The bold points on the large cycle correspond to $\mathbf{b}^{23}$, $\mathbf{b}^{34}$ and $\mathbf{b}^{24}$ while the small cycle contains the vertexes of the remaining three vectors. The bold lines are the axes of the coordinate system (4.18).

Note that the Higgs oscillator has been invented about thirty years ago and has been studied the hundreds of papers so far (see, e.g. [63] and refs therein). Nevertheless, its anisotropic version was found quite recently [64], whereas its two-center version is not known yet, up to our knowledge.

### 4.1.2 Three-particle case: circle

The simplest system is the angular part of three-particle model considered in the pioneering paper by Calogero [20]. Actually, this system was considered in the middle of XIX by Jacobi [37] (see also

[65]). For $N = 3$, we get a particle on circle $S^1$ with three force centers defined by the unit vectors $\mathbf{b}^{12}$, $\mathbf{b}^{23}$ and $\mathbf{b}^{13}$. The angles between them are equal to $\pi/3$ and $2\pi/3$ (Fig. 4.1):

$$\cos\alpha_{12,13} = \cos\alpha_{13,23} = 1/2, \qquad \cos\alpha_{12,23} = -1/2.$$

The above vectors make up the set of positive roots of $A_2 \equiv su(3)$ Lie algebra. Completing them by the oppositely directed vectors corresponding to the negative roots, we obtain a system with six force centers. The angular part of the Hamiltonian

$$\mathcal{I} = \frac{p_\varphi^2}{2} + \frac{g}{2\cos^2\varphi} + \frac{g}{2\cos^2(\varphi+\pi/3)} + \frac{g}{2\cos^2(\varphi-\pi/3)} = \frac{p_\varphi^2}{2} + \frac{9g}{1+\cos 6\varphi} \tag{4.11}$$

coincides with (4.2). It is invariant under the rotation on $\pi/3$ and the reflection $\varphi \to -\varphi$, which generate the symmetry group $D_6 \equiv S_3 \otimes Z_2$ of the hexagon (Fig. 4.1). Here $S_3$ is the symmetric group of three-particle permutations, which $\mathcal{I}$ inherits from the original Calogero Hamiltonian (4.1). The $Z_2$-symmetry corresponds to the reflection-invariance $x_i \to -x_i$ of (4.1). The integrability of this system is obvious. Note that the splitting of the three-particle Calogero Hamiltonian on the angular and radial parts has been used in Ref. [68] for the detailed study of the quantization.

Let us briefly discuss the relation of the system on circle with the

superintegrability of three-particle Calogero model. Note that its superintegrability was studied in detail (see [74] and refs. therein). In the center-of-mass system, three from the five functionally independent constants of motion of the original Calogero system survive. Namely, the Hamiltoinian of the two-particle system $\tilde{\mathcal{H}} = p_r^2/2 + \mathcal{I}/r^2$ and its constant of motion

$$\mathcal{F} = \left(p_r^2 - \frac{6\mathcal{I}}{r^2}\right) p_r \sin 3\varphi + \left(3p_r^2 - \frac{2\mathcal{I}}{r^2}\right) \frac{p_\varphi \cos 3\varphi}{r} \qquad (4.12)$$

are reduced from the second and third order (on momentum) Liouville constants of motion. Similarly, the third conserved quantity

$$\mathcal{K} = \left(p_r^2 - \frac{6\mathcal{I}}{r^2}\right) p_r p_\varphi \cos 3\varphi - \left(3p_r^2 - \frac{2\mathcal{I}}{r^2}\right) \frac{2\mathcal{I} \sin 3\varphi}{r} \qquad (4.13)$$

is inherited from the additional third order constant of motion of three-particle Calogero system. The integrals $\tilde{\mathcal{H}}$, $\mathcal{F}$, and $\mathcal{K}$ are functionally independent. We have expressed them in terms of the angular part of the Hamiltonian (4.11), which also conserves. Its Poisson bracket action maps the Liouville integral to the additional one:

$$\{\mathcal{I}, \mathcal{F}\} = 3\mathcal{K}, \qquad \{\mathcal{I}, \mathcal{K}\} = -6\mathcal{I}\mathcal{F}. \qquad (4.14)$$

The four quantities $\tilde{\mathcal{H}}$, $\mathcal{I}$, $\mathcal{F}$, and $\mathcal{K}$ form an overcompleted set of

constants of motion. They and subjected to the algebraic relation

$$\mathcal{K}^2 + 2\mathcal{I}\mathcal{F}^2 = 8\tilde{\mathcal{H}}^3(2\mathcal{I} - 9g), \qquad \text{or} \qquad \mathcal{I} = \frac{\mathcal{K}^2 + 72g\tilde{\mathcal{H}}^3}{16\tilde{\mathcal{H}}^3 - 2\mathcal{F}^2}. \quad (4.15)$$

Hence, one can choose $\tilde{\mathcal{H}}$, $\mathcal{I}$, and $\mathcal{F}$ as a complete set of functionally independent conserved quantities. The first two of them are quadratic on momenta, which ensures the separation of variables in the system.

It is easy to verify that the Poisson brackets (4.14) are in consistency with the relation (4.15). Finally, using (4.15) and the first equation in (4.14), we obtain the Poisson bracket between two third-order integrals:

$$\{\mathcal{K}, \mathcal{F}\} = 3(8\tilde{\mathcal{H}}^3 - \mathcal{F}^2) = 3\frac{\mathcal{K}^2 + 9g\mathcal{F}^2}{2\mathcal{I} - 9g}.$$

### 4.1.3 Four-particle system: sphere

In the four-particle case, everything becomes much more complicated. In the same way, we obtain a system on the sphere with *six* force centers defined by the unit vectors $\mathbf{b}^a$ with the following Cartesian coordinates of the ambient $\mathbb{R}^3$ space:

$$\mathbf{b}^{12} = \left(\sqrt{\frac{2}{3}},\ -\frac{1}{\sqrt{3}},\ 0\right), \qquad \mathbf{b}^{13} = \left(\sqrt{\frac{2}{3}},\ \frac{1}{2\sqrt{3}},\ -\frac{1}{2}\right),$$
$$\mathbf{b}^{14} = \left(\sqrt{\frac{2}{3}},\ \frac{1}{2\sqrt{3}},\ \frac{1}{2}\right), \qquad \mathbf{b}^{23} = \left(0,\ \frac{\sqrt{3}}{2},\ -\frac{1}{2}\right), \qquad (4.16)$$
$$\mathbf{b}^{24} = \left(0,\ \frac{\sqrt{3}}{2},\ \frac{1}{2}\right), \qquad \mathbf{b}^{34} = (0,\ 0,\ 1).$$

The vertexes of $\mathbf{b}^{ij}$ and their opposite vectors form an Archimedean solid called ***cuboctahedron*** (Fig.4.2). This polyhedron, like cube, has the octahedral symmetry $O_h \equiv S_4 \otimes Z_2$ of order 48. Here $S_4$ is the symmetric group of four-particle permutations, which preserve the original Calogero Hamiltonian (4.1). Note that $S_4$ is isomorphic to the Weyl group of $A_3$ Lie algebra and preserves the orientation of cuboctahedron. The $Z_2$ symmetry corresponds to the reflection $x_i \to -x_i$ of all four coordinates. In Lie algebraic description, it corresponds to the reflection symmetry of $A_3$ Dynkin diagram.

Note that the vectors $\mathbf{b}^{23}$, $\mathbf{b}^{34}$ and $\mathbf{b}^{24}$ belong to the "equatorial" plane, the angles between them are equal to $\pi/3$ and $2\pi/3$. Their vertexes and the vertexes of the opposite vectors form an hexagon (Fig. 4.2). This is precisely the same picture as in the three-particle Calogero model (see Fig. 4.1). The endpoints of the vectors $\mathbf{b}^{12}$, $\mathbf{b}^{13}$, $\mathbf{b}^{14}$ are located on a plane parallel to the equatorial one (Fig.

4.2). The distance between both planes is $\sqrt{2/3}$. They form the (regular) triangular face of the cuboctahedron, which is shifted by the angle $\pi/6$ with respect to the triangle $(\mathbf{b}^{23}, \mathbf{b}^{34}, -\mathbf{b}^{24})$.

Let us choose Cartesian coordinates with the first axis directed along $\mathbf{b}^{13}$ while the second one belonging to the plane formed by $\mathbf{b}^{12}$ and $\mathbf{b}^{13}$. The frame directions then are orthogonal to the triangles of the cuboctahedron (Fig. 4.2). In the respective spherical coordinates, the angular part of the Hamiltonian reads:

$$\mathcal{I} = \frac{p_\theta^2}{2} + \frac{p_\varphi^2}{2\sin^2\theta} + \frac{9g(8-\tan^2\theta)^2}{2(3\tan^2\theta - 8 + \tan^3\theta\cos 3\varphi)^2} + \\ + \frac{12g}{3\tan^2\theta - 8 + \tan^3\theta\cos 3\varphi} + \frac{9g}{4\sin^2\theta(1+\cos 6\varphi)}. \qquad (4.17)$$

The invariance under $Z_3$ group of the rotation on $2\pi/3$ along the third axis is apparent.

The potential (4.17) is really horrible. It is difficult to believe, that the system with such potential could be integrable, or could admit a separation of variables. However, the Hamiltonian can be represented in a much simpler form. Indeed, there are three pairs of the orthogonal vectors $\mathbf{b}^{12}\cdot\mathbf{b}^{34} = \mathbf{b}^{13}\cdot\mathbf{b}^{24} = \mathbf{b}^{14}\cdot\mathbf{b}^{23} = 0$. Taking the vector products of these pairs, one can find out that they form

an orthogonal frame:

$$\mathbf{a}_1 \equiv \mathbf{b}^{12}\times\mathbf{b}^{34}, \qquad \mathbf{a}_2 \equiv \mathbf{b}^{13}\times\mathbf{b}^{24}, \qquad \mathbf{a}_3 \equiv \mathbf{b}^{14}\times\mathbf{b}^{23} \ : \qquad \mathbf{a}_i\cdot\mathbf{a}_j = \delta_{ij}. \tag{4.18}$$

The vectors $\mathbf{a}_i$ are normal to the squares of the cuboctahedron (Fig.4.2). In this coordinate system, the Hamiltonian (4.7) looks like

$$\tilde{\mathcal{H}} = \sum_{i=1}^{3}\frac{p_i^2}{2} + \sum_{1\le i<j\le 3}\left(\frac{g}{(u_i-u_j)^2}+\frac{g}{(u_i+u_j)^2}\right), \qquad \{p_i,u_j\}=\delta_{ij}, \tag{4.19}$$

where, again, we keep the old notations for the new momenta. This is the three-particle $D_3$ Calogero model [1]. However, this is an expected result, since the diagrams $D_3$ and $A_3$ coincide and define the same algebra (in the Dynkin classification, $D_n$ is defined for $n \ge 4$).

The angular part of this Hamiltonian has the following form:

$$\mathcal{I} = \frac{p_\theta^2}{2} + \frac{p_\varphi^2}{\sin^2\theta} + \frac{4g}{\sin^2\theta}\left[\frac{1}{1+\cos 4\varphi} + \frac{k-6}{k-8+8/k-k\cos 4\varphi} + \right.$$
$$\left. + \frac{4(k-16+16/k)}{(k-8+8/k-k\cos 4\varphi)^2}\right] \tag{4.20}$$

where

$$k = \tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}.$$

In these new spherical coordinates, the invariance under $Z_4$ rotations $\varphi \to \varphi + \pi/4$ and spatial reflections $\theta \to \pi - \theta$, which are a subgroup in $O_h$, is transparent.

As was explained above and showed explicitly for three-particle system, $\mathcal{I}$ can be expressed in terms of the five integrals of the maximally superintegrable Hamiltonian (4.19). It seems that the two additional integrals of $\tilde{\mathcal{H}}$ can be obtained from the Liouville integrals by Poisson bracket action with $\mathcal{I}$ like in the three-particle case (4.14).

Since the spherical system (4.20) was obtained from the Calogero model, it is also integrable. Its constants of motion can be obtained from those of the original model.

## 4.2 Action-angle variables for dihedral systems on the circle

In this chapter we construct the action-angle variables for the dihedral systems on a circle, which are defined by the Hamiltonian

$$H(p,q) = \mathcal{I}(p_\varphi, \varphi|k) \;=\; \tfrac{1}{2}p_\varphi^2 \;+\; V_k(\varphi), \qquad (4.21)$$

with canonical variables $\{p_\varphi, \varphi\} = 1$ and the potential

$$V_k(\varphi) \;=\; \sum_{\ell=0}^{k-1} \frac{1}{(\mathbf{a}_\ell \cdot \mathbf{n})^2} \qquad \text{where} \quad \mathbf{n} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \tag{4.22}$$

and $\mathbf{a}_\ell$ are the positive roots of a two-dimensional Coxeter system $I_2(k)$ called dihedral system. The full set of roots forms a regular star shape with an angular separation of $\pi/k$. Since the symmetry relates the root lengths as $|\mathbf{a}_\ell|^2 = |\mathbf{a}_{\ell+2}|^2$, for odd $k$ all roots have the same length, say $\alpha_0$, while for even $k$ we may put $|\mathbf{a}_{\text{even}}| = \alpha_1$ and $|\mathbf{a}_{\text{odd}}| = \alpha_2$. Clearly, we have to distinguish between $k$ being even or odd. As $\mathbf{a}_\ell \cdot \mathbf{n}$ is proportional to $\cos(\phi - \frac{\ell\pi}{k})$, it is a matter of simple algebra to perform the finite sums and obtain

$$V_k(\varphi) \;=\; \frac{k^2\alpha_0^2}{2\cos^2 k\varphi} \qquad\qquad \text{for } k = 2k'+1, \tag{4.23}$$

$$V_k(\varphi) \;=\; \frac{(k'\alpha_1)^2}{2\cos^2 k'\varphi} \;+\; \frac{(k'\alpha_2)^2}{2\sin^2 k'\varphi} \qquad \text{for } k = 2k', \tag{4.24}$$

with $k' \in \mathbb{N}$. Hence, the odd systems feature one coupling ($\alpha_0$), while the even ones allow for two ($\alpha_1$, $\alpha_2$), all naturally positive. For $\alpha_1$=$\alpha_2$, the even potential attains the same form as the odd one.

Formulating these systems in terms of action-angle-variables, we shall find that both types are locally equivalent to the free particle

on the circle. Besides, we establish a global equivalence between systems (4.23) and (4.24) for $\alpha_0 = \alpha_1 + \alpha_2$ and $2k_{\rm odd} = k_{\rm even}$. We shall demonstrate that these systems are equivalently quantized in their action-angle variables or initial coordinates. We shall also present a supersymmetrization of the action-angle variable scheme for the dihedral systems and its relation to the supergeneralization of the Liouville theorem.

Finally, we shall shall enlarge the configuration space to $\mathbb{R}^2$ by adding a radial degree of freedom to the circular motion. The ensueing two-dimensional systems represent three-particle rational Calogero models after separation of their center-of-mass motion. For small values of $k$, the Coxeter roots belong to a rank-two Lie algebra $\mathcal{G}$, which labels the corresponding Calogero model [39]:

| $k$ | 2 | 3 | 4 | 6 |
|---|---|---|---|---|
| $\mathcal{G}$ | $D_2 = A_1 \oplus A_1$ | $A_2$ | $BC_2$ | $G_2$ |

(4.25)

In particular, this shall allow us to prove the global equivalence of the $A_2$ and $G_2$ rational Calogero models and their local equivalence to a free particle in the plane.

### 4.2.1 Action-angle variables

In this Section we construct the action-angle variables for the systems defined by the potentials (4.23) and (4.24). We follow the general prescription given in [2]. Being one-dimensional, our systems feature the Hamiltonian as their single constant of motion. To construct the action variable, we should fix the level surface of the Hamiltonian, $H(p_\varphi, \varphi) = \mathcal{I}(I) = h$, and introduce the generating function $S(h, \varphi)$ for the canonical transformation $(p_\varphi, \varphi) \mapsto (I, \Phi)$ via

$$S(h,\varphi) = \int_{\varphi_0}^{\varphi} p_\varphi(h,\varphi')\, d\varphi' = \int_{\varphi_0}^{\varphi} \sqrt{2(h - V_k(\varphi'))}\, d\varphi'. \qquad (4.26)$$

The full period integral yields the action variable,

$$I(h) = \frac{1}{2\pi} \oint p_\varphi(h,\varphi')\, d\varphi' = \frac{1}{2\pi} \oint \sqrt{2(h - V_k(\varphi'))}\, d\varphi', \quad (4.27)$$

while the angular variable $\Phi$ arises from

$$\Phi(h,\varphi) = \frac{\partial S}{\partial I} = \frac{dh}{dI}\frac{\partial S}{\partial h} = 2\pi \int_{\varphi_0}^{\varphi} \frac{d\varphi'}{\sqrt{2(h - V_k(\varphi'))}} \Big/ \oint \frac{d\varphi'}{\sqrt{2(h - V_k(\varphi'))}}. \qquad (4.28)$$

The parity of parameter $k$ does not play any role in our derivation.[1] Surely, in the limit $\alpha_2 \to 0$ the system (4.24) looks like

[1] Formally, $k$ need not even be an integer. In such a case, however, the system lives on the infinite cover $\mathbb{R}$ of the circle.

system (4.23). In our construction, however, it is essential to keep both $\alpha_1$ and $\alpha_2$ non-vanishing. For this reason we shall derive the action-angle variables of the even and odd systems separately.

**Systems with odd $k$**

Inserting (4.23) into (4.27), we obtain

$$I = \frac{\sqrt{2}}{\pi} \int_{\varphi_-}^{\varphi_+} d\varphi' \sqrt{h - \frac{k^2\alpha_0^2}{2\cos^2 k\varphi'}}, \tag{4.29}$$

where the reflection points $\varphi_\pm(h)$ follow from

$$2h\, \cos^2 k\varphi_\pm = k^2\alpha_0^2. \tag{4.30}$$

Calculating the definite integral (4.29), we find

$$I = \frac{1}{k}\sqrt{2h} - \alpha_0 \qquad \Rightarrow \qquad \mathcal{I} = \frac{k^2}{2}(I + \alpha_0)^2 \tag{4.31}$$

and thus get

$$\frac{dh}{dI} = k^2(I + \alpha_0) = k\sqrt{2h}. \tag{4.32}$$

At the potential mimimum ($\varphi$=0), we have $I = 0$ but $h = h_{\min} = \frac{1}{2}k^2\alpha_0^2$. To compute the angular variable $\Phi$ we employ (4.28) with

$\varphi_0$=0 and get

$$\Phi = \frac{dh}{dI}\frac{1}{k\sqrt{2h}}\int_0^{x(\varphi)}\frac{dx'}{\sqrt{1-x'^2}} = \arcsin x(\varphi) \qquad \text{where} \quad x(\varphi) := \frac{\sqrt{2h}}{\sqrt{2h-k^2\alpha_0^2}}\sin k\varphi. \tag{4.33}$$

Hence, the canonical transformation to the action-angle variables looks as follows,

$$(p_\varphi,\varphi) \mapsto \Big(I = \tfrac{1}{k}\sqrt{2\mathcal{I}(p_\varphi,\varphi)}-\alpha_0\ ,\ \Phi = \arcsin\Big\{\tfrac{\sqrt{2\mathcal{I}(p_\varphi,\varphi)}}{\sqrt{2\mathcal{I}(p_\varphi,\varphi)-k^2\alpha_0^2}}\sin k\varphi\Big\}\Big), \tag{4.34}$$

where $\mathcal{I}(p_\varphi,\varphi)$ is given by (4.21) and (4.23). When the particle makes one cycle (the variable $x$ runs from $-1$ to 1 and back), the variable $\Phi$ advances by $2\pi$ as expected. In these variables the Hamiltonian is given by the second expression in (4.31). For completeness, the inverse transformation $(I,\Phi) \mapsto (p_\phi,\phi)$ reads

$$\varphi = \frac{1}{k}\arcsin\Big\{\tfrac{\sqrt{I^2+2I\alpha_0}}{I+\alpha_0}\sin\Phi\Big\}, \qquad p_\varphi = k(I+\alpha_0)\sqrt{\tfrac{I^2+2I\alpha_0}{(I+\alpha_0)^2+(\alpha_0\tan\Phi)^2}}. \tag{4.35}$$

Performing the trivial canonical transformation $(I,\Phi) \mapsto (\tilde{I} = I+\alpha_0,\Phi)$, we get

$$\mathcal{I} = \frac{k^2}{2}\tilde{I}^2 \quad \text{with} \quad \{\tilde{I},\Phi\} = 1, \qquad \text{where} \quad \Phi\in[0,2\pi) \quad \text{and} \quad \tilde{I}\in[\alpha_0,\infty). \tag{4.36}$$

This system can be interpreted as a free particle particle of mass

$k^2$ moving on a circle with unit radius. Equivalently, it describes a free particle of unit mass moving on a circle with radius $1/k$. However, we can speak about *local* equivalence only, since the above redefinition changes the domain of the action variable from $[0,\infty)$ to $[\alpha_0,\infty)$!

**Systems with even $k$**

For the case (4.24), i.e. $k = 2k'$, the action variable is slightly harder to compute,

$$I(h) = \frac{\sqrt{2}}{\pi}\int_{\varphi_-}^{\varphi_+} d\varphi' \sqrt{h - \frac{k'^2\alpha_1^2}{2\cos^2 k'\varphi'} - \frac{k'^2\alpha_2^2}{2\sin^2 k'\varphi'}} = \frac{\sqrt{2h}a^2}{k\pi}\int_{-1}^{1}\frac{\sqrt{1-x^2}\,dx}{1-(ax+b)^2}, \tag{4.37}$$

where

$$a = \sqrt{1 - \frac{k'^2(\alpha_1^2+\alpha_2^2)}{h} + \frac{k'^4(\alpha_1^2-\alpha_2^2)^2}{4h^2}}, \qquad b = \frac{k'^2(\alpha_2^2-\alpha_1^2)}{2h},$$

$$x = \frac{1}{a}\big[\cos 2k'\varphi' - b\big], \tag{4.38}$$

and the turning points $\varphi_\pm(h)$ derive from

$$2h\,\sin k'\varphi_\pm \cos k'\varphi_\pm = k'^2(\alpha_1^2\tan k'\varphi_\pm + \alpha_2^2\cot k'\varphi_\pm) \qquad \text{with} \quad \alpha_1,\alpha_2 > 0. \tag{4.39}$$

The last integral in (4.37) can be calculated by standard methods (see Appendix) [75]:

$$\int\limits_{-1}^{1} \frac{\sqrt{1-x^2}\,dx}{1-(ax+b)^2} = \frac{\pi}{2a^2}\Big(2-\sqrt{(b-1)^2-a^2}-\sqrt{(b+1)^2-a^2}\Big), \tag{4.40}$$

thus,

$$I = \frac{1}{k'}\sqrt{2h}-(\alpha_1+\alpha_2) \qquad \Rightarrow \qquad \mathcal{I} = \frac{k'^2}{2}\big(I+(\alpha_1+\alpha_2)\big)^2. \tag{4.41}$$

Similarly, the angular variable becomes

$$\Phi = \tfrac{1}{2}\arcsin\Big\{\tfrac{1}{a}\big[\cos 2k'\varphi + b\big]\Big\}, \tag{4.42}$$

where $a$ and $b$ are defined by the expressions (4.38), and $h$ should be replaced by $\mathcal{I}(p_\varphi,\varphi)$. In these variables the Hamiltonian is displayed by the second expression in (4.41). The inverse transformations $(I,\Phi)\mapsto(p_\varphi,\varphi)$ looks as follows,

$$\phi = \frac{1}{2k'}\arccos\Big\{a\sin 2\Phi - b\Big\}, \qquad p_\varphi = k'\sqrt{(I+\alpha_1+\alpha_2)^2-\tfrac{2\alpha_1^2}{1+b+a\sin 2\Phi}-\tfrac{2\alpha_2^2}{1+b-a\sin 2\Phi}}, \tag{4.43}$$

where the quantities $a$ and $b$ take the form

$$a = \sqrt{\Big[1-\big(\tfrac{\alpha_1+\alpha_2}{I+\alpha_1+\alpha_2}\big)^2\Big]\Big[1-\big(\tfrac{\alpha_1-\alpha_2}{I+\alpha_1+\alpha_2}\big)^2\Big]}, \qquad b = \frac{k'^2(\alpha_2^2-\alpha_1^2)}{(I+\alpha_1+\alpha_2)^2}. \tag{4.44}$$

Similar to the odd case, we perform a trivial canonical transformation $(I, \Phi) \mapsto (\tilde{I} = I + \alpha_1 + \alpha_2, \Phi)$ and arrive at

$$\mathcal{I} = \frac{k'^2}{2}\tilde{I}^2 \quad \text{with} \quad \{\tilde{I}, \Phi\} = 1, \qquad \text{where} \quad \Phi \in [0, 2\pi) \quad \text{and} \quad \tilde{I} \in [\alpha_1 + \alpha_2, \infty). \tag{4.45}$$

So, also the even system (4.24) is *locally* equivalent to a free particle of mass $k'^2$ moving on the circle with unit radius (or a free particle of unit mass moving on the circle with radius $1/k'$). Like in the odd case, the equivalence is not global, since the above shift of the action variable changes its range from $[0, \infty)$ to $[\alpha_1 + \alpha_2, \infty)$.

Comparing the results (4.36) and (4.45), obtained by a canonical transformation from (4.23) and (4.24), respectively, we conclude that they differ in the "mass" of the (locally equivalent) free particle as well as in the domain of the momentum (action) variable. Thus, in general, all systems can be distinguished globally. Interestingly, however, any odd system $(k_{\text{odd}}; \alpha_0)$ matches *globally* to a one-parameter family system of even systems $(k_{\text{even}}; \alpha_1, \alpha_2)$ by the equivalence

$$(k_{\text{odd}}; \alpha_0) \;\sim\; (2k_{\text{odd}}; \beta, \alpha_0 - \beta) \qquad \text{with} \quad 0 < \beta < \alpha_0. \tag{4.46}$$

**Quantization**

In the action-angle variables $(I, \Phi)$ derived above it is quite simple to quantize the dihedral systems (4.23) and (4.24) à la Bohr-Sommerfeld:

$$I \mapsto \widehat{I} = \frac{\hbar}{\imath}\frac{\partial}{\partial\Phi}, \qquad \Psi_n = \tfrac{1}{\sqrt{2\pi}}\mathrm{e}^{\imath n\Phi} \quad \text{for} \quad n \in \mathbb{Z} \qquad \Rightarrow \qquad \widehat{I}\,\Psi_n(\Phi) = n\hbar\,\Psi_n. \tag{4.47}$$

The energy spectra of the Hamiltonians (4.31) and (4.41) then read, respectively,

$$E_n(k\,\text{odd}) = \tfrac{1}{2}k^2(n\hbar{+}\alpha_0)^2 \qquad \text{and} \qquad E_n(k\,\text{even}) = \tfrac{1}{8}k^2(n\hbar{+}\alpha_1{+}\alpha_2)^2. \tag{4.48}$$

This agrees with the literature [69, 70], where the Schrödinger equation for our potential (4.23) or (4.24) is known as the (first) Pöschl-Teller equation, whose (normalizable) solutions are given in terms of trigonometric and hypergeometric functions.

### 4.2.2 Supersymmetric extension

The supergeometric generalization of the Liouville theorem has been known for many years [71]. For our context of one-dimensional supersymmetric mechanics, we follow here the construction of action-angle (super)variables as presented in [76]. Let us have $\mathcal{N}{=}2M$

one-dimensional supersymmetric mechanics defined on a $(2|2M)$-dimensional phase superspace, coordinatized by $(p_\varphi, \varphi|\theta^\alpha, \overline{\theta}^\beta)$. The supersymmetry algebra reads

$$\{Q^\alpha, \overline{Q}^\beta\} = 2\delta^{\alpha\beta} H_s \qquad \text{and} \qquad \{Q^\alpha, H_s\} = \{\overline{Q}^\beta, H_s\} = 0$$

$$\{Q^\alpha, Q^\beta\} = \{\overline{Q}^\alpha, \overline{Q}^\beta\} = 0 \qquad \text{with} \quad \alpha, \beta = 1, \ldots, M. \tag{4.49}$$

Here, the Hamiltonian $H_s$ differs from the previous $H$ by nilpotent terms. Fixing the level super-surface, $H_s = h_s$, $Q^\alpha = q^\alpha$ and $\overline{Q}^\alpha = \overline{q}^\alpha$, we arrive at a $(1|0)$-dimensional circle in the phase superspace. On this circle, one defines bosonic action-angle variables $(\Phi_s, \tilde{I}_s)$, analogous to the non-supersymmetric case, as well as fermionic ones, $\Theta^\alpha = Q^\alpha/\sqrt{2h_s}$, with the following non-zero Poisson brackets [2]

$$\{\Phi_s, \tilde{I}_s\} = 1 \qquad \text{and} \qquad \{\Theta^\alpha, \overline{\Theta}^\beta\} = \delta^{\alpha\beta}. \tag{4.50}$$

In these variables, the Hamiltonian does not depend on $\Theta^\alpha$ or $\overline{\Theta}^\alpha$, hence $H_s = \mathcal{I}_s(\tilde{I}_s)$ just like previously. Nevertheless, the canonical transformation from the initial to the action-angle supervariables does mix bosonic and fermionic degrees of freedom.

Let us demonstrate the procedure for the simplest case of $\mathcal{N}$=2,

[2] The tilde indicates that the action variable has been shifted as in the previous section, depending on $k$ being odd or even.

given by the classical counterpart of Witten's model of supersymmetric mechanics [79]. It is defined by

$$H_s = \tfrac{1}{2}\big(p_\varphi^2 + W'^2(\varphi)\big) + \imath\theta\overline{\theta}\, W''(\varphi) \qquad \text{and} \qquad Q = \theta\,\big(p_\varphi + \imath W'(\varphi)\big), \tag{4.51}$$

$$\overline{Q} = \overline{\theta}\,\big(p_\varphi - \imath W'(\varphi)\big),$$

with a chosen superpotential function $W(\varphi)$. These functions obey the superalgebra (4.49) with $M$=1, by virtue of

$$\{p_\varphi, \varphi\} = 1 \qquad \text{and} \qquad \{\theta, \overline{\theta}\} = 1. \tag{4.52}$$

Quantization replaces $\theta$ and $\overline{\theta}$ by the Pauli matrices $\boldsymbol{\sigma}_+ = \frac{1}{2}(\boldsymbol{\sigma}_1 + \imath\boldsymbol{\sigma}_2)$ and $\boldsymbol{\sigma}_- = \frac{1}{2}(\boldsymbol{\sigma}_1 - \imath\boldsymbol{\sigma}_2)$, respectively, and $\imath\theta\overline{\theta}$ goes to $\boldsymbol{\sigma}_3$. In this way we arrive at one-dimensional $\mathcal{N}$=2 supersymmetric quantum mechanics of a spinning particle interacting with an external field. However, when passing to action-angle variables it turns out that there is no spin interaction, and the supersymmetric extension is rather trivial. On the other hand, Witten's model is quite special: its supercharges allow no momentum dependence in the nilpotent part of the Hamiltonian. For a more interesting system related to our potentials (4.23) and (4.24), let us choose a more flexible form

of the supercharges, namely

$$Q = \theta\,\tilde{k}\tilde{I}\,\mathrm{e}^{\imath\lambda(\tilde{I},\Phi)} = \sqrt{2h_s}\,\Theta \qquad \text{and} \qquad \overline{Q} = \overline{\theta}\,\tilde{k}\tilde{I}\,\mathrm{e}^{-\imath\lambda(\tilde{I},\Phi)} = \sqrt{2h_s}\,\overline{\Theta}, \tag{4.53}$$

where we defined $\tilde{k} := k$ for $k$ odd and $\tilde{k} := k/2 = k'$ for $k$ even, and $\lambda(\tilde{I},\Phi)$ is an arbitrary real function of the action-angle variables of the underlying bosonic system. By expressing $(\tilde{I},\Phi)$ through $(p_\varphi,\varphi)$, the supercharges are functions of the initial phase superspace variables. These supercharges also generate the superalgebra (4.49) (with $M$=1) and produce the Hamiltonian

$$H_s := \tfrac{1}{2}\{Q,\overline{Q}\} = \tfrac{1}{2}\tilde{k}^2\tilde{I}^2 + \imath\theta\overline{\theta}\,\tilde{k}^2\tilde{I}\,\frac{\partial\lambda(\tilde{I},\Phi)}{\partial\Phi}. \tag{4.54}$$

The freedom of an arbitrary real function $\lambda(\tilde{I},\Phi)$ leads to a variety of supersymmetric extensions of a given bosonic system. A similar freedom (of an arbitrary holomorphic function) has been observed in two-dimensional $\mathcal{N}$=4 supersymmetric mechanics [77].

To relate to the standard $\mathcal{N}$=2 supersymmetric mechanics construction (4.2.2) with $W' = \sqrt{V}$, we must choose

$$\tilde{k}\,\tilde{I}(p_\varphi,\varphi)\,\mathrm{e}^{\imath\lambda(\tilde{I}(p_\varphi,\varphi),\Phi(p_\varphi,\varphi))} = p_\varphi + \imath\sqrt{V(\varphi)}, \quad \Leftrightarrow \quad \tan\lambda = \frac{\sqrt{V(\varphi)}}{p_\varphi}, \tag{4.55}$$

where $V(\varphi)$ is defined by (4.23) or (4.24). For odd $k$ we find

$$\tan\lambda = \frac{\alpha_0}{\cos\Phi} \Big/ \sqrt{\tilde{I}^2 - \alpha_0^2}, \tag{4.56}$$

while for even $k$ the expression is more complicated. Note that $\lambda = \text{const}$ yields trivial supersymmetry, with no spin interaction. Another interesting case is $\lambda = \Phi/\tilde{I}$, which produces a coordinate-independent spin-background interaction.

Applying the (super-)Liouville theorem to the supersymmetric system given by (4.53) and (4.54), we obtain

$$\tilde{I}_s = \tilde{I} + \imath\theta\overline{\theta}\,\frac{\partial\lambda(\tilde{I},\Phi)}{\partial\Phi}, \qquad \Phi_s = \Phi + \imath\theta\overline{\theta} \tag{4.57}$$

$$\frac{\partial\lambda(\tilde{I},\Phi)}{\partial\tilde{I}}, \qquad \Theta = \mathrm{e}^{\imath\lambda(\tilde{I},\Phi)}\theta, \qquad \overline{\Theta} = \mathrm{e}^{-\imath\lambda(\tilde{I},\Phi)}\overline{\theta}.$$

As already said, the Hamiltonian in these variables is of the same form as the non-supersymmetric one, $\mathcal{I}_s = \frac{1}{2}\tilde{k}^2\tilde{I}_s^2$.

### 4.2.3 Extension to two-dimensional systems

In any conformal mechanics one may separate the radial from the angular degrees of freedom. The former part is universal, hence the such models differ only by their angular Hamiltonian systems, whose coordinates commute with the conformal algebra $so(2,1)$ [67]. Such

a splitting is useful for quantization [68] and the construction of superconformal extensions [66, 67]. For $N$-particle Calogero models it yields a separation of one more variable beyond the center of mass. Thus, their analysis becomes only complicated starting with $N$=4. For example, the angular part of the $A_{N-1}$ rational Calogero model corresponds to a $\frac{1}{2}N(N-1)$-center Higgs oscillator on $S^{N-2}$. At $N$=4 its force centers are located at the vertices of a cuboctahedron [45]. For $N$=3 however, the angular part of any rational Calogero model lives merely on a circle, and it is precisely one of the dihedral systems considered in this Note.

Therefore, by adding a radial coordinate $r \in [0,\infty)$, we may extend our one-dimensional system to a two-dimensional conformal mechanics with dihedral symmetry (a rational 3-particle Calogero model), defined by the $SO(2,1)$ generators

$$\mathcal{H}_0 = \frac{p_r^2}{2} + \frac{\mathcal{I}(\tilde{I})}{r^2} = \frac{\mathbf{p}^2}{2} + \sum_{\ell=0}^{k-1} \frac{1}{(\mathbf{a}_\ell \cdot \mathbf{r})^2},$$
$$\mathcal{D} = p_r r = \mathbf{pr}, \qquad \mathcal{K} = \tfrac{1}{2}r^2 = \tfrac{1}{2}\mathbf{r}^2, \tag{4.58}$$

where $\mathbf{a}_\ell$ run over the positive dihedral roots as before. This allows us to extend the above-established equivalence of systems with different $k$-values to these two-dimensional systems. In particular, all these Calogero models are *locally* equivalent to a free particle on the

plane, which is in agreement with the "decoupling" transformation of the quantum Calogero model to the free particle [42]. Furthermore, since for certain small values of $k$ the model is based on a Lie algebra listed in (4.25), we can also assert the *global* equivalence of the $G_2$ model with couplings $(\alpha_1, \alpha_2)$ to the $A_2$ model with coupling $\alpha_0 = \alpha_1 + \alpha_2$.

Since the radial motion is unbounded, the Hamiltonian (4.58) does non admit a formulation in terms of action-angle variables. This complication my be avoided by adding an oscillator potential,

$$\mathcal{H} = \frac{\mathbf{p}^2}{2} + \sum_{\ell=0}^{k-1} \frac{1}{(\mathbf{a}_\ell \cdot \mathbf{r})^2} + \frac{\omega^2 \mathbf{r}^2}{2} = \frac{p_r^2}{2} + \frac{\mathcal{I}(\tilde{I})}{r^2} + \frac{\omega^2 r^2}{2}. \tag{4.59}$$

The confining potential allows for the application of the Liouville theorem. Thus, in order to extend the action-angle variable formulation to the latter system, we fix the level surface of the constants of motion $\mathcal{H}$ and $\tilde{I}$ and introduce the generating function in accordance with the expression for the symplectic one-form $\tilde{I} d\Phi + p_r dr$,

$$S \;=\; \tilde{I}\,\Phi + \int_{r_0}^{r} dr\, \sqrt{2h - \tfrac{2\mathcal{I}(\tilde{I})}{r^2} - \omega^2 r^2} \;=\; \tilde{I}\,\Phi + \int_{r_0}^{r} dr\, \sqrt{2h - \tfrac{(\tilde{k}\tilde{I})^2}{r^2} - \omega^2 r^2}, \tag{4.60}$$

where $h$ is the value of the Hamiltonian $\mathcal{H}$. By the standard tech-

nique, we identify the action variables as

$$I_{\rm ang} = \tilde{I} \qquad \text{and} \qquad I_{\rm rad} = \frac{h}{2\omega} - \frac{\tilde{k}\tilde{I}}{2} \tag{4.61}$$

and find the canonically conjugated angle variables

$$\Phi_{\rm ang} = \Phi + \tilde{k}\Phi_{\rm rad} - \arcsin\frac{h - \frac{\tilde{k}\tilde{I}}{r^2}}{\sqrt{h^2 - \tilde{k}^2}} \qquad \text{and} \qquad \Phi_{\rm rad} = -\arcsin\frac{h - \omega^2 r^2}{\sqrt{h^2 - (\tilde{k}\tilde{I}\omega)^2}}. \tag{4.62}$$

Equation (4.61) gives us the Hamiltonian in terms of action variables,

$$\mathcal{I} = \omega\,(2I_{\rm rad} + \tilde{k} I_{\rm ang}). \tag{4.63}$$

One sees that our confined system (4.59) is locally equivalent to a two-dimensional anisotropic oscillator with frequencies $\omega_{\rm rad} = \omega$ and $\omega_{\rm ang} = \frac{1}{2}\widetilde{k}\omega$. Since the frequency ratio is rational, the trajectories on the two-torus are closed.

Finally, we note that the general approach for the construction of action-angle variables of Calogero models has been presented in [44]. There, the action variables are associated with the Lax constants of motion. On the other hand, due to its superintegrability, the Calogero model enjoys an additional series of constants of motion [43]. Our construction of action-angle variables is in fact related to these additional constants of motion. In the $A_2$ Calogero

model, for instance, the angular Hamiltonian $\mathcal{I}$ is a function of the Lax constants of motion as well as the Woijechowski one [45].

# Chapter 5

# Conclusion

1. We have developed the Lagrangian reduction procedure related to the first and second Hopf maps. The reduced systems describe particle moving in the field of Dirac and Yang monopoles respectively.

2. We have constructed an integrable anisotropic inharmonic Higgs oscillator.

3. Applying the reduction procedure related tro Hopf maps to anisotropic inharmonic Higgs oscillator we have constructed a (pseudo)spherical generalization of (MICZ)Kepler system with additional term describing an analogue of homogeneous electric field. The coordinate systems in which the spatial coordinates be separated was presented.

4. An $n(n-1)/2$- center integrable Higgs oscillator was con-

structed from the spherical part of $n$-particle Calogero model. For $n = 3$ the force centers are located in the vertices of hexagon and for $n = 4$- in the vertices of Archimedian solid called cuboctahedron.

5. Action-angle variables were constructed for two dimensional dihedral systems. It was shown, that choosing interaction parameters of $G_2$ Calogero model being equal to the sum of interaction parameter of $A_3$ one will establish complete correspondence of that systems.

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