ON THE GENERAL SOLUTIONS OF HILBERT-EINSTEIN FIELD EQUATIONS IN VACUUM

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The Hilbert-Einstein field equations describe the structure of space-time in the general theory of relativity. Their simplified form, in vacuum, was discovered in 1913 by Marcel Grossmann and presented in two joint publications with Einstein, which, on his side, rejected this idea and presented another hypothesis... He reverted to Grossmann idea only two years later.

The first known solution was this of Schwarzschild, that is at the origin of the notion of "black hole". Let us also quote the Kerr solution, the plane gravitational waves, the general first order solution, the successive approximations of the two black hole problem. Hundreds of rigourous solutions are known today.

The genesis of this discovery call for a series of curious coincidences. The "absolute differential calculus" was a speciality of the only School of Zürich, and an improbable meeting has been necessary, the meeting of this School, of Albert Einstein and of the mathematician Marcel Grossmann that was also able to think as a pure physicist.

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1. The Hilbert-Einstein field equation in vacuum

In the "flat" case of the special theory of relativity the proper time s is given in terms of the space-time displacements by the classical Minkowski expressi on:

$$ds^2 = dt^2 - (dx^2 + dy^2 + dz^2)/c^2 \quad , \tag{1}$$

where c is the velocity of light.

For the general case we will write as usual: $t = x^0$; $x/c = x^1$; $y/c = x^2$; $z/c = x^3$, (these superscripts are not exponents) and, with the usual Einstein summation convention (summation on the indices that appear both up and down), we will obtain:

$$ds^2 = g_{\mu\nu}(x^0, x^1, x^2, x^3)\, dx^\mu\, dx^\nu \,. \tag{2}$$

In the Minkowski case the matrix of the functions $g_{\mu\nu}$ is constant. It has twelve zeros out of the diagonal and $+1, -1, -1, -1$ in the diagonal. In the general case that matrix remains symmetrical: $g_{\mu\nu} = g_{\nu\mu}$. We will also need the inverse matrix $g^{\mu\nu}$.

$$[\ g_{\mu\nu}\ ]^{-1} = [\ g^{\mu\nu}\ ]; \qquad g_{\mu\lambda} \,.\, g^{\lambda\nu} = \delta_\mu{}^\nu . \tag{3}$$

The Christoffel brackets are then

$$\Gamma_\mu{}^\lambda{}_\nu = (1/2)\, g^{\lambda\rho}\, [\, g_{\mu\rho,\nu} + g_{\rho\nu,\mu} - g_{\nu\mu,\rho}]; \qquad \text{with} \quad g_{\alpha\beta,\gamma} = \partial g_{\alpha\beta} / \partial x^\gamma \tag{4}$$

and the components $R_{\mu\nu}$ of the Ricci tensor are

$$R_{\mu\nu} = \Gamma_\mu{}^\lambda{}_{\nu,\lambda} - \Gamma_\mu{}^\lambda{}_{\lambda,\nu} + \Gamma_\mu{}^\rho{}_\nu \Gamma_\rho{}^\lambda{}_\lambda - \Gamma_\mu{}^\rho{}_\lambda \Gamma_\nu{}^\lambda{}_\rho \,. \tag{5}$$

Notice that, if Δ is the determinant of th e matrix $g_{\mu\nu}$, the terms $\Gamma_\mu{}^\lambda{}_{\lambda,\nu}$ and $\Gamma_\rho{}^\lambda{}_\lambda$ are simple:

$$\Gamma_\alpha{}^\beta{}_\beta = \Delta_{,\alpha} \,/\, 2\Delta \quad , \tag{6}$$

hence, if we put

$$(1/2)\,\mathrm{Ln}\,|\Delta| = L \tag{7}$$

the expression (5) of the fu nction $R_{\mu\nu}$ can be simplified into:

$$R_{\mu\nu} = \Gamma_\mu{}^\lambda{}_{\nu,\lambda} - L_{,\mu\nu} + \Gamma_\mu{}^\rho{}_\nu L_{,\rho} - \Gamma_\mu{}^\rho{}_\lambda \Gamma_\nu{}^\lambda{}_\rho \,. \tag{8}$$

That latter expression allow to verify easily the symmetry of the Ricci tens or and notice that if Δ is constant half of the terms of (5) vanish.

In vacuum the Hilbert-Einstein field equation is reduce to

$$R_{\mu\nu} + \Lambda\, g_{\mu\nu} = 0; \qquad \text{with} \quad \Lambda = \text{cosmological constant.} \tag{9}$$

If Λ is equal to zero, we obtain the Grossmann equation:

$$R_{\mu\nu} = 0. \tag{10}$$

2. Some classical solutions

The Schwarzschild solution was the first known solution of the Grossmann equation (beside the Minkowski ds^2 of course...).

Scharzschild uses the spherical coordinates t, r, θ, φ (time, distance, colatitude, longitude), he considers the space-time curvature around a spherical body of mass M and obtains the following ds^2:

$$ds^2 = F(r)\, dt^2 - (1/c^2).\{ [dr^2 / F(r)] + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2 \} \tag{11}$$

with $\quad F(r) = 1 - (2m/r); \qquad m = GM/c^2$ = relativistic radius of the mass M. (12)

If M = 0, we find again the Minkowski ds^2 of (1), in spherical coordinates.

If $\Lambda \neq 0$, the function F(r) becomes:

$$F(r) = 1 - (2m/r) - (\Lambda r^2 / 3c^2). \tag{13}$$

For $\Lambda = 0$ the two most famous other presentations of the Schwarschild ds^2 are the following:

A) The Robertson ds^2. Robertson uses a radial distance ρ related to the r of Schwarzschild by:

$$r = \rho + m + (m^2/4\rho). \tag{14}$$

He also uses three Cartesian coordinates given by the usual Eulerian relations:

$$x = \rho \sin\theta \cos\varphi; \qquad y = \rho \sin\theta \sin\varphi; \qquad z = \rho \cos\theta, \tag{15}$$

then, for the same ds^2, we obtain the simple Robertson form:

$$ds^2 = f(\rho)\, dt^2 - [g(\rho)/c^2] . (dx^2 + dy^2 + dz^2) \tag{16}$$

$$\text{with} \quad f(\rho) = [(2\rho - m) / (2\rho + m)]^2; \qquad g(\rho) = [1 + (2m/\rho)]^4. \tag{17}$$

B) The Painlevé ds^2.

Only one transformation of coordinates: the Painlevé time τ that is given by:

$$\tau = t + (2m\varepsilon/c)\, \mathrm{Ln}\, [(r - 2m) / 2m]; \qquad \text{with} \quad \varepsilon = \text{constant} = \pm 1, \tag{18}$$

which gives

$$ds^2 = d\tau^2 - (1/c^2). [dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2] - (2m/r).[d\tau + (\varepsilon/c)dr]^2. \tag{19}$$

The two first terms give the usual Minkowski ds^2 and the remainder has no singularity at r = 2m, that singularity in the Schwarzschild ds^2 is only an artificial singularity.

For $\Lambda \neq 0$, the Painlevé form remains simple:

$$ds^2 = d\tau^2 - (1/c^2). [dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2] - [(2m/r) + (\Lambda r^2 / 3c^2)].[d\tau + (\varepsilon/c)dr]^2 \tag{20}$$

$$\text{with} \quad \tau = t + h(r); \qquad dh/dr = \varepsilon (1 - F) / cF; \qquad F = 1 - (2m/r) - (\Lambda r^2 / 3c^2). \tag{21}$$

The Kerr solution generalizes the Schwarzschild ds^2 to a rotating black hole, with an angular momentum A and the corresponding length a = A / Mc, that verify always $|a| \leq m$.

$$ds^2 = dt^2 - 2mr[dt - (a \sin^2\theta\, d\varphi/c)]^2 D^{-1} - (r^2 + a^2) \sin^2\theta\, d\varphi^2/c^2 - (D/c^2)[d\theta^2 + (dr^2/E)]$$

$$\text{with} \quad D = r^2 + a^2 \cos^2\theta, \qquad E = r^2 - 2mr + a^2. \tag{22}$$

A Painlevé form exists for this Kerr ds^2; let us define the new variables t_n and φ_n by:

$$dt_n = dt + (2mr\varepsilon\, dr / Ec), \qquad d\varphi_n = d\varphi + [2mra\varepsilon\, dr / E(r^2 + a^2)], \qquad \text{with } \varepsilon = \text{constant} = \pm 1. \tag{23}$$

The parameters t_n, r, θ, φ_n can be called “spheroidal parameters”, since for them the Minkowski ds^2 is written:

$$ds^2_M = dt_n^2 - (1/c^2). \{(r^2 + a^2\cos^2\theta) [d\theta^2 + dr^2/(r^2 + a^2)] + (r^2 + a^2) \sin^2\theta\, d\varphi_n^2 \}. \tag{24}$$

It is easy to verify that, with $\rho^2 = (r^2 + a^2)\sin^2\theta$ and $z = r\cos\theta$, the four parameters t_n, ρ, z, φ_n are usual cylindrical parameters, with the corresponding ds^2_M.

With D given in (22), the Kerr ds^2 is then:

$$ds^2 = ds^2_M - 2mr[cdt_n - a\sin^2\theta\, d\varphi_n + D\varepsilon\, dr / (r^2 + a^2)]^2 / Dc^2. \qquad (25)$$

The case a = 0 in (22) – (25) gives the Schwarzschild case of (11) – (19) with $t_n = \tau$ and $\varphi_n = \varphi$.

A third classical solution of the Grossmann equation are t he "plane gravitational wave", with $u = ct - x$:

$$c^2ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 - f(u,y,z).(cdt - dx)^2. \qquad (26)$$

These waves move into the x direction at the velocity of light, and the function f h as a wide freedom, it is only subjected to

$$\partial^2 f/\partial y^2 + \partial^2 f/\partial z^2 = 0. \qquad (27)$$

Notice that the expression (26) has the Painlevé form: a Minkowski ds^2 minus the product of a space-time function by the square of a sum of differentials. Furthermore these three sums of differentials given in (19), (25) and (26) have the following common point: their ds^2 can be written:

$$c^2ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 - f(t,x,y,z).(cdt - \alpha dx - \beta dy - \gamma dz)^2 \qquad (28)$$

with, for the three functions α, β and $\gamma(t,x,y,z)$:

$$\alpha^2 + \beta^2 + \gamma^2 = 1. \qquad (29)$$

This remarkable property has been observed in many of the other rigorous so lutions of the Grossmann equation.

It is possible to generalize the plane gravitational waves to the case of a non zero cosmological constant Λ.

We will use the cylindrical space-time coordinates t, x, r, θ with $y = r\cos\theta$, $z = r\sin\theta$ and $u = ct - x$, and we will obtain:

$$c^2ds^2 = f(u, r, \theta)\, du^2 + 2g(r)\, dx\, du - dr^2 - h(r)\, d\theta^2. \qquad (30)$$

The two functions g(r) and h(r) are imposed, they are only function of Λ, with the length $K = 2c\,|3\Lambda|^{-0.5}$ and with the following:

$$\left.\begin{array}{lll} \text{For } \Lambda > 0: & g(r) = \{\cos(r/K)\}^{4/3}; & h(r) = K^2 g(r)\tan^2(r/K). \\ \text{For } \Lambda < 0: & g(r) = \{\cosh(r/K)\}^{4/3}; & h(r) = K^2 g(r)\tanh^2(r/K). \end{array}\right| \qquad (31)$$

On the contrary the function $f(u, r, \theta)$ remains with a wide fredom and is only subjected to the following condition that generalizes (27):

$$2\Lambda f/c^2 + \partial^2 f/\partial r^2 + [\partial^2 f/\partial\theta^2 + 0.5\,(\partial f/\partial r).(\partial h/\partial r)] / h(r) = (\partial g/\partial r).[\partial(f/g)/\partial r]. \qquad (32)$$

Another beautiful result is the general first order solution of Grossmann equation. We will choose the velocity c as unit of velocity and obtain:

$$g_{\mu\nu} = f_{\mu,\nu} + f_{\nu,\mu} + \text{second order} + \begin{vmatrix} 1; & A_{,23}; & B_{,13}; & C_{,12} \\ A_{,23}; & -1; & -C_{,03}; & -B_{,02} \\ B_{,13}; & -C_{,03}; & -1; & -A_{,01} \\ C_{,12}; & -B_{,02}; & -A_{,01}; & -1 \end{vmatrix} \qquad (33)$$

The four functions f_μ and their partial derivatives cor responds to the modifications of referentials. The three "intrisic" functions A, B, and $C(x^0, x^1, x^2, x^3)$ verify the conditions:

$$(A + B + C)_{,0123} = 0; \qquad A = 0; \qquad B = 0; \qquad C = 0, \qquad (34)$$

where the sign is the expression of the "D'Alembertian": $F = F_{,00} - F_{,11} - F_{,22} - F_{,33}$.

For instance, for the Robertson form of the Schwarzschild ds^2, we obtain:

$$\left.\begin{array}{lll} \rho = [x^2 + y^2 + z^2]^{1/2}; & f_o = -mt/\rho; & f_1 = 0.5\, m\, \mathrm{Ln}\, [(\rho - x)/(\rho + x)] \\ & f_2 = 0.5\, m\, \mathrm{Ln}\, [(\rho - y)/(\rho + y)]; & f_3 = 0.5\, m\, \mathrm{Ln}\, [(\rho - z)/(\rho + z)] \\ A = mt\arctan(x\rho/yz); & B = mt\arctan(y\rho/xz); & C = mt\arctan(z\rho/xy). \end{array}\right| \qquad (35)$$

3. Small Historical Dackground.
A series of improbable coincidences is at the origi n of the general theory of relativity

1820–1830. Birth of non-Euclidian geometries (Gauss, Lobatchevski, Bolyai).

1826–1866. Bernhardt Riemann.

1867. Publication of Riemann general theory of "Riemann's spaces".

1829–1900. Elwin Bruno Christoffel, **professor at the Zürich polytechnicum** (1862–1869).

"On the transformation of homogemneous quadratic forms" (1869).

1853–1925. Gregorio Ricci-Curbastro.

Tensorial Analysis and absolute differential calculus.

1878–1936. Marcel Grossmann, **student of Minkowski, friend of Einstein and successor of Christoffel,** after 1907**, at the Zürich polytechnicum. Reports on the works of Christoffel and Ricci-Curbastro.**

1907 and 1911. **Einstein looks for a generalization of relativity.**

1912. **Einstein become professor at the Zürich Polytechnicum** (after two years at Prag University and the refusal of a post at Utrecht University). **In Zürich he asks for the help of Grossmann in his researches on relativity.**

1912. **With mathematical and physical considerations, Grossman reaches the conclusion that the real physical space-time is a Riemann space and that in vacuum it must obey the equation $R_{\mu\nu} = 0$. This is the simplest covariant solution for a space-time with gravitation. (Publication in June 1913, references 1 and 2).**

1913. **In the same publications (references 2 and 3), Einstein gives several reasons to refuse the Grossmann solution and presents its own solution (non covariant).**

Falling out between the two friends, they will never work together again.

The non covariant solution of Einstein gives the Newtonian curvature of light beams. The astronomers try unsucessfully to detect that curvature on their old pictures of solar eclipses.

1914 March. Einstein goes back to Berlin.

1914 Spring. Erwin Freundlich (Berlin-Babelsberg Observatory) prepares the observation of the total solar eclipse of August 21, 1914. He receives 2000 Marks (of that time) from the Royal Prussian Academy of Sciences, and then 3000 Marks from the firm Krupp. He buys the best optical equipement for parallactic measures, those coming from the specialized optical firms of Iena (and much better than Eddington's equipement of the eclipse of 1919).

1914, July 25. Freundlich and his team arrive in Theodosia (Crimea). They meet the other teams (Russian, Italian, Spanish, British, French, American) and, proudly, prepare their supermaterial.

1914 August, 1. Germany declares war to Russia. Freundlich and his team are placed under house-arrest, they give their material to be kept in Theodosia and are sent to Odessa August 5, they will be exchanged later.

1914 November, 30. The report of the Spanish team emphasizes the good meteorological conditions of observation of the eclipse at Theodosia.

1914 November. The German optical material arrives at the University of Odessa.

1915 September. Einstein leaves Berlin for two weeks and goes to Zürich. Notice, of course, the very bad wartime communications between outside and Germany, but Switzerland was neutral...

1915 October. Back to Berlin, Einstein forsakes his one year research on magnetism and hastily returns to the study of general relativity.

1915 November, 4, 11, 18. Einstein publishes three successive, and contradictory, notes on general relativity in the publication of the Royal Prussian Academy of Sciences. **These notes are based on Grossmann equation, that gives a deviation of light beams twice larger than the Newtonian one.** Unfortunately he never gave credit to Grossmann of his equation.

1915 November, 16. Hilbert presents his own results on genera l relativity at Göttingen. He had sent a letter to Einstein on the subject and had invited him. Einstein refuses to come, he answers November 18 **and modify the text of his third note that was already under press...**

The remainder is simple...

4. Comparison of Einstein and Grossmann solutions

Let us compare the Einstein and Grossmann solutions of references [1 –3].

The Grossmann solution uses the Grossmann equation $R_{\mu\nu} = 0$ and leads directly to the Schwarzschild ds^2, while the Einstein solution use the following slightly different ds^2:

$$ds^2 = [1 - (2m/r)]dt^2 - (dx^2 + dy^2 + dz^2)/c^2. \qquad (36)$$

These two ds^2 gives the same Newtonian motions of planets and they only differ by their small relativistic effects.

The corresponding relativistic effects are the following:

	Einstein	**Grossmann**
Advance of the perihelion of Mercury	29'' per century	43'' per century
Deviation of Sun grazing light beams	0.88''	1.77''

The observation of a total solar eclipse allows then to discriminate between the two hypotheses.

References

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