

Abraham-Lorentz-Dirac equation in 5D Stueckelberg electrodynamics

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Abstract. We derive the Abraham-Lorentz-Dirac (ALD) equation in the framework of the electrodynamic theory associated with Stueckelberg manifestly covariant canonical mechanics. In this framework, a particle worldline is traced out through the evolution of an event $x^\mu(\tau)$. By admitting unconstrained commutation relations between the positions and velocities, the associated electromagnetic gauge fields are in general dependent on the parameter τ , which plays the role of time in Newtonian mechanics. Standard Maxwell theory emerges from this system as a τ -independent equilibrium limit. In this paper, we calculate the τ -dependent field induced by an arbitrarily evolving event, and study the long-range radiation part, which is seen to be an on-shell plane wave of the Maxwell type. Following Dirac's method, we obtain an expression for the finite part of the self-interaction, which leads to the ALD equation that generalizes the Lorentz force. This third-order differential equation is then converted to an integro-differential equation, identical to the standard Maxwell expression, except for the τ -dependence of the field. By studying this τ -dependence in detail, we show that field can be removed from the integration, so that the Lorentz force depends only on the instantaneous external field and an integral over dynamical variables of the event evolution. In this form, pre-acceleration of the event by future values of the field is not present.

1. Introduction

In classical electrodynamics, fields accelerate charges through the Lorentz force

$$m\ddot{z}^\mu = eF^{\mu\nu}(z)\dot{z}_\nu \quad (1)$$

and current densities defined on charge worldlines

$$J^\mu(x) = \int d\tau \dot{z}^\mu(\tau) \delta^4[x - z(\tau)] \quad (2)$$

induce fields through Maxwell's equations

$$\partial_\nu F^{\mu\nu}(x) = eJ^\mu(x) \quad \partial_{[\lambda} F_{\mu\nu]}(x) = 0. \quad (3)$$

In general configurations, multiple charges interact through a complex system of mutual interactions, but as a first approximation to the motion of a single charge, the field may be treated as fixed, neglecting both its evolution under recoil of the sources producing it and the

reaction field induced by the accelerating charge itself. The manifestly covariant system (1) – (3) employs an invariant evolution parameter τ often associated with proper time; this association is at best inexact, ignoring the diversity of proper times among the many particles as well as the presumed τ -reparameterization invariance of the system.

The Abraham-Lorentz-Dirac (ALD) equation [1, 2]

$$m\ddot{z}^\mu = eF_{ext}^{\mu\nu}(z)\dot{z}_\nu + \frac{2}{3}e^2(\ddot{z}^\mu - \dot{z}^\mu\dot{z}^2) \quad (4)$$

improves the fixed field approximation, still treating the field $F_{ext}^{\mu\nu}$ as external to the dynamics, but incorporating a generic Green's function solution

$$F_{rad}^{\mu\nu}(x) = \partial^\mu A^\nu - \partial^\nu A^\mu \quad A^\nu(x) = -e \int d^4x' D_{rad}(x - x') J^\nu(x') \quad (5)$$

to the Maxwell equations to account for the transfer of energy-momentum from the charge to the field. Thus, ALD provides an effective one-particle theory for a charge in an external electromagnetic field, but being a third order differential equation, admits run-away solutions of the type

$$\dot{z}^0 = \cosh\left[\tau\left(e^{\tau/\tau_0} - 1\right)\right] \quad \dot{z}^1 = \sinh\left[\tau\left(e^{\tau/\tau_0} - 1\right)\right] \quad (6)$$

for the free particle case $F_{ext}^{\mu\nu} = 0$, where

$$\tau_0 = \frac{2}{3} \frac{e^2}{4\pi m} \sim 10^{-24} \text{ sec} \quad (7)$$

is the time scale associated with a photon crossing a classical electron radius. This unphysical solution may be eliminated under the assumption of bounded acceleration, by converting (4) to the integro-differential equation

$$m\ddot{x}^\mu(\tau) = \int_\tau^\infty ds \frac{1}{\tau_0} e^{-(s-\tau)/\tau_0} \left[eF_{ext}^{\mu\nu}(x(s)) \dot{x}_\nu(s) - \tau_0 m \ddot{x}^2(s) \dot{x}^\mu(s) \right]. \quad (8)$$

Although (8) suppresses run-away solutions, its causal behavior presents a new difficulty in the form of pre-acceleration. If the external field is a constant switched on at $\tau = 0$, so that

$$\frac{e}{\tau_0} F_{ext}^{\mu\nu}(x(\tau)) \dot{x}_\nu(\tau) \rightarrow \frac{1}{\tau_0} F^\mu(\tau) = \frac{1}{\tau_0} F_0^\mu \theta(\tau), \quad (9)$$

then one obtains a non-zero pre-acceleration

$$m\ddot{x}^\mu(\tau) \simeq \frac{1}{\tau_0} F_0^\mu \int_\tau^\infty ds e^{-(s-\tau)/\tau_0} \theta(s) = \begin{cases} F_0^\mu e^{-|\tau|/\tau_0} & , \quad \tau < 0 \\ F_0^\mu & , \quad \tau > 0 \end{cases} \quad (10)$$

effective over a period on the order of τ_0 before the field is applied. It may be argued that on this time scale, problems of classical microcausality are overtaken by quantum effects, empirically and theoretically. In particular, Johnson and Hu [3] have derived a modified ALD equation as a semiclassical limit in QED and shown that causality is enforced by the stochastic nature of the particle-field system. Nevertheless, a violation of classical causality posed in classical electrodynamics seems to raise questions about the interpretation of the underlying framework, beyond issues of approximation and measurement.

In this paper, we develop the Abraham-Lorentz-Dirac equation in a formulation of classical electrodynamics associated with a manifestly covariant canonical mechanics of interacting spacetime events. This approach, first suggested by Stueckelberg [4] in modeling pair creation/annihilation, regards the charged event $x^\mu(\tau)$ as the source of an electromagnetic field, so that event currents and fields are defined locally at a given spacetime point x^μ and given moment τ . Microscopic details of event dynamics may differ from the usual approach, but standard Maxwell theory emerges as the τ -static equilibrium limit. Because the relationships among events, particles, currents, and fields are determined at a finer resolution, this formalism leads to a generalization of (8) for which the causal structure can be given a natural and consistent interpretation. In the next section, we review the essential features of Stueckelberg electrodynamics, then characterize the induced radiation field in section 3, and derive the ALD equation for the Stueckelberg framework in section 4.

2. Stueckelberg off-shell electrodynamics

In seeking a classical description of pair creation/annihilation as a single worldline generated dynamically by the evolution of an event $x^\mu(\tau)$, Stueckelberg proposed [4] a generalized Lorentz force of the form

$$m\ddot{x}^\mu(\tau) = e_0 f^{\mu\nu}(x, \tau) \dot{x}_\nu(\tau) + e_0 \varepsilon^\mu(x, \tau) \quad (11)$$

where $f^{\mu\nu}(x, \tau)$ is the electromagnetic field tensor (made τ -dependent), and $\varepsilon^\mu(x, \tau)$ is a field strength vector introduced to permit continuous reversal of the time direction $\dot{x}^0(\tau)$. Without the additional vector field, the value of \dot{x}^2 is conserved through

$$\frac{d}{d\tau}(\dot{x}^2) = 2\dot{x}_\mu \ddot{x}^\mu = \frac{2e}{m} \dot{x}_\mu f^{\mu\nu} \dot{x}_\nu = 0 \quad (12)$$

and so the particle cannot evolve smoothly from $\dot{x}^0 > 0$ to $\dot{x}^0 < 0$ through spacelike motion with $(\dot{x}^0)^2 < \dot{\mathbf{x}}^2$. But Stueckelberg found no physical justification for the vector field and proceeded to develop the τ -parameterized covariant canonical quantum mechanics

$$i\partial_\tau \psi(x, \tau) = \frac{1}{2m} [p^\mu - eA^\mu(x)] [p_\mu - eA_\mu(x)] \psi(x, \tau) \quad (13)$$

later used by Feynman [5], Schwinger [6], DeWitt [7] and others in developing QED. Stueckelberg required an independent parameter because, by construction, Einstein coordinate time x^0 does not increase monotonically, and he regarded τ as a physical time of system evolution, playing the role of Newtonian time in nonrelativistic mechanics. The Hamiltonian system associated with (13) is then a symplectic mechanics for which manifest Poincaré covariance plays the role of Galilean covariance in Newtonian mechanics (see also [8, 9]).

Stueckelberg's generalized Lorentz force was later obtained from first principles in the framework of a fundamental gauge theory [10, 11], called off-shell electrodynamics because the non-conservation of $\dot{x}^2(\tau)$ found from (11) is associated with mass exchange between particle and field. Quite generally [12], this framework is implicitly incorporated as the underlying electrodynamics whenever one writes equations of motion

$$m\ddot{x}^\mu = F^\mu(x, \dot{x}, \tau) \quad (14)$$

on the unconstrained phase space defined by

$$[x^\mu, x^\nu] = 0 \quad m[x^\mu, \dot{x}^\nu] = -i\eta^{\mu\nu}, \quad (15)$$

with metric

$$\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1). \quad (16)$$

and $\mu, \nu = 0, \dots, 3$. The apparently naive commutation relations (15) are actually sufficient [13] to establish the self-adjointness of (14) from which one is necessarily led to the Lorentz force (13) along with homogeneous field equations

$$\partial_\mu f_{\nu\rho} + \partial_\nu f_{\rho\mu} + \partial_\rho f_{\mu\nu} = 0 \quad \partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu + \partial_\tau f_{\mu\nu} = 0. \quad (17)$$

The fields $f_{\mu\nu}(x, \tau)$ and $\varepsilon_\mu(x, \tau)$ have been called pre-Maxwell fields because of their explicit τ -dependence. Adopting formal designations

$$x^5 = \tau \quad \partial_5 = \partial_\tau \quad \varepsilon_\mu = f_{5\mu} \quad (18)$$

and

$$\mu, \nu = 0, \dots, 3 \quad \alpha, \beta = 0, 1, 2, 3, 5 \quad (19)$$

we summarize the Lorentz force (11) and field equations (17) as

$$m\ddot{x}_\mu = e_0 f_{\mu\alpha}(x, \tau) \dot{x}^\alpha \quad \Rightarrow \quad \frac{d}{d\tau}(-\frac{1}{2}m\dot{x}^2) = e_0 f_{5\alpha}(x, \tau) \dot{x}^\alpha \quad (20)$$

$$\partial_{[\alpha} f_{\beta\gamma]} = 0 \quad \Rightarrow \quad f_{\alpha\beta}(x, \tau) = \partial_\alpha a_\beta(x, \tau) - \partial_\beta a_\alpha(x, \tau) \quad (21)$$

from which the Lagrangian is uniquely determined [14] as

$$L = \frac{1}{2}m\dot{x}^\mu \dot{x}_\mu + e_0 \dot{x}^\alpha a_\alpha(x, \tau). \quad (22)$$

Writing the canonical momentum

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = m\dot{x}_\mu + e_0 a_\mu(x, \tau) \quad (23)$$

and transforming to the Hamiltonian

$$K = p_\mu \dot{x}^\mu - L \quad (24)$$

the Stueckelberg-Schrodinger equation

$$[i\partial_\tau + e_0 a_5(x, \tau)] \psi(x, \tau) = \frac{1}{2m} [p^\mu - e_0 a^\mu(x, \tau)] [p_\mu - e_0 a_\mu(x, \tau)] \psi(x, \tau) \quad (25)$$

is seen to be locally gauge invariant [11] under τ -dependent gauge transformations of the type

$$\psi \rightarrow e^{ie_0 \Lambda(x, \tau)} \psi \quad a_\mu \rightarrow a_\mu + \partial_\mu \Lambda(x, \tau) \quad a_5 \rightarrow a_5 + \partial_\tau \Lambda(x, \tau). \quad (26)$$

The electromagnetic field $f_{\alpha\beta}(x, \tau)$ is made a dynamical quantity by adding a kinetic term to the action. In analogy to standard Maxwell theory, one may adopt the formal designation $f^{\mu 5} = \eta^{55} f^\mu_5 = -f^\mu_5$ and chose the form

$$-\frac{\lambda}{4} f^{\alpha\beta}(x, \tau) f_{\alpha\beta}(x, \tau) \quad (27)$$

which is a gauge invariant Lorentz scalar and contains only first order derivatives of the fields. The electromagnetic part of the action is now

$$S_{field} = \int d^4x d\tau \left\{ e_0 \dot{z}^\alpha a_\alpha(x, \tau) \delta^4[x - z(\tau)] - \frac{\lambda}{4} f^{\alpha\beta}(x, \tau) f_{\alpha\beta}(x, \tau) \right\} \quad (28)$$

leading to inhomogeneous field equations

$$\partial_\beta f^{\alpha\beta}(x, \tau) = \frac{e_0}{\lambda} j^\alpha(x, \tau) = e j^\alpha(x, \tau) = e \dot{z}^\alpha(\tau) \delta^4[x - z(\tau)] \quad (29)$$

and a conserved 5-current

$$\partial_\mu j^\mu(x, \tau) + \partial_\tau j^5(x, \tau) = 0. \quad (30)$$

As in nonrelativistic mechanics, equation (30) suggests the interpretation of $j^5(x, \tau)$ as the probability density at τ of finding the event at x . Since $\partial_\mu j^\mu \neq 0$, $j^\mu(x, \tau)$ cannot be identified as the source current in Maxwell's equations. However, under the boundary conditions $j^5 \rightarrow 0$, pointwise, as $\tau \rightarrow \pm\infty$, integration of (30) over τ , leads to $\partial_\mu J^\mu = 0$, where

$$J^\mu(x) = \int_{-\infty}^{\infty} d\tau j^\mu(x, \tau). \quad (31)$$

This integration has been called concatenation [15] and links the event current $j^\mu(x, \tau)$ with the particle current $J^\mu(x)$ defined on the entire particle worldline. Similarly,

$$\left. \begin{aligned} \partial_\beta f^{\alpha\beta}(x, \tau) &= e j^\alpha(x, \tau) \\ \partial_{[\alpha} f_{\beta\gamma]} &= 0 \end{aligned} \right\} \xrightarrow{\int d\tau} \left\{ \begin{aligned} \partial_\nu F^{\mu\nu}(x) &= e J^\mu(x) \\ \partial_{[\mu} F_{\nu\rho]} &= 0 \end{aligned} \right. \quad (32)$$

where

$$F^{\mu\nu}(x) = \int_{-\infty}^{\infty} d\tau f^{\mu\nu}(x, \tau) \quad A^\mu(x) = \int_{-\infty}^{\infty} d\tau a^\mu(x, \tau) \quad (33)$$

extracting standard Maxwell theory as the equilibrium limit of event dynamics. It is seen from (25) and (33) that e_0 and λ must have dimensions of time, so that the dimensionless ratio $e = e_0/\lambda$ can be identified as the Maxwell charge.

As in the non-relativistic case, the two-body action-at-a-distance potential in the Horwitz-Piron theory [10] may be understood as the approximation $-e_0 a_5(x, \tau) \rightarrow V(x)$. Within this framework, solutions have been found for the generalizations of the standard central force problem, including potential scattering [16] and bound states [17, 18]. Examination of radiative transitions [19], in particular the Zeeman [20] and Stark effects [21], indicate that all five components of the gauge potential are necessary for an adequate explanation of observed phenomenology.

The wave equation derived from (29) is

$$\partial_\alpha \partial^\alpha a^\beta(x, \tau) = (\partial_\mu \partial^\mu - \partial_\tau^2) a^\beta(x, \tau) = -e j^\beta(x, \tau) \quad (34)$$

for which the Greens function [22] found from

$$(\partial_\mu \partial^\mu - \partial_\tau^2) G(x, \tau) = -\delta^4(x, \tau) \quad (35)$$

is

$$G(x, \tau) = -\frac{1}{2\pi} \delta(x^2) \delta(\tau) - \frac{1}{2\pi^2} \frac{\partial}{\partial x^2} \frac{\theta(x^2 - \tau^2)}{\sqrt{x^2 - \tau^2}} = D(x) \delta(\tau) - G_{correlation}(x, \tau). \quad (36)$$

The first term has support on the lightcone at instantaneous τ , and recovers the standard Maxwell Greens function under concatenation. The second term has spacelike support ($x^2 > \tau^2 \geq 0$) and vanishes under concatenation, so it may contribute to correlations but not to Maxwell potentials. Potentials obtained from (36) as

$$a^\beta(x, \tau) = -e \int d^4x' d\tau' G(x - x', \tau - \tau') j^\beta(x', \tau') \quad (37)$$

recover Maxwell potentials under concatenation

$$\begin{aligned} A^\mu(x) &= \int d\tau \left\{ -e \int d^4x' d\tau' G(x - x', \tau - \tau') j^\mu(x', \tau') \right\} \\ &= -e \int d^4x' D(x - x') \int d\tau' j^\mu(x', \tau') \\ &= -e \int d^4x' D(x - x') J^\mu(x'). \end{aligned} \quad (38)$$

To study the low energy Coulomb problem, the source is taken to be the “static” event

$$z(\tau) = (\tau, 0, 0, 0) \quad (39)$$

from which one calculates

$$\begin{aligned} a^0(x, \tau) &= -e \int d^4x' d\tau' D(x - x') \delta(\tau - \tau') \delta(x^0 - \tau') \delta^3(\mathbf{x}) \\ &= -\frac{e}{2\pi} \delta\left[(x - \tau)^2\right] \theta(x^0) \\ &= -\frac{e}{4\pi |\mathbf{x}|} \delta(x^0 - \tau - |\mathbf{x}|) \end{aligned} \quad (40)$$

Although concatenation of (40) recovers the correct Coulomb potential

$$A^0(x) = -\frac{e}{4\pi |\mathbf{x}|} \int d\tau \delta(x^0 - \tau - |\mathbf{x}|) = -\frac{e}{4\pi |\mathbf{x}|} \quad (41)$$

the δ -function in $a^0(x, \tau)$ results in a microscopic dynamics that cannot reproduce expected low-energy interactions between a pair of charges. In particular, a second “static” event given by

$$\zeta(\tau) = (u^0\tau + \alpha, \beta, 0, 0) \quad (42)$$

will generally not experience any interaction with $z(\tau)$, except for appropriately tuned values of $\dot{\zeta}^0 = u^0$ and the offset $\zeta^0(0) = \alpha$ that determines the τ -synchronization of the two events.

Since off-shell quantum theory does provide a correct description of Coulomb scattering for sharp asymptotic mass states, which contain no information about τ -synchronization, it was suggested in [23] that the electromagnetic interaction between events should be modified to relax the deterministic synchronization expressed in (20) and (29). Under this modification, the source $j_\varphi^\beta(x, \tau)$ for the pre-Maxwell fields is taken to be a smoothed current density induced by an ensemble of events $z^\beta(\tau + \delta\tau)$ along a particle worldline, where $\delta\tau$ is given by a normalized distribution $\varphi(\tau)$. From (29) the smoothed source current is given as the ensemble average

$$j_\varphi^\alpha(x, \tau) = \int_{-\infty}^{\infty} ds \varphi(\tau - s) \dot{z}^\alpha(s) \delta^4[x - z(s)] \quad (43)$$

or equivalently as an ensemble of sharp currents $j^\beta(x, \tau)$ induced by a single event

$$j^\alpha(x, \tau) \longrightarrow j_\varphi^\alpha(x, \tau) = \int_{-\infty}^{\infty} ds \varphi(\tau - s) j^\alpha(x, s). \quad (44)$$

The fields $f^{\alpha\beta}(x, \tau)$ and $a^\beta(x, \tau)$ are induced by the smoothed current density, so the electrodynamic system that mediates between sharp events $z^\beta(\tau)$ and $\zeta^\beta(\tau)$ explicitly introduces a statistical structure to the event-event interaction. Because Maxwell currents are defined along the entire worldline, this procedure preserves the concatenated current

$$J^\mu(x) = \int_{-\infty}^{\infty} d\tau j_\varphi^\mu(x, \tau) = \int_{-\infty}^{\infty} ds \left[\int_{-\infty}^{\infty} d\tau \varphi(\tau - s) j^\mu(x, s) \right] = \int_{-\infty}^{\infty} ds j^\mu(x, s). \quad (45)$$

Taking the distribution to be

$$\varphi(\tau) = \frac{1}{2\lambda} e^{-|\tau|/\lambda} \quad \int_{-\infty}^{\infty} d\tau \varphi(\tau) = 1 \quad (46)$$

the low energy Coulomb field becomes a Yukawa-type potential with the correct non-relativistic limit for large λ

$$m\ddot{\mathbf{x}} = -e_0 \nabla [a^0(x, \tau) + a^5(x, \tau)] \rightarrow m\ddot{\mathbf{x}} = -2e_0 \nabla a_\varphi^0(x, \tau) = e^2 \nabla \left[\frac{e^{-|\mathbf{x}|/\lambda}}{4\pi |\mathbf{x}|} \right]. \quad (47)$$

The distribution $\varphi(\tau)$ provides a cutoff for the photon mass spectrum, which we take to be the conventional experimental uncertainty in photon mass ($\Delta m_\gamma \simeq 10^{-17}$ eV [24]), leading to a value of about 400 seconds for λ . The limit $\lambda \rightarrow 0$ restores $\varphi(\tau) \rightarrow \delta(\tau)$ and the limit $\lambda \rightarrow \infty$ restores standard Maxwell theory. Since the form of $\varphi(\tau)$ given in (44) represents the distribution of interarrival times of events in a Poisson-distributed stochastic process, this choice suggests an information-theoretic interpretation for the underlying the relationship between the current density and the ensemble of events from which it is induced.

The smoothed current can be introduced through the action [25], by adding a higher τ -derivative term to the electromagnetic part. The substitution

$$S_{em} \rightarrow \int d^4x d\tau \left[e_0 j^\alpha a_\alpha - \frac{\lambda}{4} f^{\alpha\beta}(x, \tau) f_{\alpha\beta}(x, \tau) - \frac{\lambda^3}{4} [\partial_\tau f^{\alpha\beta}(x, \tau)] [\partial_\tau f_{\alpha\beta}(x, \tau)] \right] \quad (48)$$

preserves Lorentz and gauge invariance, and leaves the action first order in spacetime derivatives. Defining a field interaction kernel

$$\Phi(\tau) = \delta(\tau) - \lambda^2 \delta''(\tau) = \frac{1}{2\pi} \int d\kappa [1 + (\lambda\kappa)^2] e^{-i\kappa\tau} \quad (49)$$

which is seen from

$$\int_{-\infty}^{\infty} ds \Phi(\tau - s) \varphi(s) = \delta(\tau) \rightarrow \varphi(\tau) = \int \frac{d\kappa}{2\pi} \frac{e^{-i\kappa\tau}}{1 + (\lambda\kappa)^2} = \frac{1}{2\lambda} e^{-|\tau|/\lambda} \quad (50)$$

to be the inverse function to $\varphi(\tau)$, the action becomes

$$S_{em} = \int d^4x d\tau e_0 j^\alpha a_\alpha - \frac{\lambda}{4} \int d^4x d\tau ds f^{\alpha\beta}(x, \tau) \Phi(\tau - s) f_{\alpha\beta}(x, s). \quad (51)$$

The Euler-Lagrange equations

$$\partial_\beta f_\Phi^{\alpha\beta}(x, \tau) = \partial_\beta \int ds \Phi(\tau - s) f^{\alpha\beta}(x, s) = e j^\alpha(x, \tau) \quad (52)$$

describe a sharp field induced by a sharp event current, and using (50) can be inverted to recover

$$\partial_\beta f^{\alpha\beta}(x, \tau) = e j_\varphi^\alpha(x, \tau) = e \int ds \varphi(\tau - s) j^\alpha(x, s). \quad (53)$$

The action (51), in which the statistical synchronization performed by $\Phi(\tau - s)$ is made explicit, has the advantage of permitting the usual study of symmetries and being amenable to second quantization, where the factor $[1 + (\lambda\kappa)^2]^{-1}$ provides a natural mass cutoff for the off-shell photon that renders off-shell quantum field theory super-renormalizable at two-loop order [25].

3. Radiation fields

In this section we calculate the electromagnetic field induced by an arbitrarily evolving spacetime event, in order to identify and characterize the radiation part. In particular, we obtain the five Liénard-Wiechert potentials and the field strength tensor, which is conveniently expressed as a Clifford product of a pair orthogonal vectors. This bivector formulation simplifies calculation of field invariants, the mass-energy-momentum tensor, and the plane wave expansion. The radiation field is shown to be an equilibrium field of the standard Maxwell type.

3.1. Liénard-Wiechert potential

Beginning with a generic spacetime event $X^\mu(\tau)$ and using (29) and (44), we write the smoothed current

$$j_\varphi^\alpha(x, \tau) = \int ds \varphi(\tau - s) j^\alpha(x, s) = \int ds \varphi(\tau - s) \dot{X}^\alpha(s) \delta^4[x - X(s)] \quad (54)$$

where

$$\dot{X}^5(\tau) = \dot{\tau} = 1. \quad (55)$$

We choose the τ -instantaneous part of the Greens function (36), which recovers the Maxwell propagator under concatenation (the correlation Greens function has been studied in [26] and applied to radiation reaction in [27]). The Liénard-Wiechert potential $a^\alpha(x, \tau)$ induced by the current (54) is

$$a^\alpha(x, \tau) = -e \int d^4x' d\tau' G(x - x', \tau - \tau') j_\varphi^\alpha(x', \tau') \quad (56)$$

$$= -\frac{e}{2\pi} \int ds \varphi(\tau - s) \dot{X}^\alpha(s) \delta\left[\left(x - X(s)\right)^2\right] \theta^{ret} \quad (57)$$

$$= -\frac{e}{2\pi} \varphi(\tau - s) \frac{\dot{X}^\alpha(s)}{2(x^\mu - X^\mu(s)) \dot{X}_\mu(s)} \quad (58)$$

where we used the identity

$$\int d\tau f(\tau) \delta[g(\tau)] = \frac{f(s)}{|g'(s)|} \Big|_{s=g^{-1}(0)} \quad (59)$$

and the retarded time s satisfies

$$[x - X(s)]^2 = 0 \quad \theta^{ret} = \theta(x^0 - X^0(s)) = 1. \quad (60)$$

Introducing the timelike velocity

$$u^\alpha = \dot{X}^\alpha(s), \quad (61)$$

the vector from event $X(s)$ to observation point x

$$z^\mu = x^\mu - X^\mu(s) \Rightarrow \dot{z}^\mu = -u^\mu, \quad (62)$$

and the scalar function

$$R = \frac{1}{2} \frac{d}{ds} (x - X(s))^2 = -z^\mu u_\mu = -z \cdot u \geq 0, \quad (63)$$

we find

$$a^\alpha(x, \tau) = -e \frac{1}{2\lambda} \frac{u^\alpha}{4\pi R} e^{-|\tau-s|/\lambda} \quad (64)$$

where the nonnegativity of R follows from (60) and (61).

To calculate the field strengths, we need derivatives of the Liénard-Wiechert potential. The spacetime derivative is most conveniently found by applying the identity (59) to expression (57)

$$\partial^\mu a^\beta(x, \tau) = -\frac{e}{2\pi} \int ds \varphi(\tau - s) \dot{X}^\alpha(s) \theta^{ret} \partial^\mu \delta((x - X(s))^2) \quad (65)$$

$$= \frac{e}{2\pi} \int ds \varphi(\tau - s) \dot{X}^\beta(s) \theta^{ret} \delta'[(x - X(s))^2] [-2(x^\mu - X^\mu(s))] \quad (66)$$

$$= \frac{e}{2\pi} \int ds \varphi(\tau - s) \frac{\dot{X}^\beta(s) [x^\mu - X^\mu(s)]}{\dot{X}(s) \cdot (x - X(s))} \theta^{ret} \frac{d}{ds} \delta[(x - X(s))^2] \quad (67)$$

and integrating by parts to obtain

$$\partial^\mu a^\beta(x, \tau) = -\frac{e}{2\pi} \int ds \frac{d}{ds} \left[\varphi(\tau - s) \frac{\dot{X}^\beta(s) [x^\mu - X^\mu(s)]}{\dot{X}(s) \cdot (x - X(s))} \right] \theta^{ret} \delta[(x - X(s))^2] \quad (68)$$

$$= -\frac{e}{4\pi} \frac{1}{R} \frac{d}{ds} \left[\varphi(\tau - s) \frac{z^\mu u^\beta}{R} \right]. \quad (69)$$

Since

$$\frac{d}{d\tau} \varphi(\tau) = \frac{1}{2\lambda} \frac{d}{d\tau} e^{-|\tau|/\lambda} = -\frac{1}{2\lambda^2} \epsilon(\tau) e^{-|\tau|/\lambda} = -\frac{1}{\lambda} \epsilon(\tau) \varphi(\tau) \quad (70)$$

we obtain the τ -derivative directly from (64) as

$$\partial_\tau a_\mu(x, \tau) = -e \dot{\varphi}(\tau - s) \frac{u_\mu}{4\pi R} = e \epsilon(\tau - s) \varphi(\tau - s) \frac{u_\mu}{4\pi \lambda R}. \quad (71)$$

3.2. Field strengths

From (69) the spacetime components of the field strength tensor takes the anti-symmetric form

$$f^{\mu\nu} = -\frac{e}{4\pi} \frac{1}{R} \frac{d}{ds} \left[\varphi(\tau - s) \frac{z^\mu u^\nu - z^\nu u^\mu}{R} \right]. \quad (72)$$

Using (62), (63), and

$$\dot{R} = -\frac{d}{d\tau}(z \cdot u) = u^2 - z \cdot \dot{u} \quad (73)$$

the derivatives split into a retarded field exhibiting $|\mathbf{z}|^{-2}$ far-field behavior

$$f_{ret}^{\mu\nu} = e\varphi(\tau - s) \left[\frac{(z^\mu u^\nu - z^\nu u^\mu) u^2}{4\pi (u \cdot z)^3} - \epsilon(\tau - s) \frac{(z^\mu u^\nu - z^\nu u^\mu)}{4\pi\lambda (u \cdot z)^2} \right], \quad (74)$$

and a long-range radiation field exhibiting $|\mathbf{z}|^{-1}$ far-field behavior

$$f_{rad}^{\mu\nu} = e\varphi(\tau - s) \frac{(z^\mu \dot{u}^\nu - z^\nu \dot{u}^\mu)(u \cdot z) - (z^\mu u^\nu - z^\nu u^\mu)(\dot{u} \cdot z)}{4\pi (u \cdot z)^3}. \quad (75)$$

Because we take $\lambda \simeq 400$ seconds, we include the λ^{-1} term in the retarded field. Using (69) to find $\partial_\mu a_5$ and (71) to find $\partial_\tau a_\mu$ we calculate the fifth component fields as

$$f_{\mu 5}^{ret} = -e\varphi(\tau - s) \left[\frac{z_\mu u^2 - u_\mu (u \cdot z)}{4\pi (u \cdot z)^3} - \epsilon(\tau - s) \frac{z_\mu - u_\mu (u \cdot z)}{4\pi\lambda (u \cdot z)^2} \right] \quad (76)$$

$$f_{\mu 5}^{rad} = -e\varphi(\tau - s) \frac{(\dot{u} \cdot z) z_\mu}{4\pi (u \cdot z)^3}. \quad (77)$$

3.3. Electromagnetic two form

For notational simplicity, we will represent the field as a bivector

$$f = \frac{1}{2} f^{\alpha\beta} (\mathbf{e}_\alpha \wedge \mathbf{e}_\beta) = \frac{1}{2} f^{\alpha\beta} (\mathbf{e}_\alpha \otimes \mathbf{e}_\beta - \mathbf{e}_\beta \otimes \mathbf{e}_\alpha) \quad (78)$$

in a Clifford algebra (see [28] and references contained therein) over the formal 5D space with basis vectors

$$\mathbf{e}_\alpha \cdot \mathbf{e}_\beta = \eta_{\alpha\beta} \quad \alpha, \beta = 0, 1, 2, 3, 5. \quad (79)$$

In this notation, expression (72) for the spacetime component of the electromagnetic field becomes

$$f_{spacetime} = \frac{1}{2} f^{\mu\nu} (\mathbf{e}_\mu \wedge \mathbf{e}_\nu) = -\frac{e}{4\pi} \frac{1}{R} \frac{d}{ds} \left[\varphi(\tau - s) \frac{z \wedge u}{R} \right] \quad (80)$$

and the radiation part (75) is

$$f_{spacetime} = e\varphi(\tau - s) \frac{(z \wedge \dot{u})(u \cdot z) - (z \wedge u)(\dot{u} \cdot z)}{4\pi (u \cdot z)^3} \quad (81)$$

$$= e\varphi(\tau - s) z \wedge \frac{\dot{u}(u \cdot z) - u(\dot{u} \cdot z)}{4\pi (u \cdot z)^3}. \quad (82)$$

Similarly, the fifth component radiation field (77) takes the form

$$f_5 = \frac{1}{2} f^{\mu 5} (\mathbf{e}_\mu \wedge \mathbf{e}_5) = -e\varphi(\tau - s) \frac{(\dot{u} \cdot z) z \wedge \mathbf{e}_5}{4\pi (u \cdot z)^3}. \quad (83)$$

Introducing the scalar quantity

$$Q = -\dot{u} \cdot z \quad (84)$$

and the vector

$$w = \dot{u}R - uQ = -[\dot{u}(u \cdot z) - u(\dot{u} \cdot z)] \quad (85)$$

the radiation fields assume the form

$$f_{spacetime} = e\varphi(\tau - s) \frac{z \wedge w}{4\pi R^3} \quad (86)$$

$$f_5^{rad} = -e\varphi(\tau - s) \frac{z \wedge (Q\mathbf{e}_5)}{4\pi R^3}. \quad (87)$$

Designating formal 5-velocity and 5-acceleration as

$$U = \frac{d}{d\tau}(X(\tau), \tau) = (\dot{X}(\tau), 1) = u + \mathbf{e}_5 \quad \dot{U} = \frac{d}{d\tau}(\dot{X}(\tau), 1) = (\dot{u}, 0) = \dot{u} \quad (88)$$

so that

$$R = -z \cdot U \quad Q = -z \cdot \dot{U} \quad (89)$$

the radiation fields may be combined in the expression

$$f = f_{spacetime} + f_5 = e\varphi(\tau - s) \frac{z \wedge W}{4\pi R^3}. \quad (90)$$

where the 5D vector

$$W = \dot{U}R - UQ = w - Q\mathbf{e}_5 \quad (91)$$

depends on the velocity and acceleration of the event, characterizing the directionality of the motion inducing the radiation.

3.4. Field invariants

From (89) and the Clifford identity

$$a \cdot (b \wedge c) = (a \cdot b)c - (a \cdot c)b \quad (92)$$

the vector W can be expressed as

$$W = U(z \cdot \dot{U}) - \dot{U}(z \cdot U) = z \cdot (U \wedge \dot{U}) \quad (93)$$

describing the projection of z into the plane spanned by the vectors U and \dot{U} , and reflected through the direction of U . The vector W is orthogonal to z , which follows from the Clifford identity

$$(a \wedge b) \cdot (c \wedge d) = [(a \wedge b) \cdot c] \cdot d = (a \cdot c)(b \cdot d) - (b \cdot c)(a \cdot d) \quad (94)$$

as

$$z \cdot W = z \cdot [z \cdot (U \wedge \dot{U})] = (z \wedge z) \cdot (U \wedge \dot{U}) \quad (95)$$

or directly as

$$z \cdot W = z \cdot (\dot{U}R - UQ) = (z \cdot \dot{U})R - (z \cdot U)Q = -QR + RQ = 0. \quad (96)$$

The radiation field thus lies in the plane spanned by the orthogonal vectors z and W , where W is normal to z in the plane spanned by U and \dot{U} . It also follows from (96), (60) and the Clifford identity (92) that

$$z \cdot (z \wedge W) = z^2 W - z(z \cdot W) = 0 \quad (97)$$

so that the plane $z \wedge W$ is null with respect to the lightlike observation vector z . Writing the components of the field bivector (90) using

$$(z \wedge W)_{\beta\gamma} = z_\beta W_\gamma - z_\gamma W_\beta \quad (98)$$

the field invariant $\epsilon^{\alpha\beta\gamma\delta\epsilon} f_{\beta\gamma} f_{\delta\epsilon}$ is seen to vanish identically because the product of vectors satisfies

$$\epsilon^{\alpha\beta\gamma\delta\epsilon} (z_\beta W_\gamma - z_\gamma W_\beta) (z_\delta W_\epsilon - z_\epsilon W_\delta) = 4\epsilon^{\alpha\beta\gamma\delta\epsilon} z_\beta W_\gamma z_\delta W_\epsilon = 0. \quad (99)$$

Similarly, since $f^{\alpha\beta} f_{\alpha\beta}$ includes $(z \wedge W)^{\alpha\beta} (z \wedge W)_{\alpha\beta}$ it must vanish because using (94) we find

$$(z \wedge W) \cdot (z \wedge W) = (z \cdot W)^2 - z^2 W^2 = 0. \quad (100)$$

Thus, the radiation field (90) is seen to be a null field, satisfying

$$z \cdot f = 0 \longrightarrow z_\mu f^{\mu\alpha} = 0 \quad (101)$$

$$f \cdot f = 0 \longrightarrow f^{\alpha\beta} f_{\alpha\beta} = f^{\mu\nu} f_{\mu\nu} = f^{5\nu} f_{5\nu} = 0 \quad (102)$$

$$\epsilon^{\alpha\beta\gamma\delta\epsilon} f_{\beta\gamma} f_{\delta\epsilon} = 0 \longrightarrow \epsilon^{\mu\nu\rho\sigma} f_{\mu\nu} f_{\rho\sigma} = \epsilon^{\mu\nu\rho\sigma} f_{\sigma 5} f_{\rho\nu} = 0. \quad (103)$$

3.5. Mass-energy-momentum tensor

In [11] the mass-energy-momentum tensor was derived as the Noether current associated with the translation invariance of electromagnetic action in the form (28). Studying the translation invariance of the modified action (51), one finds the conservation law

$$\partial_\alpha T^{\alpha\beta} = e_0 f^{\alpha\beta} j_\alpha \quad (104)$$

for conserved tensor

$$T^{\alpha\beta} = -\lambda \left(g^{\alpha\beta} f^{\delta\gamma} f_{\delta\gamma}^\Phi - f_\gamma^\alpha f_\beta^{\gamma\Phi} \right), \quad (105)$$

where the modified field f_Φ is given in (52) and the sharp current j_α is defined in (29). Integrating (104) over spacetime, leads to

$$\frac{d}{d\tau} \int d^4x T^{5\alpha} = e_0 \int d^4x f^{\alpha\beta}(x, \tau) \dot{X}_\alpha \delta^4(x - X) = e_0 f^{\alpha\beta}(X, \tau) \dot{X}_\alpha \quad (106)$$

which by comparison with the Lorentz force (20) provides

$$\frac{d}{d\tau} \int d^4x \left(T^{5\mu} + m \dot{X}^\mu \right) = 0 \quad \frac{d}{d\tau} \int d^4x \left(T^{55} - \frac{1}{2} m \dot{X}^2 \right) = 0, \quad (107)$$

expressing the instantaneous conservation of total energy-momentum-mass for the combined field and event over all spacetime [29]. Since the τ -dependence of (90) resides in $\varphi(\tau - s)$, the τ -integration in (52) merely inverts $\varphi(\tau - s)$ to $\delta(\tau - s)$, and since

$$\delta(\tau - s) \varphi(\tau - s) = \delta(\tau - s) \varphi(0) = \frac{1}{2\lambda} \delta(\tau - s) \quad (108)$$

terms of the type $f^{\alpha\beta} f_{\delta\gamma}^\Phi$ become

$$f^{\alpha\beta} f_{\delta\gamma}^\Phi = \frac{1}{2\lambda} \delta(\tau - s) \left(\frac{e}{4\pi R^3} \right)^2 (z \wedge W)^{\alpha\beta} (z \wedge W)_{\delta\gamma}. \quad (109)$$

In light of (102), tensor (105) reduces to

$$T^{\alpha\beta} = \lambda f_{\gamma}^{\alpha} f_{\Phi}^{\beta\gamma} \quad (110)$$

containing the products

$$\begin{aligned} (z^{\alpha} W_{\gamma} - z_{\gamma} W^{\alpha}) (z^{\beta} W^{\gamma} - z^{\gamma} W^{\beta}) &= z^2 W^{\alpha} W^{\beta} + W^2 z^{\alpha} z^{\beta} - z \cdot W (z^{\alpha} W^{\beta} + z^{\beta} W^{\alpha}) \\ &= W^2 z^{\alpha} z^{\beta} \end{aligned} \quad (111)$$

and providing the simple expression

$$T^{\alpha\beta} = \frac{1}{2} \delta(\tau - s) \left(\frac{e}{4\pi R^3} \right)^2 W^2 z^{\alpha} z^{\beta}. \quad (112)$$

Because the observation vector z has no 5-component, this becomes

$$T^{\alpha 5} = 0 \quad T^{\mu\nu} = \frac{1}{2} \delta(\tau - s) \left(\frac{e}{4\pi R^3} \right)^2 W^2 z^{\mu} z^{\nu} \quad (113)$$

so that the total (over all spacetime) mass and energy-momentum carried by the radiation field vanishes.

Integrating (104) over space and τ — equivalent to the space integration of the concatenated dynamics — we recover an expression for the Maxwell Poynting vector

$$\frac{d}{dt} \int d^3x \, d\tau \, T^{0\mu} = e_0 \int d^3x \, d\tau \, f^{\alpha\mu}(x, \tau) \dot{X}_{\alpha}(\tau) \delta^4[x - X(\tau)] \quad (114)$$

where using (113), the energy-momentum

$$P^{\mu} = \int d^3x \, d\tau \, T^{0\mu} = \frac{1}{2} \left(\frac{e}{4\pi R^3} \right)^2 W^2 z^0 z^{\mu} \quad (115)$$

is found to be oriented along the observation vector z^{μ} . We notice that the RHS of (114) differs from the Maxwell formulation through the τ -dependence of $f^{\alpha\beta}(x, \tau)$. In the equilibrium limit $f^{\mu\nu}(x, \tau) \rightarrow \frac{1}{\lambda} F^{\mu\nu}(x)$, (114) is seen to recover the standard form of energy-momentum conservation

$$\frac{d}{dt} \int d\tau \, T^{0\mu} = e F^{\mu\nu}(x) \int d\tau \, \dot{X}_{\nu}(\tau) \delta^4[x - X(\tau)] = e F^{\nu\mu}(x) J_{\nu}(x). \quad (116)$$

3.6. Vector field picture

To compare the radiation field (90) with the standard Maxwell field, we write $f^{\alpha\beta}$ in vector components [11] as

$$e^i = f^{0i} \quad b^i = \epsilon^{ijk} f_{jk} \quad \varepsilon^{\mu} = f^{5\mu} = (\varepsilon^0, \varepsilon) \quad (117)$$

for which the field equations are

$$\begin{aligned} \nabla \cdot \mathbf{e} - \partial_{\tau} \varepsilon^0 &= e j^0 & \nabla \times \mathbf{e} + \partial_0 \mathbf{b} &= 0 \\ \nabla \times \mathbf{b} - \partial_0 \mathbf{e} - \partial_{\tau} \varepsilon &= e \mathbf{j} & \nabla \cdot \mathbf{b} &= 0 \\ \nabla \cdot \varepsilon + \partial_0 \varepsilon^0 &= e j^5 & \nabla \times \varepsilon + \partial_{\tau} \mathbf{b} &= 0 \\ \nabla \varepsilon^0 - \partial_{\tau} \mathbf{e} + \partial_0 \varepsilon &= 0 \end{aligned} \quad (118)$$

and the Poynting vectors are

$$T^{00} = \frac{\lambda}{2} [\mathbf{e} \cdot \mathbf{e}^\Phi + \mathbf{b} \cdot \mathbf{b}^\Phi - \varepsilon \cdot \varepsilon^\Phi - \varepsilon^0 \varepsilon_\Phi^0] \quad T^{0i} = \lambda [\mathbf{e} \times \mathbf{b}^\Phi - \varepsilon^0 \varepsilon^\Phi]^i \quad (119)$$

$$T^{55} = \frac{\lambda}{2} [-\mathbf{e} \cdot \mathbf{e}^\Phi + \mathbf{b} \cdot \mathbf{b}^\Phi + \varepsilon \cdot \varepsilon^\Phi - \varepsilon^0 \varepsilon_\Phi^0] \quad T^{5i} = \lambda [\varepsilon^0 \mathbf{e}^\Phi + \varepsilon \times \mathbf{b}^\Phi]^i. \quad (120)$$

From the invariants (102) and (103) it follows that

$$-\mathbf{e} \cdot \mathbf{e} + \mathbf{b} \cdot \mathbf{b} = \varepsilon \cdot \varepsilon - \varepsilon^0 \varepsilon^0 = 0 \Rightarrow \begin{cases} T^{00} = \lambda [\mathbf{e} \cdot \mathbf{e}^\Phi - \varepsilon \cdot \varepsilon^\Phi] \\ T^{55} = 0 \end{cases} \quad (121)$$

$$\mathbf{e} \times \mathbf{b} = (\mathbf{e} \cdot \mathbf{e}) \hat{\varepsilon} \quad \varepsilon^0 \mathbf{e} = -\varepsilon \times \mathbf{b} \Rightarrow \begin{cases} T^{0i} = \lambda [\mathbf{e} \cdot \mathbf{e}^\Phi - \varepsilon \cdot \varepsilon^\Phi] \hat{\varepsilon}^i \\ T^{5i} = 0 \end{cases} \quad (122)$$

where from (77)

$$\hat{\varepsilon} = \frac{\varepsilon}{|\varepsilon|} = \hat{\mathbf{z}} \quad (123)$$

again indicating that the Maxwell Poynting vector is oriented along the observation vector.

A general plane wave expansion is taken by writing the Fourier transform

$$f^{\alpha\beta}(x, \tau) = \frac{1}{(2\pi)^5} \int d^4k d\kappa e^{i(k \cdot x - \kappa \tau)} f^{\alpha\beta}(k, \kappa) \quad (124)$$

for which the wave equation (34) in vacuum gives

$$k^\alpha k_\alpha = k^\mu k_\mu - \kappa^2 = \mathbf{k}^2 - (k^0)^2 - \kappa^2 = 0 \quad (125)$$

so that concatenation of $f^{\alpha\beta}(x, \tau)$ forces the photon onto the $\kappa \rightarrow 0$ zero-mass shell. The general plane wave solution to the 3-vector field equations (118) in terms of the transverse component of \mathbf{e} and the longitudinal component of ε was given in [29] as

$$\mathbf{e} = \mathbf{e}_\perp - \frac{\kappa}{k^0} \varepsilon_\parallel \quad \mathbf{h} = \frac{1}{k^0} \mathbf{k} \times \mathbf{e}_\perp \quad \varepsilon = \varepsilon_\parallel + \frac{\kappa}{k^0} \mathbf{e}_\perp \quad \varepsilon^0 = \frac{1}{k^0} \mathbf{k} \cdot \varepsilon_\parallel. \quad (126)$$

On the zero-mass shell, the \mathbf{e} and \mathbf{h} fields take the Maxwell form — mutually orthogonal with equal amplitude, and both orthogonal to the propagation vector \mathbf{k} — and the ε field decouples from \mathbf{e} and \mathbf{h} , becoming purely longitudinal. In terms of solution (126) the components of the energy-momentum tensor are

$$\begin{aligned} T^{00} &= \lambda (\mathbf{e}_\perp \cdot \mathbf{e}_\perp^\Phi - \varepsilon_\parallel \cdot \varepsilon_\parallel^\Phi) & T^{0i} &= \frac{k^i}{k^0} T^{00} \\ T^{55} &= \left(\frac{\kappa}{k^0} \right)^2 T^{00} & T^{5\mu} &= \frac{\kappa}{k^0} \frac{k^\mu}{k^0} T^{00} \end{aligned} \quad (127)$$

and we see that $T^{5\alpha}$ vanishes on the zero-mass shell. Comparison of (113) with (127) for the $\kappa = 0$ case shows that the radiation field from the arbitrary event has the character of an equilibrium plane wave solution.

4. Abraham-Lorentz-Dirac equation

In the previous section we examined the field produced by an arbitrarily evolving event $X^\mu(\tau)$, and in particular studied the characteristics of its radiation field, identified by the far-field behavior. In this section, we sharpen the characterization of the radiation field, following Dirac's analysis [1] of the Greens functions, and obtain the ALD equation by specifically accounting for the field produced by the event's own evolution.

4.1. Radiation reaction

The Lorentz force (20) describes the motion of an event under the influence of a general electromagnetic field $f^{\alpha\beta}(x, \tau)$. Depending on the known boundary conditions, $f^{\alpha\beta}(x, \tau)$ may be described as an initial incoming field and the retarded reaction, or as a final outgoing field and the advanced reaction, according to

$$f^{\alpha\beta}(x, \tau) = f_{in}^{\alpha\beta}(x, \tau) + f_{ret}^{\alpha\beta}(x, \tau) = f_{out}^{\alpha\beta}(x, \tau) + f_{adv}^{\alpha\beta}(x, \tau) \quad (128)$$

where

$$f_{ret}^{\alpha\beta}(x, \tau) \xrightarrow{\tau \rightarrow -\infty} 0 \quad f_{adv}^{\alpha\beta}(x, \tau) \xrightarrow{\tau \rightarrow \infty} 0. \quad (129)$$

Thus, when an incoming field $f_{in}^{\alpha\beta}(x, \tau)$ impinges on the event, the Lorentz force can be expressed as

$$m\ddot{X}^\mu(\tau) = \lambda e \left[f_{in}^{\mu\beta}(X, \tau) + f_{ret}^{\mu\beta}(X, \tau) \right] \dot{X}_\beta(\tau) \quad (130)$$

where $f_{ret}^{\alpha\beta}(x, \tau)$ is produced by the event $X^\mu(\tau)$ itself through the field equation (53). However, the interaction of the event with its own field includes an infinite part, and to identify the finite part, Dirac argued that the outgoing field can be expressed as the sum of the incoming field and a radiation reaction produced when the incoming field accelerates the event. Thus,

$$f_{out}^{\alpha\beta} = f_{in}^{\alpha\beta} + f_{rad}^{\alpha\beta} \quad (131)$$

which combines with (128) to provide

$$f_{rad}^{\alpha\beta} = f_{out}^{\alpha\beta} - f_{in}^{\alpha\beta} = (f^{\alpha\beta} - f_{adv}^{\alpha\beta}) - (f^{\alpha\beta} - f_{ret}^{\alpha\beta}) = f_{ret}^{\alpha\beta} - f_{adv}^{\alpha\beta}. \quad (132)$$

Considering the combinations

$$f_{ret}^{\alpha\beta} = \frac{1}{2} (f_{ret}^{\alpha\beta} - f_{adv}^{\alpha\beta}) + \frac{1}{2} (f_{ret}^{\alpha\beta} + f_{adv}^{\alpha\beta}) = \frac{1}{2} f_{rad}^{\alpha\beta} + \frac{1}{2} (f_{ret}^{\alpha\beta} + f_{adv}^{\alpha\beta}) \quad (133)$$

the radiation reaction $f_{rad}^{\alpha\beta}$ is found to be finite, while the the infinite part of the self-interaction $f_{ret}^{\alpha\beta} + f_{adv}^{\alpha\beta}$ is treated by mass renormalization. The finite part of the Lorentz force is then

$$m\ddot{X}^\mu = \lambda e \left[\frac{1}{2} f_{rad}^{\mu\beta} + f_{in}^{\mu\beta} \right] \dot{X}_\beta. \quad (134)$$

From the prescription (132) the potential $a_{rad}^\beta(x, \tau)$ is found by making the substitution

$$\theta^{ret} \longrightarrow \theta^{rad} = \theta^{ret} - \theta^{adv} = \epsilon(x^0 - X^0) \quad (135)$$

in (57), so that the self-interaction term in (134) is the force

$$F_{self}^{\mu}(\tau) = \frac{1}{2} \lambda e f_{rad}^{\mu\beta}(X(\tau), \tau) \dot{X}_{\beta}(\tau) \quad (136)$$

evaluated at the event $X^{\mu}(\tau)$. The field must therefore be calculated from

$$\partial^{\mu} a_{rad}^{\beta}(x, \tau) = -\frac{e}{2\pi} \int ds \frac{d}{ds} \left[\varphi(\tau - s) \frac{\dot{X}^{\beta}(s) [x^{\mu} - X^{\mu}(s)]}{\dot{X}(s) \cdot [x - X(s)]} \right] \theta^{rad} \delta[(x - X(s))^2] \quad (137)$$

in the limit

$$x^{\mu} - X^{\mu}(s) \rightarrow X^{\mu}(\tau) - X^{\mu}(s) \quad (138)$$

which by conditions (62) and (60) require

$$[X(\tau) - X(s)]^2 = 0 \quad \Rightarrow \quad s \rightarrow \tau. \quad (139)$$

Dirac showed [1] that expressions of the type (137) have a finite limit as $X^{\mu}(s) \rightarrow X^{\mu}(\tau)$, found by performing a Taylor expansion along worldline $X(\tau)$, that is, evaluating

$$\partial^{\mu} a_{rad}^{\beta}(X(\tau'), \tau) \quad (140)$$

where

$$\tau' = \tau + h \quad (141)$$

$$X(\tau') - X(\tau) = h \dot{X}(\tau) + \frac{h^2}{2} \ddot{X}(\tau) + \frac{h^3}{3!} \dddot{X}(\tau) + \dots \quad (142)$$

$$\dot{X}(\tau') = \dot{X}(\tau) + h \ddot{X}(\tau) + \frac{h^2}{2} \dddot{X}(\tau) + \dots \quad (143)$$

By comparison of (137) with the expression for the field in standard Maxwell theory [1, 2]

$$\partial^{\mu} A_{rad}^{\nu}(x) = -\frac{e}{2\pi} \int ds \frac{d}{ds} \left[\frac{\dot{X}^{\nu}(s) [x^{\mu} - X^{\mu}(s)]}{\dot{X}(s) \cdot [x - X(s)]} \right] \theta^{rad} \delta[(x - X(s))^2], \quad (144)$$

the field expansion in off-shell electrodynamics is seen to take the standard form

$$f_{rad}^{\mu\nu}(X(\tau), \tau) = \frac{2}{3} \frac{e}{2\pi} \left[\dot{X}^{\mu}(\tau) \ddot{X}^{\nu}(\tau) - \dot{X}^{\nu}(\tau) \ddot{X}^{\mu}(\tau) \right] \quad (145)$$

where we use $\varphi(0) = \frac{1}{2\lambda}$. Now, the self-interaction term in (136) is

$$F_{self}^{\mu}(\tau) = \lambda \frac{2}{3} \frac{e^2}{2\pi} \left[\dot{X}^{\mu} \ddot{X}^{\nu} - \dot{X}^{\nu} \ddot{X}^{\mu} \right] \dot{X}_{\nu} = \frac{2}{3} \frac{e^2}{4\pi} \left[\dot{X}^{\mu} \ddot{X}^{\nu} \dot{X}_{\nu} - \dot{X}^2 \ddot{X}^{\mu} \right]. \quad (146)$$

Since the radiation field was seen to carry no mass, the event may be considered on-shell, so that

$$\dot{X}^2 = -1 \quad \dot{X} \cdot \ddot{X} = 0 \quad \dot{X} \cdot \ddot{X} + \ddot{X}^2 = 0, \quad (147)$$

and the Lorentz force (134) takes the form of the usual ALD equation

$$m \ddot{x}^{\mu} = e_0 f_{in}^{\mu\beta}(x, \tau) \dot{x}_{\beta} + m \tau_0 [\ddot{x}^{\mu} - \ddot{x}^2 \dot{x}^{\mu}] \quad (148)$$

where we have written τ_0 as in (7), and reverted to lower case for the event trajectory

$$X^{\mu}(\tau) \rightarrow x^{\mu}(\tau) \quad (149)$$

since the field point is understood to be the event location

$$f_{in}^{\mu\beta}(x, \tau) = f_{in}^{\mu\beta}(x(\tau), \tau). \quad (150)$$

4.2. Integro-differential equation

In order to eliminate the run-away solutions admitted by the ALD equation in the form (148), we convert it to an integro-differential equation

$$(\ddot{x}^\mu - \tau_0 \ddot{\ddot{x}}^\mu) e^{-\tau/\tau_0} = e^{-\tau/\tau_0} \left[\frac{e_0}{m} f_{in}^{\mu\beta}(x, \tau) \dot{x}_\beta - \tau_0 \ddot{x}^2 \dot{x}^\mu \right] \quad (151)$$

$$\ddot{x}^\mu e^{-\tau/\tau_0} = - \int_0^\tau ds e^{-s/\tau_0} \left[\frac{e_0}{\tau_0 m} f_{in}^{\mu\beta}(x, s) \dot{x}_\beta - \ddot{x}^2 \dot{x}^\mu \right] + \ddot{x}^\mu(0). \quad (152)$$

The run-away solutions are suppressed by imposing the requirement that velocity grow less than exponentially for long times

$$\ddot{x}^\mu(\tau) e^{-\tau/\tau_0} \xrightarrow{\tau \rightarrow \infty} 0 \quad (153)$$

so that as $\tau \rightarrow \infty$ (152) becomes

$$0 = - \int_0^\infty ds e^{-s/\tau_0} \left[\frac{e_0}{\tau_0 m} f_{in}^{\mu\beta}(x, s) \dot{x}_\beta(s) - \ddot{x}^2(s) \dot{x}^\mu(s) \right] + \ddot{x}^\mu(0) \quad (154)$$

which is incorporated back into (152) to provide the ALD equation as

$$m\ddot{x}^\mu(\tau) = \int_\tau^\infty ds \frac{1}{\tau_0} e^{-(s-\tau)/\tau_0} \left[e_0 f_{in}^{\mu\beta}(x(s), s) \dot{x}_\beta(s) - m\tau_0 \ddot{x}^2(s) \dot{x}^\mu(s) \right]. \quad (155)$$

This expression describes an acceleration at τ that depends on values of the interaction at later times, as was seen for the standard ALD equation (8), which is recovered from (155) in the Maxwell limit

$$e_0 f_{in}^{\mu\beta}(x(\tau), \tau) = e\lambda f_{in}^{\mu\beta}(x(\tau), \tau) \rightarrow eF_{in}^{\mu\beta}(x(\tau)). \quad (156)$$

In the context of Maxwell theory, the field depends only on the spacetime location of the particle on which it acts, so that the synchronization of the particle-field interaction is sharply determined by the proper time τ of the particle motion. In the off-shell formalism, however, the τ -dependence of the electromagnetic field typically introduces a statistical synchronization between interacting events, on a time scale λ much larger than τ_0 . The integration in (155) may thus be understood as analogous to this smoothed synchronization, describing a modified self-interaction between the event and a radiation field induced by an ensemble associated with the event evolution.

To clarify this interpretation, we use (52) to express $f_{in}^{\mu\beta}(x, \tau)$ in terms of $f_\Phi^{\mu\beta}(x, \tau)$, a sharp field induced by a sharp event density $j(x, \tau)$ of the type (29). From (50) and (53) the external field can be written as an ensemble average of such sharp fields

$$f_{in}^{\mu\beta}(x, \tau) = \int_{-\infty}^\infty ds f_\Phi^{\mu\beta}(x(\tau), s) \varphi(s - \tau) \quad (157)$$

leading to an integral expression for the Lorentz force (20)

$$m\ddot{x}^\mu(\tau) = e_0 f^{\mu\beta}(x(\tau), \tau) \dot{x}_\beta(\tau) = \int_{-\infty}^\infty ds e_0 f_\Phi^{\mu\beta}(x(\tau), s) \varphi(s - \tau) \dot{x}_\beta(\tau). \quad (158)$$

By comparison, the ALD equation (155) can be written as

$$m\ddot{x}^\mu(\tau) = \int_{-\infty}^\infty ds \left[e_0 f_{in}^{\mu\beta}(x(s), s) \dot{x}_\beta(s) - m\tau_0 \ddot{x}^2(s) \dot{x}^\mu(s) \right] \phi(s - \tau) \quad (159)$$

where we interpret the integrating factor as a normalized distribution

$$\phi(\tau) = \frac{1}{\tau_0} e^{-\tau/\tau_0} \theta(\tau) \quad \int_{-\infty}^{\infty} d\tau \phi(\tau) = 1. \quad (160)$$

Aside from the additional interaction term $\ddot{x}^2 \dot{x}^\mu$, equation (159) differs from (158) in that the statistical synchronization affected by the distribution $\phi(\tau)$ extends to both the field $f_{in}^{\mu\beta}(x, s)$ and the velocity $\dot{x}_\beta(s)$.

In order to examine the causal structure of (155) more closely, we shift the integration variable as $s \rightarrow s - \tau$

$$\begin{aligned} m\ddot{x}^\mu(\tau) = e_0 \int_0^\infty ds \frac{1}{\tau_0} e^{-s/\tau_0} f_{in}^{\mu\beta}(x, s + \tau) \dot{x}_\beta(s + \tau) \\ - m\tau_0 \int_0^\infty ds \frac{1}{\tau_0} e^{-s/\tau_0} \ddot{x}^2(s + \tau) \dot{x}^\mu(s + \tau). \end{aligned} \quad (161)$$

and consider the genealogy of the field. Although the incoming field $f_{in}^{\mu\beta}(x, \tau)$ is taken to be external, it must have been induced by the smoothed current $j_\varphi(x, \tau)$ associated with the sharp event densities $j(x, \tau)$ distributed along the worldlines of some configuration of evolving events. Using (157) for the field induced directly by the sharp event density $j(x, \tau)$, we rewrite (155) as

$$\begin{aligned} m\ddot{x}^\mu(\tau) = e_0 \int_0^\infty ds \frac{1}{\tau_0} e^{-s/\tau_0} \int_{-\infty}^\infty ds' \frac{1}{2\lambda} e^{-|s+\tau-s'|/\lambda} f_\Phi^{\mu\beta}(x, s') \dot{x}_\beta(s + \tau) \\ - m\tau_0 \int_0^\infty ds \frac{1}{\tau_0} e^{-s/\tau_0} \ddot{x}^2(s + \tau) \dot{x}^\mu(s + \tau) \end{aligned} \quad (162)$$

and rearrange the first line by combining the exponential functions

$$e^{-s/\tau_0} e^{-|s+\tau-s'|/\lambda} = \begin{cases} \exp\left[-s\left(\frac{1}{\tau_0} + \frac{1}{\lambda}\right) - (\tau - s')\frac{1}{\lambda}\right] & , \quad -\infty < s' \leq s + \tau \\ \exp\left[-s\left(\frac{1}{\tau_0} - \frac{1}{\lambda}\right) + (\tau - s')\frac{1}{\lambda}\right] & , \quad s + \tau \leq s' < \infty \end{cases} \quad (163)$$

and partitioning the s' -integration according to

$$\int_0^\infty ds \int_{-\infty}^\infty ds' \{\dots\} = \int_0^\infty ds \left[\int_{-\infty}^{s+\tau} ds' \{\dots\} + \int_{s+\tau}^\infty ds' \{\dots\} \right]. \quad (164)$$

The relationship of time scales $\lambda \gg \tau_0$ permits the approximation

$$\exp\left[-s\left(\frac{1}{\tau_0} \pm \frac{1}{\lambda}\right)\right] = e^{-s/\tau_0} + o\left(\frac{\tau_0}{\lambda}\right) \quad (165)$$

making no assumptions about the functional forms of events or fields, and so the integrals in the first line of (155) becomes

$$\begin{aligned} \int_0^\infty ds \frac{1}{\tau_0} e^{-s/\tau_0} \dot{x}_\beta(s + \tau) \times \\ \left\{ \int_{-\infty}^{s+\tau} ds' \frac{1}{2\lambda} e^{-(\tau-s')/\lambda} f_\Phi^{\mu\beta}(x, s') + \int_{s+\tau}^\infty ds' \frac{1}{2\lambda} e^{(\tau-s')/\lambda} f_\Phi^{\mu\beta}(x, s') \right\}. \end{aligned} \quad (166)$$

Further partitioning the s' -integrations as

$$\begin{aligned} \int_{-\infty}^{s+\tau} ds' \frac{1}{2\lambda} e^{-(\tau-s')/\lambda} f_{\Phi}^{\mu\beta}(x, s') &= \int_{-\infty}^{\tau} ds' \frac{1}{2\lambda} e^{-|\tau-s'|/\lambda} f_{\Phi}^{\mu\beta}(x, s') \\ &+ \int_{\tau}^{s+\tau} ds' \frac{1}{2\lambda} e^{-(\tau-s')/\lambda} f_{\Phi}^{\mu\beta}(x, s') \end{aligned} \quad (167)$$

and

$$\begin{aligned} \int_{s+\tau}^{\infty} ds' \frac{1}{2\lambda} e^{(\tau-s')/\lambda} f_{\Phi}^{\mu\beta}(x, s') &= \int_{\tau}^{\infty} ds' \frac{1}{2\lambda} e^{-|\tau-s'|/\lambda} f_{\Phi}^{\mu\beta}(x, s') \\ &- \int_{\tau}^{s+\tau} ds' \frac{1}{2\lambda} e^{(\tau-s')/\lambda} f_{\Phi}^{\mu\beta}(x, s') \end{aligned} \quad (168)$$

the integrals in the first line of (155) are now

$$\begin{aligned} \int_0^{\infty} ds \frac{1}{\tau_0} e^{-s/\tau_0} \dot{x}_{\beta}(s + \tau) \times \\ \left\{ \int_{-\infty}^{\infty} ds' \frac{1}{2\lambda} e^{-|\tau-s'|/\lambda} f_{\Phi}^{\mu\beta}(x, s') \right. \\ \left. + \int_{\tau}^{s+\tau} ds' \frac{1}{2\lambda} \left[e^{-(\tau-s')/\lambda} - e^{(\tau-s')/\lambda} \right] f_{\Phi}^{\mu\beta}(x, s') \right\} + o\left(\frac{\tau_0}{\lambda}\right) \end{aligned} \quad (169)$$

in which, from (157), we recognize the second line as $f_{in}^{\mu\beta}(x, \tau)$. Shifting $s' - \tau \rightarrow s'/\lambda$ in the third line of (169) leads to

$$\int_{\tau}^{s+\tau} ds' \frac{1}{\lambda} \sinh\left(\frac{s' - \tau}{\lambda}\right) f_{\Phi}^{\mu\beta}(x, s') = \int_0^{s/\lambda} ds' \sinh(s') f_{\Phi}^{\mu\beta}(x, \lambda s' + \tau) \quad (170)$$

which for smooth f_{Φ} may be estimated as being of order $o(\tau_0^2/\lambda^2)$:

$$\int_0^{s/\lambda} ds' \sinh(s') f_{\Phi}^{\mu\beta}(x, \lambda s' + \tau) \simeq \int_0^{\tau_0/\lambda} ds' \sinh(s') f_{\Phi}^{\mu\beta}(x, \lambda s' + \tau) \quad (171)$$

$$\simeq \frac{\tau_0^2}{\lambda^2} f_{\Phi}^{\mu\beta}(x, \tau) - \frac{\tau_0^3}{3\lambda^2} \frac{d}{d\tau} f_{\Phi}^{\mu\beta}(x, \tau). \quad (172)$$

Shifting the s -integral back to $s \rightarrow s - \tau$, and using the distribution $\phi(\tau)$ defined in (160) to extend the limits of integration, the ALD equation (155) now takes the form

$$\begin{aligned} m\ddot{x}^{\mu}(\tau) &= e_0 \int_{-\infty}^{\infty} ds f_{in}^{\mu\beta}(x(s), \tau) \phi(s - \tau) \dot{x}_{\beta}(s) + o\left(\frac{\tau_0}{\lambda}\right) \\ &+ e_0 \int_{-\infty}^{\infty} ds \phi(s - \tau) \dot{x}_{\beta}(s) \int_0^{(s-\tau)/\lambda} ds' \sinh(s') f_{\Phi}^{\mu\beta}(x(s), \lambda s' + \tau) \\ &- m\tau_0 \int_{-\infty}^{\infty} ds \phi(s - \tau) \ddot{x}^2(s) \dot{x}^{\mu}(s) \end{aligned} \quad (173)$$

where we have made no assumptions about the τ -dependence of events or fields. In this expression, the acceleration at time τ is given by the interaction of the field at time τ with a narrow ensemble of events described by the distribution $\phi(\tau)$.

If the field $f_{\Phi}^{\mu\beta}(x, \tau)$ varies smoothly, we see from (172) that the second line in (173) can be neglected. However, this term may be significant for external fields of the type (72), which are of the form

$$f_{in}^{\mu\beta}(x, \tau) = \varphi(\tau - \tau_{ret}) F_{Maxwell}^{\mu\beta}(x) \Big|_{\tau_{ret}} \quad (174)$$

and associated with the sharp form

$$f_{\Phi}^{\mu\beta}(x, \tau) = \delta(\tau - \tau_{ret}) F_{Maxwell}^{\mu\beta}(x) \Big|_{\tau_{ret}}. \quad (175)$$

Then, the integral s' integration is

$$\int_0^{(s-\tau)/\lambda} ds' \sinh(s') f_{\Phi}^{\mu\beta}(x, \lambda s' + \tau) = F_{Maxwell}^{\mu\beta}(x) \times \int_0^{(s-\tau)/\lambda} ds' \sinh(s') \delta(\lambda s' + \tau - \tau_{ret}) \quad (176)$$

$$= F_{Maxwell}^{\mu\beta}(x) \sinh\left(\frac{\tau - \tau_{ret}}{\lambda}\right) \theta(s - \tau_{ret}) \quad (177)$$

and the s integration becomes

$$e_0 \int_{\tau_{ret}}^{\infty} ds \frac{1}{\tau_0} e^{-(s-\tau)/\tau_0} F_{Maxwell}^{\mu\beta}(x(s)) \dot{x}_{\beta}(s) \sinh\left(\frac{\tau - \tau_{ret}}{\lambda}\right) \quad (178)$$

which can be put into the form

$$e_0 e^{-(\tau - \tau_{ret})/\tau_0} \sinh\left(\frac{\tau - \tau_{ret}}{\lambda}\right) \theta(\tau - \tau_{ret}) \times \quad (179)$$

$$\int_{\tau_{ret}}^{\infty} ds \frac{1}{\tau_0} F_{Maxwell}^{\mu\beta}(x(s)) \dot{x}_{\beta}(s) e^{-(s-\tau_{ret})/\tau_0} \quad (180)$$

so that the function

$$\psi(\tau - \tau_{ret}) = e^{-(\tau - \tau_{ret})/\tau_0} \sinh\left(\frac{\tau - \tau_{ret}}{\lambda}\right) \theta(\tau - \tau_{ret}) \sim o\left(\frac{\tau_0}{\lambda}\right) \quad (181)$$

limits the significance of this term. For fields of this type, ALD takes the form

$$\begin{aligned} m\ddot{x}^{\mu}(\tau) &= e_0 \int_{-\infty}^{\infty} ds f_{in}^{\mu\beta}(x(s), \tau) \phi(s - \tau) \dot{x}_{\beta}(s) \\ &+ e_0 \psi(\tau - \tau_{ret}) \int_{-\infty}^{\infty} ds \phi(s - \tau_{ret}) F_{Maxwell}^{\mu\beta}(x(s)) \dot{x}_{\beta}(s) \\ &- m\tau_0 \int_{-\infty}^{\infty} ds \phi(s - \tau) \ddot{x}^2(s) \dot{x}^{\mu}(s) + o\left(\frac{\tau_0}{\lambda}\right). \end{aligned} \quad (182)$$

The ALD equations, in the form (173) or (182), now appear as an interaction between the instantaneous external field and short-range ensemble averages over the specific combinations of event velocity $\dot{x}^{\mu}(\tau)$ and acceleration $\ddot{x}^{\mu}(\tau)$ that produce the radiation field (145). These averages smooth the τ -synchronization between the event and its radiation reaction, an effect that is qualitatively analogous to the statistical synchronization expressed in (53) between the event and its induced current.

5. Conclusion

Historically, the Abraham-Lorentz equation, a nonrelativistic approximation to (4), was first obtained [30, 31] by adding an effective term to the Lorentz force to account for the energy lost by an accelerating particle to Larmor radiation. In the $|\dot{\mathbf{x}}| \ll 1$ approximation (in which case $\tau \rightarrow t$), ALD reduces to

$$\ddot{\mathbf{x}}(t) - \tau_0 \ddot{\mathbf{x}}(t) = \frac{e}{m} \mathbf{E}_{ext}(t, \mathbf{x}) \quad (183)$$

which admits a runaway solution for $\mathbf{E}_{ext} = 0$ given by

$$\dot{\mathbf{x}} = \dot{\mathbf{x}}(0) e^{t/\tau_0}. \quad (184)$$

Imposing once again the boundary condition (153), which merely requires that velocity grow less than exponentially over long times, equation (183) may be converted to the integro-differential equation

$$\ddot{\mathbf{x}}(t) = \frac{e}{m} \int_{-\infty}^{\infty} dt' \left[\frac{1}{\tau_0} e^{-(t'-t)/\tau_0} \theta(t' - t) \right] \mathbf{E}_{ext}(t', \mathbf{x}(t')), \quad (185)$$

suppressing the spontaneous acceleration. However, the seemingly innocent boundary condition, without which the integration over t' could not be made sensible, provides just the pretext under which the pre-acceleration evident in (185) is granted admissibility. Conversely, to insist that one cannot give meaning to the apparent violation of classical causality under the integral is equivalent to rejecting the reasonableness of the boundary condition. Jackson [32] summarizes the conventional interpretation of this situation by accepting a possible violation of microscopic causality over time scale τ_0 , because it cannot disagree with experiment. On this view, since the external field cannot undergo a macroscopic binary transition in an interval smaller than τ_0 , and since measurements at these time scales will be dominated by quantum effects, the violation of classical causality in classical mechanics leads to no experimental contradiction. Although the relativistic ALD equation (8) may be accepted with the same qualification, an interpretation that preserves some sense of retarded causality in the classical context would be more satisfying.

As derived in the off-shell electrodynamics associated with Stueckelberg canonical covariant mechanics, the ALD equation in the form (155) was seen to differ from the standard expression in Maxwell theory (8) only in the dependence of the external field on the invariant time parameter τ . It was shown in section 2 that the general τ -dependence of off-shell fields can be understood as an expansion of the U(1) local gauge group to include the invariant evolution parameter, but in any case, follows directly from the unconstrained commutation relations (15) among coordinates and velocities of spacetime events. Thus, off-shell electrodynamics describes a microscopic interaction between spacetime events $x^\mu(\tau)$ mediated by instantaneous τ -dependent fields $f_{\alpha\beta}(x, \tau)$. However, it was seen in (53) that while the off-shell fields are τ -dependent, they are induced by an event current associated with an ensemble of events distributed in τ along the particle worldline. This distribution introduces an underlying statistical structure to the classical event-event interaction, equivalent to relaxing the sharp τ -synchronization found in expressions (8) and (185) between an event and a field with which it interacts. Detailed consideration of the resulting τ -dependence of the external field enables the transformation of equation (155) to the form (173), in which the Lorentz force depends on the instantaneous value of the external field and short-range ensemble averages over combinations of event velocity $\dot{x}^\mu(\tau)$ and acceleration $\ddot{x}^\mu(\tau)$. These remaining averages, which involve only the dynamical variables of the event evolution, can be understood as artifacts of the suppression of the runaway solutions. On the other hand, the combinations of dynamical variables in these expressions derive from the form of the radiation field (145) emitted by the event, and therefore the integrations may be seen as qualitatively analogous to the statistical synchronization expressed in (53) between the event and the field it induces.

References

- [1] Dirac P A M 1938 *Proc. R. Soc. London A* **167** 148
- [2] Barut A O 1980 *Electrodynamics and Classical Theory of Fields and Particles* (New York: Dover)
- [3] Johnson P R and Hu B L 2002 *Phys. Rev. D* **65** 065015
- [4] Stueckelberg E C G 1941 *Helv. Phys. Acta* **14** 322
 Stueckelberg E C G 1941 *Helv. Phys. Acta* **14** 588
- [5] Feynman R P 1950 *Phys. Rev.* **80** 440
 Feynman R P 1948 *Rev. Mod. Phys.* **20** 367
- [6] Schwinger J 1951 *Phys. Rev.* **82** 664
- [7] DeWitt B S 1965 *Dynamical Theory of Groups and Fields* (New York: Gordon and Breach)
- [8] Fock V A 1937 *Phys. Z. Sowjetunion* **12** 404
- [9] Nambu Y 1950 *Prog. Theor. Phys.* **5** 82
- [10] Horwitz L P and Piron C 1973 *Helv. Phys. Acta* **48** 316
- [11] Saad D, Horwitz L P and Arshansky R I 1989 *Found. of Phys.* **19** 1126
- [12] Land M C, Shnerb N and Horwitz L P 1995 *J. Math. Phys.* **36** 3263
- [13] Hojman S A and Shepley L C 1991 *J. Math. Phys.* **32** 142
- [14] Santilli R M 1990 *Foundations of Theoretical Mechanics I* (Berlin: Springer-Verlag)
- [15] Arshansky R L, Horwitz P and Lavie Y 1983 *Found. of Phys.* **13** 1167
- [16] Horwitz L P and Lavie Y 1982 *Phys. Rev. D* **26** 819
 Arshansky R I and Horwitz L P 1989 *J. Math. Phys.* **30** 213
 Arshansky R I and Horwitz L P 1988 *Phys. Lett A* **131** 222
- [17] Arshansky R I and Horwitz L P 1989 *J. Math. Phys.* **30** 66
- [18] Arshansky R I and Horwitz L P 1989 *J. Math. Phys.* **30** 380
- [19] Land M C, Arshansky R and Horwitz L P 1994 *Found. of Phys.* **24** 563
- [20] Land M C and Horwitz L P 1995 *J. Phys. A* **28** 3289
- [21] Land M C 2001 *Found. of Phys.* **31** 967
- [22] Land M C and Horwitz L P 1991 *Found. of Phys.* **21** 299
- [23] Land M C 1996 *Found. of Phys.* **27** 19
- [24] Yao W-M et. al. 2006 (Particle Data Group) *J. Phys. G* **33** 1
- [25] Land M C 2003 *Found. of Phys.* **33** 1157
- [26] Aharonovich I and Horwitz L P On the Green-Functions of the classical offshell electrodynamics under the manifestly covariant relativistic dynamics of Stueckelberg *Preprint* arXiv:1102.4840v2
- [27] Oron O and Horwitz L P 2001 *Phys. Lett. A* **280** 265-270
- [28] Land M C 2007 *Found. of Phys.* **37** 597
 Land M C 2005 *Found. of Phys.* **35** 1245
- [29] Land M C and Horwitz L P 1991 *Found. of Phys. Lett.* **4** 61
- [30] M Abraham 1905 *Theorie der Elektrizitat* Vol. II (Leipzig: Teubner)
- [31] Lorentz H A 1909 *The Theory of Electrons* (Leipzig: Teubner)
- [32] Jackson J D 1975 *Classical Electrodynamics* (New York: Wiley)