SLAC-PUB-2873
January 1982
(T/E)

# GLUON-GLUON INTERACTIONS IN THE BAG MODEL*

C. E. Carlson†
College of William and Mary
Williamsburg, Virginia 23185

T. H. Hansson‡
Institute for Theoretical Physics
University of California, Santa Barbara, California 93106

C. Peterson§
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

## ABSTRACT

An effective spin dependent interaction Hamiltonian for low lying gluon modes is calculated to $\mathcal{O}(\alpha_s)$ in the MIT Bag model. We give expressions for the energy shifts of low lying glueballs.

(Submitted to Physical Review D)

---

*Work supported in part by the Department of Energy, contract DE-AC03-76SF00515.
†Work supported in part by the National Science Foundation.
‡Work supported in part by the Swedish Natural Science Research Council, contract F-PD 4728-100. Address after 1 February 1982: Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139.
§Work supported in part by the Swedish Natural Science Research Council, contract F-PD 8207-102. On leave from NORDITA, Copenhagen, Denmark.

## I. INTRODUCTION

From considerations based on QCD one expects hadrons consisting only (or mainly) of glue.[1] The possibility of identifying as glueballs the states $\iota(1240)$ and $\theta(1660)$ recently discovered in $\psi \to \gamma X$,[2,3] has made it even more interesting to get precise predictions from QCD. Since one still cannot compute the hadron mass spectrum from first principles one must resort to phenomenological models keeping as many as possible of the properties of the full theory. In the case of glueballs, it is of special importance that the model can handle massless particles, and also treat gauge-invariance in a satisfactory way.[4] One such model, and the one to be used here, is the MIT bag. The aim of this work, which is in essence technical, is to calculate to $\mathcal{O}(\alpha_s)$ the spin-dependent energy shift due to gluon-gluon interactions in the bag. Several authors have already dealt with the properties of glueballs in the bag model,[5-8] so we shall only briefly summarize the general results concerning the mass spectrum.

Following the argument of Ref. 7, we will assume that spherical glueballs exist in the bag model. Hence we can use the static spherical cavity approximation which has been successful in the case of low-lying mesons and baryons. The effective Hamiltonian in the n-gluon sector takes the form[9,10]

$$H = \frac{4\pi R^3}{3} + \sum_i \frac{n_i x_i}{R} + H_{int} - \frac{C_{cms}}{R} + \frac{C_{cas}}{R} \qquad (1)$$

where $H_{int}$ has a nontrivial color and spin dependence. The first two terms (volume and kinetic energy) are well known, and the gluon-gluon interaction $H_{int}$, which also includes self-energies, will be dealt with in detail below. The "center of mass" term ($C_{cms}/R$) can be estimated using the method of Donoghue and Johnson,[10] but it is still not clear whether the zero point or "Casimir" energy ($C_{cas}/R$) is of importance. (In earlier works,[9] the two last terms in Eq. (1) were lumped together with the self-energy part of $H_{int}$ in a purely phenomenological term $Z_0/R$ with $Z_0 \simeq -1.8$.)

In this paper we shall derive explicit expressions for the two-gluon interaction part of $H_{int}$ in Eq. (1). However, as will become clear later, we are at this stage not ready to give any detailed predictions for the mass spectrum.

The next section outlines the calculation of the effective Hamiltonian leaving most of the technicalities to the appendices. In the last section we consider some special cases of phenomenological interest.

## II. $\mathcal{O}(\alpha_s)$ GLUON-GLUON EFFECTIVE HAMILTONIAN

The QCD interaction Hamiltonian density to $\mathcal{O}(g^2)$ is in Coulomb gauge given by,[11]

$$\begin{aligned}\mathcal{H}_I &= \mathcal{H}_I^{3g} + \mathcal{H}_I^{4g} + \mathcal{H}_I^{Coul} \qquad (2)\\ &= \frac{1}{2}\, g\, f^{abc}\, F^a_{jk}\, A^b_j\, A^c_k + \frac{1}{4}\, g^2\, f^{abc}\, f^{ade}\, A^b_j\, A^c_k\, A^d_j\, A^e_k \\ &\quad + \frac{1}{2}\, g^2\, f^{abc}\, f^{ade}\, F^b_{0k}\, A^c_k\, D_{Coul}\, F^d_{0l}\, A^e_l\end{aligned}$$

where the operator $D_{Coul}$ is defined below. The bag model interaction Hamiltonian

$$H_I = \int_{bag} d^3x \quad \mathcal{H}_I(x) \qquad (3)$$

operates on n-gluon cavity states $|1,2,\ldots,n\rangle$, which are direct products of one gluon "cavity modes" $|i\rangle = |a_i, \ell_i, m_i, \chi_i\rangle$ where a denotes color, $(\ell,m)$ orbital angular momentum, and $\chi$ radial quantum number as well as

TE or TM (transverse electric or transverse magnetic). We shall consider the $\ell = 1$ modes only, for which $|a,\ell,m,\chi\rangle \equiv |a,\alpha,\chi\rangle$, where $\alpha$ is the polarization index. A general n-gluon state built from these modes is specified by wave functions

$$\Psi^{(R)}_{(S,M)} = \eta^{(R)}_{a_1 \ldots a_N} \, \phi^{\alpha_1 \ldots \alpha_N}_{(S,M)} \, \vec{A}^{\chi_1}_{\alpha_1}(x_1) \ldots \vec{A}^{\chi_N}_{\alpha_N}(x_N) \tag{4}$$

where (R) and (S,M) denotes color and spin respectively. The relevant cavity modes A are given in Appendix A.

Now, write the effective interaction Hamiltonian $H_{int}$ in Eq. (1) as,[12]

$$H_{int} = \sum_{m<n} H_{mn} + \sum_{m} H^{self}_{m} \tag{5}$$

Although we shall only compute the effect of the interaction terms $H_{mn}$ shown in Fig. 1(a) and (b), the self energies (Fig. 1(c) and (d)) might be important as will be briefly commented upon later. Using lowest order perturbation theory Eqs. (2) and (3) immediately yields,

$$\begin{aligned} H_{mn} &= H^{3g}_{mn} + H^{4g}_{mn} + H^{Coul}_{mn} \\ &= (-i)S_B \int_{-\infty}^{0} dt \int_{bag} d^3x \, d^3y \, \langle m'n'| \mathscr{H}^{3g}(\vec{x},t) \, \mathscr{H}^{3g}(\vec{y},0) |mn\rangle \\ &\quad + S_B \int_{bag} d^3x \, \langle m'n'| \mathscr{H}^{4g} |mn\rangle + S_B \int_{bag} d^3x \, \langle m'n'| \mathscr{H}^{Coul} |mn\rangle \end{aligned} \tag{6}$$

Here $\chi = \chi'$ but in general $a' \neq a$ and $\alpha' \neq \alpha$. Thus $H_{mn}$ is still an operator in color and spin space although for notational simplicity we suppressed the corresponding indices ($\alpha_N, \alpha_{N'}, a_N$, etc.). The Bose statistics factor is $S_B = 1/2$ for identical modes, and otherwise $S_B = 1$.

The diagrams corresponding to the three terms are shown in Fig. 2. Now introduce the current and charge density operators[13]

$$j_k^a = (-i)\ \Lambda^a j_k = g\ f^{abc}\left(2\ F_{jk}^b\ A_j^c - A_j^b \partial_j A_k^c\right) \tag{7a}$$

$$\rho^a = (-i)\ \Lambda^a \rho = g\ f^{abc}\ F_{0k}^b\ A_k^c \tag{7b}$$

where $\Lambda^a$ is the $a^{th}$ color generator. The corresponding antisymmetrized matrix elements are given by

$$j_{mn} = \langle m|\vec{j}|n\rangle - \langle n|\vec{j}|m\rangle \tag{8a}$$

$$\rho_{mn} = \langle m|\rho|n\rangle - \langle n|\rho|m\rangle \tag{8b}$$

After some algebra and after carrying out the t-integration one gets

$$H_{mn}^{3g} = -\Lambda_m^a\ \Lambda_n^a\ S_B \int_{bag} d^3x\ d^3y \left[j_{mm}^k(\vec{x})\ D(\vec{x},\vec{y};\ \omega)\ j_{nn}^k(\vec{y}) + j_{mn}^k(\vec{x})\ D(\vec{x},\vec{y};\ \omega)\ j_{nm}^k(\vec{y})\right] \tag{9}$$

$$H_{mn}^{Coul} = -\Lambda_m^a\ \Lambda_n^a\ S_B \int_{bag} d^3x\ d^3y \left[\rho_{mm}(\vec{x})\ D_{Coul}(\vec{x},\vec{y};\ \omega)\ \rho_{nn}(\vec{y}) + \rho_{mn}(\vec{x})\ D_{Coul}(\vec{x},\vec{y};\ \omega)\ \rho_{nm}(\vec{y})\right] \tag{10}$$

where the "exchange" term $j_{mn}j_{nm}$ is absent for identical modes. Here the "confined" propagators D and $D_{Coul}$ differ from the "free" ones by boundary terms.[14,15] Instead of using explicit expressions for the cavity propagators, we follow the original MIT approach and directly calculate the potentials,[9,16]

$$\vec{a}_{mn}(\vec{x}) = -\int d^3y\ D(\vec{x},\vec{y};\ \omega)\ \vec{j}_{mn}(\vec{x}) \tag{11a}$$

$$\phi_{mn}(\vec{x}) = -\int d^3y\ D_{Coul}(\vec{x},\vec{y};\ \omega)\ \rho_{mn}(\vec{x}) \tag{11b}$$

subject to the boundary conditions,

$$\hat{r} \cdot (\vec{\nabla} \times \vec{a}) = 0 \tag{12a}$$

$$\hat{r} \cdot \frac{d\vec{a}}{dt} = 0 \quad \text{on the surface} \tag{12b}$$

$$\hat{r} \cdot \vec{\nabla}\phi = 0 \tag{12c}$$

We can then write $H_{mn}$ as

$$H_{mn} = -\vec{\Lambda}_m \vec{\Lambda}_n S_B \int_{bag} d^3x \left[ \vec{j}_{mm}(\vec{x}) \cdot \vec{a}_{nn}(\vec{x}) + \vec{j}_{mn}(\vec{x}) \cdot \vec{a}_{nm}(\vec{x}) \right] \tag{13}$$

$$-\vec{\Lambda}_m \vec{\Lambda}_n S_B \int_{bag} d^3x \, \langle mn|\mathscr{H}^{4g}(\vec{x},0)|mn\rangle$$

$$+\vec{\Lambda}_m \vec{\Lambda}_n S_B \int_{bag} d^3x \left[ \rho_{mm}(\vec{x}) \cdot \phi_{nn}(\vec{x}) + \rho_{mn}(\vec{x}) \cdot \phi_{nm}(\vec{x}) \right]$$

Since we consider the lowest TE and TM modes only, there are just three possible combinations (TE)(TE), (TE)(TM) and (TM)(TM). The wave functions ($\vec{A}_\alpha$) are given in Appendix A, the relevant current and charge densities ($\vec{j}$ and $\rho$) (as calculated in Appendix C) in Table I and the corresponding potentials ($\vec{a}$ and $\phi$) in Appendix D. Substituting all this in Eq. (13) we get (see Appendix E) for the spin-dependent parts in the three cases,[17]

i) (TE)(TE)

$$H_{EE} = -\frac{\alpha_s}{R} \Lambda_1 \Lambda_2 \left[ a_{EE} \vec{S}_1 \cdot \vec{S}_2 + b_{EE} T_{12} \right] \tag{14a}$$

$$a_{EE} = a_{EE}^{3g} + a_{E}^{4g} \approx 0.263 \tag{14b}$$

$$b_{EE} = b_{EE}^{Coul} \approx 0.041 \tag{14c}$$

ii) (TE)(TM)

$$H_{EM} = -\frac{\alpha_s}{R} \Lambda_1 \Lambda_2 \left[ a_{EM} \vec{S}_1 \cdot \vec{S}_2 + b_{EM} T_{12} \right] \tag{15a}$$

$$a_{EM} = a_{EM}^{3g} + a_{EM}^{4g} + a_{EM}^{Coul} \approx 0.271 \tag{15b}$$

$$b_{EM} = b_{EM}^{3g} + b_{EM}^{Coul} \approx 0.000 \tag{15c}$$

iii) (TM)(TM)

$$H_{MM} = - \frac{\alpha_s}{R} \Lambda_1 \Lambda_2 \left[ a_{MM} \vec{S}_1 \cdot \vec{S}_2 + b_{MM} T_{12} \right] \tag{16a}$$

$$a_{MM} = a_{MM}^{3g} + a_{MM}^{4g} \approx 0.247 \tag{16b}$$

$$b_{MM} = b_{MM}^{Coul} \approx 0.007 \tag{16c}$$

where $\alpha_s = g^2/4$, $\vec{S}$ is the spin-operator and $T_{12}$ a tensor operator acting in spin space given by

$$T_{12} = 2 \left[ (\vec{S}_1 \cdot \vec{S}_2)^2 - I_{12} \right] + \vec{S}_1 \cdot \vec{S}_2 \tag{17}$$

This tensor is convenient since it is the symmetric counterpart of $\vec{S}_1 \cdot \vec{S}_2$ (see Appendix B). The Eqs. (14) through (16) are the main results of this paper and we shall make some comments.

As is clear from Appendix D, the contribution from the spin independent part of the Coulomb interaction is not uniquely defined by the boundary conditions Eq. (12). To understand this, note that the arbitrariness is due to the residual gauge freedom $\phi \to \phi$ + constant. Of course, the energy-shift to $\mathcal{O}(\alpha_s)$ is gauge invariant, but that also includes the contributions from the self-energy diagrams (Fig. 1(c) and (d)). These are of the same form (const. $\alpha_s/R$) as the spin independent Coulomb contributions and have to be added to these in order to get a gauge invariant result. Without computing the self energy graphs (work in progress), we feel that nothing can be safely said about the size of the spin independent terms. Here we should mention that in Ref. 5 these contributions are calculated by putting $\phi = 0$ at infinity and neglecting the self-energies. We can see no compelling reason for this prescription.

As explained in Appendix B., the general form of the effective Hamiltonian involves three linearly independent tensors in 2 particle spin-space. The above results expressed in the tensors $\vec{S}_m \cdot \vec{S}_n$ and $T_{mn}$ can easily be transformed to any other basis by using the formulae in Appendix B.

## III. LEVEL SPLITTINGS IN LOW-LYING GLUEBALLS

Using Eqs. (14) through (16) we can compute the spin-dependent energy-shifts for any given state consisting of the lowest lying $\ell = 1$ TE and/or TM modes. Some of the phenomenologically most relevant states are the color singlets,

i) $(TE)^2$, $J^{PC} = 0^{++}, 2^{++}$

ii) $(TE)(TM)$, $J^{PC} = 0^{-+}, 2^{-+}$

iii) $(TM)^2$, $J^{PC} = 0^{++}, 2^{++}$

for which the expectation values of the operators $\Lambda_1\Lambda_2$, $\vec{S}_1\vec{S}_2$ and $T_{12}$ are listed in Table 2. Thus we have,

$$(TE)^2: \qquad \Delta E_{0^{++}} = -2.07 \frac{\alpha_s}{R}$$

$$\Delta E_{2^{++}} = 0.67 \frac{\alpha_s}{R} \tag{18}$$

$$(TE)(TM): \qquad \Delta E_{0^{-+}} = -1.63\,\frac{\alpha_s}{R}$$

$$\Delta E_{2^{-+}} = 0.82\,\frac{\alpha_s}{R} \qquad (19)$$

$$(TM)^2: \qquad \Delta E_{0^{++}} = -1.57\,\frac{\alpha_s}{R}$$

$$\Delta E_{2^{++}} = 0.72\,\frac{\alpha_s}{R} \qquad (20)$$

The self-energy contributions are not included above, but will be dealt with in a subsequent paper.[18] With an increasing amount of group theoretical labor similar calculations can be performed for a general n-gluon state. One example is the $(TE)^3$ color singlet $J^{PC} = 0^{++}$ state where $\langle \Lambda_i \Lambda_j \rangle = -3/2$ and $\left\langle \sum_{i<j} \vec{S}_i \cdot \vec{S}_j \right\rangle = -3$ and thus

$$\Delta E_{0^{++}} = -1.00\,\frac{\alpha_s}{R} \qquad (21)$$

There are two dangers in obtaining the splittings among the physical glueball states by simply adding the above energy shift to the lowest order terms and then subtract. One is that there will be mixing between the listed states and the non-glueball states with the same quantum numbers. This mixing problem is probably most severe for the $(TE)^2$ $0^{++}$ state, which is expected to mix strongly with the vacuum.[19] The second uncertainty comes from the $\mathcal{O}(\alpha_s/R)$ spin independent energy shifts and the self energies. The value of R, and hence the spin-dependent splittings (Eq. (18)), depend on these contributions. Also, the self-energies are different for different gluon modes (most successful bag-calculations for quark based hadrons have quarks only in the lowest state so the mode dependence is often not mentioned). Glueballs containing of gluons in the same mode have, of course, the same self-energies to the extent that the radii are the same. Thus one can for example predict

$$M_{2^{-+}} - M_{0^{-+}} = \Delta E_{2^{-+}} - \Delta E_{0^{-+}} = 2.45 \frac{\alpha_s}{R} \tag{22}$$

We shall not list any further predictions here.

Acknowledgement

T.H.H. and C.P. have benefitted from fruitful discussions with K. Johnson. One of us (T.H.H.) acknowledges the kind hospitality of the Institute for Theoretical Physics, Santa Barbara, where part of this work was done.

# APPENDIX A

The wavefunctions for the lowest $\ell = 1$ TE and TM modes with $P = +$ and $P = -$, respectively, are given by

$$\vec{A}^{E}_{\alpha}(\vec{r},t) = -\frac{N_E}{\omega_E} j_1(x_E\rho)\ (\hat{r} \times \hat{e}_\alpha)\ e^{-i\omega_E t} \tag{A.1a}$$

$$\vec{A}^{M}_{\alpha}(\vec{r},t) = \frac{N_M}{\omega_M}\left[j_2(x_M\rho)\left(\hat{r}_\alpha \hat{r} - \frac{1}{3}\hat{e}_\alpha\right) + \frac{2}{3} j_0(x_M\rho)\ \hat{e}_\alpha\right] e^{-i\omega_M t} \tag{A.1b}$$

where

$$N_E^2 = \frac{3}{8\pi}\frac{1}{R^4 j_0^2(x_E)}\frac{x_E}{x_E^2 - 2} \tag{A.2a}$$

$$N_M^2 = \frac{3}{8\pi}\frac{x_M}{R^4 j_0^2(x_M)} \tag{A.2b}$$

also R is the bag radius, $\omega_{E(M)} = x_{E(M)}/R$, $x_E = 2.744$, $x_M = 4.493$ and $\rho = r/R$. The spherical unit (or polarization) vectors are denoted by $\hat{e}_\alpha$ and $\hat{r}_\alpha = \hat{r} \cdot \hat{e}_\alpha$. The relation to spherical harmonics is

$$Y_1^\alpha(\Omega) = \sqrt{\frac{3}{4\pi}}\ \hat{r}_\alpha \tag{A.3}$$

The corresponding magnetic fields, $\vec{B}^{E(T)} = \nabla \times \vec{A}^{E(T)}$, are given by $(t = 0)$

$$\vec{B}^{E}_{\alpha}(\vec{r}) = \frac{N_E}{x_E\rho}\left[2j_1(x_E\rho)\hat{r}_\alpha\hat{r} - \frac{d}{d\rho}\left(\rho j_1(x_E\rho)\right)\left(\hat{r} \times (\hat{r} \times \hat{e}_\alpha)\right)\right] \tag{A.4a}$$

$$\vec{B}^{M}_{\alpha}(r) = -N_M j_1(x_M\rho)(\hat{r} \times \hat{e}_\alpha) \tag{A.4b}$$

## APPENDIX B

In this appendix we define the various operators acting in spin-space and also give some useful relations.

First consider operators $\mathcal{O}_{\alpha\beta}$ acting in one particle spin-space with (polarization) vectors $\hat{e}_\alpha$. In addition to the usual antisymmetric spin vector operator

$$\vec{S}_{\alpha\beta} = -i\,\hat{e}_\alpha \times \hat{e}_\beta \qquad (B.1)$$

we also use the symmetric pseudovector

$$\vec{T}_{\alpha\beta} = i(\hat{r}_\beta \hat{r} \times \hat{e}_\alpha + \hat{r}_\alpha \hat{r} \times \hat{e}_\beta) \qquad (B.2)$$

and the symmetric tensor operator

$$U_{\alpha\beta} = \hat{r}_\alpha \hat{r}_\beta - \frac{1}{3}\,\delta_{\alpha\beta} \qquad (B.3)$$

Since $\int d\Omega\, \vec{T}_{\alpha\beta} = 0$ and $\hat{r} \cdot \vec{T}_{\alpha\beta} = 0$, we can conclude that $\vec{T}_{\alpha\beta}$ has purely $\ell = 2$ orbital angular momentum, and the same holds up for $U_{\alpha\beta}$. On the other hand, $\vec{S}_{\alpha\beta}$ obviously has $\ell = 0$ only. A useful expression for $\vec{T}_{\alpha\beta}$ is

$$\vec{T}_{\alpha\beta} = (-i)\ [(\hat{r} \cdot \vec{S})\hat{r} \times \vec{S} + \hat{r} \times \vec{S}(\hat{r} \cdot \vec{S})]_{\alpha\beta}$$

where the order of the spin operators is important.

Next consider scalar ($\hat{r}$-independent) operators $\mathcal{O}_{(\alpha\gamma),(\beta\delta)}$ acting on the direct product spin space with vectors $\hat{e}_\alpha \hat{e}_\beta$. There are three linearly independent operators of this type, namely $\delta_{\alpha\beta}\delta_{\gamma\delta}$, $\delta_{\alpha\gamma}\delta_{\beta\delta}$ and $\delta_{\alpha\delta}\delta_{\gamma\beta}$. A more convenient basis is

$$I_{12} = \delta_{\alpha\beta}\delta_{\gamma\delta} \qquad (B.5a)$$

$$\vec{S}_1 \cdot \vec{S}_2 = S_{\alpha\beta} \cdot S_{\gamma\delta} = \delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\delta} \qquad (B.5b)$$

$$T_{12} = \delta_{\alpha\delta}\delta_{\beta\gamma} + \delta_{\alpha\gamma}\delta_{\beta\delta} \qquad (B.5c)$$

where $I_{12}'$ is the unit operator, while $T_{12}$ can be expressed as,

$$T_{12} = 2[(\vec{S}_1 \cdot \vec{S}_2)^2 - I_{12}] + \vec{S}_1 \cdot \vec{S}_2 \tag{B.6}$$

One can, of course, use other basis than Eq. (B.4). If we, e.g., use the "quadrupole-quadrupole" tensor

$$\left(S^iS^j - \frac{2}{3}\,\delta^{ij}I\right)_1 \left(S^iS^j - \frac{2}{3}\,\delta^{ij}I\right)_2 = (\vec{S}_1 \cdot \vec{S}_2)^2 - \frac{4}{3}\,I_{12} \tag{B.7}$$

we have

$$a\,\vec{S}_1 \cdot \vec{S}_2 + bT_{12} + c = (a+b)\,\vec{S}_1 \cdot \vec{S}_2 + 2b\left[(\vec{S}_1 \cdot \vec{S}_2)^2 - \frac{4}{3}\right] + c + \frac{2}{3}\,b \tag{B.8}$$

where $I_{12}$ is understood in the constant terms. In Appendix E we also need the angular integral

$$\frac{1}{4\pi} \int d\Omega\, \vec{T}_1 \cdot \vec{T}_2 = \frac{4}{15} - \frac{2}{5}\,T_{12} \tag{B.9}$$

# APPENDIX C

Here we calculate the current and charge densities $\vec{j}^{EE}$, $\rho^{EE}$, etc. The current operator in Eq. (7a) can be written as

$$\vec{j} = ig\left[\vec{A}\times\vec{B} - (\vec{A}\vec{\nabla})\vec{A}\right] \tag{C.1}$$

The antisymmetrical expectation value of j [cf., Eq. (8)] which is an operator in spin space, takes the form

$$\vec{j}_{\alpha\beta} = ig\left[\vec{A}_\alpha\times\vec{B}_\beta + \vec{B}_\alpha\times\vec{A}_\beta + \vec{\nabla}\times(\vec{A}_\alpha\times\vec{A}_\beta)\right] \tag{C.2}$$

i) The TE-TE Current

$$\vec{A}^E_\alpha\times\vec{B}^E_\beta + \vec{B}^E_\alpha\times\vec{A}^E_\beta = -\frac{2N_E^2R}{x_E^2\rho}\, j_1^{E2}\left[(\hat{r}\times\hat{e}_\alpha)\times\hat{r}\hat{r}_\beta - (\hat{r}\times\hat{e}_\beta)\times\hat{r}r_\alpha\right]$$

$$= -2i\,\frac{N_E^2R}{x_E^2\rho}\, j_1^{E2}\ \hat{r}\times\vec{S}_{\alpha\beta} \tag{C.3}$$

where the last step follows from Eq. (B.2) and we introduced the notation $j_1^E = j_1(x_E\rho)$, $j_1^M = j_1(x_M\rho)$ etc.

$$\vec{\nabla}\times(\vec{A}^E_\alpha\cdot\vec{A}^E_\beta) = \frac{N_E^2R^2}{\rho x_E^2}\ j_1^{E2}\ \nabla\times(\hat{r}\times\hat{e}_\alpha)\times(\hat{r}\times\hat{e}_\beta) = -i\,\frac{N_E^2R}{\rho x_E^2}\ j_1^{E2}\ \hat{r}\times\vec{S}_{\alpha\beta} \tag{C.4}$$

So for the current $\vec{j}^{EE}$ we get (suppressing polarization indices)

$$\vec{j}^{EE} = 3g\,\frac{N_E^2R}{\rho x_E^2}\ j_1^{E2}\ \hat{r}\times\vec{S} \tag{C.5}$$

ii) The TM-TM Current

$$\vec{A}^M_\alpha\times\vec{B}^M_\beta + \vec{B}^M_\alpha\times\vec{B}^M_\beta = -2i\,\frac{N_M^2R}{x_M^2}\, j_1^{M2}\ \hat{r}\times\vec{S}_{\alpha\beta} \tag{C.6}$$

$$\vec{\nabla}\times\vec{A}^M_\alpha\times\vec{A}^M_\beta = \frac{N_M^2 R^2}{x_M^2}\ \vec{\nabla}\times\left\{f(\rho)\left[\hat{r}(\hat{r}\cdot\vec{S}_{\alpha\beta})-\vec{S}_{\alpha\beta}\right]-ig(\rho)\ \vec{S}_{\alpha\beta}\right\} \qquad (C.7)$$

where f(ρ) and g(ρ) are defined as

$$f(\rho) = \frac{1}{3}\, j_2^M(2j_0^M - j_2^M) \qquad (C.8a)$$

$$g(\rho) = \frac{1}{9}\,(2j_0^M - j_2^M)^2 \qquad (C.8b)$$

The current $\vec{j}^{MM}$ is then given by

$$\vec{j}^{MM} = g\,\frac{N_M^2 R}{x_M^2 \rho}\,(4j_1^{M2} - j_2^{M2})\ \hat{r}\times\vec{S} \qquad (C.9)$$

iii) The TE-TM Current

$$\vec{A}^E_\alpha\times\vec{B}^M_\beta = i\,\frac{N_E N_M}{x_E}\ R\ j_1^M\, j_1^E(\hat{r}\times\vec{S}_{\alpha\beta})\hat{r} \qquad (C.10)$$

$$\vec{A}^M_\beta\times\vec{B}_\alpha{}^E = i\,\frac{N_E N_M}{3x_E}\ R\left\{a_1(\rho)\left[\hat{r}\times(\hat{r}\times\vec{S}_{\alpha\beta})\right]+a_2(\rho)\vec{T}_{\alpha\beta}+a_2(\rho)\vec{S}_{\alpha\beta}\right\} \qquad (C.11)$$

where we used the definitions Eq. (B.1), (B.3) and (B.6) and

$$a_1(\rho) = j_2^E\, j_0^M + j_2^M\, j_0^E - j_2^E\, j_2^M \qquad (C.12a)$$

$$a_2(\rho) = j_2^E\, j_0^M - j_2^M\, j_0^E \qquad (C.12b)$$

$$a_3(\rho) = -\frac{1}{3}\,(4j_0^E\, j_o^M - 2j_0^E\, j_0^M - 2j_2^E\, j_0^M + j_2^E\, j_2^M) \qquad (C.12c)$$

$$\vec{A}^E_\alpha\times\vec{A}^M_\beta = \frac{N_E N_M}{x_E x_M}\, R^2\ j_1^E\left[j_2^M\ \hat{r}_\alpha\hat{r}_\beta\hat{r}+\frac{1}{3}\,(2j_0^M - j_2^M)\delta_{\alpha\beta}-\frac{2}{3}\,(j_0^M + j_2^M)\hat{r}_\beta\hat{e}_\alpha\right] \qquad (C.13)$$

$$\vec{\nabla}\times(\vec{A}^E_\alpha\times\vec{B}^M_\beta) = i\,\frac{N_E N_M}{x_E x_M}\left[2\,\frac{j_1^E j_1^M}{x_M \rho^2}\,\vec{S}+\frac{1}{\rho}\left(j_1^E\, j_2^M+\frac{x_E}{x_M}\, j_2^E\, j_1^M\right)\hat{r}\times(\hat{r}\times\vec{S})-\frac{1}{\rho}\,\frac{x_E}{x_M}\, j_2^E\, j_1^M\vec{T}\right]_{\alpha\beta} \qquad (C.14)$$

For the current $j^{EM}$ we thus get

$$\vec{j}^{EM} = -g\,\frac{N_E N_M}{x_E x_M}\; R\left[f_2(\rho)\left(\hat{r}(\hat{r}\cdot\vec{S}) - \frac{1}{3}\,\vec{S}\right) + \tilde{f}_2(\rho)\vec{T} + f_0(\rho)\vec{S}\right] \qquad (C.15)$$

where

$$f_2(\rho) = x_M j_1^E j_1^M + x_E j_2^E j_2^M \qquad (C.16a)$$

$$\tilde{f}_2(\rho) = \frac{1}{3}\,x_E\left(j_2^M j_0^E - j_2^E j_2^M - 2 j_2^E j_0^M\right) \qquad (C.16b)$$

$$f_0(\rho) = \frac{1}{3}\left(x_M j_1^E j_1^M + 2x_E j_0^E j_0^M - x_E j_2^E j_2^M\right) \qquad (C.16c)$$

iv) The TE-TE charge

$$\rho_{\alpha\beta}^{EE} = g\,\frac{x_E}{R}\;\vec{A}_\alpha^E\cdot\vec{A}_\beta^E = -2g\,\frac{N_E^2 R}{x_E}\;j_1^{E2}\left(U_{\alpha\beta} - \frac{2}{3}\,\delta_{\alpha\beta}\right) \qquad (C.17)$$

where $U_{\alpha\beta}$ is defined in Eq. (B.3).

v) The TM-TM charge

$$\rho_{\alpha\beta}^{MM} = g\,\frac{x_M}{R}\,\vec{A}_\alpha^M\cdot\vec{A}_\beta^M = 2g\,\frac{N_M^2}{x_M}\,\frac{R}{3}\left[j_2^M\left(j_2^M + 4j_0^M\right)U_{\alpha\beta} + \frac{2}{3}\left(j_2^{M2} + j_0^{M2}\right)\delta_{\alpha\beta}\right] \qquad (C.18)$$

vi) The TE-TM charge

$$\rho_{\alpha\beta}^{EM} = g\,\frac{x_E + x_M}{R}\;\vec{A}_\alpha^E\cdot\vec{A}_\beta^M = ig\,N_E\,N_M R\;\frac{x_E + x_M}{3x_E x_M}\;j_1^E(2j_0^M - j_2^M)\;\hat{r}\cdot\vec{S}_{\alpha\beta} \qquad (C.19)$$

APPENDIX D

In this section we calculate the potentials $\vec{a}$ and $\Phi$.

I. The $\vec{a}$-potentials

Generally one has

$$\vec{a}(\vec{x}) = \int_{bag} d^3y\ D(\vec{x},\vec{y};\ \omega)\ \vec{j}(\vec{y}) \tag{D.1}$$

We use the free Green's functions and impose the boundary conditions later. The currents $j^{EE}$ and $j^{MM}$ have no time-dependence and hence the appropriate expression for $D(\vec{x},\vec{y};\ \omega)$ in Eq. (D.1) is

$$D(\vec{x},\vec{y};\ \omega) = \sum_{\ell,m} \frac{1}{2\ell+1}\ \frac{r_<^{\ell}}{r_>^{\ell+1}}\ Y_{\ell,m}(\Omega)\ Y^*_{\ell,m}(\Omega') \tag{D.2}$$

whereas in the TE-TM case one has

$$D(\vec{x},\vec{y};\ \omega) = \sum_{\ell,m} j_\ell(x\rho_<)\ n_\ell(x\rho_>)\ Y_{\ell,m}(\Omega)\ Y^*_{\ell,m}(\Omega') \tag{D.3}$$

where $\omega R = x = x_M - x_E$

i) The TE-TE Case

From Eqs. (C.5), (D.1) and (D.2) we get

$$\vec{a}^{EE} = g\ \frac{N_E^2R^3}{x_E^2} \left[ \frac{1}{\rho^2}\ J_1^E(\rho) + \rho N_1^E(\rho) \right] \hat{r}\times\vec{S} \tag{D.4}$$

where

$$J_1^E(\rho) = \int_0^\rho d\xi\, \xi^2\ j_1^2(x_E\xi) \tag{D.5a}$$

$$N_1^E(\rho) = a_1^E + \int_\rho^1 d\xi \, \frac{1}{\xi} \, j_1^2(x_E\xi) \tag{D.5b}$$

The constant $a_1^E$ is determined by the boundary condition Eq. (12) and found to be

$$a_1^E = \frac{1}{2} J_1^E(1) \tag{D.6}$$

ii) The TM-TM Case

From Eq. (C.9), (D.1) and (D.2) we get

$$\vec{a}^{MM} = g \frac{N_M^2 R^3}{3x_M^2} \left[ \frac{1}{\rho^3} j_1(\rho) + \rho N_1(\rho) \right] \hat{r} \times \vec{S} \tag{D.7}$$

where

$$J_1^M(\rho) = \int_0^\rho d\xi \, \xi^2 \left[ 4j_1^2(x_M\xi) - j_2^2(x_M\xi) \right] \tag{D.8a}$$

$$N_1^M(\rho) = a_1^M + \int_\rho^1 d\xi \, \frac{1}{\xi} \left[ 4j_1^2(x_M\xi) - j_2^2(x_M\xi) \right] \tag{D.8b}$$

The boundary condition [Eq. (12)] gives

$$a_1^M = \frac{1}{2} J_1^M(1) \tag{D.9}$$

iii) The TE-TM Case

Here we face the complication that $\vec{j}^{EM}$ is not transverse ($\vec{\nabla} \cdot \vec{j}^{EM} \neq 0$). Care must be taken because in the Coulomb gauge the vector potential $\vec{a}_T^{EM}$ satisfies the equation

$$(-\nabla^2 + \omega^2)\vec{a}_T^{EM} = \vec{j}_T^{EM} = \vec{j}^{EM} - \vec{\nabla} \frac{1}{\nabla^2} \vec{\nabla} \cdot \vec{j}^{EM} = \vec{j}^{EM} - \vec{j}_L^{EM} \quad . \tag{D.10}$$

Note that

$$\vec{j}_L^{EM} = i \omega R \vec{\nabla} \phi^{EM} \tag{D.11a}$$

where $\omega R = x_M - x_E$ and $\phi^{EM}$ is given in Eq. (D.22) below. Also

$$\hat{r} \cdot \vec{j}_L^{EM} = 0 \qquad \text{(D.11b)}$$

on the bag surface.

Rather than calculating $a_T^{EM}$ directly we first compute $\vec{a}^{EM}$ defined by

$$(-\nabla^2 + \omega^2)\vec{a}^{EM} = \vec{j}^{EM} \qquad \text{(D.12a)}$$

and then obtain $\vec{a}_T^{EM}$ from

$$\vec{a}_T^{EM} = \vec{a}^{EM} - \vec{\nabla}\frac{1}{\nabla^2}\vec{\nabla}\cdot\vec{a}^{EM} = \vec{a}^{EM} + \vec{\nabla}\frac{1}{\omega^2}\vec{\nabla}\cdot\vec{a}^{EM} + \frac{i}{\omega}\vec{\nabla}\phi^{EM} \quad . \qquad \text{(D.12b)}$$

(This procedure is equivalent to first calculating the vector potential in Lorentz gauge and then returning to Coulomb gauge using a gauge transformation.)

We get

$$\vec{a}_T^{EM} = g\,\frac{x\,N_E\,N_M}{x_E x_M}\,R^3\Big\{\Big[j_2(x\rho)\;\tilde{N}_2^{EM}(\rho) + n_2(x\rho)\;\tilde{J}_2^{EM}(\rho)\Big]\,\vec{T} \qquad \text{(D.13a)}$$

$$+\Big[j_2(x\rho)\;\hat{N}^{EM}(\rho) + n_2(x\rho)\;\hat{J}^{EM}(\rho) + \frac{1}{x^3}\Big(\frac{2}{3}\,f_2(\rho) + f_0(\rho)\Big)\Big]\Big((\hat{r}\cdot\vec{S})\;\hat{r} - \frac{1}{3}\,\vec{S}\Big)$$

$$+\frac{2}{3}\Big[j_0(x\rho)\;\hat{N}^{EM}(\rho) + n_2(x\rho)\;\hat{J}^{EM}(\rho) + \frac{1}{2x^3}\Big(\frac{2}{3}\,f_2(\rho) + f_0(\rho)\Big)\Big]\,\vec{S} + \frac{i}{\omega}\vec{\nabla}\phi^{EM}\Big\}$$

where

$$\tilde{J}_2^{EM}(\rho) = \int_0^{\rho} d\xi\;\xi^2\;j_2(x\xi)\;\tilde{f}_2(\xi) \qquad \text{(D.13b)}$$

$$\tilde{N}_2^{EM}(\rho) = \tilde{a}_2^{EM} + \int_{\rho}^{1} d\xi\;\xi^2\;n_2(x\xi)\;\tilde{f}_2(\xi) \qquad \text{(D.13c)}$$

$$\hat{J}^{EM}(\rho) = \int_0^{\rho} d\xi\;\xi^2\Big[\frac{1}{3}\,j_2(x\xi)\;f_2(\xi) + j_0(x\xi)\;f_0(\xi)\Big] \qquad \text{(D.13d)}$$

$$\hat{N}^{EM}(\rho) = \hat{a}^{EM} + \int_{\rho}^{1} d\xi\;\xi^2\Big[\frac{1}{3}\,n_2(x\xi)\;f_2(\xi) + n_0(x\xi)\;f_0(\xi)\Big] \;. \qquad \text{(D.13e)}$$

The constants $\tilde{a}_2^{EM}$ and $\hat{a}^{EM}$ are determined by Eq. (12) and given by

$$\tilde{a}_2^{EM} = -\frac{n_2(x) + xn_2'(x)}{j_2(x) + xj_2'(x)}\, \tilde{J}_2^{EM}(1) \qquad \text{(D.13f)}$$

$$\hat{a}^{EM} = -\frac{n_1(x)}{j_1(x)}\, \hat{J}^{EM}(1) \qquad \text{(D.13g)}$$

II. The $\phi$-potentials

Generally one has

$$\phi(\vec{x}) = \int_{\text{bag}} d^3y\; D_{\text{Coul}}(\vec{x},\vec{y})\; \rho(\vec{y}) \qquad \text{(D.14)}$$

i) The TE-TE Case

Here both the $\ell = 2$ and $\ell = 0$ waves in Eq. (D.14) contribute and we get

$$\phi = 2g\, \frac{N_E^2 R^3}{x_E} \left\{ \frac{1}{5}\left[\frac{1}{\rho^3}\, G_2^E(\rho) + \rho^2 H_2^E(\rho)\right] U - \left[\frac{1}{\rho}\, G_0^E(\rho) + H_0^E(\rho)\right] I \right\} \qquad \text{(D.15)}$$

where

$$G_2^E(\rho) = \int_0^\rho d\xi\, \xi^4\, j_1^2(x_E\xi) \qquad \text{(D.16a)}$$

$$H_2^E(\rho) = s_2^E + \int_\rho^1 d\xi\, \frac{1}{\xi}\, j_1^2(x_E\xi) \qquad \text{(D.16b)}$$

$$G_0^E(\rho) = \int_0^\rho d\xi\, \xi^2\, j_1^2(x_E\xi) \qquad \text{(D.16c)}$$

$$H_0^E(\rho) = s_0^E + \int_\rho^1 d\xi\, \xi\, j_1^2(x_E\xi) \qquad \text{(D.16d)}$$

Following the same procedure as above the constants $s_0^E$ and $s_2^E$ should be determined by the boundary condition [Eq. (12)].

$$\vec{\nabla}\phi = 0 \quad \text{for } \rho = 1 \qquad \text{(D.17)}$$

This equation gives for the "tensor" contribution to $\phi$

$$s_2^E = \frac{3}{2} G_2^E(1) \tag{D.18}$$

For the unit tensor term in $\phi$, the condition (D.17) is however identically fulfilled, which means that the constant $s_0^E$ remains undetermined. This is related to a residual gauge freedom as discussed in the main text.

ii) The TM-TM Case

Using Eqs. (C.9), (D.1) and (D.2) we get

$$\phi^{MM} = -2g \frac{N_M^2}{x_M} \frac{R^3}{3} \left\{ \frac{1}{5} \left[ \frac{1}{\rho^3} G_2^M(\rho) + \rho^2 H_2^M(\rho) U - \frac{1}{\rho} G_0^M(\rho) + H_0^M(\rho) \right] I \right\} \tag{D.19}$$

where

$$G_2^M(\rho) = \int_0^1 d\xi\, \xi^4\, g(x_M\xi) \tag{D.20a}$$

$$H_2^M(\rho) = s_2^M + \int_\rho^1 d\xi\, \frac{1}{\xi}\, g(x_M\xi) \tag{D.20b}$$

$$G_0^M(\rho) = \int_0^1 d\xi\, \xi^2\, h(x_M\xi) \tag{D.20c}$$

$$H_0^M(\rho) = s_0^M + \int_\rho^1 d\xi\, \xi\, h(x_M\xi) \tag{D.20d}$$

and

$$g(x_M\xi) = j_2(x_M\xi)\left[j_2(x_M\xi) + 4j_0(x_M\xi)\right] \tag{D.20e}$$

$$h(x_M\xi) = \frac{2}{3}\left[j_2^2(x_M\xi) + 2j_0^2(x_M\xi)\right] \tag{D.20f}$$

Again only $s_2^M$ can be determined by the boundary conditions [see Eq. (D.17)]. One finds

$$s_2^M = \frac{3}{2} G_2^M(1) \tag{D.21}$$

iii) The TE-TM Case

From Eqs. (C.18), (D.2) and (D.14) we get

$$\phi^{EM} = i\,\frac{g}{3}\,N_E N_M\,\frac{1}{3}\,R^3\,\frac{x_E + x_M}{x_E x_M}\left[\frac{1}{\rho^2}\;G_1^{EM}(\rho) + H_1^{EM}(\rho)\right]\hat{r}\cdot\vec{S} \qquad \text{(D.22)}$$

where

$$G_1^{EM}(\rho) = \int_0^{\rho} d\xi\,\xi^3\,j_1(x_E\xi)\left[2j_0(x_M\xi) - j_2(x_M\xi)\right] \qquad \text{(D.23a)}$$

$$H_1^{EM}(\rho) = S_1^{EM} + \int_{\rho}^{1} d\xi\;j_1(x_E\xi)\left[2j_0(x_M\xi) - j_2(x_M\xi)\right] \qquad \text{(D.23b)}$$

$a_1^{EM}$ is determined by Eq. (12) to be

$$s_1^{EM} = 2\,G_1^{EM}(1) \qquad \text{(D.24)}$$

# APPENDIX E

Here we calculate the quantities a and b occurring in Eqs. (14)-(16). In general, a and b get contributions from three sources: the 3-gluon, 4-gluon and Coulomb terms,

$$a = a^{3g} + a^{4g} + a^{Cl} \tag{E.1a}$$

$$b = b^{3g} + b^{4g} + b^{Cl} \tag{E.1b}$$

i) TE-TE

From Eqs. (13), (C.5) and (D.4) one obtains

$$H_{EE}^{3g} = -\Lambda_1\Lambda_2 \int_{bag} d^3x\ \vec{J}^{EE} \cdot \vec{a}^{EE} = -\Lambda_1\Lambda_2 \frac{\alpha_s}{R}\ a_{EE}^{3g}\ \vec{S}_1 \cdot \vec{S}_2 \tag{E.2}$$

where

$$a_{EE}^{3g} = \frac{9}{2}\gamma_E \int_0^1 d\rho\ j_1^2(x_E\rho)\left[\frac{1}{\rho}\ J_1^E(\rho) + \rho^2 N_1^E(\rho)\right] \approx 0.341 \tag{E.3a}$$

with

$$\gamma_E = \left(\frac{8\pi N_E^2 R^4}{3x_E^2}\right)^2 \tag{E.3b}$$

The Bose factor $S_B = ½$ was cancelled in Eq. (E.2) because of the two identical terms in Eq. (9). The 4-gluon contribution $H_{EE}^{4g}$ is given by

$$H_{EE}^{4g} = -\Lambda_1\Lambda_2\ 2 \int_{bag} d^3x\ \vec{\ell}_1^E(\vec{x}) \cdot \vec{\ell}_2^E(\vec{x}) = -\Lambda_1\Lambda_2 \frac{\alpha_s}{R}\ a_{EE}^{4g}\ \vec{S}_1 \cdot \vec{S}_2 \tag{E.4}$$

where

$$\vec{\ell}_1(x) = 2\vec{A}_\alpha^E \times \vec{A}_\beta^E \tag{E.5}$$

and

$$a_{EE}^{4g} = -\frac{3}{4}\gamma_E \int_0^1 d\rho\ \rho^2\ j_1^4(x_E\rho) \approx -0.078 \qquad (E.6)$$

For the Coulomb part one gets

$$H_{EE}^{Cl} = \Lambda_1\Lambda_2 \int d^3x\ \rho^{EE}\cdot\phi^{EE} = -\Lambda_1\Lambda_2 \frac{\alpha_s}{R}\left[\ b_{EE}^{C}\cdot T_{12} + c_{EE}^{Cl}\ I_{12}\ \right] \qquad (E.7)$$

where

$$b_{EE}^{Cl} = -\frac{3}{25}\gamma_E x_E^2 \int_0^1 d\rho\ \rho^2\ j_1^2(x_E\rho)\left[\frac{1}{\rho^3} G_2^E(\rho) + H_2^E(\rho)\right] \approx -0.041 \quad (E.8)$$

The constant $c_{EE}^{Cl}$ is not determined by the boundary condition [Eq. (12)], as described in the text.

ii) TM–TM

As above, one gets

$$H_{MM}^{3g} = -\Lambda_1\Lambda_2 \int_{bag} d^3x\ \vec{j}^{MM}\cdot\vec{a}^{MM} = -\Lambda_1\Lambda_2\ \frac{\alpha_s}{R}\ a_{MM}^{3g}\ \vec{S}_1\cdot\vec{S}_2 \qquad (E.9)$$

where

$$a_{MM}^{3g} = \frac{1}{2}\gamma_M \int_0^1 d\rho\left[4\,j_1^2(x_M\rho) - j_2^2(x_M\rho)\right]\left[\frac{1}{\rho} J_1^M(\rho) + \rho^2 N_1^M(\rho)\right] \approx 0.328 \qquad (E.10a)$$

with

$$\gamma_M = \left(\frac{8\pi N_E^2 R^4}{3x_M^2}\right)^2 \qquad (E.10b)$$

also

$$H_{MM}^{4g} = -\Lambda_1\Lambda_2\ 2\int_{bag} d^3x\ \vec{\ell}_1^{MM}\cdot\vec{\ell}_2^{MM} = -\Lambda_1\Lambda_2\frac{\alpha_s}{R}\ a_{MM}^{4g}\ \vec{S}_1\cdot\vec{S}_2 \qquad (E.11)$$

with

$$\vec{\ell}_1^{MM}(x) = 2\, A_\alpha^M \times A_\beta^M \tag{E.12}$$

One obtains

$$\begin{aligned} a_{MM}^{4g} = -\frac{1}{36}\gamma_M \int_0^1 d\rho\ \rho^2 \left(2\ j_0(x_M\rho) - j_2(x_M\rho)\right)^2 \\ \times \left(4\ j_0^2(x_M\rho) + 4\ j_0(x_M\rho)\ j_2(x_M\rho) + 3\ j_2^2(x_M\rho)\right) \approx -0.081 \end{aligned} \tag{E.13}$$

For the Coulomb part one obtains

$$H_{MM}^{C1} = \Lambda_1 \cdot \Lambda_2 \int_{bag} d^3x\ \rho^{MM} \cdot \phi^{MM} = -\Lambda_1\Lambda_2 \frac{\alpha_s}{R} \left[ b_{MM}^{C1}\, T_{12} + c_{MM}^{C1}\, I_{12} \right] \tag{E.14}$$

where

$$\begin{aligned} b_{MM}^{C1} = -\frac{1}{75}\gamma_M x_M^2 \int_0^1 d\rho\ \rho^2 \left[ j_2(x_M\rho) \left( j_2(x_M\rho) + 4\ j_0(x_M\rho) \right) \right] \\ \times \left[ \frac{1}{\rho^3}\ G_2^M(\rho) + \rho^2\ H_2^M(\rho) \right] \approx -0.007 \end{aligned} \tag{E.15}$$

Again the constant $c_{MM}^{C1}$ is not determined by the boundary conditions.

iii) TE-TM

From Eqs. (13), (C.15) and (D.10) one obtains

$$\begin{aligned} H_{EM}^{3g} &= -\Lambda_1\Lambda_2 \int_{bag} d^3x \left[ \vec{j}^{MM} \cdot \vec{a}^{EE} + \vec{j}_T^{EM} \cdot \vec{a}_T^{EM} \right] \\ &= -\Lambda_1\Lambda_2 \frac{\alpha_s}{R} \left[ a_{EM}^{3g}\, \vec{S}_1 \cdot \vec{S}_2 + b_{EM}^{3g}\, T_{12} \right] . \end{aligned} \tag{E.16a}$$

Here notice the equality

$$\int_{\text{bag}} d^3x\ \vec{J}_T^{EM} \cdot \vec{a}_T^{EM} = \int_{\text{bag}} d^3x\ \vec{j}^{EM} \cdot \vec{a}_T^{EM}$$

$$= \int_{\text{bag}} d^3x\ \vec{j}^{EM} \cdot \left(\vec{a}^{EM} + \vec{\nabla} \frac{1}{\omega^2} \vec{\nabla} \cdot \vec{a}^{EM}\right) \qquad \text{(E.16b)}$$

$$- \int_{\text{bag}} d^3x\ \rho^{EM} \cdot \phi^{EM} \quad .$$

We recognize the last term in Eq. (E.16b) as the Coulomb (or longitudinal electric) energy. In this case it would be simpler to directly calculate the sum of magnetic and electric energies. For consistency we quote the results separately as in the other cases. Also in this case the 3g-energy shift has a transverse electric contribution in contrast to the $(TE)^2$ and $(TM)^2$ cases.

The various contributions to $a_{EM}^{3g}$ and $b_{EM}^{3g}$ are given by

$$a_{EM}^{3g} = -\gamma_{EM}\ x \int_0^1 d\rho\ \rho^2 \Big\{\frac{1}{2}\ f_2(\rho) \Big[ j_2(x\rho)\ \hat{N}^{EM}(\rho) + n_2(x\rho)\ \hat{J}^{EM}(\rho)$$

$$+ \frac{1}{x^3}\left(\frac{2}{3}\ f_2(\rho) + f_0(\rho)\right)\Big] + \frac{3}{2}\ f_0(\rho) \Big[ j_0(x\rho)\ \hat{N}^{EM}(\rho) \qquad \text{(E.17a)}$$

$$+ n_0(x\rho)\ \hat{J}^{EM}(\rho) + \frac{1}{2x^3}\left(\frac{2}{3}\ f_2(\rho) + f_0(\rho)\right)\Big]\Big\}$$

$$+ \frac{3}{2}\ \gamma_{EM} \int_0^1 d\rho\ \left(4j_1^2(x_M\rho) - j_2^2(x_M\rho)\right)\left[\frac{1}{\rho}\ J_1^E(\rho) + \rho^2\ N_1^E(\rho)\right]$$

$$\approx 0.442$$

and

$$b_{EM}^{3g} = \frac{9}{10}\ x \int_0^1 d\rho\ \rho^2\ \tilde{f}_2(\rho) \Big[ j_2(x\rho)\ \tilde{N}_2^{EM}(\rho) \qquad \text{(E.17b)}$$

$$+ n_2(x\rho)\ \tilde{J}_2^{EM}(\rho)\Big] \approx -0.003$$

with

$$\gamma_{EM} = \left(\frac{8\pi\ N_E N_M R^4}{3x_E x_M}\right)^2 \tag{E.17c}$$

For the 4g-interaction one has

$$\begin{aligned} H_{EM}^{4g} &= -\Lambda_1\Lambda_2 \frac{\alpha_s}{R} \int_{bag} d^3x \left[\vec{\ell}_1^{EM}(x)\ \vec{\ell}_2^{EM}(x) + \vec{\ell}_1^{E}(x)\ \vec{\ell}_2^{M}(x)\right] \\ &= -\Lambda_1\Lambda_2 \frac{\alpha_s}{R}\left[a_{EM}^{4g}\ \vec{S}_1 \cdot \vec{S}_2 + b_{EM}^{4g}\ T_{12} + c_{EM}^{4g}\ \bar{I}_{12}\right] \end{aligned} \tag{E.18}$$

where

$$\ell_1^{EM}(x) = A_\alpha^E \times A_\beta^M - A_\alpha^M \times A_\beta^E \tag{E.19}$$

and

$$\begin{aligned} a_{EM}^{4g} = -\gamma_{EM} \int d\rho\ \rho^2\ j_1^2(x_E\rho)\Big[\frac{1}{12}\Big(2j_0(x_M\rho) - j_2(x_M\rho)\Big)^2 \\ + \frac{1}{8}\ j_2(x_M\rho)\ \Big(4j_0(x_M\rho) + j_2(x_M\rho)\Big)\Big] \approx -0.025 \end{aligned} \tag{E.20a}$$

$$\begin{aligned} b_{EM}^{4g} = -\gamma_{EM} \int d\rho\ \rho^2\ j_1^2(x_E\rho)\ \frac{3}{40}\ j_2(x_M\rho) \\ \cdot \Big(4j_0(x_M\rho) + j_2(x_M\rho)\Big) \approx -0.001 \end{aligned} \tag{E.20b}$$

For the Coulomb part we get

$$\begin{aligned} H_{EM}^{Cl} &= \Lambda_1\Lambda_2 \int_{bag} d^3x \left[\rho^{MM} \cdot \phi^{EE} + \rho^{EM} \cdot \phi^{EM}\right] \\ &= -\Lambda_1\Lambda_2 \frac{\alpha_s}{R}\left[a_{EM}^{Cl}\ \vec{S}_1 \cdot \vec{S}_2 + b_{EM}^{Cl}\ T_{12} + c_{EM}^{Cl}\ I_{12}\right] \end{aligned} \tag{E.21}$$

where

$$\begin{aligned} a_{EM}^{Cl} = -\frac{1}{36}\ \gamma_{EM}(x_E + x_M)^2 \int_0^1 d\rho\ j_1(x_E\rho)\ \Big(2j_0(x_M\rho) - j_2(x_M\rho)\Big) \\ \times \left[G_1^{EM}(\rho) + \rho^3\ H_1^{EM}(\rho)\right] \approx -0.146 \end{aligned} \tag{E.22a}$$

$$b_{EM}^{C1} = \frac{1}{25}\,\gamma_{EM}\,x_E\,x_M \int_0^1 d\rho\; j_2(x_M\rho)\left(4j_0(x_M\rho) + j_2(x_M\rho)\right) \times \left[\frac{1}{\rho}\,G_2^E(\rho) + \rho^4\,H_2^E(\rho)\right] \approx -0.002 \qquad \text{(E.22b)}$$

Again $c_{EM}^{C1}$ is undetermined.

REFERENCES

1. For a review with references to the literature see, e.g., M. Fishbane, Virginia University preprint, talk presented at 1981 Orbis Scientiae, Fort Lauderdale, Florida, January 1981.

2. See, e.g., D. Scharre, Proceedings of 1981 International Conference on Lepton and Photon Interactions at High Energies, Bonn, August 1981.

3. C. Edwards et al., SLAC-PUB-2822 (1981).

4. There have been several attempts (see Ref. 1) to construct non-relativistic potential models for glueballs. Such models, however, require a somewhat arbitrary introduction of an effective "constituent gluon mass." This breaks gauge invariance and special assumptions must be made about the unphysical longitudinal gluon modes.

5. During the course of this work, a paper on the same subject has appeared, T. Barnes, F. E. Close and S. Monaghan, Rutherford Laboratory preprint RL-81-086 (November 1981). As will be clear, we agree with some but not all results of this paper.

6. R. L. Jaffe and K. Johnson, Phys. Lett. 60B, 201 (1976).

7. J. F. Donoghue, K. Johnson, and B. A. Li, Phys. Lett. 99B, 416 (1981).

8. T. Barnes, F. E. Close and S. Monaghan, Rutherford Laboratory preprint, RL-81-090 (1981).

9. T. DeGrand, R. L. Jaffe, K. Johnson and J. Kiskis, Phys. Rev. D12, 2060 (1975).

10. J. F. Donoghue and K. Johnson, Phys. Rev. D21, 1975 (1980).

11. We use the conventions of E. S. Fradkin and I. V. Tyuntin, Phys. Rev. D2, 2841 (1970), for the field strengths $F^a_{jk}$ and the coupling constant g. Ghosts do not give $\mathcal{O}(g^2)$ contributions to the diagrams of interest here.

12. This form is appropriate for calculating the diagonal elements of the glueball mass matrix. For the problem of mixing, methods similar to those described in C. Carlsson and T. H. Hansson, "$\eta - \eta'$ - Glueball Mixing," to appear in Nucl. Phys. B, must be used.

13. Using this form of the current we obtain

$$\vec{\mu}^a = \frac{1}{2}\int d^3r\ \vec{r} \times \vec{j}^a = \frac{g}{2\omega}\Lambda^a(\vec{L} + 2\vec{S})$$

with obvious notation. This is the result expected for a particle with gyromagnetic ratio g = 2.

14. T. D. Lee, Phys. Rev. D19, 1802 (1979).

15. C. Peterson, T. H. Hansson and K. Johnson, SLAC-PUB-2823 (1981).

16. T. DeGrand and R. L. Jaffe, Annals Phys. 100, 425 (1976).

17. We disagree with Ref. 5 both with regard to spin structure and numerical values in the (TE)(TM) case.

18. C. E. Carlson, T. H. Hansson and C. Peterson, in preparation.

19. T. H. Hansson, K. Johnson and C. Peterson, UCSB preprint NSF-ITP-82-01 (1982).

Table I. Current and charge densities $j_{mn}$ and $\rho_{mn}$ for $\ell = 1$ gluon modes.

| | $\vec{j}$ | $\rho$ |
|---|---|---|
| TE–TE | $3g \frac{N_E^2 R}{\rho x_E^2} j_1^2(x_E\rho)\ \hat{r}\times\vec{S}$ | $-2g \frac{N_E^2 R}{x_E} j_1^2(x_E\rho)\left[U - \frac{2}{3} I\right]$ |
| TM–TM | $g \frac{N_M^2 R}{\rho x_M^2}\left[4j_1^2(x_M\rho) - j_2^2(x_M\rho)\right]\hat{r}\times\vec{S}$ | $2g \frac{N_M^2}{x_M}\frac{R}{3}\ j_2(x_M\rho)\left[j_2(x_M\rho) + 4j_0(x_M\rho)\right]U + \frac{2}{3}\left[j_2^2(x_M\rho) + 2j_0^2(x_M\rho)\right] I$ |
| TE–TM | $-g \frac{N_E N_M}{x_E x_M} R\left[f_2(\rho)'\left(\hat{r}(\hat{r}\cdot\vec{S}) - \frac{1}{3}\vec{S}\right) + \tilde{f}_2(\rho)\vec{T} + f_0(\rho)\vec{S}\right]$ | $ig \frac{N_E N_M R(x_E + x_M)}{3\, x_E x_M}\ j_1(x_E\rho) + \left[2j_0(x_M\rho) - j_2(x_M\rho)\right]\hat{r}\cdot\vec{S}$ |

Table II. Expectation values for the operators $\Lambda_1\Lambda_2$, $\vec{S}_1\cdot\vec{S}_2$ and $T_{12}$ for the lowest lying glueball states.

| | $\Lambda_1\Lambda_2$ | $\vec{S}_1\cdot\vec{S}_2$ | $T_{12}$ |
|---|---|---|---|
| $0^{(\mp)+}$ | −3 | −2 | 4 |
| $2^{(\mp)+}$ | −3 | 1 | 1 |

# FIGURE CAPTIONS

Fig. 1. Gluon-gluon interactions to $\mathcal{O}(\alpha_s)$.

Fig. 2. Two-gluon interaction diagrams in Coulomb gauge: a) one-gluon exchange, b) four-gluon interaction and c) Coulomb interaction.

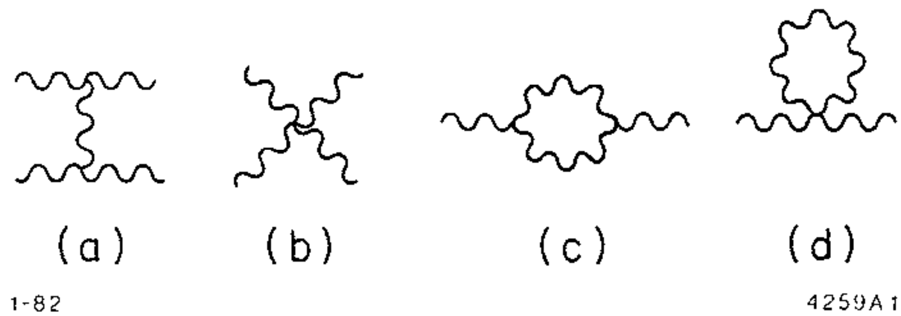


Fig. 1

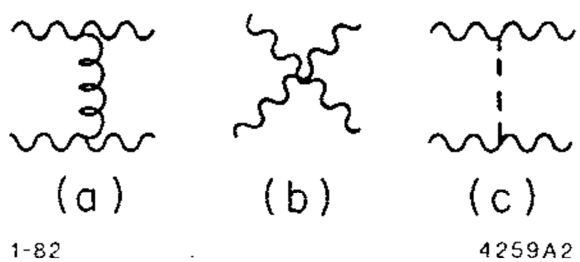


Fig. 2

SLAC-PUB-2873
April 1982
(T/E)

ADDENDUM AND ERRATA

# GLUON-GLUON INTERACTIONS IN THE BAG MODEL*

C. E. Carlson†
College of William and Mary
Williamsburg, Virginia 23185

T. H. Hansson‡
Institute for Theoretical Physics
University of California,
Santa Barbara, California 93106

and

C. Peterson§
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

Submitted to Physical Review D

* Work supported in part by the Department of Energy, contract DE-AC03-76SF00515.

† Work supported in part by the National Science Foundation.

‡ Work supported in part by the Swedish Natural Science Research Council, contract F-PD 4728-100. Address After 1 February 1982: Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139.

§ Work supported in part by the Swedish Nautral Science Research Council, contract F-PD 8207-102. On leave from NORDITA, Copenhagen, Denmark.

ADDENDUM

For states consisting of more than two gluons there is an extra contribution to the $\mathcal{O}(\alpha_s)$-energy shift for gluon pairs in color octet spin one state. One part comes from the four-gluon interaction [Fig. 1(b)] and is easily calculated. Another piece comes from the annihilation diagrams [Fig. 1(c)] which is simply related to the gluon exchange graph [Fig. 1(d)] via s-t channel crossing. We only deal with the $(TE)^2$ and $(TM)^2$ cases. The calculations are contained in Appendix F and here we just quote the results

$$H^{AN}_{mn} = -\frac{\alpha_s}{R} P_8^{col} P_1^{sp} d_{mn} \tag{1}$$

$$d_{EE} = d^{3g}_{EE} + 3a^{4g}_{EE} \approx -0.529 \tag{2a}$$

$$d_{MM} = d^{3g}_{MM} + 3a^{4g}_{MM} \approx -0.555 \tag{2b}$$

where $P_8^{col}$ is the color projection operator on a color octet state and $P_1^{sp}$ is the spin projection operator on a spin one state.

## APPENDIX F

In this Appendix we calculate the contribution to $H_{mn}$ due to the s-channel annihilation graphs Figs. 2(d) and 2(e) for the $(TE)^2$- and $(TM)^2$-cases. These are related to the t-u-channel graphs 2(a) and 2(c) by crossing. More explicitly we get the contribution of the annihilation diagram from the corresponding exchange graph by making the following changes:

(A) Use the relation

$$\left(\vec{S}_I\cdot\vec{S}_{II}\right)_{t-ch} = -\left[2I - \vec{S}_1\cdot\vec{S}_2 - \left(\vec{S}_1\cdot\vec{S}_2\right)^2\right]_{s-ch} \equiv -2P_1^{sp} , \qquad (F.1)$$

which can easily be derived using the formulae in Appendix B, to express the spin operators in terms of s-channel invariants. As expected we get the projection operator $P_1^{sp}$ on spin one states.

(B) Do the same for color, i.e., use

$$\left(\vec{\lambda}_I\cdot\vec{\lambda}_{II}\right)_{t-ch} = -\,3P_8^{col} \qquad (F.2)$$

where $P_8^{col}$ is the projection operator on color octet states. (As in the spin case there is an explicit, though more complicated, formula giving $P_8^{col}$ in terms of s-channel invariants, i.e., the quadratic and cubic SU(3) Casimir operators.)

(C) Note that the gluon exchange graphs [Figs. 2(a) and 2(c)] are the sum of t- and u-channel contributions. To get the s-channel results by crossing we must use only the t-channel piece. (For the $(TE)^2$- and $(TM)^2$-cases, the t- and u-channel contributions are equal so we can simply divide the old expressions by 2.)

(D) The energies of the propagators are for the $(TE)^2$- and $(TM)^2$-cases given by $2\omega_E$ and $2\omega_M$ respectively instead of 0 in the exchange case.

(E) For the Coulomb diagram it is easy to see that the charge densities vanish identically for the $(TE)^2$- and $(TM)^2$-cases. With these changes, the calculations proceed exactly as in Appendices C-E and we get the results:

$$H^{AN}_{mn} = -\frac{\alpha_s}{R} P^{col}_8 \, P^{sp}_1 \, d_{mn} \tag{F.3}$$

$$d_{EE} = d^{3g}_{EE} + 3a^{4g}_{EE} \approx -0.296 - 0.234 \approx -0.529 \tag{F.4a}$$

$$d_{MM} = d^{3g}_{MM} + 3a^{4g}_{MM} \approx -0.312 - 0.243 \approx -0.555 \tag{F.4b}$$

where we also added an extra term which comes from the four-gluon interaction in Fig. 1(b). Although formally not an annihilation contribution this is included here for formal convenience. As usual we neglected a spin independent constant term. Note that the diagram 1(c) gives a positive energy shift as expected from mixing with a lower lying (in this case dominantly the lowest one-gluon mode) state.

## ERRATA

The results for the TE-TM-mode are applicable only to the 2-gluon system, where all the TE-TM pairs are in color and spin symmetric states.

In Eqs. (9), (10) and (13) the u-channel graphs (containing terms $\sim j_{mn} j_{nm}$) should carry an overall ± sign where + refers to the color symmetric $(1, 8_s, 27)$ and - to the antisymmetric representation $(8_A, 10, \overline{10})$.

p. 5: The sentence "where the exchange term ..." should be deleted.

p. 6: Eq. (14c) shall read:

$$b_{EE} = b_{EE}^{Coul} \simeq -0.041 \quad .$$

p. 9: Eq. (21) shall read:

$$E_{0^{++}} = 0.59 \, \frac{\alpha_s}{R} \quad .$$

p. 18: Eq. (D.11a) shall read:

$$\vec{j}_L^{EM} = i \, \omega \, \vec{\nabla} \, \phi^{EM} \quad .$$

p. 26: Eq. (E.17a) shall read:

$$a_{EM}^{3g} = \gamma_{EM} \; \cdots$$

Eq. (E.17b) shall read:

$$b_{EM}^{3g} = \ldots \approx 0.003 \quad .$$

## FIGURE CAPTIONS

Fig. 1. Gluon-gluon interactions to $\mathcal{O}(\alpha_s)$.

Fig. 2. Two-gluon interaction diagrams in Coulomb gauge: (a) one-gluon exchange, (b) four-gluon interaction, (c) Coulomb interaction, (d) and (e) gluon-gluon annihilation into a transverse and Coulomb gluon respectively.

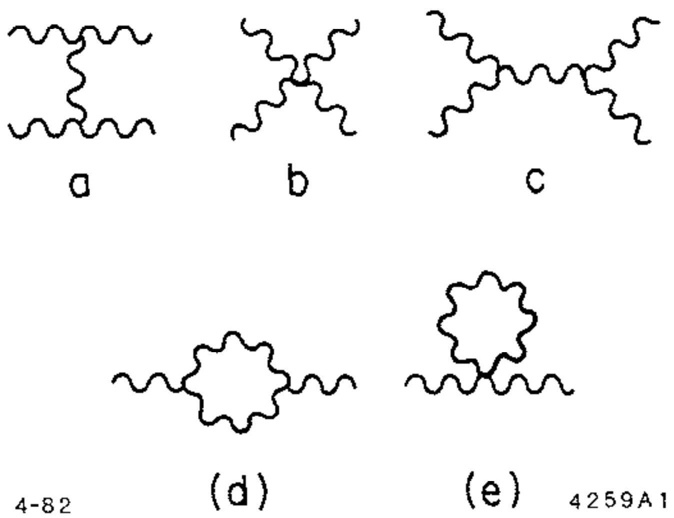


Fig. 1

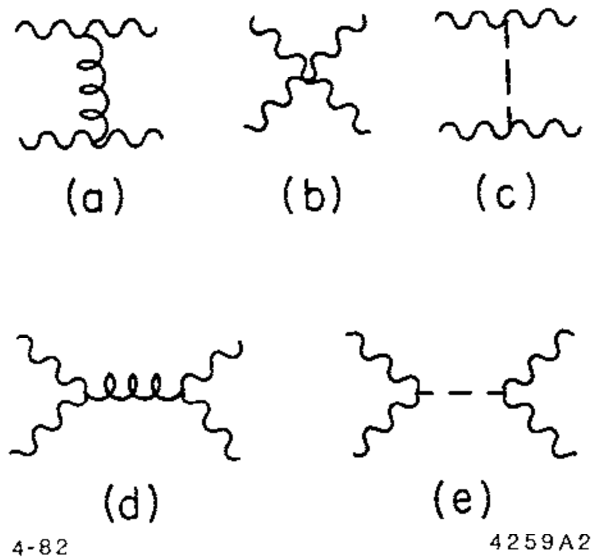


Fig. 2