

The chiral magnetic effect in a strongly-coupled anisotropic plasma

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Abstract

In this thesis we discuss a possible dependence of the chiral magnetic effect on the elliptic flow using fluid-gravity duality. We first study this in a hydrodynamic model for a static anisotropic plasma with multiple anomalous $U(1)$ currents. In the case of two charges, one axial and one vector, the CME formally appears as a first-order transport coefficient in the vector current. We compute this transport coefficient and show its dependence on v_2 . We also determine the CME coefficient from first-order corrections to the dual anti-de Sitter background using the fluid-gravity duality. For small anisotropies, we find numerical agreement with the hydrodynamic result.

In dieser Arbeit diskutieren wir eine mögliche Abhängigkeit des chiralen magnetischen Effekts vom elliptischen Fluss, indem wir die Dualität zwischen Flüssigkeit und Gravitation benutzen. Zunächst analysieren wir dies in einem hydrodynamischen Modell mit einem statischen anisotropischen Plasma welches mehrere anormale $U(1)$ Ströme besitzt. Anschliessend berechnen wir die Transportkoeffizienten und zeigen ihre Abhängigkeit von v_2 . Weiterhin bestimmen wir die CME Koeffizienten in erster Störungsordnung im dualen anti-de Sitter Hintergrund, indem wir erneut die Flüssigkeit-Gravitation-Dualität benutzen. Für kleine Anisotropien finden wir numerische Übereinstimmung mit dem hydrodynamischen Modell.

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Chapter 1

Introduction

Recently, STAR [1] and PHENIX [2] collaborations at Relativistic Heavy Ion Collider and ALICE [3] collaboration at Large Hadron Collider reported experimental observation of charge asymmetry fluctuations in heavy-ion collisions. The interpretation of the observed effect is still under intense discussion. In the last couple of years the chiral magnetic effect (CME) has attracted much attention as a candidate for the explanation of the charge asymmetry.

The chiral magnetic effect, in its simplest version, states that in chirally asymmetric quark matter in the presence of a magnetic field \vec{B} , an electric current is generated along \vec{B} [4, 5]. This is due to the fact that, the strong magnetic field aligns the quark's spins along \vec{B} . In this case right-handed positive fermions move along the direction of the magnetic field, while the negative ones move in the opposite direction (see Fig. 1.1). But since in presence of nontrivial gluonic background there are unequal densities of left- and right-handed fermions, there should be an electric current and a separation of electric charge [5, 6]. Note that analogous effects were found earlier in neutrino [7], electroweak [8] and condensed matter physics [9]. Lattice QCD results [10, 11, 12] suggest the existence of the effect, although the magnitude of the CME-induced charge asymmetry may be too small to explain the observed charge asymmetry [13].

The hydrodynamical approach to the CME and CME-related phenomena was proposed in [14, 15, 16, 17, 18, 19, 20]. There, the CME appears in form of a nonvanishing transport coefficient in the electric current, $\vec{j} = \kappa_B \vec{B}$, which measures the response of the system to an external magnetic field [14, 21]. In [20], the chiral magnetic conductivity in an isotropic fluid was determined as

$$\kappa_B = C\mu_5 \left(1 - \frac{\mu\rho}{\epsilon + P} \right). \quad (1.1)$$

The first term is the standard term for the CME and depends only on the axial anomaly coefficient C and the axial chemical potential μ_5 . The second term proportional to the

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factor $\frac{\rho}{\epsilon+P}$ depends on the dynamics of the fluid and has a chance to depend on v_2 in the anisotropic case.

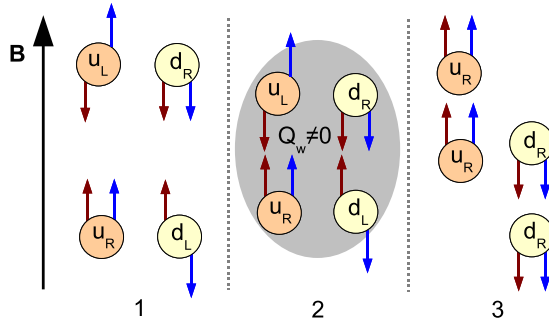


Figure 1.1: In chiral limit the right-handed particles have spin and momentum parallel to each other, while the left-handed particles have spin and momentum anti-parallel, therefore $(N_l - N_r)_\infty - (N_l - N_r)_{-\infty} = 2N_f Q$. For a detailed explanation see [4, 5]

In non-central heavy ion collisions secondary particles emerge with a nontrivial elliptic flow pattern¹ [25]. A convenient way of characterizing the various patterns of the anisotropic flow is to use the so-called elliptic flow coefficient which is the second coefficient of Fourier expansion, $v_2 = \langle \cos(2\phi) \rangle$, of the azimuthal momentum distribution

$$\frac{dN}{d\phi} \sim 1 + 2v_2 \cos(2\phi), \quad (1.2)$$

where ϕ is the azimuthal angle.

In a recent experiment, the charge separation is measured as a function of the elliptic flow coefficient v_2 [26]. The data is taken from (rare) Au+Au collisions with 20 – 40% centrality but different v_2 . In this way v_2 is varied while at the same time the number of participating nucleons (and therefore the magnetic field) is kept almost constant. The plots in [26] suggest that the charge separation is proportional to v_2 . If this holds true, the charge separation will depend on the event anisotropy.

In this thesis we discuss the question of whether and how the CME depends on the elliptic flow v_2 . We study this both in hydrodynamics and in context of a fluid/gravity duality.

We study a hydrodynamic model for an anisotropic fluid with multiple anomalous $U(1)$ charges². We compute the CME coefficient κ_B and express the result in terms of the momentum anisotropy ε_p [27] defined as

$$\varepsilon_p = \frac{\langle P_T - P_L \rangle}{\langle P_T + P_L \rangle}, \quad (1.3)$$

¹Even in central collisions there is a strong asymmetry.

²This model extends those in [22, 23, 24].

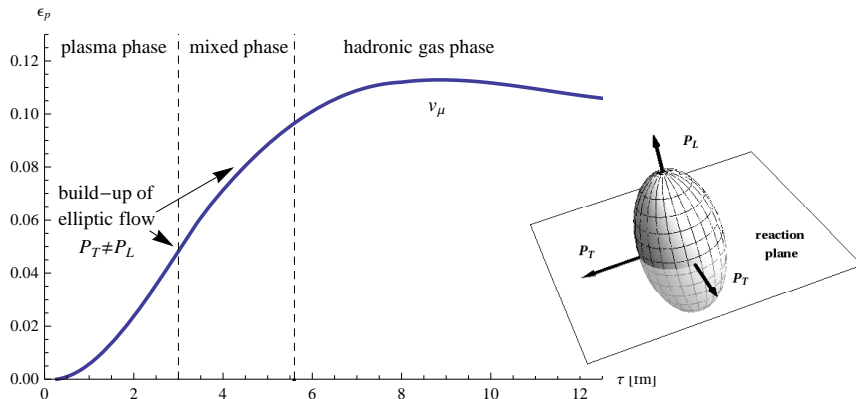


Figure 1.2: Sketch of the time evolution of the momentum anisotropy ε_p (based on [28]). The small figure shows the orientation of P_L and P_T with respect to the reaction plane.

where P_T and P_L are the pressures in the plane transverse to the beam line¹. A sketch of ε_p as a function of the proper time τ is shown in Fig. 1.2. ε_p describes the build-up of the elliptic flow in off-central collisions. Our model describes a state after thermalization with unequal pressures $P_T \neq P_L$. At freeze-out ε_p roughly equals v_2 , and we find that for small anisotropies the CME-coefficient κ_B increases linearly with v_2 .

We perform also a holographic computation of κ_B in the dual gravity model. A similar computation was previously done by Kirsch and Kalaydzhyan in [20] for the STU model [29], a string-theory-inspired prototype of an (isotropic) anti-de Sitter (AdS) black hole solution with three $U(1)$ charges. Other holographic approaches to the CME can be found in [30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40].

In the anisotropic case, we first need to construct an appropriate gravity background. As an ansatz, we choose a multiply charged AdS black hole solution with some additional functions w_L and w_T inserted which will make the background anisotropic and ε_p -dependent. Since analytical solutions for charged anisotropic backgrounds are notoriously difficult to find, we will use shooting techniques to find a numerical solution. Other AdS backgrounds dual to anisotropic fluids are constructed in [41, 42, 43, 44, 45].

As the AdS solution in [43], the background is static and does not describe the process of isotropization. Even though such models have some limitations [43], they are nevertheless useful for the computation of transport coefficients. We show this, following [20], by determining κ_B from the first-order corrections to this background using the fluid-gravity duality [46]. For small anisotropies, we find numerical agreement

¹In our conventions the indices L and T refer to the longitudinal and transverse direction with respect to an anisotropy vector v_μ normal to the reaction plane, see Fig. 1.2.

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with the hydrodynamic result for κ_B . Other (dissipative) transport coefficients in strongly-coupled anisotropic plasmas are discussed in [47, 48, 49, 50].

The thesis is organized as follows:

- Chapter 2 consists largely of a necessary review of relativistic hydrodynamics. In particular we review the hydrodynamics of an anisotropic fluid and derive some thermodynamical identities for this case.
- In chapter 3 we review the hydrodynamics of an isotropic relativistic fluid with triangle anomalies. We extend our consideration to the anisotropic case and compute the CME coefficient.
- In chapter 4 we briefly present basics of the *AdS/CFT* correspondence and discuss the fluid/gravity duality.
- In the last chapter we construct the holographic dual of the anisotropic fluid dynamics and present a numerical solution. We use this background to perform a holographic computation of the vortical and magnetic conductivities of an anisotropic fluid.

The work in chapter 5 and the part of the work in chapters 2 and 3 were performed in collaboration with Tigran Kalaydzhyan and Ingo Kirsch [51], and results have been published in the journal Physical Review D.

Chapter 2

Review of relativistic hydrodynamics

In this chapter we will give some necessary information about relativistic fluid dynamics for the study of fluid/gravity dual models. There is a huge amount of review articles on relativistic hydrodynamics (see e.g. [52], [53]), on applications of fluid dynamics to heavy-ion collisions [54]. We also refer to [55] for details of the subject.

2.1 Ideal hydrodynamics

We will work in four-dimensional Minkowski space with the metric $\eta^{\mu\nu} = \text{diag}\{-1,1,1,1\}$. The fluid is described by its energy density $\epsilon(x, t)$, its pressure field $P(t, x)$ and its four-velocity u^μ defined as:

$$u^\mu(t, x) = \frac{dx^\mu}{d\tau}, \quad (2.1)$$

where $x^\mu = (t, x, y, z)$ and τ is the proper time. The four-velocity is expressed as

$$u^\mu = \gamma(1, v), \quad (2.2)$$

where $\gamma = \sqrt{1 - v^2}$ and v is the three-velocity of the fluid element. The four-velocity is a time-like, normalized vector

$$u_\mu u^\mu = -1 \quad (2.3)$$

In the local rest frame it has the only non-vanishing time component

$$u^\mu = (1, 0, 0, 0) . \quad (2.4)$$

Also one can define the local rest frame of the fluid as the frame in which u^μ has the form of (2.4).

2. REVIEW OF RELATIVISTIC HYDRODYNAMICS

We consider a fluid with n global $U(1)$ charges. The hydrodynamic equations are simply the conservation of energy-momentum and $U(1)$ currents

$$\partial_\mu T^{\mu\nu} = 0, \quad (2.5)$$

$$\partial_\mu j_a^\mu = 0. \quad (2.6)$$

In thermal equilibrium the energy-momentum tensor $T^{\mu\nu}$ and the current are given by

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + Pg^{\mu\nu}, \quad (2.7)$$

$$j_a^\mu = \rho_a u^\mu, \quad (2.8)$$

where ρ_a are the global $U(1)$ charge densities and $a = 1, \dots, n$. In the local rest frame the energy-momentum tensor has a diagonal form:

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}. \quad (2.9)$$

2.2 Thermodynamics

Let us consider the combined first and second laws of thermodynamics of a fluid with a single charge ($n = 1$):

$$dE = TdS - PdV + \mu dN, \quad (2.10)$$

where E , S , V and N are total energy, entropy, volume, particle number, respectively. As we will show now, one can derive this equation from the equation of state

$$E = E(S, V, N) \quad (2.11)$$

using the following definitions of the temperature, pressure and chemical potential:

$$T = \left. \frac{\partial E}{\partial S} \right|_{V,N}, \quad P = - \left. \frac{\partial E}{\partial V} \right|_{S,N}, \quad \mu = \left. \frac{\partial E}{\partial N} \right|_{S,V}. \quad (2.12)$$

Let us rescale thermodynamic variables when S , V , and N in the following way

$$\tilde{S} = \lambda S, \quad \tilde{V} = \lambda V, \quad \tilde{N} = \lambda N. \quad (2.13)$$

From thermodynamics we know [52]

$$\tilde{E}(\tilde{S}, \tilde{V}, \tilde{N}) = \lambda E(S, V, N). \quad (2.14)$$

2.3 First order corrections to hydrodynamics

Then

$$\begin{aligned} d\tilde{E} &= \tilde{T}d\tilde{S} - \tilde{p}d\tilde{V} + \tilde{\mu}d\tilde{N} \\ &= \lambda(TdS - PdV + \mu dN) + (TS - PV + \mu N) d\lambda \\ &= \lambda dE + E d\lambda, \end{aligned} \tag{2.15}$$

and we obtain the Euler relation

$$E = TS - pV + \mu N . \tag{2.16}$$

If $\epsilon = E/V$ denotes the total energy density, $s = S/V$ the total entropy density, and $\rho = N/V$ the total particle number density, then

$$P + \epsilon = Ts + \mu\rho . \tag{2.17}$$

One can reduce by one the number of parameters required for a complete specification of the thermodynamic state by setting $\lambda = 1/V$, then $\tilde{S} = s$, $\tilde{V} = 1$, $\tilde{N} = \rho$ and

$$\tilde{E} = E/V = \epsilon. \tag{2.18}$$

The first law of thermodynamics thus becomes

$$d\tilde{E} = \tilde{T}d\tilde{S} - \tilde{P}d\tilde{V} + \tilde{\mu}d\tilde{N} \tag{2.19}$$

$$= Tds + \mu d\rho , \tag{2.20}$$

and we find that

$$d\epsilon = Tds + \mu d\rho. \tag{2.21}$$

2.3 First order corrections to hydrodynamics

If the fluid is dissipative, one must modify the energy-momentum tensor and entropy current:

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - P g^{\mu\nu} + \tau^{\mu\nu}, \tag{2.22}$$

$$j^\mu = \rho u^\mu + \nu^\mu, \tag{2.23}$$

where $\tau^{\mu\nu}$ and ν^μ are the first order corrections (dissipative part).

In presence of the dissipative corrections, the definition of the flow velocity is more arbitrary. In other words, the form of the dissipative terms $\tau^{\mu\nu}$ and ν^μ depends on the definition of the local rest frame of the fluid [56]. One natural definition of the rest frame is the so called Landau frame¹ [55], where we have

$$u_\mu \tau^{\mu\nu} = 0, \quad u_\mu \nu^\mu = 0. \tag{2.24}$$

¹Sometimes this frame is called also Landau-Lifshitz frame.

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With the first constraint and with requirement that entropy increases with time, the most general form of $\tau^{\mu\nu}$ is

$$\tau^{\mu\nu} = -\eta P^{\mu\alpha} P^{\nu\beta} (\partial_\alpha u_\beta + \partial_\beta u_\alpha) - \left(\zeta - \frac{2}{3}\eta\right) P^{\mu\nu} \partial \cdot u, \quad (2.25)$$

where $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$.

There are also other frames, particularly, the Eckart definition of the rest frame. Note that the Eckart frame is ill-defined for the systems with vanishing net baryon number [56], so we use the Landau frame.

2.4 Anisotropic fluid

The energy-momentum tensor of an ideal anisotropic fluid has the general form [57]

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P_x x^\mu x^\nu + P_y y^\mu y^\nu + P_z z^\mu z^\nu, \quad (2.26)$$

where P_x, P_y, P_z are three different pressure components. The four vectors u^μ, x^μ, y^μ and z^μ satisfy the following normalization conditions

$$u_\mu u^\mu = -x_\mu x^\mu = -y_\mu y^\mu = -z_\mu z^\mu = -1, \quad (2.27)$$

$$u_\mu x^\mu = u_\mu y^\mu = u_\mu z^\mu = x_\mu y^\mu = x_\mu z^\mu = y_\mu z^\mu = 0. \quad (2.28)$$

In the local rest frame of the fluid element these vectors have the following forms,

$$u^\mu = (1, 0, 0, 0), \quad (2.29)$$

$$x^\mu = (0, 1, 0, 0), \quad (2.30)$$

$$y^\mu = (0, 0, 1, 0), \quad (2.31)$$

$$z^\mu = (0, 0, 0, 1), \quad (2.32)$$

and the energy-momentum tensor has the following diagonal form

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P_x & 0 & 0 \\ 0 & 0 & P_y & 0 \\ 0 & 0 & 0 & P_z \end{pmatrix}. \quad (2.33)$$

For boost-invariant and cylindrically symmetric systems the fluid four-velocity u_μ has the following parametrization

$$\begin{aligned} u^0 &= \cosh \theta_\perp \cosh \eta_\parallel, \\ u^1 &= \sinh \theta_\perp \cos \phi, \\ u^2 &= \sinh \theta_\perp \sin \phi, \\ u^3 &= \cosh \theta_\perp \sinh \eta_\parallel, \end{aligned} \quad (2.34)$$

where θ_\perp is the transverse fluid rapidity defined by the formula

$$v_\perp = \sqrt{v_x^2 + v_y^2} = \tanh \theta_\perp, \quad (2.35)$$

η_\parallel is the space-time rapidity,

$$\eta_\parallel = \frac{1}{2} \ln \frac{t+z}{t-z}, \quad (2.36)$$

and ϕ is the azimuthal angle

$$\phi = \arctan \frac{y}{x}. \quad (2.37)$$

For the longitudinal direction z_μ , the transverse direction to the beam x_μ and the second transverse direction y_μ we may use the following parametrization

$$z^0 = \sinh \eta_\parallel, \quad x^0 = \sinh \theta_\perp \cosh \eta_\parallel, \quad y^0 = 0, \quad (2.38)$$

$$z^1 = 0, \quad x^1 = \cosh \theta_\perp \cos \phi, \quad y^1 = -\sin \phi, \quad (2.39)$$

$$z^2 = 0, \quad x^2 = \cosh \theta_\perp \sin \phi, \quad y^2 = \cos \phi, \quad (2.40)$$

$$z^3 = \cosh \eta_\parallel, \quad x^3 = \sinh \theta_\perp \sinh \eta_\parallel, \quad y^3 = 0. \quad (2.41)$$

$g_{\mu\nu}$ is the metric with signature $(-, +, +, +)$. It can easily be shown that

$$g^{\mu\nu} + u^{\mu\nu} = x^\mu x^\nu + y^\mu y^\nu + z^\mu z^\nu. \quad (2.42)$$

Using this expression one can rewrite the energy-momentum tensor (2.26) as follows:

$$T^{\mu\nu} = (\epsilon + P_x)u^\mu u^\nu + P_x g^{\mu\nu} - \Delta_1 v^\mu v^\nu - \Delta_2 w^\mu w^\nu, \quad (2.43)$$

with

$$\Delta_1 = P_x - P_y, \quad \Delta_2 = P_x - P_z, \quad (2.44)$$

and here we introduced new variables $v^\mu = y^\mu$ and $w^\mu = z^\mu$.

For simplicity we will consider an anisotropic relativistic fluid with $P_z = P_L$ and $P_x = P_y = P_T$. The hydrodynamic equations are the same, while the stress-energy tensor $T^{\mu\nu}$ and $U(1)$ currents $j^{a\mu}$ now have the following form

$$T^{\mu\nu} = (\epsilon + P_T)u^\mu u^\nu + P_T g^{\mu\nu} - \Delta v^\mu v^\nu + \tau^{\mu\nu}, \quad (2.45)$$

$$j^{a\mu} = \rho^a u^\mu + \nu^{a\mu}, \quad (2.46)$$

where $\Delta = P_T - P_L$, and P_T and P_L denote the transverse and longitudinal pressures, respectively [22, 23, 24]. $\tau^{\mu\nu}$ and $\nu^{a\mu}$ denote higher-gradient corrections, for which we require $u_\mu \tau^{\mu\nu} = 0$ and $u_\mu \nu^{a\mu} = 0$. For $P_L = P_T = P$ we recover the hydrodynamic equations of the isotropic fluid (2.7)–(2.8).

2. REVIEW OF RELATIVISTIC HYDRODYNAMICS

The four-vectors u^μ and v^μ describe the flow of the fluid and the direction of the longitudinal axis, respectively. The vector v^μ is space-like and orthogonal to u^μ ,

$$u_\mu u^\mu = -1, \quad v_\mu v^\mu = 1, \quad u_\mu v^\mu = 0. \quad (2.47)$$

It is convenient to define the proper time τ by $\partial^\nu \ln \tau \equiv v^\mu \partial_\mu v^\nu$ [23]. In the rest frame of the fluid, $u^\mu = (1, 0, 0, 0)$ and $v^\mu = (0, 0, 0, 1)$, the stress-energy tensor becomes diagonal,

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P_T & 0 & 0 \\ 0 & 0 & P_T & 0 \\ 0 & 0 & 0 & P_L \end{pmatrix}. \quad (2.48)$$

In conformal fluids, the stress-energy tensor is traceless, $T^\mu{}_\mu = 0$, and $\epsilon = 2P_T + P_L$.

The thermodynamical identities for an anisotropic fluid can be found [51] by computing $I_0 = u_\nu \partial_\mu T^{\mu\nu} + \mu \partial_\mu j^\mu$ which is zero because of conservation laws. Using $\partial_\mu (s u^\mu) = 0$, we get

$$\begin{aligned} u_\nu \partial_\mu T^{\mu\nu} &= -u_\nu \partial_\mu \epsilon - (\epsilon + P_T) \partial_\mu u^\mu - \Delta u_\nu \partial^\nu \ln \tau \\ &= -u^\mu \partial_\mu \epsilon + \frac{\epsilon + P_T}{s} u^\mu \partial_\mu s - \frac{\Delta}{\tau} u^\mu \partial_\mu \tau, \end{aligned} \quad (2.49)$$

$$\mu \partial_\mu j^\mu = \mu (\partial_\mu \rho) u^\mu - \frac{\mu \rho}{s} u^\mu \partial_\mu s. \quad (2.50)$$

As in [23], we consider a generalized energy density $\epsilon = \epsilon(s, \rho, \tau)$, which depends not only on the entropy density s and particle density ρ but also on the new variable τ . Its differential is

$$d\epsilon = \left(\frac{\partial \epsilon}{\partial s} \right)_{\rho, \tau} ds + \left(\frac{\partial \epsilon}{\partial \rho} \right)_{s, \tau} d\rho + \left(\frac{\partial \epsilon}{\partial \tau} \right)_{s, \rho} d\tau, \quad (2.51)$$

with

$$\left(\frac{\partial \epsilon}{\partial s} \right)_{\rho, \tau} = T, \quad \left(\frac{\partial \epsilon}{\partial \rho} \right)_{s, \tau} = \mu, \quad \left(\frac{\partial \epsilon}{\partial \tau} \right)_{s, \rho} = -\frac{\Delta}{\tau}. \quad (2.52)$$

The temperature and the chemical potential are defined in the usual way. If we also impose $(\partial \epsilon / \partial \tau)_{s, \rho} = -\Delta / \tau$ and substitute (2.51) into (2.49), then $I_0 = 0$ implies the following thermodynamical identities for an anisotropic fluid:

$$\epsilon + P_T = Ts + \mu \rho, \quad (2.53)$$

$$dP_T = \frac{\Delta}{\tau} d\tau + s dT + \rho d\mu, \quad (2.54)$$

$$d\epsilon = T ds + \mu d\rho - \frac{\Delta}{\tau} d\tau, \quad (2.55)$$

in agreement with [23] for $\mu = 0$.

Chapter 3

Fluids with triangle anomalies

The chiral magnetic effect can be derived in several ways, in particular by using holographic models. In their remarkable paper [14] Dam Son and Piotr Surowka modified the hydrodynamics equations in order to take into account quantum triangle anomalies. These additional kinetic transport coefficients in the case of real electromagnetic fields describe the chiral magnetic and chiral vortical effects [16, 20]. We have extended the works [14, 20] to the anisotropic case and rederived the transport coefficients, relevant for the chiral magnetic effect.

This chapter provides a review of triangle anomalies, including the so-called Adler–Bell–Jackiw anomaly, in quantum field theory and in relativistic hydrodynamics. We follow [14] and discuss how the triangle anomalies arise in relativistic hydrodynamics. We then compute the vortical and magnetic conductivities of an anisotropic fluid and show the dependence of the transport coefficients on the elliptic flow v_2 [51].

3.1 Adler-Bell-Jackiw anomaly

We briefly derive the Adler-Bell-Jackiw anomaly, for detailed discussion, see [58] and also [59] for the mathematical aspects of anomalies.

Let ψ be a massless Dirac field interacting with a non-abelian gauge field A_μ . We start with the generating functional for the fermions:

$$Z[\eta, \bar{\eta}] = \int D\psi D\bar{\psi} e^{i \int (\bar{\psi} i \gamma^\mu D_\mu \psi) d^4x}. \quad (3.1)$$

Here $\psi, \bar{\psi}$ are Dirac spinors and $D_\mu = \partial_\mu - igA_\mu$ is a covariant derivative.

Now let us make a chiral transformation of variables

$$\psi(x) \rightarrow e^{i\alpha\gamma^5} \psi, \quad (3.2)$$

$$\bar{\psi}(x) \rightarrow \bar{\psi} e^{i\alpha\gamma^5}, \quad (3.3)$$

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where α is an infinitesimal parameter. Under these transformations the action changes in the following way

$$\int d^4x \bar{\psi}'(i\gamma^\mu D_\mu)\psi' = \int d^4x [\bar{\psi}(i\gamma^\mu D_\mu)\psi - \partial_\mu \alpha(x) \bar{\psi} \gamma^\mu \gamma^5 \psi] \quad (3.4)$$

$$= \int d^4x [\bar{\psi}(i\gamma^\mu D_\mu)\psi + \alpha(x) \partial_\mu (\bar{\psi} \gamma^\mu \gamma^5 \psi)]. \quad (3.5)$$

The path–integral measure for gauge–invariant fermion theory is not invariant under γ^5 transformations. From integration measure comes an extra Jacobian factor which gives rise to the ABJ anomaly

$$Z[\eta, \bar{\eta}] = \int D\psi D\bar{\psi} |J| \exp\{i \int d^4x [\bar{\psi}(i\gamma^\mu D_\mu)\psi + \alpha(x) \partial_\mu (\bar{\psi} \gamma^\mu \gamma^5 \psi)]\}. \quad (3.6)$$

One can calculate the Jacobian and find

$$|J| = \exp \left[-i \int d^4x \alpha(x) \left(\frac{e^2}{32\pi^2} \epsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} F_{\lambda\sigma} \right) \right]. \quad (3.7)$$

If we now replace the Jacobian in the functional integral by the above expression, we get

$$Z[\eta, \bar{\eta}] = \int D\psi D\bar{\psi} e^{i \int d^4x [\bar{\psi}(i\gamma^\mu D_\mu)\psi + \alpha(x) (\partial_\mu j^{\mu 5} + \frac{e^2}{16\pi^2} \epsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} F_{\lambda\sigma})]} \quad (3.8)$$

Varying the exponent with respect to $\alpha(x)$ we find the Adler–Bell–Jackiw anomaly

$$\partial_\mu j^{\mu 5} = -\frac{e^2}{16\pi^2} \epsilon^{\alpha\nu\beta\lambda} F_{\alpha\nu} F_{\beta\lambda}. \quad (3.9)$$

3.2 Hydrodynamics of isotropic fluids with anomalies

The hydrodynamic regime of isotropic relativistic fluids with triangle anomalies we discussed above has been studied in [14, 15, 16, 17, 18, 19], and much can be taken over to the anisotropic case. Such fluids typically contain n anomalous $U(1)$ charges which commute with each other. The anomaly coefficients are given by a totally symmetric rank-3 tensor C^{abc} . The hydrodynamic description exhibits interesting effects when a global symmetry is broken by anomalies of QFT. As explained in the previous section, in presence of the triangle anomalies the hydrodynamic equations are

$$\partial_\mu T^{\mu\nu} = F^{\alpha\nu\lambda} j_\lambda^a, \quad \partial_\mu j^{a\mu} = C^{abc} E^b \cdot B^c, \quad (3.10)$$

where $E^{a\mu} = F^{a\mu\nu} u_\nu$, $B^{a\mu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta}^a$ ($a = 1, \dots, n$) are electric and magnetic fields, and $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a$ denotes the gauge field strengths. This leads to modification of the currents [14, 21]

$$j^{a\mu} = \rho^a u^\mu + \sigma^{ab} \left(E_b^\mu - TP^{\mu\nu} \partial_\nu \frac{\mu_b}{T} \right) + \xi^a \omega^\mu + \xi_B^{ab} B_{b\mu}, \quad (3.11)$$

3.3 Hydrodynamics of anisotropic fluids with triangle anomalies

where $E_a^\mu = F_a^{\mu\nu} u_\nu$, $B^{a\mu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho} u_\nu F_{\lambda\rho}^a$, $\omega^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\lambda\rho} u_\nu \partial_\lambda u_\rho$ and ρ_a , T , μ_a and σ_a^b are the charge densities, temperature, chemical potentials and electrical conductivities of the medium. One can compute the new coefficients ξ_a and ξ_B^{ab} exactly from the requirement that the entropy current has a nonnegative divergence, $\partial_\mu s^\mu \geq 0$ [14, 18, 16, 20]. They read

$$\begin{aligned}\xi^a &= C^{abc} \mu_b \mu_c + 2\beta^a T^2 - \frac{2\rho^a}{\epsilon + p} \left(\frac{1}{3} C^{bcd} \mu_b \mu_c \mu_d + 2\beta^b \mu_b T^2 \right), \\ \xi_B^{ab} &= C^{abc} \mu_c - \frac{\rho^a}{\epsilon + p} \left(\frac{1}{2} C^{bcd} \mu_c \mu_d + \beta^b T^2 \right).\end{aligned}\tag{3.12}$$

The coefficients β_a are presumably related to gravitational anomaly [60]. Remarkably, these coefficients have been discovered in the context of the fluid/gravity duality [14, 18, 21, 20]. In the case of two charges ($n = 2$) the new hydrodynamic terms describe the physical chiral magnetic and chiral vortical effects in heavy ion collisions, as will be explain in section (3.3.3).

Note that the anomaly-induced effects arise also in the superfluids [39, 61, 62, 63] and in the Fermi liquids [64].

3.3 Hydrodynamics of anisotropic fluids with triangle anomalies

We now repeat the computation of Son and Surowka [14] for the case of an anisotropic fluid. Recall that in *anisotropic* relativistic fluids, the hydrodynamic equations are again given by (3.10) but the stress-energy tensor $T^{\mu\nu}$ and $U(1)$ currents $j^{a\mu}$ now have the more general form¹

$$T^{\mu\nu} = (\epsilon + P_T) u^\mu u^\nu + P_T g^{\mu\nu} - \Delta v^\mu v^\nu + \tau^{\mu\nu},\tag{3.13}$$

$$j^{a\mu} = \rho^a u^\mu + \nu^{a\mu}.\tag{3.14}$$

For simplicity, we restrict to the case of a single charge in Sec. 3.3.1, $n = 1$. In Secs. 3.3.2 and 3.3.3 we generalize our findings to arbitrary n and discuss the case $n = 2$, which is relevant for the CME.

¹The symmetries allow in principle for more general currents $j^{a\mu} = \rho^a u^\mu + c^a v^\mu + \nu^{a\mu}$ with some coefficients c^a . Here we switch off all the ‘electric’ background currents, $c^a = 0$.

3. FLUIDS WITH TRIANGLE ANOMALIES

3.3.1 Vortical and magnetic coefficients ($n = 1$)

We now discuss corrections to the $U(1)$ current $j^\mu \equiv j^{1\mu}$ ($n = 1$). In anisotropic fluids the transport coefficients are usually promoted to tensors such that one should consider first-derivative corrections of the type

$$\nu^\mu = (\xi_\omega)^\mu{}_\nu \omega^\nu + (\xi_B)^\mu{}_\nu B^\nu, \quad (3.15)$$

where $\omega^\mu = \frac{1}{2}\epsilon^{\nu\rho\sigma\mu}u_\nu\partial_\rho u_\sigma$ is the vorticity, and B^μ is an external magnetic field. In Landau frame $u_\mu\nu^\mu = 0$ and therefore $u_\mu(\xi_\omega)^\mu{}_\nu\omega^\nu = 0$ (and similar for $(\xi_B)^\mu{}_\nu$). This is satisfied *e.g.* for $(\xi_\omega)^\mu{}_\nu = \xi_\omega\delta^\mu{}_\nu$, since $u_\mu\omega^\mu = 0$ (We do not consider other components of ξ_ω here). We therefore restrict to consider corrections of the type

$$\nu^\mu = \xi_\omega\omega^\mu + \xi_B B^\mu, \quad (3.16)$$

as in the isotropic case [14]. Our goal is to compute the vortical and magnetic conductivities ξ_ω and ξ_B . These transport coefficients can be found by assuming the existence of an entropy current s^μ with a non-negative derivative, $\partial_\mu s^\mu \geq 0$. The computation closely follows that of [14].

The hydrodynamic Eqs. (3.10) imply that the quantity

$$I_1 = u_\nu\partial_\mu T^{\mu\nu} + \mu\partial_\mu j^\mu + E^\mu\nu_\mu - \mu C E^\mu B_\mu \quad (3.17)$$

vanishes at first order, $I_1 = 0$. Substituting the explicit expressions for the stress-energy tensor and $U(1)$ currents into I_1 and using the thermodynamical identities (2.53) and (2.55), we find

$$\begin{aligned} \partial_\mu \left(s u^\mu - \frac{\mu}{T} \nu^\mu \right) &= -\frac{1}{T} \partial_\mu u_\nu \tau^{\mu\nu} - \nu^\mu \left(\partial_\mu \frac{\mu}{T} - \frac{E_\mu}{T} \right) \\ &\quad - C \frac{\mu}{T} E \cdot B, \end{aligned} \quad (3.18)$$

which is exactly the same equation for the entropy production as in the isotropic case [14].

In the following, we will need the identities

$$\partial_\mu \omega^\mu = -\frac{2}{\epsilon + P_T} \omega^\mu (\partial_\mu P_T - \Delta \partial_\mu \ln \tau - \rho E_\mu), \quad (3.19)$$

$$\partial_\mu B^\mu = -2\omega^\mu E_\mu - \frac{B^\mu}{\epsilon + P_T} (\partial_\mu P_T - \Delta \partial_\mu \ln \tau - \rho E_\mu). \quad (3.20)$$

Let us derive them. To find an explicit expression for $\partial_\mu \omega^\mu$, we compute the term $\omega_\nu \partial_\mu T^{\mu\nu}$ in two ways. First, using the hydrodynamic equations, we get

$$\omega_\nu \partial_\mu T^{\mu\nu} = \omega_\nu F^{\nu\mu} j_\mu = \rho \omega_\nu F^{\nu\mu} u_\mu = \rho \omega_\nu E^\nu. \quad (3.21)$$

3.3 Hydrodynamics of anisotropic fluids with triangle anomalies

Next, substituting the stress-energy tensor (3.13) in this expression, we find

$$\begin{aligned}
\omega_\nu \partial_\mu T^{\mu\nu} &= (\epsilon + P_T) u^\mu \omega_\nu \partial_\mu u^\nu + \omega_\nu g^{\mu\nu} \partial_\mu P_T - \Delta \omega_\nu v^\mu \partial_\mu v^\nu \\
&\quad - v^\nu \omega_\nu v^\mu \partial_\mu \Delta - \Delta v^\nu \omega_\nu \partial_\mu v^\mu \\
&= -(\epsilon + P_T) u^\mu u^\nu \partial_\mu \omega_\nu + \omega^\mu \partial_\mu P_T - \Delta \omega_\nu \partial^\nu \ln \tau \\
&\quad - v^\nu \omega_\nu v^\mu \partial_\mu \Delta - \Delta v^\nu \omega_\nu \partial_\mu v^\mu.
\end{aligned} \tag{3.22}$$

Using the identity

$$u^\mu u^\lambda \partial_\mu \omega_\lambda = -\frac{1}{2} \partial_\mu \omega^\mu, \tag{3.23}$$

we find

$$\begin{aligned}
\partial_\mu \omega^\mu &= -\frac{2}{\epsilon + P_T} \omega^\mu (\partial_\mu P_T - \Delta \partial_\mu \ln \tau - \rho E_\mu \\
&\quad - v_\mu v^\nu \partial_\nu \Delta - \Delta v_\mu \partial_\nu v^\nu).
\end{aligned} \tag{3.24}$$

Similar manipulations of the term $B_\nu \partial_\mu T^{\mu\nu}$ lead to

$$\begin{aligned}
B_\nu \partial_\mu T^{\mu\nu} &= B_\nu F^{\nu\mu} j_\mu = \rho B_\mu E^\mu, \\
B_\nu \partial_\mu T^{\mu\nu} &= -(\epsilon + P_T) u^\mu u^\nu \partial_\mu B_\nu + B^\mu \partial_\mu P_T \\
&\quad - \Delta B_\nu v^\mu \partial_\mu v^\nu - B_\nu v^\nu v^\mu \partial_\mu \Delta - \Delta B_\nu v^\nu \partial_\mu v^\mu \\
&= -(\epsilon + P_T) (\partial_\mu B^\mu - 2\omega^\mu E_\mu) - \Delta B_\mu \partial^\mu \ln \tau \\
&\quad - B_\nu v^\nu v^\mu \partial_\mu \Delta - \Delta B_\nu v^\nu \partial_\mu v^\mu,
\end{aligned} \tag{3.26}$$

where we used

$$u^\mu u^\lambda \partial_\mu B_\lambda = \partial_\mu B^\mu + 2\omega^\rho E_\rho. \tag{3.27}$$

From (3.25) and (3.26) we obtain the following expression:

$$\begin{aligned}
\partial_\mu B^\mu &= -2\omega^\mu E_\mu - \frac{B^\mu}{\epsilon + P_T} (\partial_\mu P_T - \Delta \partial_\mu \ln \tau - \rho E_\mu \\
&\quad - v_\mu v^\nu \partial_\nu \Delta - \Delta v_\mu \partial_\nu v^\nu).
\end{aligned} \tag{3.28}$$

To simplify the computation we assume that the fluid satisfies

$$\partial_\mu v^\mu = 0, \quad v^\mu \partial_\mu \Delta = 0. \tag{3.29}$$

The first equation is basically a ‘‘continuity equation’’ for the vector v^μ . There are no sources for the generation of anisotropy. The second equation imposes an orthogonality relation between the gradient of the pressure difference $\Delta = P_T - P_L$ and v^μ . Substituting (3.29) into (3.24) and (3.28) we get (3.19) and (3.20), respectively.

3. FLUIDS WITH TRIANGLE ANOMALIES

Now we have all the ingredients to complete our computation. As in [14], we assume a generalized entropy current of the form

$$s^\mu = su^\mu - \frac{\mu}{T} \nu^\mu + D\omega^\mu + D_B B^\mu, \quad (3.30)$$

where ξ_ω , ξ_B , D , and D_B are functions of T , μ and τ . We now compute $\partial_\mu s^\mu$, using (3.18) and (3.19) and impose $\partial_\mu s^\mu \geq 0$. Since the coefficients in front of ω^μ , B^μ , $\omega_\mu E^\mu$ and $E_\mu B^\mu$ inside $\partial_\mu s^\mu$ can have either sign, we require them to vanish and obtain the following four differential equations:

$$\partial_\mu D - \frac{2D}{\epsilon + P_T} (\partial_\mu P_T - \Delta \partial_\mu \ln \tau) - \xi_\omega \partial_\mu \frac{\mu}{T} = 0, \quad (3.31)$$

$$\partial_\mu D_B - \frac{D_B}{\epsilon + P_T} (\partial_\mu P_T - \Delta \partial_\mu \ln \tau) - \xi_B \partial_\mu \frac{\mu}{T} = 0, \quad (3.32)$$

$$\frac{2\rho D}{\epsilon + P_T} - 2D_B + \frac{\xi_\omega}{T} = 0, \quad (3.33)$$

$$\frac{\rho D_B}{\epsilon + P_T} + \frac{\xi_B}{T} - C \frac{\mu}{T} = 0. \quad (3.34)$$

For $\Delta = 0$, these equations reduce to those in the isotropic case [14].

Let us solve (3.31)–(3.34) for D , D_B , ξ_ω and ξ_B . Following [14], we change variables from $\ln \tau$, μ , T to $\ln \tau$, $\bar{\mu} = \mu/T$ and P_T . From (2.53) and (2.54), we derive the thermodynamic expressions

$$\left(\frac{\partial \bar{\mu}}{\partial T} \right)_{P_T, \ln \tau} = -\frac{\epsilon + P_T}{\rho T^2}, \quad (3.35)$$

$$\left(\frac{\partial P_T}{\partial T} \right)_{\bar{\mu}, \ln \tau} = \frac{\epsilon + P_T}{T}, \quad (3.36)$$

$$\left(\frac{\partial \ln \tau}{\partial T} \right)_{\bar{\mu}, P_T} = -\frac{1}{\Delta} \frac{\epsilon + P_T}{T}. \quad (3.37)$$

Using

$$\partial_\mu D = \frac{\partial D}{\partial P_T} \partial_\mu P_T + \frac{\partial D}{\partial \bar{\mu}} \partial_\mu \bar{\mu} + \frac{\partial D}{\partial \ln \tau} \partial_\mu \ln \tau, \quad (3.38)$$

$$\partial_\mu D_B = \frac{\partial D_B}{\partial P_T} \partial_\mu P_T + \frac{\partial D_B}{\partial \bar{\mu}} \partial_\mu \bar{\mu} + \frac{\partial D_B}{\partial \ln \tau} \partial_\mu \ln \tau, \quad (3.39)$$

the first two equations, (3.31) and (3.32), can be rewritten as

$$-\xi_\omega + \frac{\partial D}{\partial \bar{\mu}} = 0, \quad -\xi_B + \frac{\partial D_B}{\partial \bar{\mu}} = 0, \quad (3.40)$$

$$\frac{\partial D}{\partial P_T} - \frac{2D}{\epsilon + P_T} = 0, \quad \frac{\partial D_B}{\partial P_T} - \frac{D_B}{\epsilon + P_T} = 0, \quad (3.41)$$

$$\frac{\partial D}{\partial \ln \tau} + \frac{2\Delta D}{\epsilon + P_T} = 0, \quad \frac{\partial D_B}{\partial \ln \tau} + \frac{\Delta D_B}{\epsilon + P_T} = 0. \quad (3.42)$$

3.3 Hydrodynamics of anisotropic fluids with triangle anomalies

Note that (3.41) and (3.42) are related by the thermodynamic identities (3.36) and (3.37). Using the ansatz

$$D = T^2 d(\bar{\mu}, \ln \tau), \quad D_B = T d_B(\bar{\mu}, \ln \tau), \quad (3.43)$$

and (3.35), we obtain two differential equations from (3.33) and (3.34),

$$\begin{aligned} 0 &= \frac{2\rho D}{\epsilon + P_T} - 2D_B + \frac{\xi_\omega}{T} \\ &= T (\partial_{\bar{\mu}} d(\bar{\mu}, \ln \tau) - 2d_B(\bar{\mu}, \ln \tau)), \end{aligned} \quad (3.44)$$

$$\begin{aligned} 0 &= \frac{\rho D_B}{\epsilon + P_T} + \frac{\xi_B}{T} - C\bar{\mu} \\ &= \partial_{\bar{\mu}} d_B(\bar{\mu}, \ln \tau) - C\bar{\mu}. \end{aligned} \quad (3.45)$$

These equations can be integrated to give

$$d_B(\bar{\mu}, \ln \tau) = \frac{1}{2} C \bar{\mu}^2 + \beta(\ln \tau), \quad (3.46)$$

$$d(\bar{\mu}, \ln \tau) = \frac{1}{3} C \bar{\mu}^3 + 2\bar{\mu}\beta(\ln \tau) + \gamma(\ln \tau), \quad (3.47)$$

where $\beta(\ln \tau)$ and $\gamma(\ln \tau)$ are arbitrary functions of $\ln \tau$. Substituting this back into (3.33), (3.34), we obtain

$$\begin{aligned} \xi_\omega &= C \left(\mu^2 - \frac{2}{3} \frac{\rho \mu^3}{\epsilon + P_T} \right) + \mathcal{O}(T^2), \\ \xi_B &= C \left(\mu - \frac{1}{2} \frac{\rho \mu^2}{\epsilon + P_T} \right) + \mathcal{O}(T^2), \end{aligned} \quad (3.48)$$

where $\mathcal{O}(T^2)$ denotes terms proportional to T^2 . These terms are presumably related to gravitational triangle anomalies [18, 60] and may, in the anisotropic case, depend on the proper time τ . In the absence of gravitational anomalies, which we do not discuss in this thesis, the conductivities do not depend on τ . Apart from these changes in $\mathcal{O}(T^2)$, the relations have the same form as in the isotropic case but with P replaced by the transverse pressure P_T .

3.3.2 Multiple charge case (arbitrary n)

The generalization of the previous computation to a fluid with multiple anomalous $U(1)$ charges is straightforward, and we only state the result here. The corrections $\nu^{a\mu}$ of the currents $j^{a\mu}$ in (3.14) are

$$\nu^{a\mu} = \xi_\omega^a \omega^\mu + \xi_B^{ab} B^{b\mu}, \quad (3.49)$$

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with [terms of order $\mathcal{O}(T^2)$ ignored]

$$\xi_\omega^a = C^{abc} \mu^b \mu^c - \frac{2}{3} \rho^a C^{bcd} \frac{\mu^b \mu^c \mu^d}{\epsilon + P_T}, \quad (3.50)$$

$$\xi_B^{ab} = C^{abc} \mu^c - \frac{1}{2} \rho^a C^{bcd} \frac{\mu^c \mu^d}{\epsilon + P_T}. \quad (3.51)$$

These are simple generalizations of the corresponding conductivities in the isotropic case [14, 18]. If $P = P_T = P_L$, then (3.51) reduces to (3.12)

3.3.3 Chiral magnetic and vortical effect ($n = 2$)

Physically, the most interesting case is that involving two charges ($n = 2$) [16, 17, 20]. The chiral magnetic effect [4] can be described by one axial and one vector $U(1)$, denoted by $U(1)_A \times U(1)_V$. A convenient notation for the gauge fields and currents is ($a, b, \dots = 1, 2$)

$$\begin{aligned} A_\mu^A &= A_\mu^1, & A_\mu^V &= A_\mu^2, \\ j_5^\mu &= j^{1\mu}, & j^\mu &= j^{2\mu}. \end{aligned} \quad (3.52)$$

Let us now derive the chiral magnetic and vortical effects from (3.50) and (3.51). C -parity allows for two anomalous triangle diagrams, (AAA) and (AVV), shown in Fig. 3.1, while diagrams of the type (VVV) and (VAA) vanish. Accordingly, the anomaly coefficients are

$$\begin{aligned} C^{121} &= C^{211} = C^{112} = 0, & (VAA) \\ C^{222} &= 0, & (VVV) \\ C^{111} &\neq 0, & (AAA) \\ C^{122} &= C^{221} = C^{212} \neq 0. & (AVV) \end{aligned} \quad (3.53)$$

The hydrodynamic Eqs. (3.10) then imply nonconserved vector and axial currents

$$\begin{aligned} \partial_\mu j^\mu &= -\frac{1}{4} (C^{212} F_{\mu\nu}^A \tilde{F}^{V\mu\nu} + C^{221} F_{\mu\nu}^V \tilde{F}^{A\mu\nu}), \\ \partial_\mu j_5^\mu &= -\frac{1}{4} (C^{111} F_{\mu\nu}^A \tilde{F}^{A\mu\nu} + C^{122} F_{\mu\nu}^V \tilde{F}^{V\mu\nu}), \end{aligned} \quad (3.54)$$

where we rewrote $E^b \cdot B^c = -\frac{1}{4} F_{\mu\nu}^b \tilde{F}^{c\mu\nu}$ (with $\tilde{F}^{a\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}^a$).

To restore conservation of the vector current, we add the (topological) Bardeen term to the gauge theory,

$$S_B = c_B \int d^4x \epsilon^{\mu\nu\lambda\rho} A_\mu^A A_\nu^V F_{\lambda\rho}^V. \quad (3.55)$$

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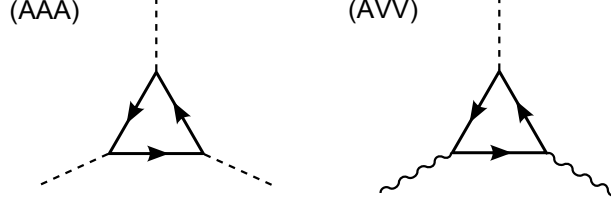


Figure 3.1: Anomalous diagrams corresponding to C^{111} (left) and to $C^{122} = C^{221} = C^{212}$ (right). Dashed (wavy) lines denote the axial (vector) currents/fields.

Combining the corresponding Bardeen currents

$$\begin{aligned} j_B^\mu &= c_B \varepsilon^{\mu\nu\lambda\rho} (A_\nu^V F_{\lambda\rho}^A - 2A_\nu^A F_{\lambda\rho}^V), \\ j_{5,B}^\mu &= c_B \varepsilon^{\mu\nu\lambda\rho} A_\nu^V F_{\lambda\rho}^V, \end{aligned} \quad (3.56)$$

with the vector and axial currents,

$$j'^\mu \equiv j^\mu + j_B^\mu, \quad j_5'^\mu \equiv j_5^\mu + j_{5,B}^\mu, \quad (3.57)$$

we obtain the anomaly equations

$$\begin{aligned} \partial_\mu j'^\mu &= - \left(\frac{C^{122}}{2} + c_B \right) F_{\alpha\beta}^V \tilde{F}^{A\alpha\beta}, \\ \partial_\mu j_5'^\mu &= - \frac{C^{111}}{4} F_{\alpha\beta}^A \tilde{F}^{A\alpha\beta} - \left(\frac{C^{122}}{4} - c_B \right) F_{\alpha\beta}^V \tilde{F}^{V\alpha\beta}. \end{aligned} \quad (3.58)$$

The electric current j'^μ is conserved if $c_B = -C^{122}/2$. Setting $C^{111} = C^{122} \equiv C/3$, the hydrodynamic Eqs. (3.10) become

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= F^{V\nu\lambda} j'_\lambda + F^{A\nu\lambda} j_{5,\lambda}', \\ \partial_\mu j'^\mu &= 0, \\ \partial_\mu j_5'^\mu &= CE \cdot B + (C/3) E_5 \cdot B_5. \end{aligned} \quad (3.59)$$

Using the derivative expansion

$$j'^\mu = \rho u^\mu + \kappa_\omega \omega^\mu + \kappa_B B^\mu + \kappa_{5,B} B_5^\mu, \quad (3.60)$$

where $\kappa_\omega \equiv \xi_\omega^2$, $\kappa_B \equiv \xi_B^2$ and $\kappa_{5,B} \equiv \xi_B^{21}$, we obtain from (3.50) and (3.51) the conductivities ($\mu_5 \equiv \mu^1$, $\mu \equiv \mu^2$)

$$\begin{aligned} \kappa_\omega &= 2C\mu_5 \left(\mu - \frac{\rho}{\epsilon + P_T} \left[\mu^2 + \frac{\mu_5^2}{3} \right] \right), \\ \kappa_B &= C\mu_5 \left(1 - \frac{\mu\rho}{\epsilon + P_T} \right), \\ \kappa_{5,B} &= C\mu \left(1 - \frac{1}{2} \frac{\mu\rho}{\epsilon + P_T} \left[1 + \frac{\mu_5^2}{3\mu^2} \right] \right). \end{aligned} \quad (3.61)$$

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There are analogous transport coefficients in the axial current j_5^μ [20]. The axial fields $E_{5\mu}$ and $B_{5\mu}$ are not needed and can now be switched off. The first term in κ_B and κ_ω , $\kappa_B = C\mu_5$ and $\kappa_\omega = 2C\mu\mu_5$, is the leading term in the *chiral magnetic* (CME) [4, 5] and *chiral vortical effect* [65], respectively.¹ They are in agreement with those found in the isotropic case [16, 17, 20]. The second term proportional to $\rho/(\epsilon + P_T)$ actually depends on the dynamics of the fluid² and therefore on ε_p .

Recall that our goal in this section is to show how the transport coefficients depend on the elliptic flow v_2 . The dependence of κ_B on ε_p can be made more visible by introducing an average pressure $\bar{P} = (2P_T + P_L)/3$ such that $\epsilon = 3\bar{P}$. Assuming ε_p to be small (see Fig. 1.2), we expand the CME-coefficient κ_B to linear order in ε_p ,

$$\kappa_B \approx C\mu_5 \left(1 - \frac{\mu\rho}{\epsilon + \bar{P}} \left[1 - \frac{\varepsilon_p}{6} \right] \right). \quad (3.62)$$

At freeze-out the elliptic flow coefficient $v_2 \approx \varepsilon_p/2$ [27]. Now we see that for small momentum anisotropies, the CME thus increases linearly in v_2 .

One should keep in mind that we consider an oversimplified case of a fluid filling the entire space³. Nevertheless, the result (3.62) is the first and hence valuable example of anisotropic corrections to CME.

¹ $\kappa_{5,B}$ represents another effect, which we added for completeness, but it seems to be not realized in heavy-ion collisions.

²In [15] this term was considered as a one-loop correction in an effective theory and $(\epsilon + P)/\rho$ was interpreted as the corresponding infrared cutoff in the energy/momentum integration.

³We ignore the finiteness of the fireball, as well as possible pressure inhomogeneities and details of the initial conditions.

Chapter 4

Holographic duality

To understand the rest of the thesis we take a birds eye view to explain the *AdS/CFT* correspondence [66, 67, 68]. Further information can be found in the following reviews and lectures [69, 70, 71, 72].

The *AdS/CFT* correspondence is a great tool to study strongly coupled gauge theories [73, 74]. There is one important caveat that this duality was found for QCD-like theories, not for real QCD. Nevertheless there is a possible connection between gravity/gauge duality and QCD [75]. In order to achieve a description of QCD-like theories it is necessary to deform and modify the standard *AdS/CFT* correspondence. There are two ways to do this: top-down models and bottom-up models. The former corresponds to the models derived directly from the string theory constructions, while the latter refers to giving a gravity dual by hand. We do not discuss here¹ these two models because we will not use them, except the fact that they exist and our gravity construction in Chapter 5 is a bottom-up model.

4.1 Basics of AdS/CFT Correspondence

The gauge/gravity duality is initially formulated for maximally supersymmetric four-dimensional conformal $\mathcal{N}=4$ gauge theory. The $\mathcal{N} = 4$ Super Yang–Mills theory contains a vector field, six real scalars and four fermions. All fields are in the adjoint representation of $SU(N)$. The action of the theory can be schematically written as

$$S_{\mathcal{N}=4} = \frac{1}{g_{YM}^2} \int d^4x \text{Tr} [F_{\mu\nu}^2 + (D\Phi_i)^2 + [\Phi_i, \Phi_j]^2] + \text{fermions} \quad (4.1)$$

It has a vanishing beta function and is a conformal field theory. The action of the theory is uniquely determined by the coupling constant g_{YM} and the rank of the gauge

¹We refer to [76] [77] for details.

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group $SU(N)$. When the number of colors is large, i.e. $N \rightarrow \infty$ it is the 't Hooft coupling $\lambda = g_{YM}^2 N$ that controls the perturbative expansion.

On the string theory side, we have type IIB string theory, which contains a finite number of massless fields, including the graviton, the dilaton Φ , some other fields (forms) and their fermionic superpartners, and an infinite number of massive string excitations. On the string theory side, the parameters are g_s , l_s , and radius R of the AdS space. The AdS/CFT parameters are related through the following relations:

$$g^2 = 4\pi g_s, \quad (4.2)$$

$$g^2 N_c = \frac{R^4}{l_s^4}. \quad (4.3)$$

Equations (4.2) and (4.3) tell us that, at $\lambda \ll 1$ the gauge theory is perturbatively calculable, while the dual string theory is defined in $AdS_5 \times S^5$ with $R \ll l_s$. In the long-wavelength limit, when all fields vary over length scales much larger than l_s , the massive modes decouple and string theory is well approximated by supergravity, which can be described by an action [78]

$$S_{\text{SUGRA}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} e^{-2\Phi} \left(\mathcal{R} + 4 \partial^\mu \Phi \partial_\mu \Phi \right) \quad (4.4)$$

$$+ \text{contributions from the other fields}, \quad (4.5)$$

where κ_{10} is the 10-dimensional gravitational constant,

$$\kappa_{10} = \sqrt{8\pi G} = 8\pi^{7/2} g_s l_s^4, \quad (4.6)$$

and \mathcal{R} is the curvature scalar. The $AdS_5 \times S^5$ solution is given by the metric

$$ds^2 = \frac{r^2}{R^2} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + R^2 \frac{dr^2}{r^2} + R^2 d\Omega_5^2 \quad (4.7)$$

where $d\Omega_5^2$ is the metric of S^5 .

4.2 Black hole solutions

The *AdS/CFT* correspondence says that a thermal state of the gauge theory in the regime when $\lambda \rightarrow \infty$ is the same as those of the black hole [79]. We begin with a brief review of the black hole solutions in general relativity and proceed with discussion of the black hole in *AdS* background and a charged black hole¹.

¹In particular we have followed [80] and [81] in this section.

One of the simplest black hole solutions is the Schwarzschild black hole in four dimensions [80]. It is a spherically symmetric solution of the free Einstein-Hilbert equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi GT_{\mu\nu} \equiv 0 \quad (4.8)$$

with asymptotically flat boundary conditions. It corresponds to the line element:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2d\Omega_2^2, \quad (4.9)$$

where

$$d\Omega_2^2 = d\theta^2 + \sin^2(\theta)d\phi^2, \quad (4.10)$$

and M is the black hole mass measured by asymptotics of the energy-momentum tensor at infinity.

The Schwarzschild solution has a coordinate singularity at $r = r_0 = 2GM$. This singularity can be avoided by a coordinate change, because the curvature is finite at $r = r_0$. Also there is a curvature singularity at $r = 0$, which cannot be avoided by any coordinate transformation.

While pure empty AdS is a ground state for gravity, finite-temperature states correspond to black holes inside AdS . The simplest asymptotically- AdS black hole is the AdS -Schwarzschild black brane,

$$ds^2 = \frac{R^2}{z^2} \left(-f(z)dt^2 + d\vec{x}^2 + \frac{1}{f(z)}dr^2 \right), \quad (4.11)$$

with factor

$$f(z) = 1 - \frac{z^d}{z_H^d}. \quad (4.12)$$

Near the asymptotic boundary at $z \rightarrow 0$, we have $f \rightarrow 1$, so this metric is asymptotically AdS_{d+1} . At $z = z_H$, however, $f \rightarrow 0$, signaling the presence of a black hole horizon. This horizon in turn shields us from a physical singularity at $z \rightarrow \infty$ by ensuring that nothing which is inside the horizon, and thus sensitive to the singularity, can ever escape to influence events in the rest of the spacetime. Since the entire solution is translationally invariant in the $(d-1)$ spatial directions, \vec{x} , this black brane is not a compact object, but rather extended in all directions other than z .

A black hole in AdS as a classical black hole is a thermodynamic object with a definite temperature, energy and entropy, as shown by Bekenstein and Hawking (see [82] and references therein). The Hawking temperature T , energy density ϵ and entropy

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density s of the black hole (4.11) are calculated in terms of the inverse of the period in the corresponding Euclidean solution

$$T = \frac{d}{4\pi z_H}, \quad (4.13)$$

$$\epsilon = \frac{d-1}{16\pi z_H^d} \left(\frac{R}{\ell_p}\right)^{d-1}, \quad (4.14)$$

$$s = \frac{1}{4z_H^{d-1}} \left(\frac{R}{\ell_p}\right)^{d-1}, \quad (4.15)$$

where ℓ_p is the Planck length and $\frac{R}{\ell_p}$ measures the *AdS*-scale in Planck units.

Now let us consider the charged (Reissner–Nordstrøm) *AdS* black hole which plays a significant role in the applied holography. The Reissner–Nordstrøm black hole is the most general solution of the Einstein–Maxwell action

$$I_{ME} = \frac{1}{16\pi G_N} \int d^d x \sqrt{-g} \left(-2\Lambda + \mathcal{R} - \frac{R^2}{4e^2} F^2 \right), \quad (4.16)$$

The action admits the following charged black hole solutions for the bulk metric

$$ds^2 = R^2 \frac{-f(z)dt^2 + d\vec{x}^2 + \frac{1}{f(z)}dz^2}{z^2}, \quad (4.17)$$

$$A = A_t(z)dt, \quad (4.18)$$

with

$$f = 1 - M z^d + Q^2 z^{2(d-1)}, \quad (4.19)$$

and electromagnetic scalar potential

$$A_t(z) = \mu \left(1 - \left(\frac{z}{z_H} \right)^{d-2} \right), \quad (4.20)$$

where $\mu = \frac{2Q}{\mathcal{C}} z_H^{d-2}$ and $\mathcal{C} = \sqrt{\frac{2(d-2)}{d-1}}$.

In this geometry, the horizon lies at the radial position $z = z_H$ implicitly defined as the value of z where $f(z)$ vanishes. M and Q then determine the Hawking temperature of the horizon,

$$T = \frac{d}{4\pi r_H} \left(1 - \frac{d-2}{d} Q^2 z_H^{2d-2} \right), \quad (4.21)$$

as well as its energy, entropy and charge densities,

$$\epsilon = M \frac{d-1}{16\pi} \left(\frac{R}{\ell_p} \right)^{d-1}, \quad (4.22)$$

$$s = \frac{1}{4z_H^{d-1}} \left(\frac{R}{\ell_p} \right)^{d-1}, \quad (4.23)$$

$$\rho = Q \frac{d-1}{8\pi\mathcal{C}} \left(\frac{R}{\ell_p} \right)^{d-1}. \quad (4.24)$$

One can show that these variables satisfy the first law of thermodynamics (see (2.21)), i.e.

$$d\epsilon = T ds + \mu d\rho. \quad (4.25)$$

4.3 Fluid–Gravity duality

The fluid/gravity duality is a map between black brane solutions of Einstein equations with a negative cosmological constant and conformal fluid flows in one lower dimension¹ [84, 85, 86, 87]. A useful starting point is a so-called holographic renormalization which links the boundary energy–momentum tensor to the behavior of the bulk metric near the *AdS* boundary.

Let us consider a general background involving an asymptotic *AdS* metric in Fefferman–Graham coordinates,

$$ds^2 = \frac{g_{\mu\nu} dx^\mu dx^\nu + dz^2}{z^2}. \quad (4.26)$$

Following [88] we look for solutions of the vacuum Einstein equations with negative cosmological constant $\Lambda = -6$ and the large z –expansion of $g_{\mu\nu}(z, x)$

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + g_{\mu\nu}^{(2)} z^2 + g_{\mu\nu}^{(4)} z^4 + \dots, \quad (4.27)$$

where $g_{\mu\nu}^{(0)}$ is the 4-dimensional metric for the gauge theory on the boundary, $g_{\mu\nu}^{(2)}$ is equal to zero and $g_{\mu\nu}^{(4)}$ is proportional to the VEV of the energy–momentum tensor² [88, 89]:

$$T_{\mu\nu} = \frac{1}{4\pi G_5} g_{\mu\nu}^{(4)} \quad (4.28)$$

Now let us see how the fluid/gravity prescription works in practice. We give here only a sketch of the fluid/gravity calculation which we will do in the next chapter. The basic idea is that for a given black-brane solution with certain parameters such

¹Note that this map originally motivated by string theory becomes independent of it. The gravity side is nothing but solutions of Einstein’s equations [46, 83].

²One can consider also a gauge field A , and find the boundary current in same way $j_a^\mu = \frac{\eta^{\mu\nu} A_{a\nu}^{(2)}}{8\pi G_5} + \hat{j}_a^\mu$.

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as temperature, charges etc, one simply considers those parameters as slowly-varying. Say we know the black brane solution with multiple charges

$$ds^2 = -A(r)dt^2 + 2B(r)dtdr + C(r)(dx^i)^2 \quad , \quad (4.29)$$

$$A^I = D(m, q_I, r)u_\mu dx^\mu, \quad (4.30)$$

where for simplicity of explanation we write $A(r)$, $B(r)$, $C(r)$ and $D(m, q_I, r)$ as arbitrary functions and ($I = 1, \dots, n$).

Now we slowly vary parameters u_μ , m , and q_I up to first order in derivatives,

$$u_\mu = (-1, x^\mu \partial_\mu u_i), \quad (4.31)$$

$$m = m^{(0)} + x^\mu \partial_\mu m, \quad (4.32)$$

$$q_I = q_I^{(0)} + x^\mu \partial_\mu q_I. \quad (4.33)$$

Of course, if we now replace u_μ , m and q_I in the metric by (4.31)-(4.33) the above charged black brane solution will no longer be a solution of the Einstein–Maxwell equations. To be a solution, we have to add corrections $g_{MN}^{(1)}$ and $A_M^{I(1)}$ to the zero'th order solution with varying parameters, which should be chosen to satisfy the equations of motion. Then the metric up to the first order in derivatives including the correction $g_{MN}^{(1)}$ looks as¹

$$\begin{aligned} ds^2 = & -A(r)dt^2 + 2B(r)dtdr + C(r)(dx^i)^2 \\ & + \left[-x^\mu (\partial_\mu A) + g_{tt}^{(1)}(r) \right] dt^2 + 2 \left[x^\mu (\partial_\mu B) + g_{tr}^{(1)}(r) \right] dtdr \\ & + 2 \left[-x^\mu (\partial_\mu u_i) B(r) \right] dr dx^i + 2 \left[x^\mu (\partial_\mu u_i) (A(r) - C(r)) + g_{ti}^{(1)}(r) \right] dt dx^i \\ & + \left[x^\mu (\partial_\mu C) \delta_{ij} + g_{ij}^{(1)}(r) \right] dx^i dx^j \quad , \end{aligned} \quad (4.34)$$

and the gauge fields become

$$\begin{aligned} A^I = & -D(m, q_I, r)dt \\ & + \left[-x^\mu \partial_\mu D(m, q_I, r) + A_t^{I(1)}(r) \right] dt \\ & + \left[x^\mu (\partial_\mu u_i) D(m, q_I, r) + A_i^{I(1)}(r) \right] dx^i, \end{aligned} \quad (4.35)$$

The task is to insert the above into the original equations of motion to obtain the equations for the first order corrections $g_{MN}^{(1)}$, $A_M^{I(1)}$ and then to solve them. After obtaining the full solution at first order in derivatives, one can read off physical quantities at that order via AdS/CFT dictionary, such as energy-momentum tensor and charge currents². In principle one can go to an arbitrary order in derivative expansion systematically.

¹Following [90], we choose a gauge, such that $g_{rr}^{(1)} = 0$, $g_{r\mu}^{(1)} \sim u_\mu$, $A_r^{I(1)} = 0$ and $\sum_{i=1}^3 g_{ii}^{(1)} = 0$.

²We will discuss this in detail in Chapter 5.

Chapter 5

Gravity model for an anisotropic hydrodynamics

In this chapter we construct the dual gravity background and present a numerical solution for its gauge field and metric functions. The holographic computation of the transport coefficients is very similar to that in the isotropic case.

5.1 Fluid-gravity model

We construct the gravity dual of a static anisotropic plasma with diagonal stress-energy momentum $T_{\mu\nu} = \text{diag}(\epsilon, P_T, P_T, P_L)$ and charge densities ρ^a .

We start from a five-dimensional $U(1)^n$ Einstein-Maxwell theory in an asymptotic AdS space. The action is

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left[R - 2\Lambda - F_{MN}^a F^{aMN} + \frac{S_{abc}}{6\sqrt{-g}} \varepsilon^{PKLMN} A_P^a F_{KL}^b F_{MN}^c \right], \quad (5.1)$$

where $\Lambda = -6$ is the cosmological constant. As usual, the $U(1)$ field strengths are defined by

$$F_{MN}^a = \partial_M A_N^a - \partial_N A_M^a, \quad (5.2)$$

where $M, N, \dots = 0, \dots, 4$ and $a = 1, \dots, n$. The Chern-Simons term $A \wedge F \wedge F$ encodes the information of the triangle anomalies in the field theory [14]. In fact, the Chern-Simons coefficients S_{abc} are related to the anomaly coefficients C_{abc} by

$$C_{abc} = S_{abc}/(4\pi G_5). \quad (5.3)$$

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The corresponding equations of motion are given by the combined system of Einstein–Maxwell equations,

$$G_{MN} - 6g_{MN} = T_{MN}, \quad (5.4)$$

$$\nabla_M F^{aMP} = -\frac{S_{abc}}{8\sqrt{-g}} \varepsilon^{PMNKL} F_{MN}^b F_{KL}^c, \quad (5.5)$$

where the energy-momentum tensor T_{MN} is

$$T_{MN} = -2 \left(F_{MR}^a F_N^{aR} + \frac{1}{4} g_{MN} F_{SR}^a F^{aSR} \right). \quad (5.6)$$

5.1.1 AdS black hole with multiple U(1) charges

A gravity dual to an *isotropic* fluid ($\epsilon = 3P$) with multiple chemical potentials μ_a ($a = 1, \dots, n$) at finite temperature T is given by an AdS black hole solution with mass m and multiple U(1) charges q^a . In Eddington-Finkelstein coordinates, the metric and U(1) gauge fields of this solution are

$$\begin{aligned} ds^2 &= -f(r)dt^2 + 2drdt + r^2 d\vec{x}^2, \\ A^a &= -A_0^a(r)dt, \end{aligned} \quad (5.7)$$

where

$$\begin{aligned} f(r) &= r^2 - \frac{m}{r^2} + \sum_a \frac{(q^a)^2}{r^4}, \\ A_0^a(r) &= \mu_\infty^a + \frac{\sqrt{3}q^a}{2r^2}. \end{aligned} \quad (5.8)$$

The constants μ_∞^a can be fixed such that the gauge fields vanish at the horizon. In case of a single charge ($n = 1$), the background reduces to an ordinary Reissner-Nordström black hole solution in AdS_5 [91].

The temperature T and chemical potentials μ^a of the fluid are defined by

$$T = \frac{\kappa}{2\pi} = \frac{f'(r_+)}{4\pi} = \frac{2r_+^6 - \sum_a (q_a)^2}{2\pi r_+^5}, \quad (5.9)$$

$$\mu^a = A_0^a(r_+) - A_0^a(r_\infty), \quad (5.10)$$

where r_+ is the outer horizon defined by the maximal solution of $f(r) = 0$, and r_∞ indicates the location of the boundary. The temperature of the fluid is the Hawking temperature of the black hole and is computed from the surface gravity $\kappa = \sqrt{\partial_M |\chi| \partial^M |\chi|}|_{r_+}$, where $|\chi| = (-\chi^M \chi_M)^{(1/2)}$ is the norm of the timelike Killing vector $\chi^M = \delta_0^M$ [here $|\chi| = \sqrt{f(r)}$].

5.1.2 Anisotropic AdS geometry with multiple U(1) charges

We now construct a solution for an *anisotropic* fluid ($\epsilon = 2P_T + P_L$). An ansatz for an anisotropic AdS black hole solution is given by

$$\begin{aligned} ds^2 &= -f(r)dt^2 + 2drdt \\ &\quad + r^2(w_T(r)dx^2 + w_T(r)dy^2 + w_L(r)dz^2), \\ A^a &= -A_0^a(r)dt. \end{aligned} \tag{5.11}$$

The anisotropies are realized via $w_T(r)$ and $w_L(r)$, which are functions of the momentum anisotropy ε_p as defined in (1.3),

$$\varepsilon_p = \frac{\langle P_T - P_L \rangle}{\langle P_T + P_L \rangle}. \tag{5.12}$$

In the isotropic case ($\varepsilon_p = 0$), these functions are required to be one, $w_T(r) = w_L(r) = 1$, and the background reduces to the AdS black hole geometry (5.7).

An analytical solution of the type (5.11) is difficult to find, and we resort to numerics in the next subsection. For this, we need to know the solution close to the boundary. An asymptotic solution ($r \rightarrow \infty$) is given by the four functions

$$\begin{aligned} A_0^a(r) &= \mu_\infty^a + \frac{\sqrt{3}q^a}{2r^2} + \mathcal{O}(r^{-8}), \\ f(r)/r^2 &= 1 - \frac{m}{r^4} + \sum_a \frac{(q^a)^2}{r^6} + \mathcal{O}(r^{-8}), \\ w_T(r) &= 1 + \frac{w_T^{(4)}}{r^4} + \mathcal{O}(r^{-8}), \\ w_L(r) &= 1 + \frac{w_L^{(4)}}{r^4} + \mathcal{O}(r^{-8}), \end{aligned} \tag{5.13}$$

where $w_L^{(4)} = -2w_T^{(4)} = -m\zeta/2$, $\mu_\infty^a = \text{const.}$, and ζ is related to the momentum anisotropy ε_p by

$$\zeta = \frac{2\varepsilon_p}{\varepsilon_p + 3}. \tag{5.14}$$

The functions $w_T(r)$ and $w_L(r)$ have been introduced in view of the structure of the anisotropic fluid stress-energy tensor. More precisely, in (5.13) we fixed the r^{-4} coefficients $w_T^{(4)}$ and $w_L^{(4)}$ such that the fluid stress-energy tensor is of the diagonal form (2.48), $T^{\mu\nu} = \text{diag}(\epsilon, P_T, P_T, P_L)$ with $\epsilon = 2P_T + P_L$. Computing the stress-energy tensor in the standard way from the asymptotic solution (5.13) via the extrinsic

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curvature, see *e.g.* [92], we find the transverse and longitudinal pressures

$$P_T = \frac{m - 4w_T^{(4)} - 4w_L^{(4)}}{16\pi G_5} = \frac{m(1 + \zeta)}{16\pi G_5}, \quad (5.15)$$

$$P_L = \frac{m - 8w_T^{(4)}}{16\pi G_5} = \frac{m(1 - 2\zeta)}{16\pi G_5}. \quad (5.16)$$

Note that if (5.14) holds true, the pressures P_T and P_L satisfy (5.12). Likewise, the charge densities are

$$\rho^a = \frac{\sqrt{3}q^a}{16\pi G_5}. \quad (5.17)$$

From these relations, we find the useful identity

$$\frac{\rho^a}{\epsilon + P_T} = \frac{\sqrt{3}q^a}{4m(1 + \frac{1}{4}\zeta)}, \quad (5.18)$$

which we will need later.

Numerical solution

We now use shooting techniques to solve the system of ordinary differential equations (ODE) which follows from the equations of motion (5.4) and (5.5) upon substituting the ansatz (5.11). The idea is to vary the metric and gauge fields at some minimal value r_+ in the radial direction, integrate outwards and find solutions with the correct asymptotic behavior (5.13). A similar method was previously applied in [42].

We first need to study the asymptotic solution near r_+ and near the boundary at $r_\infty \gg r_+$ (we choose $r_\infty = 50$ in our numerics). We define r_+ by the maximal solution of

$$f(r_+) = 0 \quad (5.19)$$

and use scale invariance to set $r_+ = 1$. We then expand the functions in the metric and gauge fields near r_+ in powers of the parameter $\varepsilon = \frac{r}{r_+} - 1 \ll 1$ and substitute them into the equations of motion. In this way, we find that the only independent variables are $\{f'(r_+), w_T(r_+), w_L(r_+), w'_L(r_+)\}$ since the gauge field parameters $A_0^a(r_+)$ can be set to zero using gauge invariance, $A_0^a(r_+) = 0$. The other parameters at r_+ can be expressed in terms of these four parameters, *e.g.* $w'_T(r_+) = w_T(r_+)w'_L(r_+)/w_L(r_+)$.

The near-boundary solution is given by (5.11) with (5.13) and is parameterized by the values $(\zeta, m, q^a, \mu_\infty^a)$. The final set of data is summarized in the following table:

$r = r_+ = 1$	$r = r_\infty \gg r_+$
$A_0^a(r_+) = 0$	μ_∞^a
$f(r_+) = 0$	$f(r_\infty)$
$f'(r_+) = \text{fixed}$	$A_0^{a'}(r_\infty)$
$w_L(r_+) = \text{var}$	$w_L(r_\infty)$
$w_T(r_+) = \text{var}$	$w_T(r_\infty)$
$w'_L(r_+) = \text{var}$	

Parameters not listed are related to those in the table by the equations of motion.

To integrate the equations we proceed as follows. We fix ζ and vary three parameters at r_+ , namely $w_T(r_+)$, $w_L(r_+)$ and $w'_L(r_+)$, by choosing a grid with suitable number of sites (in our case $20^3 - 40^3$). The value $f'(r_+)$ can be thought of as the temperature of the system and will simply be fixed to some value. It turns out that the form of the functions $w_{L,T}(r)$ does not depend on this parameter. For each site in the grid we numerically solve the system of ODEs and determine the pair (m, q^a) from the known asymptotics of $A_0^{a'}(r = r_\infty)$ and $f(r = r_\infty)$. This ensures that the analytical and numerical values for these quantities coincide.

We then calculate the combined residual

$$\begin{aligned} \text{res}_\infty[w_T(r_+), w_L(r_+), w'_L(r_+)] \\ = (w_L^\#(r_\infty) - w_L^*(r_\infty))^2 + (w_T^\#(r_\infty) - w_T^*(r_\infty))^2, \end{aligned} \quad (5.20)$$

where $w_{L,T}^\#(r_\infty)$ are the numerical values, and $w_{L,T}^*(r_\infty)$ are the analytical values given by (5.13). We interpolate the residual by a piecewise linear function and find its global minimum by the simulated annealing method [93]. The result of the minimization is shown in Fig. 5.1, which depicts numerical plots of $f(r)$, $A_0(r)$, $w_T(r)$ and $w_L(r)$ for $n = 1$.

We conclude this section with a comment on r_+ . In the isotropic case, r_+ is simply the size of the horizon of the AdS black hole geometry. For nonvanishing anisotropies and vanishing $U(1)$ charges, a naked singularity was found at r_+ [43], implying that the static background does not exist indefinitely. The singularity is mild in the sense that there is a notion of ingoing boundary conditions and possible instabilities are absent at the linear level in the anisotropy parameter [43]. This behavior may persist even for nonvanishing $U(1)$ charges, even though it was difficult to see the singularity in our numerics, *cf.* Figure 5.2. Despite this subtlety, we show in the next section that, at least for small anisotropies where the bulk geometry approximates a black hole solution, the singular geometry may be used to compute some transport coefficients of the fluid.

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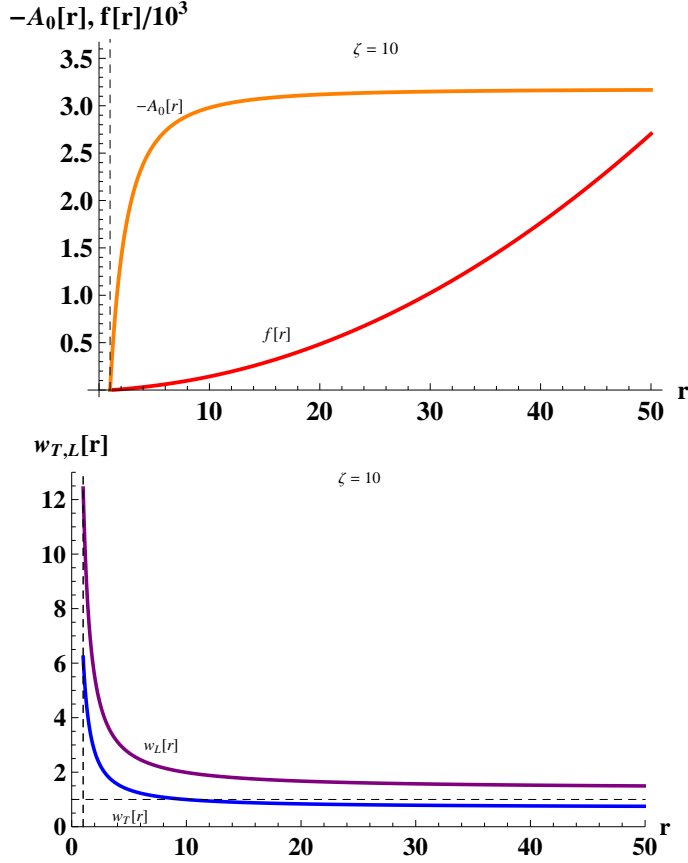


Figure 5.1: Numerical plots of $f(r)$, $A_0(r)$, $w_T(r)$ and $w_L(r)$ for $\zeta = 10$ ($r_+ = 1$). We get $w_L(r_+) = 12.42$.

5.2 Holographic vortical and magnetic conductivities

We will now compute the chiral vortical and magnetic conductivities ξ_ω^a and ξ_B^{ab} from first-order corrections to the numerical AdS geometry (5.11) using the fluid-gravity correspondence [46].

5.2.1 First-order corrected background

In order to become a dual to a multiply charged fluid, the AdS geometry (5.11) must be boosted along the four-velocity of the fluid u_μ ($\mu = 0, \dots, 3$). The boosted version of (5.11) is

$$\begin{aligned}
 ds^2 &= (r^2 w_T(r) P_{\mu\nu} - f(r) u_\mu u_\nu) dx^\mu dx^\nu - 2u_\mu dx^\mu dr \\
 &\quad - r^2 (w_T(r) - w_L(r)) v_\mu v_\nu dx^\mu dx^\nu, \\
 A^a &= (A_0^a(r) u_\mu + \mathcal{A}_\mu^a) dx^\mu,
 \end{aligned} \tag{5.21}$$

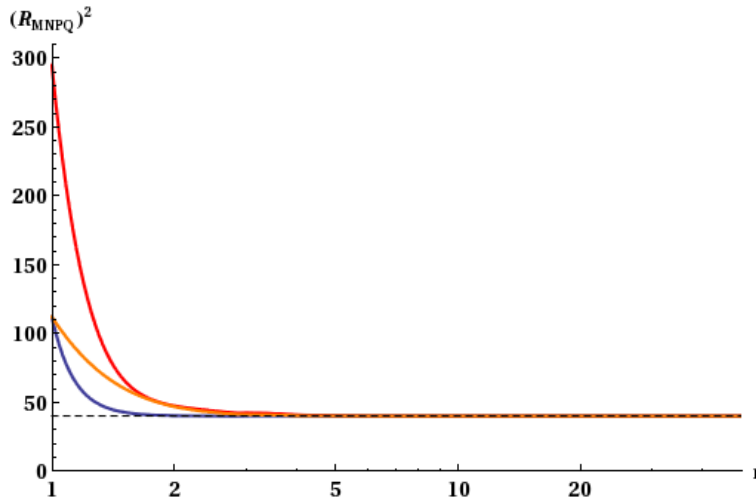


Figure 5.2: Numerical plots of $(R_{MNPQ})^2$ for $\zeta = 10$, $q \neq 0$ (red), $\zeta = 10$, $q = 0$ (orange), and $\zeta = 0$, $q = 0$ (blue).

where $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$, and $f(r)$, $A_0^a(r)$, $w_T(r)$ and $w_L(r)$ are numerically known functions. As in hydrodynamics, the four-vector v^μ determines the direction of the longitudinal axis, cf. Sec. 2. Following [14, 20], we have formally introduced constant background gauge fields \mathcal{A}_μ^a to model external electromagnetic fields, such as the magnetic fields $B^{a\mu}$ needed for the chiral magnetic effect.

The transport coefficients ξ_ω^a and ξ_B^{ab} can now be computed using standard fluid-gravity techniques [46]. We closely follow [14, 90, 20], in which these transport coefficients were determined for an isotropic fluid with one and three charges ($n = 1, 3$). We work in the static frame $u_\mu = (-1, 0, 0, 0)$, $v_\mu = (0, 0, 0, 1)$, and consider vanishing background fields \mathcal{A}_μ^a (at $x^\mu = 0$). The transport coefficients ξ_ω^a and ξ_B^{ab} measure the response of the system to rotation and the perturbation by an external magnetic field. We therefore slowly vary the velocity u_μ and the background fields \mathcal{A}_μ^a up to first order as

$$u_\mu = (-1, x^\nu \partial_\nu u_i), \quad \mathcal{A}_\mu^a = (0, x^\nu \partial_\nu \mathcal{A}_i^a). \quad (5.22)$$

We may also vary m and q in this way, but it turns out that varying these parameters has no influence on the transport coefficients ξ_ω^a and ξ_B^{ab} .

Because of the dependence on x^μ , the background (5.21) is no longer an exact solution of the equations of motion. Instead with varying parameters the solution (5.21) receives higher-order corrections, which are in this case of first order in the derivatives.

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An ansatz for the first-order corrected metric and gauge fields is given by

$$\begin{aligned}
 ds^2 &= (-f(r) + \tilde{g}_{tt}) dt^2 + 2(1 + \tilde{g}_{tr}) dt dr \\
 &\quad + r^2(w_T(r)dx^2 + w_T(r)dy^2 + w_L(r)dz^2) \\
 &\quad + \tilde{g}_{ij}dx^i dx^j - 2x^\nu \partial_\nu u_i dr dx^i \\
 &\quad + 2((f(r) - r^2) x^\nu \partial_\nu u_i + \tilde{g}_{ti}) dt dx^i, \\
 A^a &= \left(-A_0^a(r) + \tilde{A}_t^a\right) dt \\
 &\quad + \left(A_0^a(r)x^\nu \partial_\nu u_i + x^\nu \partial_\nu A_i^a + \tilde{A}_i^a\right) dx^i,
 \end{aligned} \tag{5.23}$$

where the first-order corrections are denoted by

$$\tilde{g}_{MN} = \tilde{g}_{MN}(r), \quad \tilde{A}_M^a = \tilde{A}_M^a(r). \tag{5.24}$$

As in [90], we work in the gauge

$$\tilde{g}_{rr} = 0, \quad \tilde{g}_{r\mu} \sim u_\mu, \quad \tilde{A}_r^a = 0, \quad \sum_{i=1}^3 \tilde{g}_{ii} = 0. \tag{5.25}$$

The first-order corrections can be obtained by substituting the ansatz (5.23) into the equations of motion (5.4) and (5.5). We denote the resulting Maxwell equations, Eqs. (5.5) by M_N^a ($a = 1, \dots, n$) and the components of the Einstein equation, Eqn. (5.4) by E_{MN} $M, N = 0, \dots, 4$ [$x^M = (t, x^1, x^2, x^3, r)$]. Then, from $g^{rt}E_{ti} + g^{rr}E_{ri} = 0$, we find $\partial_t u_i = 0$, and E_{tt} , E_{rt} , E_{rr} , E_{tt} , M_t^a , and M_r^a are solved by

$$\partial_i u_i = \tilde{g}_{tr} = \tilde{g}_{tt} = \tilde{A}_t^a = 0. \tag{5.26}$$

The remaining equations are E_{ij} , E_{ti} , M_i^a .

From E_{ij} we get

$$-\partial_r \left(r^3 f(r) \partial_r \left(\frac{\tilde{g}_{ij}(r)}{r^2} \right) \right) = 3r^2 (\partial_i u_j + \partial_j u_i). \tag{5.27}$$

From E_{ti} we get

$$\begin{aligned}
 &\left[\frac{f'(r)}{f(r)} \left(\frac{2}{r} + \frac{w'_T(r)}{w_T(r)} \right) + \frac{4}{3f(r)} \left(\sum_{a=1}^n A_0^{a'}(r)^2 - 6 \right) \right] \tilde{g}_{ti}(r) \\
 &\quad + \left(\frac{1}{r} + \frac{w'_L(r)}{2w_L(r)} \right) \tilde{g}'_{ti}(r) + \tilde{g}''_{ti}(r) = 4 \sum_{a=1}^n A_0^{a'}(r) \tilde{A}_i^{a'}(r),
 \end{aligned} \tag{5.28}$$

where a prime denotes the partial derivative ∂_r with respect to r .

From M_i^a we get

$$\begin{aligned} & \partial_r \left[w_L(r)^{1/2} r \left(f(r) \tilde{A}_i^{a'} - \tilde{g}_{ti}(r) A_0^{a'} \right) \right] \\ &= \partial_r \left(\frac{1}{2} S_{abc} A_0^b A_0^c \epsilon^{ijk} (\partial_j u_k) + S_{abc} A_0^b \epsilon^{ijk} (\partial_j \mathcal{A}_k^c) \right) \\ &\equiv \partial_r Q_i^a(r). \end{aligned} \quad (5.29)$$

Equation (5.27) depends only on $\tilde{g}_{ij}(r)$ and can easily be solved. The integration of (5.29) leads to

$$\begin{aligned} & w_L(r)^{1/2} \left(r f(r) \tilde{A}_i^{a'}(r) - r \tilde{g}_{ti}(r) A_0^{a'}(r) \right) \\ &= Q_i^a(r) + C_i^a. \end{aligned} \quad (5.30)$$

Here C_i^a are some integration constants, which can be fixed as

$$C_i^a = -Q_i^a(r_+) - C_i w_L(r_+)^{1/2} r_+ A_0^{a'}(r_+), \quad (5.31)$$

with r_+ as in (5.19) and $C_i = \tilde{g}_{ti}(r_+)$. This can be solved for $\tilde{A}_i^a(r)$,

$$\begin{aligned} \tilde{A}_i^a(r) &= \int_{\infty}^r dr' \frac{1}{r' f(r') w_L(r')^{1/2}} \left[Q_i^a(r') - Q_i^a(r_+) \right. \\ &\quad \left. - C_i r_+ A_0^{a'}(r_+) w_L(r_+)^{1/2} + r' \tilde{g}_{ti}(r') A_0^{a'}(r') \right]. \end{aligned} \quad (5.32)$$

We still need to determine the constants C_i . Using (5.30), we replace $\tilde{A}_i^{a'}$ in (5.28) and obtain

$$\begin{aligned} & \left[\frac{f'(r)}{f(r)} \left(\frac{2}{r} + \frac{w'_T(r)}{w_T(r)} \right) - \frac{8}{3f(r)} \left(\sum_{a=1}^n A_0^{a'}(r)^2 + 3 \right) \right] \tilde{g}_{ti}(r) \\ &+ \left(\frac{1}{r} + \frac{w'_L(r)}{2w_L(r)} \right) \tilde{g}'_{ti}(r) + \tilde{g}''_{ti}(r) = \frac{1}{w_L(r)^{1/2} r f(r)} I(r), \end{aligned} \quad (5.33)$$

where

$$\begin{aligned} I(r) &= \sum_{a=1}^n 4A_0^{a'}(r) \left(Q_i^a(r) - Q_i^a(r_+) \right. \\ &\quad \left. - C_i r_+ w_L(r_+)^{1/2} A_0^{a'}(r_+) \right). \end{aligned} \quad (5.34)$$

A homogeneous solution of this equation $\tilde{g}_{ti}(r) = g_{tt}^{(0)}(r) = f(r)$ can be generated by the infinitesimal coordinate transformation

$$\begin{aligned} dt &\rightarrow dt - \epsilon(dx + dy + dz), \quad dz \rightarrow dz + \epsilon \frac{dr}{r^2 w_L}, \\ dx &\rightarrow dx + \epsilon \frac{dr}{r^2 w_T}, \quad dy \rightarrow dy + \epsilon \frac{dr}{r^2 w_T}. \end{aligned} \quad (5.35)$$

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Then, using this homogeneous solution and techniques used in Appendix D of [51] [$P(r) = f(r)$ and $E(r) = rw_L(r)^{1/2}$ there], we bring (5.33) to the integrable form

$$\partial_r \left(w_L(r)^{1/2} r f^2(r) \partial_r \left(\frac{\tilde{g}_{ti}(r)}{f(r)} \right) \right) = I(r). \quad (5.36)$$

Solving this equation for $\tilde{g}_{ti}(r)$ and fixing the integration constants at r_+ , we get

$$\begin{aligned} \tilde{g}_{ti}(r) = f(r) \int_{\infty}^r dr' \frac{1}{w_L(r')^{1/2} r' (f(r'))^2} & \left(\int_{r_+}^{r'} dr'' I(r'') \right. \\ & \left. - w_L(r_+)^{1/2} r_+ f'(r_+) C_i \right). \end{aligned} \quad (5.37)$$

In the Landau frame we require $u_\mu \tau^{\mu\nu} = 0$, which in particular implies the absence of corrections to T^{ti} . Holographic renormalization [94] translates this into a constraint for the r^{-2} coefficient of $\tilde{g}_{ti}(r)$ which is proportional to the first correction of T^{ti} ,

$$\lim_{r \rightarrow \infty} r^2 \tilde{g}_{ti}(r) = 0. \quad (5.38)$$

In the limit $r \rightarrow \infty$, we have the asymptotics

$$\begin{aligned} f(r) = O(r^2), \quad w_L(r) = O(1), \\ \int_{r_+}^r dr' I(r') = O(1), \end{aligned} \quad (5.39)$$

and, from the vanishing of the r^{-2} -coefficient of $\tilde{g}_{ti}(r)$, we obtain the following equation for C_i :

$$w_L(r_+)^{1/2} r_+ f'(r_+) C_i = \int_{r_+}^{\infty} dr' I(r') \equiv \mathcal{J}_1 + \mathcal{J}_2 \cdot C_i, \quad (5.40)$$

where we defined the integrals

$$\begin{aligned} \mathcal{J}_1 & \equiv 4 \int_{r_+}^{\infty} dr' \sum_{a=1}^n A_0^{a'}(r') (Q_i^a(r') - Q_i^a(r_+)) \\ & = \frac{4}{3} S_{abc} A_0^a(r_+) A_0^b(r_+) A_0^c(r_+) \epsilon^{ijk} (\partial_j u_k) \\ & \quad + 2 S_{abc} A_0^a(r_+) A_0^b(r_+) \epsilon^{ijk} (\partial_j \mathcal{A}_k^c) \end{aligned} \quad (5.41)$$

and

$$\begin{aligned} \mathcal{J}_2 & \equiv 4 \int_{r_+}^{\infty} dr' \sum_{a=1}^n A_0^{a'}(r') \left(-w_L(r_+)^{1/2} r_+ A_0^{a'}(r_+) \right) \\ & = 4 w_L(r_+)^{1/2} r_+ \sum_{a=1}^n A_0^a(r_+) A_0^{a'}(r_+). \end{aligned} \quad (5.42)$$

Solving this for C_i , we eventually get

$$\begin{aligned}
 C^i &= \frac{4}{r_+(f'(r_+) - 4 \sum_a A_0^a(r_+) A_0^{a'}(r_+))} \cdot \frac{1}{w_L(r_+)^{1/2}} \\
 &\times \left(\frac{1}{3} S_{abc} A_0^a(r_+) A_0^b(r_+) A_0^c(r_+) \epsilon^{ijk} (\partial_j u_k) \right. \\
 &\quad \left. + \frac{1}{2} S_{abc} A_0^a(r_+) A_0^b(r_+) \epsilon^{ijk} (\partial_j A_k^c) \right). \tag{5.43}
 \end{aligned}$$

5.2.2 Holographic conductivities

On the boundary of the asymptotic AdS space (5.23), the metric and gauge fields couple to the fluid stress-energy tensor and $U(1)$ currents, respectively. Holographic renormalization [94] provides relations between these currents and the near-boundary behavior of their dual bulk fields. For the magnetic and vortical effects, we need the $U(1)$ currents $j^{a\mu}$, which are related to the bulk gauge fields $A^{a\mu}$ by [94, 95]

$$j^{a\mu} = \lim_{r \rightarrow \infty} \frac{r^2}{8\pi G_5} \eta^{\mu\nu} A_\nu^a(r). \tag{5.44}$$

Expanding the solution in $\frac{1}{r}$ and substituting only the corrections \tilde{A}_μ^a , we get the currents

$$\begin{aligned}
 \tilde{j}^{a\mu} &= \lim_{r \rightarrow \infty} \frac{r^2}{8\pi G_5} \eta^{\mu\nu} \tilde{A}_\nu^a(r) \\
 &= \frac{1}{16\pi G_5} \eta^{\mu\nu} (Q_\nu^a(r_+) + r_+ A_0^{a'}(r_+) C_\nu). \tag{5.45}
 \end{aligned}$$

Note that, in the isotropic case ($w_L = 1$, $P_T = P_L = P$), the prefactor of the second term of (5.45) is simply

$$r_+ A_0^{a'}(r_+) c(r_+) = \frac{\sqrt{3}}{4m} q^a, \tag{5.46}$$

as can be seen by substituting the Reissner-Nordström solution (5.8) into the left-hand-side of this equation. In the anisotropic case, we need to show that

$$r_+ A_0^{a'}(r_+) c(r_+) \cdot w_L(r_+)^{-1/2} = \frac{\sqrt{3} q^a}{4m} \cdot \frac{1}{1 + \frac{1}{4}\zeta}, \tag{5.47}$$

which, by (5.18), is equivalent to $\rho^a/(\epsilon + P_T)$. This equation holds in particular if the first and second factors on both sides agree individually. The first factors correspond to (5.46), which is expected to hold, at least approximately for small anisotropies ζ . The second factors are identical if $w_L(r_+, \zeta) = (1 + \frac{1}{4}\zeta)^2$. We find numerically (for $n = 1$)

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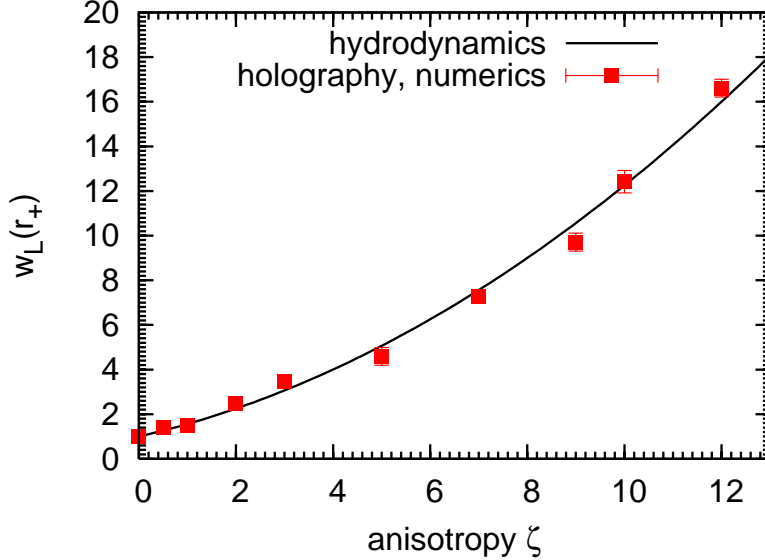


Figure 5.3: Values of $w_L(r_+)$ as a function of the anisotropy ζ . The numerically determined values for $w_L(r_+)$ lie on the solid curve, which represents the function $(1 + \frac{1}{4}\zeta)^2$.

that $w_L(r_+)$ indeed satisfies this equation, see Fig. 5.3. Thus (5.47) holds numerically, at least in the limit of small ζ .

Comparing (5.45) with the general expansion

$$\begin{aligned} \tilde{j}^{a\mu} &= \xi_\omega^a \omega^\mu + \xi_B^{ab} B^{b\mu} \\ &= \xi_\omega^a \frac{1}{2} \epsilon^{\nu\rho\sigma\mu} u_\nu \partial_\rho u_\sigma + \xi_B^{ab} \epsilon^{\nu\rho\sigma\mu} u_\nu \partial_\rho \mathcal{A}_\sigma^b, \end{aligned} \quad (5.48)$$

we finally obtain the coefficients

$$\xi_\omega^a = \frac{4}{16\pi G_5} \left(S^{abc} \mu^b \mu^c - \frac{2}{3} \frac{\rho^a}{\epsilon + P_T} S^{bcd} \mu^b \mu^c \mu^d \right), \quad (5.49)$$

$$\xi_B^{ab} = \frac{4}{16\pi G_5} \left(S^{abc} \mu^c - \frac{1}{2} \frac{\rho^a}{\epsilon + P_T} S^{bcd} \mu^c \mu^d \right), \quad (5.50)$$

with $\mu^a \equiv A_0^a(r_+)$ [since $A_0^a(\infty) = 0$]. Using the relation (5.3), we find that the holographically computed transport coefficients (5.49) and (5.50) coincide exactly with those found in hydrodynamics, (3.50) and (3.51).

5.2.3 Subtleties in holographic descriptions of the CME

The conservation of the electromagnetic current requires the introduction of the Bardeen counterterm into the action. In AdS/QCD models of the CME, this typically leads to a vanishing result for the electromagnetic current [33, 35]. The problem is related to

5.2 Holographic vortical and magnetic conductivities

the difficulty of introducing a chemical potential conjugated to a nonconserved chiral charge [33, 34]. It is possible to modify the action to obtain a conserved chiral charge [33]. This charge is however only gauge-invariant when integrated over all space in homogeneous configurations.

In AdS black hole models of the CME, one usually introduces a chiral chemical potential dual to a gauge-invariant current, despite it being anomalous [34, 20]. The prize to pay is the appearance of a singular bulk gauge field at the horizon, a phenomenon which seems to be generic in AdS black hole models of the CME.

Careful holographic renormalization shows that, in the presence of Chern-Simons terms, there is an additional term on the right-hand side of (5.44) [95]. This term is of the form

$$\hat{j}_a^\mu = -\frac{S_{abc}}{8\pi G_5} \epsilon^{\mu\nu\rho\sigma} A_{b\nu}^{(0)}(x) \partial_\rho A_{c\sigma}^{(0)}(x), \quad (5.51)$$

where $A_{a\mu}^{(0)}(x)$ are the 0th-order coefficients in a $\frac{1}{r}$ expansion of the bulk gauge fields $A_{a\mu}(r, x)$. In (5.22) we expanded the background gauge fields \mathcal{A}_μ^a around zero and set $A_\nu^{a(0)} = \mu_\infty^a u_\nu = 0$. This allowed us to ignore terms in (5.44) coming from (5.51) (at least to first order in the derivatives).

Problems arise if $\mu_\infty^a \neq 0$. To see this, let us restrict again to two charges ($n = 2$) as in Sec. 3.3.2 and define axial and vector gauge fields by $A_\mu^A = A_\mu^1$ and $A_\mu^V = A_\mu^2$. Then $\hat{j}^\mu = \hat{j}_2^\mu$ gives rise to additional contributions of the type

$$\hat{j}^\mu \supset \epsilon^{\mu\nu\rho\sigma} A_\nu^{A(0)}(x) F_{\rho\sigma}^{V(0)}(x), \quad (5.52)$$

which are forbidden by electromagnetic gauge invariance [33], unless $A_\nu^{A(0)}(x) = 0$. However, in general $A_\nu^{A(0)}(x) = \mu_5^\infty u_\nu$ (at $x = 0$) with some constant μ_5^∞ . We should thus set $\mu_5^\infty = 0$ [Note that this does not imply $\mu_5 = A_0^A(r_\infty) - A_0^A(r_+) = 0$]. This corresponds to a nonvanishing gauge field at the horizon, as noticed also in [34, 20].

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Chapter 6

Summary and Outlook

The chiral magnetic effect can be derived in several ways, in particular by using fluid-gravity dual models. Using holographic duality we studied an anisotropic hydrodynamics with multiple anomalous $U(1)$ charges and found the dependence of CME coefficient on the momentum anisotropy.

We discussed two descriptions of the chiral magnetic effect in the anisotropic quark-gluon plasma. We first computed the vortical and magnetic conductivities of the anisotropic fluid. We found that CME coefficient increases linearly with elliptic flow coefficient. We then constructed the dual gravity background of the anisotropic fluid. Finally, we used this background to perform a holographic computation of the vortical and magnetic conductivities and found numerical agreement with the hydrodynamic result for small anisotropies.

6. SUMMARY AND OUTLOOK

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