Noncommutative Solitons

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Lectures given at the
Spring School on Superstrings and Related Matters
Trieste, 2 – 10 April 2001

LNS027006

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Abstract

These are pedagogical lectures on solitons in noncommutative field theories delivered at the Spring School, March'01.

Contents

1 Introduction

Though noncommutative field theories have been explored for several years, a resurgence of interest in it was sparked off after it was realised that they arise very naturally as limits of string theory in certain background fields [1]. It became more plausible (at least to string theorists) that these nonlocal deformations of usual quantum field theories are consistent theories in themselves. This led to a detailed exploration of many of their classical and quantum properties. I will elaborate further on the string theory context in the next section.

One of the consequences of this exploration was the discovery of novel classical solutions in noncommutative field theories [2]. Since then much work has been done in exploring many of their novel properties. My lectures focussed on some specific aspects of these noncommutative solitons. They primarily reflect the topics that I have worked on and are not intended to be a survey of the large amount of work on this topic. Some reviews that give a more comprehensive list of references are [3],[4].

2 The context

Here we will try to provide the context in which the study of noncommutative field theories and their classical solutions assume importance.

The Importance of Open Strings: The understanding of the role of open strings in string theory via D-branes, has proven to be a development of overwhelming importance. This understanding was instrumental in correctly counting black hole microstates, one of the dramatic successes of string theory. One of the surprises was the manner in which a purely gravitational phenomenon, like black hole entropy, was described in terms of open strings, which at least classically don't contain closed string excitations like the graviton.

This connection between open and closed strings was sought to be further exploited in the Matrix theory proposal for a DLCQ description of M-Theory. But it's most striking manifestation was the AdS/CFT duality of Maldacena relating large N gauge theories to purely closed string theories. This conjecture is a reflection of an underlying duality between open and closed strings which is yet to be completely understood.

Decoupling limits: In the AdS/CFT duality, one takes a certain scaling limit of open string theories living on D-branes in which only the massless gauge theory modes survive and are described by a (super) Yang-Mills lagrangian. The massive open string states are effectively decoupled by taking the string scale to infinity. This scaling limit of open string theories is conjectured to describe pure closed strings propagating in the near horizon geometry of the D-branes. The fact that one can gain nontrivial information from studying a simple field theory limit of string theory has led one to examine more closely the various decoupling limits of string theory. (Cf. Kutasov's lectures.) Taking decoupling limits of different sorts also help one to focus more sharply on various aspects of string theory. The idea is to get a limit which is easier to analyse than the full theory, but which nevertheless retains enough of the complexity.

Noncommutativity and String Field Theory: In parallel with these developments, and at first sight unrelated to it, is an ambitious program initiated by Sen which attempts, among other things, to understand closed strings in terms of open strings. The idea is to use the formulation of open string interactions in terms of a cubic string field theory as a complete description of string theory. This formulation relies on a representation of open string interactions which consists of gluing them in a fundamentally noncommutative way [5]. This defines an associative but noncommutative product of string fields in terms of which the string field action is expressed. D-branes are nontrivial classical solutions of this action while closed strings could arise as some kind of quantum excitations.

Since noncommutativity is thus in some sense intrinsic to string theory (and not just a property of some backgrounds) and perhaps plays a crucial role in understanding the notions that replace classical geometry, it is worthwhile to try and understand it better.

However, when one takes the conventional field theory limit of open string theory, the remnant of the noncommutativity is the somewhat trivial matrix algebra of the Chan-Paton indices. It does not involve the noncommutativity that comes from the extended nature of the open string.

Noncommutative Field theories: One might therefore ask if there is a limit of string theory which has the relative simplicity of keeping only a field theoretic number of degrees of freedom and yet displays the extended nature of strings. In particular, it should capture some of the nontrivial noncommutativity of open string interactions. It turns out that the answer is yes. One can obtain a nonlocal deformation of field theories by taking a decoupling limit of open strings in a large magnetic field [1], [6]. The massive string modes decouple leaving a kind of elastic dipole object.

These resulting noncommutative field theories will be the main topic of these lectures.

Noncommutative solitons: More specifically, we will study the classical limit of these noncommutative field theories and find finite energy soliton solutions that have no counterpart in local field theories. Among the nice features of these solitons is that they are fairly universal and more or less insensitive to the details of the theory. They exhibit various novel features like nonabelian enhancement of symmetry when they are coincident.

In fact, these solitons are really the D-branes of string theory manifested in a field theory. This is somewhat surprising as it does not happen that you can find D-branes as finite energy excitations in a conventional field theory limit of string theory. The simplicity of noncommutative solitons implies that one can study many properties of D-branes very explicitly in this context.

Therefore the motivation for studying these solitons will be to use them as a simple set of probes of stringy behaviour in a well controlled manner. Much of the applications have been in the context of issues of tachyon condensation in open string theory. We can however also use these solitons to probe issues of how D-branes see space time, for instance.

Finally, the field theoretic aspects of these solitons are interesting in themselves and might perhaps have applications in very different contexts such as in the Quantum Hall effect.

3 Strings in a large magnetic field

As a prelude to studying strings in a large magnetic field, let us look at point particles in a large magnetic field.

The action for (nonrelativistic) point particles reads as

$$S = \int dt \left(\frac{1}{2} m \dot{x}_\mu \dot{x}^\mu + e B_{\mu\nu} x^\mu \dot{x}^\nu \right). \tag{3.1}$$

The conjugate momentum Π_μ to x^μ is

$$\Pi_\mu = m\dot{x}_\mu + eB_{\mu\nu}x^\nu. \tag{3.2}$$

In the limit where the energy $\omega \ll \frac{e|B|}{m}$, the canonical commutation relations become simply

$$[x^\mu, x^\nu] = i(B^{-1})^{\mu\nu}\frac{m}{e}. \tag{3.3}$$

Thus at energies much less than the cyclotron frequency $\frac{e|B|}{m}$, when one is in the lowest Landau level, one effectively has noncommuting coordinates. This is why the physics of the quantum hall effect displays some features of noncommutativity.

Now write the action for an open string in a constant magnetic field. We assume that the open string ends on a p brane in some of whose worldvolume directions the magnetic field is switched on.

$$S = \frac{1}{4\pi\alpha'}\int_{\Sigma} d^2\sigma \left(g_{\mu\nu}\partial_\alpha X^\mu \partial^\alpha X^\nu - 2\pi i\alpha' B_{\mu\nu}\epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu\right). \qquad (3.4)$$

The additional term involving B is really a boundary term which couples to the charges at the end of the open string like a constant magnetic field.

It leads to boundary conditions in the directions along the brane which are mixed.

$$\left(g_{\mu\nu}\partial_n X^\nu + 2\pi i\alpha' B_{\mu\nu}\partial_t X^\nu\right)|_{\partial\Sigma} = 0. \qquad (3.5)$$

One can write down the Green's functions on the disc worldsheet with these boundary conditions. What we will need is the particular case when the X's are at on the boundary of the disc (parametrised by τ).

$$< X^\mu(\tau)X^\nu(\tau') >= -\alpha' G^{\mu\nu}\ln(\tau-\tau')^2 + \frac{i}{2}\Theta^{\mu\nu}\epsilon(\tau-\tau'). \qquad (3.6)$$

Here

$$\begin{aligned} G^{\mu\nu} &= \left(\frac{1}{g+2\pi\alpha' B}g\frac{1}{g-2\pi\alpha' B}\right)^{\mu\nu} \\ \Theta^{\mu\nu} &= -(2\pi\alpha')^2\left(\frac{1}{g+2\pi\alpha' B}B\frac{1}{g-2\pi\alpha' B}\right)^{\mu\nu} \end{aligned} \qquad (3.7)$$

are usually called the open string metric and the noncommutativity parameter [6]. The open string metric is what determines the mass shell condition for open string states. Θ is called the noncommutativity parameter since the above OPE essentially implies that

$$[X^\mu(\tau), X^\nu(\tau)] = i\Theta^{\mu\nu}. \qquad (3.8)$$

Note that Θ has dimensions of length^2.

There is one more ingredient, namely that the effective coupling of open string modes is also rescaled by a factor that depends on the magnetic field. We will not need the exact expression until later.

The noncommutativity parameter leads to an extra term in the OPE of open string vertex operators $e^{ik\cdot X}$:

$$e^{ik_1\cdot X}(\tau)e^{ik_2\cdot X}(\tau') \sim (\tau-\tau')^{2\alpha' G^{\mu\nu}k_{1\mu}k_{2\nu}}e^{-i\frac{1}{2}\Theta^{\mu\nu}k_{1\mu}k_{2\nu}}e^{i(k_1+k_2)\cdot X}(\tau') + \ldots \qquad (3.9)$$

The additional term $e^{-i\frac{1}{2}\Theta^{\mu\nu}k_{1\mu}k_{2\nu}}$ can be understood in position space as giving a nonlocal interaction which is expressed in terms of the Moyal product.

$$(f \star g)(x) = e^{i\frac{1}{2}\Theta^{\mu\nu}\partial_\mu \partial'_\nu} f(x)g(x')|_{x=x'}. \tag{3.10}$$

In general, there will be such a phase factor for all vertex operators implying that the effect of the magnetic field on the effective action for open string modes in spacetime is completely captured by replacing all local products by the Moyal products, if we additionally remember to make all metric contractions with the open string metric (note that it is the open string metric that appears in (3.9) in the anomalous dimension of the vertex operators).

We can now take the equivalent of the limit of a large magnetic field, namely take $\alpha'|B| \gg 1$. Here $|B|^2 = B_{\mu\nu}B_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}$. We will in addition demand that this limit is taken keeping the open string metric $G^{\mu\nu}$ and $\Theta^{\mu\nu}$ finite. This requires taking the string scale to infinity ($\alpha' \to 0$). In the absence of the magnetic field this would mean decoupling all the massive string modes giving a field theory of the zero mode (if we keep the coupling constant finite).

With the magnetic field, as we have seen the only effect is to replace local products with the moyal product. The terms involving massive modes (both open and closed) then decouple for the same reason as in the case without a magnetic field. The lowest open string modes then interact via a nonlocal deformation of ordinary field theory.

4 Scalar noncommutative solitons

With these motivations we will start our study of semiclassical noncommutative field theories. This section closely follows the discussion in [2]. The simplest example is a theory of a single scalar field in $2+1$ dimensions with noncommutativity in the two spatial directions. Though this does not necessarily arise as any decoupling limit of string theory, the classical solutions we find are generic to noncommutative field theories.

We will parametrize the spatial R^2 by complex coordinates $z, \bar{z}$. The energy functional for static configurations is

$$E = \frac{1}{g^2}\int d^2z \left(\partial_z\phi\partial_{\bar{z}}\phi + V(\phi)_\star\right), \tag{4.1}$$

where $d^2z = dxdy$. Fields in the action are multiplied using the Moyal star product (which reads in complex form as),

$$\left(f \star g\right)(z,\bar{z}) = e^{\frac{\theta}{2}(\partial_z\partial_{\bar{z}'} - \partial_{z'}\partial_{\bar{z}})} f(z,\bar{z})g(z',\bar{z}')|_{z=z'}. \tag{4.2}$$

Note that since $\int f \star g = \int fg$, the Moyal product drops out of the quadratic term in the action.

Before we look for classical solutions to this action, let us recall that the scalar theory without noncommutativity does not have any lump solutions. This is actually true for any bounded potential in spatial dimension greater than one, and follows from a simple scaling argument of Derrick [7]. If $\phi_0(x)$ be an extremum of the energy functional (4.1) (with $\theta = 0$), then consider the energy of the field configurations $\phi_\lambda(x) = \phi_0(\lambda x)$.

$$\begin{aligned} E(\lambda) &= \frac{1}{g^2}\int d^Dx \left(\frac{1}{2}(\partial\phi_0(\lambda x))^2 + V(\phi_0(\lambda x))\right) \\ &= \frac{1}{g^2}\int d^Dx \left(\frac{1}{2}\lambda^{2-D}(\partial\phi_0(x))^2 + \lambda^{-D}V(\phi_0(x))\right). \end{aligned} \tag{4.3}$$

Since $\phi_0(x)$ is an extremum, we require $\frac{\partial E(\lambda)}{\partial \lambda}|_{\lambda=1} = 0$. that is,

$$\int d^Dx \left(\frac{1}{2}(D-2)(\partial\phi_0(x))^2 + DV(\phi_0(x))\right) = 0.$$

For spatial dimension $D \geq 2$, for a potential bounded from below by zero, the only way this can be true is for the kinetic and the potential terms to separately vanish. There are therefore no nontrivial configurations. Note that this argument fails once one includes higher derivative terms.

We now seek finite energy (localized) solitons of (4.1) for nonzero θ. Since no solutions exist for $\theta = 0$ (4.3), we begin our search in the limit of large noncommutativity, $\theta \to \infty$. It is useful to non-dimensionalize the coordinates $z \to z\sqrt{\theta}$, $\bar{z} \to \bar{z}\sqrt{\theta}$. As a result, the $\star$ product will henceforth have no θ; i.e. it will be given by (4.2) with $\theta = 1$. Written in rescaled coordinates, the dependence on θ in the energy is entirely in front of the potential term:

$$E = \frac{1}{g^2}\int d^2z \left(\frac{1}{2}(\partial\phi)^2 + \theta V(\phi)_\star\right) \tag{4.4}$$

In the limit $\theta \to \infty$, with V held fixed, the kinetic term in (4.4) is negligible in comparison to $V(\phi)$, at least for field configurations varying over sizes of order one in our new coordinates.

Our considerations apply to generic potentials $V(\phi)$, but we will, for definiteness, mostly discuss those of polynomial form

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \sum_{j=3}^{r}\frac{b_j}{j}\phi^j. \tag{4.5}$$

4.1 Solutions in the $\theta = \infty$ limit

After neglecting the kinetic term, the energy

$$E = \frac{\theta}{g^2} \int d^2 z V(\phi)_\star, \tag{4.6}$$

is extremised by solving the equation

$$\left(\frac{\partial V}{\partial \phi} \right)_\star = 0. \tag{4.7}$$

For instance, for a cubic potential one has to solve an equation of the form

$$m^2 \phi + b_3 \phi \star \phi = 0. \tag{4.8}$$

If $V(\phi)$ were the potential in a commutative scalar field theory, the only solutions to (4.7) would be the constant configurations

$$\phi = \lambda_i, \tag{4.9}$$

where $\lambda_i \in \{\lambda_1, \lambda_2, \cdots, \lambda_k\}$ are the various real extrema of the function $V(x)$. The derivatives in the definition of the star product allow for more interesting solutions of (4.7).

In order to find all solutions of (4.7) we will exploit the connection between Moyal products and quantization. Given a C^∞ function $f(q,p)$ on R^2 (thought of as the phase space of a one-dimensional particle), there is a prescription which uniquely assigns to it an operator $\widehat{f}(\widehat{q}, \widehat{p})$, acting on the corresponding single particle quantum mechanical Hilbert space, $\mathcal{H}$. It is convenient for our purposes to choose the Weyl or symmetric ordering prescription

$$\widehat{f}(\widehat{q}, \widehat{p}) = \frac{1}{(2\pi)^2} \int d^2 k \widetilde{f}(k) e^{-i(k_q \widehat{q} + k_p \widehat{p})}, \tag{4.10}$$

where

$$\widetilde{f}(k) = \int d^2 x e^{i(k_q q + k_p p)} f(q,p), \tag{4.11}$$

and

$$[\widehat{q}, \widehat{p}] = i. \tag{4.12}$$

With this prescription, it may be verified that

$$\frac{1}{2\pi}\int dpdq f(q,p) = \mathrm{Tr}_{\mathcal{H}}\widehat{f}, \tag{4.13}$$

and that the Moyal product of functions is isomorphic to ordinary operator multiplication

$$\widehat{f}\cdot\widehat{g} = \widehat{f\star g}. \tag{4.14}$$

In order to solve any algebraic equation involving the star product, it is thus sufficient to determine all operator solutions to the equation in $\mathcal{H}$. The functions on phase space corresponding to each of these operators may then be read off from (4.10). We will now employ this procedure to find all solutions of (4.7).

It is easy to see that $\widehat{\phi} = \lambda_i P$ is a solution to $V'(\widehat{\phi}) = 0$, if P is an arbitrary projection operator on some subspace of $\mathcal{H}$ and if λ_i is an extremum of $V(x)$. The energy of this solution is, using (4.13),

$$E = \frac{2\pi\theta}{g^2}\mathrm{Tr}V(\widehat{\phi}) = \frac{2\pi\theta}{g^2}V(\lambda_i)\mathrm{Tr}P. \tag{4.15}$$

Thus the energy is finite if P is projector onto a finite dimensional subspace of $\mathcal{H}$.

In fact, you can convince yourself that the most general solution to (4.7) takes the form

$$\widehat{\phi} = \sum_j a_j P_j \tag{4.16}$$

where $\{P_j\}$ are mutually orthogonal projection operators onto one dimensional subspaces,

$$P_i P_j = \delta_{ij} P_j; \qquad Tr_{\mathcal{H}} P_i = 1, \tag{4.17}$$

with a_j taking values in the set $\{\lambda_i\}$ of real extrema of $V(x)$.

From now on we will restrict ourselves to a potential with one nontrivial minimum λ other than the one at the origin.

We have a huge infinity of solutions of the form λP. To see what they mean, let us translate them into position space. It will be convenient for this purpose to choose a particular basis in $\mathcal{H}$. Let $|n\rangle$ represent the energy eigenstates of the one dimensional harmonic oscillator whose creation and annihilation operators are defined by

$$a = \frac{\widehat{q}+i\widehat{p}}{\sqrt{2}}; \quad a^\dagger = \frac{\widehat{q}-i\widehat{p}}{\sqrt{2}}. \tag{4.18}$$

Note that $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. Any operator may be written as a linear combination of the basis operators $|m\rangle\langle n|$'s, which, in turn, may be expressed in terms of a and $a^\dagger$ as

$$|m\rangle\langle n| =: \frac{a^{\dagger m}}{\sqrt{m!}} e^{-a^\dagger a} \frac{a^n}{\sqrt{n!}} : \tag{4.19}$$

where double dots denote normal ordering. We will first describe operators of the form (4.16) that correspond to radially symmetric functions in space. As $a^\dagger a \approx \frac{r^2}{2}$, operators corresponding to radially symmetric wavefunctions are functions of $a^\dagger a$. From (4.19), the only such operators are linear combinations of the diagonal projection operators $|n\rangle\langle n| = \frac{1}{n!} : a^{\dagger n} e^{-a^\dagger a} a^n :$. Hence all radially symmetric solutions of (4.7) correspond to operators of the form $\widehat{\phi} = \lambda \sum a_n |n\rangle\langle n|$, where the numbers a_n can take values 0 or 1.

It is not difficult to translate these operators back to position space [2]. One finds

$$|n\rangle\langle n| = \frac{1}{(2\pi)} \int d^2k e^{\frac{-k^2}{4}} L_n(\frac{k^2}{2}) e^{-i\left(k_{\bar{z}} a + k_z a^\dagger\right)} \tag{4.20}$$

where $L_n(x)$ is the n^{th} Laguerre polynomial. The field $\phi_n(x,y)$ that corresponds to the operator $\widehat{\phi}_n = |n\rangle\langle n|$ is, therefore,

$$\phi_n(r^2 = x^2 + y^2) = \frac{1}{(2\pi)} \int d^2k e^{\frac{-k^2}{4}} L_n(\frac{k^2}{2}) e^{-ik.x} = 2(-1)^n e^{-r^2} L_n(2r^2). \tag{4.21}$$

Note that $\phi_0(r^2)$ is the simple gaussian $2e^{-r^2}$. In summary, (4.7) has an infinite number of real radial solutions, given by

$$\sum_{n=0}^{\infty} a_n \phi_n(r^2) \tag{4.22}$$

where $\phi_n(r^2)$ is given by (4.21) and each a_n takes values either 0 or 1. These solutions will have finite energy if only a finite number of the a_n are nonzero, as is evident from (4.15).

We also see from (4.15) that the action at $\theta = \infty$ has a large symmetry $\widehat{\phi} \to U\widehat{\phi}U^\dagger$, where U is any unitary operator acting on $\mathcal{H}$. This $U(\infty)$ global symmetry generates new nonradially symmetric solutions out of the radially symmetric ones. The most general projection operator $\widehat{\phi} = \lambda P$, of rank k, is unitarily related to a projection operator which is diagonal (in the

SHO basis), that is of the form $\lambda(\sum_{i=0}^{k-1}|i><i|)$. And the corresponding solutions are all degenerate in energy. In fact, their energy $E=\frac{2\pi k\theta}{g^2}V(\lambda)$ is k times the energy of the minimal energy soliton $k=1$. This suggests an interpretation as k solitons which will become clearer as we proceed.

It is remarkable that the energy of the soliton is completely insensitive to the value of the scalar potential at any point except $\phi=\lambda$. Thus the mass of the soliton is unchanged if the height of the barrier in $V(\phi)$ (between $\phi=\lambda$ and $\phi=0$) is taken to infinity while $V(\lambda)$ is kept fixed. This is true even though $\phi_0(r)$, the solitonic field configuration corresponding to $\lambda|0\rangle\langle 0|$, decreases continuously from $\phi=2\lambda$ at $r=0$ to $\phi=0$ at $r=\infty$! It is also striking that the form of the solutions themselves are remarkably universal too, more or less independent of the details of the potential.

4.2 Stability and Moduli Space at $\theta=\infty$

Because of the $U(\infty)$ symmetry it suffices to examine the stability of radial solutions of the form

$$\phi(r^2)=\lambda\sum_{n=0}^{k-1}\phi_n(r^2) \tag{4.23}$$

to small fluctuations. Since any $U(\infty)$ rotation does not change the energy of our solution (4.23) it is sufficient to study the stability to radially symmetric fluctuations. These are most conveniently parameterized as deformations of the eigenvalues. The energy for an arbitrary radially symmetric state $\phi(r^2)=\sum_{n=0}^{\infty}c_n\phi_n(r^2)$ is

$$E=\frac{2\pi\theta}{g^2}\sum_{n=0}^{\infty}V(c_n).$$

The solutions with $c_n\in\{\lambda,0\}$ are manifestly local minima of E, as λ and 0 are minima of the function $V(x)$. Thus the solution of the form (4.23) (and all solutions unitarily related to it) are stable to small fluctuations. (If any of the c_n took the value of a local maximum of $V(x)$, then it is equally easy to see that while the corresponding $\phi(r^2)$ would be a solution to (4.7) it is not stable to small radial fluctuations.)

The stability of the gaussian soliton $\lambda\phi_0(r^2)$ may qualitatively be understood as follows. Since $\lambda\phi_0(r^2)=2\lambda e^{-r^2}$ is a Gaussian of height 2λ, far away from the origin, $\phi_0(x)=0$, but near $x=0$, it is in the vicinity of the second vacuum. In other words, the static solution corresponds to a bubble of the "false" vacuum. The area of the bubble is of order one (or θ in our original coordinates), the non-commutativity scale. In a commutative theory such a bubble would decay by shrinking to zero size. Noncommutativity prevents the bubble from shrinking to a spatial size smaller than $\sqrt{\theta}$. In order to decay, ϕ_0 actually has to scale to zero - but that process involves going over the hump in the potential and so is classically forbidden.

The $U(\infty)$ symmetry of (4.7) results in there being an infinite number of zero modes for a given solution with energy $2\pi k V(\lambda)$. This infinite dimensional moduli space can be mathematically characterised as follows. The rank k hermitian projection operators on $\mathcal{H}$ (or equivalently, the k-dimensional hyperplanes in $\mathcal{H}$) form a manifold known as the Grassmannian $\mathrm{Gr}(k, \mathcal{H})$, which can also be described as the coset space

$$\frac{\mathrm{U}(\infty)}{\mathrm{U}(k) \times \mathrm{U}(\infty - k)}, \tag{4.24}$$

where $\mathrm{U}(\infty)$ acts on the entire space, while $\mathrm{U}(\infty - k)$ acts only on the orthogonal complement of a k-dimensional hyperplane.

5 Scalar solitons at finite θ

So far, by working at infinite θ, we have found an infinite number of solutions. This is because we have neglected the kinetic energy. As we will now see, the kinetic energy breaks the $U(\infty)$ symmetry that the potential term possessed. Including it in a systematic expansion in powers of $\frac{1}{\theta}$, we will find that most of the solutions no longer remain. However, we will find to leading order in $\frac{1}{\theta}$, that there is an interesting finite dimensional (approximate) moduli space. These will in some limits correspond to separated gaussian solitons. Apparent singularities in this moduli space are resolved in a very stringy way. The discussion in this section is largely based on [8].

The kinetic term can also be written in terms of operators if we use the Weyl-Moyal correspondence for derivatives

$$\partial_z \to -\frac{1}{\sqrt{\theta}}[a^\dagger, \cdot]. \tag{5.1}$$

The energy functional then reads as

$$E = \frac{2\pi}{g^2} \mathrm{Tr}_{\mathcal{H}} \left([a, \widehat{\phi}][\widehat{\phi}, a^\dagger] + \theta V(\widehat{\phi}) \right). \tag{5.2}$$

This no longer has the symmetry under $\widehat{\phi} \to U \widehat{\phi} U^\dagger$.

5.1 The expansion in $\frac{1}{\theta}$

If m^2 is a typical mass scale of the theory one can define a perturbation expansion in $1/(\theta m^2)$ for the energy and solutions of the equations of motion of (5.2),

$$\begin{aligned}\widehat{\phi} &= \widehat{\phi}_0 + \frac{1}{\theta m^2}\widehat{\phi}_1 + \cdots\ , \\ E &= \theta m^2 E_0 + E_1 + \frac{1}{\theta m^2}E_2 + \cdots\ ,\end{aligned} \tag{5.3}$$

where $\widehat{\phi}_0 = \lambda P$ is a solution at infinite θ. The first correction to the energy is just the kinetic term:

$$E_1[\widehat{\phi}_0] = 2\pi\,\mathrm{Tr}[a,\widehat{\phi}_0][\widehat{\phi}_0,a^\dagger]. \tag{5.4}$$

Due to the fact that $V'(\widehat{\phi}_0) = 0$, E_1 is independent of the correction $\widehat{\phi}_1$.

A reasonable guess would be that a minimum of the kinetic energy is achieved only by rotationally symmetric solutions. However, the story is not so simple. It turns out that there are non-rotationally symmetric minima of E_1 [8]. Indeed, the full moduli space $\mathcal{M}_k$ has an interesting structure, large enough to allow non-trivial dynamics. This unexpected fact is a consequence, not of any symmetry possessed by E_1, but rather of a Bogomolnyi-like bound that it satisfies:

$$E_1[\widehat{\phi}_0] = 2\pi\lambda^2\,\mathrm{Tr}[a,P][P,a^\dagger] = 2\pi\lambda^2\,\mathrm{Tr}\left(P + 2F(P)^\dagger F(P)\right) \geq 2\pi\lambda^2 k \tag{5.5}$$

where

$$F(P) \equiv (1-P)aP. \tag{5.6}$$

The bound is saturated when $F(P) = 0$, in other words when the image of P is an invariant subspace of the operator a. The projection operators satisfying this condition define a finite dimensional subspace $\mathcal{M}_k$ of the space of all projection operators. The field configurations corresponding to these projection operators will have a natural interpretation in terms of separated solitons.

Starting with the simplest case of $k = 1$, it is clear that any 1-dimensional invariant subspace of a must be spanned by an eigenstate of that operator, i.e. by a coherent state. We'll use an unnormalised version of coherent states defined by $|z\rangle = \exp a^\dagger z|0\rangle$ obeying $a|z\rangle = z|z\rangle$. The corresponding projector $\widehat{\phi}_z = |z\rangle\langle z|$ is a gaussian soliton localised around z when viewed in position space.

For higher k the one can similarly construct invariant subspaces spanned by k different coherent states $|z_i\rangle$. We can think of this as k solitons, each with independent collective coordinate z_i. (Indeed, if the z_i are far from each other, then the respective coherent states are nearly orthogonal and the corresponding field configuration is approximately the sum of distant Gaussian solitons.) And the moduli space is, at least naively, the k-fold symmetric product of the single-soliton moduli space, $\mathrm{Sym}^k(\mathbb{C}) \equiv \mathbb{C}^k/S_k$ (symmetric because permuting the z_i leaves the configuration unchanged; the solitons are like identical particles).

Examining the potential singularities when the solitons come together will give us some intuition about how these solitons see space time. Consider the case $k=2$ after factoring out the centre of mass degrees of freedom. The description in terms of coherent states $\{|z\rangle, |-z\rangle\}$ becomes bad when $z \to 0$. But this is the fault of our description. A basis which has a smooth limit as $z \to 0$ is $\{|z\rangle, \frac{|z\rangle - |-z\rangle}{z-(-z)}\}$ which approaches $\{|0\rangle, a^\dagger|0\rangle = |1\rangle\}$. Thus there is no singularity when two solitons come together – one ends up in the radially symmetric configuration $|0\rangle\langle 0| + |1\rangle\langle 1|$. One can study the metric on the k soliton moduli space and find that it is Kahler. An explicit form of the Kahler potential can also be given.

The situation becomes more interesting in higher dimensions. For instance, consider noncommutativity in four spatial directions. Then the Moyal-Weyl correspondence maps the fields in four dimensions to operators in the hilbert space of a particle in $2d$. Again the SHO basis spanned by the $2d$ oscillators $a_1^\dagger, a_2^\dagger$ is the useful one to work in.

As before, projection operators (in this $2d$ hilbert space), are solutions to the equations of motion at $\theta = \infty$. Inclusion of the kinetic energy at leading order in $\frac{1}{\theta}$ leads to a lifting of the degernacy. Nevertheless, a bogomolnyi bound similar to (5.5) implies that there is a finite dimensional moduli space parametrised by projection operators satisfying $Pa_rP = a_rP$ $(r = 1,2)$.

Again, such projectors can be parametrised by the subspaces spanned by $\{|\vec{z}_i\rangle\}$ $(i = 1\ldots k)$. The moduli space is naively $\mathrm{Sym}^k(\mathbb{C}^2)$. What happens at the coincidence locus is very interesting. When two solitons come together,

$$\lim_{\vec{z}\to\vec{0}} \mathrm{span}\{|\vec{z}\rangle, |-\vec{z}\rangle\} = \mathrm{span}\left\{|0,0\rangle, \vec{\gamma}\cdot\vec{a}^\dagger|0,0\rangle\right\}, \qquad \text{where } \vec{\gamma} = \lim_{\vec{z}\to\vec{0}} \frac{\vec{z}}{|\vec{z}|}. \tag{5.7}$$

Thus the "origin" of the relative moduli space is not a single point, but rather a $\mathbb{P}^1$ parametrized by the complex direction $\vec{\gamma}$ along which the two solitons came together. This is in contrast to the $d=1$ case. This is also exactly how string theory resolves the $\mathbb{C}^2/\mathbb{Z}_2$ singularity. Namely, by introducing an S^2 at the singular point. Here we see that the geometry seen by the noncommutative algebra of projection operators is very different from that seen by functions. Somehow, this is perhaps a hint of how noncommutative structures in string theory will modify our notion of geometry. Before closing this discussion, it should be mentioned that going to higher than 4 spatial dimensions introduces even more weird behaviour. The moduli spaces seen by the solitons are not even smooth – they are spaces known to mathematicians as Hilbert schemes.

The moduli space $\mathcal{M}_k$ is also useful in constructing solitons on quotient spaces. For example, in two dimensions one can write down stable solitons on $\mathbb{R}^2/\mathbb{Z}_k$, the cylinder and torus. One small surprise is that stable noncommutative solitons do not exist when the area of the torus is smaller than $2\pi\theta$. The torus becomes too small for the solitons to fit on.

6 Noncommutative solitons as D-branes

In this section we provide a brief sketch of how noncommutative solitons in the scalar theories show up as D-branes in studies of tachyon condensation [9], [10].

In the bosonic string theory there are D-branes of all dimensions which are however unstable – they have a real tachyon on their world volume. In particular, the space filling $D25$ brane is unstable and reflects the instability of the bosonic open string theory in 26 dimensions. Ashoke Sen has made a series of definite conjectures [11],[12] about the fate of the tachyon. Firstly, the vacuum that the tachyon rolls down to, is expected to contain no open strings. Secondly, the difference in energy per unit volume of this vacuum to the original unstable one is expected to be equal to the tension of the $D-25$ brane. Thirdly, the lower dimensional D-branes are solitonic excitations of the tachyon potential.

To make the connection with the noncommutative scalar field theories we have studied thus far, we consider the effective action for the tachyon field, obtained by integrating out the massive string fields. It is expected to take the form

$$S = \frac{C}{g_s}\int d^{26}x\sqrt{g}\left(\frac{1}{2}f(T)g^{\mu\nu}\partial_\mu T\partial_\nu T - V(T) + \cdots\right). \tag{6.1}$$

Here $V(T)$ is a general potential having an unstable extremum at $T = T_0$ (the unstable vacuum) and a minimum chosen to be $V(0) = 0$. The constant $C = g_s T_{25}$ is independent of g_s. Sen's conjecture then requires $V(T_0) = 1$. The terms that are omitted are higher derivative terms and terms involving the massless modes.

Let us now turn on a B field in some of the spatial directions of the theory. In the presence of a B-field, as mentioned in section 3, the action is modified to

$$S = \frac{C}{G_s} \int d^{26}x \sqrt{G} \left(\frac{1}{2} f(T) G^{\mu\nu} \partial_\mu T \partial_\nu T - V(T) + \cdots \right)_\star . \qquad (6.2)$$

Here it is understood that there is noncommutativity only in the directions where the B field has been turned on. Now, the advantage of taking the limit of a large B-field is that derivative terms can be neglected. The solitons of this theory are precisely the noncommutative solitons we constructed earlier. According to Sen's conjecture, these should be the D-branes of the bosonic string theory.

The simplest NC soliton solution takes the form of $T = T_0 \phi_0(r^2)$. Where $\phi_0(r^2)$ is the gaussian localised in two of the noncommutative directions. This would be a codimension two object and a candidate $D23$ brane. It's energy is given by $\frac{2\pi\theta C}{G_s} V(T_0) \int d^{24}x \sqrt{G}$. We can now use Sen's conjecture which implies, in our convention, that $V(T_0) = 1$. In terms of this, using the dictionary between open and closed string quantities, it is then possible to verify that the energy density of the above solution is exactly that of the $D23$ brane. It is lucky that the only information needed to obtain the energy of the noncommutative soliton is the value of V at the extremum T_0 which is one piece of the potential which we have some information about from Sen's conjecture. Using noncommutativity in additional spatial directions, it is also possible to obtain branes of all even codimensions as noncommutative solitons, together with the right tension.

Moreover, by considering a projector of rank k one obtains multiple solitons which have the interpretation as multiple D-branes. The fact that their energy is k times that of a single soliton is a reflection of the fact that, in classical open string theory, D-branes exert no force on each other.

The structure of multiple noncommutative solitons now gives a nice realisation of the nonabelian spectrum of fluctuations around coincident D-branes. This essentially follows from the fact that a projector like $P_k = \sum_{i=0}^{k-1} |i\rangle\langle i|$ leaves an unbroken $U(k)$ group. The reader is referred to [10] for details.

The noncommutative solitons also exhibit the instability of the corresponding D-branes. Since the solitons correspond to an extremum of $V(T)$ which is a maximum, one finds for a rank k soliton, a tachyonic mode which transforms in the adjoint of $U(k)$. It's mass can again be compared with that on the $D23$ brane and one finds agreement.

Thus the identification of D-branes in the bosonic, as also in the type II theories, with noncommutative solitons provides a potentially powerful tool to study many properties of D-branes in a easily controlled manner. In the next section we will see another application of this philosophy.

7 Noncommutative solitons in gauge theories

We will now consider noncommutative Yang-Mills theories in which there are unstable solitons which have no counterpart in the commutative theory. The simplest theory to study will again be one in $(2+1)$ dimensions. The soliton in this system will have an interpretation as a $D0$ brane localised on a noncommutative $D2$ brane worldvolume. The process of the $D0$ brane dissolving into the $D2$ brane can be explicitly studied in the NCYM theory. There is a quartic tachyon potential which allows one to follow the condensation of the tachyon.

7.1 Noncommutative Yang-Mills

The noncommutative version of Yang-Mills theory (we will restrict ourselves to the $U(1)$ version for simplicity) can be written as

$$S = -\frac{1}{4g_{YM}^2}\int G^{\mu\rho}G^{\nu\sigma}\widehat{F}_{\mu\nu}\star\widehat{F}_{\rho\sigma}, \tag{7.1}$$

where

$$\widehat{F}_{\mu\nu} = \partial_\mu\widehat{A}_\nu - \partial_\nu\widehat{A}_\mu + i(\widehat{A}_\mu\star\widehat{A}_\nu - \widehat{A}_\nu\star\widehat{A}_\mu) \tag{7.2}$$

is the noncommutative field strength, expressed in terms of $\widehat{A}$, the noncommutative gauge field. This theory has noncommutative gauge transformations

$$\delta\widehat{A}_\mu = \partial_\mu\epsilon + i(\epsilon\star\widehat{A}_\mu - \widehat{A}_\mu\star\epsilon). \tag{7.3}$$

Note the similarity to nonabelian gauge theories and the fact that the $U(1)$ noncommutative theory is not free.

This Lagrangian can be written in terms of operators using the Weyl-Moyal correspondence. In the (2+1) dimensional case, it is useful to define

$$C_z = C = -i\widehat{A}_z + \frac{a^\dagger}{\sqrt{\theta}}, \qquad C_{\bar{z}} = \bar{C}. \tag{7.4}$$

These shifted gauge fields have the nice property that they transform covariantly under gauge transformations $\delta C = i[\epsilon, C]$ (just like covariant derivatives).

The action takes the simple form (fixing temporal gauge $A_0 = 0$)

$$-S = \frac{2\pi\theta}{g_{YM}^2} \int dt \mathrm{Tr}\left[-\partial_t \bar{C} \partial_t C + \left([C, \bar{C}] + \frac{1}{\theta} \right)^2 \right]. \tag{7.5}$$

The field strength component $F_{z\bar{z}}$ is proportional to the combination $([C, \bar{C}] + \frac{1}{\theta})$. We have "shifted away" the spatial derivative terms and obtained a matrix model-like action.

7.2 Flux lumps

The equation of motion for (7.5), for static solutions, is simply

$$[C, [C, \bar{C}]] = 0. \tag{7.6}$$

In the gauge that we are working in, we should also ensure that the Gauss' law constraint is satisfied

$$[\bar{C}, \partial_t C] + [C, \partial_t \bar{C}] = 0 \tag{7.7}$$

obtained by varying (7.5) with respect to A_0 prior to gauge fixing, and then setting $A_0 = 0$. A trivial solution to both is $C = a^\dagger$. This corresponds to the vacuum with zero gauge field. We would like to find nontrivial solutions.

A class of such solutions labelled by a positive integer $m > 0$ were found in [13] and [14].

$$C = C_0 \equiv (S^\dagger)^m a^\dagger S^m. \tag{7.8}$$

where the shift operator $S = \sum_{i=0}^{\infty} |i\rangle\langle i+1|$. S obeys

$$\begin{aligned} &SS^\dagger = 1, \quad S^\dagger S = 1 - P_0 \\ &S^m (S^\dagger)^m = 1, (S^\dagger)^m S^m = 1 - P_{m-1} \end{aligned} \tag{7.9}$$

where $P_{m-1} = \sum_{i=0}^{m-1} |i\rangle\langle i|$ is the rank m projector. One can actually can easily see that there is a generalisation of (7.8) where one adds $\sum_{a=0}^{m-1} c^a |a\rangle\langle a|$ to C_0. The c_a are arbitrary complex numbers. The matrix C_0 then takes the block form

$$C_0 = \begin{pmatrix} c_0 & 0 \\ 0 & a^\dagger \end{pmatrix} \tag{7.10}$$

where c_0 is an $m \times m$ diagonal matrix with eigenvalues c^a.

The flux operator $-iF_{z\bar{z}} = F_0$ evaluated on C_0 is given by

$$\theta F_0 = 1 + \theta[C_0, \bar{C}_0] = 1 + \theta(S^\dagger)^m [a^\dagger, a] S^m = P_{m-1}. \tag{7.11}$$

For an arbitrary configuration, the normalized integral of the flux over the z plane may be rewritten as a trace over the operator F

$$c_1 = \frac{1}{2\pi}\int F = \theta \mathrm{Tr} F; \tag{7.12}$$

from (7.11) and (7.12) C_0 carries m units of flux. Since $S^m|a\rangle = 0 = \langle a|(S^\dagger)^m$ $(a = 0 \ldots m-1)$

$$[C_0, F_0] = 0, \tag{7.13}$$

and C_0 is a static solution to the equation of motion (7.6). Its energy is

$$E = \frac{2\pi\theta}{2g_{YM}^2}\mathrm{Tr} F_0^2 = \frac{m\pi}{g_{YM}^2\theta}. \tag{7.14}$$

(If one has a nontrivial open string metric $G = G_{z\bar{z}}$, then there is an additional factor of G in the denominator.) Thus the solution we have found is finite energy with $m > 0$ units of flux.

The solution can be generalised to the supersymmetric version of the noncommutative gauge theory. The fermions are not excited in the solution. The additional scalars corresponding to transverse fluctuations can take arbitrary diagonal values. See [15] for details (also [16]), [17]).

7.3 Fluctuation spectrum

The spectrum of quadratic fluctuations about the solution C_0 is easy to compute. It is useful to parametrise the fluctuation δC in $C = C_0 + \delta C$ as

$$\delta C = \begin{pmatrix} A & W \\ \bar{T} & D \end{pmatrix}. \tag{7.15}$$

We then expand (7.5) to quadratic order in the fields A, W, T, D and diagonalize this quadratic form. We find that A is a massless field, while the spectrum of D is exactly that of the vector about the vacuum – these can be identified with the bulk modes on the D2-brane.

The interesting case is that of the offdiagonal fluctuations. One set of linear combinations of the of the W and T are pure gauge, and are set to zero by the Gauss Law constraint. The other set form a tower of states.

Each energy level of the tower has m complex fields in the fundamental of $U(m)$. The base of the tower is tachyonic with the m complex modes $\langle a|T|m\rangle$ $(a = 0 \ldots m-1)$ of $m^2 = -\frac{1}{\theta}$. Thus the solution is unstable as one might intuitively have guessed from the fact that flux prefers to be spread out to reduce energy.

The rest of the tower has a harmonic oscillator spectrum, with energies

$$m_k^2 = \frac{(2k+1)}{\theta}, \quad k = 1, 2, \ldots \tag{7.16}$$

All these modes can be identified with the modes of 0-2 strings.

Again, the transverse scalars and fermions in the supersymmetric case can also be taken into account.

7.4 Comparison with string theory

Since the $(2+1)$ dimensional noncommutative Yang-Mills theory describes the worldvolume theory of a $D2$ brane in the presence of a large B field, we expect a flux solution to be a zero brane. We can make precise tests of this hypothesis.

Firstly, one can compare the energy of a localised zero brane on the two brane with (7.14) obtained from the gauge theory. We will determine the difference between the energy of this configuration and one in which the 0-brane is completely dissolved in the 2-brane. For this purpose we work in commutative variables. Let the constant value of F be equal to B after the 0-brane has dissolved into the 2-brane. The energy of this dissolved state is

$$\begin{aligned} E &= \frac{1}{g_{str}(2\pi)^2(\alpha')^{\frac{3}{2}}} \int d^2x \sqrt{\det(g + 2\pi\alpha' B)} \\ &= \frac{1}{(2\pi)^2 g_{str}(\alpha')^{\frac{3}{2}}} \int d^2x \sqrt{g^2 + (2\pi\alpha' B)^2}. \end{aligned} \tag{7.17}$$

In the limit of large B field (7.17) may be expanded as

$$E = \frac{1}{g_{str}\sqrt{\alpha'}} \frac{1}{2\pi} \int d^2x B \left(1 + \frac{1}{2}\frac{g^2}{(2\pi\alpha' B)^2} + \ldots\right). \tag{7.18}$$

Removing a unit of D0-brane charge from the constant value of the background F field on the brane, $(\frac{1}{2\pi}\int d^2x \Delta F = -1)$ lowers the energy of the 2-brane by

$$\Delta E = \frac{1}{g_{str}\sqrt{\alpha'}} \left(1 - \frac{1}{2}\frac{g^2}{(2\pi\alpha' B)^2}\right). \tag{7.19}$$

Thus

$$E_{bind} = E(D0) - \Delta E = \frac{1}{g_{str}\sqrt{\alpha'}} - \Delta E = \frac{1}{2g_{str}\sqrt{\alpha'}}\frac{g^2}{(2\pi\alpha' B)^2}. \qquad (7.20)$$

On using the dictionary between open and closed string quantities,

$$\begin{aligned} \theta &= \frac{1}{B} \\ G &= \frac{(2\pi\alpha' B)^2}{g} \\ g_{YM}^2 &= \frac{g_{str} 2\pi(\alpha')^{\frac{1}{2}} B}{g}. \end{aligned} \qquad (7.21)$$

one finds (7.20) is precisely the same as (7.14).

In fact, the spectra of fluctuations we found in the previous subsection can be exactly matched to string theory as well. The free 0-2 conformal field theory has a hagedorn spectrum of stringy states. The moding of the $0-2$ oscillators is shifted by

$$\nu = 1 - \frac{1}{\pi b}, \qquad b = \frac{2\pi\alpha' B}{g} \qquad (7.22)$$

in the scaling limit of a large B-field. Most of the oscillator states have masses of order the string scale and thus decouple. There is however a single tower of massive string states generated by the oscillator $\alpha_{-1+\nu}$ whose energy spacing $\frac{1}{\pi\alpha' b} = \frac{1}{2G\theta}$, is the spacing of the states we found in the gauge theory. A careful analysis reveals a tachyon of exactly the expected mass, as well as the massive tower $m_k^2 = \frac{(2k+1)}{2\pi\alpha' b}$ which matches with (7.16).

7.5 Tachyon condensation

The 0-2 system we have studied in this paper also has a world-volume tachyon, and can be regarded as a toy laboratory for the more difficult and interesting $D\,\bar{D}$ system. In the 0-2 context there is a small parameter, namely the ratio of the string scale to the noncommutativity scale, which can be used to control the analysis. A similar logic was used in [18] to study tachyon condensation of a large number of $D0$ branes in a $D2$ brane.

Let us consider the case of a single flux $m = 1$. The initial state $C = (S^\dagger)a^\dagger S$ decays to the final state $C = a^\dagger$ on exciting the tachyonic mode $T = C_{1,0}$. Note that the tachyonic mode and the nonzero matrix elements in the initial and final state are all of the form $C_{i+1,i}$. One might thus suspect that it is possible to set all C matrix elements not of this form to zero through the entire process of tachyon condensation. This is indeed the case[1], as (7.5) admits a consistent truncation to these modes.

We can then easily expand the action out in terms of the fluctuations T and the $2-2$ modes $C_{i+1,i} = D_{i,i-1}$ (in the notation of (7.15)). Since the gauge theory action is quartic in the fields, the potential for T and $D's$ will also be quartic. In fact, it takes the relatively simple form (setting $\theta = 1$ temporarily for convenience: we can reinstate it at the end using dimensional analysis)

$$\begin{aligned} V &= \frac{\pi}{g_{YM}^2}\Bigg[\left[|T|^2 - 1\right]^2 + \left[|T|^2 - |D_{1,0}+1|^2 + 1\right]^2 \\ &\quad + \sum_{i=1} \left[|D_{i,i-1} + \sqrt{i}|^2 - |D_{i+1,i} + \sqrt{i+1}|^2 + 1\right]^2 \Bigg]. \end{aligned} \tag{7.23}$$

An unstable extremum of (7.23) that corresponds to an undissolved 0-brane on the 2-brane is

$$T = D_{i,i-1} = 0. \tag{7.24}$$

It decays into the stable extremum

$$|T| = 1, \qquad D_{i,i-1} = \sqrt{i+1} - \sqrt{i}. \tag{7.25}$$

It is easy to see that (7.25) corresponds to $C = a^\dagger$, the 2-brane vacuum.

It is possible to integrate out the fields $D_{i,i-1}$ and obtain the potential V as a function of the tachyon alone. Minimizing V w.r.t $D_{i,i-1}$ we find

$$|D_{i,i-1} + \sqrt{i}|^2 = |T|^2 + i \tag{7.26}$$

which sets all except the first term (7.23) to zero. Restoring θ, the potential thus takes the simple form[2]

$$V = \frac{\pi\theta}{g_{YM}^2}\left[|T|^2 - \frac{1}{\theta}\right]^2. \tag{7.27}$$

[1]In order to demonstrate this we assign the the fields C_{ij} and C^*_{km} 'angular momentum' quantum numbers $i-j$ and $m-k$ respectively. With this assignment the potential conserves angular momentum. All terms in the potential are the product of an equal number of C and a C^* fields. Angular momentum conservation prohibits linear coupling of 'other fields' to C (C^*) fields of angular momentum 1 (-1).

[2]A quartic tachyon potential was also obtained in [18], (see equation 2.8) using scattering calculations in string theory, strengthening our identification of the fluctuation modes of subsection 2.3 with 0-2 strings.

Thus we can accurately study tachyon condensation in this decoupling limit of string theory.

But as a caveat we should note that the 2-2 string modes in the CFT after tachyon condensation include all the 0-0, 0-2, 2-0, and 2-2 strings of the CFT prior to tachyon condensation. Thus, in the process of tachyon condensation, 0-0 and 0-2 modes are absorbed into the 2-2 continuum. In this respect tachyon condensation in the 0-2 system appears qualitatively different from tachyon condensation in a $p\,\bar{p}$ system. In the latter case there appears to be no continuum for the $p-\bar{p}$ modes to disappear into. Restated, the decay of the 0-brane into 'nothing' in the 0-2 system is not mysterious once the 0-brane is constructed as a soliton on the 2-brane.

8 Conclusion

We have tried to give a flavour of the physics that can be captured by the relatively elementary classical solutions of noncommutative field theories. We have seen in different contexts how these solitons are really simple manifestations of D-branes, possessing many of their important features. Though they have been primarily studied in the context of tachyon condensation, we saw that they can also shed some light on the resolution of singularities in spacetime by D-brane probes. In addition to other applications in string theory it is important at this stage to explore their presence in other systems with a strong magnetic field like the quantum hall effect.

Acknowledgments. I must thank S. Minwalla, A. Strominger, M. Aganagic, M. Headrick and M. Spradlin – my collaborators, with whom much of the work presented here was done.

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