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NEW CONSISTENCY CONDITIONS ON PION-PION AMPLITUDES AND THEIR DETERMINATION TO FOURTH ORDER IN EXTERNAL MOMENTA*

by

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ABSTRACT

We derive a set of new consistency conditions for the pion-pion scattering amplitude. These conditions hold for any s, t, u in the cube, $0 \leq s, t, u \leq \mu^2$, with the four external mass variables off-mass-shell and restricted such that $q_1^2 = 0$, $q_2^2 = s$, $q_3^2 = t$, and $q_4^2 = u$. Using these consistency conditions, we determine the coefficients of the power series expansion of the pion-pion amplitude up to and including second order terms in the variables s, t, u, and q_i^2. We use this expansion to calculate the pion-pion S-wave scattering lengths and thus check the consistency of Weinberg's recent calculation of these numbers to one higher order. The final result is to within 10% the same as that obtained by Weinberg.

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I. INTRODUCTION

In a recent paper Weinberg[1] has used current algebra to calculate the pion-pion, S-wave scattering lengths. The answer he obtained is smaller by at least a factor of five from what had been believed to be reasonable estimates of the $\pi\pi$ scattering lengths from dispersion theory or comparison of peripheral models with experiment.

Weinberg's result does not follow from current algebra alone. The restrictions given by current algebra and PCAC on the $\pi\pi$ amplitude give us information at unphysical points and unphysical external masses. The problem is to extrapolate these results to the physical threshold. This is relatively easy in the case of πN scattering where there is a small number, μ/M, and where one neglects terms of order μ^2/M^2, etc. For $\pi\pi$ scattering there is no such number. What Weinberg does to effect an extrapolation is to expand the amplitude in a power series of s, t, u, and the external mass variables, q_i^2 , $i = 1,2,3,4$, and keep terms only up to first order in these variables. One can then determine the three coefficients in the expansion from Adler's consistency condition and a low energy theorem for $\pi\pi$ scattering. Once the coefficients are known one assumes the expansion is still good up to threshold and calculates the scattering lengths.

Such a method of extrapolation is rather dangerous. It is known that the expansion used is divergent at threshold. One can get around this difficulty by assuming that the unitarity branch point is a weak singularity which allows us to use the expansion at least as an asymptotic expansion up to and maybe a little beyond threshold. Since Weinberg gets small scattering lengths in the end his argument is self-consistent, but it does not in fact prove that the scattering lengths are indeed small. Even if one accepts the asymptotic nature of the expansion one does not a priori know at what order does it give a good approximation to the amplitude near threshold. There is no a priori reason for example to assume that the second order terms, s^2, st, u^2, etc. are smaller than the first order terms. One would feel much more at ease with Weinberg's results if one is able to calculate these higher order terms and compare them with the lower order ones. This becomes even more pertinent when we recall that the results of Ref. 1 give much smaller scattering lengths than had been expected from previous arguments.

In this paper we shall derive a set of new consistency conditions on the $\pi\pi$ amplitude that hold in addition to the Adler[2] consistency condition. We shall then use these consistency conditions to determine the coefficients of the expansion of the $\pi\pi$-amplitude to second order in the variables s, t, u, and q_i^2. The remarkable result is that the second order terms turn out to be negligible and Weinberg's results are essentially unchanged within our approximations.

Adler has derived consistency conditions on πN and $\pi\pi$ scattering which hold with one pion taken off-mass shell.[2] If one tries to derive a consistency condition for πN scattering with two pions off the mass shell then one has to estimate the matrix element of a scalar density between two nucleon states.[3] This scalar density essentially arises from the equal time commutator of the axial vector charge with the divergence of the axial vector current. Thus as in Ref. 3 one does not get a new consistency condition but a relation between the scalar matrix element and πN scattering.

In the case of $\pi\pi$ scattering it turns out that one can essentially eliminate the matrix element of the scalar density between two single pion states and get new and stronger consistency conditions. The main new tool that one needs to do this is to know the equal time commutator of the axial vector charge with the scalar density. There are several ways to do this all leading to the same answer for our purposes. One can use directly the commutators of the axial vector charge with the scalar densities, u_i, and the pseudoscalar densities, v_i, given by Gell-Mann.[4] We can then use the densities v_i, $i = 1,2,3$, as an interpolating field for the pion. It is reasonable to assume that these pseudoscalar densities are smooth interpolating fields like $\partial_\mu A_i^\mu$ and allow us to make extrapolations off the mass-shell of the order of the pion mass without introducing large errors. In fact in some models like the quark-model (or the σ-model), $\partial_\mu A_i^\mu$ is just proportional to v_i. In a general quark model $\partial_\mu A_i^\mu$ is proportional to v_i plus SU(3) breaking terms. Anyway, no one ever proved that $\partial_\mu A_i^\mu$ was a good interpolating field. This was just verified by experience starting with the success of the Goldberger-Treiman formula. In the same way one can only verify whether v_i are good interpolating fields by the results of using them as such. One can easily see, for example, that the Adler consistency condition for πN scattering follows also from using v_i as an interpolating field for one of the pions and $\partial_\mu A_i^\mu$ for the other and the commutation relation (1).

If one does not like to introduce a new interpolating field one can get results identical to ours in the following way: First, one uses the commutator of $A_i^o(\vec{x},t)$ with $\partial_\mu A_j^\mu$ to define a scalar density. One assumes this scalar density is a local field. To compute the commutator of the scalar density with the axial charge one uses the Jacobi-identity to get a result essentially identical to our Eq. (2'). In this way one would just have to replace v_i by $\partial_\mu A_i^\mu$ wherever it appears in our paper and the results will be the same.

In Section II we shall derive a new consistency condition on $\pi\pi$ scattering with two pions taken with zero external mass. We also show how one can get Adler's consistency condition using our methods. These two consistency conditions are used to calculate the coefficients in the Weinberg expansion up to first order, to verify that our method gives the same results.

In Section III, starting with a reduction formula for the $\pi\pi$ amplitude in which all four pions are reduced out, we derive a general consistency condition on the amplitude. This consistency condition does not only hold at one point in the six dimensional space of the off-shell $\pi\pi$ variables, but holds for all s, t, u in the domain $0 \leq s, t, u \leq \mu^2$, with the external masses restricted such that $q_1^2 = 0$, $q_2^2 = s$, $q_3^2 = t$, $q_2^2 = u$. All four external mass variables are taken off-mass shell.

Finally, in Section IV we use this general consistency condition to evaluate all but one of the coefficients of the expansion of the $\pi\pi$ amplitudes up to and including second order in s, t, u, and q_i^2. We then give arguments to show that the one coefficient left undetermined is small. Our final result is that all the first order coefficients remain the same as in Ref. 1, and all the second order ones are within our approximations negligible. Even if we carry over some correction terms to our main approximation we find that they only change Weinberg's value for the scattering lengths by five present.

II. A NEW CONSISTENCY CONDITION ON THE PION-PION AMPLITUDE

In order to clarify our method we shall in this section derive a consistency condition on the $\pi\pi$ amplitude with two pions taken with zero external mass. Our main point is to show how one can get a consistency condition on all three $\pi\pi$ amplitudes which unlike the πN case does not depend on the matrix elements of the scalar densities. We then show how this consistency condition when coupled with Adler's consistency condition will lead to Weinberg's scattering lengths.

Our starting point shall be the commutation relations of the axial-vector charge with the scalar and pseudoscalar densities given by Gell-Mann in Ref. 4,

$$\left[Q_i^A(t),\ v_j(\vec{x},t)\right] = i\, d_{ijk}\, u_k\,(\vec{x},t), \tag{1}$$

$$\left[Q_i^A(t),\ u_j(\vec{x},t)\right] = -\,i\, d_{ijk}\, v_k(\vec{x},t)\ ;\ i,j,k = 0,1,\ldots,8; \tag{2}$$

where

$$Q_i^A = \int d^3x\, A_i^o\,(\vec{x},t)\ , \tag{3}$$

and $A_i^\mu(x)$ is the usual axial vector current. In a quark model u_i and v_i are given by

$$u_i = \frac{1}{2}\bar{t}\ \lambda_i t\ ;\quad v_i = -\frac{i}{2}\,\bar{t}\ \gamma_5\lambda_i t,\ i = 0,1,\ldots,8\ . \tag{4}$$

Most of the results obtained from PCAC or current algebra follow from using $\partial_\mu A_\alpha^\mu$ as an interpolating field for the pion. The success of PCAC strongly suggests that $\partial_\mu A_\alpha^\mu$ is a good interpolating field in the sense that it allows us to go off the mass shell by an amount of the order of the mass of the pion without introducing large errors. One can also use v_α, $\alpha = 1,2,3,$ as an interpolating field for the pion. This would not make any fundamental difference for the results derived in this paper, but it will we think make certain points clearer. We would expect v_α to be also a good interpolating like $\partial_\mu A_\alpha^\mu$ since in models like the quark model[4] $\partial_\mu A_\alpha^\mu$ is proportional to v_α plus SU(3) breaking terms. (In one specific quark model where the symmetry breaking Hamiltonian is proportional to u_8, $\partial_\mu A_\alpha^\mu$ is proportional to v_α for $\alpha = 1,2,3$.) The only problem with using v_α is that we do not know its normalization to the one pion state. We shall see how we can get around this problem by using Eqs. (1) and (2) together and getting a relation in which the unknown normalization of the v_α's is cancelled by the unknown $\pi\pi$ scalar vertex.

Since in this paper we shall deal only with pions, i, j, k = 1, 2, 3, we simplify Eq. (1) and Eq. (2) by first defining the scalar density $\sigma(x)$ as

$$\sigma(x) \equiv \sqrt{\frac{2}{3}}\, u_o + \sqrt{\frac{1}{3}}\, u_8 \, . \tag{5}$$

Then instead of Eq. (1) and Eq. (2) we have

$$\left[Q^A_\alpha(t),\ v_\beta(\vec{x},t)\right] = i\, \delta_{\alpha\beta}\, \sigma(\vec{x},t)\ , \tag{1'}$$

$$\left[Q^A_\alpha(t),\ \sigma(\vec{x},t)\right] = -i\, \delta_{\alpha\beta}\, v_\beta(\vec{x},t),\ \alpha,\beta = 1,2,3 \tag{2'}$$

These last two commutation relations are the only ones we shall use in this paper. We should perhaps remind the reader that Eqs. (1') and (2') are also true in the σ-model if one identifies v_α with the unrenormalized pion field and σ with the unrenormalized σ field. We stress here that our final results will not depend on the σ field or its matrix elements.

As we mentioned in the introduction one can avoid using the v_α's and use $\partial_\mu A^\mu_\alpha$ in their place in the following way. First, one <u>defines</u> a new σ' from the commutation relation, $\left[Q^A_\alpha(t),\ \partial_\mu A^\mu_\beta(x,t)\right] \equiv i\, \delta_{\alpha\beta}\, \sigma'(x,t)$, and assumes that this σ' is a local field. To calculate the commutator $\left[Q^A_\alpha(t),\ \sigma'(x,t)\right]$ one now uses the Jacobi identity and the known commutator $\left[Q^A_\alpha(t),\ Q^A_\beta(t)\right]$ to get a result similar to Eq. (2), $\left[Q^A_\alpha(t),\ \sigma'(x,t)\right] = -i\, \delta_{\alpha\beta}\, \partial_\mu A^\mu_\beta(x,t)$.[5]

We define the normalization constant, a_π, of the v_α field as

$$<0\left|v_\alpha(0)\right|\pi_\beta(q)> = \frac{1}{(2\pi)^{3/2}}\,\frac{1}{\sqrt{2q^o}}\, a_\pi\, \delta_{\alpha\beta} \tag{6}$$

In our reduction formulae we shall make both the replacements

$$\partial_\mu A^\mu_\alpha(x) \longrightarrow c_\pi \mu^2 \phi_\alpha(x)\ , \qquad c_\pi = \frac{M_N\, g^A}{G_{\pi NN}} \tag{7}$$

$$v_\alpha(x) \longrightarrow a_\pi \phi_\alpha(x) \tag{8}$$

If we identify v_α with $\partial_\mu A^\mu_\alpha$ as in Ref. 3, then in that case $a_\pi = c_\pi m_\pi^2$. In that case Eq. (1) and Eq. (2) will remain unchanged with σ replaced by some σ'. Since we are only interested in the relative normalizations of v_α and σ we shall not worry about cases where a_π is zero and deal with a_π as if it were finite. This will not affect our final answers.

Our first step is to relate a_π to the $\sigma\pi\pi$ vertex by the usual Fubini-Furlan trick.[6] We write

$$< \pi_\alpha(k) \left|\sigma(0)\right| \pi_\beta(q) > = \frac{1}{(2\pi)^3} \frac{1}{\sqrt{4k^o q^o}} \delta_{\alpha\beta} f^\sigma(q^2, k^2; (q-k)^2). \tag{9}$$

From LSZ we can express f^σ as

$$f^\sigma\left(q^2, k^2; (q-k)^2\right) \delta_{\alpha\beta} \frac{c_\pi \mu^2}{(2\pi)^{3/2}\sqrt{2q^o}}$$

$$= i(\mu^2 - k^2)\int d^4x\, e^{ik\cdot x} < 0\left|T\left(\partial_\mu A^\mu_\alpha(x)\, \sigma(0)\right)\right| \pi_\beta(q) > \tag{10}$$

This is an identity as $k^2 \to \mu^2$ and the usual PCAC tells us that f^σ is a slowly varying function as k^2 varies from $k^2 = \mu^2$ to $k^2 = 0$. Integrating Eq. (10) by parts we get the identity

$$\delta_{\alpha\beta} f^\sigma\left(q^2, k^2; (q-k)^2\right) \frac{c_\pi \mu^2}{(2\pi)^{3/2}\sqrt{2q^o}}$$

$$= k_\mu(\mu^2 - k^2)\int d^4x\, e^{ik\cdot x} < 0\left|T(A^\mu_\alpha(x)\, \sigma(0)\right| \pi_\beta(q) >$$

$$- i(\mu^2 - k^2)\int d^4x\, e^{ik\cdot x}\, \delta(x_o) < 0\left|\left[A^o_\alpha(x), \sigma(0)\right]\right| \pi_\beta(q) > \tag{11}$$

In the limit as $k_\mu \to 0$ the first term on the right is zero. The second term using Eq. (2) gives as $k_\mu \to 0$ and q remains on shell,

$$f^\sigma(\mu^2, 0; \mu^2) = - a_\pi / c_\pi \tag{12}$$

where the first two variables in f^σ always refer to the external masses of the pions in the $\sigma\pi\pi$ vertex and the third variable is the momentum transfer variable. The constant a_π was defined in Eq. (6) and c_π is the pion decay form factor which if one uses the Goldberger-Treiman formula is $c_\pi = M_N g^A / G_{\pi NN}$. Both a_π and f^σ are in general unknown but the relation Eq. (12) will help us eliminate them from our final answers as seen below. (If one chooses $v_\alpha \equiv \partial_\mu A^\mu_\alpha$ then in that specific case $f^\sigma = - m_\pi^2$.)

To get our consistency condition we define the off-mass-shell invariant $\pi\pi$ amplitude by

$$i\, M(q_4\ \delta, q_3\gamma; q_2\beta, q_1\alpha)\ \frac{c_\pi \mu^2 a_\pi}{(2\pi)^3 \sqrt{4 q_4^0\, q_2^0}} \tag{13}$$

$$= (\mu^2 - q_3^2)(\mu^2 - q_1^2) \int d^4x\, e^{-iq_1 \cdot x} < \pi_\delta(q_4) \left| T\left(\partial_\mu A^\mu_\alpha(x) v_\gamma(0)\right) \right| \pi_\beta(q_2) >$$

Here $q_2^2 = q_4^2 = \mu^2$ and will not be varied in this section. As $q_3^2 \to \mu^2$ and $q_1^2 \to \mu^2$, M as defined in Eq. (13) is guaranteed by the LSZ formalism to give the correct $\pi\pi$ amplitude, assuming we have chosen a $\partial_\mu A^\mu_\alpha$ and v_γ that are relatively local. (We have factored out the energy momentum conserving δ function and $q_3 \equiv q_1 + q_2 - q_4$.)

Integrating the right hand side of Eq. (13) by parts, we get

$$i\, M(q_4\delta,\ q_3\gamma; q_2\beta,\ q_1\alpha)\ \frac{c_\pi \mu^2 a_\pi}{(2\pi)^3 \sqrt{4q_4^o\, q_2^o}} = i\, q_{1\mu}(\mu^2 - q_1^2)\,(\mu^2 - q_3^2)$$

$$\int d^4x\, e^{-iq_1 \cdot x} < \pi_\delta(q_4) \left| T\left(A_\alpha^\mu(x) v_\gamma(0)\right)\right| \pi_\beta(q_2) >$$

$$- (\mu^2 - q_1^2)\,(\mu^2 - q_3^2) \int d^4x\, e^{-iq_1 \cdot x} < \pi_\delta(q_4)$$

$$\cdot \left| \left[A_\alpha^o(x),\ v_\gamma(0)\right]\right| \pi_\beta(q_2) > \delta(x_o) \ . \tag{14}$$

We now let both $q_1 \rightarrow 0$ and $q_3 \rightarrow 0$. The first term will be zero and the second term will after using Eq. (2) and Eq. (9) give us,

$$\lim_{q_1 \rightarrow 0}\ \lim_{q_3 \rightarrow 0}\ M\,(q_4\delta,\ q_3\gamma;\ q_2\beta,\ q_1\alpha)\ c_\pi \mu^2 a_\pi = - \mu^4\, f^\sigma(\mu^2,\ \mu^2;\ 0)\ \delta_{\alpha\gamma}\ \delta_{\beta\delta} \ . \tag{15}$$

Let us assume that f^σ is a slowly varying function of the external pion masses as q^2 varies between zero and μ^2 and the same for the transfer variable, and write

$$f^\sigma(\mu^2,\ 0; \mu^2) \cong f^\sigma(\mu^2,\ \mu^2;\ 0)\ . \tag{16}$$

We shall justify this approximation in detail at the end of the next section. As far as varying the external mass variables are concerned this is just the usual PCAC assumption. Varying the third variable, i.e. - the one in the σ channel, could be more dangerous and we shall study it in detail later.

With Eq. (16) we can use Eq. (12) to eliminate f^{σ} and a_{π} from Eq. (15) and get

$$\lim_{q_1 \to 0} \lim_{q_3 \to 0} M\,(q_4\delta,\ q_3\gamma;\ q_2\beta,\ q_1\alpha) = \frac{\mu^2}{c_\pi^2} \cdot \delta_{\alpha\gamma}\,\delta_{\beta\delta} \tag{17}$$

We recall the iso-spin decomposition of M into the three amplitudes A, B, and C given by

$$M\,(q_4\delta,\ q_3\gamma;\ q_2\beta,\ q_1\alpha) = A\ \delta_{\alpha\beta}\,\delta_{\gamma\delta} + B\ \delta_{\alpha\gamma}\,\delta_{\beta\delta} + C\ \delta_{\alpha\delta}\,\delta_{\beta\gamma} \tag{18}$$

where

$$A = A\left(s,t,u;\ q_1^2,\ q_2^2,\ q_3^2,\ q_4^2\right)\ ,\ \text{etc.}, \tag{19}$$

and

$$\begin{aligned} s &= (q_1 + q_2)^2\ , \\ t &= (q_1 - q_3)^2\ , \\ u &= (q_1 - q_4)^2\ , \\ s + t + u &= \sum_{i=1}^{4} q_i^2 \end{aligned} \tag{20}$$

In terms of A, B. and C our consistency condition in Eq. (17) becomes

$$\begin{aligned} &A\left(s = \mu^2,\ t = 0,\ u = \mu^2;\ q_1^2 = 0,\ q_2^2 = \mu^2,\ q_3^2 = 0,\ q_4^2 = \mu^2\right) = 0\ , \\ &B\left(\mu^2,\ 0,\ \mu^2;\ 0,\ \mu^2,\ 0,\ \mu^2\right) = \mu^2 / c_\pi^2 \\ &C\left(\mu^2,\ 0,\ \mu^2;\ 0,\ \mu^2,\ 0,\ \mu^2\right) = 0\ . \end{aligned} \tag{21}$$

The Adler-Weisberger sum rule for $\pi\pi$ scattering also has two external pion momenta taken to zero. However, it essentially gives a consistency condition on the derivative of the add $\pi\pi$ amplitude at $\nu = 0$.

One could easily repeat our calculation to get Adler's consistency condition[2] for the $\pi\pi$ amplitude with $q_1 \to 0$ and q_2, q_3, q_4 all on the mass shell. We shall not do this here since the Adler consistency condition will be a special case of the general consistency condition to be derived in the next section. Adler's consistency condition gives

$$A\ (\mu^2, \mu^2, \mu^2; 0, \mu^2, \mu^2, \mu^2) = B = C = 0 \tag{22}$$

To go from Eq. (21) and Eq. (22) to a statement about physical quantities such as scattering lengths one has to go through extrapolations which at first sight would seem quite dangerous. Weinberg's method of extrapolation consisted of expanding A, B. and C in powers of s, t, u, and q_i^2 and keeping terms only up to first order in these variables. Crossing symmetry and Bose statistics require the off-mass-shell amplitude to have an expansion of the form

$$\begin{aligned} A &= a + b(t+u) + cs + 0(s^2, st, \ldots, q_i^2 q_j^2, \ldots, q_i^4, \ldots) \ , \\ B &= a + b(s+u) + ct + \ldots \ , \\ C &= a + b(s+t) + cu + \ldots \ , \end{aligned} \tag{23}$$

The main point here is that in Eq. (23) there could be no first order terms in the q_i^2 variables.

In this approximation one can use Eq. (21) and Eq. (22) to determine a, b, and c. From Eq. (21) we get two equations

$$\begin{aligned} a + \mu^2 b + \mu^2 c &= 0 \ , \\ a + 2\mu^2 b &= \mu^2 / c_\pi^2 \ , \end{aligned} \tag{24}$$

and from Adler's consistency condition , Eq. (22), we have

$$a + 2\mu^2 b + \mu^2 c = 0 . \tag{25}$$

The solution of Eq. (24) and (25) is

$$a = \mu^2/c_\pi^2 \; ; \; b = 0 \; ; \; c = -1/c_\pi^2 \; ; \tag{26}$$

where $c_\pi = M_N g^A/G_{\pi NN}$. This is the same as the result obtained by Weinberg,[1] where in his notation $c_\pi = F_\pi/2$. If one uses Eq. (23) to give the amplitude at threshold one gets the scattering lengths given in Ref. 1.

However, there are several troubles with the expansion in Eq. (23). First, it is known to be divergent at threshold. Weinberg gets around this difficulty by assuming that the unitarity branch point is a weak singularity which allows him to use Eq. (23) at least as an asymptotic expansion up to and somewhat beyond threshold. Since he gets small scattering lengths in the end this shows that his argument is self-consistent, but does not prove that the scattering lengths are indeed small.

The strong consistency condition which we shall obtain in the next section will enable us to calculate the coefficients of the power series expansion up to second order in s,t,u, and q_i^2 . The remarkable result is that all the second order coefficients are not only small but also negligible within our approximation.

III. A GENERAL CONSISTENCY CONDITION ON THE PION PION AMPLITUDE

In this section we shall extend our method to get a general consistency condition on the $\pi\pi$ amplitude which gives restrictions not only at one point in the six dimensional space of the $\pi\pi$ scattering off-shell variables, but in a three dimensional region.

We write for the off-shell π-π amplitude the following reduction formula

$$- i (2\pi)^4 \delta(q_1 + q_2 - q_3 - q_4) M (q_4\delta, q_3\gamma; q_2\beta, q_1\alpha) a_\pi^3 c_\pi \mu^2$$

$$= \left(\prod_{i=1}^{4} (\mu^2 - q_i^2) \right) \int d^4x_1 \ldots d^4x_4 \exp(-iq_1 \cdot x_1 - iq_2 \cdot x_2 + iq_3 \cdot x_3 + iq_4 \cdot x_4)$$

$$\times < 0 \left| T\left(\partial_\mu A_\alpha^\mu (x_1) v_\beta(x_2) v_\gamma(x_3) v_\delta(x_4) \right) \right| 0 > . \tag{27}$$

Again in the limit where all $q_i^2 \to \mu^2$, M as defined above gives the exact $\pi\pi$ amplitude.

If we integrate Eq. (27) by parts we get the identity

$$- i (2\pi)^4 \delta(q_1 + q_2 - q_3 - q_4) M (q_4\delta, q_3\gamma; q_2\beta, q_1\alpha) a_\pi^3 c_\pi \mu^2$$

$$= iq_{1\mu} \left(\prod_{i=1}^{4} \left(\mu^2 - q_i^2\right) \right) \int d^4x_1 \ldots d^4x_4 \exp(-iq_1 \cdot x_1 - iq_2 \cdot x_2 + iq_3 \cdot x_3 + iq_4 \cdot x_4)$$

$$\times < 0 \left| T\left(A_\alpha^\mu (x_1) v_\beta(x_2) v_\gamma(x_3) v_\delta(x_4) \right) \right| 0 >$$

$$- \left(\prod_{i=1}^{4} \left(\mu^2 - q_i^2\right) \right) \int d^4x_1 \ldots d^4x_4 \, \delta(x_1^o - x_2^o) \exp(-iq_1 \cdot x_1 - iq_2 \cdot x_2 + iq_3 \cdot x_3 + iq_4 \cdot x_4)$$

$$\cdot < 0 \left| T\left(\left[A_\alpha^o (x_1), v_\beta(x_2) \right] v_\gamma(x_3) v_\delta(x_4) \right) \right| 0 > \tag{28}$$

- permutations of the last term over the v's.

In the limit $q_1 \to 0$ the first term in Eq. (28) vanishes. The other three terms, after using the equal time commutation relation, Eq. (1'), will give us three terms proportional to the $\sigma\pi\pi$ vertex.

We obtain

$$-i \lim_{q_1 \to 0} M(q_4\delta, q_3\gamma; q_2\beta, q_1\alpha)\, a_\pi^3 c_\pi$$

$$= i(\mu^2 - q_2^2)\, a_\pi^2 f^\sigma\left(q_3^2, q_4^2; (q_3+q_4)^2\right)\delta_{\alpha\beta}\delta_{\gamma\delta}$$

$$+ i(\mu^2 - q_3^2)\, a_\pi^2 f^\sigma\left(q_2^2, q_4^2; (q_2-q_4)^2\right)\delta_{\alpha\gamma}\delta_{\beta\delta} \tag{29}$$

$$+ i(\mu^2 - q_4^2)\, a_\pi^2 f^\sigma\left(q_2^2, q_3^2; (q_2-q_3)^2\right)\delta_{\alpha\delta}\delta_{\beta\gamma}\ ,$$

where to obtain Eq. (29) we have used the identity

$$a_\pi^2\ f^\sigma\left(q^2, k^2; (q-k)^2\right)\delta_{\alpha\beta}$$

$$= -(\mu^2 - k^2)(\mu^2 - q^2)\int d^4x\, d^4y\ e^{-iq\cdot x}\, e^{+ik\cdot y} < 0\left|T\left(\sigma(0) v_\alpha(x) v_\beta(y)\right)\right| 0 > . \tag{30}$$

This follows from applying the reduction formula directly to Eq. (9).[7]

In the limit as $q_1^\mu \to 0$ we have the following relations between the six variables of $\pi\pi$ scattering,

$$q_2 = q_3 + q_4\ ; \tag{31}$$

and hence when $q_1^\mu \equiv 0$,

$$s = (q_3 + q_4)^2 = q_2^2\ ,$$

$$t = (q_2 - q_4)^2 = q_3^2 \tag{32}$$

$$u = (q_2 - q_3)^2 = q_4^2\ .$$

Thus Eq. (20) becomes

$$\lim_{q_1 \to 0} M(q_4\delta, q_3\gamma; q_2\beta, q_1\alpha) = -\frac{1}{c_\pi}(\mu^2 - s)\frac{f^\sigma(t,u;s)}{a_\pi}\delta_{\alpha\beta}\delta_{\gamma\delta}$$

$$-\frac{1}{c_\pi}(\mu^2 - t)\frac{f^\sigma(s,u;t)}{a_\pi}\delta_{\alpha\gamma}\delta_{\beta\gamma} \tag{33}$$

$$-\frac{1}{c_\pi}(\mu^2 - u)\frac{f^\sigma(s,t;u)}{a_\pi}\delta_{\alpha\delta}\delta_{\beta\gamma} \quad .$$

We now use Eq. (12) to eliminate a_π from Eq. (33) and get a relation between the off-shell $\pi\pi$ amplitudes and the $\sigma\pi\pi$ vertex. In terms of the amplitudes A, B, and C we now have

$$A(s,t,u; q_1^2 = 0, q_2^2 = s, q_3^2 = t, q_4^2 = u) = \frac{1}{c_\pi^2}(\mu^2 - s)\frac{f^\sigma(t,u;s)}{f^\sigma(\mu^2, 0; \mu^2)} ,$$

$$B(s,t,u; q_1^2 = 0, q_2^2 = s, q_3^2 = t, q_4^2 = u) = \frac{1}{c_\pi^2}(\mu^2 - t)\frac{f^\sigma(s,u;t)}{f^\sigma(\mu^2, 0; \mu^2)} , \tag{34}$$

$$C(s,t,u; q_1^2 = 0, q_2^2 = s, q_3^2 = t, q_4^2 = u) = \frac{1}{c_\pi^2}(\mu^2 - u)\frac{f^\sigma(s,t;u)}{f^\sigma(\mu^2, 0; \mu^2)} .$$

The functions f^σ are by definition symmetric in the first two variables, i.e. - the external pion masses, so Eq. (34) is manifestly crossing symmetric. What we have succeeded in doing so far is to show that when $q_1^\mu \equiv 0$, then if one sets the other three external mass variables equal to s,t,u respectively, one gets a relation between the off-shell amplitude and a ratio of the $\sigma\pi\pi$ vertex at two different points. Thus the problem reduces to a study of how fast f^σ varies in all three of its variables.

We shall restrict ourselves to the domain $0 \leq s,t,u \leq \mu^2$ and show that in this region

$$\frac{f^{\sigma}(t,u;s)}{f^{\sigma}\left(\mu^2, 0;\mu^2\right)} \cong 1 \ ; \ 0 \leq s,t,u \leq \mu^2 . \tag{35}$$

The fact that $f^{\sigma}(x,y;z)$ is slowly varying in the first two variables is actually part of the PCAC assumption (or the assumption that v is a smooth interpolating field) as long as x and y do not vary much from their on mass-shell value, $x = y = \mu^2$. This can be justified by pion pole dominance arguments similar to those used by Weisberger.[8] For example if we let g^{σ} be given by

$$g^{\sigma}\left(\mu^2, k^2; (q-k)^2\right)\frac{\delta_{\alpha\beta}}{(2\pi)^{3/2}\sqrt{2q^0}} = i\int e^{ik\cdot x}\, d^4x \ < 0\left|T\left(\sigma(0)\phi_{\alpha}(x)\right)\right| \pi_{\beta}(q) > ,$$

$$q^2 = \mu^2 . \tag{36}$$

Then as a function of k^2, for fixed $(q-k)^2$, g^{σ} has a pole at $k^2 = \mu^2$ and the residue of that pole is just $f^{\sigma}(\mu^2, \mu^2; s)$, where $s \equiv (q-k)^2$. The PCAC assumption tells us that for $0 < k^2 < \mu^2$, and s fixed and small, the pion pole term dominates over contributions from other singularities in the k^2 plane. We get

$$g^{\sigma}(\mu^2, k^2; s) \cong \frac{f^{\sigma}(\mu^2, \mu^2; s)}{k^2 - \mu^2} , \quad 0 \leq k^2 \leq \mu^2 . \tag{37}$$

But comparing Eq. (36) with Eq. (10) we get

$$(\mu^2 - x)\, g^{\sigma}(\mu^2, x; s) = f^{\sigma}(\mu^2, x; s) ,$$

and hence

$$f^{\sigma}(\mu^2, x; s) \cong f^{\sigma}(\mu^2, \mu^2; s); \ 0 \leq x \leq \mu^2 . \tag{38}$$

Extrapolation in the other pion mass variable can be handled in the same way. To a good approximation we can therefore write

$$f^{\sigma}(x,y;s) \cong f^{\sigma}(\mu^2,\mu^2;s), \; 0 \leq x, \; y \leq \mu^2 . \tag{39}$$

The behavior of f^{σ} in the third variable, the one corresponding to the square of the σ four momentum, could in principle be much more dangerous. Indeed one would argue that a strong $\pi\pi$ S-wave, I = 0, interaction could give the vertex $f^{\sigma}(\mu^2, \mu^2; s)$ a large derivative in s at s = 0. Fortunately, dispersion theory gives us a fairly reliable way of estimating the effect of rescattering on a vertex. The Omnes formula for f^{σ} would give us

$$\frac{f^{\sigma}(\mu^2, \mu^2; s)}{f^{\sigma}(\mu^2, \mu^2; 0)} = \exp\left[\frac{s}{\pi}\int_{4\mu^2}^{\infty} \frac{\delta_0^0(s')}{s'(s'-s)}\, ds'\right] , \tag{40}$$

where δ_0^0 is the S-wave, I = 0, $\pi\pi$ phase shift. The slope of $f^{\sigma}(\mu^2,\mu^2;s)$ at $s = 0$ could be large either because of a large scattering length or because of a low mass resonance in the $\ell = 0$, I = 0, channel. Let us first estimate the effect of a scattering length on the slope. Starting with $\sqrt{\frac{s-4\mu^2}{s}} \cot \delta_0^0 = 1/a_0\mu$, we use the expression $\delta_0^0(s) \cong a_0\mu \sqrt{\frac{s-4\mu^2}{s}}$ in Eq. (40) and obtain for the derivative of f^{σ},

$$\frac{1}{f^{\sigma}(\mu^2,\mu^2,0)} \cdot \frac{df^{\sigma}}{ds}(\mu^2,\mu^2;s)\Big|_{s=0} \cong \frac{a_0\mu}{\pi}\int_{4\mu^2}^{\infty} \frac{\sqrt{s'-4\mu^2}}{\sqrt{s'}\;s'^2}\, ds' \cong \frac{a_0}{6\pi\mu} .$$

Thus the ratio in Eq. (35) is approximately given by

$$\frac{f^{\sigma}(t,u;s)}{f^{\sigma}(\mu^2,0;\mu^2)} \cong \frac{f^{\sigma}(\mu^2,\mu^2;s)}{f^{\sigma}(\mu^2,\mu^2;\mu^2)} \cong 1 + \frac{a_0}{6\pi\mu}(s-\mu^2) \; ; \; 0 \leq s \leq \mu^2 . \tag{42}$$

We note that the form we have used for δ_0^0 in Eq. (41) does not vanish as $s \to \infty$ as it would have if we had included an effective range. This makes our correction term in Eq. (42) larger than it actually is. Nevertheless, we easily see that even if a_0 is as large as μ^{-1}, the correction term in Eq. (42) is at most $1/6\pi = 0.05$, as s varies in the interval $0 \leq s \leq \mu^2$. In the next section we shall keep the second term on the right in Eq. (42) in our calculation of the scattering lengths and show that it only changes Weinberg's result by a few percent. Even including these corrections our final result for a_0 will still be $a_0 \cong 0.20\, \mu^{-1}$. For the region $0 \leq s,t,u \leq \mu^2$ one can thus safely neglect the second term in Eq. (42).

If there exists an actual σ-resonance, in the $\ell = 0$, $I = 0$ channel, then the correction to Eq. (35) will be of the form

$$\frac{f^{\sigma}(\mu^2,\mu^2;s)}{f^{\sigma}(\mu^2,\mu^2;0)} \cong 1 + (s-\mu^2)\, 0\left(\frac{\mu^2}{m_\sigma^2}\right);\ 0 \leq s,\ t,\ u \leq \mu^2 . \tag{43}$$

There seems to be no evidence for a narrow ($\Gamma < 100$ MeV) σ particle with mass lower than 600 MeV.[9] Thus we can also neglect the correction term in Eq. (43). The only possibility left is for a very broad $\pi\pi$ resonance in the region below 600 MeV. But the effect of such a broad resonance ($\Gamma > 200$ MeV) on the slope of f^σ at $s = 0$ will be very similar to that of a large scattering length which we have already shown does not affect our results appreciably.

The consistency condition in Eq. (34) can now be written as

$$\begin{aligned} &A\,(s,t,u;\ q_1^2 = 0,\ q_2^2 = s,\ q_3^2 = t,\ q_4^2 = u) \cong \frac{1}{c_\pi^2}(\mu^2 - s);\ 0 \leq s,t,u \leq \mu^2 ; \\ &B\,(s,t,u;\ q_1^2 = 0,\ q_2^2 = s,\ q_3^2 = t,\ q_4^2 = u) \cong \frac{1}{c_\pi^2}(\mu^2 - t)\ , \\ &C\,(s,t,u;\ q_1^2 = 0,\ q_2^2 = s,\ q_3^2 = t,\ q_4^2 = u) \cong \frac{1}{c_\pi^2}(\mu^2 - u)\ . \end{aligned} \tag{44}$$

As we have mentioned earlier these consistency conditions are much stronger than the usual ones which hold only for one point; these hold for any s, t, u that lie in the cube $0 \leq s, t, u \leq \mu^2$, if the masses are restricted as in (44).

IV. THE POWER SERIES EXPANSION OF THE PION PION AMPLITUDE

We use the consistency condition (44) to calculate the $\pi\pi$ amplitude up to second order in the variables s, t, u and q_i^2.

We expand A, B, and C in a power series of the variables s, t, u, q_i^2, where $u = \sum q_i^2 - s - t$. To second order in these variables, crossing symmetry and Bose statistics require the expansion to take the form

$$A(s, t, u; q_1^2, q_2^2, q_3^2, q_4^2)$$

$$= a + b(t+u) + cs + d(t+u)^2 + etu + fs^2 + g(t+u)\, s + h \sum_{\substack{i \neq j \\ i > j}} q_i^2\, q_j^2; \tag{45}$$

and B and C are obtained by exchanging s and t in (45) or s and u respectively. No terms linear in the q_i^2 variables appear. Also terms of the form q_i^2 s, q_i^2 t, etc., can after using crossing and Bose symmetry be reduced to forms already in (45). Before applying (44), we note that there is one remark we can make in general about the coefficients a, b, c, ..., g, h. There is no a-priori reason to assume that any of the second order coefficients are small except for h. For if h is not small then the amplitude will vary strongly with the external pion masses a situation which is in contradiction to the PCAC philosophy. For example, if this were the case and h was large then the Adler-Weisberger sum rule for $\pi\pi$ scattering would be practically useless even if we were someday able to measure the $\pi\pi$ total cross sections exactly.

Let us use (44) to determine the coefficients a, b, c, ..., h. We restrict ourselves to the domain $0 \leq s, t, u \leq \mu^2$. Comparing (45) with $q_1^2 = 0$, $q_2^2 = s$, $q_3^2 = t$, $q_4^2 = u$, with (44) we get

$$a + b(t+u) + cs + d(t+u)^2 + etu + fs^2 + g(t+u)\, s + h(st+tu+su)$$
$$= \frac{1}{c_\pi^2}(\mu^2 - s)\, ; \quad 0 \leq s,\ t,\ u \leq \mu^2. \tag{46}$$

This gives us

$$a = \mu^2/c_\pi^2 \quad ,$$

$$b = 0 \quad ,$$

$$c = -\frac{1}{c_\pi^2} \quad , \tag{47}$$

$$d = f = 0; \ \text{and} \ \ h = -e = -g \ .$$

Note that a, b, and c still have the same value obtained by expanding only up to first order. Only one constant is left undetermined in the second order terms; and that one is h which as we mentioned earlier we expect to be small.

In order to estimate the scattering lengths a_o, and a_2 we need to assume that the expansion in (45) is at least numerically good up to $s = 4\mu^2$. In extending $s \to 4\mu^2$ we shall keep track of the correction terms in (42) in order to make sure that they do not make important contributions.

If we keep the correction terms from (42) in the consistency condition (44), then instead of (47) we obtain for the coefficients

$$a = \frac{\mu^2}{c_\pi^2}\left(1 - \frac{a_o \mu}{6\pi}\right) \quad ,$$

$$b = 0 ,$$

$$c = - \frac{1}{c_\pi^2} \left(1 - \frac{a_o \mu}{3\pi}\right), \tag{48}$$

$$f = - \frac{1}{c_\pi^2} \left(\frac{a_o}{6\pi\mu}\right) ,$$

and $d = 0$; $e = g = - h$.

We have mentioned earlier that h must be small compared to the dominant lower order terms. This indeed has to be so if we are to be consistent with the approximation used in (37) and (38). For example, let us consider in detail the $\pi^o \pi^o \rightarrow \pi^o \pi^o$ amplitude, F , given by

$$\begin{aligned} F &= A + B + C \\ &= \frac{1}{3} A^{I=0} + \frac{2}{3} A^{I=2} \end{aligned} \tag{49}$$

Let us fix our attention on the symmetry point $s = u$, and $t = 0$. Then by using the arguments of reference 8 and considering dispersion relations in the external mass variables q_1^2 and q_3^2 and assuming dominance by the double pion pole we get

$$F(s{=}u,\ t{=}0;\ 0,\ \mu^2,\ 0,\ \mu^2) \cong F(s{=}u,\ t{=}0;\ \mu^2,\ \mu^2,\ \mu^2,\ \mu^2) . \tag{50}$$

A similar result was written down for the even πN amplitude in Ref. 3. The argument is very similar to that used in (36) - (39) above and one can refer to Ref. 8 for details. What we have done here is to keep $t = 0$ fixed, and $s = u$, ($\nu = 0$, fixed) and extrapolate two external mass variables, q_1^2 and q_3^2 , from

μ^2 to zero. On the other hand we can compute both sides of (50) from our expansion (45). For F, using the coefficients in (47), we have

$$F(s,t,u;q_1^2,q_2^2,q_3^2,q_4^2) = 3a + 4\mu^2 c - 3h(st + tu + su) + 3h \sum_{\substack{i>j \\ i\neq j}} q_i^2 \cdot q_j^2 , \tag{51}$$

where always $u = \sum_i q_i^2 - s - t$. We now use (51) to calculate the difference

$$F(s{=}u,\ t{=}0;\ \mu^2,\mu^2,\mu^2,\mu^2) - F(s{=}u,\ t{=}0;\ 0,\mu^2,0,\mu^2) \cong 6h\mu^4. \tag{52}$$

Thus to the extent that (50) is a good extrapolation we conclude that $6h\mu^4$ must be small when compared to the dominant term in (51) which is $(3a + 4\mu^2 c) = -\mu^2/c_\pi^2$. Here

$$\frac{\mu^2}{c_\pi^2} = \frac{\mu^2 G_{\pi NN}^2}{M_N^2 g_A^2} \cong \frac{8}{9}\pi .$$

At the end of this section we shall write down a sum rule for h and discuss its magnitude further, however it is clear that to be consistent with our approximations on f^σ earlier we must neglect h. We can now compute the scattering lengths.

The S-wave scattering lengths are related to our expansion coefficients by

$$a_0 \cong -\frac{1}{32\pi\mu}\left[5a + 12\mu^2 c + 48f\mu^4 + 30h\mu^4\right] ,$$

$$a_2 \cong -\frac{1}{32\pi\mu}\left[2a + 12h\mu^4\right] . \tag{53}$$

We have kept both the terms proportional to f and h in (53). Following our estimate of $6h\mu^4$ when compared with μ^2/c_π^2, we see that $30h\mu^4$ is also negligible when compared with $(5a + 12\mu^2 c) \cong -7 \frac{\mu^2}{c_\pi^2}$. Even if $6h\mu^4$ was as large as 20% of μ^2/c_π^2 keeping the term $30h\mu^4$ in (53) will only change Weinberg's result by 13% and raise the scattering length at most to $a_o \cong 0.23\ \mu^{-1}$.

We thus have, setting $h \approx 0$

$$a_o \cong -\frac{1}{32\pi\mu}\left[5a + 12\mu^2 c + 48f\mu^4\right] . \qquad (53')$$

Substituting the values (48) for a, c, and f we get an equation for a_o which we can solve and obtain

$$a_o \cong \frac{\frac{7}{4}\left(\frac{1}{8\pi} \cdot \frac{\mu}{c_\pi^2}\right)}{\left[1 - \frac{29\mu^2}{192\pi^2 c_\pi^2}\right]} . \qquad (54)$$

The numerator of this last expression is exactly Weinberg's result. The quantity $(29\mu^2/192\pi^2 c_\pi^2)$ is about 0.04. Therefore keeping the correction terms in (48) will only change the result by 4%. We get

$$a_o \cong 0.2\mu^{-1} . \qquad (55)$$

This means that the ratio in (42) is indeed close to unity and the quantity $a_o\mu/6\pi$ is of the order of 1%. The coefficients are therefore given by (47) to a good approximation.

In a similar way the corrections terms do not affect a_2 in any appreciable way and one still gets

$$a_2 \cong - \frac{1}{2}\left(\frac{\mu}{8\pi c_\pi^2}\right) \cong 0.06\,\mu^{-1} \quad . \tag{56}$$

In closing we shall write a dispersion relation for the forward $\pi^0\pi^0 \to \pi^0\pi^0$ amplitude F and show how it can be used to give a sum rule for h. It is more convenient to use the laboratory energy ν instead of s as a variable, where

$$s = 2\mu^2 + 2\nu\mu \quad . \tag{57}$$

For $t = 0$ the expansion for $F(\nu)$ for physical masses, keeping h, is

$$\begin{aligned} F(\nu) &\cong -\left(\frac{\mu^2}{c_\pi^2}\right) + 18h\mu^4 - 3h\,(4\mu^2 - s)\ s \\ &\cong -\left(\frac{\mu^2}{c_\pi^2}\right) + 12h\mu^2\,(\nu^2 + \frac{\mu^2}{2})\ , \quad |\nu| < \mu \quad . \end{aligned} \tag{58}$$

This expansion is good, even convergent, for $|\nu| < \mu$. We note that at the points $\nu = \pm\, i\,\mu/\sqrt{2}$, $F(\nu)$ is through (58) given by $-\mu^2/c_\pi^2$ and not dependent on h. We can therefore write a twice subtracted dispersion relation for $F(\nu)$ and if we choose the subtraction points to be $\nu = \pm\, i\,\mu/\sqrt{2}$, the subtractions will not depend on h. We get

$$F(\nu) = -\frac{\mu^2}{c_\pi^2} + \frac{2\left(\nu^2 + \frac{\mu^2}{2}\right)}{\pi} \int_\mu^\infty \frac{\mathrm{Im}\ F(\nu')\ \nu'}{(\nu'^2 + \mu^2/2)(\nu'^2 - \nu^2)}\ d\nu' \quad . \tag{59}$$

The expansion (58) is certainly good at $\nu = 0$, and it gives

$$F(0) \cong -\mu^2/c_\pi^2 + 6\,h\mu^4 \quad . \tag{60}$$

We see that $6h\mu^4$ is just the difference between $F(\nu = 0)$ and $F(\nu = i\mu/\sqrt{2})$. Our assumption is that this is small compared to the value of F at either of these two points. Comparing (60) with (59) we get a sum rule for h

$$6h\mu^4 = \frac{\mu^2}{\pi} \int_\mu^\infty \frac{\mathrm{Im}\,F(\nu')}{\nu'(v'^2 + \mu^2/2)}\, d\nu' \quad . \tag{61}$$

We recall that F is the physical forward fully symmetric amplitude and only $I = 0$ or $I = 2$ contribute to ImF.

The first thing we learn from (61) is that h is negative; ImF in our normalization is negative. The contribution of resonances like the f^0 to h through (61) will certainly be negligible for our purposes, so will that of any high mass, i.e. > 500 MeV, resonance. If a low energy narrow resonance exists say in the $\ell = 0$, $I = 0$ channel it could change our result appreciably but it is hard to see how it can increase the scattering length up to more than $a_0 = 0.3\mu^{-1}$ at worst. Such a resonance would make the Weisberger extrapolation quite bad for $\pi\pi$ scattering, and it has of course not been established experimentally.[9] Many of the theoretical arguments for its existence like the analyses of $K_{\ell 4}$ decay[10] and τ-decay[11] have lately been rendered unnecessary. The only remaining question is the saturation of the Adler-Weisberger $\pi\pi$-sum rule.[12] That sum rule has one less power of ν in the denominator than in (61) and it could easily be saturated with, in addition to known resonances, an $\ell = 0$, $I = 0$ resonance of mass > 600 MeV. It does not necessarily force us to predict a low lying resonance.

There is one contribution to (61) which might be dangerous and whose effect we can approximately check. Namely, the contribution from $\mathrm{Im}F(\nu')$ near threshold that are related to the $\ell = 0$, $I = 0$ scattering length. This will give a contribution proportional to a_o^2 from the low energy part in (61) and that when substituted in (53) will change the functional form of our resulting equation for a_o. To make sure that this will not appreciably change our results we divide the integration range in (61) into two parts $a \leq \nu \leq 6\mu$, and $6\mu \leq \nu \leq \infty$. In the first interval we approximate ImF by the contribution from $\ell = 0$, $I = 0$ channel and use $\delta_o^o \simeq \sqrt{\frac{s-4}{s}}\, a_o \mu$ and get

$$6h\mu^4 \cong -2a_o^2\,\mu^2 + \mu^2/\pi \int_{6\mu}^{\infty} \frac{\mathrm{Im}F\,(\nu')}{\nu'\,(\nu'^2 + \mu^2/2)}\, d\nu' \quad . \tag{62}$$

If we ignore the second term in (62) and assume it to be a fraction of $\mu^2/c_\pi^2 \cong 8\pi/9$, we obtain on substituting (62) into (53)

$$a_o = -\frac{1}{32\pi\mu}\left[-7\,\frac{\mu^2}{c_\pi^2} - 10\,a_o^2\,\mu^2\right] . \tag{63}$$

This last equation has two roots for a_o. One will, to within 2%, give us back the same answer as before, $a_o \cong 0.2\mu^{-1}$. The other root is ridiculously large, $a_o \cong 10\mu^{-1}$, and clearly unphysical. The latter root will also give a very large value for a_2.

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7. With both pions on the mass shell it does not matter whether we use the definition (30)or (10) for f^σ. In principle, when we go off the mass shell f^σ defined with $\partial_\mu A^\mu_\alpha$ as an interpolating field could be different than f^σ defined by the v_α's. However, we have assumed that both v_α and $\partial_\mu A^\mu_\alpha$ are good smooth interpolating fields and we shall only use (30) for $0 \leq q^2, k^2 \leq \mu^2$. So as long as we do not go too far off the mass shell f^σ as defined in (30) and that defined in (10) are within our approximations the same. This problem of course does not arise if we had used $\partial_\mu A^\mu_\alpha$ instead of the v_α's all through.

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