Non-Perturbative String Theory from the Gauge/Gravity Correspondence

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ABSTRACT

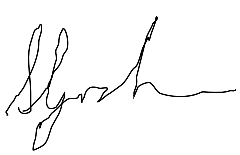
In this dissertation we study the action of the one loop dilatation operator on operators with a classical dimension of order N. We consider the $su(3)$ and $su(2)$ sectors. The operators in the $su(3)$ sector are constructed using three complex fields X, Y and Z, while operators in the $su(2)$ sector are constructed from only the two complex fields Y and Z. For the operators in these sectors non-planar diagrams contribute already at the leading order in N and the planar and large N limits are distinct.

Although the spectrum of anomalous dimensions in $su(3)$ has been computed for this class of operators, previous studies have neglected certain terms which were argued to be small. After dropping these terms diagonalizing the dilatation operator reduces to diagonalizing a set of decoupled oscillators. In this dissertation we explicitly compute the terms which were neglected previously and show that diagonalizing the dilatation operator still reduces to diagonalizing a set of decoupled oscillators.

In the $su(2)$ sector the action of the one loop and the two loop dilatation operator reduces to a set of decoupled oscillators and factorizes into an action on the Z fields and an action on the Y fields. Direct computation has shown that the action on the Y fields is the same at one and two loops. In this dissertation, using the $su(2)$ symmetry algebra as well as structural features of field theory, we give compelling evidence that the factor in the dilatation operator that acts on the Ys is given by the one loop expression, at any loop order.

I hereby declare that the content of this dissertation is based on my following original works:

- R. de Mello Koch, S. Graham and W. Mabanga, "Subleading corrections to the Double Coset Ansatz preserve integrability" (2013) [arXiv:1312.6230v1 [hep-th]]
- R. de Mello Koch, S. Graham and I. Messamah, "Higher Loop Nonplanar Anomalous Dimensions from Symmetry" (2013) [arXiv:1312.6227v1 [hep-th]].



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Contents

1 Introduction

The two great pillars of modern theoretical physics, Einstein's general theory of relativity (GR) and quantum mechanics (QM) have long been at apparent odds with one another. GR provides an understanding of gravity through the realisation that space and time should be unified. QM and its Lorentz invariant improvement, quantum field theory (QFT) describe physics on small scales. In situations where large amounts of matter are found in small volumes of space a unification of these two great theories is required. This unification - a quantisation of gravity - has proved exceedingly difficult to construct. While quantisation of theories such as electrodynamics resulted in very successful renormalisable QFTs, the quantisation of gravity resulted in a non-renormalisable QFT. In other words, these attempts at unifying GR and QFT resulted in a theory with no predictive power. A theory which provides a consistent quantisation of gravity is string theory. Although much is still unknown about string theory it has provided many fascinating results. Perhaps the greatest of these is the AdS/CFT correspondence[1, 2, 3], which postulates a duality between string theory and a gauge theory. To be precise, it states that type IIB string theory defined on $AdS_5 \times S_5$ is dual to $\mathcal{N} = 4$ super Yang-Mills theory (SYM) defined on the boundary. The first indication that such a duality may exist was due to t'Hooft[4], although it was Maldacena who first made the duality concrete[1].

Quantum Chromo Dynamics (QCD) is an asymptotically free theory - the coupling tends to zero as we increase energy. Since $\mathcal{N} = 4$ SYM theory is closely related to QCD, we hope to learn about QCD by studying $\mathcal{N} = 4$ SYM. Understanding the strong coupling dynamics of $\mathcal{N} = 4$ SYM theory is a formidable problem that can't be tackled using perturbation theory. On the other hand, at high energies in string theory we have large curvature corrections corresponding to strong coupling worldsheet physics, which is again beyond the scope of current QFT methods. The beauty of the AdS/CFT correspondence is that it provides a manner in which to make predictions in these situations - highly curved gravity backgrounds are dual to weak coupling QFT, while nearly flat gravity backgrounds are dual to strongly coupled QFT. As such, the AdS/CFT correspondence provides a tool with which to delve deeply into both theories in a manner which was certainly not possible before. For instance, at the classical level, the Dirac-Born-Infeld action provides a suitable description of giant gravitons. However, since this action is not renormalizable, it can't possibly provide a correct starting point for a fundamental quantum theory. Using the AdS/CFT correspondence, it should be possible to study giant graviton physics with both quantum and stringy corrections included in the description.

An important point we have not yet mentioned is that everything we have described above obviously relies on the premise that the AdS/CFT correspondence is in fact correct. By the very nature of the correspondence the task of proving it is a difficult one. Nonetheless, many checks have been done and as of yet no exceptions have been found. The task of computing quantities on both sides of the correspondence and ensuring they match remains an active area of research.

In order to utilise the AdS/CFT correspondence a dictionary between the gauge theory and the string theory is required. In $\mathcal{N} = 4$ SYM there are six scalar Higgs fields ϕ_i $i = 1, 2, \ldots, 6$ that transform in the adjoint of the $U(N)$ gauge group. Form the three complex linear combinations

$$X = \phi_1 + i\phi_2, \qquad Y = \phi_3 + i\phi_4, \qquad Z = \phi_5 + i\phi_6 \,. \tag{1.1}$$

The non-zero free field two point functions are

$$\langle Z^i_j (Z^\dagger)^k_l \rangle = \delta^i_l \delta^k_j = \langle Y^i_j (Y^\dagger)^k_l \rangle = \langle X^i_j (X^\dagger)^k_l \rangle \,. \tag{1.2}$$

Consider the $\frac{1}{2}$-BPS sector of the theory - that is the sector with operators built only from the Z field. The gauge invariant operators in this sector are products of traces of powers of Z. The dictionary relates these operators to different objects in the string theory depending on the number of fields comprising each operator. Operators built from $\mathcal{O}(1)$ fields are dual to gravitons. Operators built from $\mathcal{O}(\sqrt{N})$ fields are dual to strings. Operators built from $\mathcal{O}(N)$ fields are dual to giant gravitons. Finally, operators built from $\mathcal{O}(N^2)$ fields are dual to new spacetime geometries.

Another important element in the dictionary is the mapping of anomalous dimensions into string state energies. Consequently, much work has recently been put into the computation of the spectrum of anomalous dimensions. This is done by computing the action of the dilatation operator. The dilatation operator can be defined perturbatively and as such, the computation of its action amounts to summing Feynman diagrams. This is a highly non-trivial task. Dramatic progress, however, was made through the discovery of integrability in the planar limit[5, 6, 7]. The fate of integrability beyond the planar limit, however, is not clear. In this dissertation we will investigate this issue.

Let us delve somewhat deeper into the operators of $\mathcal{N} = 4$ SYM and argue why the existence of integrability in the non-planar limit is by no means obvious. Consider the $\frac{1}{2}$-BPS operators

$$O_J \equiv \frac{\mathrm{Tr} Z^J}{\sqrt{J N^J}} \tag{1.3}$$

which are normalized to have a unit two point function

$$\langle O_J O^\dagger_J \rangle = 1 \,. \tag{1.4}$$

To obtain (1.4) we have summed only the planar diagrams. This is perfectly accurate at large N as long as $J^2 \ll N$. Now consider the two point function between a double trace structure given by $O_{J_1}O_{J_2}$ and the single trace $O_{J_1+J_2}$

$$\langle O_{J_1}O_{J_2}O^\dagger_{J_1+J_2}\rangle = \frac{\sqrt{J_1J_2(J_1+J_2)}}{N}\,. \tag{1.5}$$

If we take $N \to \infty$ holding J_1 and J_2 fixed, it is clear that the above two point function vanishes. There is no conservation law forcing this correlator to vanish - its a nontrivial statement about the dynamics. The two point function in the planar limit, between two operators that have different multitrace structures, vanishes. Although we have described this only in the $\frac{1}{2}$-BPS sector and for a specific example, this is a general property of matrix models. Thus, if we want to compute anomalous dimensions in the planar limit of the theory, we can focus on single trace operators since these will not mix with operators that have a different trace structure. This property of the planar limit is a crucial ingredient in the arguments for the integrability of the planar limit. Indeed, integrability follows because the planar dilatation operator can be identified with the Hamiltonian of an integrable spin chain. A single trace operator containing K fields can be identified with a spin chain state, where the spin chain lives on a lattice that has K sites. The fields in the single trace operator determine the states of the spins in the lattice. In this way, there is a bijection between single trace operators and the states of a spin chain. If we scale J_1, J_2 as $N^{\frac{2}{3}}$, the right hand side of (1.5) scales as N^0 at large N and different trace structures start to mix. This mixing sets in even sooner: if we had computed the left hand side of (1.5) exactly, we would find that mixing between different trace structures is no longer suppressed if $J_1, J_2 \gtrsim \sqrt{N}$[23]. For the case of interest to us $J_i \sim N$ and there is uncontrolled mixing. Consequently, the bijection between single trace operators and the states of a spin chain is not useful at all and the link to the dynamics of a spin chain is lost.

Despite these drawbacks, recent evidence suggests that integrability may in fact be a feature of the non-planar limit. Working in an appropriate basis, the restricted Schur basis, and using group representation theory it has been shown that in the $su(2)$ sector integrability is present at one[25, 26, 27, 28] and two loops[31]. In this process a conservation law was discovered. In the $su(3)$ sector, however, this conservation law was found to be broken, suggesting that integrability may not be present in this sector. In this dissertation we will show that despite this broken conservation law integrability is a feature of the $su(3)$ sector. The natural interpretation of this result is that the conservation law that was found in the $su(2)$ sector is replaced by a new conservation law in the $su(3)$ sector, which reduces to the $su(2)$ conservation law when suitably restricted. Further, we will extend the $su(2)$ result to all loops. This is achieved by using symmetry arguments and not by explicit computation, suggesting that similar arguments can be used to study other sectors of the theory.

In chapter 2 we discuss the restricted Schur polynomial technology. In chapter 3 we consider the $su(3)$ sector of $\mathcal{N} = 4$ SYM and show that this sector is integrable. In chapter 4 we consider the $su(2)$ sector and extend the result of integrability to all loops. We conclude in chapter 5.

The content in chapters 3 and 4 is based on original work which appears on the arXiv and has been submitted to JHEP for publication. Specifically, chapter 3 is based on the paper “Subleading corrections to the Double Coset Ansatz preserve integrability” - arXiv:1312.6230v1, authored by Robert de Mello Koch, Wandile Mabanga and Stuart Graham. Chapter 4 is based on the paper “Higher Loop Nonplanar Anomalous Dimensions from Symmetry” - arXiv:1312:6227v1, authored by Robert de Mello Koch, Ilies Messamah and Stuart Graham.

2 Restricted Schur Polynomials

As we have seen above, the mixing of traces in the non-planar limit is unconstrained. As such, the trace basis is no longer suitable and we need a convenient description that easily allows us to simultaneously talk about all possible trace structures. The operators we consider are constructed using n Z fields, m Y fields and p X fields. We are primarily interested in the case that n, m, p all scale as N in the large N limit, but $n \gg m + p$ and $\frac{m}{p} \sim 1$. Thus, the operators that we consider are a small perturbation of a $\frac{1}{2}$-BPS operator. Z^i_j is an operator acting on N dimensional vector space V. By tensoring n copies of Z we obtain an operator $Z^{\otimes n} \equiv Z \otimes Z \otimes \cdots \otimes Z$ which acts on the space $V^{\otimes n}$. $V^{\otimes n}$ admits a natural action of the symmetric group S_n obtained by allowing $\sigma \in S_n$ to permute the factors of V in $V^{\otimes n}$. Concretely, for $\sigma \in S_n$ we have

$$(\sigma)^I_J = \delta^{i_1}_{j_{\sigma(1)}} \delta^{i_2}_{j_{\sigma(2)}} \cdots \delta^{i_n}_{j_{\sigma(n)}} . \tag{2.1}$$

Using this action

$$\mathrm{Tr}(\sigma Z^{\otimes n}) = \sigma^I_J (Z^{\otimes n})^J_I = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \cdots Z^{i_n}_{i_{\sigma(n)}} . \tag{2.2}$$

We can obtain every possible multi-trace structure by choosing the correct permutation σ. For example, consider the simplest non-trivial case $n = 2$. The possible permutations, in cycle notation, are $\sigma = \{(1)(2), (12)\}$ which gives

$$\begin{aligned} \sigma = (1)(2) \qquad & \mathrm{Tr}(\sigma Z^{\otimes 2}) = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} = Z^{i_1}_{i_1} Z^{i_2}_{i_2} = \mathrm{Tr}Z\mathrm{Tr}Z \,, \\ \sigma = (12) \qquad & \mathrm{Tr}(\sigma Z^{\otimes 2}) = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} = Z^{i_1}_{i_2} Z^{i_2}_{i_1} = \mathrm{Tr}Z^2 \,. \end{aligned} \tag{2.3}$$

At $n = 2$ each permutation corresponds to a different gauge invariant operator. This is not generic. In general permutations in the same conjugacy class determine the same operator. The set up we have just outlined allows us to trade gauge invariant operators for permutations. Thus the different multi-trace structures can now be discussed on an equal footing - each corresponds to a permutation. For the large N limit we consider there is a particularly useful set of gauge invariant operators, known as the Schur polynomials, given by [9]

$$\chi_R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \mathrm{Tr}(\sigma Z^{\otimes\, n}) . \tag{2.4}$$

Notice that the right hand side includes a sum over all possible permutations which implies that the Schur polynomials are a sum of all possible multi-trace structures. $\chi_R(\sigma)$ is the character of symmetric group element σ in irreducible representation R. The irreducible representations of the symmetric group S_n are labeled by Young diagrams with n boxes. The set of Young diagrams with n boxes correspond to the partitions of n, so that the number of Schur polynomials matches the number of gauge invariant operators. The Schur polynomials simply provide an alternative basis to the trace basis. The fact that the matching of gauge invariant operators matches is more subtle than our discussion above suggests. Imagine that $N = 2$. It is easy to check that at $n = 3$ there are only two independent gauge invariant operators because (just write the two sides of this equation out in the basis in which Z is diagonal)

$$\mathrm{Tr}(Z^3) = \frac{1}{2}\left[3\mathrm{Tr}Z^2\mathrm{Tr}Z - \mathrm{Tr}Z\mathrm{Tr}Z\mathrm{Tr}Z\right] . \tag{2.5}$$

This is called a trace relation and there will be relations of this type whenever $n > N$ as is the case here. The Schur polynomials naturally take the trace relations into account, because the Schur polynomial $\chi_R(Z)$ vanishes as soon as the Young diagram R has more than N rows. Thus, for $N = 2$ and $n = 3$ there are only two Schur polynomials, given by $\chi_{\square\square\square}(Z)$ and $\chi_{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}}(Z)$. Since we are going to take $N \to \infty$ one might expect that the trace relations never apply. This is the case in the planar limit where the number of fields in our operator is held fixed as we scale $N \to \infty$. However, for the problems of interest in this dissertation, the number of fields in each operator is also scaled as the limit is taken so that the number of fields in each operator generically exceeds N and the trace relations must be respected. Another important property of the Schur polynomials is that they diagonalize the two point function

$$\langle \chi_R(Z)\chi_S(Z^\dagger)\rangle = f_R\delta_{RS} \tag{2.6}$$

where f_R is a product of factors, one for each box in the Young diagram R. A box in column j and row i has factor $N - i + j$.

The dilatation operator annihilates all operators in the $\frac{1}{2}$-BPS sector. Thus, to obtain a non-trivial anomalous dimension problem we need to move beyond the $\frac{1}{2}$-BPS sector, by taking $p \neq 0$ and/or $m \neq 0$.

Our discussion above of the $\frac{1}{2}$-BPS sector generalizes nicely to this more general setting. Multitrace operators can again be associated to permutations $\sigma \in S_{m+n+p}$

$$\begin{aligned}\mathrm{Tr}(\sigma X^{\otimes p} \otimes Y^{\otimes m} \otimes Z^{\otimes n}) = X^{i_1}_{i_{\sigma(1)}} \cdots X^{i_p}_{i_{\sigma(p)}} Y^{i_{p+1}}_{i_{\sigma(p+1)}} \cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}} \\ \times Z^{i_{m+p+1}}_{i_{\sigma(m+p+1)}} \cdots Z^{i_{m+p+n}}_{i_{\sigma(m+p+n)}} .\end{aligned} \tag{2.7}$$

Permutations that are conjugate, with respect to the $S_p \times S_m \times S_n$ subgroup

$$\gamma\sigma_1\gamma^{-1} = \sigma_2 \qquad \gamma \in S_p \times S_m \times S_n \tag{2.8}$$

give rise to the same operator. The Schur polynomials generalize to the restricted Schur polynomials[24, 15]

$$\begin{aligned}\chi_{R,(t,s,r)\vec{\mu}\vec{\nu}} = \frac{1}{n!m!p!} \sum_{\sigma \in S_{n+m+p}} \mathrm{Tr}_{(t,s,r)\vec{\mu}\vec{\nu}}(\Gamma^R(\sigma)) X^{i_1}_{i_{\sigma(1)}} \cdots X^{i_p}_{i_{\sigma(p)}} Y^{i_{p+1}}_{i_{\sigma(p+1)}} \cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}} \\ \times Z^{i_{m+p+1}}_{i_{\sigma(m+p+1)}} \cdots Z^{i_{m+p+n}}_{i_{\sigma(m+p+n)}} .\end{aligned} \tag{2.9}$$

We call $\mathrm{Tr}_{(t,s,r)\vec{\mu}\vec{\nu}}(\Gamma^R(\sigma))$ the restricted trace of $\Gamma^R(\sigma)$[10]. When computing this trace, we trace over a subspace of the carrier space of R. R is an irreducible representation of S_{n+m+p}, that is, it is a Young diagram with $m+n+p$ boxes. We write $R \vdash m+n+p$. This subspace we trace over is a carrier space of the subgroup $S_n \times S_m \times S_p$. It is labeled by three Young diagrams $t \vdash p$, $s \vdash m$ and $r \vdash n$. $\vec{\mu}$ and $\vec{\nu}$ are degeneracy labels; they are each two dimensional vectors. Their two components resolve different copies of the two representations $s \vdash m$ and $t \vdash p$. To properly understand the role of the degeneracy labels and what they label, we note that the restricted trace can be written as

$$\mathrm{Tr}_{(t,s,r)\vec{\mu}\vec{\nu}}(\cdots) = \mathrm{Tr}_R(P_{(t,s,r)\vec{\mu}\vec{\nu}}\cdots) \tag{2.10}$$

where $P_{(t,s,r)\vec{\mu}\vec{\nu}}$ is an intertwining map. The degenaracy labels $\vec{\mu}$ and $\vec{\nu}$ play an important role in constructing this intertwining map as we now explain. The first step in constructing $P_{(t,s,r)\vec{\mu}\vec{\nu}}$ entails constructing a basis for the (t,s,r) irreducible representation of $S_n \times S_m \times S_p$. To do this start from the Young diagram for irreducible representation R. Remove p boxes in any order such that everytime a box is removed what remains is a valid Young diagram and we remove p_i boxes from row i. Assemble the p_i into a vector $\vec{p}$; this vector will play an important role in what follows. Now remove m boxes in any order such that everytime a box is removed what remains is a valid Young diagram and we remove m_i boxes from row i. Assemble the m_i into a vector $\vec{m}$; again, this vector will play an important role in what follows. The boxes are labeled according to the order in which they are removed so that the first box removed is box 1, the second box removed is box 2, and so on. In this way we land up with a partly labeled Young diagram R. The unlabeled boxes have the shape r and each partly labeled Young diagram is a distinct subspace of R that carries the irreducible representation r under the S_n subgroup. Now assemble the vectors with first p boxes labeled into an irrep t of S_p, resolving multiplicities that arise with ν_1. In this process, the labels of the next m boxes are simply ignored. For each state in a given S_p irreducible representation specified by both t and ν_1, one has all possible labelings of the next m boxes. Assemble these into vectors in an irreducible representation s of S_m, resolving multiplicities with ν_2. The two multiplicity labels are assembled to produce the vector $\vec{\nu} = (\nu_1, \nu_2)$. The result of this exercise is a set of vectors labeled with two irreducible representations $t \vdash p$ and $s \vdash m$ each with a multiplicity label ν_1 and ν_2, and two state labels, a, b, one for each state $|t, \nu_1, a; s, \nu_2, b\rangle$. The boxes that are not labeled stand for vectors that belong to a unique irreducible representation r of S_n. Use c to label states in r. We can make this explicit and write our state as $|t, \nu_1, a; s, \nu_2, b; r, c\rangle$. This gives a basis for the (t,s,r) irreducible representation of $S_n \times S_m \times S_p$. Now, the intertwining map is a matrix so that it has both a row label and a column label. We can use different copies of the (t,s,r) irreducible representation for the rows and columns of the intertwining map. Consequently

$$P_{(t,s,r)\vec{\mu}\vec{\nu}} = \sum_{a,b,c} |t, \mu_1, a; s, \mu_2, b; r, c\rangle\langle t, \nu_1, a; s, \nu_2, b; r, c| \tag{2.11}$$

Since the S_m and S_p actions commute it is clear that

$$|t, \mu_1, a; s, \mu_2, b; r, c\rangle = |t, \mu_1, a\rangle \otimes |s, \mu_2, b\rangle \otimes |r, c\rangle \tag{2.12}$$

where $\otimes$ is the usual tensor product on a vector space. It then also follows that the intertwining maps can be written as a tensor product

$$\begin{aligned} P_{(t,s,r)\vec{\mu}\vec{\nu}} &= \sum_a |t, \mu_1, a\rangle\langle t, \nu_1, a| \otimes \sum_b |s, \mu_2, b\rangle\langle s, \nu_2, b| \otimes \sum_c |r, c\rangle\langle r, c| \\ &\equiv p_{t\mu_1\nu_1} \otimes p_{s\mu_2\nu_2} \otimes \mathbf{1}_r \end{aligned} \tag{2.13}$$

The last factor in this product is always a genuine projector.

The restricted Schur polynomials share many of the nice properties that make the Schur polynomials so useful. In particular, the restricted Schur polynomials respect the trace relations and the two point function of the restricted Schur polynomials[15]

$$\langle \chi_{R,(t,s,r)\vec{\mu}\vec{\nu}} \chi^{\dagger}_{T,(y,x,w)\vec{\beta}\vec{\alpha}} \rangle = \frac{f_R \mathrm{hooks}_R}{\mathrm{hooks}_r \mathrm{hooks}_s \mathrm{hooks}_t} \delta_{RT}\delta_{rw}\delta_{sx}\delta_{ty}\delta_{\vec{\mu}\vec{\beta}}\delta_{\vec{\nu}\vec{\alpha}} \tag{2.14}$$

again diagonalize the free field two point function. The number hooks_R is a product of the hook lengths in Young diagram R. We will often find it convenient to work with operators $\hat{O}_{R,(t,s,r)\vec{\mu}\vec{\nu}}$ normalized to have a unit two point function. These operators are related to the restricted Schur polynomials $\chi_{R,(t,s,r)\vec{\mu}\vec{\nu}}$ as

$$\hat{O}_{R,(t,s,r)\vec{\mu}\vec{\nu}} = \sqrt{\frac{\mathrm{hooks}_r \mathrm{hooks}_s \mathrm{hooks}_t}{f_R \mathrm{hooks}_R}} \chi_{R,(t,s,r)\vec{\mu}\vec{\nu}} \,. \tag{2.15}$$

The key difficulty with working with the restricted Schur polynomials, is in constructing and working with the intertwining maps $P_{(t,s,r)\vec{\mu}\vec{\nu}}$. Convenient methods to accomplish this have been developed for two rows in [25] and in general in [26]. Using these methods, the one loop dilatation operator has been diagonalized in the $su(2)$ sector (obtained by setting $p = 0$)[25, 26, 27, 28]. In this sector, the one loop dilatation operator reduces to a set of decoupled oscillators, which is an integrable system. These results provided perfect confirmation of earlier numerical studies[29, 30]. At two loops the system remains integrable in the $su(2)$ sector[31]. The one loop results were generalized to $p \neq 0$ in [32], but the interactions between the X and Y fields were argued to be subleading and were neglected. The subleading terms are of order $\frac{m}{n}$ relative to the leading terms[32]. It is precisely these terms that we will evaluate in chapter 3 of this dissertation.

When interactions between the X and Y fields are neglected, the vectors $\vec{p}$ and $\vec{m}$ defined above are conserved[26]. The dilatation operator only mixes operators that have the same $\vec{p}$ and $\vec{m}$ values. This is not at all surprising - integrable systems are always accompanied with higher conserved quantities. What makes the interaction between the X and Y fields so interesting is that they spoil the conservation of $\vec{p}$ and $\vec{m}$. This can mean one of two things: either, integrability does not persist beyond the $su(2)$ sector and this large N but non-planar limit is not integrable, or the dynamics remains integrable but the conservation of $\vec{p}$ and $\vec{m}$ is not one of the conservation laws of this (extended) integrable system. Our results are unambiguous - the second case is realized and the one loop dilatation operator continues to be integrable when extended to act on operators built using all three complex scalars. Indeed, we are able to identify the new terms we have evaluated with elements of the Lie algebra of a unitary group. Diagonalizing the complete dilatation operator then reduces to a solved problem in representation theory.

3 Subleading corrections to the Double Coset Ansatz preserve Integrability

3.1 Dilatation Operator

The one loop dilatation operator in the $su(3)$ sector is given by[5]

$$D = -g_{YM}^2 \mathrm{Tr}\left([Y,Z]\left[\frac{d}{dY},\frac{d}{dZ}\right] + [X,Z]\left[\frac{d}{dX},\frac{d}{dZ}\right] + [Y,X]\left[\frac{d}{dY},\frac{d}{dX}\right]\right). \tag{3.1}$$

To be completely explicit, the index structure is

$$\mathrm{Tr}\left([Y,X]\left[\frac{d}{dY},\frac{d}{dX}\right]\right) = (Y^i_l X^l_j - X^i_l Y^l_j)\left(\frac{d}{dY^k_j}\frac{d}{dX^i_k} - \frac{d}{dX^k_j}\frac{d}{dY^i_k}\right). \tag{3.2}$$

Our first task is to consider the action of D on restricted Schur polynomials. In what follows we will often need the identity [33]

$$\mathrm{Tr}(\sigma Y^{\otimes m}\otimes Z^{\otimes n}) = \sum_{T,t,u,\vec{\nu}} \frac{d_T n! m!}{d_t d_u (n+m)!}\chi_{T,(t,u)\vec{\nu}^*}(\sigma^{-1})\chi_{T,(t,u)\vec{\nu}}(Z,Y) \tag{3.3}$$

where if $\vec{\nu}=(\nu_1,\nu_2)$ then $\vec{\nu}^*=(\nu_2,\nu_1)$. With a suitable choice of σ, the right hand side above gives any desired multitrace operator. Thus, the above equation expresses an arbitrary multitrace operator as a linear combination of restricted Schur polynomials. The sum above runs over all Young diagrams $T\vdash m+n$, $t\vdash n$ and $u\vdash m$ as well as over the multplicity labels $\vec{\nu}$. d_T denotes the dimension of the irreducible representation T of S_{n+m}. Similarly, d_t denotes the dimension of irreducible representation t of S_n and d_u the dimension of irreducible representation u of S_m. Finally, $\chi_{T,(t,u)\vec{\nu}^*}(\sigma^{-1})$ is the restricted character obtained by tracing $\Gamma_R(\sigma^{-1})$ over the (t,u) subspace, i.e. $\chi_{T,(t,u)\vec{\nu}^*}(\sigma^{-1}) = \mathrm{Tr}_{(t,u)\vec{\nu}^*}(\Gamma_T(\sigma^{-1}))$. The multiplicity index $\vec{\nu}^*=(\nu_2,\nu_1)$ tells us to trace the row index over the ν_2 copy of (r,s) and the column index over the ν_1 copy. We will consider in detail the subleading term which mixes Y and X. The remaining terms can be evaluated in an identical way. A straight forward computation gives

$$\begin{aligned}
&[Y,X]^i_j\left(\frac{d}{dY^k_j}\frac{d}{dX^i_k} - \frac{d}{dX^k_j}\frac{d}{dY^i_k}\right)\chi_{R,(t,s,r)\vec{\mu}\vec{\nu}}\\
&= [Y,X]^i_j\left(\frac{d}{dY^k_j}\frac{d}{dX^i_k} - \frac{d}{dX^k_j}\frac{d}{dY^i_k}\right)\frac{1}{n!m!p!}\sum_{\sigma\in S_{n+m+p}}\mathrm{Tr}_{(t,s,r)\vec{\mu}\vec{\nu}}(\Gamma^R(\sigma))X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}\\
&\qquad\times Z^{i_{m+p+1}}_{i_{\sigma(m+p+1)}}\cdots Z^{i_{m+p+n}}_{i_{\sigma(m+p+n)}}\\
&= \frac{mp}{n!m!p!}\sum_{\sigma\in S_{n+m+p}}\mathrm{Tr}_{(t,s,r)\vec{\mu}\vec{\nu}}(\Gamma^R(\sigma))(\delta^{i_{p+1}}_{i_{\sigma(1)}}[Y,X]^{i_1}_{i_{\sigma(p+1)}} - \delta^{i_1}_{i_{\sigma(p+1)}}[Y,X]^{i_{p+1}}_{i_{\sigma(1)}})\\
&\qquad\times X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}Z^{i_{m+p+1}}_{i_{\sigma(m+p+1)}}\cdots Z^{i_{m+p+n}}_{i_{\sigma(m+p+n)}}\\
&= \frac{mp}{n!m!p!}\sum_{\sigma\in S_{n+m+p}}\mathrm{Tr}_{(t,s,r)\vec{\mu}\vec{\nu}}(\Gamma^R([(1,p+1),\sigma]))\delta^{i_1}_{i_{\sigma(1)}}X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}
\end{aligned}$$

$$\times[Y,X]^{i_{p+1}}_{i_{\sigma(p+1)}} Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}} Z^{i_{m+p+1}}_{i_{\sigma(m+p+1)}}\cdots Z^{i_{m+p+n}}_{i_{\sigma(m+p+n)}}$$

The delta function in the summand will restrict the sum over S_{n+m+p} to a sum over the $S_{n+m+p-1}$ subgroup. The $S_{n+m+p-1}$ subgroup is obtained by retaining those elements that hold i_1 inert, i.e. $\sigma(1)=1$. To see how this happens, introduce the notation $\rho_i=\sigma(i,1)$ and rewrite the above sum as a sum over $S_{n+m+p-1}$ and its cosets. The result is

$$[Y,X]^i_j\left(\frac{d}{dY^k_j}\frac{d}{dX^i_k}-\frac{d}{dX^k_j}\frac{d}{dY^i_k}\right)\chi_{R,(t,s,r)\vec{\mu}\vec{\nu}}$$

$$=\frac{mp}{n!m!p!}\sum_{\sigma\in S_{n+m+p-1}}\sum_{i=1}^{n+m+p}\mathrm{Tr}_{(t,s,r)\vec{\mu}\vec{\nu}}(\Gamma^R([(1,p+1),\rho_i]))\delta^{i_1}_{i_{\rho_i(1)}}X^{i_2}_{i_{\rho_i(2)}}\cdots X^{i_p}_{i_{\rho_i(p)}}$$
$$\times[Y,X]^{i_{p+1}}_{i_{\rho_i(p+1)}}Y^{i_{p+2}}_{i_{\rho_i(p+2)}}\cdots Y^{i_{p+m}}_{i_{\rho_i(p+m)}}Z^{i_{m+p+1}}_{i_{\rho_i(m+p+1)}}\cdots Z^{i_{m+p+n}}_{i_{\rho_i(m+p+n)}}$$

$$=\frac{mp}{n!m!p!}\sum_{\sigma\in S_{n+m+p-1}}\mathrm{Tr}_{(t,s,r)\vec{\mu}\vec{\nu}}(\Gamma^R([(1,p+1),\{N+\sum_{i=2}^{n+m+p}(i,1)\}]))$$
$$\times\mathrm{Tr}_{V^{\otimes\, n+m+p-1}}(\sigma\cdot X^{\otimes\, p-1}\otimes[Y,X]\otimes Y^{\otimes\, m-1}\otimes Z^{\otimes\, n})$$

$$=\frac{mp}{n!m!p!}\sum_{R'}c_{RR'}\sum_{\sigma\in S_{n+m+p-1}}\mathrm{Tr}_{(t,s,r)\vec{\mu}\vec{\nu}}(([\Gamma^R((1,p+1)),\Gamma^{R'}(\sigma)])$$
$$\times\mathrm{Tr}_{V^{\otimes\, n+m+p-1}}(\sigma\cdot X^{\otimes\, p-1}\otimes[Y,X]\otimes Y^{\otimes\, m-1}\otimes Z^{\otimes\, n})$$

The sum over R' runs over all irreducible representations R' of the $S_{n+m+p-1}$ subgroup that can be subduced from the irreducible representation R of the S_{n+m+p} subgroup. As a Young diagram R' is obtained from R by dropping a single box. A prime on a letter denoting a Young diagram will always indicate that we drop a box. To obtain the last line above, use the fact that $N+\sum_{i=2}^{n+m+p}(i,1)$ when acting on any state within the subspace R' subduced by R gives $c_{RR'}$. This is proved by noting that $\sum_{i=2}^{n+m+p}(i,1)$ is a Jucys-Murphy element; see [10] for the details.

$$[Y,X]^i_j\left(\frac{d}{dY^k_j}\frac{d}{dX^i_k}-\frac{d}{dX^k_j}\frac{d}{dY^i_k}\right)\chi_{R,(t,s,r)\vec{\mu}\vec{\nu}}$$

$$=\frac{mp}{n!m!p!}\sum_{R'}c_{RR'}\sum_{\sigma\in S_{n+m+p-1}}\mathrm{Tr}_{(t,s,r)\vec{\mu}\vec{\nu}}(([\Gamma^R((1,p+1)),\Gamma^{R'}(\sigma)])$$
$$\times\mathrm{Tr}_{V^{\otimes\, n+m+p-1}}([(1,p+1),\sigma]\cdot X^{\otimes\, p}\otimes Y^{\otimes\, m}\otimes Z^{\otimes\, n})$$

$$=\frac{mp}{n!m!p!}\sum_{T,(y,x,w)\vec{\alpha}\vec{\beta}}\frac{d_Tn!m!p!}{d_wd_xd_y(n+m+p)!}\sum_{R'}c_{RR'}\sum_{\sigma\in S_{n+m+p-1}}\mathrm{Tr}_{(t,s,r)\vec{\mu}\vec{\nu}}(([\Gamma^R((1,p+1)),\Gamma^{R'}(\sigma)])$$
$$\mathrm{Tr}_{(y,x,w)\vec{\alpha}\vec{\beta}}(\Gamma_T([(1,p+1),\sigma]))\chi_{T,(y,x,w)\vec{\beta}\vec{\alpha}}(X,Y,Z)$$

$$=\sum_{R'}c_{RR'}\sum_{T,(y,x,w)\vec{\alpha}\vec{\beta}}\frac{d_Tmp}{d_wd_xd_y(n+m+p)d_{R'}}$$
$$\mathrm{Tr}_{R\oplus T}([P_{R,(t,s,r)\vec{\mu}\vec{\nu}},\Gamma^R(1,p+1)]I_{R'T'}[P_{T,(y,x,w)\vec{\alpha}\vec{\beta}},\Gamma^T(1,p+1)]I_{T'R'})\chi_{T,(y,x,w)\vec{\beta}\vec{\alpha}}(X,Y,Z)\,.$$

To get to the last line sum over $S_{n+m+p-1}$ using the fundamental orthogonality relation. Now consider the second term, which is treated in exactly the same way

$$[Y,Z]^i_j\left(\frac{d}{dY^k_j}\frac{d}{dZ^i_k}-\frac{d}{dZ^k_j}\frac{d}{dY^i_k}\right)\frac{1}{n!m!p!}\sum_{\sigma\in S_{n+m+p}}\mathrm{Tr}_{(t,s,r)\vec{\mu}\vec{\nu}}(\Gamma^R(\sigma))X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}$$

$$\times Z^{i_{m+p+1}}_{i_{\sigma(m+p+1)}} \cdots Z^{i_{m+p+n}}_{i_{\sigma(m+p+n)}}$$

$$= \frac{mn}{n!m!p!} \sum_{\sigma \in S_{n+m+p}} \mathrm{Tr}_{(t,s,r)\vec{\mu}\vec{\nu}}(\Gamma^R(\sigma))(\delta^{i_{p+1}}_{i_{\sigma(p+m+1)}}[Y,Z]^{i_{p+m+1}}_{i_{\sigma(p+1)}} - \delta^{i_{p+m+1}}_{i_{\sigma(p+1)}}[Y,Z]^{i_{p+1}}_{i_{\sigma(p+m+1)}})$$
$$\times X^{i_1}_{i_{\sigma(1)}} \cdots X^{i_p}_{i_{\sigma(p)}} Y^{i_{p+2}}_{i_{\sigma(p+2)}} \cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}} Z^{i_{m+p+2}}_{i_{\sigma(m+p+2)}} \cdots Z^{i_{m+p+n}}_{i_{\sigma(m+p+n)}}$$

$$= \frac{mn}{n!m!p!} \sum_{\sigma \in S_{n+m+p}} \mathrm{Tr}_{(t,s,r)\vec{\mu}\vec{\nu}}(\Gamma^R((1,p+1)\sigma(1,p+1)))(\delta^{i_1}_{i_{\sigma(p+m+1)}}[Y,Z]^{i_{p+m+1}}_{i_{\sigma(1)}} - \delta^{i_{p+m+1}}_{i_{\sigma(1)}}[Y,Z]^{i_1}_{i_{\sigma(p+m+1)}})$$
$$\times X^{i_2}_{i_{\sigma(2)}} \cdots X^{i_{p+1}}_{i_{\sigma(p+1)}} Y^{i_{p+2}}_{i_{\sigma(p+2)}} \cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}} Z^{i_{m+p+2}}_{i_{\sigma(m+p+2)}} \cdots Z^{i_{m+p+n}}_{i_{\sigma(m+p+n)}}$$

$$= \frac{mn}{n!m!p!} \sum_{\sigma \in S_{n+m+p}} \mathrm{Tr}_{(t,s,r)\vec{\mu}\vec{\nu}}(\Gamma^R((1,p+1)[\sigma,(1,p+m+1)](1,p+1)))\delta^{i_1}_{i_{\sigma(1)}}[Y,Z]^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}$$
$$\times X^{i_2}_{i_{\sigma(2)}} \cdots X^{i_{p+1}}_{i_{\sigma(p+1)}} Y^{i_{p+2}}_{i_{\sigma(p+2)}} \cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}} Z^{i_{m+p+2}}_{i_{\sigma(m+p+2)}} \cdots Z^{i_{m+p+n}}_{i_{\sigma(m+p+n)}}$$

$$= \frac{mn}{n!m!p!} \sum_{R'} c_{RR'} \sum_{\sigma \in S_{n+m+p-1}} \mathrm{Tr}_{(t,s,r)\vec{\mu}\vec{\nu}}(\Gamma^R(1,p+1)[\Gamma^{R'}(\sigma),\Gamma^R(1,p+m+1)]\Gamma^R(1,p+1)))$$
$$\times \mathrm{Tr}_{V^{\otimes n+m+p-1}}([\sigma,(p+1,p+m+1)] \cdot X^{\otimes p} \otimes Y^{\otimes m-1} \otimes [Y,Z] \otimes Z^{\otimes n-1})$$

$$= \frac{mn}{n!m!p!} \sum_{T,(y,x,w)\vec{\alpha}\vec{\beta}} \frac{d_T n!m!p!}{d_w d_x d_y (n+m+p)!} \sum_{R'} c_{RR'} \sum_{\sigma \in S_{n+m+p-1}}$$
$$\mathrm{Tr}_{(t,s,r)\vec{\mu}\vec{\nu}}(\Gamma^R(1,p+1)[\Gamma^{R'}(\sigma),\Gamma^R(1,p+m+1)]\Gamma^R(1,p+1)))$$
$$\times \mathrm{Tr}_{(y,x,w)\vec{\alpha}\vec{\beta}}(\Gamma_T((1,p+1)[\sigma,(1,p+m+1)](1,p+1)))\chi_{T,(y,x,w)\vec{\beta}\vec{\alpha}}(X,Y,Z)$$

$$= \sum_{R'} c_{RR'} \sum_{T,(y,x,w)\vec{\alpha}\vec{\beta}} \frac{d_T mn}{d_w d_x d_y (n+m+p) d_{R'}}$$
$$\mathrm{Tr}_{R\oplus T}([\Gamma^R(1,p+1)P_{R,(t,s,r)\vec{\mu}\vec{\nu}}\Gamma^R(1,p+1),\Gamma^R(1,p+m+1)]I_{R'T'}$$
$$\times[\Gamma^T(1,p+1)P_{T,(y,x,w)\vec{\alpha}\vec{\beta}}\Gamma^T(1,p+1),\Gamma^T(1,p+m+1)]I_{T'R'})\chi_{T,(y,x,w)\vec{\beta}\vec{\alpha}}(X,Y,Z)$$

Finally, consider the third term

$$[X,Z]^i_j \left(\frac{d}{dX^k_j}\frac{d}{dZ^i_k} - \frac{d}{dZ^k_j}\frac{d}{dX^i_k}\right) \frac{1}{n!m!p!} \sum_{\sigma \in S_{n+m+p}} \mathrm{Tr}_{(t,s,r)\vec{\mu}\vec{\nu}}(\Gamma^R(\sigma)) X^{i_1}_{i_{\sigma(1)}} \cdots X^{i_p}_{i_{\sigma(p)}} Y^{i_{p+1}}_{i_{\sigma(p+1)}} \cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}$$
$$\times Z^{i_{m+p+1}}_{i_{\sigma(m+p+1)}} \cdots Z^{i_{m+p+n}}_{i_{\sigma(m+p+n)}}$$

$$= \frac{pn}{n!m!p!} \sum_{\sigma \in S_{n+m+p}} \mathrm{Tr}_{(t,s,r)\vec{\mu}\vec{\nu}}(\Gamma^R(\sigma))(\delta^{i_1}_{i_{\sigma(p+m+1)}}[X,Z]^{i_{p+m+1}}_{i_{\sigma(1)}} - \delta^{i_{p+m+1}}_{i_{\sigma(1)}}[X,Z]^{i_1}_{i_{\sigma(p+m+1)}})$$
$$\times X^{i_2}_{i_{\sigma(2)}} \cdots X^{i_p}_{i_{\sigma(p)}} Y^{i_{p+1}}_{i_{\sigma(p+1)}} \cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}} Z^{i_{m+p+2}}_{i_{\sigma(m+p+2)}} \cdots Z^{i_{m+p+n}}_{i_{\sigma(m+p+n)}}$$

$$= \frac{pn}{n!m!p!} \sum_{\sigma \in S_{n+m+p}} \mathrm{Tr}_{(t,s,r)\vec{\mu}\vec{\nu}}(\Gamma^R([\sigma,(1,p+m+1)]))\delta^{i_1}_{i_{\sigma(1)}}[X,Z]^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}$$
$$\times X^{i_2}_{i_{\sigma(2)}} \cdots X^{i_p}_{i_{\sigma(p)}} Y^{i_{p+1}}_{i_{\sigma(p+1)}} \cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}} Z^{i_{m+p+2}}_{i_{\sigma(m+p+2)}} \cdots Z^{i_{m+p+n}}_{i_{\sigma(m+p+n)}}$$

$$= \frac{pn}{n!m!p!} \sum_{R'} c_{RR'} \sum_{\sigma \in S_{n+m+p-1}} \mathrm{Tr}_{(t,s,r)\vec{\mu}\vec{\nu}}([\Gamma^{R'}(\sigma),\Gamma^R(1,p+m+1)])$$
$$\times \mathrm{Tr}_{V^{\otimes m+n+p-1}}(\sigma \cdot X^{\otimes p-1} \otimes Y^{\otimes m} \otimes [X,Z] \otimes Z^{\otimes n-1})$$

$$= \frac{pn}{n!m!p!} \sum_{R'} c_{RR'} \sum_{T,(y,x,w)\vec{\alpha}\vec{\beta}} \frac{d_T n!m!p!}{d_w d_x d_y (n+m+p)!}$$
$$\mathrm{Tr}_{R\oplus T}([P_{R,(t,s,r)\vec{\mu}\vec{\nu}},\Gamma^R(1,p+m+1)]I_{R'T'}[P_{T,(y,x,w)\vec{\alpha}\vec{\beta}},\Gamma^T(1,p+m+1)]I_{T'R'})\chi_{T,(y,x,w)\vec{\beta}\vec{\alpha}}(X,Y,Z)$$

$$
= \sum_{R'} c_{RR'} \sum_{T,(y,x,w)\vec{\alpha}\vec{\beta}} \frac{d_T np}{d_w d_x d_y (n+m+p) d_{R'}}
$$
$$
\mathrm{Tr}_{R\oplus T}([P_{R,(t,s,r)\vec{\mu}\vec{\nu}}, \Gamma^R(1,p+m+1)] I_{R'T'} [P_{T,(y,x,w)\vec{\alpha}\vec{\beta}}, \Gamma^T(1,p+m+1)] I_{T'R'}) \chi_{T,(y,x,w)\vec{\beta}\vec{\alpha}}(X,Y,Z)
$$

$$
[Y,X]^i_j \left(\frac{d}{dY^k_j} \frac{d}{dX^i_k} - \frac{d}{dX^k_j} \frac{d}{dY^i_k} \right) \hat{O}_{R,(t,s,r)\vec{\mu}\vec{\nu}}
$$
$$
= \sum_{R'} c_{RR'} \sum_{T,(y,x,w)\vec{\alpha}\vec{\beta}} \frac{d_T mp}{d_w d_x d_y (n+m+p) d_{R'}} \sqrt{\frac{f_T \mathrm{hooks}_T \mathrm{hooks}_r \mathrm{hooks}_s \mathrm{hooks}_t}{f_R \mathrm{hooks}_R \mathrm{hooks}_w \mathrm{hooks}_x \mathrm{hooks}_y}}
$$
$$
\mathrm{Tr}_{R\oplus T}([P_{R,(t,s,r)\vec{\mu}\vec{\nu}}, \Gamma^R(1,p+1)] I_{R'T'} [P_{T,(y,x,w)\vec{\alpha}\vec{\beta}}, \Gamma^T(1,p+1)] I_{T'R'}) \hat{O}_{T,(y,x,w)\vec{\beta}\vec{\alpha}} \, . \tag{3.4}
$$

Using identical methods it is straight forward to find

$$
[Y,Z]^i_j \left(\frac{d}{dY^k_j} \frac{d}{dZ^i_k} - \frac{d}{dZ^k_j} \frac{d}{dY^i_k} \right) \hat{O}_{R,(t,s,r)\vec{\mu}\vec{\nu}}
$$
$$
= \sum_{R'} c_{RR'} \sum_{T,(y,x,w)\vec{\alpha}\vec{\beta}} \frac{d_T mn}{d_w d_x d_y (n+m+p) d_{R'}} \sqrt{\frac{f_T \mathrm{hooks}_T \mathrm{hooks}_r \mathrm{hooks}_s \mathrm{hooks}_t}{f_R \mathrm{hooks}_R \mathrm{hooks}_w \mathrm{hooks}_x \mathrm{hooks}_y}}
$$
$$
\mathrm{Tr}_{R\oplus T}([\Gamma^R(1,p+1) P_{R,(t,s,r)\vec{\mu}\vec{\nu}} \Gamma^R(1,p+1), \Gamma^R(1,p+m+1)] I_{R'T'}
$$
$$
\times [\Gamma^T(1,p+1) P_{T,(y,x,w)\vec{\alpha}\vec{\beta}} \Gamma^T(1,p+1), \Gamma^T(1,p+m+1)] I_{T'R'}) \hat{O}_{T,(y,x,w)\vec{\beta}\vec{\alpha}} \, , \tag{3.5}
$$

$$
[X,Z]^i_j \left(\frac{d}{dX^k_j} \frac{d}{dZ^i_k} - \frac{d}{dZ^k_j} \frac{d}{dX^i_k} \right) \hat{O}_{R,(t,s,r)\vec{\mu}\vec{\nu}}
$$
$$
= \sum_{R'} c_{RR'} \sum_{T,(y,x,w)\vec{\alpha}\vec{\beta}} \frac{d_T pn}{d_w d_x d_y (n+m+p) d_{R'}} \sqrt{\frac{f_T \mathrm{hooks}_T \mathrm{hooks}_r \mathrm{hooks}_s \mathrm{hooks}_t}{f_R \mathrm{hooks}_R \mathrm{hooks}_w \mathrm{hooks}_x \mathrm{hooks}_y}}
$$
$$
\mathrm{Tr}_{R\oplus T}([P_{R,(t,s,r)\vec{\mu}\vec{\nu}}, \Gamma^R(1,p+m+1)] I_{R'T'} [P_{T,(y,x,w)\vec{\alpha}\vec{\beta}}, \Gamma^T(1,p+m+1)] I_{T'R'}) \hat{O}_{T,(y,x,w)\vec{\beta}\vec{\alpha}} \, . \tag{3.6}
$$

The next step in the evaluation of the action of the dilatation operator entails computing the traces over $R \oplus T$ that have appeared in our results above. Our results for the action of the one loop dilatation operator given above are exact. From this point on we assume the displaced corners approximation so that our answers for the traces are only valid in the large N limit. The background information used in this section can be found in [26]. For the term in the one loop dilatation operator that mixes X and Y the trace that needs to be computed is

$$
\mathbb{T} = \mathrm{Tr}_{R\oplus T}([P_{R,(t,s,r)\vec{\mu}\vec{\nu}}, \Gamma^R(1,p+1)] I_{R'T'} [P_{T,(y,x,w)\vec{\alpha}\vec{\beta}}, \Gamma^T(1,p+1)] I_{T'R'}) \, . \tag{3.7}
$$

To ease the notation we will use the following shorthand

$$
P_{T,(y,x,w)\vec{\alpha}\vec{\beta}} \equiv p_y \otimes p_x \otimes \mathbf{1}_w \, . \tag{3.8}
$$

Consider the case that R' is obtained from R by dropping a box in row i and that T' is obtained from T by dropping a box from row j. The intertwiner is only non-zero if $T' = R'$. In this case the intertwiners are

$$I_{R'T'} = E^{(1)}_{ij}\,, \qquad I_{T'R'} = E^{(1)}_{ji}\,. \tag{3.9}$$

Since the trace $\mathbb{T}$ is a product of two commutators, when we expand things out we get a total of four terms. Since both the swaps $\Gamma^R(1,p+1)$ and $\Gamma^T(1,p+1)$ have a trivial action on the Z indices, we know that the result will be proportional to δ_{rw} and that the trace over the Z indices produce a factor d_r. Thus, after tracing over the Z indices we have

$$\begin{aligned}\mathbb{T} = \Big(&\mathrm{Tr}(p_t\otimes p_s\Gamma^R(1,p+1)E^{(1)}_{ij}p_y\otimes p_x\Gamma^T(1,p+1)E^{(1)}_{ji})\\ &-\mathrm{Tr}(p_t\otimes p_s\Gamma^R(1,p+1)E^{(1)}_{ij}\Gamma^T(1,p+1)p_y\otimes p_xE^{(1)}_{ji})\\ &-\mathrm{Tr}(\Gamma^R(1,p+1)p_t\otimes p_sE^{(1)}_{ij}p_y\otimes p_x\Gamma^T(1,p+1)E^{(1)}_{ji})\\ &+\mathrm{Tr}(\Gamma^R(1,p+1)p_t\otimes p_sE^{(1)}_{ij}\Gamma^T(1,p+1)p_y\otimes p_xE^{(1)}_{ji})\Big)\,\delta_{rw}d_r\,.\end{aligned} \tag{3.10}$$

Allow the swaps to act on the intertwiners

$$(1,p+1)E^{(1)}_{ij} = E^{(1)}_{lj}E^{(p+1)}_{il}\,, \qquad (1,p+1)E^{(1)}_{ji} = E^{(1)}_{li}E^{(p+1)}_{jl} \tag{3.11}$$

to obtain

$$\begin{aligned}\mathbb{T} = \Big(&\langle\vec{p},t,\nu_1;a|E^{(1)}_{lj}|\vec{p}',y,\alpha_1;b\rangle\langle\vec{p}',y,\beta_1;b|E^{(1)}_{ki}|\vec{p},t,\mu_1;a\rangle\\ &\quad\times\langle\vec{m},s,\nu_2;c|E^{(p+1)}_{il}|\vec{m}',x,\alpha_2;d\rangle\langle\vec{m}',x,\beta_2;d|E^{(p+1)}_{jk}|\vec{m},s,\mu_2;c\rangle\\ &-\langle\vec{p},t,\nu_1;a|\vec{p}',y,\alpha_1;b\rangle\langle\vec{p}',y,\beta_1;b|E^{(1)}_{ji}|\vec{p},t,\mu_1;a\rangle\\ &\quad\times\langle\vec{m},s,\nu_2;c|E^{(p+1)}_{ij}|\vec{m}',x,\alpha_2;d\rangle\langle\vec{m}',x,\beta_2;d|\vec{m},s,\mu_2;c\rangle\\ &-\langle\vec{p},t,\nu_1;a|E^{(1)}_{ij}|\vec{p}',y,\alpha_1;b\rangle\langle\vec{p}',y,\beta_1;b|\vec{p},t,\mu_1;a\rangle\\ &\quad\times\langle\vec{m},s,\nu_2;c|\vec{m}',x,\alpha_2;d\rangle\langle\vec{m}',x,\beta_2;d|E^{(p+1)}_{ji}|\vec{m},s,\mu_2;c\rangle\\ &+\langle\vec{p},t,\nu_1;a|E^{(1)}_{il}|\vec{p}',y,\alpha_1;b\rangle\langle\vec{p}',y,\beta_1;b|E^{(1)}_{jk}|\vec{p},t,\mu_1;a\rangle\\ &\quad\times\langle\vec{m},s,\nu_2;c|E^{(p+1)}_{lj}|\vec{m}',x,\alpha_2;d\rangle\langle\vec{m}',x,\beta_2;d|E^{(p+1)}_{ki}|\vec{m},s,\mu_2;c\rangle\Big)\delta_{rw}d_r\\[1em] = \Big(&\langle\vec{p},t,\nu_1;a|E^{(1)}_{lj}|\vec{p}',y,\alpha_1;b\rangle\langle\vec{p}',y,\beta_1;b|E^{(1)}_{ki}|\vec{p},t,\mu_1;a\rangle\\ &\quad\times\langle\vec{m},s,\nu_2;c|E^{(p+1)}_{il}|\vec{m}',x,\alpha_2;d\rangle\langle\vec{m}',x,\beta_2;d|E^{(p+1)}_{jk}|\vec{m},s,\mu_2;c\rangle\\ &-\delta_{\vec{p}\vec{p}'}\delta_{yt}\delta_{\nu_1\alpha_1}\delta_{\vec{m}\vec{m}'}\delta_{sx}\delta_{\beta_2\mu_2}\langle\vec{p}',y,\beta_1;b|E^{(1)}_{ji}|\vec{p},t,\mu_1;a\rangle\\ &\quad\times\langle\vec{m},s,\nu_2;c|E^{(p+1)}_{ij}|\vec{m}',x,\alpha_2;d\rangle\\ &-\delta_{\vec{p}\vec{p}'}\delta_{yt}\delta_{\mu_1\beta_1}\delta_{\vec{m}\vec{m}'}\delta_{sx}\delta_{\alpha_2\nu_2}\langle\vec{p},t,\nu_1;a|E^{(1)}_{ij}|\vec{p}',y,\alpha_1;b\rangle\\ &\quad\times\langle\vec{m}',x,\beta_2;d|E^{(p+1)}_{ji}|\vec{m},s,\mu_2;c\rangle\\ &+\langle\vec{p},t,\nu_1;a|E^{(1)}_{il}|\vec{p}',y,\alpha_1;b\rangle\langle\vec{p}',y,\beta_1;b|E^{(1)}_{jk}|\vec{p},t,\mu_1;a\rangle\\ &\quad\times\langle\vec{m},s,\nu_2;c|E^{(p+1)}_{lj}|\vec{m}',x,\alpha_2;d\rangle\langle\vec{m}',x,\beta_2;d|E^{(p+1)}_{ki}|\vec{m},s,\mu_2;c\rangle\Big)\delta_{rw}d_r\,.\end{aligned} \tag{3.12}$$

In a similar way we obtain

$$
\begin{aligned}
&\mathrm{Tr}_{R\oplus T}([\Gamma^R(1,p+1)P_{R,(t,s,r)\vec\mu\vec\nu}\Gamma^R(1,p+1),\Gamma^R(1,p+m+1)]I_{R'T'}\\
&\times[\Gamma^T(1,p+1)P_{T,(y,x,w)\vec\alpha\vec\beta}\Gamma^T(1,p+1),\Gamma^T(1,p+m+1)]I_{T'R'})\\
&=\delta_{ty}d_t\delta_{\vec p\vec p'}\delta_{\nu_1\alpha_1}\delta_{\beta_1\mu_1}d_{r'_i}\delta_{r'_iw'_j}\delta_{\vec m\vec m'}\Big[\langle\vec m',x,\beta_2;d|E^{(p+1)}_{ii}|\vec m,s,\mu_2;c\rangle\langle\vec m,s\nu_2;c|E^{(p+1)}_{jj}|\vec m',x,\alpha_2;d\rangle\\
&+\langle\vec m',x,\beta_2;d|E^{(p+1)}_{jj}|\vec m,s,\mu_2;c\rangle\langle\vec m,s\nu_2;c|E^{(p+1)}_{ii}|\vec m',x,\alpha_2;d\rangle\Big]\\
&-\delta_{ij}\delta_{ty}d_t\delta_{\vec p\vec p'}\delta_{\nu_1\alpha_1}\delta_{\beta_1,\mu_1}d_{r'_i}\delta_{rw}\delta_{sx}\delta_{\vec m\vec m'}\Big[\delta_{\nu_2\alpha_2}\langle\vec m,x,\beta_2;c|E^{(p+1)}_{jj}|\vec m,s,\mu_2;c\rangle\\
&+\delta_{\beta_2\mu_2}\langle\vec m,s\nu_2;c|E^{(p+1)}_{ii}|\vec m,x,\alpha_2;c\rangle\Big]
\end{aligned}
\tag{3.13}
$$

relevant for the term in the one loop dilatation operator that mixes Z and Y and

$$
\begin{aligned}
&\mathrm{Tr}_{R\oplus T}([P_{R,(t,s,r)\vec\mu\vec\nu},\Gamma^R(1,p+m+1)]I_{R'T'}[P_{T,(y,x,w)\vec\alpha\vec\beta},\Gamma^T(1,p+m+1)]I_{T'R'})\\
&=\delta_{sx}d_s\delta_{\vec p\vec p'}\delta_{\nu_2\alpha_2}\delta_{\beta_2\mu_2}d_{r'_i}\delta_{r'_iw'_j}\delta_{\vec m\vec m'}\Big[\langle\vec p',y,\beta_1;d|E^{(1)}_{ii}|\vec p,t,\mu_1;c\rangle\langle\vec p,t,\nu_1;c|E^{(1)}_{jj}|\vec p',y,\alpha_1;d\rangle\\
&+\langle\vec p',y,\beta_1;d|E^{(1)}_{jj}|\vec p,t,\mu_1;c\rangle\langle\vec p,t,\nu_1;c|E^{(1)}_{ii}|\vec p',y,\alpha_1;d\rangle\Big]\\
&-\delta_{ij}\delta_{ty}d_s\delta_{\vec p\vec p'}\delta_{\nu_2\alpha_2}\delta_{\beta_2\mu_2}d_{r'_i}\delta_{rw}\delta_{sx}\delta_{\vec m\vec m'}\Big[\delta_{\nu_1\alpha_1}\langle\vec p,y,\beta_1;c|E^{(1)}_{jj}|\vec p,t,\mu_1;c\rangle\\
&+\delta_{\beta_1\mu_1}\langle\vec p,t,\nu_1;c|E^{(1)}_{ii}|\vec p,y,\alpha_1;c\rangle\Big]
\end{aligned}
\tag{3.14}
$$

which is relevant for the term in the one loop dilatation operator that mixes X and Z.

This completes our discussion of the action of the one loop dilatation operator.

3.2 Gauss Operators

The problem of diagonalizing the terms in the dilatation operator that mix the X and Z fields and the terms that mix the Y and Z fields has been solved[25, 26, 27, 28]. The operators that have a good scaling dimension are the Gauss operators. Our ultimate goal is to write the action of the terms in the dilatation operator that mix X and Y fields, on the Gauss operators, which amounts to a change of basis from restricted Schur polynomials to Gauss operators. Towards this end we describe how to construct Gauss operators for operators built from three complex scalar fields and develop the tools we will need to change basis. The results of this section are a simple generalization of [28].

Natural hints for the construction of the Gauss operators come from the AdS/CFT correspondence. Indeed, the correspondence implies an equivalence between quantum states in the quantum gravity and operators in the $\mathcal{N}=4$ SYM theory. In particular, the restricted Schur polynomials $\chi_{R,(t,s,r),\vec\mu\vec\nu}(X,Y,Z)$ are dual to multiple giant graviton systems [34, 35, 36] consisting of large branes in the AdS_5 space when R has order one rows each of length order N, or to systems consisting of large branes in the S^5 space when R has order one columns each of length order N. A giant graviton has a compact world volume so that the Gauss Law forces the total charge on the giant's world volume to vanish. Since the string end points are charged, this gives a constraint on the possible open string configurations that are allowed: the number of strings leaving the giant must equal the number of strings arriving at the giant. The matrices X and Y generate two species of 1-bit strings [37, 38, 39, 40]. Each row of R corresponds to a giant graviton. Each open string configuration corresponds to a pair of graphs - one for each open string species. We will refer to these as the X graph and the Y graph. The vertices of the graph represent the branes and the directed links represent the (oriented) strings. Motivated by [41] a useful combinatoric description of these graphs is to divide each string into two halves and label each half. Using the orientation of the string, label the outgoing ends with numbers $\{1, \cdots, p\}$ for the X graph or $\{1, \cdots, m\}$ for the Y graph and the ingoing ends with these same numbers. A permutation $\sigma \in S_p \times S_m$ is then determined by how the halves are joined. We will often decompose $\sigma = \sigma^X \circ \sigma^Y$ with $\sigma^X \in S_p$ and $\sigma^Y \in S_m$. Given a permutation, we can reconstruct the graphs. A graph is not associated to a unique permutation because the strings leaving the i'th vertex are indistinguishable, and the strings arriving at the i'th vertex are indistinguishable.

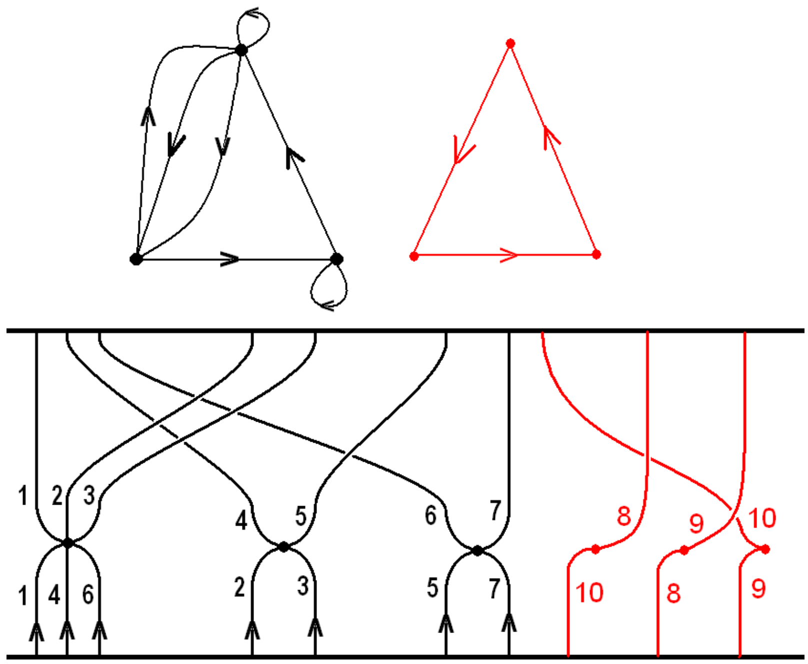


Figure 1: Any open string configuration can be mapped to a pair of labeled graphs. The black graph describes the X matrices and the red graph the Y matrices. The two bold horizontal lines are identified. The graphs determine a permutation, so each open string configuration is mapped to a permutation. For the graph shown the permutation in cycle notation is $\sigma = (2,4)(5,3,6)(8,10,9)$. The figure shows a configuration for a three giant system with ten open strings attached. Equivalently, this is an operator whose Young diagram describing the Z fields has 3 long rows/columns and $p = 7$, $m = 3$. The vectors $\vec{p}$ and $\vec{m}$ describe the number of strings leaving each node. Thus, $\vec{p} = (3,2,2)$, $\vec{m} = (1,1,1)$.

We will make use of two subgroups in what follows

$$H_Y = S_{m_1} \times \cdots \times S_{m_g} \qquad H_X = S_{p_1} \times \cdots \times S_{p_g} \, . \tag{3.15}$$

H_X acts on boxes in the partly labeled Young diagrams that are labeled with an integer $i < p+1$, i.e. on the boxes associated to Xs. H_Y acts on boxes associated to Ys. These two subgroups leave all partly labeled Young diagrams invariant. Consequently, the partly labeled Young diagrams belong to $S_p \times S_m / H_X \times H_Y$. The Gauss graphs themselves are in one-to-one correspondence with elements of the double coset

$$H_X \times H_Y \setminus S_p \times S_m / H_X \times H_Y \tag{3.16}$$

Introduce the states (these states span $V^{\otimes p+m}$)

$$|v, \vec{p}, \vec{m}\rangle \equiv |v_1^{\otimes p_1} \otimes v_2^{\otimes p_2} \otimes \cdots v_g^{\otimes p_g} \otimes v_{g+1}^{\otimes m_1} \otimes v_{g+2}^{\otimes m_2} \otimes \cdots v_{2g}^{\otimes m_g}\rangle \, . \tag{3.17}$$

There is an action of the $S_p \times S_m$ group defined on this space by

$$\sigma |v_{i_1} \otimes \cdots \otimes v_{i_{m+p}}\rangle = |v_{i_{\sigma(1)}} \otimes \cdots \otimes v_{i_{\sigma(m+p)}}\rangle \, . \tag{3.18}$$

This can trivially be enlarged to obtain an action of S_{p+m}, but we want to consider only permutations that mix X indices with each other and Y indices with each other, but not X and Y indices. Introduce the notation $|v_\sigma\rangle \equiv \sigma|v, \vec{p}, \vec{m}\rangle$. Invariance under the $H_X \times H_Y$ subgroup can be written as

$$|v_\sigma\rangle = |v_{\sigma\gamma}\rangle \quad \gamma \in H_X \times H_Y \tag{3.19}$$

or even

$$|v_\sigma\rangle = \frac{1}{|H_X \times H_Y|} \sum_{\gamma \in H_X \times H_Y} |v_{\sigma\gamma}\rangle \, . \tag{3.20}$$

Recall that the operator that projects onto representation r of a group $\mathcal{G}$ is given by[42]

$$P_r = \frac{d_r}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \chi_r(g) g \, . \tag{3.21}$$

By the identity representation we mean the representation for which all the elements of $H_X \times H_Y$ are represented by 1. We want to project onto the identity representation of $H_X \times H_Y$ within the carrier space (s,t) organizing the Xs and Ys. Recall that $t \vdash p$ and $s \vdash m$. The characters in the identity representation are of course all equal to 1. The identity representation may appear more than once in (s,t). Resolve these different copies with a multiplicity label $\vec{\mu}$. The multiplicity label has two components, one that refers to s and one that refers to t. Introduce branching coefficients that resolve these projectors into a set of projectors onto each of the one dimensional spaces labeled by $\vec{\mu}$

$$\frac{1}{|H_X \times H_Y|} \sum_{\gamma \in H_X \times H_Y} \Gamma^{(s,t)}(\gamma)_{ik} = \sum_{\vec{\mu}} B_{i\vec{\mu}}^{(s,t) \to 1_{H_X \times H_Y}} B_{k\vec{\mu}}^{s \to 1_{H_X \times H_Y}} \, . \tag{3.22}$$

Thus, for example, $B^{(s,t)\to 1_H}_{i\vec{\nu}} B^{(s,t)\to 1_H}_{k\vec{\nu}}$ projects onto the copy ν_1 of the identity representation of H_X in s and onto the copy ν_2 of the identity representation of H_Y in t. The branching coefficient $B^{(s,t)\to 1_{H_X\times H_Y}}_{i\vec{\mu}}$ can be understood as the one dimensional vector that spans the $\vec{\nu}$ copy of $1_{H_X\times H_Y}$ inside the carrier space of (s,t)

$$|\vec{v}\rangle_i = B^{(s,t)\to 1_{H_X\times H_Y}}_{i\vec{\nu}} \ . \tag{3.23}$$

Vector orthogonality says

$$\langle\vec{\nu}|\vec{\mu}\rangle = \delta_{\vec{\mu}\vec{\nu}} = \sum_i B^{(s,t)\to 1_{H_X\times H_Y}}_{i\vec{\mu}} B^{(s,t)\to 1_{H_X\times H_Y}}_{i\vec{\nu}} \tag{3.24}$$

whilst vector completeness says

$$\sum_{\vec{\mu}} |\vec{\mu}\rangle\langle\vec{\mu}| = 1_{H_X\times H_Y} \tag{3.25}$$

or, displaying all indices,

$$\sum_{\vec{\mu}} B^{(s,t)\to 1_{H_X\times H_Y}}_{i\vec{\mu}} B^{(s,t)\to 1_{H_X\times H_Y}}_{j\vec{\mu}} = (1_{H_X\times H_Y})_{ij} \ . \tag{3.26}$$

Together (3.24) and (3.26) allow us to think of the branching coefficients $B^{(s,t)\to 1_{H_X\times H_Y}}_{i\vec{\mu}}$ as a matrix that implements a change of basis

$$|i\rangle = \sum_{\vec{\mu}} B^{(s,t)\to 1_{H_X\times H_Y}}_{i\vec{\mu}} |\vec{\mu}\rangle \qquad |\vec{\mu}\rangle = \sum_i B^{(s,t)\to 1_{H_X\times H_Y}}_{i\vec{\mu}} |i\rangle \ . \tag{3.27}$$

We are now ready to argue that the Gauss operators are simply an alternative basis to the restricted Schur polynomials. First, following [28], we will show that the number of restricted Schur polynmials is equal to the number of Gauss operators. Towards this end consider

$$|v_{(s,t)}, i, j\rangle = \sum_{\sigma\in S_m\times S_p} \Gamma^{(s,t)}_{ij}(\sigma)\sigma|v,\vec{p},\vec{m}\rangle \ . \tag{3.28}$$

Above we have projected onto the representation (s,t) of $S_m \times S_p$. You can think of j as a label for different vectors and of i as the components of the vector. According to [26] this space is organized by the Schur-Weyl duality between $S_m\times S_p$ and $U(m)\times U(p)$. Concretely, we can trade the index j for a Gelfand-Tsetlin pattern. Thus, using [26] we know that we can decompose this space as

$$\begin{aligned} V_g^{\otimes p+m} \qquad &= \oplus_{\substack{s\vdash m \ t\vdash p \\ c_1(s)\le g \ c_1(t)\le g}} V^{U(m)\times U(p)}_{(s,t)} \otimes V^{S_m\times S_p}_{(s,t)} \\ &= \oplus_{\substack{s\vdash m \ t\vdash p \\ c_1(s)\le g \ c_1(t)\le g}} \oplus_{\vec{m}\ \vec{p}} V^{U(g)\times U(g)\to U(1)^g\times U(1)^g}_{(s,t)\to(\vec{m},\vec{p})} \otimes V^{S_m\times S_p}_{(s,t)} \ . \end{aligned} \tag{3.29}$$

The first factor in the last line above is the space of Gelfand-Tsetlin patterns. Now, lets consider a different decomposition of this space, as follows[28]

$$
\begin{aligned}
|v_{(s,t)}, i, j\rangle &= \sum_{\sigma\in S_m\times S_p} \Gamma^{(s,t)}_{ij}(\sigma)\sigma|v,\vec{p},\vec{m}\rangle \\
&= \sum_{\sigma\in S_m\times S_p} \Gamma^{(s,t)}_{ij}(\sigma)|v_\sigma\rangle \\
&= \frac{1}{|H_X\times H_Y|}\sum_{\sigma\in S_m\times S_p}\sum_{\gamma\in H_X\times H_Y}\Gamma^{(s,t)}(\sigma\gamma)_{ij}|v_\sigma\rangle \\
&= \frac{1}{|H_X\times H_Y|}\sum_{\sigma\in S_m\times S_p}\sum_{\gamma\in H_X\times H_Y}\Gamma^{(s,t)}(\sigma)_{ik}\Gamma^{(s,t)}(\gamma)_{kj}|v_\sigma\rangle \\
&= \sum_{\sigma\in S_m\times S_p}\Gamma^{(s,t)}(\sigma)_{ik}\sum_{\vec{\mu}} B^{(s,t)\to 1_{H_X\times H_Y}}_{k\vec{\mu}} B^{(s,t)\to 1_{H_X\times H_Y}}_{j\vec{\mu}}|v_\sigma\rangle\,.
\end{aligned}
\tag{3.30}
$$

As we have already discussed, the branching coefficients provide a natural change of basis from one space to the other

$$
|\vec{m},\vec{p},(s,t),\vec{\mu};i\rangle = \sum_j B^{(s,t)\to 1_{H_X\times H_Y}}_{j\vec{\mu}} \sum_{\sigma\in S_m\times S_p}\Gamma^{(s,t)}_{ij}(\sigma)\sigma|v_\sigma\rangle\,. \tag{3.31}
$$

This decomposition is[28]

$$
\begin{aligned}
V_g^{\otimes p+m} &= \oplus_{\substack{s\vdash m\ t\vdash p \\ c_1(s)\le g\ c_1(t)\le g}} V^{U(m)\times U(p)}_{(s,t)}\otimes V^{S_m\times S_p}_{(s,t)} \\
&= \oplus_{\substack{s\vdash m\ t\vdash p \\ c_1(s)\le g\ c_1(t)\le g}} \oplus_{\vec{m}\ \vec{p}} V^{S_m\times S_p\to H_X\times H_Y}_{(s,t)\to 1_{H_X\times H_Y}}\otimes V^{S_m\times S_p}_{(s,t)}\,.
\end{aligned}
\tag{3.32}
$$

Comparing (3.29) to (3.32) we conclude that

$$
|V^{U(g)\times U(g)\to U(1)^g\times U(1)^g}_{(s,t)\to(\vec{m},\vec{p})}| = |V^{S_m\times S_p\to H_X\times H_Y}_{(s,t)\to 1_{H_X\times H_Y}}|\,. \tag{3.33}
$$

Using the idea that the branching coefficients provide a transformation between two bases, we easily write the Gauss operators

$$
\begin{aligned}
O_{R,r}(\sigma_X,\sigma_Y) &= \frac{|H_X\times H_Y|}{\sqrt{m!p!}}\sum_{jk}\sum_{s\vdash m}\sum_{t\vdash p}\sum_{\vec{\mu}}\sum_{\vec{\nu}}\sqrt{d_sd_t}\Gamma^{(s,t)}(\sigma_X\circ\sigma_Y)_{jk} \\
&\times B^{(s,t)\to 1_{H_X\times H_Y}}_{j\vec{\mu}} B^{(s,t)\to 1_{H_X\times H_Y}}_{k\vec{\nu}} O_{R,(t,s,r)\vec{\mu}\vec{\nu}}\,.
\end{aligned}
\tag{3.34}
$$

Note that the factor $\sqrt{d_sd_t}$ can not be determined by group theory alone. It is chosen so that the group theoretic coefficients

$$
C^{(s,t)}_{\vec{\mu}\vec{\nu}}(\sigma_X\circ\sigma_Y) = \frac{|H_X\times H_Y|}{\sqrt{m!p!}}\sum_{jk}\sum_{s\vdash m}\sum_{t\vdash p}\sqrt{d_sd_t}\Gamma^{(s,t)}(\sigma_X\circ\sigma_Y)_{jk}B^{(s,t)\to 1_{H_X\times H_Y}}_{j\vec{\mu}} B^{(s,t)\to 1_{H_X\times H_Y}}_{k\vec{\nu}} \tag{3.35}
$$

provide an orthogonal transformation between the restricted Schur polynomials and the Gauss graph basis. Indeed,

$$\sum_{(s,t)}\sum_{\vec\mu}\sum_{\vec\nu} C^{(s,t)}_{\vec\mu\vec\nu}(\sigma_X\circ\sigma_Y)C^{(s,t)}_{\vec\mu\vec\nu}(\tau_X\circ\tau_Y) = \sum_{\gamma\in H_X\times H_Y}\delta(\gamma_1\,\sigma_X\circ\sigma_Y\,\gamma_2\,\tau_X^{-1}\circ\tau_Y^{-1})\,. \quad (3.36)$$

There is an important point that is worth stressing here: Our Gauss operators are normalized as

$$\langle O_{R,r}(\sigma_X,\sigma_Y)O_{R,r}(\tau_X,\tau_Y)^\dagger\rangle = \sum_{\gamma\in H_X\times H_Y}\delta(\gamma_1\,\sigma_X\circ\sigma_Y\,\gamma_2\,\tau_X^{-1}\circ\tau_Y^{-1})\,. \quad (3.37)$$

These operators certainly do not have unit two point function. For example, if we set both σ_X,σ_Y and τ_X,τ_Y equal to the identity permutation, the right hand side evaluates to $|H_X\times H_Y|$. Our final answer is simplest when expressed in terms of normalized operators

$$\hat O_{R,r}(\sigma_X,\sigma_Y)\equiv\frac{1}{N_{\sigma_X,\sigma_Y}}O_{R,r}(\sigma_X,\sigma_Y)\,, \quad (3.38)$$

$$N^2_{\sigma_X,\sigma_Y} = \langle O_{R,r}(\sigma_X,\sigma_Y)O_{R,r}(\sigma_X,\sigma_Y)^\dagger\rangle\,. \quad (3.39)$$

We will not obtain or need the explicit form of N_{σ_X,σ_Y}.

3.3 Dilatation Operator in the Gauss Graph Basis

We will now write the term in the dilatation operator that mixes X and Y in the Gauss graph basis, i.e. we will write this term in the basis provided by (3.34). We already know that the other two terms are diagonal in this basis and we know their detailed form[26, 28].

Towards this end, transform the intertwining operator used to construct the restricted Schur polynomial

$$P_{R,(t,s,r)\vec\mu\vec\nu} = \sum_i\ |\vec m,\vec p,(s,t),\vec\mu;i\rangle\langle\vec m,\vec p,(s,t),\vec\nu;i|\otimes\mathbf{1}_r \quad (3.40)$$

to the Gauss graph basis. Of course, it is only

$$p^{\vec m\vec p}_{(t,s)\vec\mu\vec\nu} = \sum_i\ |\vec m,\vec p,(s,t),\vec\mu;i\rangle\langle\vec m,\vec p,(s,t),\vec\nu;i| \quad (3.41)$$

that we need to consider. The transformation is a simple computation

$$\sum_{(s,t)}|\vec m,\vec p,(s,t),\vec\mu;i\rangle\langle\vec m,\vec p,(s,t),\vec\nu;i|B^{(s,t)\to 1_{H_X\times H_Y}}_{l\vec\nu}\,B^{(s,t)\to 1_{H_X\times H_Y}}_{m\vec\mu}\,\Gamma^{(s,t)}_{lm}(\sigma_2)$$
$$=\frac{1}{|H_X\times H_Y|m!p!}\sum_{(s,t)}\sum_{\sigma,\tau\in S_m\times S_p} d_sd_t\ \ B^{(s,t)\to 1_{H_X\times H_Y}}_{j\vec\mu}\Gamma^{(s,t)}_{bj}(\sigma)|v_\sigma\rangle\langle v_\tau|\Gamma^{(s,t)}_{bk}(\tau)B^{(s,t)\to 1_{H_X\times H_Y}}_{k\vec\nu}$$

$$
\begin{aligned}
&\quad \times B^{(s,t)\to 1_{H_X\times H_Y}}_{l\vec{\nu}} B^{(s,t)\to 1_{H_X\times H_Y}}_{m\vec{\mu}} \Gamma^{(s,t)}_{lm}(\sigma_2)\\
&= \frac{1}{|H_X\times H_Y|m!p!}\sum_{(s,t)}\sum_{\sigma,\tau\in S_m\times S_p} d_s d_t \quad |v_\sigma\rangle\langle v_\tau|\Gamma^{(s,t)}_{jk}(\sigma^{-1}\tau) B^{(s,t)\to 1_{H_X\times H_Y}}_{j\vec{\mu}} B^{(s,t)\to 1_{H_X\times H_Y}}_{k\vec{\nu}}\\
&\quad B^{(s,t)\to 1_{H_X\times H_Y}}_{l\vec{\nu}} B^{(s,t)\to 1_{H_X\times H_Y}}_{m\vec{\mu}} \Gamma_{lm}(\sigma_2)\\
&= \frac{1}{|H_X\times H_Y|m!p!}\sum_{(s,t)}\sum_{\sigma,\tau\in S_m\times S_p} \frac{1}{|H_X\times H_Y|^2}\sum_{\gamma_1,\gamma_2\in H_X\times H_Y} d_s d_t \Gamma^{(s,t)}_{jm}(\gamma_1)\Gamma^{(s,t)}_{kl}(\gamma_2)\Gamma^{(s,t)}_{lm}(\sigma_2)\\
&\quad \times \Gamma^{(s,t)}_{jk}(\sigma^{-1}\tau)|v_\sigma\rangle\langle v_\tau|\\
&= \frac{1}{m!p!|H_X\times H_Y|}\sum_{(s,t)}\sum_{\sigma,\tau\in S_m\times S_p} \frac{1}{|H_X\times H_Y|^2}\sum_{\gamma_1,\gamma_2\in H_X\times H_Y} d_s d_t \chi_u(\gamma_1\sigma_2^{-1}\gamma_2^{-1}\tau^{-1}\sigma)|v_\sigma\rangle\langle v_\tau|\\
&= \frac{1}{|H_X\times H_Y|^3}\sum_{\sigma,\tau\in S_m\times S_p}\sum_{\gamma_1,\gamma_2\in H_X\times H_Y} \delta(\gamma_1\sigma_2^{-1}\gamma_2^{-1}\tau^{-1}\sigma)|v_\sigma\rangle\langle v_\tau|\,. \qquad (3.42)
\end{aligned}
$$

Notice that up to normalization this is a sum over all $\sigma,\tau\in S_m\times S_p$ of $|v_\sigma\rangle\langle v_\tau|$ with the condition that $\tau^{-1}\sigma$ belongs to the same coset as σ_2 does. With this result in hand, we easily find

$$
\begin{aligned}
&\langle O^\dagger_{T,w}(\sigma_2) D_{XY} O_{R,r}(\sigma_1)\rangle =\\
&= \frac{|H_X\times H_Y||H'_X\times H'_Y|}{m!p!}\sum_{jk}\sum_{s\vdash m}\sum_{t\vdash p}\sum_{\vec{\mu}\vec{\nu}}\sum_{lm}\sum_{x\vdash m}\sum_{y\vdash p}\sum_{\vec{\alpha}\vec{\beta}}\sqrt{d_s d_t}\sqrt{d_x d_y}\,\Gamma^{(s,t)}(\sigma_1)_{jk}\Gamma^{(x,y)}(\sigma_2)_{lm}\\
&\quad \times B^{(s,t)\to 1_{H_X\times H_Y}}_{j\vec{\mu}} B^{(s,t)\to 1_{H_X\times H_Y}}_{k\vec{\nu}} B^{(x,y)\to 1_{H'_X\times H'_Y}}_{l\vec{\alpha}} B^{(x,y)\to 1_{H'_X\times H'_Y}}_{m\vec{\beta}} \langle O^\dagger_{T,(y,x,w)\vec{\alpha}\vec{\beta}} D_{XY} O_{R,(t,s,r)\vec{\mu}\vec{\nu}}\rangle\\
&= \frac{|H_X\times H_Y||H'_X\times H'_Y|}{m!p!}\sum_{jk}\sum_{s\vdash m}\sum_{t\vdash p}\sum_{\vec{\mu}\vec{\nu}}\sum_{lm}\sum_{x\vdash m}\sum_{y\vdash p}\sum_{\vec{\alpha}\vec{\beta}}\sqrt{d_s d_t}\sqrt{d_x d_y}\,\Gamma^{(s,t)}(\sigma_1)_{jk}\Gamma^{(x,y)}(\sigma_2)_{lm}\\
&\quad \times B^{(s,t)\to 1_{H_X\times H_Y}}_{j\vec{\mu}} B^{(s,t)\to 1_{H_X\times H_Y}}_{k\vec{\nu}} B^{(x,y)\to 1_{H'_X\times H'_Y}}_{l\vec{\alpha}} B^{(x,y)\to 1_{H'_X\times H'_Y}}_{m\vec{\beta}}\\
&\quad \times \sum_{R'} c_{RR'}\frac{d_T mp}{d_x d_y (n+m+p) d_{R'}}\sqrt{\frac{f_T \mathrm{hooks}_T \mathrm{hooks}_s \mathrm{hooks}_t}{f_R \mathrm{hooks}_R \mathrm{hooks}_x \mathrm{hooks}_y}}\delta_{rw}\\
&\quad \times \Big(\langle \vec{p},t,\nu_1;a|E^{(1)}_{lj}|\vec{p}',y,\alpha_1;b\rangle\langle \vec{p}',y,\beta_1;b|E^{(1)}_{ki}|\vec{p},t,\mu_1;a\rangle\\
&\quad \times\langle \vec{m},s,\nu_2;c|E^{(p+1)}_{il}|\vec{m}',x,\alpha_2;d\rangle\langle \vec{m}',x,\beta_2;d|E^{(p+1)}_{jk}|\vec{m},s,\mu_2;c\rangle\\
&\quad -\delta_{\vec{p}\vec{p}'}\delta_{yt}\delta_{\nu_1\alpha_1}\delta_{\vec{m}\vec{m}'}\delta_{sx}\delta_{\beta_2\mu_2}\langle \vec{p}',y,\beta_1;b|E^{(1)}_{ji}|\vec{p},t,\mu_1;a\rangle\langle \vec{m},s,\nu_2;c|E^{(p+1)}_{ij}|\vec{m}',x,\alpha_2;d\rangle\\
&\quad -\delta_{\vec{p}\vec{p}'}\delta_{yt}\delta_{\mu_1\beta_1}\delta_{\vec{m}\vec{m}'}\delta_{sx}\delta_{\alpha_2\nu_2}\langle \vec{p},t,\nu_1;a|E^{(1)}_{ij}|\vec{p}',y,\alpha_1;b\rangle\langle \vec{m}',x,\beta_2;d|E^{(p+1)}_{ji}|\vec{m},s,\mu_2;c\rangle\\
&\quad +\langle \vec{p},t,\nu_1;a|E^{(1)}_{il}|\vec{p}',y,\alpha_1;b\rangle\langle \vec{p}',y,\beta_1;b|E^{(1)}_{jk}|\vec{p},t,\mu_1;a\rangle\\
&\quad \times\langle \vec{m},s,\nu_2;c|E^{(p+1)}_{lj}|\vec{m}',x,\alpha_2;d\rangle\langle \vec{m}',x,\beta_2;d|E^{(p+1)}_{ki}|\vec{m},s,\mu_2;c\rangle\Big)\\
&= \frac{|H_X\times H_Y||H'_X\times H'_Y|}{m!p!}\sum_{jk}\sum_{s\vdash m}\sum_{t\vdash p}\sum_{\vec{\mu}\vec{\nu}}\sum_{lm}\sum_{x\vdash m}\sum_{y\vdash p}\sum_{\vec{\alpha}\vec{\beta}}\Gamma^{(s,t)}(\sigma_1)_{jk}\Gamma^{(x,y)}(\sigma_2)_{lm}\\
&\quad \times B^{(s,t)\to 1_{H_X\times H_Y}}_{j\vec{\mu}} B^{(s,t)\to 1_{H_X\times H_Y}}_{k\vec{\nu}} B^{(x,y)\to 1_{H'_X\times H'_Y}}_{l\vec{\alpha}} B^{(x,y)\to 1_{H'_X\times H'_Y}}_{m\vec{\beta}}
\end{aligned}
$$

$$\sum_{R'} c_{RR'} \frac{d_T mp}{(n+m+p)d_{R'}} \sqrt{\frac{f_T \text{hooks}_T}{f_R \text{hooks}_R}} \delta_{rw} \Big(\langle \vec{p}, t, \nu_1; a | E^{(1)}_{lj} | \vec{p}', y, \alpha_1; b \rangle \langle \vec{p}', y, \beta_1; b | E^{(1)}_{ki} | \vec{p}, t, \mu_1; a \rangle$$
$$\times \langle \vec{m}, s, \nu_2; c | E^{(p+1)}_{il} | \vec{m}', x, \alpha_2; d \rangle \langle \vec{m}', x, \beta_2; d | E^{(p+1)}_{jk} | \vec{m}, s, \mu_2; c \rangle$$
$$- \delta_{\vec{p}\vec{p}'} \delta_{yt} \delta_{\nu_1\alpha_1} \delta_{\vec{m}\vec{m}'} \delta_{sx} \delta_{\beta_2\mu_2} \langle \vec{p}', y, \beta_1; b | E^{(1)}_{ji} | \vec{p}, t, \mu_1; a \rangle \langle \vec{m}, s, \nu_2; c | E^{(p+1)}_{ij} | \vec{m}', x, \alpha_2; d \rangle$$
$$- \delta_{\vec{p}\vec{p}'} \delta_{yt} \delta_{\mu_1\beta_1} \delta_{\vec{m}\vec{m}'} \delta_{sx} \delta_{\alpha_2\nu_2} \langle \vec{p}, t, \nu_1; a | E^{(1)}_{ij} | \vec{p}', y, \alpha_1; b \rangle \langle \vec{m}', x, \beta_2; d | E^{(p+1)}_{ji} | \vec{m}, s, \mu_2; c \rangle$$
$$+ \langle \vec{p}, t, \nu_1; a | E^{(1)}_{il} | \vec{p}', y, \alpha_1; b \rangle \langle \vec{p}', y, \beta_1; b | E^{(1)}_{jk} | \vec{p}, t, \mu_1; a \rangle$$
$$\times \langle \vec{m}, s, \nu_2; c | E^{(p+1)}_{lj} | \vec{m}', x, \alpha_2; d \rangle \langle \vec{m}', x, \beta_2; d | E^{(p+1)}_{ki} | \vec{m}, s, \mu_2; c \rangle \Big) .$$

There are four terms in the above expression. We will deal with each term, one at a time.

3.3.1 First term

Focus on the first term for now

$$= \frac{|H_X \times H_Y||H'_X \times H'_Y|}{m!p!} \sum_{jk} \sum_{s\vdash m} \sum_{t\vdash p} \sum_{\vec{\mu}\vec{\nu}} \sum_{lm} \sum_{x\vdash m} \sum_{y\vdash p} \sum_{\vec{\alpha}\vec{\beta}} \Gamma^{(s,t)}(\sigma_1)_{jk} \Gamma^{(x,y)}(\sigma_2)_{lm}$$
$$\times B^{(s,t)\to 1_{H_X\times H_Y}}_{j\vec{\mu}} B^{(s,t)\to 1_{H_X\times H_Y}}_{k\vec{\nu}} B^{(x,y)\to 1_{H'_X\times H'_Y}}_{l\vec{\alpha}} B^{(x,y)\to 1_{H'_X\times H'_Y}}_{m\vec{\beta}}$$
$$\sum_{R'} c_{RR'} \frac{d_T mp}{(n+m+p)d_{R'}} \sqrt{\frac{f_T \text{hooks}_T}{f_R \text{hooks}_R}} \delta_{rw} \langle \vec{p}, t, \nu_1; a | E^{(1)}_{lj} | \vec{p}', y, \alpha_1; b \rangle \langle \vec{p}', y, \beta_1; b | E^{(1)}_{ki} | \vec{p}, t, \mu_1; a \rangle$$
$$\times \langle \vec{m}, s, \nu_2; c | E^{(p+1)}_{il} | \vec{m}', x, \alpha_2; d \rangle \langle \vec{m}', x, \beta_2; d | E^{(p+1)}_{jk} | \vec{m}, s, \mu_2; c \rangle$$
$$= \frac{1}{|H_X \times H_Y|^2 |H'_X \times H'_Y|^2 m! p!}$$
$$\sum_{R'} c_{RR'} \text{hooks}_{R'} mp \sqrt{\frac{f_T}{f_R \text{hooks}_R \text{hooks}_T}} \delta_{rw} \sum_{\tau\in S_m\times S_p} \sum_{\phi\in S_m\times S_p} \sum_{\gamma_1,\gamma_2\in H'_X\times H'_Y} \sum_{\beta_1,\beta_2\in H_X\times H_Y}$$
$$\times \langle v, \vec{p}', \vec{m}' | E^{\tau^{-1}(p+1)}_{ji} \tau^{-1} (1, p+1) \phi \beta_2 \sigma_2 \beta_1^{-1} | v, \vec{p}, \vec{m} \rangle \langle v, \vec{p}, \vec{m} | E^{\phi^{-1}(p+1)}_{ij} \phi^{-1} (1, p+1) \tau \gamma_2 \sigma_1 \gamma_1^{-1} | v, \vec{p}', \vec{m}' \rangle$$

Now, lets study the case that $i = j$. To find a simple condition on $\vec{p}', \vec{m}'$ and $\vec{p}, \vec{m}$ that tells us when this matrix element is non-zero, focus on

$$\langle v_\tau | E^{(p+1)}_{ji} (1, p+1) | v_\psi \rangle \langle v_\phi | (1, p+1) E^{(1)}_{ij} | v_\sigma \rangle . \tag{3.43}$$

If $i = j$, the matrix element $\langle v, \vec{p}', \vec{m}' | \tau^{-1} E^{(p+1)}_{ji} (1, p+1) \psi | v, \vec{p}, \vec{m} \rangle$ forces $\vec{p} + \vec{m} = \vec{p}' + \vec{m}'$. Indeed, $E^{(p+1)}_{ii}$ is one if the vector in the first slot of $\psi | v, \vec{p}, \vec{m} \rangle$ is v_1 and it is zero otherwise, so it clearly does not change the identity of any vectors. The remaining elements between the two states (i.e. τ^{-1} and $(1, p+1) \psi$) can swap vectors around but not change the identity of any vector. Thus, the identity of the collection of vectors used to construct $|v, \vec{p}, \vec{m}\rangle$ must match the identity of the collection of vectors used to construct $|v, \vec{p}', \vec{m}'\rangle$. This then proves that $\vec{p} + \vec{m} = \vec{p}' + \vec{m}'$. We can argue for this conclusion in a second way: recall that we obtain R' from R by dropping box in row i and we obtain T' from T by dropping a box in row j. Thus, if $i = j$, since $R' = T'$ we are saying that $R = T$. We already know that $r = w$.

$\vec{p}+\vec{m}$ tells us the collection of boxes that needs to be dropped from R to get r and $\vec{p}'+\vec{m}'$ tells us the collection of boxes that needs to be dropped from T to get w. Since $R=T$ and $r=w$, this then again proves that $\vec{p}+\vec{m}=\vec{p}'+\vec{m}'$. We can say a bit more. Consider

$$\langle v,\vec{p},\vec{m}|\phi^{-1}\,(1,p+1)\,E^{(1)}_{ii}\sigma|v,\vec{p}',\vec{m}'\rangle\,. \tag{3.44}$$

This tells you that if you take the state $|v,\vec{p}',\vec{m}'\rangle$ and shuffle some of the X slots amongst each other and some of the Y slots amongst each other (σ does this shuffling) keeping only states with vector v_i in their first slot, and then swapping the vectors in slots 1 and $p+1$, we can get the vector $|v,\vec{p},\vec{m}\rangle$ by shuffling (according to ϕ^{-1}) what we have. **Thus, to get $|v,\vec{p},\vec{m}\rangle$ from $|v,\vec{p}',\vec{m}'\rangle$ we removed v_i from an X slot of $|v,\vec{p}',\vec{m}'\rangle$ and inserted it into a Y slot of $|v,\vec{p},\vec{m}\rangle$.**

Now consider

$$\langle v,\vec{p}',\vec{m}'|\tau^{-1}E^{(p+1)}_{ii}\,(1,p+1)\psi|v,\vec{p},\vec{m}\rangle\,. \tag{3.45}$$

This tells you that if you take the state $|v,\vec{p},\vec{m}\rangle$ and shuffle some of the X slots amongst each other and some of the Y slots amongst each other (ψ does this shuffling) keeping only states with vector v_i in their first slot, and then, swapping the vectors in slots 1 and $p+1$, we can get the vector $|v,\vec{p}',\vec{m}'\rangle$ by shuffling (according to τ^{-1}) what we have. **Thus, to get $|v,\vec{p}',\vec{m}'\rangle$ from $|v,\vec{p},\vec{m}\rangle$ we removed v_i from an X slot of $|v,\vec{p},\vec{m}\rangle$ and inserted it into a Y slot of $|v,\vec{p},\vec{m}\rangle$.**

Thus, the two vectors we are swapping have the *same* identity. This implies that we must have $\vec{p}=\vec{p}'$ and $\vec{m}=\vec{m}'$. Since we must have $\vec{p}=\vec{p}'$ and $\vec{m}=\vec{m}'$ we find

$$\begin{aligned}
&=\frac{1}{|H_X\times H_Y|^4 m!p!}\sum_{R'}c_{RR'}\mathrm{hooks}_{R'}mp\sqrt{\frac{f_T}{f_R\mathrm{hooks}_R\mathrm{hooks}_T}}\delta_{rw}\sum_{\tau\in S_m\times S_p}\sum_{\phi\in S_m\times S_p}\sum_{\gamma_1,\gamma_2,\beta_1,\beta_2\in H_X\times H_Y}\\
&\quad\times\langle v,\vec{p},\vec{m}|E^{\tau^{-1}(p+1)}_{ji}\tau^{-1}\,(1,p+1)\,\phi\beta_2\sigma_2\beta_1^{-1}|v,\vec{p},\vec{m}\rangle\langle v,\vec{p},\vec{m}|E^{\phi^{-1}(p+1)}_{ij}\phi^{-1}\,(1,p+1)\,\tau\gamma_2\sigma_1\gamma_1^{-1}|v,\vec{p},\vec{m}\rangle\\
&=\frac{1}{|H_X\times H_Y|^4 m!p!}\sum_{R'}c_{RR'}\mathrm{hooks}_{R'}mp\sqrt{\frac{f_T}{f_R\mathrm{hooks}_R\mathrm{hooks}_T}}\delta_{rw}\sum_{\tau\in S_m\times S_p}\sum_{\phi\in S_m\times S_p}\sum_{\gamma_1,\gamma_2,\beta_1,\beta_2,\rho_1,\rho_2\in H_X\times H_Y}\\
&\quad\times\delta(\tau^{-1}\,\phi\beta_2\sigma_2\beta_1^{-1}\rho_1)\delta(\phi^{-1}\,\tau\gamma_2\sigma_1\gamma_1^{-1}\rho_2)\\
&\quad\times\sum_{k,q\in S_{i,m}}\sum_{l,r\in S_{i,p}}\delta(\tau^{-1}(p+1),k)\delta(\tau^{-1}(1),l)\delta(\phi^{-1}(p+1),q)\delta(\phi^{-1}(1),r)\,.
\end{aligned}$$

Now, set $\tau=\alpha\tilde{\tau}$ and $\beta=\alpha\tilde{\beta}$ with $\alpha\in Z_m\times Z_p$, with $Z_m\times Z_p$ a product of cyclic groups. The above expression becomes

$$\begin{aligned}
&=\frac{1}{|H_X\times H_Y|^4 m!p!}\sum_{R'}c_{RR'}\mathrm{hooks}_{R'}mp\sqrt{\frac{f_T}{f_R\mathrm{hooks}_R\mathrm{hooks}_T}}\delta_{rw}\sum_{\tau\in S_m\times S_p}\sum_{\phi\in S_m\times S_p}\\
&\quad\times\sum_{\gamma_1,\gamma_2,\beta_1,\beta_2,\rho_1,\rho_2\in H_X\times H_Y}\delta(\tau^{-1}\,\phi\beta_2\sigma_2\beta_1^{-1}\rho_1)\delta(\phi^{-1}\,\tau\gamma_2\sigma_1\gamma_1^{-1}\rho_2)\\
&\quad\times\sum_{k,q\in S_{i,m}}\sum_{l,r\in S_{i,p}}\delta(\tau^{-1}(\alpha(p+1)),k)\delta(\tau^{-1}(\alpha(1)),l)\delta(\phi^{-1}(\alpha(p+1)),q)\delta(\phi^{-1}(\alpha(1)),r)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\alpha\in Z_m\times Z_p} \frac{1}{|H_X\times H_Y|^4 m!p!} \sum_{R'} c_{RR'}\mathrm{hooks}_{R'} \sqrt{\frac{f_T}{f_R \mathrm{hooks}_R \mathrm{hooks}_T}}\delta_{rw} \sum_{\tau\in S_m\times S_p}\sum_{\phi\in S_m\times S_p} \\
&\qquad \times \sum_{\gamma_1,\gamma_2,\beta_1,\beta_2,\rho_1,\rho_2\in H_X\times H_Y} \delta(\tau^{-1}\,\phi\beta_2\sigma_2\beta_1^{-1}\rho_1)\delta(\phi^{-1}\,\tau\gamma_2\sigma_1\gamma_1^{-1}\rho_2) \\
&\times \sum_{k,q\in S_{i,m}}\sum_{l,r\in S_{i,p}} \delta(\tau^{-1}(\alpha(p+1)),k)\delta(\tau^{-1}(\alpha(1)),l)\delta(\phi^{-1}(\alpha(p+1)),q)\delta(\phi^{-1}(\alpha(1)),r) \\
&= \frac{1}{|H_X\times H_Y|^4 m!p!} \sum_{R'} c_{RR'}\mathrm{hooks}_{R'} \sqrt{\frac{f_T}{f_R \mathrm{hooks}_R \mathrm{hooks}_T}}\delta_{rw} \sum_{\tau\in S_m\times S_p}\sum_{\phi\in S_m\times S_p} \\
&\qquad \times \sum_{\gamma_1,\gamma_2,\beta_1,\beta_2,\rho_1,\rho_2\in H_X\times H_Y} \delta(\tau^{-1}\,\phi\beta_2\sigma_2\beta_1^{-1}\rho_1)\delta(\phi^{-1}\,\tau\gamma_2\sigma_1\gamma_1^{-1}\rho_2) \\
&\qquad\qquad \times \sum_{k,q\in S_{i,m}}\sum_{l,r\in S_{i,p}} \delta(\tau^{-1}(\phi(q)),k)\delta(\tau^{-1}(\phi(r)),l) \\
&= \frac{1}{|H_X\times H_Y|^4} \sum_{R'} c_{RR'}\mathrm{hooks}_{R'} \sqrt{\frac{f_T}{f_R \mathrm{hooks}_R \mathrm{hooks}_T}}\delta_{rw} \sum_{\phi\in S_m\times S_p} \\
&\times \sum_{\gamma_1,\gamma_2,\beta_1,\beta_2,\rho_1,\rho_2\in H_X\times H_Y} \delta(\phi\beta_2\sigma_2\beta_1^{-1}\rho_1)\delta(\phi^{-1}\,\gamma_2\sigma_1\gamma_1^{-1}\rho_2) \sum_{k,q\in S_{i,m}}\sum_{l,r\in S_{i,p}} \delta(\phi(q),k)\delta(\phi(r),l) \\
&= \frac{1}{|H_X\times H_Y|^2} \sum_{R'} c_{RR'}\mathrm{hooks}_{R'} \sqrt{\frac{f_T}{f_R \mathrm{hooks}_R \mathrm{hooks}_T}}\delta_{rw} \sum_{\phi\in S_m\times S_p}\sum_{\gamma_1,\gamma_2,\beta_1,\beta_2\in H_X\times H_Y} \\
&\qquad \times\delta(\phi\beta_2\sigma_2\beta_1^{-1})\delta(\phi^{-1}\,\gamma_2\sigma_1\gamma_1^{-1}) \sum_{k,q\in S_{i,m}}\sum_{l,r\in S_{i,p}} \delta(\phi(q),k)\delta(\phi(r),l) \\
&= \frac{1}{|H_X\times H_Y|^2} \sum_{R'} c_{RR'}\mathrm{hooks}_{R'} \sqrt{\frac{f_T}{f_R \mathrm{hooks}_R \mathrm{hooks}_T}}\delta_{rw} \sum_{\gamma_1,\gamma_2,\beta_1,\beta_2\in H_X\times H_Y} \\
&\qquad \times\delta(\gamma_2\sigma_1\gamma_1^{-1}\beta_2\sigma_2\beta_1^{-1}) n_{ii}^X(\sigma_1) n_{ii}^Y(\sigma_1) \\
&= \sum_{R'} c_{RR'}\mathrm{hooks}_{R'} \sqrt{\frac{f_T}{f_R \mathrm{hooks}_R \mathrm{hooks}_T}}\delta_{rw} \sum_{\beta_1,\beta_2\in H_X\times H_Y} \delta(\sigma_1\beta_2\sigma_2\beta_1^{-1}) n_{ii}^X(\sigma_1) n_{ii}^Y(\sigma_1)\,.
\end{aligned} \tag{3.46}
$$

Now, return to the case that $i\neq j$. The matrix element $\langle v,\vec{p}',\vec{m}'|\tau^{-1}E_{ji}^{(p+1)}(1,p+1)\,\psi|v,\vec{p},\vec{m}\rangle$ forces $\vec{p}+\vec{m}\neq\vec{p}'+\vec{m}'$. Indeed, $(1,p+1)\,\psi$ shuffles vectors, $E_{ji}^{(p+1)}$ removes v_i and inserts v_j and τ^{-1} does some more shuffling. Thus, using an obvious notation, we have

$$
\vec{p}+\vec{m}-\hat{i}=\vec{p}'+\vec{m}'-\hat{j}\,. \tag{3.47}
$$

We can also see this by noting that since $i\neq j$ we know that $R\neq T$. We still have $r=w$ so that the collection of boxes that needs to be dropped from R to get r (described by $\vec{p}+\vec{m}$) and the collection of boxes that needs to be dropped from T to get w (described by $\vec{p}'+\vec{m}'$) can't possibly be equal.

Again, we can say more. Consider

$$
\langle v,\vec{p},\vec{m}|\phi^{-1}\,(1,p+1)\,E_{ij}^{(1)}\sigma|v,\vec{p}',\vec{m}'\rangle\,. \tag{3.48}
$$

This tells you that if you take the state $|v, \vec{p}', \vec{m}'\rangle$ and shuffle some of the X slots amongst each other and some of the Y slots amongst each other (σ does this shuffling) keeping only states with vector v_j in their first slot, replacing this vector v_j with another vector v_i and then swapping the vectors in slots 1 and $p+1$, we can get the vector $|v, \vec{p}, \vec{m}\rangle$ by shuffling (according to ϕ^{-1}) what we have. We can summarize this as

$$\begin{aligned} &\vec{p}' - \hat{j} = \vec{p} - \hat{a} \\ &\vec{m}' - \hat{a} = \vec{m} - \hat{i}\,. \end{aligned} \tag{3.49}$$

Now consider

$$\langle v, \vec{p}', \vec{m}'|\tau^{-1} E^{(p+1)}_{ji}\,(1, p+1)\psi|v, \vec{p}, \vec{m}\rangle\,. \tag{3.50}$$

This tells you that if you take the state $|v, \vec{p}, \vec{m}\rangle$ and shuffle some of the X slots amongst each other and some of the Y slots amongst each other (ψ does this shuffling) keeping only states with vector v_i in their first slot, replacing this vector v_i with v_j and then, swapping the vectors in slots 1 and $p+1$, we can get the vector $|v, \vec{p}', \vec{m}'\rangle$ by shuffling (according to τ^{-1}) what we have. We can summarize this as

$$\begin{aligned} &\vec{p} - \hat{i} = \vec{p}' - \hat{b} \\ &\vec{m} - \hat{b} = \vec{m}' - \hat{j}\,. \end{aligned} \tag{3.51}$$

The equations (3.49) and (3.51) only have two solutions. If we choose $\hat{a} = \hat{i}$, we must have $\hat{b} = \hat{j}$ and then

$$\begin{aligned} &\vec{m} = \vec{m}' \\ &\vec{p} - \hat{i} = \vec{p}' - \hat{j}\,. \end{aligned} \tag{3.52}$$

If we choose $\hat{a} = \hat{j}$, we must have $\hat{b} = \hat{i}$ and then

$$\begin{aligned} &\vec{p} = \vec{p}' \\ &\vec{m} - \hat{i} = \vec{m}' - \hat{j}\,. \end{aligned} \tag{3.53}$$

Thus, only $\vec{m}$ or $\vec{p}$ can change - but not both. In fact, only one of the Gauss graphs (there is one graph for the Xs and one for the Ys) change - but not both.

It is now rather simple to write the relation between $|v, \vec{p}', \vec{m}'\rangle$ and $|v, \vec{p}, \vec{m}\rangle$. Consider for example, (3.52). Let $S_{j,p}$ denote the collection of slots that (i) are X slots and (ii) are occupied by v_j. There are similar definitions for $S_{j,m}$, $S'_{j,p}$ and $S'_{j,m}$. To go from $\vec{p}$ to $\vec{p}'$, we want to remove a v_i and replace it with a v_j and then reorder the slots into the order prescribed by (3.17). We can do this as

$$|v, \vec{p}', \vec{m}'\rangle = \zeta E^{(q)}_{ji}|v, \vec{p}, \vec{m}\rangle \qquad q \in S_{i,p} \quad \zeta \in S_m \times S_p\,. \tag{3.54}$$

Consequently we can again write a definite relation between $|v,\vec{p}',\vec{m}'\rangle$ and $|v,\vec{p},\vec{m}\rangle$. This allows us to simplify the matrix element expressions to the structure of elements we have already evaluated. Now, consider

$$\begin{aligned}
A = &\frac{1}{|H_X\times H_Y|^2|H'_X\times H'_Y|^2 m!p!}\sum_{R'} c_{RR'}\text{hooks}_{R'}mp \\
&\sqrt{\frac{f_T}{f_R\text{hooks}_R\text{hooks}_T}}\delta_{rw}\sum_{\tau\in S_m\times S_p}\sum_{\phi\in S_m\times S_p}\sum_{\gamma_1,\gamma_2\in H'_X\times H'_Y}\sum_{\beta_1,\beta_2\in H_X\times H_Y} \\
&\times\langle v,\vec{p}',\vec{m}'|E_{ji}^{\tau^{-1}(p+1)}\tau^{-1}\,(1,p+1)\,\phi\beta_2\sigma_2\beta_1^{-1}|v,\vec{p},\vec{m}\rangle \\
&\times\langle v,\vec{p},\vec{m}|E_{ij}^{\phi^{-1}(p+1)}\phi^{-1}\,(1,p+1)\,\tau\gamma_2\sigma_1\gamma_1^{-1}|v,\vec{p}',\vec{m}'\rangle \\
= &\frac{1}{|H_X\times H_Y|^2|H'_X\times H'_Y|^2 m!p!}\sum_{R'}\sqrt{c_{RR'}c_{TR'}} \\
&\times\frac{\text{hooks}_{R'}}{\sqrt{\text{hooks}_R\text{hooks}_T}}mp\delta_{rw}\sum_{\tau,\phi\in S_m\times S_p}\sum_{\gamma_1,\gamma_2\in H'_X\times H'_Y}\sum_{\beta_1,\beta_2\in H_X\times H_Y} \\
&\times\langle v,\vec{p}',\vec{m}'|E_{ji}^{\tau^{-1}(p+1)}\tau^{-1}\,(1,p+1)\,\phi\beta_2\sigma_2\beta_1^{-1}|v,\vec{p},\vec{m}\rangle \\
&\times\langle v,\vec{p},\vec{m}|E_{ij}^{\phi^{-1}(p+1)}\phi^{-1}\,(1,p+1)\,\tau\gamma_2\sigma_1\gamma_1^{-1}|v,\vec{p}',\vec{m}'\rangle\,.
\end{aligned}\qquad (3.55)$$

To start, study

$$\begin{aligned}
&\langle v,\vec{p}',\vec{m}'|E_{ji}^{\tau^{-1}(p+1)}\tau^{-1}\,(1,p+1)\,\phi\beta_2\sigma_2\beta_1^{-1}|v,\vec{p},\vec{m}\rangle \\
&=\langle v,\vec{p}',\vec{m}'|\tau^{-1}E_{ji}^{(p+1)}\,(1,p+1)\,\phi\beta_2\sigma_2\beta_1^{-1}|v,\vec{p},\vec{m}\rangle
\end{aligned}\qquad (3.56)$$

and consider a matrix element for which $\vec{m}=\vec{m}'$ and $\vec{p}=\vec{p}'-\hat{j}+\hat{i}$. In this case, we know that we can write

$$|v,\vec{p}',\vec{m}'\rangle=\zeta E_{ji}^{(q)}|v,\vec{p},\vec{m}\rangle\qquad \zeta\in S_p\qquad q\in S_{i,p}\qquad (3.57)$$

We can choose any basis for the vectors $|v,\vec{p},\vec{m}\rangle$, $|v,\vec{p}',\vec{m}'\rangle$ that we like - the result will be independent of the choice we make. In (3.17) choose the i and j vectors to sit in adjacent slots, and always choose q to lie on the border between the two. In this case we can always choose ζ_q to be the identity. With this choice understood, we have

$$|v,\vec{p}',\vec{m}'\rangle=E_{ji}^{(q)}|v,\vec{p},\vec{m}\rangle\qquad q\in S_{i,p}\,.\qquad (3.58)$$

In a similar way

$$\begin{aligned}
&\langle v,\vec{p},\vec{m}|E_{ij}^{\phi^{-1}(p+1)}\phi^{-1}\,(1,p+1)\,\tau\gamma_2\sigma_1\gamma_1^{-1}|v,\vec{p}',\vec{m}'\rangle \\
&=\langle v,\vec{p},\vec{m}|\phi^{-1}E_{ij}^{(p+1)}\,(1,p+1)\,\tau\gamma_2\sigma_1\gamma_1^{-1}|v,\vec{p}',\vec{m}'\rangle
\end{aligned}\qquad (3.59)$$

and, from (3.58) we have

$$|v,\vec{p},\vec{m}\rangle=E_{ij}^{(q)}|v,\vec{p}',\vec{m}'\rangle\,.\qquad (3.60)$$

Consequently

$$
\begin{aligned}
A = & \frac{1}{|H_X \times H_Y|^2 |H'_X \times H'_Y|^2 m! p!} \sum_{R'} \sqrt{c_{RR'} c_{TR'}} \frac{\text{hooks}_{R'}}{\sqrt{\text{hooks}_R \text{hooks}_T}} m p \delta_{rw} \sum_{\gamma_1,\gamma_2 \in H'_X \times H'_Y} \\
& \times \sum_{\beta_1,\beta_2 \in H_X \times H_Y} \sum_{\tau,\phi \in S_m \times S_p} \langle v, \vec{p}', \vec{m}' | \tau^{-1} E^{(p+1)}_{ji} (1, p+1) \, \phi \beta_2 \sigma_2 \beta_1^{-1} | v, \vec{p}, \vec{m} \rangle \\
& \times \langle v, \vec{p}, \vec{m} | \phi^{-1} E^{(p+1)}_{ij} (1, p+1) \, \tau \gamma_2 \sigma_1 \gamma_1^{-1} | v, \vec{p}', \vec{m}' \rangle \\
= & \frac{1}{|H_X \times H_Y|^2 |H'_X \times H'_Y|^2 m! p!} \sum_{R'} \sqrt{c_{RR'} c_{TR'}} \frac{\text{hooks}_{R'}}{\sqrt{\text{hooks}_R \text{hooks}_T}} m p \delta_{rw} \sum_{\gamma_1,\gamma_2 \in H'_X \times H'_Y} \\
& \times \sum_{\beta_1,\beta_2 \in H_X \times H_Y} \sum_{\tau,\phi \in S_m \times S_p} \langle v, \vec{p}, \vec{m} | E^{(q)}_{ij} \tau^{-1} E^{(p+1)}_{ji} (1, p+1) \, \phi \beta_2 \sigma_2 \beta_1^{-1} | v, \vec{p}, \vec{m} \rangle \\
& \times \langle v, \vec{p}', \vec{m}' | E^{(q)}_{ji} \phi^{-1} E^{(p+1)}_{ij} (1, p+1) \, \tau \gamma_2 \sigma_1 \gamma_1^{-1} | v, \vec{p}', \vec{m}' \rangle \\
= & \frac{1}{|H_X \times H_Y|^2 |H'_X \times H'_Y|^2 m! p!} \sum_{R'} \sqrt{c_{RR'} c_{TR'}} \frac{\text{hooks}_{R'}}{\sqrt{\text{hooks}_R \text{hooks}_T}} m p \delta_{rw} \sum_{\gamma_1,\gamma_2 \in H'_X \times H'_Y} \\
& \times \sum_{\beta_1,\beta_2 \in H_X \times H_Y} \sum_{\tau,\phi \in S_m \times S_p} \langle v, \vec{p}, \vec{m} | \tau^{-1} E^{\tau(q)}_{ij} E^{(p+1)}_{ji} (1, p+1) \, \phi \beta_2 \sigma_2 \beta_1^{-1} | v, \vec{p}, \vec{m} \rangle \\
& \times \langle v, \vec{p}', \vec{m}' | \phi^{-1} E^{\phi(q)}_{ji} E^{(p+1)}_{ij} (1, p+1) \, \tau \gamma_2 \sigma_1 \gamma_1^{-1} | v, \vec{p}', \vec{m}' \rangle \\
= & \frac{1}{|H_X \times H_Y|^2 |H'_X \times H'_Y|^2 m! p!} \sum_{R'} \sqrt{c_{RR'} c_{TR'}} \frac{\text{hooks}_{R'}}{\sqrt{\text{hooks}_R \text{hooks}_T}} m p \delta_{rw} \sum_{\gamma_1,\gamma_2 \in H'_X \times H'_Y} \\
& \sum_{\beta_1,\beta_2 \in H_X \times H_Y} \sum_{\tau,\phi \in S_m \times S_p} \langle v_\tau | E^{\tau(q)}_{ij} E^{(p+1)}_{ji} (1, p+1) \, \phi \beta_2 \sigma_2 \beta_1^{-1} \tau^{-1} | v_\tau \rangle) \\
& \times \langle v'_\phi | E^{\phi(q)}_{ji} E^{(p+1)}_{ij} (1, p+1) \, \tau \gamma_2 \sigma_1 \gamma_1^{-1} \phi^{-1} | v'_\phi \rangle
\end{aligned} \tag{3.61}
$$

We need to understand $\langle v_\tau | E^{\tau(q)}_{ij} E^{(p+1)}_{ji} (1,p+1)$ and $\langle v'_\phi | E^{\phi(q)}_{ji} E^{(p+1)}_{ij} (1,p+1)$ better. Consider $\langle v_\tau | E^{\tau(q)}_{ij} E^{(p+1)}_{ji} (1, p+1)$ first. Turn this into a ket state

$$
\begin{aligned}
(1, p+1) \, E^{(p+1)}_{ij} E^{\tau(q)}_{ji} \tau | v \rangle &= (1, p+1) \, \tau \, E^{\tau^{-1}(p+1)}_{ij} E^{(q)}_{ji} | v \rangle \\
&= \tau \, (\tau^{-1}(1), \tau^{-1}(p+1)) \, E^{\tau^{-1}(p+1)}_{ij} E^{(q)}_{ji} | v \rangle
\end{aligned} \tag{3.62}
$$

Now, there are a few things we should note. First, recall that $i \neq j$. In the matrix element

$$
\langle v_\tau | E^{\tau(q)}_{ij} E^{(p+1)}_{ji} (1, p+1) \, \phi \beta_2 \sigma_2 \beta_1^{-1} \tau^{-1} | v_\tau \rangle \tag{3.63}
$$

we know that $\phi \beta_2 \sigma_2 \beta_1^{-1} \tau^{-1}$ is an element in $S_m \times S_p$ and thus it is not able to swap vectors between the Y and X slots. The product $E^{\tau(q)}_{ij} E^{(p+1)}_{ji}$ makes an X slot change as (imagine acting to the right) $j \to i$ and a Y slot change as $i \to j$. This amounts to exchanging an i vector from X with a j vector from Y. The only way that the above matrix element can be non-zero, is if $(1, p+1)$ is able to swap these two back again. Thus, we can write

$$
\tau \, (\tau^{-1}(1), \tau^{-1}(p+1)) \, E^{\tau^{-1}(p+1)}_{ij} E^{(q)}_{ji} | v \rangle = \tau \, E^{\tau^{-1}(1)}_{ij} E^{\tau^{-1}(p+1)}_{ji} \, E^{\tau^{-1}(p+1)}_{ij} E^{(q)}_{ji} | v \rangle
$$

$$= \sum_{l\in S_{j,m}} \delta(\tau^{-1}(p+1),l)\tau\, E_{ij}^{\tau^{-1}(1)} E_{ji}^{(q)}|v\rangle$$

$$= \sum_{l\in S_{j,m}} \delta(\tau^{-1}(p+1),l) \left(\delta(\tau^{-1}(1),q) + \sum_{r\in S_{j,p}} \delta(\tau^{-1}(1),r)\,(q,r) \right) \tau\,|v\rangle\,. \qquad (3.64)$$

Now, each of the terms in round brackets for which index r belongs to a string that loops back to node j above makes the same contribution so that we have

$$(1,p+1)\, E_{ij}^{(p+1)} E_{ji}^{\tau(q)}|v_\tau\rangle = n^X_{jj}(\sigma_2) \sum_{l\in S_{j,m}} \delta(\tau^{-1}(p+1),l)\delta(\tau^{-1}(1),q)|v_\tau\rangle\,. \qquad (3.65)$$

The above equation is not exactly true (certain terms on the RHS have been dropped) but it gives the correct result when plugged into (3.61). A very similar argument implies that we can replace

$$(1,p+1)\, E_{ji}^{(p+1)} E_{ij}^{(\phi\zeta_q)(q)}|v'_\phi\rangle = n^X_{ii}(\sigma_1) \sum_{w\in S'_{i,m}} \delta(\phi^{-1}(p+1),w)\delta(\phi^{-1}(1),\zeta_q(q))|v'_\phi\rangle \qquad (3.66)$$

We can now use these results to compute

$$\begin{aligned}
A = {} & \frac{n^X_{ii}(\sigma_1)n^X_{jj}(\sigma_2)}{|H_X\times H_Y|^2|H'_X\times H'_Y|^2 m!p!} \sum_{R'} \sqrt{c_{RR'}c_{TR'}} \frac{\mathrm{hooks}_{R'}}{\sqrt{\mathrm{hooks}_R \mathrm{hooks}_T}} mp\delta_{rw} \\
& \times \sum_{\gamma_1,\gamma_2\in H'_X\times H'_Y} \sum_{\beta_1,\beta_2\in H_X\times H_Y} \sum_{\tau\in S_m\times S_p} \sum_{\phi\in S_m\times S_p} \sum_{l\in S_{j,m}} \sum_{w\in S'_{i,m}} \delta(\tau^{-1}(p+1),l)\delta(\phi^{-1}(p+1),w) \\
& \delta(\tau^{-1}(1),q)\delta(\phi^{-1}(1),\zeta_q(q))\langle v_\tau|\phi\beta_2\sigma_2\beta_1^{-1}\tau^{-1}|v_\tau\rangle\, \langle v'_\phi|\tau\gamma_2\sigma_1\gamma_1^{-1}\phi^{-1}|v'_\phi\rangle \\
= {} & \frac{n^X_{ii}(\sigma_1)n^X_{jj}(\sigma_2)}{|H_X\times H_Y|^2|H'_X\times H'_Y|^2 m!p!} \sum_{R'} \sqrt{c_{RR'}c_{TR'}} \frac{\mathrm{hooks}_{R'}}{\sqrt{\mathrm{hooks}_R \mathrm{hooks}_T}} mp\delta_{rw} \\
& \times \sum_{\gamma_1,\gamma_2,\gamma\in H'_X\times H'_Y} \sum_{\beta_1,\beta_2,\beta\in H_X\times H_Y} \sum_{\tau\in S_m\times S_p} \sum_{\phi\in S_m\times S_p} \sum_{l\in S_{j,m}} \sum_{w\in S'_{i,m}} \delta(\tau^{-1}(p+1),l)\delta(\phi^{-1}(p+1),w) \\
& \delta(\tau^{-1}(1),q)\delta(\phi^{-1}(1),q)\delta(\tau^{-1}\phi\beta_2\sigma_2\beta_1^{-1}\beta)\, \delta(\phi^{-1}\tau\gamma_2\sigma_1\gamma_1^{-1}\gamma) \\
= {} & \frac{n^X_{ii}(\sigma_1)n^X_{jj}(\sigma_2)}{|H_X\times H_Y||H'_X\times H'_Y| m!p!} \sum_{R'} \sqrt{c_{RR'}c_{TR'}} \frac{\mathrm{hooks}_{R'}}{\sqrt{\mathrm{hooks}_R \mathrm{hooks}_T}} mp\delta_{rw} \\
& \times \sum_{\gamma_1,\gamma_2\in H'_X\times H'_Y} \sum_{\beta_1,\beta_2\in H_X\times H_Y} \sum_{\tau\in S_m\times S_p} \sum_{\phi\in S_m\times S_p} \sum_{l\in S_{j,m}} \sum_{w\in S'_{i,m}} \delta(\tau^{-1}(p+1),l)\delta(\phi^{-1}(p+1),w) \\
& \delta(\tau^{-1}(1),q)\delta(\phi^{-1}(1),q)\delta(\tau^{-1}\phi\beta_2\sigma_2\beta_1^{-1})\, \delta(\phi^{-1}\tau\gamma_2\sigma_1\gamma_1^{-1})
\end{aligned} \qquad (3.67)$$

Now, we do the same trick as before, setting $\phi = \alpha\tilde{\phi}$ and $\tau = \alpha\tilde{\tau}$. After performing manipulations just like we did before we find

$$A = \frac{n^X_{ii}(\sigma_1)n^X_{jj}(\sigma_2)}{|H_X\times H_Y||H'_X\times H'_Y| m!p!} \sum_{R'} \sqrt{c_{RR'}c_{TR'}} \frac{\mathrm{hooks}_{R'}}{\sqrt{\mathrm{hooks}_R \mathrm{hooks}_T}} \delta_{rw}$$

$$
\begin{aligned}
&\times \sum_{\gamma_1,\gamma_2\in H'_X\times H'_Y}\ \sum_{\beta_1,\beta_2\in H_X\times H_Y}\ \sum_{\tau\in S_m\times S_p}\ \sum_{\phi\in S_m\times S_p}\\
&\times \sum_{l\in S_{j,m}}\sum_{w\in S'_{i,m}} \delta(\tau^{-1}\phi(w),l)\delta(\tau^{-1}\phi(q),q)\delta(\tau^{-1}\phi\beta_2\sigma_2\beta_1^{-1})\,\delta(\phi^{-1}\tau\gamma_2\sigma_1\gamma_1^{-1})\\
&= \frac{n^X_{ii}(\sigma_1)n^X_{jj}(\sigma_2)}{|H_X\times H_Y||H'_X\times H'_Y|}\sum_{R'}\sqrt{c_{RR'}c_{TR'}}\frac{\text{hooks}_{R'}}{\sqrt{\text{hooks}_R\text{hooks}_T}}\delta_{rw}\sum_{\gamma_1,\gamma_2\in H'_X\times H'_Y}\ \sum_{\beta_1,\beta_2\in H_X\times H_Y}\ \sum_{\tau\in S_m\times S_p}\\
&\times \sum_{l\in S_{j,m}}\sum_{w\in S'_{i,m}} \delta(\tau^{-1}(w),l)\delta(\tau^{-1}(q),q)\delta(\tau^{-1}\beta_2\sigma_2\beta_1^{-1})\,\delta(\tau\gamma_2\sigma_1\gamma_1^{-1})\\
&= \frac{n^X_{ii}(\sigma_1)n^X_{jj}(\sigma_2)}{|H_X\times H_Y||H'_X\times H'_Y|}\sum_{R'}\sqrt{c_{RR'}c_{TR'}}\frac{\text{hooks}_{R'}}{\sqrt{\text{hooks}_R\text{hooks}_T}}\delta_{rw}\sum_{\gamma_1,\gamma_2\in H'_X\times H'_Y}\ \sum_{\beta_1,\beta_2\in H_X\times H_Y}\\
&\times \sum_{l\in S_{j,m}}\sum_{w\in S'_{i,m}} \delta(\gamma_2\sigma_1\gamma_1^{-1}(w),l)\delta(\gamma_2\sigma_1\gamma_1^{-1}(q),q)\,\delta(\beta_2\sigma_2\beta_1^{-1}\gamma_2\sigma_1\gamma_1^{-1})\,.
\end{aligned}
\tag{3.68}
$$

Consider

$$
\sum_{l\in S_{j,m}}\sum_{w\in S'_{i,m}} \delta(\gamma_2\sigma_1\gamma_1^{-1}(w),l) = n^{Y+}_{ij}(\sigma_1) \tag{3.69}
$$

where $n^{Y+}_{ij}(\sigma_2) = n^{Y+}_{ij}(\sigma_1)$ is the number of strings going from i to j in the Gauss graph associated to the Ys. The fact that this $n^{Y+}_{ij}(\sigma_2)$ appears suggests that the strings stretching between i and j in the Gauss graph associated to the Ys are participating, even though it is the X Gauss graph that undergoes the transition. Thus

$$
\begin{aligned}
A = \frac{n^X_{ii}(\sigma_1)n^X_{jj}(\sigma_2)}{|H_X\times H_Y||H'_X\times H'_Y|}\sum_{R'}\sqrt{c_{RR'}c_{TR'}}\frac{\text{hooks}_{R'}}{\sqrt{\text{hooks}_R\text{hooks}_T}}\delta_{rw}\sum_{\gamma_1,\gamma_2\in H'_X\times H'_Y}\ \sum_{\beta_1,\beta_2\in H_X\times H_Y}\\
n^{Y+}_{ij}(\sigma_2)\delta(\gamma_2\sigma_1\gamma_1^{-1}(q),q)\,\delta(\beta_2\sigma_2\beta_1^{-1}\gamma_2\sigma_1\gamma_1^{-1})
\end{aligned}
\tag{3.70}
$$

Next, consider the role of $\delta(\gamma_2\sigma_1\gamma_1^{-1}(q),q)$. This tells us that a single string, which loops back to the same brane, is plucked from brane i (or j) and reattached to brane j (or i). This follows because the string which is plucked has startpoint q and end point $\gamma_2\sigma_1\gamma_1^{-1}(q)$. So the delta function is setting the start point equal to the end point. Another way to say it is that states with different values for the n^Y_{ij} or n^X_{ij} don't mix - and this is why the terms in the dilatation operator that mix X and Y commute with terms that mix X and Z and the terms that mix Y and Z. The role of this delta function is also easy to interpret in terms of the Gauss graph: the two Gauss graphs that mix, σ_1 and σ_2, are related by peeling a closed loop from node i of σ_1^X (or σ_1^Y) and reattaching it to node j of σ_2^X (or σ_2^Y). This implies that, as permutations, σ_1 and σ_2 are identical (recall that closed loops are 1 cycles).

If we peel a string from node i of σ_1^X and reattach it to node j of σ_2^X, the factor $n^X_{ii}(\sigma_1)n^X_{jj}(\sigma_2) = n^X_{ii}(\sigma_1)(n^X_{jj}(\sigma_1)+1) = (n^X_{ii}(\sigma_2)+1)n^X_{jj}(\sigma_2)$ is the number of strings starting and ending at node i before we peel a string off, multiplied by the number of strings starting and ending at node j after we have attached the string.

Notice that the delta function $\delta(\gamma_2\sigma_1\gamma_1^{-1}(q),q)$ reduces the full sum over γ_1 and γ_2 to those elements of $H'_X\times H'_Y$ that leave q inert. This is a subgroup of $(H'_X\times H'_Y)\cap(H_X\times H_Y)$ that we will denote $\mathcal{G}_{\sigma_1,q}$. Consequently, the size of this matrix element is

$$n^X_{ii}(\sigma_1)n^X_{jj}(\sigma_2)n^{Y+}_{ij}(\sigma_2)|\mathcal{G}_{\sigma_1,q}| \tag{3.71}$$

Notice that

$$\frac{|\mathcal{G}_{\sigma_1,q}|}{N_{\sigma_1}}=n^X_{ii}(\sigma_1)\qquad \frac{|\mathcal{G}_{\sigma_1,q}|}{N_{\sigma_2}}=n^X_{jj}(\sigma_2)\,. \tag{3.72}$$

These two formulas follow because N_σ counts the number of symmetries of the Gauss graph, while $|\mathcal{G}_{\sigma,q}|$ counts the number of symmetries that don't include permutations of the closed loop corresponding to q. Thus, we finally see that the normalized matrix element is nothing but

$$\sqrt{c_{RR'}c_{TR'}}\frac{\text{hooks}_{R'}}{\sqrt{\text{hooks}_R\text{hooks}_T}}\delta_{rw}\sqrt{n^X_{ii}(\sigma_1)n^X_{jj}(\sigma_2)}n^{Y+}_{ij}(\sigma_2)\,. \tag{3.73}$$

The evaluation of the fourth term is practically identical and will not be discussed.

3.3.2 Second Term

Now consider the second term

$$\begin{aligned}B=\frac{|H_X\times H_Y||H'_X\times H'_Y|}{m!p!}&\sum_{jk}\sum_{s\vdash m}\sum_{t\vdash p}\sum_{\vec\mu\vec\nu}\sum_{lm}\sum_{x\vdash m}\sum_{y\vdash p}\sum_{\vec\alpha\vec\beta}\Gamma^{(s,t)}(\sigma_1)_{jk}\Gamma^{(x,y)}(\sigma_2)_{lm}\\ &\times B^{(s,t)\to 1_{H_X\times H_Y}}_{j\vec\mu}B^{(s,t)\to 1_{H_X\times H_Y}}_{k\vec\nu}B^{(x,y)\to 1_{H'_X\times H'_Y}}_{l\vec\alpha}B^{(x,y)\to 1_{H'_X\times H'_Y}}_{m\vec\beta}\\ \sum_{R'}c_{RR'}\frac{d_Tmp}{(n+m+p)d_{R'}}\sqrt{\frac{f_T\text{hooks}_T}{f_R\text{hooks}_R}}&\delta_{rw}\langle\vec p,t,\nu_1;a|\vec p',y,\alpha_1;b\rangle\langle\vec p',y,\beta_1;b|E^{(1)}_{ji}|\vec p,t,\mu_1;a\rangle\\ &\times\langle\vec m,s,\nu_2;c|E^{(p+1)}_{ij}|\vec m',x,\alpha_2;d\rangle\langle\vec m',x,\beta_2;d|\vec m,s,\mu_2;c\rangle\\ &=\frac{1}{|H_X\times H_Y|^2|H'_X\times H'_Y|^2m!p!}\\ \sum_{R'}c_{RR'}\frac{d_Tmp}{(n+m+p)d_{R'}}\sqrt{\frac{f_T\text{hooks}_T}{f_R\text{hooks}_R}}\delta_{rw}&\sum_{\tau\in S_m\times S_p}\sum_{\phi\in S_m\times S_p}\sum_{\gamma_1,\gamma_2\in H'_X\times H'_Y}\sum_{\beta_1,\beta_2\in H_X\times H_Y}\\ &\times\delta_{\vec m\vec m'}\delta_{\vec p\vec p'}\langle v,\vec p',\vec m'|\tau^{-1}E^{(1)}_{ji}\,\phi\beta_2\sigma_2\beta_1^{-1}|v,\vec p,\vec m\rangle\\ &\times\langle v,\vec p,\vec m|\phi^{-1}E^{(p+1)}_{ij}\,\tau\gamma_2\sigma_1\gamma_1^{-1}|v,\vec p',\vec m'\rangle\,.\end{aligned} \tag{3.74}$$

For the above result to be nonzero it is clear that we need

$$\vec p=\vec p'\qquad \vec m-\hat i+\hat j=\vec m' \tag{3.75}$$

as well as

$$\vec{p} = \vec{p}' \qquad \vec{m} = \vec{m}'\,. \tag{3.76}$$

Consequently, this term is only non-zero when $i = j$. In this case

$$B = \frac{1}{|H_X \times H_Y|^4 m!p!}$$

$$\sum_{R'} c_{RR'} \frac{d_T mp}{(n+m+p)d_{R'}} \sqrt{\frac{f_T \text{hooks}_T}{f_R \text{hooks}_R}} \delta_{rw} \sum_{\tau \in S_m \times S_p} \sum_{\phi \in S_m \times S_p} \sum_{\gamma_1,\gamma_2,\gamma \in H_X \times H_Y} \sum_{\beta_1,\beta_2,\beta \in H_X \times H_Y}$$

$$\times \sum_{k \in S_{j,m}} \delta(\tau^{-1}(1), k) \delta(\tau^{-1}\phi\beta_2\sigma_2\beta_1^{-1}\beta)$$

$$\times \sum_{l \in S_{i,m}} \delta(\phi^{-1}(p+1), l) \delta(\phi^{-1}\tau\gamma_2\sigma_1\gamma_1^{-1}\gamma)$$

$$= \frac{1}{|H_X \times H_Y|^2 m!p!}$$

$$\sum_{R'} c_{RR'} \frac{d_T mp}{(n+m+p)d_{R'}} \sqrt{\frac{f_T \text{hooks}_T}{f_R \text{hooks}_R}} \delta_{rw} \sum_{\tau \in S_m \times S_p} \sum_{\phi \in S_m \times S_p} \sum_{\gamma_1,\gamma_2 \in H_X \times H_Y} \sum_{\beta_1,\beta_2 \in H_X \times H_Y}$$

$$\times \sum_{k \in S_{j,m}} \delta(\tau^{-1}(1), k) \delta(\tau^{-1}\phi\beta_2\sigma_2\beta_1^{-1})$$

$$\times \sum_{l \in S_{i,m}} \delta(\phi^{-1}(p+1), l) \delta(\phi^{-1}\tau\gamma_2\sigma_1\gamma_1^{-1})$$

$$= \frac{1}{|H_X \times H_Y|^2 m!p!}$$

$$\sum_{R'} c_{RR'} \frac{d_T}{(n+m+p)d_{R'}} \sqrt{\frac{f_T \text{hooks}_T}{f_R \text{hooks}_R}} \delta_{rw} \sum_{\tau \in S_m \times S_p} \sum_{\phi \in S_m \times S_p} \sum_{\gamma_1,\gamma_2 \in H_X \times H_Y} \sum_{\beta_1,\beta_2 \in H_X \times H_Y}$$

$$\times \sum_{k \in S_{j,m}} \sum_{r=1}^{m} \delta(\tau^{-1}(m), k) \sum_{l \in S_{i,m}} \sum_{s=p+1}^{p+m} \delta(\phi^{-1}(s), l) \delta(\tau^{-1}\phi\beta_2\sigma_2\beta_1^{-1}) \delta(\phi^{-1}\tau\gamma_2\sigma_1\gamma_1^{-1})$$

$$= \frac{1}{|H_X \times H_Y|^2}$$

$$\sum_{R'} c_{RR'} \frac{d_T p}{(n+m+p)d_{R'}} \sqrt{\frac{f_T \text{hooks}_T}{f_R \text{hooks}_R}} \delta_{rw} \sum_{\tau \in S_m \times S_p} \sum_{\gamma_1,\gamma_2 \in H_X \times H_Y} \sum_{\beta_1,\beta_2 \in H_X \times H_Y}$$

$$\times n_i^{+,X} n_i^{+,Y} \delta(\tau^{-1}\beta_2\sigma_2\beta_1^{-1}) \delta(\tau\gamma_2\sigma_1\gamma_1^{-1})$$

$$= \frac{1}{|H_X \times H_Y|^2}$$

$$\sum_{R'} c_{RR'} \frac{d_T}{(n+m+p)d_{R'}} \sqrt{\frac{f_T \text{hooks}_T}{f_R \text{hooks}_R}} \delta_{rw} \sum_{\gamma_1,\gamma_2 \in H_X \times H_Y} \sum_{\beta_1,\beta_2 \in H_X \times H_Y}$$

$$\times n_i^{+,X} n_i^{+,Y} \delta(\beta_2\sigma_2\beta_1^{-1}\gamma_2\sigma_1\gamma_1^{-1})$$

$$= \sum_{R'} c_{RR'} \frac{d_T}{(n+m+p)d_{R'}} \sqrt{\frac{f_T \text{hooks}_T}{f_R \text{hooks}_R}} \delta_{rw} \sum_{\beta_1,\beta_2 \in H_X \times H_Y} n_{ii}^{+,X} n_{ii}^{+,Y} \delta(\beta_2\sigma_2\beta_1^{-1}\sigma_1)$$

$$(3.77)$$

In summary, and perhaps writing it a bit more clearly, we have

$$
\begin{aligned}
B = \frac{|H_X \times H_Y||H'_X \times H'_Y|}{m!p!} & \sum_{jk}\sum_{s\vdash m}\sum_{t\vdash p}\sum_{\vec{\mu}\vec{\nu}}\sum_{lm}\sum_{x\vdash m}\sum_{y\vdash p}\sum_{\vec{\alpha}\vec{\beta}} \Gamma^{(s,t)}(\sigma_1)_{jk}\Gamma^{(x,y)}(\sigma_2)_{lm} \\
& \times B^{(s,t)\to 1_{H_X\times H_Y}}_{j\vec{\mu}} B^{(s,t)\to 1_{H_X\times H_Y}}_{k\vec{\nu}} B^{(x,y)\to 1_{H'_X\times H'_Y}}_{l\vec{\alpha}} B^{(x,y)\to 1_{H'_X\times H'_Y}}_{m\vec{\beta}} \\
\sum_{R'} c_{RR'} \frac{d_T mp}{(n+m+p)d_{R'}} \sqrt{\frac{f_T \mathrm{hooks}_T}{f_R \mathrm{hooks}_R}} & \delta_{rw} \langle \vec{p}, t, \nu_1; a|\vec{p}', y, \alpha_1; b\rangle \langle \vec{p}', y, \beta_1; b|E^{(1)}_{ji}|\vec{p}, t, \mu_1; a\rangle \\
& \times \langle \vec{m}, s, \nu_2; c|E^{(p+1)}_{ij}|\vec{m}', x, \alpha_2; d\rangle \langle \vec{m}', x, \beta_2; d|\vec{m}, s, \mu_2; c\rangle \\
= \delta_{\vec{p}\vec{p}'}\delta_{\vec{m}\vec{m}'} \sum_{R'_i} c_{RR'_i} \frac{d_T}{(n+m+p)d_{R'}} & \sqrt{\frac{f_T \mathrm{hooks}_T}{f_R \mathrm{hooks}_R}} \delta_{rw} \sum_{\beta_1,\beta_2 \in H_X\times H_Y} n_i^{+,X} n_i^{+,Y} \delta(\beta_2\sigma_2\beta_1^{-1}\sigma_1)
\end{aligned}
\tag{3.78}
$$

Notice that this term is already diagonal in the Gauss graph basis. Notation: $n_i^{+,X}$ is the number of strings ending on node i of the X Gauss graph; n_{ii}^X is the number of strings starting on and then looping back to end on node i of the X Gauss graph.

The evaluation of the third term is practically identical and will not be discussed.

3.3.3 Final Answer

In this section we will summarize the action of the term in the dilatation operator that mixes Xs and Ys on the Gauss operators.

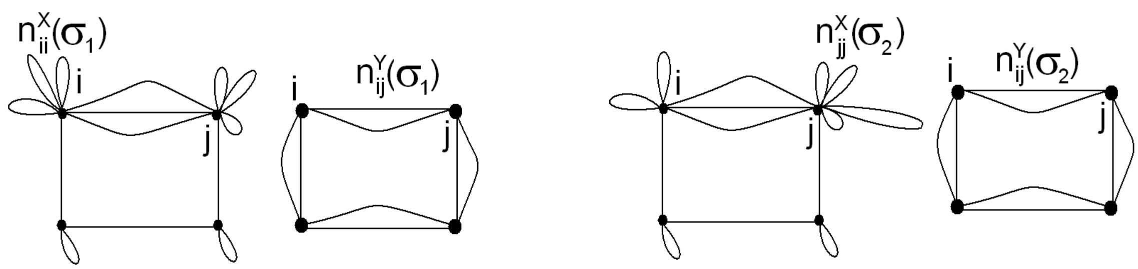


Figure 2: The Gauss graph on the left is described by σ_1, while the Gauss graph on the right is described by σ_2. To make a transition between the two pairs of Gauss graphs shown, we pluck a string from node i of the X graph on the left and attach it to node j of the X graph on the right. The numbers which participate are (i) the number of strings n_{ij}^Y stretching between nodes i and j of the Y graph, (ii) the number of strings attached to node i of the X graph before a string is removed $n_{ii}^X(\sigma_1) = n_{ii}^X(\sigma_2) + 1$ and (iii) the number of strings attached to the node j of the X graph after a string is attached $n_{ii}^X(\sigma_1) + 1 = n_{ii}^X(\sigma_2)$.

Here is the final answer for matrix elements of D taken with *normalized operators*. The diagonal terms are

$$\langle O^\dagger_{R,r}(\sigma) D_{XY} O_{R,r}(\sigma)\rangle = 2\sum_{i=1}^p \frac{c_{RR'_i}}{l_i}\left(n(\sigma)_i^{+X} n(\sigma)_i^{+Y} - n(\sigma)_{ii}^{+X} n(\sigma)_{ii}^{+Y}\right) \tag{3.79}$$

Now, consider an off diagonal term. One possible non-zero matrix element corresponds to the case that the X Gauss graph changes, by detaching a loop from node i of the σ_1^X Gauss graph and reattaching it to node j. The matrix element describing this process is (recall that we only ever get a non-zero matrix element if $n^Y_{ij}(\sigma_1) = n^Y_{ij}(\sigma_2)$ and $n^X_{ij}(\sigma_1) = n^X_{ij}(\sigma_2)$)

$$\langle O^\dagger_{R,r}(\sigma_1) D_{XY} O_{R,r}(\sigma_2)\rangle \;=\; -\sqrt{\frac{c_{RR'}c_{TR'}}{l_i l_j}} n^Y_{ij}(\sigma_1)\sqrt{n^X_{ii}(\sigma_1)(n^X_{jj}(\sigma_1)+1)} \tag{3.80}$$

Another non-zero matrix element is obtained when the Y Gauss graph changes, by detaching a loop from node i of the σ_1^Y Gauss graph and reattaching it to node j. The matrix element describing this process is

$$\langle O^\dagger_{R,r}(\sigma_1) D_{XY} O_{R,r}(\sigma_2)\rangle \;=\; -\sqrt{\frac{c_{RR'}c_{TR'}}{l_i l_j}} n^X_{ij}\sqrt{n^Y_{ii}(\sigma_1)(n^Y_{jj}(\sigma_1)+1)} \tag{3.81}$$

This gives a complete description of the action of the term in the dilatation operator that mixes Xs and Ys on the Gauss operators.

3.4 Diagonalization

To understand the structure of the diagonalization problem, lets start off with a warm up problem. This will also be an example of the use of the formulas (3.79), (3.80) and (3.81), which will allow the reader to test her understanding of our result. Consider the Gauss graphs shown in figure 3. Using the formulas from the previous section, there is a transition between $|1\rangle$ and $|2\rangle$. To understand how we have labeled the dots, we must detach a loop from black node 3 of $|1\rangle$ and attach it to black node 2 of $|2\rangle$. Denote the Gauss graph correspodning to $|1\rangle$ by σ_1 and the Gauss graph of $|2\rangle$ by σ_2. We have $n^X_{23}(\sigma_1) = 1$ (read from the red Gauss graph), $n^Y_{22}(\sigma_1) + 1 = 2$ read from $|1\rangle$ and $n^Y_{33}(\sigma_1) = 1$ read from $|1\rangle$. Thus, in total the matrix element is

$$-\sqrt{\frac{(N+l_2)(N+l_3)}{l_2 l_3}}\sqrt{2} \tag{3.82}$$

As a second example, the matrix element for the transition between $|2\rangle$ and $|3\rangle$ is

$$-\sqrt{\frac{(N+l_1)(N+l_3)}{l_1 l_3}} \tag{3.83}$$

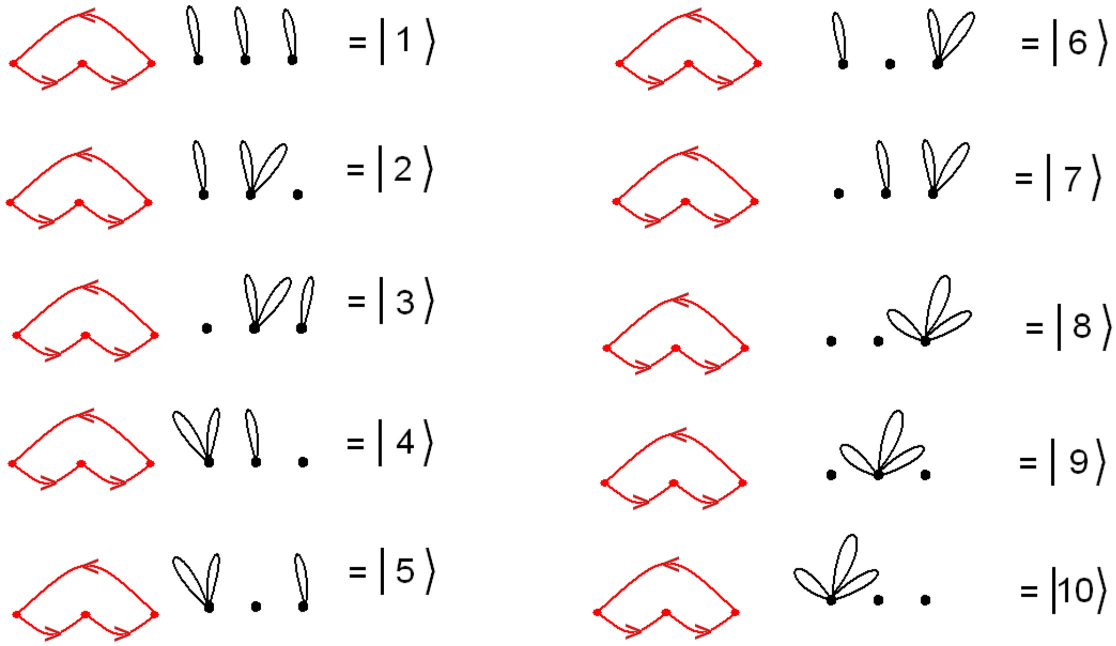


Figure 3: The 10 states that appear in our first example are defined in the figure above.

For the 10 states shown, we have the off diagonal piece of the dilatation operator given by

$$-\sqrt{\frac{(N+l_1)(N+l_2)}{l_1 l_2}}M_{12}-\sqrt{\frac{(N+l_2)(N+l_3)}{l_2 l_3}}M_{23}-\sqrt{\frac{(N+l_1)(N+l_3)}{l_1 l_3}}M_{13} \qquad (3.84)$$

where

$$M_{12}=\begin{bmatrix} 0 & 0 & -\sqrt{2} & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 \\ -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{3} \\ -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (3.85)$$

$$
M_{23} = \begin{bmatrix}
0 & -\sqrt{2} & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 \\
-\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -\sqrt{3} & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 & 0 \\
0 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{3.86}
$$

$$
M_{13} = \begin{bmatrix}
0 & 0 & 0 & -\sqrt{2} & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & -\sqrt{3} \\
0 & 0 & 0 & 0 & -2 & 0 & 0 & -\sqrt{3} & 0 & 0 \\
-\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{3.87}
$$

and we have the on diagonal piece of the dilatation operator given by

$$
\frac{(N+l_1)}{l_1} M_{11} + \frac{(N+l_2)}{l_2} M_{22} + \frac{(N+l_3)}{l_3} M_{13} \tag{3.88}
$$

where

$$
M_{11} = \begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6
\end{bmatrix} \tag{3.89}
$$

$$M_{22} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.90}$$

$$M_{33} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.91}$$

To get some insight into the structure of these matrices, note that the matrix

$$M = M_{11} + M_{22} + M_{33} + M_{12} + M_{23} + M_{13} \tag{3.92}$$

has eigenvalues $0, 3, 3, 6, 6, 6, 9, 9, 9, 9$. The even spacing and the degeneracy of the eigenvalues matches the weights of the symmetric representation $\square\!\square\!\square$ of $SU(3)$. This strongly suggests that, we can understand the off diagonal pieces of the dilatation operator as raising/lowering operators of some $SU(k)$ representations, with $k \leq g$. Recall that g is the number of rows in our restricted Schur polynomials. This guess turns out to be correct as we now explain.

First, we need to define a bijection between the Gauss graphs that mix and the states of a particular unitary group representation. Lets start by considering a situation for which the Y Gauss graph is fixed and we have transitions between different X Gauss graphs. We can only have transitions of closed loops between nodes i and j in the X Gauss graph if $n^Y_{ij}(\sigma) \neq 0$. Denote the number of connected components of σ^Y by C. Each connected component is a set of directed line segments running between nodes. Let c denote the number of connected components that have more than a single node. Let the number of nodes in each of these connected components be n_i, $i = 1, ..., c$. The irreducible representation that organizes the σ^X graphs is an irreducible representation of the group

$$SU(n_1) \times SU(n_2) \times \cdots \times SU(n_c) \tag{3.93}$$

Focus on one of the connected components, say the j^{th} connected component. Assume that there are a total of $\tilde{n}$ closed loops attached to nodes of σ^X that belong to this connected component. The irreducible representation of the $SU(n_j)$ factor in the above group that plays a role is labeled by a Young diagram that has a single row containing $\tilde{n}$ boxes. We now want to give the map between different X Gauss graphs and states of this representation. Number the nodes in the j^{th} connected component from 1 up to n_j. Consider an X Gauss graph that has n_{11} strings attached to node 1, n_{22} to node 2, and so on up to $n_{n_j n_j}$ attached to node n_j. The Gelfand-Tsetlin pattern for this state

$$|M\rangle = \begin{bmatrix} m_{1,n_j} & & m_{2,n_j} & & \cdots & & m_{n_j-1,n_j} & & & & m_{n_j n_j} \\ & m_{1,n_j-1} & & & & m_{2,n_j-1} & & \cdots & & m_{n_j-1,n_j-1} & \\ & & \cdots & & \cdots & & \cdots & & & & \\ & & m_{1,2} & & & & m_{2,2} & & & & \\ & & & & m_{1,1} & & & & & & \end{bmatrix}$$

has $m_{p,q} = 0$ for $p > 1$ and

$$m_{1,q} = \sum_{i=1}^{q} n_{ii} \qquad (3.94)$$

This completes our discussion of how the Gauss graphs are organized, for a fixed σ^Y. To complete the discussion note that there is a completely parallel argument with the roles of σ^X and σ^Y switched.

As a concrete example, the Gauss graph in figure 4 corresponds to the Gelfand-Tsetlin pattern

$$|M\rangle = \begin{bmatrix} n_{11}+n_{22}+n_{33} & & 0 & & 0 \\ & n_{11}+n_{22} & & 0 & \\ & & n_{11} & & \end{bmatrix}$$

This map between Gelfand-Tsetlin patterns and operators labeled by Gauss graphs turns out to be useful because we know the matrix elements of the Lie algebra elements in the Gelfand-Tsetlin basis. For example, let us consider the lowering operator $E_{i,i+1}$. This will shift $n_{ii} \to n_{ii} - 1$ and $n_{i+1,i+1} \to n_{i+1,i+1} + 1$. The net effect of these shifts in the Gelfand-Tsetlin pattern is to replace $m_{i,k} \to m_{i,k} - 1$; we will denote this pattern by M_i^-. The Gauss graph corresponding to M_i^- is obtained from the Gauss graph corresponding to M by peeling a closed loop from node i and reattaching it to node $i+1$. We have already studied the matrix element of the dilatation operator that mixes these two Gauss graphs and have found

$$-\sqrt{\frac{c_{RR'}c_{TR'}}{l_i l_{i+1}}} n^Y_{i,i+1}(\sigma)\sqrt{n^X_{ii}(\sigma)(n^X_{i+1,i+1}(\sigma)+1)} \qquad (3.95)$$

where σ describes the state with Gelfand-Tsetlin pattern M. According to [43] the matrix element for the lowering operator, written in terms of the entries of the Gelfand-Tsetlin pattern, is

$$\langle M_i^- | E_{i,i+1} | M \rangle = \sqrt{ - \frac{\prod_{k'=1}^{l+1}(m_{k',l+1} - m_{k,l} + k - k' + 1) \prod_{k'=1}^{l-1}(m_{k',l-1} - m_{k,l} + k - k')}{\prod_{k'=1,k'\neq k}^{l}(m_{k',l+1} - m_{k,l} + k - k' + 1)(m_{k',l+1} - m_{k,l} + k - k')} } \tag{3.96}$$

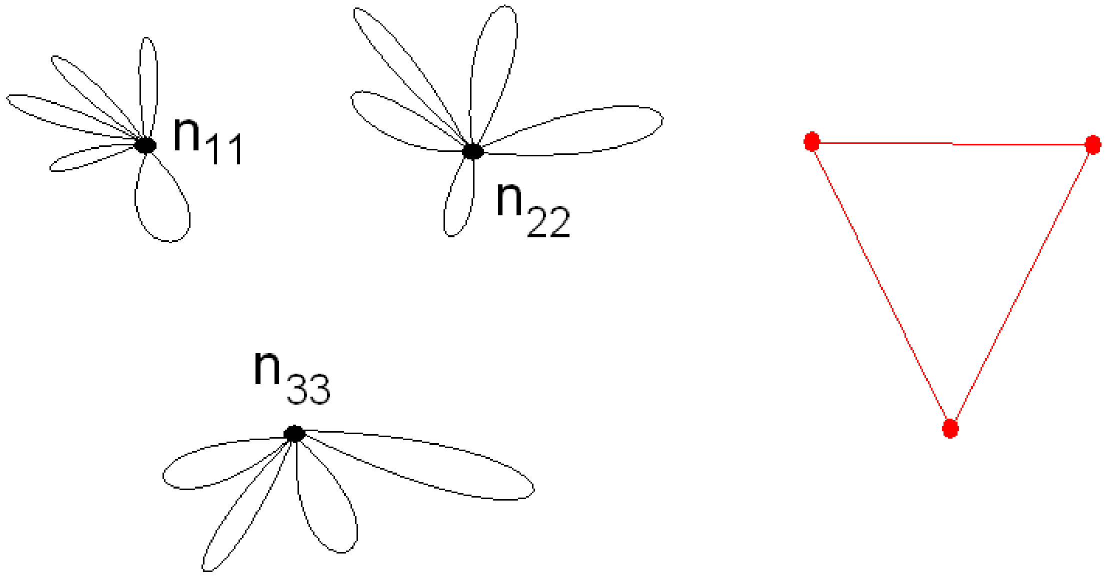


Figure 4: The Gauss graph is shown in black. Closed loops can detach from a node and reattach to another node.

Plugging in the patterns for the two Gauss graphs that mix, it is straight forward to see that (3.96) evaluates to

$$\sqrt{n_{ii}^X(\sigma)(n_{i+1,i+1}^X(\sigma) + 1)} \tag{3.97}$$

Comparing to (3.95) we see that the off diagonal term of the dilatation operator that we are considering is in fact

$$-\sqrt{\frac{c_{RR'}c_{TR'}}{l_i l_{i+1}}} n_{i,i+1}^Y(\sigma) E_{i,i+1} \tag{3.98}$$

We will state the result for the general case, for which loops move on both the X and Y Gauss graphs, using an example for illustration. The Gauss graph relevant for this example is show in figure 5.

Note that σ^X has two connected components, one which has 2 nodes and one which has 4 nodes. Consequently the group relevant for the organization of the Ys is $SU(2) \times SU(4)$. Counting closed loops on the nodes in σ^Y grouped by the connected components of σ^X we

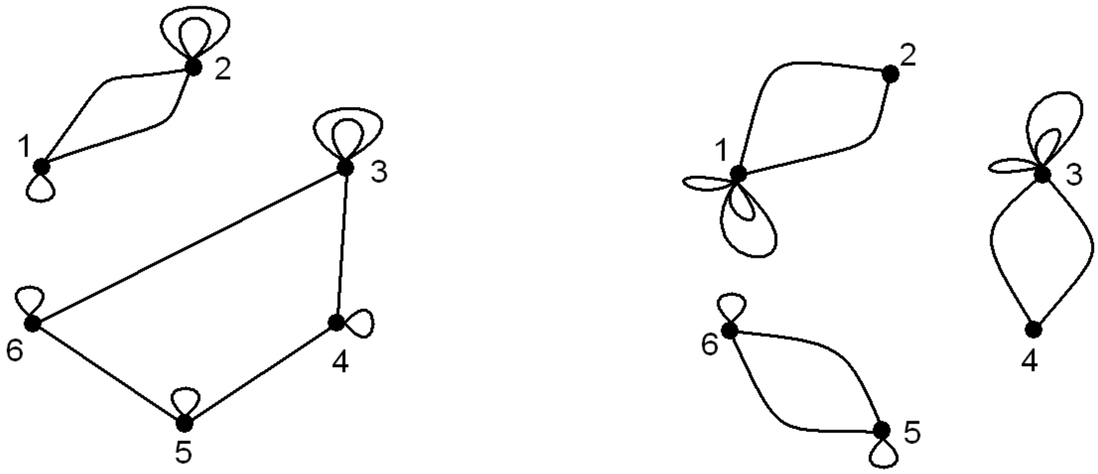


Figure 5: The graph on the left is σ^X. The graph on the right is σ^Y. Each node label in the above diagrams corresponds to a row number of Young diagram R in the restricted Schur polynomial $\chi_{R,(t,s,r)\vec{\mu}\vec{\nu}}$. The Gauss graphs shown correspond to an R with 6 long rows.

find that the representation of $SU(2)$ we need is $\square\square$ while the representation of $SU(4)$ we need is $\square\square\square\square\square$. Also, σ^Y has three connected components, each of which has 2 nodes. Consequently the group relevant for the organization of the Ys is $SU(2) \times SU(2) \times SU(2)$. Counting closed loops on the nodes in σ^X grouped by the connected components of σ^Y we find that the three representations for the three different $SU(2)$ groups we have are $\square\square\square$, $\square\square\square$ and $\square\square$. Denoting the groups that appear with a superscript

$$G^{(1)} \times G^{(2)} \times G^{(3)} \times G^{(4)} \times G^{(5)} = SU(2) \times SU(4) \times SU(2) \times SU(2) \times SU(2) \tag{3.99}$$

we can write the off diagonal terms in the dilatation operator as (the superscript on the Lie algebra element tells you which group it belongs to)

$$\begin{aligned} D_{\text{off diagonal}} = &-\sqrt{\frac{(N+l_3)(N+l_4)}{l_3 l_4}}(E^{(2)}_{12}+E^{(2)}_{21}) - \sqrt{\frac{(N+l_4)(N+l_5)}{l_4 l_5}}(E^{(2)}_{23}+E^{(2)}_{32}) \\ &-\sqrt{\frac{(N+l_5)(N+l_6)}{l_5 l_6}}(E^{(2)}_{34}+E^{(2)}_{43}) - \sqrt{\frac{(N+l_6)(N+l_3)}{l_6 l_3}}(E^{(2)}_{14}+E^{(2)}_{41}) \\ &-2\sqrt{\frac{(N+l_1)(N+l_2)}{l_1 l_2}}(E^{(1)}_{12}+E^{(1)}_{21}) - 2\sqrt{\frac{(N+l_3)(N+l_4)}{l_3 l_4}}(E^{(3)}_{12}+E^{(3)}_{21}) \\ &-2\sqrt{\frac{(N+l_3)(N+l_4)}{l_3 l_4}}(E^{(4)}_{12}+E^{(4)}_{21}) - 2\sqrt{\frac{(N+l_3)(N+l_4)}{l_3 l_4}}(E^{(5)}_{12}+E^{(5)}_{21}) \end{aligned} \tag{3.100}$$

The specific representation we should use for each Lie algebra has been spelt out above.

3.5 Conclusions and Discussion

In this chapter we have evaluated certain subleading terms in the action of the dilatation operator in the $su(3)$ sector. The operators we have studied have a classical dimension that scales as N. Consequently, even at large N, non-planar diagrams need to be summed and the limit we study is quite distinct from the planar limit. There is by now growing evidence that the dilatation operator in the large N but non-planar limit can be mapped into the Hamiltonian of a set of decoupled oscillators and hence that this limit of the theory continues to enjoy integrability. In the $su(2)$ sector, a new conservation law has been found. The corrections that we have evaluated spoil this new conservation law and consequently, these terms may be the first indiactions that the limit we consider is not integrable.

Our results clearly show that although the new terms do spoil the old conservation law, the system that emerges continues to be integrable. Indeed, the terms in the Hamiltonian that mix X and Z or Y and Z commute with the terms that mix X and Y, so that we simply need to change basis inside eigenspaces of fixed anomalous dimension. This change of basis has been reduced to the problem of diagonalizing certain elements in the Lie algebra of a well defined representation of a definite product of special unitary groups (the specific representation and product can be read off of the Gauss graphs as we explained in the last section). This is a solved problem in group theory.

The term in the dilatation operator that mixes X and Y does not act on the Z labels. The eigenproblem in the Z label, after moving to the Gauss operator basis, reduces to an oscillator problem[27]. The eigenvalues of the term in the dilatation operator that mixes X and Y sets the ground state energy of these oscillators. Note however that the BPS operators, which correspond to Gauss graphs with loops that start and end on the same node but no directed segments between nodes, are annihilated by the term in the dilatation operator that mixes X and Y, so that these operators remain BPS even when the corrections we have computed are included.

4 Higher Loop Non-planar Anomalous Dimensions

In this chapter we consider the action of the dilatation operator at higher loops in the $su(2)$ sector. The restricted Schur polynomials we consider are built only from the two adjoint scalars Z and Y. In other words we set $p = 0$ in (2.9). The one loop and two loop answers for the spectrum of anomalous dimensions show an interesting pattern. The action of the dilatation operator at one loop and at two loops factorizes into a piece that acts only on the r label - i.e. on the Z fields and a piece that acts only on the s and $\vec{\mu}$ labels, i.e. on the Y fields. Further, at one loop and at two loop the factor that acts on the Y fields is identical[31]. This prompts a very natural question: does this persist at higher loops? In this chapter we will argue that it does.

A brute force field theoretic approach to this problem seems hopeless. Here however, we can take some guidance from progress made in the planar sector of the theory [44]. Indeed, working in the $su(2|3)$ sector of theory and using the symmetry algebra as well as structural features from field theory, a great deal of information was obtained about higher loop corrections to the dilatation operator[44]. In the $su(2)$ sector that we study, we have operators $\vec{J}$ that generate an $SU(2)$ subgroup of the full $SU(4)$ $\mathcal{R}$ symmetry enjoyed by the theory. The $\vec{J}$ rotate the Y and Z fields amongst each other. Since their eigenvalues are fixed by the $su(2)$ algebra, we know that these generators do not receive quantum corrections. One of our results is a concrete expression for the action of these generators, in the large N limit, on restricted Schur polynomials. This is described in section 4.1. In contrast to the operators $\vec{J}$ the dilatation operator does receive quantum corrections. Since the operators $\vec{J}$ commute with the dilatation operator, we do have some information about higher loop corrections. Using this algebra, together with the large N limit and the constraints that follow from the fact that the dilatation operator is constructed by summing Feynman diagrams, we will give compelling evidence that the factor in the dilatation operator that acts on the Ys is given by the one loop expression at any loop order. Concretely, the algebra $\left[\vec{J}, D\right] = 0$ implies a set of recursion relations, hermitticity of the dilatation operator equates certain matrix elements of D and the fact that we work at large N implies that we can neglect changes in Young diagram r and further that the relation between R and r is preserved by D[1]. The derivation of these recursion relations and the structure of the dilatation operator and a demonstration that they determine the one loop dilatation operator is carried out in section 4.2. This analysis is most easily extended to higher loops by employing a continuum limit. The structure of this continuum limit is developed in section 4.3. In section 4.4 we demonstrate that the recursion relations derived in section 4.2 are replaced by partial differential equations. These partial differential equations describe all higher loops corrections to the dilatation operator. As we explain in section 4.4, they can be solved rather completely.

[1] r is obtained by removing boxes from R. When we say that the relation between R and r is preserved by D, we mean that D will only mix operators that are obtained by pulling the same number of boxes from each row of the big Young diagram R to obtain r.

The fact that the factor in the dilatation operator that acts on the Ys is given by the one loop expression at any loop order is not completely unexpected. Indeed, the diagonalization of this factor, achieved in general in [28], gives the set of states that is consistent with the Gauss Law constraints on a compact giant graviton world volume[24]. We expect these constraints to be satisfied at any order in the loop expansion, because the Gauss Law is an exact statement.

For simplicity we have restricted ourselves to the sector of the theory that is dual to a system of two giant gravitons. It would be straight forward but rather tedious to extend this to systems of more than two giant gravitons. A much more interesting generalization is to go beyond the $su(2)$ sector, because symmetry is not very constraining in the $su(2)$ sector. This follows because the dilatation operator is abelian and not part of a bigger algebra. Restricted Schur polynomials for the $su(2|3)$ sector have been derived in [45] and the use of symmetry in this sector would represent a very interesting generalization.

Another problem that should be tackled is to determine the factor in the dilatation operator that acts on the Z label. Understanding this factor, together with the results of this chapter, would allow a determination of the exact large N anomalous dimensions. This is not as unexpected as one might expect. Indeed, the operators we study are dual to giant gravitons. One expects the local relativistic invariant world volume theory dynamics to emerge from the sector of the theory we are considering. This picture suggests a relatively simple expression for the anomalous dimensions, determined by relativistic dispersion relations. The simplicity we find in this chapter is the first signal that this expectation is correct. For closely related discussions see [46, 47].

4.1 Action of $su(2)$ elements on restricted Schur polynomials

In this section our goal is to compute the action of the generators $J_\pm$ and J_3 on restricted Schur polynomials. We will freely make use of the results obtained in [26] in this section. Recall that in terms of the complex coordinates z and y, we can realize the $su(2)$ algebra as follows

$$J_+ = y\frac{\partial}{\partial z}, \quad J_- = z\frac{\partial}{\partial y}, \quad J_3 = y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}\,. \tag{4.1}$$

This follows because $SU(2)$ rotates the complex coordinates into each other. These generators close the usual algebra

$$\big[J_+, J_-\big] = J_3, \qquad \big[J_3, J_\pm\big] = \pm 2J_\pm\,. \tag{4.2}$$

When acting on the restricted Schur polynomials the generators are

$$J_+ = \mathrm{Tr}\left(Y\frac{d}{dZ}\right), \quad J_- = \mathrm{Tr}\left(Z\frac{d}{dY}\right), \quad J_3 = \mathrm{Tr}\left(Y\frac{d}{dY}\right) - \mathrm{Tr}\left(Z\frac{d}{dZ}\right)\,. \tag{4.3}$$

This follows because the $SU(2)$ $\mathcal{R}$-symmetry rotates the matrices Z and Y into each other. In what follows we will again make use of the identity (3.3).

Consider a system of g giant gravitons, i.e. the Young diagrams labeling the restricted Schur polynomials have a total of g rows. Our operators are built using n Zs and m Ys, with $n \gg m$. With $p = 0$ (2.9) becomes

$$\chi_{R,(r,s)\vec{\mu}}(Z,Y) = \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \mathrm{Tr}_{(r,s)\vec{\mu}} \left(\Gamma^R(\sigma)\right) \mathrm{Tr}(\sigma Y^{\otimes m} \otimes Z^{\otimes n}) \,. \tag{4.4}$$

We now have only one multiplicity label $\vec{\mu}$, which resolves the different copies of $s \vdash m$. As before, the restricted trace can be written in terms of an intertwining map $P_{R,(r,s)\vec{\mu}}$ as

$$\mathrm{Tr}_{(r,s)\vec{\mu}}(\cdots) = \mathrm{Tr}\left(P_{R,(r,s)\vec{\mu}} \cdots\right) \tag{4.5}$$

which factorizes as[26]

$$P_{R,(r,s)\vec{\mu}} = p_{s\vec{\mu}} \otimes \mathbf{1}_r \tag{4.6}$$

It is possible to compute $P_{R,(r,s)\vec{\mu}}$ explicitly for restricted Schur polynomials that are labeled by Young diagrams R with long rows and well separated corners[26]. We call this the displaced corners approximation. Recall that $n \gg m$ and that R has g long rows. We hold g fixed and order 1 as we take $N \to \infty$. In this limit the difference in the lengths of the corresponding rows of R and r can be neglected. Let V_g be a g dimensional vector space. In the construction of the projectors we removed m boxes from R to produce r with each box represented by a vector in V_g. The matrix E_{ij} acting in V_g is a $g \times g$ matrix with a 1 in the i^{th} row and j^{th} column, and zeros elsewhere. The space $V_g^{\otimes k}$ obtained by tensoring k copies of V_g will also play a role in what follows. The matrix $E_{ij}^{(a)}$ acts as E_{ij} on the a^{th} copy of V_g in $V_g^{\otimes k}$ and as the identity on all other copies. In the displaced corners approximation the multiplicity label is a pair of Gelfand-Tsetlin patterns. Both the space $V_g^{\otimes k}$ as well as the $E_{ij}^{(a)}$ will play an important role in the computations that follow. For more details and background see [26]. Consider the action of J_-

$$\begin{aligned}
&J_-\chi_{R,(r,s)\vec{\mu}}(Z,Y) = \mathrm{Tr}\left(Z\frac{d}{dY}\right)\chi_{R,(r,s)\vec{\mu}}(Z,Y)\\
&= \frac{m}{n!m!}\sum_{\sigma\in S_{n+m}} \mathrm{Tr}_{(r,s)\vec{\mu}}\left(\Gamma^R(\sigma)\right)\mathrm{Tr}(\sigma Y^{\otimes m-1} \otimes Z^{\otimes n+1})\\
&= \frac{m}{n!m!}\sum_{\sigma\in S_{n+m}} \mathrm{Tr}_{(r,s)\vec{\mu}}\left(\Gamma^R(\sigma)\right)\sum_{T,(t^+,u^-)\vec{\nu}} \frac{d_T(n+1)!(m-1)!}{d_{t^+}d_{u^-}(n+m)!}\chi_{T,(t^+,u^-)\vec{\nu}^*}(\sigma^{-1})\chi_{T,(t^+,u^-)\vec{\nu}}(Z,Y)\\
&= \sum_{T,(t^+,u^-)\vec{\nu}} \frac{d_T(n+1)}{d_{t^+}d_{u^-}(n+m)!}\frac{(n+m)!}{d_T}\delta_{RT}\mathrm{Tr}_{R\oplus T}(P_{R,(r,s)\vec{\mu}}P_{T,(t^+,u^-)\vec{\nu}^*})\chi_{T,(t^+,u^-)\vec{\nu}}(Z,Y)\\
&= \sum_{(t^+,u^-)\vec{\nu}} \frac{n+1}{d_{t^+}d_{u^-}}\mathrm{Tr}_R(P_{R,(r,s)\vec{\mu}}P_{R,(t^+,u^-)\vec{\nu}^*})\chi_{R,(t^+,u^-)\vec{\nu}}(Z,Y)\,.
\end{aligned} \tag{4.7}$$

In the above expression t^+ is a Young diagram with $n+1$ boxes, $t^+ \vdash n+1$. The $+$ superscript indicates that a box has been added to t. Similarly $u^- \vdash m-1$ with the $-$ superscript indicating that a box has been removed from u. Let us now discuss how to perform the trace in the above expression. Using the factorized form of the intertwining map in (4.6), we have[26]

$$\mathrm{Tr}_R(P_{R,(r,s)\vec{\mu}} P_{R,(t^+,u^-)\vec{\nu}^*}) = \mathrm{Tr}_R(p_{s\vec{\mu}} \otimes \mathbf{1}_r \, \cdot \, p_{u^-\vec{\nu}^*} \otimes \mathbf{1}_{t^+}) \,. \tag{4.8}$$

The only way that this trace can be non-zero is if it is possible for t^+ to subduce r. Write the projector $\mathbf{1}_{t^+}$ in terms of its action on the m$^{\text{th}}$ slot and $\mathbf{1}_r$. As an example to illustrate the idea, consider

$$\mathbf{1}_{\yng(3,2)} = E_{11}^{(m)} \otimes \mathbf{1}_{\yng(2,2)} + E_{22}^{(m)} \otimes \mathbf{1}_{\yng(3,1)} \,. \tag{4.9}$$

In the same way, if $t_i^{+\prime} = r$ we have[2]

$$\mathbf{1}_{t^+} = E_{ii}^{(m)} \otimes \mathbf{1}_r + \cdots \tag{4.10}$$

where $\cdots$ collects the terms that don't contribute to the value of the trace. Consequently, in the displaced corners approximation we find[26]

$$\begin{aligned} \mathrm{Tr}_R(P_{R,(r,s)\vec{\mu}} P_{R,(t^+,u^-)\vec{\nu}^*}) &= \mathrm{Tr}_R(p_{s\vec{\mu}} \otimes \mathbf{1}_r \, \cdot \, p_{u^-\vec{\nu}^*} \otimes \mathbf{1}_{t^+}) \\ &= \sum_i d_r \mathrm{Tr}_{V_g^{\otimes m}}(p_{s\vec{\mu}} \, \cdot \, p_{u^-\vec{\nu}^*} \otimes E_{ii}^{(m)}) \delta_{t_i^{+\prime} r} \,. \end{aligned} \tag{4.11}$$

To proceed further, recall that the multiplicity labels $\vec{\mu}$ and $\vec{\nu}$ stand for Gelfand-Tsetlin patterns, that is, states of $U(g)$. In addition, $E_{ii} = |\vec{v}(i)\rangle\langle\vec{v}(i)|$ and there is no sum on i. The state $|\vec{v}(i)\rangle$ is a state in the fundamental of $U(g)$ - it is a g dimensional vector of zeros except for the ith entry which is a 1. The projector $p_{s\vec{\mu}}$ is[26]

$$p_{s\vec{\mu}} = \sum_{a=1}^{d_s} |M_s^{\mu_1}, a\rangle\langle M_s^{\mu_2}, a| \tag{4.12}$$

where $|M_s^{\mu_1}, a\rangle$ is a state labeled by a Gelfand-Tsetlin pattern. $M_s^{\mu_1}$ is the pattern and a labels states inside symmetric group irreducible representation s. This state is obtained by taking a suitable linear combination of tensor products of m copies (one for each slot) of the fundamental representation of $U(g)$. Rewrite this state as a linear combination of states which are each the tensor product of the fundamental representation for the m^{th} slot, with a state obtained by taking the tensor product of states of the remaining $m-1$ slots[3]

$$|M_s^{\mu_1}, a\rangle = \sum_{M_{s'}^{\alpha_1}, M_F^l} C^{M_s^{\mu_1}}_{M_{s'}^{\alpha_1}, M_F^l} |M_{s'}^{\alpha_1}, b\rangle \otimes |M_F^l\rangle \,. \tag{4.13}$$

[2] $t_i^{+\prime}$ is the Young diagram obtained by dropping a box from the i$^{\text{th}}$ row of t^+.

[3] It is useful to spell out the index structure of the next equation. The index a runs over states in S_m irreducible representation s. The index b runs over states in irreducible representations s' subduced by s when S_m is restricted to S_{m-1}. We can thus put a and the sets of different b indices (one for every s') into correspondence.

$|M_F^l\rangle$ stands for a state in the fundamental representation of $U(g)$, $|M_F^l\rangle = |\vec{v}(l)\rangle$. When $E_{ii}^{(m)}$ acts on $|M_s^{\mu_1}, a\rangle$ it will pick out the piece with $l = i$. Thus,

$$\begin{aligned}
\mathrm{Tr}_{V_g^{\otimes m}}(p_{s\vec{\mu}} \cdot p_{u^-\vec{\nu}^*} \otimes E_{ii}^{(m)}) \qquad &= C^{M_s^{\mu_1}}_{M_{s'}^{\alpha_1}, M_F^i} C^{M_s^{\mu_2}}_{M_{s'}^{\alpha_2}, M_F^i} \mathrm{Tr}_{V_g^{\otimes m-1}}(p_{s'\vec{\alpha}} \cdot p_{u^-\vec{\nu}^*}) \\
&= d_{u^-} C^{M_s^{\mu_1}}_{M_{u^-}^{\nu_1}, M_F^i} C^{M_s^{\mu_2}}_{M_{u^-}^{\nu_2}, M_F^i} \,. \qquad (4.14)
\end{aligned}$$

The Clebsch-Gordan coefficient can be written is in terms of bras and kets as follows

$$C^{M_s^{\mu_1}}_{M_{u^-}^{\nu_1}, M_F^i} = \langle \nu_1 \otimes \vec{v}(i) | \mu_1 \rangle \,. \qquad (4.15)$$

Using this notation we finally have

$$\mathrm{Tr}_R(P_{R,(r,s)\vec{\mu}} P_{R,(t^+,u^-)\vec{\nu}^*}) = \sum_i d_r d_{u^-} \langle \mu_2 | \nu_2 \otimes \vec{v}(i) \rangle \langle \nu_1 \otimes \vec{v}(i) | \mu_1 \rangle \delta_{t_i^{+\prime} r} \,. \qquad (4.16)$$

Thus,

$$\begin{aligned}
J_- \chi_{R,(r,s)\vec{\mu}}(Z,Y) &= \sum_{(t^+,u^-)\vec{\nu}} \frac{n+1}{d_{t^+} d_{u^-}} \mathrm{Tr}_R(P_{R,(r,s)\vec{\mu}} P_{R,(t^+,u^-)\vec{\nu}^*}) \chi_{R,(t^+,u^-)\vec{\nu}} \\
&= \sum_{(t^+,u^-)\vec{\nu}} \sum_i \delta_{RT} \delta_{t_i^{+\prime} r} \frac{(n+1) d_r}{d_{t^+}} \langle \mu_2 | \nu_2 \otimes \vec{v}(i) \rangle \langle \nu_1 \otimes \vec{v}(i) | \mu_1 \rangle \chi_{R,(t^+,u^-)\vec{\nu}} \,. \qquad (4.17)
\end{aligned}$$

We want the action on normalized operators. The two point function of our operators are[15]

$$\langle \chi_{R,(r,s)\vec{\mu}}(Z,Y) \chi^\dagger_{T,(t,u)\vec{\nu}}(Z,Y) \rangle = \delta_{RT} \delta_{rt} \delta_{su} \delta_{\vec{\mu}\vec{\nu}} \frac{f_R \mathrm{hooks}_R}{\mathrm{hooks}_r \mathrm{hooks}_s} \,. \qquad (4.18)$$

By rescaling we can get operators with two point function equal to 1. Denote these by $O_{R,(r,s)\vec{\mu}}(Z,Y)$. Acting on the normalized operators we have

$$J_- O_{R,(r,s)\vec{\mu}}(Z,Y) = \sum_{T,(t^+,u^-)\vec{\nu}} (J_-)_{T,(t^+,u^-)\vec{\nu}\,,\,R,(r,s)\vec{\mu}} O_{T,(t^+,u^-)\vec{\nu}}(Z,Y) \qquad (4.19)$$

where

$$\begin{aligned}
(J_-)_{T,(t^+,u^-)\vec{\nu}\,,\,R,(r,s)\vec{\mu}} &= \sqrt{\frac{\mathrm{hooks}_r \mathrm{hooks}_s}{\mathrm{hooks}_{t^+} \mathrm{hooks}_{u^-}}} \\
&\times \sum_i \delta_{RT} \delta_{t_i^{+\prime} r} \frac{(n+1) d_r}{d_{t^+}} \langle \mu_2 | \nu_2 \otimes \vec{v}(i) \rangle \langle \nu_1 \otimes \vec{v}(i) | \mu_1 \rangle \\
&= \sqrt{\frac{\mathrm{hooks}_{t^+} \mathrm{hooks}_s}{\mathrm{hooks}_r \mathrm{hooks}_{u^-}}} \sum_i \delta_{RT} \delta_{t_i^{+\prime} r} \langle \mu_2 | \nu_2 \otimes \vec{v}(i) \rangle \langle \nu_1 \otimes \vec{v}(i) | \mu_1 \rangle \,. \qquad (4.20)
\end{aligned}$$

Very similar arguments give

$$J_+ O_{R,(r,s)\vec{\mu}}(Z,Y) = \sum_{T,(t^-,u^+)\vec{\nu}} (J_+)_{T,(t^-,u^+)\vec{\nu}\,,\,R,(r,s)\vec{\mu}} O_{T,(t^-,u^+)\vec{\nu}}(Z,Y) \qquad (4.21)$$

where

$$
\begin{aligned}
(J_+)_{T,(t^-,u^+)\vec{\nu}\,,\,R,(r,s)\vec{\mu}} &= \sqrt{\frac{\text{hooks}_r \text{hooks}_s}{\text{hooks}_{t^-} \text{hooks}_{u^+}}} \\
&\times \sum_i \delta_{RT} \delta_{t^- r_i'} (m+1) \frac{d_s}{d_{u^+}} \langle \mu_2 \otimes \vec{v}(i) | \nu_2 \rangle \langle \nu_1 | \mu_1 \otimes \vec{v}(i) \rangle \\
&= \sqrt{\frac{\text{hooks}_r \text{hooks}_{u^+}}{\text{hooks}_{t^-} \text{hooks}_s}} \sum_i \delta_{RT} \delta_{t^- r_i'} \langle \mu_2 \otimes \vec{v}(i) | \nu_2 \rangle \langle \nu_1 | \mu_1 \otimes \vec{v}(i) \rangle
\end{aligned} \tag{4.22}
$$

and

$$
J_3 O_{R,(r,s)\vec{\mu}}(Z,Y) = \sum_{T,(t,u)\vec{\nu}} (J_3)_{T,(t,u)\vec{\nu}\,,\,R,(r,s)\vec{\mu}} O_{T,(t,u)\vec{\nu}}(Z,Y) \tag{4.23}
$$

where

$$
(J_3)_{T,(t,u)\vec{\nu}\,,\,R,(r,s)\vec{\mu}} = \delta_{RT} \delta_{tr} \delta_{us} \delta_{\vec{\mu}\vec{\nu}} (m-n)\,. \tag{4.24}
$$

Our main interest is in the case of 2 rows. This is the simplest setting in which to develop our arguments because in this case there are no multiplicities for the irreducible representations that organize the Y fields. We will make use of a vector $\vec{m}$ which summarizes how to obtain r from R. Consider $O_{R,(r,s)}$. The vector $\vec{m} = (m_1, m_2)$ tells us how boxes should be removed from R to obtain r. Denoting the row lengths of R by (R_1, R_2) and of r by (r_1, r_2), we have $R_1 = r_1 + m_1$ and $R_2 = r_2 + m_2$. As explained in Appendix E.1 of [26], we can trade the irreducible representation s organizing the Y fields and $\vec{m}$ for an $SU(2)$ state. In the new labelling, we specify an operator (which belongs to the sector of the theory constructed using n Zs and m Ys) by giving the Young diagram r and an $SU(2)$ state with labels (j, j_3) where[4]

$$
s = (j, j_3) \qquad \longleftrightarrow \qquad s_1 = \frac{m+2j}{2}, \quad s_2 = \frac{m-2j}{2}, \quad j_3 = \frac{m_1 - m_2}{2}\,. \tag{4.25}
$$

We will use the j, j^3 notation in what follows.

We know that J_+ removes a Z box and adds a Y box. Thus, it could have the following possible actions on r, the irreducible representation organizing the Zs (the box to be removed has a $-$ sign in it - i.e. drop the box with the $-$ sign)

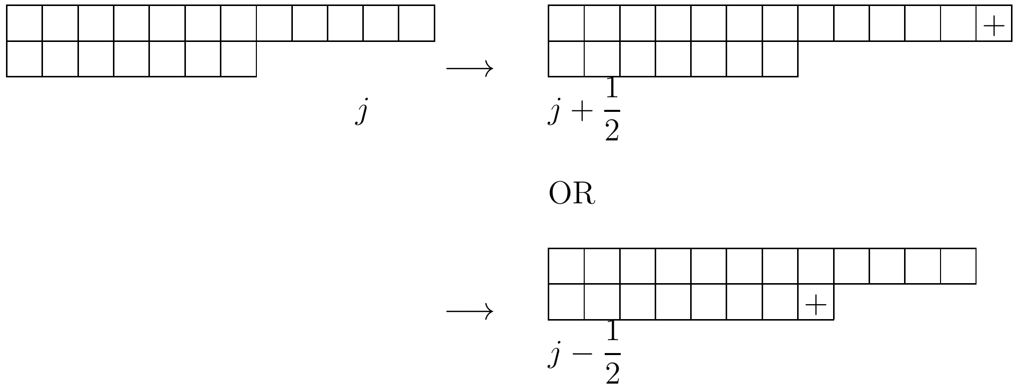
$j^3, r_1, r_2 \qquad j^3 + \frac{1}{2}, r_1 - 1, r_2$

OR

[4] s_i denote the row lengths of s.

$$\longrightarrow \quad \underset{j^3 - \frac{1}{2}, r_1, r_2 - 1}{\begin{ytableau} \; & \; & \; & \; & \; & \; & \; & \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; & \; & - \end{ytableau}} \tag{4.26}$$

It is trivial to understand how the row lengths r_1 and r_2 change when the box shown is dropped. To understand the changes in j^3, note the following: J_+ does not change the shape of R so that if we know how r changes, we know how $\vec{m}$ changes. In the first possibility above we remove a box from the first row of r which implies that m_1 grows by 1 and hence that j^3 grows by $\frac{1}{2}$. In the second possibility above we remove a box from the second row of r which implies that m_2 grows by 1 and hence that j^3 decreases by $\frac{1}{2}$. Since we have added a Y box, J_+ can have the following action on s (the box that has been addded has a $+$ in it)



(4.27)

Consequently we have[5]

$$
\begin{aligned}
&J_+O^{(n,m)}(r_1,j,j^3) = A_+O^{(n-1,m+1)}(r_1-1,j+\frac{1}{2},j^3+\frac{1}{2}) + B_+O^{(n-1,m+1)}(r_1-1,j-\frac{1}{2},j^3+\frac{1}{2})\\
&+C_+O^{(n-1,m+1)}(r_1,j+\frac{1}{2},j^3-\frac{1}{2}) + D_+O^{(n-1,m+1)}(r_1,j-\frac{1}{2},j^3-\frac{1}{2})\,. \qquad (4.28)
\end{aligned}
$$

We will describe the computation of A_+ in detail. From (4.22) we have

$$
A_+ = \sqrt{\frac{\text{hooks}_r}{\text{hooks}_{t^-}}}\sqrt{\frac{\text{hooks}_{u^+}}{\text{hooks}_s}}\left(\langle j,j^3;\frac{1}{2},\frac{1}{2}|j+\frac{1}{2},j^3+\frac{1}{2}\rangle\right)^2 \qquad (4.29)
$$

where

$$
\begin{aligned}
\sqrt{\frac{\text{hooks}_r}{\text{hooks}_{t^-}}} &= \sqrt{\frac{(r_1+1)(r_1-r_2)}{(r_1-r_2+1)}}\\
\sqrt{\frac{\text{hooks}_{u^+}}{\text{hooks}_s}} &= \sqrt{\frac{m+2j+4}{2}\frac{2j+1}{2j+2}}\\
\left(\langle j,j^3;\frac{1}{2},\frac{1}{2}|j+\frac{1}{2},j^3+\frac{1}{2}\rangle\right)^2 &= \frac{j+j^3+1}{2j+1}\,. \qquad (4.30)
\end{aligned}
$$

Putting the above factors together, we find

$$
A_+ = \sqrt{\frac{(r_1+1)(r_1-r_2)}{(r_1-r_2+1)}}\sqrt{\frac{m+2j+4}{2}\frac{2j+1}{2j+2}}\frac{j+j^3+1}{2j+1}\,. \qquad (4.31)
$$

In the large N limit this simplifies to

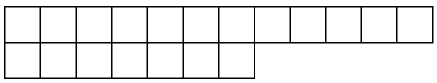
$$
A_+ = \sqrt{r_1}\sqrt{\frac{m+2j+4}{2}\frac{2j+1}{2j+2}}\frac{j+j^3+1}{2j+1}\,. \qquad (4.32)
$$

[5]Note that we don't need to display r_2 since $r_2 = n - r_1$.

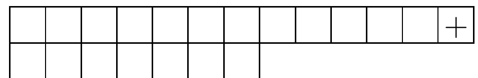
Very similar arguments imply that

$$
\begin{aligned}
B_+ &= \sqrt{r_1}\sqrt{\frac{m-2j+2}{2}\frac{2j+1}{2j}\frac{j-j^3}{2j+1}}\,,\\
C_+ &= \sqrt{r_2}\sqrt{\frac{m+2j+4}{2}\frac{2j+1}{2j+2}\frac{j-j^3+1}{2j+1}}\,,\\
D_+ &= \sqrt{r_2}\sqrt{\frac{m+2j+4}{2}\frac{2j+1}{2j}\frac{j+j^3}{2j+1}}\,.
\end{aligned}
\tag{4.33}
$$

Next, consider the action of J_-. We know that J_- removes a Y box and adds a Z box. Thus, it could have the following possible actions on r, the irreducible representation organizing the Zs (the box added has a $+$ sign in it)



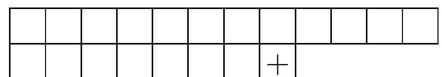
$\longrightarrow$



j^3, r_1, r_2 $\qquad$ $j^3-\frac{1}{2}, r_1+1, r_2$

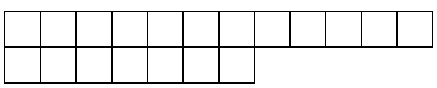
OR

$\longrightarrow$

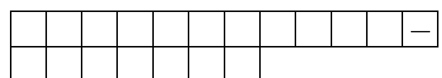


$j^3+\frac{1}{2}, r_1, r_2+1$ $\qquad$ (4.34)

Since we have removed a Y box, J_- can have the following action on s (the box removed has a $-$ in it)



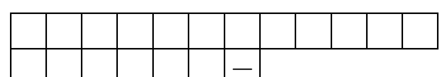
$\longrightarrow$



j $\qquad$ $j-\frac{1}{2}$

OR

$\longrightarrow$



$j+\frac{1}{2}$ $\qquad$ (4.35)

Consequently we have

$$
\begin{aligned}
J_-O^{(n,m)}(r_1,j,j^3) &= A_-O^{(n+1,m-1)}(r_1+1,j+\frac{1}{2},j^3-\frac{1}{2}) + B_-O^{(n+1,m-1)}(r_1+1,j-\frac{1}{2},j^3-\frac{1}{2})\\
&+C_-O^{(n+1,m-1)}(r_1,j+\frac{1}{2},j^3+\frac{1}{2}) + D_-O^{(n+1,m-1)}(r_1,j-\frac{1}{2},j^3+\frac{1}{2})\,.
\end{aligned}
\tag{4.36}
$$

To compute A_-, note that (4.20) implies that

$$
A_- = \sqrt{\frac{\text{hooks}_{t^+}}{\text{hooks}_r}}\sqrt{\frac{\text{hooks}_s}{\text{hooks}_{u^-}}}\left(\langle j,j^3|\frac{1}{2},\frac{1}{2};j+\frac{1}{2},j^3-\frac{1}{2}\rangle\right)^2 \tag{4.37}
$$

where

$$\sqrt{\frac{\text{hooks}_{t^+}}{\text{hooks}_r}} = \sqrt{\frac{(r_1+2)(r_1-r_2+1)}{(r_1-r_2+2)}}$$
$$\sqrt{\frac{\text{hooks}_s}{\text{hooks}_{u^-}}} = \sqrt{\frac{m-2j}{2}\frac{2j+2}{2j+1}}$$
$$\left(\langle j,j^3|\frac{1}{2},\frac{1}{2};j+\frac{1}{2},j^3-\frac{1}{2}\rangle\right)^2 = \frac{j-j^3+1}{2j+2}\,. \tag{4.38}$$

Thus, we find

$$A_- = \sqrt{\frac{(r_1+2)(r_1-r_2+1)}{(r_1-r_2+2)}}\sqrt{\frac{m-2j}{2}\frac{2j+2}{2j+1}}\frac{j-j^3+1}{2j+2}\,. \tag{4.39}$$

In the large N limit this becomes

$$A_- = \sqrt{r_1}\sqrt{\frac{m-2j}{2}\frac{2j+2}{2j+1}}\frac{j-j^3+1}{2j+2}\,. \tag{4.40}$$

Very similar arguments imply that

$$B_- = \sqrt{r_1}\sqrt{\frac{m+2j+2}{2}\frac{2j}{2j+1}}\frac{j+j^3}{2j}\,,$$
$$C_- = \sqrt{r_2}\sqrt{\frac{m-2j}{2}\frac{2j+2}{2j+1}}\frac{j+j^3+1}{2j+2}\,,$$
$$D_- = \sqrt{r_2}\sqrt{\frac{m+2j+2}{2}\frac{2j}{2j+1}}\frac{j-j^3}{2j}\,. \tag{4.41}$$

Using these results it is straight forward to find

$$[J_+,J_-]O^{(n,m)}(r_1,j,j^3) = -nO^{(n,m)}(r_1,j,j^3)\,. \tag{4.42}$$

Noting that $J_3O^{(n,m)}(r_1,j,j^3) = (m-n)O^{(n,m)}(r_1,j,j^3)$, this is indeed the correct large N limit of (4.2).

4.2 Recursion relations and one loop dilatation operator

The one loop dilatation operator in the $su(2)$ sector[6]

$$D_2 = -g_{\rm YM}^2 \text{Tr}\left[Y,Z\right]\left[\partial_Y,\partial_Z\right] \tag{4.43}$$

acting on two giant graviton systems, is given by [25, 26]

$$\begin{aligned} D_2 O^{(n,m)}(r_1,j,j^3) \quad &= g_{YM}^2 \Bigg[-\frac{1}{2}\left(m - \frac{(m+2)(j^3)^2}{j(j+1)}\right)\Delta O^{(n,m)}(r_1,j,j^3) \\ &+\sqrt{\frac{(m+2j+4)(m-2j)}{(2j+1)(2j+3)}}\frac{(j+j^3+1)(j-j^3+1)}{2(j+1)}\Delta O^{(n,m)}(r_1,j+1,j^3) \\ &+\sqrt{\frac{(m+2j+2)(m-2j+2)}{(2j+1)(2j-1)}}\frac{(j+j^3)(j-j^3)}{2j}\Delta O^{(n,m)}(r_1,j-1,j^3)\Bigg] \end{aligned} \tag{4.44}$$

where $(r_2 = n - r_1)$

$$\begin{aligned} \Delta O^{(n,m)}(r_1,j,j^3) \quad &= \sqrt{(N+r_1)(N+r_2)}(O^{(n,m)}(r_1+1,j,j^3) + O^{(n,m)}(r_1-1,j,j^3)) \\ &-(2N+r_1+r_2)O^{(n,m)}(r_1,j,j^3)\,. \end{aligned} \tag{4.45}$$

Our goal in this section is to argue that we can recover (4.44) by requiring that the correct algebra

$$\left[D_2, J_\pm\right] = 0 = \left[D_2, J_3\right] \tag{4.46}$$

is obeyed. We have already obtained a formula for the action of $J_\pm$ and J_3 on restricted Schur polynomials. Our first task is thus to obtain a similar result for the action of D_2, that can be used in (4.46). We are not trying to write down a detailed formula for D_2, but rather, want to write the general structure of this action that is consistent with the fact that it is derived by summing Feynman diagrams, we are working at large N and the dilatation operator is a hermittian operator. Given this general form, we will derive the detailed matrix elements by requiring (4.46).

There is a pair of derivatives in the one loop dilatation operator (4.43). Since they share an index, their action on the restricted Schur polynomials produces a Kronecker delta function. Equivalently, at one loop our Feynman diagrams have a single interaction vertex and this vertex has two pairs of adjacent fields, Z, Y and $Z^\dagger, Y^\dagger$. Wick contraction with the vertex will thus set a pair of indices equal, producing a Kronecker delta function. The net consequence of this Kronecker delta function is that the sum over S_{n+m} appearing in the evaluation of D_2 is reduced to a sum over the subgroup S_{n+m-1}[30]. When we sum over the S_{n+m-1} subgroup, the fundamental orthogonality relation forces one of the representations of S_{n+m-1} subduced by T to be equal to one of the representations subduced by R. This allows D_2 to shift the position of a single box in each of the Young diagram labels of the restricted Schur polynomial. This is precisely the process we saw in chapter 3. At p-loops we will have p insertions of the interaction vertex producing (at most) p Kronecker delta functions, thereby reducing the sum over S_{n+m} to a sum over S_{n+m-p}. This allows the p-loop dilatation operator to shift the position of (at most) p-boxes in each of the Young diagram labels of the restricted Schur polynomial. Returning to one loop, a single box shifts position under the action of D_2. This implies that we can have the following changes in the labels of our operators

$$\begin{aligned} j^3 &\to j^3, j^3 \pm 1\,, \\ j &\to j, j \pm 1\,, \\ r_1 &\to r_1, r_1 \pm 1\,. \end{aligned} \tag{4.47}$$

This change of labels implies a total of 27 possible terms under the action of D_2

$$D_2 O^{(n,m)}(r_1, j, j^3) = \sum_{c=-1}^{1} \sum_{d=-1}^{1} \sum_{e=-1}^{1} \beta^{(n,m)}_{r_1,j,j^3}(c,d,e) O^{(n,m)}(r_1 + c, j + d, j^3 + e) \tag{4.48}$$

This is slightly too general, as we have not yet put in the constraint that only 1 box can move, i.e. that even if R and T don't agree, by removing a single box from R and a single box from T we can get Young diagrams which do agree. The boxes that must be moved between R and T can be deduced from the boxes moving between r and t and the number of Y boxes that move between the rows (determined by j^3). The matrix element of the dilatation operator that takes

$$O^{(n,m)}_{r_1,j,j^3} \longrightarrow O^{(n,m)}_{r_1+a,j+b,j^3+c} \equiv O^{(n,m)}_{t_1,j',j^{3\prime}} \tag{4.49}$$

is $\beta^{(n,m)}_{r_1,j,j^3}(a,b,c)$. The integer a determines how r_1 changes, $t_1 - r_1 = a$. The integer c determines how j^3 changes, $j^{3\prime} - j^3 = c$. From the definition of j^3 we have

$$2j^3 = (R_1 - r_1) - (R_2 - r_2)\,, \tag{4.50}$$

$$2j^{3\prime} = (T_1 - t_1) - (T_2 - t_2)\,. \tag{4.51}$$

We also know that $T_1 + T_2 = R_1 + R_2 = m + n$ and $t_1 + t_2 = r_1 + r_2 = n$ so that

$$2j^3 = 2R_1 - (m+n) - 2r_1 + n\,, \tag{4.52}$$

$$2j^{3\prime} = 2T_1 - (m+n) - 2t_1 + n\,. \tag{4.53}$$

Subtracting these last two equations gives

$$2(j^{3\prime} - j^3) = 2c = 2(T_1 - R_1) + 2(r_1 - t_1) = 2(T_1 - R_1) - 2a\,. \tag{4.54}$$

Thus, $T_1 - R_1 = a + c$ and we must have $|a + c| \leq 1$. This forces

$$\begin{aligned} \beta^{(n,m)}_{r_1,j,j^3}(1,0,1) &= 0 \qquad \beta^{(n,m)}_{r_1,j,j^3}(-1,0,-1) = 0 \\ \beta^{(n,m)}_{r_1,j,j^3}(1,1,1) &= 0 \qquad \beta^{(n,m)}_{r_1,j,j^3}(-1,1,-1) = 0 \\ \beta^{(n,m)}_{r_1,j,j^3}(1,-1,1) &= 0 \qquad \beta^{(n,m)}_{r_1,j,j^3}(-1,-1,-1) = 0\,. \end{aligned} \tag{4.55}$$

This reduces the number of terms in the action of D_2 to 21. Next, we know that the dilatation operator is hermittian $D_2 = D_2^\dagger$. This implies that

$$\langle r_1 + a, j + b, j^3 + c|D_2|r_1, j, j^3\rangle = \langle r_1, j, j^3|D_2|r_1 + a, j + b, j^3 + c\rangle . \tag{4.56}$$

Further, since

$$\langle r_1 + a, j + b, j^3 + c|D_2|r_1, j, j^3\rangle = \beta^{(n,m)}_{r_1,j,j^3}(a, b, c) \tag{4.57}$$

and

$$\langle r_1, j, j^3|D_2|r_1 + a, j + b, j^3 + c\rangle = \beta^{(n,m)}_{r_1+a,j+b,j^3+c}(-a, -b, -c) \tag{4.58}$$

we find that the condition $D_2 = D_2^\dagger$ implies that

$$\beta^{(n,m)}_{r_1,j,j^3}(a, b, c) = \beta^{(n,m)}_{r_1+a,j+b,j^3+c}(-a, -b, -c) . \tag{4.59}$$

This reduces the number of unknown terms to be determined to 11.

In the large N limit, the string coupling $g_s = \frac{1}{N}$ goes to zero. Consequently there is no string splitting or joining. Since each trace in the SYM theory corresponds to a closed string state, this translates into the fact that, in the planar limit in the SYM theory, different multi-trace structures do not mix. For the open string sector, when the string coupling goes to zero there is again no splitting and joining so that the open string Chan-Paton factors are frozen. Recall the translation of a giant graviton system into an operator in the field theory[39, 24, 48, 49, 10, 11, 13, 26, 28]: in the operator $O^{(n,m)}_{r,j,j^3}$ each row of r corresponds to a giant graviton and each impurity Y corresponds to an open string (this last interpretation is proved in [26, 28]). j^3 tells us the number of open string end points attached to each giant. Since the Chan-Paton factors are frozen, j^3 is not changed by the action of the dilatation operator and

$$\beta(a, b, \pm 1) = 0 . \tag{4.60}$$

This now leaves 4 unknown terms to be determined.

Another consequence of working at large N in the displaced corners approximation, is

$$\beta^{(n,m)}_{r_1+\alpha,j,j^3}(a, b, c) = \beta^{(n,m)}_{r_1,j,j^3}(a, b, c) \tag{4.61}$$

with α any number of order 1. This follows because r_1 is order N and the matrix elements of the dilatation operator depend smoothly on the parameters r_1, j, j^3, so we can replace $r_1+\alpha$ by r_1 making negligible error in the large N limit. There is one point that deserves attention. In general our results depend on r_1, r_2 and on $r_1 - r_2$. Even if $r_1 = O(N)$ and $r_2 = O(N)$, if $r_1 - r_2 = O(1)$, replacing $r_1 + \alpha \to r_1$ can result in errors that do not vanish as $N \to \infty$. In the displaced corners approximation all r row lengths are well separated and this does not happen. It then follows that the r_i are conserved and that the coefficients of $\sqrt{r_1}$ and $\sqrt{r_2}$ in

(4.46) must separately vanish. This has a very natural physical interpretation: the r_i set the momenta of the giant gravitons and the back reaction on each giant graviton is negligible.

Although we are mainly interested in the dependence of the dilatation operator on j, j^3, we do know that

$$
\begin{aligned}
\beta^{(n,m)}_{r_1,j,j^3}(\pm 1, d, e) &= \sqrt{(N+r_1)(N+r_2)} f(j, j^3, d, e) \\
&= \sqrt{(N+r_1)(N+n-r_1)} f(j, j^3, d, e) \\
\beta^{(n,m)}_{r_1,j,j^3}(0, d, e) &= (2N + r_1 + r_2) g(j, j^3, d, e) \\
&= (2N+n) g(j, j^3, d, e) \, .
\end{aligned}
\tag{4.62}
$$

These formulas deserve some discussion. The dependence of matrix elements on factors[6] of boxes in the Young diagram labels has two sources:

1. There is an overall normalization $\sqrt{\frac{f_T}{f_R}}$. The factors of any boxes that are common to R and T will cancel so that we are left with
$$
F_1 = \sqrt{\frac{\prod_{i\in \text{boxes in } T \text{ that are not in } R} \; c_i}{\prod_{j\in \text{boxes in } R \text{ that are not in } T} \; c_j}}
\tag{4.63}
$$
2. When evaluating the dilatation operator, we need to sum over S_{n+m}. As discussed above, derivatives with respect to Y and Z produce Kronecker delta functions that restrict the sum to the subgroup S_{n+m-1}. The original trace over $R \vdash m+n$ then becomes a trace over an irreducible representation of the subgroup $R' \vdash m+n-1$. The sum then produces the factor of the box that must be removed from R to obtain R'. The trace splits into a trace over $r' \vdash n-1$ which sets $r' = t'$ and a trace over s which depends only on j, j^3. This dependence is summarized in the functions $f(j, j^3, d, e)$ and $g(j, j^3, d, e)$ above and it is these functions that we want to constrain using the su(2) invariance.

For the first term in (4.62) we have

$$
F_1 = \sqrt{\frac{N+r_1}{N+r_2}} \quad F_2 = N + r_2 \qquad \text{or} \qquad F_1 = \sqrt{\frac{N+r_2}{N+r_1}} \quad F_2 = N + r_1
\tag{4.64}
$$

so that

$$
F_1 \cdot F_2 = \sqrt{(N+r_1)(N+r_2)}
\tag{4.65}
$$

For the second term in (4.62) we have two contributions which both have $F_1 = 1$ and

$$
F_2 = N + r_1 \qquad \text{or} \qquad F_2 = N + r_2
\tag{4.66}
$$

[6]Recall that a box in row i and column j has a factor $N - i + j$.

Thus, the total coefficient of this term is

$$N + r_1 + N + r_2 = 2N + r_1 + r_2 = 2N + n \tag{4.67}$$

Since we are computing a commutator, the answer for D_2 will not be unique. Indeed, replacing $D_2 \to D_2 + \alpha \mathbf{1}$ with α a constant, will not change the value of the commutator. To fix the value of α note that there are BPS operators belonging to the $su(2)$ sector. These operators are annihilated by D_2, so that the smallest eigenvalue of D_2 is zero. This fixes α.

Now, use

$$J_+ O^{(n,m)}_{r_1,j,j^3} = \sum_{a=-1}^{0} \sum_{b=-\frac{1}{2}}^{\frac{1}{2}} \alpha^{(n,m)}_{r_1,j,j^3}(a,b) O^{(n-1,m+1)}_{r_1+a,j+b,j^3-\frac{1}{2}-a} \tag{4.68}$$

where (using the results of the last section)

$$\alpha^{(n,m)}_{r_1,j,j^3}(-1,\frac{1}{2}) = \sqrt{r_1}\sqrt{\frac{m+2j+4}{2}} \frac{j+j^3+1}{\sqrt{(2j+2)(2j+1)}} \tag{4.69}$$

$$\alpha^{(n,m)}_{r_1,j,j^3}(-1,-\frac{1}{2}) = \sqrt{r_1}\sqrt{\frac{m-2j+2}{2}} \frac{j-j^3}{\sqrt{2j(2j+1)}} \tag{4.70}$$

$$\alpha^{(n,m)}_{r_1,j,j^3}(0,\frac{1}{2}) = \sqrt{r_2}\sqrt{\frac{m+2j+4}{2}} \frac{j-j^3+1}{\sqrt{(2j+2)(2j+1)}} \tag{4.71}$$

$$\alpha^{(n,m)}_{r_1,j,j^3}(0,-\frac{1}{2}) = \sqrt{r_2}\sqrt{\frac{m-2j+2}{2}} \frac{j+j^3}{\sqrt{2j(2j+1)}} \tag{4.72}$$

and use

$$D_2 O^{(n,m)}_{r_1,j,j^3} = \sum_{a=1}^{-1}\sum_{b=1}^{-1}\sum_{c=1}^{-1} \beta^{(n,m)}_{r_1,j,j^3}(a,b,c) O^{(n,m)}_{r_1+a,j+b,j^3+c} \tag{4.73}$$

to evaluate

$$\big[J_+, D_2\big] O^{(n,m)}_{r_1,j,j^3} = 0\,. \tag{4.74}$$

The result is

$$\sum_{a=-1}^{0} \sum_{b=-\frac{1}{2}}^{\frac{1}{2}} \sum_{c=-1}^{1} \sum_{d=-1}^{1} \sum_{e=-1}^{1} \Big(\beta^{(n,m)}_{r_1,j,j^3}(c,d,e) \alpha^{(n,m)}_{r_1+c,j+d,j^3+e}(a,b) \tag{4.75}$$
$$- \alpha^{(n,m)}_{r_1,j,j^3}(a,b) \beta^{(n-1,m+1)}_{r_1+a,j+b,j^3-\frac{1}{2}-a}(c,d,e) \Big) O^{(n-1,m+1)}_{r_1+a+c,j+d+b,j^3+e-\frac{1}{2}-a} = 0\,.$$

The operators $O^{(n,m)}_{r_1,j,j^3}$ are all linearly independent, so that the coefficient of each term must vanish separately. Further, since $\alpha^{(n,m)}_{r_1,j,j^3}(-1,\cdot) \propto \sqrt{r_1}$ and $\alpha^{(n,m)}_{r_1,j,j^3}(0,\cdot) \propto \sqrt{r_2}$, terms with different values of a in $\alpha^{(n,m)}_{r_1,j,j^3}(a,\cdot)$ must separately vanish.

To illustrate some of the details, we will discuss some examples of equations that we obtain from (4.75). In particular, we will explain how the $\beta_{r_1,j,j^3}(0,1,0)$ matrix element is determined. Set $a=0$, $c=0$, $e=0$, $d+b=-\frac{3}{2} \Rightarrow (d,b)=(-1,-\frac{1}{2})$ to obtain

$$\beta^{(n,m)}_{r_1,j,j^3}(0,-1,0)\alpha^{(n,m)}_{r_1,j-1,j^3}(0,-\frac{1}{2}) - \alpha^{(n,m)}_{r_1,j,j^3}(0,-\frac{1}{2})\beta^{(n-1,m+1)}_{r_1,j-\frac{1}{2},j^3-\frac{1}{2}}(0,-1,0) = 0\,,$$

$$\begin{aligned} &\sqrt{\frac{m-2j+4}{2}}\frac{j+j^3-1}{\sqrt{(2j-2)(2j-1)}}\beta^{(n,m)}_{r_1,j,j^3}(0,-1,0) \\ &-\sqrt{\frac{m-2j+2}{2}}\frac{j+j^3}{\sqrt{2j(2j+1)}}\beta^{(n-1,m+1)}_{r_1,j-\frac{1}{2},j^3-\frac{1}{2}}(0,-1,0) = 0\,. \end{aligned} \tag{4.76}$$

Next, set $a=-1$, $c=0$, $e=0$, $d+b=-\frac{3}{2} \Rightarrow (d,b)=(-1,-\frac{1}{2})$ to obtain

$$\beta^{(n,m)}_{r_1,j,j^3}(0,-1,0)\alpha^{(n,m)}_{r_1,j-1,j^3}(-1,-\frac{1}{2}) - \alpha^{(n,m)}_{r_1,j,j^3}(-1,-\frac{1}{2})\beta^{(n-1,m+1)}_{r_1-1,j-\frac{1}{2},j^3+\frac{1}{2}}(0,-1,0) = 0\,,$$

$$\begin{aligned} &\sqrt{\frac{m-2j+4}{2}}\frac{j-j^3-1}{\sqrt{(2j-2)(2j-1)}}\beta^{(n,m)}_{r_1,j,j^3}(0,-1,0) \\ &-\sqrt{\frac{m-2j+2}{2}}\frac{j-j^3}{\sqrt{(2j+1)2j}}\beta^{(n-1,m+1)}_{r_1-1,j-\frac{1}{2},j^3+\frac{1}{2}}(0,-1,0) = 0\,. \end{aligned} \tag{4.77}$$

Combining (4.76) and (4.77) we find

$$\beta^{(n,m)}_{r_1,j,j^3}(0,-1,0) = \frac{j+j^3}{j+j^3-1}\frac{j-j^3}{j-j^3+1}\beta^{(n,m)}_{r_1,j,j^3-1}(0,-1,0) \tag{4.78}$$

which implies that

$$\beta^{(n,m)}_{r_1,j,j^3}(0,-1,0) \propto (j+j^3)(j-j^3)\,. \tag{4.79}$$

Daggering equation (4.78) we find

$$\beta^{(n,m)}_{r_1,j,j^3}(0,1,0) = \frac{j+j^3+1}{j+j^3}\frac{j-j^3+1}{j-j^3+2}\beta^{(n,m)}_{r_1,j,j^3-1}(0,1,0) \tag{4.80}$$

which implies that

$$\beta^{(n,m)}_{r_1,j,j^3}(0,1,0) \propto (j+j^3+1)(j-j^3+1)\,. \tag{4.81}$$

Now, set $a=0$, $b=\frac{1}{2}$, $c=0$, $d=1$ and $e=0$ to obtain

$$\beta^{(n,m)}_{r_1,j,j^3}(0,1,0)\alpha^{(n,m)}_{r_1,j+1,j^3}(0,\frac{1}{2})-\alpha^{(n,m)}_{r_1,j,j^3}(0,\frac{1}{2})\beta^{(n-1,m+1)}_{r_1,j+\frac{1}{2},j^3-\frac{1}{2}}(0,1,0)=0\,,$$

$$\sqrt{\frac{m+2j+6}{2}}\frac{j-j^3+2}{\sqrt{(2j+4)(2j+3)}}\beta^{(n,m)}_{r_1,j,j^3}(0,1,0) - \sqrt{\frac{m+2j+4}{2}}\frac{j-j^3+1}{\sqrt{(2j+1)(2j+2)}}\beta^{(n-1,m+1)}_{r_1,j+\frac{1}{2},j^3-\frac{1}{2}}(0,1,0)=0\,. \tag{4.82}$$

Daggering this we find

$$\beta^{(n-1,m+1)}_{r_1,j-\frac{1}{2},j^3-\frac{1}{2}}(0,-1,0)=\sqrt{\frac{m+2j+2}{m+2j}}\frac{j-j^3}{j-j^3-1}\sqrt{\frac{(2j-3)(2j-2)}{(2j-1)2j}}\beta^{(n,m)}_{r_1,j-1,j^3}(0,-1,0)\,.$$

Combining this with (4.76) we find

$$\beta^{(n,m)}_{r_1,j,j^3}(0,-1,0)=\sqrt{\frac{(m-2j+2)(m+2j+2)}{(m-2j+4)(m+2j)}}\frac{2j-2}{2j}\sqrt{\frac{(2j-1)(2j-3)}{(2j+1)(2j-1)}} \times\frac{j+j^3}{j+j^3-1}\frac{j-j^3}{j-j^3-1}\beta^{(n,m)}_{r_1,j-1,j^3}(0,-1,0)$$

which implies that

$$\beta^{(n,m)}_{r_1,j,j^3}(0,-1,0)\propto\sqrt{\frac{(m+2j+2)(m-2j+2)}{(2j+1)(2j-1)}}\frac{(j+j^3)(j-j^3)}{2j} \tag{4.83}$$

which is indeed the correct result. Daggering, we find

$$\beta^{(n,m)}_{r_1,j,j^3}(0,1,0)\propto\sqrt{\frac{(m+2j+4)(m-2j)}{(2j+1)(2j+3)}}\frac{(j+j^3+1)(j-j^3+1)}{2(j+1)} \tag{4.84}$$

which is also correct. Solving the complete set of recursion relations we find

$$\begin{aligned}
&D_2O^{(n,m)}(r_1,j,j^3)=\\
&\sqrt{\frac{(m-2j+2)(m+2j+2)}{(2j+1)(2j-1)}}\frac{(j+j^3)(j-j^3)}{2j}\Big[c_{010}(2N+r_1+r_2)O^{(n,m)}(r_1,j-1,j^3)\\
&+c_{110}\sqrt{(N+r_1)(N+r_2)}(O^{(n,m)}(r_1-1,j-1,j^3)+O^{(n,m)}(r_1+1,j-1,j^3))\Big]\\
&+\sqrt{\frac{(m+2j+4)(m-2j)}{(2j+3)(2j+1)}}\frac{(j+j^3+1)(j-j^3+1)}{2j+2}\Big[c_{010}(2N+r_1+r_2)O^{(n,m)}(r_1,j+1,j^3)\\
&+c_{110}\sqrt{(N+r_1)(N+r_2)}(O^{(n,m)}(r_1-1,j+1,j^3)+O^{(n,m)}(r_1+1,j+1,j^3))\Big]\\
&+\left(-\frac{1}{2}\left(m-\frac{(m+2)(j^3)^2}{j(j+1)}\right)\right)\Big[c_{010}(2N+r_1+r_2)O^{(n,m)}(r_1,j,j^3)\\
&+c_{110}\sqrt{(N+r_1)(N+r_2)}(O^{(n,m)}(r_1-1,j,j^3)+O^{(n,m)}(r_1+1,j,j^3))\Big]
\end{aligned}$$

where c_{010} and c_{110} are arbitrary constants, independent of j, j^3 and r_1. Thus, we have determined the j, j^3 dependence of the matrix elements of the one loop dilatation operator. Achieving this at higher loops is one of the main goals of this dissertation. To completely determine the spectrum of anomalous dimensions, we need to determine the constants c_{010} and c_{110} in the above expression. These constants are tightly constrained as we now explain. In the large N regime, we can take a continuum limit of the action of the dilatation operator. Towards this end, introduce the continuous variable $\rho = \frac{r_1 - r_2}{2\sqrt{N+r_2}}$ and replace $O^{(r,m)}(r_1, j, j^3)$ with $O^{(r,m)}(\rho, j, j^3)$. r_1 is the longer (top) row and r_2 is the shorter bottom row. When ρ is order 1 the dilatation operator becomes an N independent differential operator[27]. Expanding we have

$$\sqrt{(N+r_1)(N+r_2)} = (N+r_2)\left(1 + \frac{1}{2}\frac{r_1 - r_2}{N+r_2} - \frac{1}{8}\frac{(r_1-r_2)^2}{(N+r_2)^2} + ...\right)$$

The first term above is $O(N)$, the second $O(\sqrt{N})$ and the third $O(1)$.

$$O^{(n,m)}\left(\rho - \frac{1}{\sqrt{N+r_2}}, j, j^3\right) = O^{(n,m)}(\rho, j, j^3) - \frac{1}{\sqrt{N+r_2}}\frac{\partial O^{(n,m)}}{\partial \rho}\Big|_{\rho,j,j^3} + \frac{1}{N+r_2}\frac{\partial^2 O^{(n,m)}}{\partial \rho^2}\Big|_{\rho,j,j^3} + ...$$

These expansions are only valid if $r_1 - r_2 \ll N + r_2$, which is certainly not always the case. However, we will learn something about the relation between the coeficients c_{110} and c_{010} by studying this situation. Using these expansions we have

$$\begin{aligned}
&c_{010}(2N + r_1 + r_2)O^{(n,m)}(r_1, j, j^3) \\
&\quad + c_{110}\sqrt{(N+r_1)(N+r_2)}(O^{(n,m)}(r_1 - 1, j, j^3) + O^{(n,m)}(r_1 + 1, j, j^3)) \\
&= \left[c_{110} + 2c_{010}\right](N + r_2)O^{(n,m)}(r_1, j, j^3) + \frac{1}{2}\left[c_{110} + 2c_{010}\right]\sqrt{N+r_2}O^{(n,m)}(r_1, j, j^3) + O(1)
\end{aligned}$$

Again, the lowest eigenvalue of this operator is zero, reflecting a BPS operator. To achieve this, the $O(N)$ and $O(\sqrt{N})$ pieces of this expansion must cancel which determines $c_{110} + 2c_{010} = 0$. Thus, up to an overall normalization which our argument can't determine, we have reproduced (4.44).

4.3 Continuum Limit

We have demonstrated that the requirement that the one loop dilatation operator closes the correct Lie algebra when commuted with an $su(2)$ subgroup of the $\mathcal{R}$-symmetry group determines a set of recursion relations. Solving these recursion relations we have recovered the formula for the one loop dilatation operator derived in [25, 26] by detailed computation. We are interested in carrying this analysis out at higher loops. The resulting recursion relations become very clumsy to solve. To overcome this difficulty, we will now pursue a continuum approach to the problem, replacing the discrete variables j, j^3 by continuous variables x_j, x_{j^3}. The advantage of considering a continuum limit is that our recursion relations will be replaced by partial differential equations and we are able to explicitly determine the general solution of these partial differential equations. In this section we will motivate the continuum limit we study by considering the dilatation operator eigenproblem at one loop.

The structure of the action of the one loop dilatation operator problem given in (4.44) exhibits an interesting factorization. There is an action of Δ which acts only on the r label times an action that is only on the j, j^3 labels. The continuum limit we consider here is concerned with the action on the j, j^3 labels. Recall that we take m to be $O(\sqrt{N})$. The discrete eigenproblem that we consider is[25, 26]

$$-\lambda\psi(j,j^3) = \sqrt{\frac{(m+2j+4)(m-2j)}{(2j+1)(2j+3)}}\frac{(j+j^3+1)(j-j^3+1)}{2(j+1)}\psi(j+1,j^3)$$
$$\sqrt{\frac{(m+2j+2)(m-2j+2)}{(2j+1)(2j-1)}}\frac{(j+j^3)(j-j^3)}{2j}\psi(j-1,j^3) - \frac{1}{2}\left(m - \frac{(m+2)(j^3)^2}{j(j+1)}\right)\psi(j,j^3)\,. \tag{4.85}$$

The variables that become continuous as we take $N \to \infty$ are

$$x_j = \frac{j}{\sqrt{m}}, \qquad x_{j^3} = \frac{j^3}{\sqrt{m}} \tag{4.86}$$

Replace $\psi(j,j^3)$ by $\psi(x_j, x_{j^3})$ and use the expansions

$$-\frac{1}{2}\left(m - \frac{(m+2)(j^3)^2}{j(j+1)}\right) = -\frac{m}{2} + \frac{m}{2}\frac{x_{j^3}^2}{x_j^2} - \frac{\sqrt{m}}{2}\frac{x_{j^3}^2}{x_j^3} + \frac{x_{j^3}^2}{2x_j^4} + \frac{x_{j^3}^2}{x_j^2} \tag{4.87}$$

$$\begin{aligned}\sqrt{\frac{(m+2j+4)(m-2j)}{(2j+1)(2j+3)}}\frac{(j+j^3+1)(j-j^3+1)}{2(j+1)} &= \frac{m}{4} + \frac{1}{2} - \frac{x_j^2}{2} + \frac{1}{32x_j^2} \\ &\quad -\frac{m}{4}\frac{x_{j^3}^2}{x_j^2} - \frac{1}{2}\frac{x_{j^3}^2}{x_j^2} + \frac{x_{j^3}^2}{2} - \frac{25x_{j^3}^2}{32x_j^4} + \frac{\sqrt{m}x_{j^3}^2}{2x_j^3}\end{aligned} \tag{4.88}$$

$$\begin{aligned}\sqrt{\frac{(m+2j+2)(m-2j+2)}{(2j+1)(2j-1)}}\frac{(j+j^3)(j-j^3)}{2j} &= \frac{m}{4} + \frac{1}{2} - \frac{x_j^2}{2} + \frac{1}{32x_j^2} \\ &\quad -\frac{m}{4}\frac{x_{j^3}^2}{x_j^2} - \frac{1}{2}\frac{x_{j^3}^2}{x_j^2} + \frac{x_{j^3}^2}{2} - \frac{x_{j^3}^2}{32x_j^4}\,.\end{aligned} \tag{4.89}$$

It is now a simple matter to find the following eigenproblem in the continuum

$$\frac{1}{4}\left(1 - \frac{x_{j^3}^2}{x_j^2}\right)\frac{d^2\psi}{dx_j^2} + \frac{x_{j^3}^2}{2x_j^3}\frac{d\psi}{dx_j} + \left[-\frac{5x_{j^3}^2}{16x_j^4} + 1 - x_j^2 + \frac{1}{16x_j^2} + x_{j^3}^2\right]\psi = -\lambda\psi\,. \tag{4.90}$$

In obtaining this result the form for our continuum limit, as spelled out in (4.86) is crucial. Indeed, if one sets $x_j = j/m^\alpha$ the "kinetic" and "harmonic potential" terms on the LHS are only the same size if $\alpha = \frac{1}{2}$. Now, set $\psi = \sqrt{x_j} g$ to obtain

$$\frac{1}{4}\left(1-\frac{x_{j^3}^2}{x_j^2}\right)\frac{d^2 g}{dx_j^2}+\frac{1}{4x_j}\left(1+\frac{x_{j^3}^2}{2x_j^3}\right)\frac{dg}{dx_j}+\left[1-x_j^2+x_{j^3}^2\right]g=-\lambda g\,. \tag{4.91}$$

Finally, in terms of the new variable u defined by $u^2 = x_j^2 - x_{j^3}^2$ we find

$$\frac{1}{4}\frac{d^2 g}{du^2}+\frac{1}{4u}\frac{dg}{du}+(1-u^2)g=-\lambda g\,. \tag{4.92}$$

If we set $r = 2u$ we find the eigenproblem of the 2-dimensional oscillator with zero angular momentum. The energy spacing is 2 (recall $j \geq 0$ to see this). This is exactly the spectrum obtained by solving the discrete problem[25, 26]. It is also easy to check that the eigenvectors of the discrete problem are in perfect agreement with the eigenfunctions of (4.92). Thus, the continuum problem contains the same information as the discrete problem.

To get the correct spectrum we must obtain the $O(m)$, $O(\sqrt{m})$ and $O(1)$ pieces of the matrix elements of the dilatation operator. Writing things schematically, we should expand our dilatation operator matrix elements as

$$\beta = mf^{(0)}+\sqrt{m}f^{(1)}+f^{(2)}+\frac{f^{(3)}}{\sqrt{m}}+O(\frac{1}{m}) \tag{4.93}$$

and we should expand

$$\alpha=\sqrt{m}\alpha^{(0)}+\alpha^{(1)}+\frac{1}{\sqrt{m}}\alpha^{(2)}+\frac{1}{m}\alpha^{(3)}+O(\frac{1}{m^{\frac{3}{2}}}) \tag{4.94}$$

After expansion (4.46) gives 3 sets of non-trivial equations, and these three equations are the complete content of the recursion relations. They are obtained by plugging the above expansions into (4.46) and setting the coefficients of m, $\sqrt{m}$ and 1 to zero. The terms with coefficient $m^{\frac{3}{2}}$ trivially vanish. The terms with negative powers of m also do not give new equations: they vanish automatically because we are working in the $m = \sqrt{N} \to \infty$ limit.

At one loop, solving the partial differential equations that arise from (4.46) must reproduce the following expansions

$$\beta^{(n,m)}_{r_1,j,j^3}(c,0,0)=-\frac{m}{2}+\frac{m}{2}\frac{x_{j^3}^2}{x_j^2}-\frac{\sqrt{m}}{2}\frac{x_{j^3}^2}{x_j^3}+\frac{x_{j^3}^2}{2x_j^4}+\frac{x_{j^3}^2}{x_j^2} \tag{4.95}$$

$$\beta^{(n,m)}_{r_1,j,j^3}(c,1,0)=\frac{m}{4}+\frac{1}{2}-\frac{x_j^2}{2}+\frac{1}{32x_j^2}-\frac{m}{4}\frac{x_{j^3}^2}{x_j^2}-\frac{1}{2}\frac{x_{j^3}^2}{x_j^2}+\frac{x_{j^3}^2}{2}-\frac{25x_{j^3}^2}{32x_j^4}+\frac{\sqrt{m}x_{j^3}^2}{2x_j^3} \tag{4.96}$$

$$\beta^{(n,m)}_{r_1,j,j^3}(c,-1,0)=\frac{m}{4}+\frac{1}{2}-\frac{x_j^2}{2}+\frac{1}{32x_j^2}-\frac{m}{4}\frac{x_{j^3}^2}{x_j^2}-\frac{1}{2}\frac{x_{j^3}^2}{x_j^2}+\frac{x_{j^3}^2}{2}-\frac{x_{j^3}^2}{32x_j^4} \tag{4.97}$$

Given these continuum results, we can immediately claim that we have reproduced (4.44). Indeed, the ambiguity in reconstructing the exact functions $\beta^{(n,m)}_{r_1,j,j^3}(c,d,0)$ of the discrete variables j, j^3 from the continuum expressions above is order $\frac{1}{m}$ and we are working in the $m = \sqrt{N} \to \infty$ limit.

Finally, it is important to note that the solutions to our continuum differential equations are not unique. Indeed, we are finding a dilatation operator D that obeys

$$[J_\pm, D] = 0 = [J_3, D] \ . \tag{4.98}$$

Given a first solution, another solution is easily constructed by rescaling and shifting

$$D \to \kappa_1 D + 2k_0 \mathbf{1} \tag{4.99}$$

where $\mathbf{1}$ is the identity. Thus, there will always be two arbitrary constants in our solutions. This has important implications for us, particularly when it comes to finding the most general solution to the partial differential equations we will derive. For example, by choosing $\kappa_1 = \frac{1}{\sqrt{m}}\gamma$ we see that we shift

$$\beta = mf^{(0)} + \sqrt{m}f^{(1)} + f^{(2)} + \frac{f^{(3)}}{\sqrt{m}} + O(\frac{1}{m}) \longrightarrow$$
$$\beta' = mf^{(0)} + \sqrt{m}(f^{(1)} + \gamma f^{(0)}) + f^{(2)} + \gamma f^{(1)} + \frac{f^{(3)} + \gamma f^{(2)}}{\sqrt{m}} + O(\frac{1}{m})$$

In what follows, we will construct the solution that has $\gamma = 0$ and say that "we have the most general solution up to symmetry". Note that by choosing $\kappa_1 = \frac{1}{m}\gamma$ we would have

$$\beta' = mf^{(0)} + \sqrt{m}f^{(1)} + f^{(2)} + \gamma f^{(0)} + \frac{f^{(3)} + \gamma f^{(1)}}{\sqrt{m}} + O(\frac{1}{m}) \, .$$

We will thus also not include terms $\propto f^{(0)}$ when solving the partial differential equations that determine $f^{(2)}$. This completes out discussion of the continuum limit.

4.4 Differential Equations and Higher Loop Anomalous Dimensions

The main goal of this section is to study the constraints implied by (4.46) on the p-loop dilatation operator. As we discussed above, the p-loop dilatation operator allows a total of p boxes on the Young diagram labels of the restricted Schur polynomial to move. In this case, the requirement that J_+ commutes with D implies that

$$\sum_{b=-\frac{1}{2}}^{\frac{1}{2}} \sum_{d=-p}^{p} \Big[\beta^{(n,m)}_{r_1,j,j^3}(c,d,0)\alpha^{(n,m)}_{r_1+c,j+d,j^3}(a,b)$$

$$-\alpha^{(n,m)}_{r_1,j,j^3}(a,b)\beta^{(n-1,m+1)}_{r_1+a,j+b,j^3-\frac{1}{2}-a}(c,d,0)\Big]\, O^{(n-1,m+1)}_{r_1+a+c,j+d+b,j^3-\frac{1}{2}-a} = 0 \tag{4.100}$$

which can be rewritten as

$$\begin{aligned}&\beta^{(n,m)}_{r_1,j,j^3}(c,d,0)\alpha^{(n,m)}_{r_1+c,j+d,j^3}(a,\frac{1}{2}) + \beta^{(n,m)}_{r_1,j,j^3}(c,d+1,0)\alpha^{(n,m)}_{r_1+c,j+d+1,j^3}(a,-\frac{1}{2})\\ &-\alpha^{(n,m)}_{r_1,j,j^3}(a,\frac{1}{2})\beta^{(n-1,m+1)}_{r_1+a,j+\frac{1}{2},j^3-\frac{1}{2}-a}(c,d,0) - \alpha^{(n,m)}_{r_1,j,j^3}(a,-\frac{1}{2})\beta^{(n-1,m+1)}_{r_1+a,j-\frac{1}{2},j^3-\frac{1}{2}-a}(c,d+1,0) = 0\,.\end{aligned} \tag{4.101}$$

Recall that $\alpha(-1,\cdot) \propto \sqrt{r_1}$ and $\alpha(0,\cdot) \propto \sqrt{r_2}$ so that we get independent equations from (4.100) for each value of $a = \{-1, 0\}$, $c = \{-p, -p+1, \cdots, p-1, p\}$, and $d+b$ where $b = \pm\frac{1}{2}$ and $d = \{-p, -p+1, \cdots, p-1, p\}$. We will freely make use of the result of the Chapter Appendix in this section.

To begin we will consider $a = 0$ in (4.101). A few words on how we perform the expansion of the $\alpha_{r_1,j,j^3}(a,\pm\frac{1}{2})$ is in order. After rewriting j, j^3 in terms of x_j, x_{j^3}

$$\begin{aligned}\alpha_{r_1,j,j^3}(0,\frac{1}{2}) &= \sqrt{r_2}\sqrt{\frac{m+2j+4}{2}}\frac{j-j^3+1}{\sqrt{2j+2}\sqrt{2j+1}}\\ &= \sqrt{r_2}\sqrt{\frac{m}{2}}\sqrt{1+2\frac{x_j}{\sqrt{m}}+\frac{4}{m}}\frac{x_j - x_{j^3} + \frac{1}{\sqrt{m}}}{\sqrt{2x_j+\frac{2}{\sqrt{m}}}\sqrt{2x_j+\frac{1}{\sqrt{m}}}}\end{aligned} \tag{4.102}$$

we perform an expansion treating $\frac{1}{\sqrt{m}}$ and $\frac{1}{m}$ as small numbers. Using these expansions, after equating the coefficients of $m^{\frac{3}{2}}$ to zero, in

$$\begin{aligned}&\beta^{(n,m)}_{r_1,j,j^3}(c,d,0)\alpha^{(n,m)}_{r_1+c,j+d,j^3}(0,\frac{1}{2}) - \beta^{(n-1,m+1)}_{r_1,j+\frac{1}{2},j^3-\frac{1}{2}}(c,d,0)\alpha^{(n,m)}_{r_1,j,j}(0,\frac{1}{2})\\ &+\beta^{(n,m)}_{r_1,j,j^3}(c,d+1,0)\alpha^{(n,m)}_{r_1+c,j+d+1,j^3}(0,-\frac{1}{2}) - \beta^{(n-1,m+1)}_{r_1,j-\frac{1}{2},j^3-\frac{1}{2}}(c,d+1,0)\alpha^{(n,m)}_{r_1,j,j^3}(0,-\frac{1}{2}) = 0\end{aligned} \tag{4.103}$$

we find

$$\begin{aligned}&\frac{(x_j - x_{j^3})}{2\sqrt{2}x_j}f^{(0)}_{c,d}(x_j,x_{j^3}) + \frac{(x_j + x_{j^3})}{2\sqrt{2}x_j}f^{(0)}_{c,d+1}(x_j,x_{j^3})\\ &-\frac{(x_j - x_{j^3})}{2\sqrt{2}x_j}f^{(0)}_{c,d}(x_j,x_{j^3}) - \frac{(x_j + x_{j^3})}{2\sqrt{2}x_j}f^{(0)}_{c,d+1}(x_j,x_{j^3}) = 0\end{aligned} \tag{4.104}$$

which is trivially obeyed. By equating the $O(m)$ term to zero we have

$$\begin{aligned}2x_{j^3}(df^{(0)}_{c,d} - (d+1)f^{(0)}_{c,d+1}) + x_j\left(x_j\left(\frac{\partial f^{(0)}_{c,d}}{\partial x_{j^3}} + \frac{\partial f^{(0)}_{c,d+1}}{\partial x_{j^3}} - \frac{\partial f^{(0)}_{c,d}}{\partial x_j} + \frac{\partial f^{(0)}_{c,d+1}}{\partial x_j}\right)\right.\\ \left.+x_{j^3}\left(-\frac{\partial f^{(0)}_{c,d}}{\partial x_{j^3}} + \frac{\partial f^{(0)}_{c,d+1}}{\partial x_{j^3}} + \frac{\partial f^{(0)}_{c,d}}{\partial x_j} + \frac{\partial f^{(0)}_{c,d+1}}{\partial x_j}\right)\right) = 0\end{aligned} \tag{4.105}$$

Equating the $O(\sqrt{m})$ term to zero gives

$$x_j\left(2d\left(4x_j^2-1\right)(f^{(0)}_{c,d} - f^{(0)}_{c,d+1}) - x_j\left[x_j\left(\frac{\partial^2 f^{(0)}_{c,d}}{\partial x_{j^3}^2} + \frac{\partial^2 f^{(0)}_{c,d+1}}{\partial x_{j^3}^2} - 4\frac{\partial f^{(1)}_{c,d}}{\partial x_{j^3}} - 4\frac{\partial f^{(1)}_{c,d+1}}{\partial x_{j^3}}\right.\right.\right.$$

$$
\begin{aligned}
-2\frac{\partial^2 f^{(0)}_{c,d}}{\partial x_j \partial x_{j^3}} + 2\frac{\partial^2 f^{(0)}_{c,d+1}}{\partial x_j \partial x_{j^3}}\Bigg) - \frac{\partial f^{(0)}_{c,d}}{\partial x_{j^3}} + \frac{\partial f^{(0)}_{c,d+1}}{\partial x_{j^3}} + \frac{\partial f^{(0)}_{c,d}}{\partial x_j} + \frac{\partial f^{(0)}_{c,d+1}}{\partial x_j}\Bigg] + x_j^2 \Bigg(-\Bigg[4x_j\Bigg(-\frac{\partial f^{(0)}_{c,d}}{\partial x_{j^3}} \\
+\frac{\partial f^{(0)}_{c,d+1}}{\partial x_{j^3}} + \frac{\partial f^{(0)}_{c,d}}{\partial x_j} + \frac{\partial f^{(0)}_{c,d+1}}{\partial x_j}\Bigg) + 8f^{(0)}_{c,d} + 16 f^{(0)}_{c,d+1} + 4\frac{\partial f^{(1)}_{c,d}}{\partial x_j} - 4\frac{\partial f^{(1)}_{c,d+1}}{\partial x_j} + \frac{\partial^2 f^{(0)}_{c,d}}{\partial x_j^2} \\
+\frac{\partial^2 f^{(0)}_{c,d+1}}{\partial x_j^2}\Bigg]\Bigg) + 2f^{(0)}_{c,d+1}\Bigg) - x_{j^3}\Bigg(x_j\Bigg[-8df^{(1)}_{c,d} + 8df^{(1)}_{c,d+1} - 3\frac{\partial f^{(0)}_{c,d}}{\partial x_{j^3}} + \frac{\partial f^{(0)}_{c,d+1}}{\partial x_{j^3}} + 8f^{(1)}_{c,d+1} \\
+3\frac{\partial f^{(0)}_{c,d}}{\partial x_j} + \frac{\partial f^{(0)}_{c,d+1}}{\partial x_j}\Bigg] + 4(2d+3)\left[df^{(0)}_{c,d} - (d+1)f^{(0)}_{c,d+1}\right] + x_j^2\Bigg[-\frac{\partial^2 f^{(0)}_{c,d}}{\partial x_{j^3}^2} + \frac{\partial^2 f^{(0)}_{c,d+1}}{\partial x_{j^3}^2} + 4\frac{\partial f^{(1)}_{c,d}}{\partial x_{j^3}} \\
-4\frac{\partial f^{(1)}_{c,d+1}}{\partial x_{j^3}} + 2\frac{\partial^2 f^{(0)}_{c,d}}{\partial x_j \partial x_{j^3}} + 2\frac{\partial^2 f^{(0)}_{c,d+1}}{\partial x_j \partial x_{j^3}} - 8f^{(0)}_{c,d} + 8f^{(0)}_{c,d+1} - 4\frac{\partial f^{(1)}_{c,d}}{\partial x_j} - 4\frac{\partial f^{(1)}_{c,d+1}}{\partial x_j} \\
-\frac{\partial^2 f^{(0)}_{c,d}}{\partial x_j^2} + \frac{\partial^2 f^{(0)}_{c,d+1}}{\partial x_j^2}\Bigg] + 4x_j^3\left(\frac{\partial f^{(0)}_{c,d}}{\partial x_{j^3}} + \frac{\partial f^{(0)}_{c,d+1}}{\partial x_{j^3}} - \frac{\partial f^{(0)}_{c,d}}{\partial x_j} + \frac{\partial f^{(0)}_{c,d+1}}{\partial x_j}\right)\Bigg) = 0
\end{aligned}
\tag{4.106}
$$

Finally, equating the $O(1)$ term to zero we find another equation that is rather long and hence we will not quote it here. We will also study the equations obtained by plugging $a = -1$ into (4.101). Equating the term in

$$
\begin{aligned}
\beta^{(n,m)}_{r_1,j,j^3}(c,d,0)\alpha^{(n,m)}_{r_1+c,j+d,j^3}(-1,\frac{1}{2}) - \beta^{(n-1,m+1)}_{r_1-1,j+\frac{1}{2},j^3+\frac{1}{2}}(c,d,0)\alpha^{(n,m)}_{r_1,j,j}(-1,\frac{1}{2}) \\
+\beta^{(n,m)}_{r_1,j,j^3}(c,d+1,0)\alpha^{(n,m)}_{r_1+c,j+d+1,j^3}(-1,-\frac{1}{2}) - \beta^{(n-1,m+1)}_{r_1-1,j-\frac{1}{2},j^3+\frac{1}{2}}(c,d+1,0)\alpha^{(n,m)}_{r_1,j,j^3}(-1,-\frac{1}{2}) = 0
\end{aligned}
\tag{4.107}
$$

of order $m^{3/2}$ to zero, we find the equation

$$
\begin{aligned}
\frac{(x_j + x_{j^3})}{2\sqrt{2}x_j} f^{(0)}_{c,d}(x_j, x_{j^3}) + \frac{(x_j - x_{j^3})}{2\sqrt{2}x_j} f^{(0)}_{c,d+1}(x_j, x_{j^3}) \\
-\frac{(x_j + x_{j^3})}{2\sqrt{2}x_j} f^{(0)}_{c,d}(x_j, x_{j^3}) - \frac{(x_j - x_{j^3})}{2\sqrt{2}x_j} f^{(0)}_{c,d+1}(x_j, x_{j^3}) = 0
\end{aligned}
\tag{4.108}
$$

that is again trivially obeyed. The coefficient of the term of order m is

$$
\begin{aligned}
- x_j^2\left(\frac{\partial f^{(0)}_{c,d}}{\partial x_{j^3}} + \frac{\partial f^{(0)}_{c,d+1}}{\partial x_{j^3}} + \frac{\partial f^{(0)}_{c,d}}{\partial x_j} - \frac{\partial f^{(0)}_{c,d+1}}{\partial x_j}\right) \\
-x_{j^3}\left(2df^{(0)}_{c,d} - 2(d+1)f^{(0)}_{c,d+1} + x_j\left(\frac{\partial f^{(0)}_{c,d}}{\partial x_{j^3}} - \frac{\partial f^{(0)}_{c,d+1}}{\partial x_{j^3}} + \frac{\partial f^{(0)}_{c,d}}{\partial x_j} + \frac{\partial f^{(0)}_{c,d+1}}{\partial x_j}\right)\right) = 0
\end{aligned}
\tag{4.109}
$$

From the coefficient of the $O(\sqrt{m})$ term we find

$$
x_j\Bigg(-x_j^2\Bigg[-8(d-1)f^{(0)}_{c,d} + 8(d+2)f^{(0)}_{c,d+1} + \frac{\partial^2 f^{(0)}_{c,d}}{\partial x_{j^3}^2} + \frac{\partial^2 f^{(0)}_{c,d+1}}{\partial x_{j^3}^2} + 4\frac{\partial f^{(1)}_{c,d}}{\partial x_{j^3}} + 4\frac{\partial f^{(1)}_{c,d+1}}{\partial x_{j^3}}
$$

$$
\begin{aligned}
&+2\frac{\partial^2 f^{(0)}_{c,d}}{\partial x_j\partial x_{j^3}}-2\frac{\partial^2 f^{(0)}_{c,d+1}}{\partial x_j\partial x_{j^3}}+4\frac{\partial f^{(1)}_{c,d}}{\partial x_j}-4\frac{\partial f^{(1)}_{c,d+1}}{\partial x_j}+\frac{\partial^2 f^{(0)}_{c,d}}{\partial x_j^2}+\frac{\partial^2 f^{(0)}_{c,d+1}}{\partial x_j^2}\Bigg]\\
&+2(d(f^{(0)}_{c,d+1}-f^{(0)}_{c,d})+f^{(0)}_{c,d+1})-4x_j^3\left[\frac{\partial f^{(0)}_{c,d}}{\partial x_{j^3}}-\frac{\partial f^{(0)}_{c,d+1}}{\partial x_{j^3}}+\frac{\partial f^{(0)}_{c,d}}{\partial x_j}+\frac{\partial f^{(0)}_{c,d+1}}{\partial x_j}\right]\\
&-x_j\left[\frac{\partial f^{(0)}_{c,d}}{\partial x_{j^3}}-\frac{\partial f^{(0)}_{c,d+1}}{\partial x_{j^3}}+\frac{\partial f^{(0)}_{c,d}}{\partial x_j}+\frac{\partial f^{(0)}_{c,d+1}}{\partial x_j}\right]\Bigg)-x_{j^3}\Bigg(-x_j\Bigg[-8df^{(1)}_{c,d}+8df^{(1)}_{c,d+1}+3\frac{\partial f^{(0)}_{c,d}}{\partial x_{j^3}}\\
&-\frac{\partial f^{(0)}_{c,d+1}}{\partial x_{j^3}}+8f^{(1)}_{c,d+1}+3\frac{\partial f^{(0)}_{c,d}}{\partial x_j}+\frac{\partial f^{(0)}_{c,d+1}}{\partial x_j}\Bigg]-4(2d+3)(df^{(0)}_{c,d}-(d+1)f^{(0)}_{c,d+1})\\
&+x_j^2\Bigg(\frac{\partial^2 f^{(0)}_{c,d}}{\partial x_{j^3}^2}-\frac{\partial^2 f^{(0)}_{c,d+1}}{\partial x_{j^3}^2}+4\frac{\partial f^{(1)}_{c,d}}{\partial x_{j^3}}-4\frac{\partial f^{(1)}_{c,d+1}}{\partial x_{j^3}}+2\frac{\partial^2 f^{(0)}_{c,d}}{\partial x_j\partial x_{j^3}}+2\frac{\partial^2 f^{(0)}_{c,d+1}}{\partial x_j\partial x_{j^3}}+8f^{(0)}_{c,d}-8f^{(0)}_{c,d+1}\\
&+4\frac{\partial f^{(1)}_{c,d}}{\partial x_j}+4\frac{\partial f^{(1)}_{c,d+1}}{\partial x_j}+\frac{\partial^2 f^{(0)}_{c,d}}{\partial x_j^2}-\frac{\partial^2 f^{(0)}_{c,d+1}}{\partial x_j^2}\Bigg)+4x_j^3\left(\frac{\partial f^{(0)}_{c,d}}{\partial x_{j^3}}+\frac{\partial f^{(0)}_{c,d+1}}{\partial x_{j^3}}+\frac{\partial f^{(0)}_{c,d}}{\partial x_j}-\frac{\partial f^{(0)}_{c,d+1}}{\partial x_j}\right)\Bigg)=0
\end{aligned}
\tag{4.110}
$$

Finally, the coefficient of the $O(1)$ term gives another long equation that we will again not quote.

Apart from the partial differential equations obtained above, we also need to require that the dilatation operator is hermittian. Recall that

$$(\beta^\dagger)^{(n,m)}_{r_1,j,j^3}(a,b,c)=\beta^{(n,m)}_{r_1+c,j+b,j^3+c}(-a,-b,-c) \tag{4.111}$$

Thus, we require

$$\beta^{(n,m)}_{r_1,j,j^3}(c,q,0)=\beta^{(n,m)}_{r_1,j+q,j^3}(-c,-q,0)=\beta^{(n,m)}_{r_1,j+q,j^3}(c,-q,0) \tag{4.112}$$

which implies that

$$
\begin{aligned}
&mf^{(0)}_{c,a}(x_j,x_{j^3})+\sqrt{m}f^{(1)}_{c,a}(x_j,x_{j^3})+f^{(2)}_{c,a}(x_j,x_{j^3})+\frac{1}{\sqrt{m}}f^{(3)}_{(c,a)}(x_j,x_{j^3})\\
&=mf^{(0)}_{c,-a}(x_j+\frac{a}{\sqrt{m}},x_{j^3})+\sqrt{m}f^{(1)}_{c,-a}(x_j+\frac{a}{\sqrt{m}},x_{j^3})+f^{(2)}_{c,-a}(x_j+\frac{a}{\sqrt{m}},x_{j^3})\\
&+\frac{1}{\sqrt{m}}f^{(3)}_{(c,-a)}(x_j+\frac{a}{\sqrt{m}},x_{j^3})\,.
\end{aligned}
\tag{4.113}
$$

Our goal now is to solve the equations given above for the leading order of the functions introduced. There are two equations we will use: (4.105) and (4.109). Introduce the functions

$$F_+\equiv f^{(0)}_{c,d}+f^{(0)}_{c,d+1}\qquad F_-\equiv f^{(0)}_{c,d}-f^{(0)}_{c,d+1} \tag{4.114}$$

In terms of these functions (4.105) becomes

$$2x_{j^3}\left[dF_-+\frac{F_--F_+}{2}\right]+x_j^2\left[\frac{\partial F_+}{\partial x_{j^3}}-\frac{\partial F_-}{\partial x_j}\right]+x_jx_{j^3}\left[\frac{\partial F_+}{\partial x_j}-\frac{\partial F_-}{\partial x_{j^3}}\right]=0 \tag{4.115}$$

and (4.109) becomes

$$2x_{j^3}\left[dF_- + \frac{F_- - F_+}{2}\right] + x_j^2\left[\frac{\partial F_+}{\partial x_{j^3}} + \frac{\partial F_-}{\partial x_j}\right] + x_j x_{j^3}\left[\frac{\partial F_+}{\partial x_j} + \frac{\partial F_-}{\partial x_{j^3}}\right] = 0\,. \tag{4.116}$$

Suming these two equations we learn that

$$x_j\frac{\partial F_-}{\partial x_j} + x_{j^3}\frac{\partial F_-}{\partial x_{j^3}} = 0 \tag{4.117}$$

which implies that

$$F_- = F_-(u) \qquad u = \frac{x_{j^3}}{x_j}\,. \tag{4.118}$$

Note that this holds for any d. If we set $d = p$, since $F_- = f^{(0)}_{c,p}$ we learn that $f^{(0)}_{c,p} = f^{(0)}_{c,p}(u)$. If we set $d = p-1$, since $F_- = f^{(0)}_{c,p-1} - f^{(0)}_{c,p}$ depends only on u and we already argued that $f^{(0)}_{c,p}$ depends only on u, we learn that $f^{(0)}_{c,p-1} = f^{(0)}_{c,p-1}(u)$. We can keep going in this way and consequently we have actually proved that

$$f^{(0)}_{cd} = f^{(0)}_{cd}(u) \tag{4.119}$$

for *any* d. This is a dramatic simplification - we had a collection of functions of two variables and now we have a collection of functions that depend only on one variable.

Now, again set $d = p$. In this case $F_+ = F_- = F(u)$. We find that (4.105) becomes

$$x_j\frac{\partial F}{\partial x_{j^3}} + x_{j^3}\frac{\partial F}{\partial x_j} = -2\frac{x_{j^3}}{x_j}pF \tag{4.120}$$

which has the general solution

$$F = f^{(0)}_{c,p} = \kappa_p(1-u^2)^p = \kappa_p\left(1 - \frac{x_{j^3}^2}{x_j^2}\right)^p \tag{4.121}$$

where κ_p is a constant. This has reproduced the correct answer for one loop when $p = 1$ and has determined the leading order to an infinite number of higher loop dilatation operator coefficients.

Now, return to (4.115), and rewrite it using the new variable $y = 1 - u^2$ to obtain the simple form

$$y\frac{df^{(0)}_{c,d}}{dy} + y\frac{df^{(0)}_{c,d+1}}{dy} = df^{(0)}_{c,d} - (d+1)df^{(0)}_{c,d+1}\,. \tag{4.122}$$

If we now, set $d = p-1$ in (4.122) we can solve to obtain

$$f^{(0)}_{c,p-1} = -2p\kappa_p y^p + \kappa_{p-1}y^{p-1}\,. \tag{4.123}$$

Next, set $d = p-2$ in (4.122) and again solve to obtain

$$f^{(0)}_{c,p-2} = p(2p-1)\kappa_p y^p - 2(p-1)\kappa_{p-1}y^{p-1} + \kappa_{p-2}y^{p-2}\,. \tag{4.124}$$

It is clear that we could continue with this process and determine all of the $f^{(0)}_{c,d}$. We have however determined all that we will need about the leading order. We will now show that we can determine the one loop answer and then return to the general p-loop analysis.

4.5 One Loop

To determine the next to leading order, plug $d = 1$ and the known leading order functions into (4.106) to obtain

$$\frac{2}{x_{j^3}}\left(\frac{\partial f^{(1)}_{c,1}}{\partial x_{j^3}} - \frac{\partial f^{(1)}_{c,1}}{\partial x_j}\right) x_j^4(x_j - x_{j^3}) + 4 f^{(1)}_{c,1} x_j^3 - \kappa_1(2x_j - x_{j^3})(x_j + x_{j^3}) = 0\,, \tag{4.125}$$

plug $d = 0$ and the known leading order functions into (4.106) to obtain

$$\begin{aligned} &x_j^3 x_{j^3}\left(x_j\left[-\frac{\partial f^{(1)}_{c,0}}{\partial x_{j^3}} + \frac{\partial f^{(1)}_{c,1}}{\partial x_{j^3}} + \frac{\partial f^{(1)}_{c,0}}{\partial x_j} + \frac{\partial f^{(1)}_{c,1}}{\partial x_j}\right] - 2f^{(1)}_{c,1}\right) \\ &+x_j^5\left(\frac{\partial f^{(1)}_{c,0}}{\partial x_{j^3}} + \frac{\partial f^{(1)}_{c,1}}{\partial x_{j^3}} - \frac{\partial f^{(1)}_{c,0}}{\partial x_j} + \frac{\partial f^{(1)}_{c,1}}{\partial x_j}\right) + \kappa_1 x_j x_{j^3}^2 + \kappa_1 x_{j^3}^3 = 0\,, \end{aligned} \tag{4.126}$$

and finally, plug $d = -2$ and the known leading order functions into (4.106) to obtain

$$x_j\left(\frac{\partial f^{(1)}_{c,-1}}{\partial x_{j^3}} + \frac{\partial f^{(1)}_{c,-1}}{\partial x_j}\right)(x_j + x_{j^3}) + 2f^{(1)}_{c,-1} x_{j^3} = 0\,. \tag{4.127}$$

Next, plug $d = 1$ and the known leading order functions into (4.110) to obtain

$$-\frac{2}{x_{j^3}}\left(\frac{\partial f^{(1)}_{c,1}}{\partial x_{j^3}} + \frac{\partial f^{(1)}_{c,1}}{\partial x_j}\right) x_j^4(x_j + x_{j^3}) - \left(4f^{(1)}_{c,1} x_j^3 + \kappa_1\left(-2x_j^2 + x_j x_{j^3} + x_{j^3}^2\right)\right) = 0\,, \tag{4.128}$$

plug $d = 0$ and the known leading order functions into (4.110) to obtain

$$\begin{aligned} &- x_j^3 x_{j^3}\left(x_j\left[\frac{\partial f^{(1)}_{c,0}}{\partial x_{j^3}} - \frac{\partial f^{(1)}_{c,1}}{\partial x_{j^3}} + \frac{\partial f^{(1)}_{c,0}}{\partial x_j} + \frac{\partial f^{(1)}_{c,1}}{\partial x_j}\right] - 2f^{(1)}_{c,1}\right) \\ &-x_j^5\left(\frac{\partial f^{(1)}_{c,0}}{\partial x_{j^3}} + \frac{\partial f^{(1)}_{c,1}}{\partial x_{j^3}} + \frac{\partial f^{(1)}_{c,0}}{\partial x_j} - \frac{\partial f^{(1)}_{c,1}}{\partial x_j}\right) + \kappa_1 x_j x_{j^3}^2 - \kappa_1 x_{j^3}^3 = 0\,, \end{aligned} \tag{4.129}$$

and finally, plug $d = -2$ and the known leading order functions into (4.110) to obtain

$$- x_j\left(\frac{\partial f^{(1)}_{c,-1}}{\partial x_{j^3}} - \frac{\partial f^{(1)}_{c,-1}}{\partial x_j}\right)(x_j - x_{j^3}) - 2f^{(1)}_{c,-1} x_{j^3} = 0\,. \tag{4.130}$$

We will now solve the above 6 partial differential equations simultaneously. To start, sum (4.125) and (4.128) which leads to

$$4\left(x_j \frac{\partial f^{(1)}_{c,1}}{\partial x_j} + x_{j^3}\frac{\partial f^{(1)}_{c,1}}{\partial x_{j^3}}\right) + 2\kappa_1 \frac{x_{j^3}^2}{x_j^3} = 0\,. \tag{4.131}$$

The most general solution, regular at $x_{j^3} = 0$ is

$$f_{c,1}^{(1)} = \frac{\kappa_1}{2} \frac{x_{j^3}^2}{x_j^3} + \sum_{n=0}^{\infty} c_n \frac{x_{j^3}^n}{x_j^n} \,. \tag{4.132}$$

Inserting this solution into (4.125) we find

$$\sum_n 2c_n x_j^{3-n} x_{j^3}^{n-2} \left(n x_j^2 - (n-2) x_{j^3}^2 \right) = 0 \,. \tag{4.133}$$

Rearranging a little we find

$$\sum_{m=-2}^{\infty} 2c_{m+2}(m+2) x_j^{3-m} x_{j^3}^m - \sum_{n=0}^{\infty} 2c_n (n-2) x_j^{3-n} x_{j^3}^n = 0 \,.$$

From the coefficient of $x_j x_{j^3}^2$ we have $4c_4 = 0$. From the coefficient of $x_j^{-1-2k} x_{j^3}^{4+2k}$ we have $(6+2k)c_{6+2k} = (4+2k)c_{4+2k}$ which together implies $c_{2k} = 0$ for $k \geq 2$. From the coefficient of x_j^3 we have $2c_2 = -2c_0$. This just shifts the constant κ_1 appearing in $f_{c,1}^{(0)}$ by a term of $O(\frac{1}{\sqrt{m}})$ and we may as well set it to zero. We shoud have expected this - as we described in the last section, this is one of the symmetries that are present in our equations. By setting the coefficient of $x_j^4 x_{j^3}^{-1}$ to zero we find $c_1 = 0$ and from the coefficient of $x_j^{4-2k} x_{j^3}^{-1+2k}$ we find $c_{2k+1} = 0$ for $k > 1$. Putting everything together we only get a solution if all the coefficients $c_n = 0$. Thus, we finally obtain

$$f_{c,1}^{(1)} = \frac{\kappa_1}{2} \frac{x_{j^3}^2}{x_j^3} \tag{4.134}$$

which is indeed the correct answer.

Now, consider (4.127) and (4.130). From these two equations we can solve for $\frac{\partial f_{c,-1}^{(1)}}{\partial x_j}$ and for $\frac{\partial f_{c,-1}^{(1)}}{\partial x_j}$

$$\frac{\partial f_{c,-1}^{(1)}}{\partial x_j} = -\frac{4 x_{j^3}^2 f_{c,-1}^{(1)}}{x_j (x_j^2 - x_{j^3}^2)} \,, \qquad \frac{\partial f_{c,-1}^{(1)}}{\partial x_{j^3}} = \frac{4 x_{j^3} f_{c,-1}^{(1)}}{x_j^2 - x_{j^3}^2} \,. \tag{4.135}$$

These two equations are integrable - they give the same answer for $\frac{\partial^2 f_{c,1}^{(1)}}{\partial x_j \partial x_{j^3}}$. The only solution again corresponds to shifting κ_1, so that up to symmetry the most general solution is

$$f_{c,-1}^{(1)} = 0 \tag{4.136}$$

which is again the correct answer.

Finally, consider (4.126) and (4.129). After plugging in the solution we found for $f_{c,1}^{(1)}$ we find

$$2x_j^4 \left(\frac{\partial f_{c,0}^{(1)}}{\partial x_{j^3}} - \frac{\partial f_{c,0}^{(1)}}{\partial x_j} \right) + 2\kappa_1 x_j x_{j^3} + 3\kappa_1 x_{j^3}^2 = 0 \tag{4.137}$$

and

$$2x_j^4\left(\frac{\partial f^{(1)}_{c,0}}{\partial x_{j^3}}+\frac{\partial f^{(1)}_{c,0}}{\partial x_j}\right)+2\kappa_1 x_j x_{j^3}-3\kappa_1 x_{j^3}^2=0\,. \tag{4.138}$$

It is trivial to obtain the unique solution up to symmetry

$$f^{(1)}_{c,0}=-\frac{\kappa_1}{2}\frac{x_{j^3}^2}{x_j^3} \tag{4.139}$$

which is again correct. This reproduces the complete leading correction at one loop.

We can now check if our solution is hermittian which implies the following two conditions

$$f^{(0)}_{c,a}(x_j,x_{j^3})=f^{(0)}_{c,-a}(x_j,x_{j^3}) \tag{4.140}$$

and

$$f^{(1)}_{c,a}(x_j,x_{j^3})=f^{(1)}_{c,-a}(x_j,x_{j^3})+a\frac{\partial f^{(0)}_{c,-a}}{\partial x_j}\,. \tag{4.141}$$

Recall that at one loop we have

$$f^{(0)}_{c,\pm 1}=\frac{\kappa_1}{4}\left(1-\frac{x_{j^3}^2}{x_j^2}\right),\qquad f^{(1)}_{c,1}=\frac{\kappa_1}{2}\frac{x_{j^3}^2}{x_j^3},\qquad f^{(1)}_{c,-1}=0\,. \tag{4.142}$$

It is a non-trivial fact that

$$f^{(1)}_{c,1}(x_j,x_{j^3})=f^{(1)}_{c,-1}(x_j,x_{j^3})+\frac{\partial f^{(0)}_{c,-1}}{\partial x_j} \tag{4.143}$$

so that our one loop solution is indeed Hermittian.

Finally, the next order is determined by the requirement that the $O(1)$ piece of (4.103) vanishes. Plugging in the solutions for $f^{(0)}$, $f^{(1)}$ as well as $d=1$, we find

$$\begin{aligned}&16x_j^6\left(\frac{\partial f^{(2)}_{c,1}}{\partial x_{j^3}}-\frac{\partial f^{(2)}_{c,1}}{\partial x_j}+x_{j^3}\kappa_1\right)+16x_j^5x_{j^3}\left(-\frac{\partial f^{(2)}_{c,1}}{\partial x_{j^3}}+\frac{\partial f^{(2)}_{c,1}}{\partial x_j}+x_{j^3}\kappa_1\right)\\&-16x_j^4x_{j^3}\left(-2f^{(2)}_{c,1}+x_{j^3}^2\kappa_1\right)-16x_j^7\kappa_1-x_j^3\kappa_1+25x_j^2x_{j^3}\kappa_1+25x_jx_{j^3}^2\kappa_1-25x_{j^3}^3\kappa_1=0\end{aligned} \tag{4.144}$$

and

$$\begin{aligned}&-16x_j^6\left(\frac{\partial f^{(2)}_{c,1}}{\partial x_{j^3}}+\frac{\partial f^{(2)}_{c,1}}{\partial x_j}+x_{j^3}\kappa_1\right)-16x_j^5x_{j^3}\left(\frac{\partial f^{(2)}_{c,1}}{\partial x_{j^3}}+\frac{\partial f^{(2)}_{c,1}}{\partial x_j}-x_{j^3}\kappa_1\right)\\&+16x_j^4x_{j^3}\left(-2f^{(2)}_{c,1}+x_{j^3}^2\kappa_1\right)-16x_j^7\kappa_1-x_j^3\kappa_1-25x_j^2x_{j^3}\kappa_1+25x_jx_{j^3}^2\kappa_1+25x_{j^3}^3\kappa_1=0\,.\end{aligned} \tag{4.145}$$

Summing (4.144) and (4.145) we find

$$x_j\frac{\partial f^{(2)}_{c,1}}{\partial x_j}+x_{j^3}\frac{\partial f^{(2)}_{c,1}}{\partial x_{j^3}}-x_{j^3}^2\kappa_1+x_j^2\kappa_1+\frac{\kappa_1}{16x_j^2}-\frac{25x_{j^3}^2}{16x_j^4}\kappa_1=0\,. \tag{4.146}$$

The general solution to this equation is (again we have required that the solution is regular at $x_{j^3}=0$)

$$f^{(2)}_{c,1}=\frac{x_{j^3}^2}{2}\kappa_1-\frac{x_j^2}{2}\kappa_1+\frac{\kappa_1}{32x_j^2}-\frac{25x_{j^3}^2}{32x_j^4}\kappa_1+\sum_{n=0}a_n\frac{x_{j^3}^n}{x_j^n}\,. \tag{4.147}$$

Plugging this into (4.144) we find

$$\sum_{n=0}^{\infty}a_n x_j^{-n-3}x_{j^3}^{n-1}\left(nx_j^2-(n-2)x_{j^3}^2\right)=0\,. \tag{4.148}$$

The most general solution to this equation is $a_0=-a_2$ and $a_n=0$ for $n\neq 0,2$. You reach precisely the same conclusion if you use (4.145) instead of (4.144). Thus, our solution is

$$f^{(2)}_{c,1}=\frac{x_{j^3}^2}{2}\kappa_1-\frac{x_j^2}{2}\kappa_1+\frac{\kappa_1}{32x_j^2}-\frac{25x_{j^3}^2}{32x_j^4}\kappa_1+k_0-k_0\frac{x_{j^3}^2}{x_j^2}\,. \tag{4.149}$$

Setting $k_0=\frac{1}{2}$ and $\kappa_1=1$ we recover the answer from expanding the known dilatation operator coefficients.

Plugging in the solutions for $f^{(0)},f^{(1)}$ as well as $d=0$ we find

$$\begin{aligned}
16x_j^6\left(\frac{\partial f^{(2)}_{c,0}}{\partial x_{j^3}}+\frac{\partial f^{(2)}_{c,1}}{\partial x_{j^3}}-\frac{\partial f^{(2)}_{c,0}}{\partial x_j}+\frac{\partial f^{(2)}_{c,1}}{\partial x_j}-x_{j^3}\kappa_1\right)&\\
-16x_j^5x_{j^3}\left(\frac{\partial f^{(2)}_{c,0}}{\partial x_{j^3}}-\frac{\partial f^{(2)}_{c,1}}{\partial x_{j^3}}-\frac{\partial f^{(2)}_{c,0}}{\partial x_j}-\frac{\partial f^{(2)}_{c,1}}{\partial x_j}+x_{j^3}\kappa_1\right)&\\
+16x_j^4x_{j^3}\left(x_{j^3}^2\kappa_1-2f^{(2)}_{c,1}\right)+16x_j^7\kappa_1+x_j^3\kappa_1+11x_j^2x_{j^3}\kappa_1-41x_jx_{j^3}^2\kappa_1-43x_{j^3}^3\kappa_1&=0
\end{aligned} \tag{4.150}$$

and

$$\begin{aligned}
-16x_j^6\left(\frac{\partial f^{(2)}_{c,0}}{\partial x_{j^3}}+\frac{\partial f^{(2)}_{c,1}}{\partial x_{j^3}}+\frac{\partial f^{(2)}_{c,0}}{\partial x_j}-\frac{\partial f^{(2)}_{c,1}}{\partial x_j}-x_{j^3}\kappa_1\right)&\\
-16x_j^5x_{j^3}\left(\frac{\partial f^{(2)}_{c,0}}{\partial x_{j^3}}-\frac{\partial f^{(2)}_{c,1}}{\partial x_{j^3}}+\frac{\partial f^{(2)}_{c,0}}{\partial x_j}+\frac{\partial f^{(2)}_{c,1}}{\partial x_j}+x_{j^3}\kappa_1\right)&\\
-16x_j^4x_{j^3}\left(x_{j^3}^2\kappa_1-2f^{(2)}_{c,1}\right)+16x_j^7\kappa_1+x_j^3\kappa_1-11x_j^2x_{j^3}\kappa_1-41x_jx_{j^3}^2\kappa_1+43x_{j^3}^3\kappa_1&=0\,.
\end{aligned} \tag{4.151}$$

Now, summing (4.150) and (4.151) we find

$$-x_j\frac{\partial f^{(2)}_{c,0}}{\partial x_j}-x_{j^3}\frac{\partial f^{(2)}_{c,0}}{\partial x_{j^3}}+x_j\frac{\partial f^{(2)}_{c,1}}{\partial x_j}+x_{j^3}\frac{\partial f^{(2)}_{c,1}}{\partial x_{j^3}}-x_{j^3}^2\kappa_1+x_j^2\kappa_1+\frac{1}{16x_j^2}\kappa_1-\frac{41x_{j^3}^2}{16x_j^4}\kappa_1=0\,. \tag{4.152}$$

Plugging in the solution for $f^{(2)}_{c,1}$ that we constructed above, we find

$$x_j\frac{\partial f^{(2)}_{c,0}}{\partial x_j}+x_{j^3}\frac{\partial f^{(2)}_{c,0}}{\partial x_{j^3}}+\frac{x_{j^3}^2}{x_j^4}\kappa_1=0 \tag{4.153}$$

which has the general solution

$$f^{(2)}_{c,0}=\frac{x_{j^3}^2}{2x_j^4}\kappa_1+\sum_{n=0}a_n\frac{x_{j^3}^n}{x_j^n}\,. \tag{4.154}$$

Inserting this solution into (4.150) we finally find

$$f^{(2)}_{c,0}=\frac{x_{j^3}^2}{2x_j^4}\kappa_1+2k_0\frac{x_{j^3}^2}{x_j^2} \tag{4.155}$$

where k_0 is the same constant that appeared above.

Finally, plugging in the solutions for $f^{(0)}, f^{(1)}$ as well as $d=-2$ we find

$$\begin{aligned}16x_j^6\left(\frac{\partial f^{(2)}_{c,-1}}{\partial x_{j^3}}+\frac{\partial f^{(2)}_{c,-1}}{\partial x_j}+x_{j^3}\kappa_1\right)-16x_j^5x_{j^3}\left(-\frac{\partial f^{(2)}_{c,-1}}{\partial x_{j^3}}-\frac{\partial f^{(2)}_{c,-1}}{\partial x_j}+x_{j^3}\kappa_1\right)\\ -16x_j^4x_{j^3}\left(-2f^{(2)}_{c,-1}+x_{j^3}^2\kappa_1\right)+16x_j^7\kappa_1+x_j^3\kappa_1+x_j^2x_{j^3}\kappa_1-x_jx_{j^3}^2\kappa_1-x_{j^3}^3\kappa_1=0\end{aligned} \tag{4.156}$$

and

$$\begin{aligned}-16x_j^6\left(\frac{\partial f^{(2)}_{c,-1}}{\partial x_{j^3}}-\frac{\partial f^{(2)}_{c,-1}}{\partial x_j}+x_{j^3}\kappa_1\right)-16x_j^5x_{j^3}\left(-\frac{\partial f^{(2)}_{c,-1}}{\partial x_{j^3}}+\frac{\partial f^{(2)}_{c,-1}}{\partial x_j}+x_{j^3}\kappa_1\right)\\ +16x_j^4x_{j^3}\left(-2f^{(2)}_{c,-1}+x_{j^3}^2\kappa_1\right)+16x_j^7\kappa_1+x_j^3\kappa_1-x_j^2x_{j^3}\kappa_1-x_jx_{j^3}^2\kappa_1+x_{j^3}^3\kappa_1=0\,.\end{aligned} \tag{4.157}$$

Summing (4.156) and (4.157) we find

$$x_j\frac{\partial f^{(2)}_{c,-1}}{\partial x_j}+x_{j^3}\frac{\partial f^{(2)}_{c,-1}}{\partial x_{j^3}}-x_{j^3}^2\kappa_1+x_j^2\kappa_1+\frac{\kappa_1}{16x_j^2}-\frac{x_{j^3}^2}{16x_j^4}\kappa_1=0\,. \tag{4.158}$$

The general solution to this equation is (again we have required that the solution is regular at $x_{j^3}=0$)

$$f^{(2)}_{c,-1}=\frac{x_{j^3}^2}{2}\kappa_1-\frac{x_j^2}{2}\kappa_1+\frac{\kappa_1}{32x_j^2}-\frac{x_{j^3}^2}{32x_j^4}\kappa_1+\sum_{n=0}a_n\frac{x_{j^3}^n}{x_j^n}\,. \tag{4.159}$$

Plugging this into (4.156) we find

$$\sum_{n=0}^{\infty} a_n x_j^{-n-3} x_{j^3}^{n-1} \left(n x_j^2 - (n-2) x_{j^3}^2 \right) = 0 \,. \tag{4.160}$$

This is the equation we obtained above; the most general solution is $a_0 = -a_2$ and $a_n = 0$ for $n \neq 0, 2$. Thus, our solution is

$$f^{(2)}_{c,-1} = \frac{x_{j^3}^2}{2}\kappa_1 - \frac{x_j^2}{2}\kappa_1 + \frac{\kappa_1}{32 x_j^2} - \frac{x_{j^3}^2}{32 x_j^4}\kappa_1 + \tilde{k}_0 - \tilde{k}_0 \frac{x_{j^3}^2}{x_j^2} \,. \tag{4.161}$$

Setting $\tilde{k}_0 = \frac{1}{2}$ we recover the answer from expanding the known dilatation operator coefficients.

If we now study the $d = -1$ equation we can prove that $k_0 = \tilde{k}_0$. Thus, in summary we have

$$f^{(2)}_{c,1} = \frac{x_{j^3}^2}{2}\kappa_1 - \frac{x_j^2}{2}\kappa_1 + \frac{\kappa_1}{32 x_j^2} - \frac{25 x_{j^3}^2}{32 x_j^4}\kappa_1 + k_0 - k_0 \frac{x_{j^3}^2}{x_j^2} \,, \tag{4.162}$$

$$f^{(2)}_{c,0} = \frac{x_{j^3}^2}{2 x_j^4}\kappa_1 + 2 k_0 \frac{x_{j^3}^2}{x_j^2} \,, \tag{4.163}$$

$$f^{(2)}_{c,-1} = \frac{x_{j^3}^2}{2}\kappa_1 - \frac{x_j^2}{2}\kappa_1 + \frac{1}{32 x_j^2}\kappa_1 - \frac{x_{j^3}^2}{32 x_j^4}\kappa_1 + k_0 - k_0 \frac{x_{j^3}^2}{x_j^2} \,. \tag{4.164}$$

Collecting the results we have found above, we have the three functions above as well as

$$f^{(0)}_{c,1} = \frac{\kappa_1}{4}\left(1 - \frac{x_{j^3}^2}{x_j^2}\right), \qquad f^{(0)}_{c,0} = -\frac{\kappa_1}{2}\left(1 - \frac{x_{j^3}^2}{x_j^2}\right), \qquad f^{(0)}_{c,-1} = \frac{\kappa_1}{4}\left(1 - \frac{x_{j^3}^2}{x_j^2}\right) \tag{4.165}$$

$$f^{(1)}_{c,1} = \frac{\kappa_1}{2}\frac{x_{j^3}^2}{x_j^3}, \qquad f^{(1)}_{c,0} = -\frac{\kappa_1}{2}\frac{x_{j^3}^2}{x_j^3}, \qquad f^{(1)}_{c,-1} = 0 \,. \tag{4.166}$$

Requiring that the smallest eigenvalue of the one loop dilatation operator is zero determines $k_0 = 0$. Thus, up to an overall normalization which our argument can't determine, we have again reproduced (4.44).

4.6 General Discussion

In this section we will extended our arguments to higher loops. More specifically, in the language of the discussion towards the end of section 4.3, our goal is to construct the most general solution up to symmetry. Recall that we have already determined (see (4.121) and (4.123) above)

$$f_{c,p}^{(0)} = \kappa_p \left(1 - \frac{x_{j^3}^2}{x_j^2}\right)^p , \tag{4.167}$$

$$f_{c,p-1}^{(0)} = -2p\kappa_p \left(1 - \frac{x_{j^3}^2}{x_j^2}\right)^p + \kappa_{p-1} \left(1 - \frac{x_{j^3}^2}{x_j^2}\right)^{p-1} . \tag{4.168}$$

Plug $d = p$ and the known leading order functions into (4.106) to obtain

$$\begin{aligned} &x_j(x_j - x_{j^3}) \left(x_j \left[\frac{\partial f_{c,p}^{(1)}}{\partial x_{j^3}} - \frac{\partial f_{c,p}^{(1)}}{\partial x_j} \right] (x_j - x_{j^3}) + 2 f_{c,p}^{(1)} p x_{j^3} \right) \\ &+\kappa_p \left(2(p-1)x_j^4 + x_{j^3}^2 \left(2(p-1)x_j^2 + p(p+1)\right) - 2x_j x_{j^3} \left(2(p-1)x_j^2 + p(p+1)\right)\right) \left(1 - \frac{x_{j^3}^2}{x_j^2}\right)^p = 0 \,. \end{aligned}$$

Plug $d = p$ and the known leading order functions into (4.110) to obtain

$$\begin{aligned} &\kappa_p \left(1 - \frac{x_{j^3}^2}{x_j^2}\right)^p \left(2(p-1)x_j^4 + 4(p-1)x_j^3 x_{j^3} + 2(p-1)x_j^2 x_{j^3}^2 + 2p(p+1)x_j x_{j^3} + p(p+1)x_{j^3}^2\right) \\ &-x_j(x_j + x_{j^3}) \left(x_j \left[\frac{\partial f_{c,p}^{(1)}}{\partial x_{j^3}} + \frac{\partial f_{c,p}^{(1)}}{\partial x_j} \right] (x_j + x_{j^3}) + 2 f_{c,p}^{(1)} p x_{j^3} \right) = 0 \,. \end{aligned}$$

Summing these two we obtain

$$x_j \frac{\partial f_{c,p}^{(1)}}{\partial x_j} + x_{j^3} \frac{\partial f_{c,p}^{(1)}}{\partial x_{j^3}} - \kappa_p \left(1 - \frac{x_{j^3}^2}{x_j^2}\right)^{p-1} \left[2(p-1)x_j - p(p+1)\frac{x_{j^3}^2}{x_j^3} - 2(p-1)\frac{x_{j^3}^2}{x_j}\right] = 0 \,.$$

The general solution to this equation, that is regular at $x_{j^3} = 0$ is

$$f_{c,p}^{(1)} = \kappa_p \left(1 - \frac{x_{j^3}^2}{x_j^2}\right)^{p-1} \left[2(p-1)x_j + p(p+1)\frac{x_{j^3}^2}{x_j^3} - 2(p-1)\frac{x_{j^3}^2}{x_j}\right] + \sum_{n=0}^{\infty} a_n \frac{x_{j^3}^n}{x_j^n} \,.$$

Plugging this back into the first of our equations we find

$$\sum_{n=0}^{\infty} a_n x_j^{-n} x_{j^3}^n \left[n(x_j - x_{j^3})(x_j + x_{j^3}) + 2p x_{j^3}^2\right] = 0 \,. \tag{4.169}$$

The only solution is $a_n = 0$ so that

$$f_{c,p}^{(1)} = \kappa_p \left(1 - \frac{x_{j^3}^2}{x_j^2}\right)^{p-1} \left[2(p-1)x_j + p(p+1)\frac{x_{j^3}^2}{x_j^3} - 2(p-1)\frac{x_{j^3}^2}{x_j}\right] . \tag{4.170}$$

Now, plug $d = p$ and the known leading order functions and $f_{c,p}^{(1)}$ into (4.106) to obtain

$$\begin{aligned}
&x_j(x_j - x_{j^3})^2(x_j + x_{j^3})\left[x_j\left(\frac{\partial f_{c,p-1}^{(1)}}{\partial x_{j^3}} - \frac{\partial f_{c,p-1}^{(1)}}{\partial x_j}\right)(x_j - x_{j^3}) + 2f_{c,p-1}^{(1)}(p-1)x_{j^3}\right] \\
&\quad - \left(1 - \frac{x_{j^3}^2}{x_j^2}\right)^p \Big(x_j^2 x_{j^3}^2\,(p(\kappa_p(6p+3) + \kappa_{p-1}(-p) + \kappa_{p-1} + 2(p-3)p - 3) \\
&\qquad -2x_j^2\left(\kappa_p\left(4p^2 - 6p + 4\right) + \kappa_{p-1}p - 2(\kappa_{p-1} + p)\right)) \\
&\quad +2x_j^6(2\kappa_p - (p-2)(\kappa_{p-1} - 2p)) + x_j^3 x_{j^3}\left(4x_j^2((\kappa_p - 1)p(2p-3) + \kappa_{p-1}(p-2))\right. \\
&\qquad \left. -p(4\kappa_p p + \kappa_p - 2\kappa_{p-1}(p-1) + 4(p-1)p - 1)\right) \\
&\quad +p x_j x_{j^3}^3\left(4\kappa_p p^2 - 4(\kappa_p - 1)(2p-3)x_j^2 + 4\kappa_p p + \kappa_p + 4p^2 - 4p - 1\right) \\
&\qquad -(\kappa_p - 1)p x_j^4 - 2x_{j^3}^4\,(p(\kappa_p(p+1)(2p+1) + (p-3)p - 1) \\
&\qquad -2(p-1)x_j^2(\kappa_p(2p-1) - p))\Big) = 0
\end{aligned} \tag{4.171}$$

and plug $d = p$ and the known leading order functions and $f_{c,p}^{(1)}$ into (4.110) to obtain

$$\begin{aligned}
&- x_j(x_j - x_{j^3})(x_j + x_{j^3})^2\left[x_j\left(\frac{\partial f_{c,p-1}^{(1)}}{\partial x_{j^3}} + \frac{\partial f_{c,p-1}^{(1)}}{\partial x_j}\right)(x_j + x_{j^3}) + 2f_{c,p-1}^{(1)}(p-1)x_{j^3}\right] \\
&\qquad \left(1 - \frac{x_{j^3}^2}{x_j^2}\right)^p \Big(x_j^2 x_{j^3}^2\,\big(2x_j^2\left[\kappa_p\left(4p^2 - 6p + 4\right)\right. \\
&\quad \left. +\kappa_{p-1}p - 2(\kappa_{p-1} + p)\right] - p\left[\kappa_p(6p+3) + \kappa_{p-1}(-p) + \kappa_{p-1} + 2(p-3)p - 3\right]\big) \\
&\quad +2x_j^6((p-2)(\kappa_{p-1} - 2p) - 2\kappa_p) + x_j^3 x_{j^3}\left(4x_j^2((\kappa_p - 1)p(2p-3) + \kappa_{p-1}(p-2))\right. \\
&\qquad \left. -p(4\kappa_p p + \kappa_p - 2\kappa_{p-1}(p-1) + 4(p-1)p - 1)\right) + p x_j x_{j^3}^3\left(4\kappa_p p^2\right. \\
&\qquad \left. -4(\kappa_p - 1)(2p-3)x_j^2 + 4\kappa_p p + \kappa_p + 4p^2 - 4p - 1\right) + (\kappa_p - 1)p x_j^4 \\
&\quad +2x_{j^3}^4\left(p(\kappa_p(p+1)(2p+1) + (p-3)p - 1) - 2(p-1)x_j^2(\kappa_p(2p-1) - p)\right)\Big) = 0\,.
\end{aligned} \tag{4.172}$$

Summing these two we obtain

$$x_{j^3}\frac{\partial f_{c,p-1}^{(1)}}{\partial x_{j^3}} + x_j\frac{\partial f_{c,p-1}^{(1)}}{\partial x_j} + \left(1 - \frac{x_{j^3}^2}{x_j^2}\right)^{p-2} F(x_j, x_{j^3}) = 0\,, \tag{4.173}$$

where

$$\begin{aligned}
F(x_j, x_{j^3}) &= \frac{x_{j^3}^2}{x_j}\left(-8\kappa_p + 2\kappa_{p-1}p - 4\kappa_{p-1} - 8p^2\kappa_p + 16p\kappa_p\right) \\
&\quad +x_j\left(4\kappa_p - 2\kappa_{p-1}p + 4\kappa_{p-1} + 4p^2\kappa_p - 8p\kappa_p\right) + 2p^3\kappa_p\frac{x_{j^3}^4}{x_j^5} \\
&+\frac{x_{j^3}^2}{x_j^3}\left(\kappa_{p-1}p^2 - \kappa_{p-1}p - 2p^3\kappa_p\right) + \left(4\kappa_p + 4p^2\kappa_p - 8p\kappa_p\right)\frac{x_{j^3}^4}{x_j^3}\,.
\end{aligned}$$

The general solution to this equation, which is regular at $x_{j^3} = 0$ is

$$f_{c,p-1}^{(1)} = \left(1 - \frac{x_{j^3}^2}{x_j^2}\right)^{p-2} G(x_j, x_{j^3}) + \sum_{n=0}^{\infty} a_n \frac{x_{j^3}^n}{x_j^n} \tag{4.174}$$

where

$$
\begin{aligned}
G(x_j, x_{j^3}) = &-\frac{x_{j^3}^2}{x_j}\left(-8\kappa_p + 2\kappa_{p-1}p - 4\kappa_{p-1} - 8p^2\kappa_p + 16p\kappa_p\right) \\
&-x_j\left(4\kappa_p - 2\kappa_{p-1}p + 4\kappa_{p-1} + 4p^2\kappa_p - 8p\kappa_p\right) + 2p^3\kappa_p\frac{x_{j^3}^4}{x_j^5} \\
&+\frac{x_{j^3}^2}{x_j^3}\left(\kappa_{p-1}p^2 - \kappa_{p-1}p - 2p^3\kappa_p\right) - \left(4\kappa_p + 4p^2\kappa_p - 8p\kappa_p\right)\frac{x_{j^3}^4}{x_j^3}\,.
\end{aligned}
$$

Plugging this back into the first equation above we find

$$
x_j^{-n-1}x_{j^3}^{n-1}\left[a_n n x_j^2 - a_n x_{j^3}^2(n-2p+2)\right] = 0 \tag{4.175}
$$

which forces $a_n = 0$.

Now, study the equation obtained by plugging $d = -p-1$ and the known leading order functions into (4.106) to obtain

$$
\begin{aligned}
&x_j(x_j + x_{j^3})\left(x_j\left[\frac{\partial f^{(1)}_{c,-p}}{\partial x_{j^3}} + \frac{\partial f^{(1)}_{c,-p}}{\partial x_j}\right](x_j + x_{j^3}) + 2f^{(1)}_{c,-p}p x_{j^3}\right) \\
&+\kappa_p(p-1)\left[x_{j^3}^2\left(p + 2x_j^2\right) + 2x_j x_{j^3}\left(p + 2x_j^2\right) + 2x_j^4\right]\left(1 - \frac{x_{j^3}^2}{x_j^2}\right)^p = 0\,.
\end{aligned}
$$

Plug $d = p$ and the known leading order functions into (4.110) to obtain

$$
\begin{aligned}
&\kappa_p(p-1)\left(1 - \frac{x_{j^3}^2}{x_j^2}\right)^p\left[x_{j^3}^2\left(p + 2x_j^2\right) - 2x_j x_{j^3}\left(p + 2x_j^2\right) + 2x_j^4\right] \\
&-x_j(x_j - x_{j^3})\left(x_j\left[\frac{\partial f^{(1)}_{c,-p}}{\partial x_{j^3}} - \frac{\partial f^{(1)}_{c,-p}}{\partial x_j}\right](x_j - x_{j^3}) + 2f^{(1)}_{c,-p}p x_{j^3}\right) = 0\,.
\end{aligned}
$$

Summing these two we obtain

$$
x_j^3\left(\frac{\partial f^{(1)}_{c,-p}}{\partial x_{j^3}}x_{j^3} + \frac{\partial f^{(1)}_{c,-p}}{\partial x_j}x_j\right) - \kappa_p(p-1)\left(1 - \frac{x_{j^3}^2}{x_j^2}\right)^{p-2}\left[x_{j^3}^2\left(p + 2x_j^2\right) - 2x_j^4\right] = 0\,.
$$

The general solution to this equation, that is regular at $x_{j^3} = 0$ is

$$
f^{(1)}_{c,-p} = \kappa_p(p-1)\left(1 - \frac{x_{j^3}^2}{x_j^2}\right)^{p-1}\left(-p\frac{x_{j^3}^2}{x_j^3} + \frac{2x_{j^3}^2}{x_j} - 2x_j\right) + \sum_{n=0}^{\infty} a_n\frac{x_{j^3}^n}{x_j^n}\,. \tag{4.176}
$$

Plugging this back into the first equation above we learn that $a_n = 0$.

The only results we need from the above analysis are

$$
f^{(0)}_{c,\pm p} = \kappa_p\left(1 - \frac{x_{j^3}^2}{x_j^2}\right)^p\,, \tag{4.177}
$$

$$f_{c,p}^{(1)} = \kappa_p \left(1 - \frac{x_{j^3}^2}{x_j^2}\right)^{p-1} \left[2(p-1)x_j + p(p+1)\frac{x_{j^3}^2}{x_j^3} - 2(p-1)\frac{x_{j^3}^2}{x_j}\right], \tag{4.178}$$

$$f_{c,-p}^{(1)} = \kappa_p(p-1) \left(1 - \frac{x_{j^3}^2}{x_j^2}\right)^{p-1} \left(-p\frac{x_{j^3}^2}{x_j^3} + \frac{2x_{j^3}^2}{x_j} - 2x_j\right) + \sum_{n=0}^{\infty} a_n \frac{x_{j^3}^n}{x_j^n}. \tag{4.179}$$

Now, computing

$$f_{c,p}^{(1)}(x_j, x_{j^3}) - f_{c,-p}^{(1)}(x_j, x_{j^3}) - p\frac{\partial f_{c,-p}^{(0)}}{\partial x_j} = 4\kappa_p(p-1)x_j \left(1 - \frac{x_{j^3}^2}{x_j^2}\right)^p \tag{4.180}$$

we see that the only time that we get a Hermittian solution is when $p = 1$. Thus we are forced to set $f_{c,\pm p}^{(0)} = f_{c,\pm p}^{(1)} = 0$ which then implies that $f_{c,\pm p}^{(2)} = 0$. We now apply the same argument to conclude that $f_{c,\pm p\mp 1}^{(0)} = f_{c,\pm p\mp 1}^{(1)} = f_{c,\pm p\mp 1}^{(2)} = 0$ and keep going. Finally, when we get to $f_{c,\pm 1}^{(0)}, f_{c,\pm 1}^{(1)}, f_{c,\pm 1}^{(2)}$, we will find the one loop answer. This proves that the form of the piece of the dilatation operator that acts on the Y fields is not corrected at any higher loop order.

4.7 Chapter Appendix: The relation between $f_{c,d}^{(n,m)}(x_j, x_{j^3})$ and $f_{c,d}^{(n-1,m+1)}(x_j, x_{j^3})$

In this appendix we derive a relation between $f_{c,d}^{(n,m)}(x_j, x_{j^3})$ and $f_{c,d}^{(n-1,m+1)}(x_j, x_{j^3})$ that is used extensively in section 4.4. To make the discussion concrete we will study $f_{c,0}^{(n,m)}(x_j, x_{j^3})$ which is the continuum limit function corresponding to the following dilatation operator matrix element

$$-\frac{1}{2}\left[m - \frac{(m+2)(j^3)^2}{j(j+1)}\right] \tag{4.181}$$

This becomes the following function

$$f_{c,0}^{(n,m)}(x_j, x_{j^3}) = -\frac{1}{2}\left[m - \frac{(m+2)(\sqrt{m}x_{j^3})^2}{\sqrt{m}x_j(\sqrt{m}x_j + 1)}\right] \tag{4.182}$$

We have the series expansion

$$f_{c,0}^{(n,m)}(x_j, x_{j^3}) = \sum_{q=0}^{\infty} m^{1-\frac{q}{2}} f_{c,0}^{(m)}(x_j, x_{j^3}) \tag{4.183}$$

When we replace $m \to m+1$, we do so *without* changing j and j^3 - it is the expression (4.181) with $m \to m+1$ that solves the correct recursion relation. We must use the same definition of x_j and x_{j^3} for both $f_{c,0}^{(n,m)}(x_j, x_{j^3})$ and $f_{c,0}^{(n-1,m+1)}(x_j, x_{j^3})$, which implies that the new dilatation operator matrix element

$$-\frac{1}{2}\left[m+1-\frac{(m+1+2)(j^3)^2}{j(j+1)}\right] \tag{4.184}$$

leads to the following function

$$f_{c,0}^{(n-1,m+1)}(x_j, x_{j^3}) = -\frac{1}{2}\left[m+1-\frac{(m+1+2)(\sqrt{m}x_{j^3})^2}{\sqrt{m}x_j(\sqrt{m}x_j+1)}\right]\,. \tag{4.185}$$

We can get this function from $f_{c,0}^{(n,m)}(x_j, x_{j^3})$ by (i) shifting every $m \to m+1$ and then (ii) rescaling $x_j \to \sqrt{\frac{m}{m+1}}x_j$ and $x_{j^3} \to \sqrt{\frac{m}{m+1}}x_{j^3}$. In summary

$$\begin{aligned} f_{c,d}^{(n,m)}(x_j, x_{j^3}) &= \sum_{q=0}^{\infty} m^{1-\frac{q}{2}} f_{c,d}^{(m)}(x_j, x_{j^3})\,, \\ f_{c,d}^{(n-1,m+1)}(x_j, x_{j^3}) &= \sum_{q=0}^{\infty}(m+1)^{1-\frac{q}{2}} f_{c,d}^{(m)}(\sqrt{\frac{m}{m+1}}x_j, \sqrt{\frac{m}{m+1}}x_{j^3})\,. \end{aligned} \tag{4.186}$$

Finally, note that

$$\sqrt{\frac{m}{m+1}} = 1 - \frac{1}{2m} + \frac{3}{8m^2} + ... \tag{4.187}$$

5 Conclusion

In this dissertation we have presented further evidence for integrability in the large N but non-planar limit of $\mathcal{N} = 4$ SYM. In the $su(3)$ sector we have shown that integrability is present at one loop. The term we have focused on in this dissertation was argued to be subleading in [32] and as such was neglected. In this approximation the conservation laws found in the $su(2)$ sector still hold and integrability was clearly still a feature. Inclusion of this subleading term, however, sees the conservation law broken suggesting that perhaps integrability is not exact in this sector. We have argued that this is not the case. Even when the subleading term is included integrability is still present and as such integrability in the $su(3)$ sector is exact at one loop.

In the $su(2)$ sector we have shown that integrability is present at all loops. Previous explicit computations[25, 26, 27, 28, 31] have shown that integrability is present at one and two loops. Rather than continuing with an explicit computation - indeed the precise form of the dilatation operator is not even know beyond two loops - we have employed a powerful symmetry argument. This results in a set of recursion relations which are readily solved at one loop, reproducing the previous results. To proceed to higher loops the recursion relations proved very tedious and as such a continuum limit was taken which resulted in a set of partial differential equations. These were solved fully at all loops and brought us to the conclusion that in this sector integrability is present at all loops.

An interesting extension of this work would be to consider the $su(2|3)$ sector - the sector containing operators built from all three adjoint scalar fields as well as the two fermionic fields we have neglected in this work. Recently[46], it has been suggested that by computing the spectrum of anomalous dimensions in this sector the relativistic dispersion relation would emerge in the string theory. Application of the symmetry arguments presented in chapter 4 to the $su(2|3)$ sector would provide an effective means of proving this conjecture definitively.

The focus in this dissertation has been on operators built from $\mathcal{O}(N)$ fields. Further work would see this extended to operators built from $\mathcal{O}(N^2)$ fields. Recall that these operators are dual to new spacetime geometries. This sector of the theory remains rather unexplored. The restricted Schur polynomial technology provides an effective tool for delving into this sector. After further development of this technology the hope is that some limit of Einstein's field equations may emerge.

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