Particle Accelerators
1984 Vol. 15 pp. 67–82
0031-2460/84/1502–0067$18.50/0

APPLICATION OF THE GREEN'S FUNCTION METHOD TO SOME NONLINEAR PROBLEMS OF AN ELECTRON STORAGE RING†

PART I THE GREEN'S FUNCTION FOR THE FOKKER-PLANCK EQUATION

S. KHEIFETS

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

(Received August 12, 1983)

1. INTRODUCTION

One of the most important characteristics of an electron storage ring is the size of the beam. Not only is the efficiency of the ring a direct function of this parameter, but also the choice of almost all important characteristics of the ring, including its cost and power consumption, depends on the proper and accurate evaluation of the beam size.

In an electron storage ring, the horizontal size of the beam (at least for single-beam operation) is determined mainly by two different competing physical processes. The quantum radiation noise, being a classic example of a pure stochastic process, increases the phase space occupied by a bunch of particles. That causes the size of the beam to grow proportionally to the square root of time (or conversely the rate of the beam emittance growth is a constant, depending on the ring parameters).

On the other hand, the damping of particle oscillations diminishes the phase volume (the rate of the beam emittance decrease is another constant, depending on the ring parameters).

The combined effect of these two processes produces an equilibrium (Gaussian) particle distribution in phase space and an equilibrium size of the whole ensemble of the particles (beam size). In a small storage ring (in the energy range of several hundred MeV), the processes described indeed comprise the main effect. The calculations of the horizontal beam size for such a ring are quite simple, are in good agreement with experiment and give sound and reliable foundation for the design of a ring.[1] The vertical size can then be found provided the coupling coefficient of the ring is known. The beam size of a linear storage ring, established due to the phenomena described above is usually referred to as the natural beam size.

The situation is more complicated for colliding-beam operation. The beam-beam interaction changes the particle distribution and usually increases the vertical size of

† Work supported by the Department of Energy, Contract DE-AC03-76SF00515.

the beams, thus reducing the luminosity of the ring, the beam lifetime and the signal-to-noise ratio of an experimental apparatus. There exist no good theoretical ways for beam-size calculation including the beam-beam interaction.

Further complications arise for larger storage rings. The need for chromatic corrections, which are larger the bigger is the size and the energy of the storage ring, makes it a more and more nonlinear machine. The presence in the lattice of nonlinear magnetic fields (sextupoles) changes the character of the distribution function. It is no longer Gaussian. In fact, the actual distribution function for nonlinear lattice has not been known until now. The same is also true for the beam size.

For example, in a machine such as SPEAR, the distribution is not Gaussian (for large amplitudes) and the size of the bunch is also not the natural one. These effects are more pronounced for the vertical plane (cf. the results of measurements on SPEAR made by H. Wiedemann[2]). Of course, the size of the beam at any particular point of the lattice might be changed by other reasons then the presence of the sextupoles. For example, a spurious dispersion function together with the energy spread in the beam can change the effective beam size. But this does not change the main fact—one needs to be able to evaluate the beam size taking the nonlinear magnets into account.

The absence of an analytical approach for the beam-size calculations is circumvented at the present time by using different computational methods and programs. Although they proved to be extremely useful both in the design and operational stages of modern storage rings, it is obvious how limited such methods are for certain problems. For example, they are completely inadequate for all problems in which stochastic noise is an essential part.

For an electron storage ring, beam-size evaluation including beam-beam interaction gives an example of such a problem. Another good example is finding the beam size for a nonlinear machine. The list of such problems can probably be made much longer.

I hope the present work gives a way to solve some of these problems, at least in principle. The approach described here is an application of the well-known Green's function method, which in this case is applied to the Fokker-Planck equation governing the distribution function in the phase space of particle motion.[3]

A similar approach proved to be useful in solving other problems, such as estimating the particle losses in an electron accelerator due to the presence of a boundary[4–6] or finding the threshold of a longitudinal instability due to the longitudinal impedance of the ring.[7]

The new step made in this paper is to consider the particle motion in two degrees of freedom rather than in one dimension, a characteristic of all the previous work. This step seems to be necessary for an adequate description of the problem, at least for the class of problems which are considered below.

Part I of this work consists of the formal solution of the Fokker-Planck equation in terms of its Green's function[3] and describing the Green's function itself. The Green's function and the description of some of its properties can be found in the Appendices. In Section 2 of Part I, I discuss the distribution function in the transverse phase space of a particle and its Fokker-Planck equation for a simple case of a weak-focusing machine. Sections 3 and 4 are devoted to describing the Green's function and solution of this equation. Then in Section 5 this technique is applied to a strong-focusing machine and finally in Section 6 there is a discussion of applicability of this method, its limitations and relation to other methods (such as the Hamiltonian method).

Subsequent parts of this work contain examples of applications of the method. In particular, the calculations of the beam-size enhancement due to the presence of sextupoles (Part III) and of the weak-beam blowup due to collisions with a strong counter-rotating beam (Part IV) are performed there.

2. FOKKER-PLANCK EQUATION

Let us consider first the lateral motion of a particle in a weak-focusing machine. I assume that nonlinear forces can be represented in the form of nonlinear "kicks," i.e., sudden but regular jumps in the transverse velocity components of the particle. Apart from these nonlinear forces, horizontal (x) and vertical (y) components of particle motion are assumed to be uncoupled. If one neglects quantum radiation noise for a moment, the transverse motion can be described as a damped two-dimensional oscillation

$$d^2x/dt^2 + 2\alpha\, dx/dt + \omega^2 x = \sum_k \delta(t - t_k)F_x(x, y) \tag{2.1}$$

$$d^2y/dt^2 + 2\delta\, dy/dt + \Omega^2 y = \sum_k \delta(t - t_k)F_y(x, y) \tag{2.2}$$

In Eqs. (2.1) and (2.2), ω and Ω denote horizontal and vertical oscillation frequencies, α and δ the corresponding damping constants and $F_x(x, y)$ and $F_y(x, y)$ the corresponding components of the nonlinear force. The definition of the sequence of the times t_k at which the particle experiences the sudden jumps in its velocities depends on the particulars of the problem. In the case, for example, when there are N equally spaced kicks around the machine,

$$t_k = T \cdot k/n = 2\pi k/n\omega_0, \tag{2.3}$$

where T is the particle revolution period and $\omega_0 = 2\pi/T$.

In the presence of radiation noise, stochastic terms should be added to the right-hand sides of Eqs. (2.1) and (2.2) describing the effect of the noise.

Even if one would be able to solve such a system of stochastic equations, there would still remain the problem of averaging the solution over the initial particle distribution. Indeed, one is hardly interested in knowing the trajectories of each of many particles comprising a bunch. It is much more useful to have information on the average behavior of the whole particle ensemble.

We can get such information directly by considering from the start a distribution function for the ensemble of the particles in transverse space

$$\psi = \psi(x, \dot{x}, y, \dot{y}, t). \tag{2.4}$$

The distribution function ψ can be seen as a particle density in the four-dimensional phase space of the coordinates and velocities. Since there is no loss of particles in the problems we consider, the integral of ψ over all space should be constant. It is convenient to normalize ψ to unity

$$\int dV\, \psi(V, t) = 1. \tag{2.5}$$

In formulae (2.5) and throughout this paper $V \equiv (x, \dot{x}, y, \dot{y})$ and $dV = dx\, d\dot{x}\, dy\, d\dot{y}$. The integration in (2.5) is performed over the whole space, i.e., from $-\infty$ to $+\infty$ in each of the four coordinates. The infinite limits of integrals for integration over the whole space V will be omitted below.

The equation describing the development of the distribution function in time and its dependence on V in the presence of stochastic jumps of particle velocities is called the Fokker-Planck equation.[8] In our case it has the following form

$$\frac{\partial\psi}{\partial t} + \frac{\partial}{\partial x}(\dot{x}\psi) + \frac{\partial}{\partial \dot{x}}(\ddot{x}\psi) + \frac{\partial}{\partial y}(\dot{y}\psi) + \frac{\partial}{\partial \dot{y}}(\ddot{y}\psi) = q_x\partial^2\psi/\partial\dot{x}^2 + q_y\partial^2\psi/\partial\dot{y}^2 \tag{2.6}$$

The left-hand side of Eq. (2.6) is simply the full time derivative of the distribution function $d\psi/dt$. If one neglects the noise, $d\psi/dt = 0$ showing that the density calculated along the phase trajectory of the particle is constant (Liouville's theorem[9]).

The right-hand side of this equation represents the change of the distribution function due to the radiation noise. The constants q_x and q_y represent the rates of this noise for the x and y planes respectively.

One can argue that for the betatron oscillation the quantum emission causes the jump in the particle coordinate rather then in its lateral velocity component. This would change the second derivative in $\dot{x}$ on the right-hand side of Eq. (2.6) to the second derivative in x.

It is easy to show that such a change in the equation does not make any difference in the long run. Indeed, the betatron oscillation moves the phase-space point representing the particle motion around so rapidly that the actual direction of small jumps of its trajectory appears to be irrelevant. Formally, it can be shown that the Green's function (see next Section) of the equation with the jumps in x is the same as the Green's function for the equation with the jumps in $\dot{x}$ (with some redefinitions of its coefficients). Furthermore, there is a contribution to the transverse size of the beam from energy oscillations not taken into account here. For the vertical oscillations, the latter effect is practically negligible, the main contribution coming from coupling to horizontal oscillations.

We can consider Eq. (2.6) as an effective tool to describe the actual size of the beam. By properly choosing coefficients in Eq. (2.6), specifically q_x and q_y, one is able to obtain the distribution function that produces correct values for beam size. In particular, that takes care of the horizontal-vertical and horizontal-longitudinal couplings. As we will see below, the coefficients q_x and q_y do not enter into any final result separately, but always in a ratio with damping constants in such a way as to give the unperturbed rms value of the corresponding beam size.

Now, substitute into (2.6) $\ddot{x}$ and $\ddot{y}$ from Eqs. (2.1) and (2.2) to get

$$\frac{\partial\psi}{\partial t} + D_x\psi + D_y\psi = -\sum_k \delta(t - t_k)\left(F_x\frac{\partial\psi}{\partial\dot{x}} + F_y\frac{\partial\psi}{\partial\dot{y}}\right), \tag{2.7}$$

where I use the following abbreviations for differential operators in $x, \dot{x}, y, \dot{y}$ acting on ψ

$$D_x\psi = \dot{x}\frac{\partial\psi}{\partial x} - 2\alpha\psi - 2\alpha\dot{x}\frac{\partial\psi}{\partial\dot{x}} - \omega^2 x\frac{\partial\psi}{\partial\dot{x}} - q_x\frac{\partial^2\psi}{\partial\dot{x}^2} \tag{2.8}$$

$$D_y\psi = \dot{y}\frac{\partial\psi}{\partial y} - 2\delta\psi - 2\delta\dot{y}\frac{\partial\psi}{\partial\dot{y}} - \Omega^2 y\frac{\partial\psi}{\partial\dot{y}} - q_y\frac{\partial^2\psi}{\partial\dot{y}^2}. \tag{2.9}$$

The solution of Eq. (2.7) is the aim of this paper.

3. THE GREEN'S FUNCTION AND SOLUTION OF THE FOKKER-PLANCK EQUATION

By the Green's function G for Eq. (2.7) we understand a function of two phase-space points V, V_0 and two times t, t_0 which satisfies the equation

$$\frac{\partial G}{\partial t} + D_x G + D_y G = 0 \tag{3.1}$$

and the initial condition

$$G(V, t_0, V_0, t_0) = \delta(x - x_0)\,\delta(\dot{x} - \dot{x}_0)\,\delta(y - y_0)\,\delta(\dot{y} - \dot{y}_0). \tag{3.2}$$

In Eq. (3.1) the differential operators D_x and D_y act on the variables V.

The physical meaning of the Green's function is the evolution in time $t > t_0$ and space V of the phase-space density, which initially (at time $t = t_0$) was represented by a δ-function distribution positioned at the point V_0.

It is clear from this remark that the integral of the Green's function over all space should be 1 for all times $t \geqslant t_0$

$$\int dV\, G(V, t, V_0, t_0) = 1. \tag{3.3}$$

Since all the coefficients in Eq. (3.1) are constants in time, the Green's function actually depends on the difference $t - t_0$, rather than the two time variables separately.

Since both operators D_x (2.8) and D_y (2.9) only act on one set of coordinates $(x, \dot{x})$ and $(y, \dot{y})$ respectively, the Green's function is a product of the two simpler Green's functions G_x and G_y, each depending on the corresponding coordinates and time

$$G(V, t, V_0, t_0) = G_x(x, \dot{x}, x_0, \dot{x}_0, t - t_0)G_y(y, \dot{y}, y_0, \dot{y}_0, t - t_0). \tag{3.4}$$

The Green's function G_x for a one-dimensional damped linear stochastic oscillator has been obtained by S. Chandrasekhar.[3] For the readers' convenience, the description of the Green's function and some of its properties can be found in the Appendices. As soon as we know the Green's function of Eq. (2.7), its solution can be written right away. Let us denote the expression on the right-hand side of Eq. (2.7) as $\Pi(V, t)$. Then

$$\psi(V, t) = \int dV_0\, G(V, t, V_0, t_0)\tilde{\psi}(V_0, t_0) + \int_{t_0}^{t} d\tau \int dV_0\, G(V, t, V_0, \tau)\Pi(V_0, \tau) \tag{3.5}$$

is a formal solution of (2.7). The first term in (3.5) depends on the initial distribution $\tilde{\psi}(V, t_0)$. To check that (3.5) satisfies Eq. (2.7), one may find $\partial\psi/\partial t$, $D_x\psi$ and $D_y\psi$ of (3.5) and sum them all up. Then, using Eq. (3.1) and (3.2), it is easy to see that (3.5) is indeed the solution of (2.7).

In our case, the function $\Pi(V, t)$ itself depends on ψ. Hence (3.5) is an integral equation for ψ rather than a real solution of (2.7)

$$\psi(V, t) = \psi_0(V, t) - \sum_k \int dV_0 \left(F_x \frac{\partial\psi}{\partial\dot{x}} + F_y \frac{\partial\psi}{\partial\dot{y}}\right)_{V_0} G(V, t, V_0, t_k) \tag{3.6}$$

The summation in (3.7) should be performed for all $t_k \leqslant t$ for any given t.

Let us now consider the case where the nonlinear force can be treated as small in some sense. As we will see later, this condition is realized in many practical problems. In this case, we can search for ψ in terms of a perturbation theory

$$\psi = \psi_0 + \psi_1 + \psi_2 + \cdots \tag{3.7}$$

Here ψ_0 is the unperturbed solution of (2.7), i.e., the solution of the uniform equation with $\Pi = 0$. ψ_1 then is the solution of (2.7) where ψ is substituted by ψ_0 in the right-hand side. ψ_2 is the solution of (2.7) with ψ in Π substituted by ψ_1 and so on

$$\psi_1(V_1, t_k) = -\sum_m^k \int dV_0\, G(V_1, t_k, V_0, t_m)\left(F_x \frac{\partial\psi_0}{\partial\dot{x}} + F_y \frac{\partial\psi_0}{\partial\dot{y}}\right)_{V_0} \tag{3.8}$$

$$\begin{aligned}\psi_2(V, t) &= -\sum_k \int dV_1\, G(V, t, V_1, t_k)\left(F_x \frac{\partial\psi_1}{\partial\dot{x}} + F_y \frac{\partial\psi_1}{\partial\dot{y}}\right)_{V_1, t_k} \\ &= \sum_k \int dV_1\, G(V, t, V_1, t_k) \sum_m^k \left(F_x(x_1, y_1)\frac{\partial}{\partial\dot{x}_1} + F_y(x_1, y_1)\frac{\partial\psi_0}{\partial\dot{y}}\right)_{V_0} \\ &\quad \times \int dV_0\, G(V_1, t_k, V_0, t_m)\left(F_x \frac{\partial\psi_0}{\partial\dot{x}} + F_y \frac{\partial\psi_0}{\partial\dot{y}}\right)_{V_0}\end{aligned} \tag{3.9}$$

Finding the average values of the different parameters from the known distribution function (3.7) is now a matter of integration. For example, the rms value of the vertical size of the beam is given by

$$\Sigma_y^2(t) = \int dV\, \psi(V, t) y^2 = \sigma_y{}^2(1 + \Delta_1 + \Delta_2 + \cdots), \tag{3.10}$$

where

$$\sigma_y{}^2 = \int dV\, \psi_0(V, t) y^2$$

is by definition the square of the rms vertical size of the unperturbed beam.

Suppose that the nonlinear force F is proportional to some small parameter λ. It is seen from (3.8) and (3.9) that ψ_1 is then of order λ, ψ_2 is of order λ^2 and so on. In other words (3.7) is then a power series in the parameter λ.

In the same way, the second expression in (3.10) is also a power-series expansion in the parameter λ. For example, for the weak beam-strong beam interaction, the bunch current of the strong beam plays the role of the parameter λ. Hence, for the beam-beam interaction, the beam blowup can be found as an expansion in powers of the beam current.

4. UNPERTURBED DISTRIBUTION FUNCTION

The unperturbed ($\Pi = 0$) distribution function $\psi_0(V, t)$ satisfies the equation

$$\psi_0(V, t) = \int dV_0\, G(V, t, V_0, t_0)\tilde{\psi}(V_0, t_0), \tag{4.1}$$

where $\tilde{\psi}(V_0, t_0)$ is the initial distribution function. As soon as it is given we can find from (4.1) the distribution function at any later time $t \geqslant t_0$.

One example of the application of formula (4.1) is to solve the following problem.[10] A Gaussian bunch of particles is injected at time 0. The rms values of x and $\dot{x}$ are different from the proper values of the accelerator under consideration. The distribution function is found describing the transition from the injection distribution to the proper distribution with the eigen rms values.

Here I consider another example. Suppose that at time 0 a bunch of particles with a δ-function-like distribution in V is injected at the four-dimensional point V_0. The development in time and space of the particle distribution is described now by the Green's function itself

$$\psi_0(V, t) = G(V, V_0, t). \tag{4.2}$$

The behavior of this function is considered in Appendix A. In particular, for large times ($\alpha t > 1$ and $\delta t > 1$). $\psi_0(V, t)$ contains no information on the initial point V_0, nor does it depend on time. The distribution function becomes the equilibrium distribution function

$$\psi_0(x, \dot{x}, y, \dot{y}) = \frac{\exp\left\{-\dfrac{x^2}{2\sigma_x{}^2} - \dfrac{\dot{x}^2}{2\sigma_{\dot{x}}{}^2} - \dfrac{y^2}{2\sigma_y{}^2} - \dfrac{\dot{y}^2}{2\sigma_{\dot{y}}{}^2}\right\}}{(2\pi)^2\sigma_x\sigma_{\dot{x}}\sigma_y\sigma_{\dot{y}}}, \tag{4.3}$$

where σ_x, $\sigma_{\dot{x}}$, σ_y and $\sigma_{\dot{y}}$ are the rms values of the corresponding coordinates and velocities. From formula (A23), (A24) of Appendix A one finds

$$\sigma_x{}^2 = q_x/2\omega^2\alpha \tag{4.4}$$

$$\sigma_{\dot{x}}{}^2 = q_x/2\alpha \tag{4.5}$$

$$\sigma_y{}^2 = q_y/2\Omega^2\delta \tag{4.6}$$

$$\sigma_{\dot{y}}{}^2 = q_y/2\delta \tag{4.7}$$

The same equilibrium distribution function (4.3) will of course emerge at large time for any initial distribution function.

5. EXTENSION OF THE FORMALISM FOR A STRONG-FOCUSING MACHINE

To be really useful, the formalism developed should be shown to be applicable to a periodic lattice. To learn how to use it in this case, we start from equations similar to (2.1), (2.2) for the transverse motion in a periodic lattice, which is characterized by two periodic focusing functions $K_x(s)$ and $K_y(s)$

$$d^2x/ds^2 + 2\alpha\, dx/ds + K_x x = \sum_k \delta(s - s_k)F_x(x, y) \tag{5.1}$$

$$d^2y/ds^2 + 2\delta\, dy/ds + K_y y = \sum_k \delta(s - s_k)F_y(x, y). \tag{5.2}$$

I have changed the independent variable here to $s = ct$, where c is longitudinal particle velocity. Let us make now the transformation to Courant-Snyder variables[11]:

$$x = \sqrt{\beta_x}\, u \tag{5.3}$$

$$d\phi = ds/\nu_x\beta_x \tag{5.4}$$

$$y = \sqrt{\beta_y}\, \upsilon \tag{5.5}$$

$$d\theta = ds/\nu_y\beta_y \tag{5.6}$$

In these variables, the equations of motion are

$$u'' + 2\tilde{\alpha}u' + \nu_x{}^2 u = \sum_k \delta(\phi - \phi_k)\tilde{F}_x \tag{5.7}$$

$$\upsilon'' + 2\tilde{\delta}\upsilon' + \nu_y{}^2 \upsilon = \sum_k \delta(\theta - \theta_k)\tilde{F}_y \tag{5.8}$$

Here and below primes denote derivation with respect to ϕ for u and θ for υ, and

$$\tilde{F}_x(u, \upsilon) = \nu_x \sqrt{\beta_x}\, F_x[x(u), y(\upsilon)] \tag{5.9}$$

$$\tilde{F}_y(u, \upsilon) = \nu_y \sqrt{\beta_y}\, F_y[x(u), y(\upsilon)]. \tag{5.10}$$

In the last two functions, x and y should be substituted by u and υ according to (5.3) and (5.5).

The corresponding Fokker-Planck equation in this case can be written in the form

$$\begin{aligned} \frac{\partial\psi}{\partial s} + \frac{1}{\nu_x\beta_x} D_u\psi + \frac{1}{\nu_y\beta_y} D_\upsilon\psi = &-\sum_k \frac{1}{\nu_x\beta_x}\delta(\phi - \phi_k)\tilde{F}_x \frac{\partial\psi}{\partial u'} \\ &-\sum_k \frac{1}{\nu_y\beta_y}\delta(\theta - \theta_k)\tilde{F}_y \frac{\partial\psi}{\partial \upsilon'}, \end{aligned} \tag{5.11}$$

where

$$D_u\psi = u'\frac{\partial\psi}{\partial u} - 2\tilde{\alpha}\psi - 2\tilde{\alpha}u'\frac{\partial\psi}{\partial u'} - \nu_x{}^2 u\frac{\partial\psi}{\partial u'} - \tilde{q}_x\frac{\partial^2\psi}{\partial u'^2} \tag{5.12}$$

$$D_\upsilon\psi = \upsilon'\frac{\partial\psi}{\partial \upsilon} - 2\tilde{\delta}\psi - 2\tilde{\delta}\upsilon'\frac{\partial\psi}{\partial \upsilon'} - \nu_y{}^2 \upsilon\frac{\partial\psi}{\partial \upsilon'} - \tilde{q}_y\frac{\partial^2\psi}{\partial \upsilon'^2} \tag{5.13}$$

The constants $\tilde{\alpha}$, $\tilde{\delta}$, $\tilde{q}_x$ and $\tilde{q}_y$ cannot be obtained without considering the coupling of horizontal to vertical and horizontal to longitudinal motions. The corresponding expressions for them can be found, for example, in Sands' book.[1]

Here I should make a comment similar to that made about Eq. (2.6). The form of the stochastic term in (5.12) implies a jump in x' at the moment of a quantum emission. In a strong-focusing machine, such jumps occur both in x' and x. But, as discussed in

Section 2, we can keep Eq. (5.12), as long as the coefficients in it are chosen in such a way as to produce the correct effective size of the beam including the effect of the horizontal-longitudinal coupling. The same argument applies also to Eq. (5.13), where coefficients should be chosen in such a way as to give the effective unperturbed vertical size of the beam due to coupling to the horizontal motion.

The Green's function of Eq. (5.11) G is the product $G_u G_\upsilon$ of the Green's functions of the two equations

$$\frac{\partial G_u}{\partial \phi} + D_u G_u = 0 \tag{5.14}$$

and

$$\frac{\partial G_\upsilon}{\partial \theta} + D_\upsilon G_\upsilon = 0. \tag{5.15}$$

G_u is the same function of the variables $(u, u', \phi, u_0, u_0', \phi_0)$ as G_x is of the variables $(x, \dot{x}, t, x_0, \dot{x}_0, t_0)$.

$$G(V, s, V_0, s_0) = G_u(u, u', \phi(s), u_0, u_0', \phi(s_0))G_\upsilon(\upsilon, \upsilon', \theta(s), \upsilon_0, \upsilon_0', \theta(s_0)) \tag{5.16}$$

Again we denote the right-hand side of Eq. (5.11) by $\Pi(V, s)$.

$$\Pi(V, s) = -\sum_k \delta(s - s_k)\left(\tilde{F}_x \frac{\partial \psi}{\partial u'} + \tilde{F}_y \frac{\partial \psi}{\partial \upsilon'}\right). \tag{5.17}$$

The formal solution of Eq. (5.11) is

$$\psi(V, s) = \int dV_0\, G(V, s, V_0, s_0)\tilde{\psi}(V_0, s_0) + \int_{s_0}^{s} d\sigma \int dV_0\, G(V, s, V_0, \sigma)\Pi(V_0, \sigma) \tag{5.18}$$

To check that (5.18) satisfies Eq. (5.11), one may find $\partial\psi/\partial s$, $D_u\psi$, $D_\upsilon\psi$ and sum them up with the proper coefficients. For example,

$$\frac{\partial G}{\partial s} = \frac{G_\upsilon}{\nu_x \beta_x}\frac{\partial G_u}{\partial \phi} + \frac{G_u}{\nu_y \beta_y}\frac{\partial G_\upsilon}{\partial \theta}, \tag{5.19}$$

All the rest of Section 3 can now be repeated with almost no change. For example, the unperturbed equilibrium function ψ_0 is now

$$\psi_0(u, u', \upsilon, \upsilon') = \frac{\exp\left\{-\dfrac{u^2}{2\epsilon_x} - \dfrac{u'^2}{2\epsilon_x \nu_x{}^2} - \dfrac{\upsilon^2}{2\epsilon_y} - \dfrac{\upsilon'^2}{2\epsilon_y \nu_y{}^2}\right\}}{(2\pi)^2 \epsilon_x \nu_x \epsilon_y \nu_y}, \tag{5.16}$$

where ϵ_x, ϵ_y are horizontal and vertical emittances of the bunch

$$\epsilon_x = \tilde{q}_x / 2\tilde{\alpha}\nu_x{}^2 \tag{5.17}$$

$$\epsilon_y = \tilde{q}_y / 2\tilde{\delta}\nu_y{}^2. \tag{5.18}$$

6. DISCUSSION

Let us summarize the features of the approach developed in this paper. First, I consider transverse two-dimensional linear oscillations. These oscillations are treated as uncoupled, apart from the coupling brought up by a nonlinear force, which is treated as a perturbation.

The use of perturbation theory certainly limits the applicability of this method. One limitation is that only those problems in which the perturbation is small can be treated. For example, the beam-beam interaction may be treated for the case of a bunch current that is not too high. The quantitative limits depend of course on the particulars of the problem. Another limitation is that even for a problem where the perturbation is small, there could arise conditions where the approximation breaks down. For example, on or close to a particular resonance, the perturbed distribution function may deviate too far from its unperturbed value to be considered in the frame of a perturbation theory.

Still such an approach might be useful in some cases. Usually, a storage ring operates far from all major resonances and to know the behavior of the beam size around the working point might be useful.

Further, the action of the nonlinear force is approximated by a 'kick' in transverse particle velocities. This approximation is rather good for all cases where the effective interaction length is much smaller than the wavelength of the betatron oscillations.

For the strong-focusing machine, I use a smooth approximation to describe the betatron motion. Thus I am neglecting all the changes of the machine functions (such as the beta-function) caused by the nonlinear force. I believe that such effects may be important for the description of the behavior of a single particle, but they should be of less importance for a bunch of particles. At least the changes of beta-function are different for different particles.

The attractive feature of the method is that both the oscillation damping and the quantum noise are taken into consideration. There is no doubt that both effects are important for the correct evaluation of the beam size. There is a weak point, though, in the method developed where 'effective' constants are used in equations governing the evolution of the distribution function. Here more work is needed to demonstrate the validity of such an approach.

The inclusion of damping and radiation noise into the treatment prevents the Hamiltonian formalism from being used. The perturbation treatment of the distribution function differs significantly from perturbation treatment in the Hamiltonian theory. In a sense, the infinite sum (3.7) of the higher-order corrections to the distribution function is equivalent to the first-order correction in a Hamiltonian, if the Hamiltonian theory were applicable. Indeed, the unperturbed distribution function [cf. for example, (5.16)] can be written in the form (Boltzmann's distribution)

$$\psi_0 \sim \exp\left\{-\frac{H_x}{\epsilon_x} - \frac{H_y}{\epsilon_y}\right\}, \tag{6.1}$$

where

$$H_x = \frac{u^2}{2} + \frac{1}{2}\left(\frac{u'}{\nu_x}\right)^2, \tag{6.2}$$

with a similar expression for H_y.

The last comment concerns Eqs. (2.7) and (5.11). The treatment of the beam-beam interaction as a diffusion process has been attempted by myself.[12] There are similarities between the method presented here and the diffusion-like calculations. For both methods, the Fokker-Planck equation is the core. But there is a fundamental difference between the equations used. In the diffusion approach, the beam-beam force (or at least some part of it) is treated as stochastic, giving rise to noise in the particle motion (in additional to the radiation noise). That makes it necessary to introduce into the theory an arbitrary and not well-defined subtraction and a fitting parameter.

In contrast, the present method treats the beam-beam force as a fully deterministic one. The method does not need and does not contain any undefined parameters.

The formal difference of the two methods can be seen from the way in which the perturbing force appears in the equations. In the previous work,[12] it appears as a coefficient in the term $\partial^2\psi/\partial\dot{y}^2$, while in the present work it appears before $\partial\psi/\partial\dot{y}$.

ACKNOWLEDGMENTS

I am grateful to J. Rees who inspired and supported this work. I appreciate many useful discussions with A. Chao, P. Morton, and M. Sands. My thanks are also to all the participants of the machine physics seminar for their interest to this work. The comments by A. Hutton, who read the manuscript, are appreciated.

REFERENCES

1. M. Sands, "The Physics of Electron Storage Rings", SLAC-121, UC-28 (ACC), November 1970.
2. H. Wiedemann, "Experiments on the Beam-Beam Effect in e^+e^- Storage Rings", in *Nonlinear Dynamics and the Beam-Beam Interaction*, AIP Conference Proceedings, **57**, p. 84 (1979).
3. S. Chandrasekhar, *Rev. Mod. Phys.*, **15**, 1 (1943).
4. L. Goldin and D. Koshkarev, *Nuovo Cimento*, **6**, 286 (1957).
5. S. Kheifets, *Prib. Tekh. Exper.*, **5**, 17 (1960).
6. A. Chao, *IEEE Trans. Nucl. Sci.*, **NS-24**, 1885 (1977).
7. P. Channell, "Strong Turbulence and the Anamolous Length of Stored Particle Beams", Ph.D Thesis, LBL-4433, November 1975.
8. A. D. Fokker, *Ann. d. Physik.*, **43**, 812 (1914); M. Planck, Sitz, d. Preuss. Akad., p. 324 (1917).
9. H. Goldstein, *Classical Mechanics* (Addison-Wesley, N. Y., 1953).
10. W. Ebeling and S. Kheifets, Kohärente Betatronschwingungen bei Injektion, DESY Sl-77/03, DESY PET 77/06, June 1977.
11. E. Courant and H. Snyder, *Ann. Phys.*, **3**, 1 (1958).
12. S. Kheifets, *IEEE Trans. Nucl. Sci.*, **NS-26**, 3615 (1979).
13. S. Kheifets, "Particle Losses in Modern Circular Accelerators", Ph.D. Thesis, Yerevan Physics Institute, 1960.

APPENDIX A

The Green's Function for a Stochastic Oscillator

I present here the Green's function for a one-dimensional oscillator. The solution is due to S. Chandrasekhar.[3] A different method of obtaining the same solution has also been developed.[13]

The solution of the equation

$$\frac{\partial G}{\partial t} + \dot{x}\frac{\partial G}{\partial x} - 2\alpha G - 2\alpha\dot{x}\frac{\partial G}{\partial \dot{x}} - \omega^2 x\frac{\partial G}{\partial \dot{x}} - q\frac{\partial^2 G}{\partial \dot{x}^2} = 0, \qquad \text{(A1)}$$

which has a source at $x = x_0, \dot{x} = \dot{x}_0$ at $t = 0$

$$G(x, \dot{x}, x_0, \dot{x}_0, 0) = \delta(x - x_0)\,\delta(\dot{x} - \dot{x}_0) \qquad \text{(A2)}$$

has the form

$$G = \frac{\omega'}{\pi\sqrt{\Delta}}\exp\left\{2\alpha t - \frac{a(\xi - \xi_0)^2 + b(\eta - \eta_0)^2 + 2l(\xi - \xi_0)(\eta - \eta_0)}{2\Delta}\right\}, \qquad \text{(A3)}$$

where

$$a = \frac{q}{\mu_1}(e^{-2\mu_1 t} - 1) \qquad \text{(A4)}$$

$$b = \frac{q}{\mu_2}(e^{-2\mu_2 t} - 1) \qquad \text{(A5)}$$

$$l = -\frac{2q}{\mu_1 + \mu_2}(e^{-(\mu_1 + \mu_2)t} - 1) \qquad \text{(A6)}$$

$$\Delta = l^2 - ab \qquad \text{(A7)}$$

$$\xi = (\mu_1 x - \dot{x})e^{-\mu_2 t} \qquad \text{(A8)}$$

$$\eta = (\mu_2 x - \dot{x})e^{-\mu_1 t} \qquad \text{(A9)}$$

$$\xi_0 = \mu_1 x_0 - \dot{x}_0 \qquad \text{(A8')}$$

$$\eta_0 = \mu_2 x_0 - \dot{x}_0 \qquad \text{(A9')}$$

$$\mu_1 = -\alpha + i\omega' \qquad \text{(A10)}$$

$$\mu_2 = -\alpha - i\omega' \qquad \text{(A11)}$$

$$\omega' = \sqrt{\omega^2 - \alpha^2} \qquad \text{(A12)}$$

The Green's function (A3) expressed in terms of $x, x_0, \dot{x}, \dot{x}_0$ is

$$G = \frac{\omega'}{\pi\sqrt{\Delta}}\exp\{2\alpha t - A_1 x^2 - A_2\dot{x}^2 - A_3 x\dot{x} - A_4 x_0{}^2 - A_5\dot{x}_0{}^2$$
$$- A_6 x_0\dot{x}_0 - A_7 x x_0 - A_8 x\dot{x}_0 - A_9\dot{x}x_0 - A_{10}\dot{x}\dot{x}_0\}, \qquad \text{(A3')}$$

where

$$A_1 = \frac{1}{2\Delta}(a\mu_1{}^2 e^{-2\mu_2 t} + b\mu_2{}^2 e^{-2\mu_1 t} + 2l\mu_1\mu_2 e^{-(\mu_1+\mu_2)t}) \tag{A13}$$

$$A_2 = \frac{1}{2\Delta}(ae^{-2\mu_2 t} + be^{-2\mu_1 t} + 2le^{-(\mu_1+\mu_2)t}) \tag{A14}$$

$$A_3 = -\frac{1}{\Delta}(a\mu_1 e^{-2\mu_2 t} + b\mu_2 e^{-2\mu_1 t} + l(\mu_1 + \mu_2)e^{-(\mu_1+\mu_2)t}) \tag{A15}$$

$$A_4 = \frac{1}{2\Delta}(a\mu_1{}^2 + b\mu_2{}^2 + 2l\mu_1\mu_2) \tag{A16}$$

$$A_5 = \frac{1}{2\Delta}(a + b + 2l) \tag{A17}$$

$$A_6 = -\frac{1}{\Delta}(a\mu_1 + b\mu_2 + l(\mu_1 + \mu_2)) \tag{A18}$$

$$A_7 = -\frac{1}{\Delta}(a\mu_1{}^2 e^{-\mu_2 t} + b\mu_2{}^2 e^{-\mu_1 t} + l\mu_1\mu_2(e^{-\mu_1 t} + e^{-\mu_2 t})) \tag{A19}$$

$$A_8 = \frac{1}{\Delta}(a\mu_1 e^{-\mu_2 t} + b\mu_2 e^{-\mu_1 t} + l\mu_2 e^{-\mu_1 t} + l\mu_1 e^{-\mu_2 t}) \tag{A20}$$

$$A_9 = \frac{1}{\Delta}(a\mu_1 e^{-\mu_2 t} + b\mu_2 e^{-\mu_1 t} + l\mu_1 e^{-\mu_1 t} + l\mu_2 e^{-\mu_2 t}) \tag{A21}$$

$$A_{10} = -\frac{1}{\Delta}(ae^{-\mu_2 t} + be^{-\mu_1 t} + l(e^{-\mu_1 t} + e^{-\mu_2 t})). \tag{A22}$$

For large time ($t \to \infty$),

$$a \to \frac{q}{\mu_1} e^{-2\mu_1 t} \tag{A4$'$}$$

$$b \to \frac{q}{\mu_2} e^{-2\mu_2 t} \tag{A5$'$}$$

$$l \to -\frac{q}{\alpha} e^{2\alpha t} \tag{A6$'$}$$

$$\Delta \to \frac{q^2\omega'^2}{\omega^2\alpha^2} e^{4\alpha t} \tag{A7$'$}$$

$$G \to \frac{\omega\alpha}{\pi q} \exp\left\{-\frac{x^2}{q/\omega^2\alpha} - \frac{\dot{x}^2}{q/\alpha}\right\}, \tag{A3$''$}$$

Hence, we get

$$\sigma_x{}^2 = q/2\omega^2\alpha \tag{A23}$$

$$\sigma_{\dot{x}}{}^2 = q/2\alpha \tag{A24}$$

One notices how the Green's function "forgets" its initial conditions x_0, $\dot{x}_0$ as time goes on.

APPENDIX B

Several Lower Moments of the Green's Function

It is useful for future applications to find several first moments of the Green's function. I start from the calculation of the normalization integral (zeroth moment). We know that this integral equals unity

$$\int dx\, d\dot{x}\, G(x, \dot{x}, x_0, \dot{x}_0, t) = 1. \tag{B1}$$

On the other hand, performing the actual integration over the form (A3′) we find

$$\frac{\omega' e^{2\alpha t}}{\sqrt{\Delta \cdot A_2 \bar{A}_1}} \exp\left\{-\left[\left(A_4 - \frac{A_9{}^2}{4A_2} - \frac{\bar{A}_7{}^2}{4\bar{A}_1}\right)x_0{}^2 + \left(A_5 - \frac{A_{10}^2}{4A_2} - \frac{\bar{A}_8{}^2}{4\bar{A}_1}\right)\dot{x}_0{}^2 \right.\right.$$
$$\left.\left. + \left(A_6 - \frac{2A_9A_{10}}{4A_2} - \frac{2\bar{A}_7\bar{A}_8}{4\bar{A}_1}\right)x_0\dot{x}_0\right]\right\} = 1, \tag{B2}$$

where we introduce the abbreviations

$$\bar{A}_1 = A_1 - A_3{}^2/4A_2 \tag{B3}$$

$$\bar{A}_7 = A_7 - A_3A_9/2A_2 \tag{B4}$$

$$\bar{A}_8 = A_8 - A_3A_{10}/2A_2 \tag{B5}$$

Equation (B2) immediately suggests

$$\sqrt{\Delta \cdot A_2 \cdot \bar{A}_1} = \omega' \cdot e^{2\alpha t} \tag{B6}$$

$$A_4 - A_9{}^2/4A_2 = \bar{A}_7{}^2/4\bar{A}_1 \tag{B7}$$

$$A_5 - A_{10}^2/4A_2 = \bar{A}_8{}^2/4\bar{A}_1 \tag{B8}$$

$$A_6 - 2A_9A_{10}/4A_2 = 2\bar{A}_7\bar{A}_8/4\bar{A}_1 \tag{B9}$$

These equations will be extensively used in further parts of this work. Expressions (B6–B9) are also checked by direct calculations using Eqs. (A13–A22).

Let us now consider the second moment x^2.

$$P_2(x_0, \dot{x}_0, t) = \int dx\, d\dot{x}\, x^2 G \tag{B10}$$

The integration yields

$$P_2 = p_0 + p_1{}^2 x_0{}^2 + p_2{}^2 \dot{x}_0{}^2 + 2p_3 x_0 \dot{x}_0, \tag{B11}$$

where

$$\begin{aligned} p_0 &= 1/2\bar{A}_1 \\ &= \frac{\sigma \dot{x}^2}{\omega'^2}\left[1 - e^{-2\alpha t} - \frac{\alpha\omega'}{\omega^2} e^{-2\alpha t} \sin 2\omega' t - \frac{\alpha^2}{\omega^2}(1 - e^{-2\alpha t} \cos 2\omega' t)\right] \end{aligned} \tag{B12}$$

$$p_1 = \bar{A}_7/2\bar{A}_1 = -\frac{1}{\omega'} e^{-\alpha t} (\alpha \sin \omega' t + \omega' \cos \omega' t) \tag{B13}$$

$$p_2 = \bar{A}_8/2\bar{A}_1 = -\frac{1}{\omega'} e^{-\alpha t} \sin \omega' t \tag{B14}$$

$$\begin{aligned} p_3 &= \bar{A}_7 \bar{A}_8 / 4\bar{A}_1{}^2 \\ &= \frac{1}{\omega'^2} e^{-2\alpha t}(\alpha \sin^2 \omega' t + \omega' \sin \omega' t \cos \omega' t). \end{aligned} \tag{B15}$$

There is a useful relationship between the coefficients p_1, p_2 and p_3

$$p_1 p_2 - p_3 = 0. \tag{B16}$$

The first moment x can also be expressed through the coefficients p_1 and p_2

$$P_1 = \int dx\, d\dot{x}\, xG = -p_1 x_0 - p_2 \dot{x}_0. \tag{B17}$$

Last, the second moment $x\dot{x}$ is a bit more complicated

$$\dot{P}_2 = \int dx\, d\dot{x}\, x\dot{x} G = r_0 + r_1 x_0{}^2 + r_2 \dot{x}_0{}^2 + 2r_3 x_0 \dot{x}_0. \tag{B18}$$

Here

$$r_0 = -\frac{A_3}{2A_2} p_0 \tag{B19}$$

$$r_1 = \frac{A_9}{2A_2} p_1 - \frac{A_3}{2A_2} p_1{}^2 \tag{B20}$$

$$r_2 = \frac{A_{10}}{2A_2} p_2 - \frac{A_3}{2A_2} p_2{}^2 \tag{B21}$$

$$r_3 = \frac{A_9}{4A_2} p_2 + \frac{A_{10}}{4A_2} p_1 - \frac{A_3}{2A_2} p_3 \tag{B22}$$

It is easy (although cumbersome) to find the coefficients A_2, A_3, A_9 and A_{10}. Formulae for them, neglecting terms of order $(\alpha/\omega)^2$ and higher are

$$A_2 = \frac{2\sigma_{\dot{x}}^2}{\Delta} e^{4\alpha t}\left(1 - e^{-2\alpha t} - \frac{\alpha}{\omega} e^{-2\alpha t} \sin 2\omega t\right) \tag{B23}$$

$$A_3 = -\frac{8\sigma_{\dot{x}}^2}{\Delta} e^{2\alpha t} \alpha \sin^2 \omega t \tag{B24}$$

$$A_9 = \frac{4\sigma_{\dot{x}}^2}{\Delta} e^{3\alpha t}(1 - e^{-2\alpha t})\omega \sin \omega t \tag{B25}$$

$$A_{10} = -\frac{4\sigma_{\dot{x}}^2}{\Delta} e^{3\alpha t}\left((1 - e^{-2\alpha t}) \cos \omega t - \frac{\alpha}{\omega}(1 + e^{-2\alpha t}) \sin \omega t\right) \tag{B26}$$

There is no need to find Δ, since the coefficients considered enter all formulae only as a ratio.