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Colored Group Field Theory

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Abstract Random matrix models generalize to Group Field Theories (GFT) whose Feynman graphs are dual to higher dimensional topological spaces. The perturbative development of the usual GFT's is rather involved combinatorially and plagued by topological singularities (which we discuss in great detail in this paper), thus very difficult to control and unsatisfactory.

Both these problems simplify greatly for the "colored" GFT (CGFT) model we introduce in this paper. Not only this model is combinatorially simpler but also it is free from the worst topological singularities. We establish that the Feynman graphs of our model are combinatorial cellular complexes dual to manifolds or pseudomanifolds, and study their cellular homology. We also relate the amplitude of CGFT graphs to their fundamental group.

1 Introduction: Group Field Theory

Group Field Theories (GFT) (1; 2; 3; 4; 5) are quantum field theories over group manifolds. They generalize random matrix models and random tensor models (6; 7; 8; 9; 10). GFT's arise naturally in several discrete approaches to quantum gravity, like Regge calculus (11), dynamical triangulations (12; 13) or spin foam models (14; 15) (see also (16) for further details).

To define a GFT we choose a group G, and the quatum field ϕ, a real scalar function over n copies of the group $\phi : G^{\otimes n} \to \mathbb{R}$. Furthermore we require that ϕ is invariant under arbitrary permutations σ and simultaneous multiplication to the left of all its arguments,

$$\begin{aligned} \phi(hg_{\alpha_1},\dots,hg_{\alpha_n}) &= \phi(g_{\alpha_1},\dots,g_{\alpha_n}), \quad \forall h\in G, \\ \phi(g_{\alpha_{\sigma(1)}},\dots,g_{\alpha_{\sigma(n)}}) &= \phi(g_{\alpha_1},\dots,g_{\alpha_n}), \quad \forall \sigma \in \mathfrak{S}_n. \end{aligned} \tag{1}$$

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As a quantum field theory, a GFT in n dimensions is defined by the action functional (17)

$$S = \frac{1}{2}\int [dg]\, \phi_{\alpha_0\dots\alpha_{n-1}} K^{-1}(g_{\alpha_0},\dots,g_{\alpha_{n-1}};g_{\alpha'_0},\dots g_{\alpha'_{n-1}})\phi_{\alpha'_0\dots\alpha'_{n-1}} + \frac{\lambda}{n+1}\int [dg]\, V(g_{\alpha^0_0},\dots,g_{\alpha^{n+1}_{n-1}})\phi_{\alpha^0_0\dots\alpha^0_{n-1}}\cdots\phi_{\alpha^{n+1}_0\dots\alpha^{n+1}_{n-1}}\,, \tag{2}$$

where we used the shorthand notations $\phi(g_{\alpha_0},\dots g_{\alpha_{n-1}}) = \phi_{\alpha_0\dots\alpha_{n-1}}$, and $\int [dg]$ for the integral over the group manifold with Haar measure of all group elements appearing in the arguments of the integrand.

The vertex operator V encodes the connectivity dual to a n simplex, whereas the propagator K encodes the connectivity dual to the gluing of two n simplices along $(n-1)$ simplices[1] of their boundary. A Feynman graph with vertices V and propagators K is dual to a "gluing of simplices". From a mathematical standpoint, GFT is a tool to investigate the properties of this gluing of simplices. For instance, if the gluing represents a manifold one can define topological invariants (18) using GFT's.

In discrete approaches to quantum gravity the gluing of simplices is interpreted as a space-time background, making GFT a combinatorial, background independent theory whose perturbative development generates space-times. This is further supported as, for the simplest choice of V and K, the Feynman amplitude of a graph reproduces the partition function of a BF theory discretized on the gluing of simplices. BF theory becomes Einstein gravity after imposing the Plebanski constraints, hence it is natural to suppose that a choice of operators K and V exists which reproduces the partition function of the latter. This line of research has been explored in (19; 20; 21; 22; 23), to define more realistic GFT's whose semiclassical limit (24; 25) exhibits the expected behavior.

In this paper we are interested in the combinatorial and topological aspects of the Feynman diagrams of the action (2). As these do not depend on the specifics of K and V we will opportunistically make the simplest choices. Our study is motivated by the following analogy with matrix models. Even though only identically distributed matrix models are topological in some scaling limit ((6)), it turns out that the power counting of more involved models (like for instance the Gross-Wulkenhaar model (26; 27)) is again governed by topological data. In fact, from a quantum field theory perspective one can argue that the only matrix models for which some renormalization procedure can be defined are those with topological power counting. Pursuing this analogy one step further, the non trivial fixed point of the Gross-Wulkenhaar model (28; 29) opens up the intriguing possibility that GFTs are UV complete quantum field theories.

However a perturbative development of the usual GFT runs almost immediately into a very subtle and extremely serious problem: some gluings of simplices dual to GFT graphs present "wrapping singularities". Such gluings are *not* manifolds or pseudo-manifolds! This is extremely surprising, especially as it appears to contradict the most basic results of (30) concerning n-dimensional rectilinear gluings of simplices! This apparent contradiction is explained by the subtle differences between

[1] That is the field ϕ is associated to $(n-1)$ simplices, and its arguments g to $(n-2)$ simplices.

Fig. 1 GFT vertex in three dimensions and dual tetrahedron

arbitrary gluings of simplices dual to GFT graphs and *rectilinear gluings* used in (30). In Sect. 2 we give a very detailed discussion of these singularities, explain the precise relation between GFT graphs and the classical results of (30) and provide abundant examples.

The pathological "wrapping singularities" plague the perturbative development of the usual GFT models and virtually any result about the topology of gluings of simplices dual to GFT graphs can be established only under some restrictions (obviously the most common is to require that the gluing *is* rectilinear). However up to this work none of these restrictions could be promoted at the level of the action to ensure that *all* GFT graphs respect it.

In Sect. 3 we introduce the new "colored" GFT (CGFT) model. In Sect. 4 we prove that all CGFT graphs are free of the infamous wrapping singularities and that they always correspond to pseudo-manifolds. We then undertake the first steps in a systematic study of the topology of CGFT graphs by means of the graph combinatorial cellular complex and cellular homology. The amplitudes of CGFT graphs are subsequently related to the first homotopy group of the graph complex in Sect. 6, and a necessary condition for a graph to be homotopically trivial is derived. Section 7 draws the conclusion of our work and Appendix A presents explicit examples of homology groups for several CGFT graphs.

2 GFT Graphs in Three Dimensions

In this section[2] we will discuss in detail the Feynman graphs of the familiar GFT's in three dimensions and describe precisely the "wrapping singularities" mentioned in the Introduction.

The usual GFT action in three dimensions is

$$\begin{aligned} S &= \frac{1}{2}\int [dg]\ \phi_{\alpha_0\alpha_1\alpha_2}\ \phi_{\alpha_0\alpha_1\alpha_2} + S_{int}\,, \\ S_{int} &= \frac{\lambda}{4}\int [dg]\ \phi_{\alpha_{03}\alpha_{02}\alpha_{01}}\ \phi_{\alpha_{10}\alpha_{13}\alpha_{12}}\ \phi_{\alpha_{21}\alpha_{20}\alpha_{23}}\ \phi_{\alpha_{32}\alpha_{31}\alpha_{30}}\ , \end{aligned} \tag{3}$$

where $\phi_{\alpha_0\alpha_1\alpha_2} \equiv \phi(g_{\alpha_0}, g_{\alpha_1}, g_{\alpha_2})$, and $g_{\alpha_{ij}} = g_{\alpha_{ji}}$ in S_{int}. The GFT vertex generated by S_{int}, is represented on the left in Fig. 1.

Each field ϕ in S_{int} is associated to a halfline of the GFT vertex. Every two fields in S_{int} share a group element, consequently every two halflines of the GFT vertex share a *strand* (depicted as a solid line in Fig. 1). We label the halflines of the GFT vertex $0,1,2$ and 3 and each strand by the (unordered) couple of labels of the two halflines which share it. The halflines 0 and 1 share the strand 01 (or 10), the halflines 0 and 2 share the strand 02 (or 20), etc.

The GFT vertex is dual to a tetrahedron represented on the right in Fig. 1. The half lines of the vertex are dual to the triangles bounding the tetrahedron. The edge shared by two triangles is dual to the strand shared by the corresponding halflines.

[2] We would like to thank the anonymous referee for suggesting the addition of this section.

Fig. 2 GFT lines in three dimensions

Fig. 3 The two tadpoles in three dimensional GFT

Fig. 4 Second order graphs in three dimensional GFT

For simplicity, the labels of the triangles have been omitted in Fig. 1. Instead, we labeled the vertex of the tetrahedron opposite to the triangle 0 by A_0, the one opposite the triangle 1 by A_1, etc. In the sequel the GFT vertex will be called a *stranded vertex*, to emphasize its internal strand structure.

The quadratic part of the action 3 connects two GFT vertices via an arbitrary permutation of the strands, yielding the six possible choices for the lines, presented in Fig. 2. The GFT lines are dual to the identification of two triangles and each permutation of the strands encodes one of the six possible ways to identify two triangles. The GFT lines will sometimes be referred to as *stranded lines* to emphasize their internal strand structure. The perturbative development of GFT is indexed by the Feynman graphs of the action 3, dual to arbitrary gluings of tetrahedra, built with stranded vertices and stranded lines. They will be called *stranded graphs*. Stranded graphs are the higher dimensional generalization of the ribbon graphs of matrix models.

2.1 Combinatorics of GFT graphs

Due to the strand structure of the GFT lines and vertices the combinatorics of GFT graphs is somewhat involved. One needs to track carefully which halflines are contracted into the GFT lines, as well as count every line with a factor six.

To illustrate this consider the double tadpole graph made of one vertex and two lines. Choose a halfline (say 0 in Fig. 1). It has three choices of a partner to contract into a line ($1,2$ and 3 respectively). The remaining halflines necessarily connect into a line. Thus a first rough count gives 3×6^2 different graphs. This is of course a gross overestimate. For instance choosing 1 or 3 yields the same graph (on the left in Fig. 3) while choosing 2 yields the graph on the right in Fig. 3. The number of distinct graphs drops to 2×6^2. This is still a large overestimate as, for specific choices of the permutations of the strands some of these tadpoles are still related by symmetry (we will see an example later on).

At second order the very naive counting of contraction yields $[3!+2\binom{4}{2}^2]\times 6^4$ (corresponding to the two cases schematically drawn in Fig. 4). This number is again an extreme overestimate and can be reduced drastically.

Fixing precisely the symmetry factors of the graphs requires tracking carefully the structure of the strands and consequently is rather involved. In fact, to our knowledge this has not yet been done even for the first and second order graphs.

Fig. 5 A GFT graph $\mathscr{G}^1$ dual to a gluing with negative Euler characteristic

2.2 Wrapping singularities in GFT graphs

We now come to the second part of our analysis, namely the topology of the gluing of tetrahedra dual to a GFT graph.

As mentioned in the Introduction, the results of Sect. 3.2 in (30)[3] are proved for *rectilinear* gluings of simplices. A rectilinear gluing (Definition 3.2.1. of (30)) is, crucially, *non-branching*, namely any two triangles identified by a gluing belong to exactly two *distinct* tetrahedra.

The gluings of tetrahedra dual to all the GFT graphs at first order (Fig. 3) and some GFT graphs at second order (on the right in Fig. 4) are *not rectilinear*: there exist triangles belonging to the same tetrahedron which are identified. One can not apply directly the results of (30). When reconsidering carefully the topology of arbitrary gluings we will discover that some of them present wrapping singularities and are *not* manifolds or pseudo manifolds.

Of utmost importance in the sequel is the notion of *link of a vertex* (see Sect. 3.2 in (30)). Any vertex v in a gluing of tetrahedra belongs to several tetrahedra τ_i. For each tetrahedron τ_i, we denote σ_i the triangle of the tetrahedron τ_i *opposite* to v (that is not containing v). The *link* of v, denoted lk$v$ is the gluing of the σ_i. A neighborhood of a vertex is the topological cone over its link, $C|\mathrm{lk}v| = \frac{\mathrm{lk}v\times[0,1]}{\mathrm{lk}v\times\{1\}}$.

The main result in Sect. 3.2 of (30) is Proposition 3.2.8. The Euler characteristic $\chi(X)$ of any closed connected three dimensional rectilinear gluing X, with k vertices $v_1 \dots v_k$ and links $\mathrm{lk}(v_1)$ obeys

$$\chi(X) = k - \frac{1}{2}\sum_i \chi(\mathrm{lk}v_i), \tag{4}$$

where $\chi(\mathrm{lk}v_i)$ is the Euler characteristic of the link lkv_i. In particular, as the links are two dimensional surfaces, their Euler characteristic is at most 2 and one recovers the classical result that the Euler characteristic of a three dimensional closed connected pseudo-manifold is positive $\chi(X) \geq 0$.

Consider now the GFT graph represented on the left in Fig. 5. It is dual to the *non-rectilinear* gluing (using the labeling in Fig. 1):

- the gluing of the triangles $A_1A_2A_3 \equiv A_0A_3A_2$, by identifying the vertices $A_1 \equiv A_0, A_2 \equiv A_3, A_3 \equiv A_2$ and the edges $01 \equiv 10, 02 \equiv 13, 03 \equiv 12$.
- the gluing of the triangles $A_1A_0A_3 \equiv A_0A_1A_2$ by identifying $A_1 \equiv A_0, A_0 \equiv A_1, A_3 \equiv A_2$ and the edges $21 \equiv 30, 20 \equiv 31, 23 \equiv 32$.

This gluing has 2 vertices ($A_0 = A_1$ and $A_2 = A_3$), 4 edges ($01, 02 = 13, 03 = 12$ and 23), 2 triangles ($A_1A_2A_3 = A_0A_3A_2$ and $A_1A_0A_3 = A_0A_1A_2$) and one tetrahedron ($A_0A_1A_2A_3$), hence an Euler characteristic $\chi(X) = -1 < 0$. Therefore this gluing is *not* a pseudo-manifold. Even worse, this gluing does corespond to an abstract simplicial complex (Definition 2.1 in (31)) and not even a trisp (Definition 2.44 in (31)).

[3] We thank the anonymous referee for pointing us to this excellent reference which we will use extensively in the sequel.

Fig. 6 Link graphs in GFT

In Appendix B we show that Eq. (4) is a *necessary* condition that a gluing of tetrahedra actually is a pseudo-manifold (we do not claim that it is sufficient). In our quest to find a GFT such that all its graphs are pseudo-manifolds, Eq. (4) will be our guide. We will first try to understand the profound origin of its failure for general GFT graphs, and then we will try to build a model such that Eq. (4) holds for *all* its graphs.

The proof of Lemma 3.2.8 in (30) relies on the following detailed balance valid for a rectilinear gluing

- each edge of the gluing accounts for two vertices of the links,
- each triangle of the gluing accounts for three edges of the links,
- each tetrahedron of the gluing accounts for four triangles of the links.

Denoting V, E, T and Θ the number of vertices, edges, triangles and tetrahedra of the gluing X, and v, e, and t the number of vertices, edges and triangles of the links, the total Euler characteristic of the links is written:

$$\sum_i \chi(\text{lk} v_i) = v - e + t = 2E - 3T + 4\Theta, \tag{5}$$

but, as every tetrahedron is bounded by four triangles, $4\Theta = 2T$, hence

$$\frac{1}{2}\sum_i \chi(\text{lk} v_i) = E - T + \Theta = V - \chi(X). \tag{6}$$

The links defined for rectilinear gluings generalize to arbitrary gluings dual to GFT graphs. A link is a gluing of triangles, hence it is dual to a ribbon graph, called *the link graph*. The link graphs can be accessed directly starting from the GFT graph by the following algorithm (see (32)):

- for all GFT vertices erase all strands belonging to a halfline. For instance, erasing the strands belonging to the halfline 0 in Fig. 1, we obtain the ribbon vertex with strands 12, 13 and 23. Then repeat this for all halflines of the GFT vertex to obtain four "descendent" ribbon vertices.
- connect the strands of the ribbon vertices as dictated by the GFT lines.

An example is presented in Fig. 6, where each connected ribbon graph is dual to the link of a vertex in the gluing of tetrahedra.

The detailed balance would translate for GFT graphs as

- each face (closed strand) of a GFT graph accounts for two faces in the link graphs.
- each line in the GFT graph accounts for three lines in the link graphs.
- each GFT vertex accounts for four vertices in the link graphs.

The first item above *does not hold* in arbitrary GFT graphs. For the graph $\mathcal{G}^1$ in Fig. 5 we drawn the two link graphs on the right. The faces F_1 and F_2 of the GFT graph account each for *only one face* in the link graphs, f_1 and f_2 (that is an edge of the tetrahedron contributes only one vertex on a link, and not two). Thus

Fig. 7 A GFT graph $\mathscr{G}^2$ related by symmetry to $\mathscr{G}^1$

Fig. 8 A third singular graph $\mathscr{G}^3$

Fig. 9 Wrapping singularities in arbitrary GFT graphs

although the link graphs have four vertices and six lines, they only have a total of six faces, and not eight.

The faces of the links, f_1 and f_2, consist each of two lines l_1, respectively l_2, while the faces F_1 and F_2 consist of only one line L_1 respectively L_2. That is f_1 and f_2 wrap twice around F_1 and F_2, hence the name "wrapping singularity". The graph of Fig. 7 presents the same phenomenon. This illustrates the difficulty to accurately compute symmetry factors: surprisingly $\mathscr{G}^1$ and $\mathscr{G}^2$ are related by symmetry.

A somewhat different example of a graph for which Eq. 4 fails is given in Fig. 8. In this case one of the links is planar, thus its Euler characteristic is 2, while the second is non-orientable, and its Euler characteristic is 1 (the link is homeomorphic with $\mathbb{R}P^2$). Hence $k - \frac{1}{2}\sum_i \chi(\mathrm{lk} v_i) = 1/2$ which is not even an integer.

The wrapping singularities are generic in the usual GFT: any graph having a tadpole insertion like in Fig. 9 will have a wrapping singularity.

Lemma 1 *The gluing dual to a GFT graph with a wrapping singularity will not obey Eq.* (4) *hence, by Appendix* B*, is not a pseudo-manifold.*

Proof Every GFT vertex always gives four ribbon vertices hence (in the notation of
Eq. (5)) $t = 4\Theta$ always holds. Any halfline of the GFT vertex will contribute to three of the four descendent ribbon vertices, thus $e = 3T$ always holds also.

A strand in the GFT vertex will belong to two different ribbon vertices (for instance the strand 12 in Fig. 1 belongs to the ribbon vertices $12, 13, 23$ and $12, 01, 02$). A face in the link corresponding to a strand either passes through only one of these two vertices (in which case there exists a second, distinct face in some link graph which passes through the second vertex) or it passes through both and generates a wrapping singularity. Hence $v \leq 2E$ and $v = 2E$ if and only if a graph has no wrapping singularity, hence

$$\chi(X) \leq k - \frac{1}{2}\sum_i \chi(\mathrm{lk} v_i), \tag{7}$$

and the equality holds only if the graph is free of wrapping singularities. □

It is therefore a very important question whether there exists a GFT action such that all its graphs are free of wrapping singularities. The central point of this paper is that indeed such an action exists: it is the colored GFT model we define in Sect. 3.

The CGFT model brings also many other advantages. In generates fewer graphs than the usual GFT model (no graph at first order and only one graph at second order) simplifying the combinatorics. Not only are its graphs free of wrapping singularities but also we will prove that all graphs are dual to pseudo-manifolds. It is, to this day, the *only* example of a GFT action whose graphs are *all* dual to pseudo-manifolds. The faces and link graphs are easily identified. All links are orientable surfaces (of arbitrary genus). The homology groups and fundamental group of graphs can easily be defined related to the Feynman amplitudes of the graphs.

3 The Colored GFT Model

We will define a fermionic colored GFT model invariant under a global color transformation. One can alternatively consider a bosonic CGFT model, but the latter does not exhibit this tantalizing invariance. Instead of an unique bosonic field, consider $n+1$ **Grassmann** fields, $\psi^0,\dots,\psi^n : G^{\otimes n} \to \mathbb{G}$,

$$\{\psi^i,\psi^j\} = 0, \tag{8}$$

with hermitian conjugation

$$\psi \to \bar{\psi} \text{ such that } \overline{\psi^1\psi^2} = -\bar{\psi}^2\bar{\psi}^1, \quad \bar{\bar{\psi}} = -\psi. \tag{9}$$

The fields ψ have *no* symmetry properties under permutations of the arguments, but are all invariant under simultaneous left action of the group on all their arguments. The upper index p denotes the color of the field ψ^p.

A (global) color transformation is an internal rotation $U \in SU(n+1)$ on the grassmann fields

$$(\psi^i)' = U^{ij}\psi^j, \quad (\bar{\psi}^i)' = \bar{\psi}^j (U^{ij})^* = \bar{\psi}^j (U^{-1})^{ji}. \tag{10}$$

The only quadratic form invariant under color transformation is

$$\sum_p \bar{\psi}^p \psi^p. \tag{11}$$

The interaction is a monomial in the fields. The only monomial in ψ invariant under color rotation is

$$\psi^0 \dots \psi^n, \tag{12}$$

as

$$(\psi^0)' \dots (\psi^n)' = U^{0i_0}\dots U^{ni_n}\psi^{i_0}\dots\psi^{i_n} = \det(U)\,\psi^0\dots\psi^n \,. \tag{13}$$

The hermitian GFT action of minimal degree, invariant under (global) color rotation is

$$S = \sum_p \int [dg]\ \bar{\psi}^p \psi^p + \int [dg]\ \psi^0\psi^1 \dots \psi^p + \int [dg]\ \bar{\psi}^0\bar{\psi}^1 \dots \bar{\psi}^p, \tag{14}$$

Fig. 10 nD vertex

Fig. 11 The second order CGFT graph

where the arguments of ψ^p and $\bar{\psi}^p$ in the interaction terms are chosen to reproduce the combinatorics of the GFT vertex (see (17)), that is, denoting $g_{pq} = g_{qp}$ the group element associated to the strand connecting the halflines p and q,

$$\psi^p(g_{p-1p}, g_{p-2p}, \ldots, g_{0p}, g_{pn}, \ldots g_{pp+1}). \tag{15}$$

The interaction part of the action of Eq. (14) has two terms. We call the vertex involving only ψ's the positive vertex and represent it like in Fig. 10. The second term, involving only $\bar{\psi}$ is similar, but the colors turn anticlockwise around it. The vertices have a detailed internal structure encoding the connectivity of the arguments g

From Eq. (14) we conclude that the propagator of the model is formed of n parallel strands and always connects two halflines of the same color, one on a positive vertex and one on a negative vertex. We orient all lines from positive to negative vertices.

The strand structure of the vertex and propagator is rigid. A stranded colored GFT graph admits therefore a simplified representation as a *colored* graph. The colored graph is obtained by collapsing all the strands of the lines in "thin" lines, and all the strands of the vertices in point vertices. Conversely, given a colored graph with thin lines and point vertices one can reconstruct the stranded graph associated to it. Figure 11 depicts a CGFT graph either as a stranded graph (on the right) or as colored graph (on the left).

4 Bubble Homology in Colored GFT

The strand structure of the vertices and propagators render the Feynman graphs of our model topologically very rich. This should come as no surprise, as in three dimensions for instance, gluings dual to CGFT graphs include not only all orientable piecewise linear three dimensional manifolds (see (33) and references therein), but also pseudo-manifolds. The topology of the dual gluing of simplices is encoded in the topology of CGFT graphs and in the rest of this paper we will study the latter.

An important notion associated to CGFT graphs is that of *p-bubble*. Represent the graph $\mathscr{G}$ as colored graph with thin lines and point vertices. Let $\mathscr{C}$ be an ordered set of colors $\mathscr{C} \subset \{0, 1, \ldots, n\}$ of cardinality p. A p-bubble of colors $\mathscr{C}$, denoted $\mathscr{B}^{\mathscr{C}}_{\mathscr{V}}$, is the *connected* subgraph of $\mathscr{G}$ made only of lines of colors $\mathscr{C}$ and with vertex set $\mathscr{V}$. The graph itself is not considered a $(n+1)$-bubble.

The graph in Fig. 11 has 3-bubbles $\mathscr{B}^{012}_{v_1v_2}, \mathscr{B}^{013}_{v_1v_2}, \mathscr{B}^{023}_{v_1v_2}$ and $\mathscr{B}^{123}_{v_1v_2}$, 2-bubbles $\mathscr{B}^{01}_{v_1v_2}$,
$\mathscr{B}^{02}_{v_1v_2}, \mathscr{B}^{03}_{v_1v_2}, \mathscr{B}^{12}_{v_1v_2}, \mathscr{B}^{13}_{v_1v_2}, \mathscr{B}^{23}_{v_1v_2}$, 1-bubbles $\mathscr{B}^{0}_{v_1v_2}, \mathscr{B}^{1}_{v_1v_2}, \mathscr{B}^{2}_{v_1v_2}, \mathscr{B}^{3}_{v_1v_2}$, and finally 0-bubbles $\mathscr{B}_{v_1}, \mathscr{B}_{v_2}$.

The bubbles are the combinatorial encoding of the vertices, lines, faces and link graphs discussed in Sect. 2 for CGFT graphs. Trivially

Fig. 12 Stranded and colored graph

Remark 1 The 0-bubbles of $\mathcal{G}$ are its vertices and the 1-bubbles are its lines.

The following remarks are a bit more interesting.

Remark 2 The 2-bubbles of the colored CGFT graph $\mathcal{G}$ are the faces (closed strands) of the associated stranded graph.

Proof Consider three successive vertices v_p, v_{p+1} and v_{p+2} along a strand, as depicted in Fig. 12. Their halflines have a color index $0,1,2$ or 3. The strands are identified, in each vertex, by the couple of colors of the halflines they belong to. The lines have only parallel strands and connect (respecting the colorings of the halflines) two vertices of opposite orientation.

This drawing essentially proves the result. The strand 12, common to the halflines of colors 1 and 2 on v_p necessarily connects with the strand 12 on v_{p+1} which in turn connects to the strand 12 on v_{p+2}. Thus the labels (12) are *conserved* all along the strand. This is the fundamental difference between the usual GFT graphs and the CGFT graphs and render the latter much better behaved.

The 2-bubble 12 is obtained by deleting all the lines of colors 0 and 3 of the colored graph. In the stranded graph representation one deletes all the lines of color 0 and 3 *together with all their strands* (as these strands have at least a label 0 or 3). Therefore the 2-bubble 12 is the closed strand (face) 12 in Fig. 12. □

Remark 3 The 3-bubbles of the colored CGFT graph $\mathcal{G}$ are the link graphs of the associated stranded graph.

Proof Again Fig. 12 essentially proves the result. Consider the vertex v_{p+1} in Fig. 12 and the ribbon vertex (of a link graph) obtained by deleting all the strands belonging to the halfline 0. This vertex is made of strands of colors 12, 13 and 23 of v_{p+1}. We will call it a 123 ribbon vertex.

The labels of the strands are conserved along the lines, thus the vertex 123 coming from v_{p+1} connects, via the line of color 2 with the ribbon vertex (coming from v_p) containing *both* strands 12 and 23, hence necessarily the 123 ribbon vertex coming from v_p. Consequently, all the ribbon vertices of the link graph are 123 vertices coming from various GFT vertices. The link graph is then a connected component obtained from the stranded graph by deleting all lines of color 0 *together with all their strands*, hence a 3-bubble.

Note that this algorithm to identify link graphs in a colored GFT graph is a drastic simplification with respect to the one valid for arbitrary GFT graphs presented in Sect. 1.

□

Taking a careful look at Fig. 12 and the algorithm giving the link graphs we notice that the latter are necessarily ribbon graphs with no twists along the ribbons, hence

Remark 4 The link graphs of a colored GFT graph are dual to gluings representing orientable, closed surfaces.

A face, say 12, will always contribute two *different faces*, belonging to two different link graphs, one with colors 123 and the other with colors 012.

Remark 5 A CGFT graph has no wrapping singularities, hence the dual gluing respects Eq. 4.

Most importantly, the main topological property of CGFT graphs is

Theorem 1 *Any CGFT is dual to a simplicial pseudo-manifold.*

Proof Step 1. In this proof we use (31). We first prove that the dual of any CGFT graphs is a finite abstract simplicial complex.

A finite abstract simplicial complex (Definition 2.1 (31)) is a finite set A together with a collection Δ of subsets such that if $X \in \Delta$ and $Y \subseteq X$ then $Y \in \Delta$.

Consider the dual gluing of tetrahedra. Its vertices are dual to the 3-bubbles of the CGFT graph, hence its set of vertices is written $A = \{\mathcal{B}^{ijk}_{\mathcal{V}}\}$. The edges of the gluing are dual to the 2-bubbles of the graph. Every 2-bubble is shared by exactly two
3-bubbles, and we denote it by the couple of the two 3-bubbles $e = \mathcal{B}^{ij}_{\mathcal{V}} = \{\mathcal{B}^{ijk}_{\mathcal{V}_1}, \mathcal{B}^{ijl}_{\mathcal{V}_2}\}$ and $\mathcal{V} \subset \mathcal{V}_1 \cap \mathcal{V}_2$ every triangle is dual to a line, and every line belongs to exactly three
3-bubbles, $l = \mathcal{B}^{i}_{v_1 v_2} = \{\mathcal{B}^{ijk}_{\mathcal{V}_1}, \mathcal{B}^{ijl}_{\mathcal{V}_2}, \mathcal{B}^{ikl}_{\mathcal{V}_3}\}$ and $\{v_1 v_2\} \subset \mathcal{V}_1 \cap \mathcal{V}_2 \cap \mathcal{V}_3$. Finally, every tetrahedron is dual to a vertex shared by exactly four 3-bubbles $v_1 = \mathcal{B}_{v_1} = \{\mathcal{B}^{ijk}_{\mathcal{V}_1}, \mathcal{B}^{ijl}_{\mathcal{V}_2}, \mathcal{B}^{ikl}_{\mathcal{V}_3}, \mathcal{B}^{jkl}_{\mathcal{V}_4}\}$ and $v_1 \in \mathcal{V}_1 \cap \mathcal{V}_2 \cap \mathcal{V}_3 \cap \mathcal{V}_4$. Δ is the set of all vertices edges triangles and tetrahedra (supplemented by the empty set).

Consider the subset $\{\mathcal{B}^{ijk}_{\mathcal{V}_1}, \mathcal{B}^{ijl}_{\mathcal{V}_2}, \mathcal{B}^{ikl}_{\mathcal{V}_3}\} \subset \{\mathcal{B}^{ijk}_{\mathcal{V}_1}, \mathcal{B}^{ijl}_{\mathcal{V}_2}, \mathcal{B}^{ikl}_{\mathcal{V}_3}, \mathcal{B}^{jkl}_{\mathcal{V}_4}\}$, and $v_1 \in \mathcal{V}_1 \cap \mathcal{V}_2 \cap \mathcal{V}_3 \cap \mathcal{V}_4$. Then $v_1 \in \mathcal{V}_1 \cap \mathcal{V}_2 \cap \mathcal{V}_3$, and necessarily the line of color i touching $v_1, \mathcal{B}^{i}_{v_1 v_2}$, is a line in all subgraphs $\mathcal{B}^{ijk}_{\mathcal{V}_1}, \mathcal{B}^{ijl}_{\mathcal{V}_2}$ and $\mathcal{B}^{ikl}_{\mathcal{V}_3}$. Hence $\{v_1, v_2\} \subset \mathcal{V}_1 \cap \mathcal{V}_2 \cap \mathcal{V}_3$, and $\mathcal{B}^{i}_{v_1 v_2} = \{\mathcal{B}^{ijk}_{\mathcal{V}_1}, \mathcal{B}^{ijl}_{\mathcal{V}_2}, \mathcal{B}^{ikl}_{\mathcal{V}_3}\} \subset \Delta$.

Consider now the subsets $\{\mathcal{B}^{ijk}_{\mathcal{V}_1}, \mathcal{B}^{ijl}_{\mathcal{V}_2}\} \subset \{\mathcal{B}^{ijk}_{\mathcal{V}_1}, \mathcal{B}^{ijl}_{\mathcal{V}_2}, \mathcal{B}^{ikl}_{\mathcal{V}_3}, \mathcal{B}^{jkl}_{\mathcal{V}_4}\}, v_1 \in \mathcal{V}_1 \cap \mathcal{V}_2 \cap \mathcal{V}_3 \cap \mathcal{V}_4$. Then $v_1 \in \mathcal{V}_1 \cap \mathcal{V}_2$, and necessarily the entire face $\mathcal{B}^{ij}_{\mathcal{V}}$, with $v_1 \in \mathcal{V}$ is a face in both subgraphs $\mathcal{B}^{ijk}_{\mathcal{V}_1}$ and $\mathcal{B}^{ijl}_{\mathcal{V}_2}$. Hence $\mathcal{V} \subset \mathcal{V}_1 \cap \mathcal{V}_2$ and $\mathcal{B}^{ij}_{\mathcal{V}} = \{\mathcal{B}^{ijk}_{\mathcal{V}_1}, \mathcal{B}^{ijl}_{\mathcal{V}_2}\} \subset \Delta$. The same reasoning applies for the subsets $\{\mathcal{B}^{ijk}_{\mathcal{V}_1}, \mathcal{B}^{ijl}_{\mathcal{V}_2}\} \subset \{\mathcal{B}^{ijk}_{\mathcal{V}_1}, \mathcal{B}^{ijl}_{\mathcal{V}_2}, \mathcal{B}^{ikl}_{\mathcal{V}_3}\}$ with $\{v_1 v_2\} \subset \mathcal{V}_1 \cap \mathcal{V}_2 \cap \mathcal{V}_3$.

The subsets with one element are in the vertex set A hence always in Δ consequently Δ an abstract simplicial complex.

On a more abstract note, the CGFT graph itself is an abstract complex ((34), p. 125). An abstract complex is a partially ordered set Θ, to each of whose elements θ is associated a nonnegative integer $d\theta$ (the dimension of the element θ) such that $\theta_1 < \theta_2 \Rightarrow d\theta_1 < d\theta_2$. The dimension zero elements are the 0-bubbles, $\{\mathcal{B}_v\}$. The dimension one elements are the 1-bubbles seen as sets of vertices, $\mathcal{B}^{i}_{v_1 v_2} = \{\mathcal{B}_{v_1}, \mathcal{B}_{v_2}\}$. The dimension two elements are the 2-bubbles seen as sets of lines (thus sets of sets of vertices) $\mathcal{B}^{ij}_{\mathcal{V}} = \{\mathcal{B}^{i}_{v_1 v_2}, \dots\}$ such that $\{v_1, v_2\}, \dots \subset \mathcal{V}$. The dimension three elements are the 3-bubbles, seen as sets of faces $\mathscr{B}^{ijk}_{\mathscr{V}} = \{\mathscr{B}^{ij}_{\mathscr{V}'}, \dots\}$ such that $\mathscr{V} \in \mathscr{V}'$. The complex Θ is the set of all elements partially ordered by $\in$.

Step 2. One needs just to check the definition. A 3-dimensional simplicial pseudo-manifold is a finite abstract simplicial complex with the following properties:

- it is non-branching: Each 2-dimensional simplex is a face of precisely two 3-dimensional simplices. This is obviously respected as every GFT line connects exactly two distinct GFT vertices in a graph (note that this condition is broken by generic GFT graphs).
- it is strongly connected: Any two 3-dimensional simplices can be joined by a "chain" of 3-dimensional simplices in which each pair of neighboring simplices have a common 2-dimensional simplex. Again trivial: in any connected CGFT graph any path connecting two vertices is dual to a "chain".
- it has dimensional homogeneity: Each simplex is a face of some 3-dimensional simplex. This is easy to check by following the proof that Δ is an abstract simplicial complex. □

The p-bubbles are the building blocks of a combinatorial graph complex and generate an associated homology. Denote the set of p-bubbles by $\mathfrak{B}^p$, and define the p^{th} chain group as the finitely generated group $C_p(\mathscr{G}) = \{\alpha_p\}$,

$$\alpha_p = \sum_{\mathscr{B}^{\mathscr{C}}_{\mathscr{V}} \in \mathfrak{B}^p} c^{\mathscr{C}}_{\mathscr{V}} \mathscr{B}^{\mathscr{C}}_{\mathscr{V}} \quad , c^{\mathscr{C}}_{\mathscr{V}} \in \mathbb{Z}. \tag{16}$$

The chain groups define a homology groups via a boundary operator,

Definition 1 *The p^{th} boundary operator d_p acting on a p-bubble $\mathscr{B}^{\mathscr{C}}_{\mathscr{V}}$ with colors $\mathscr{C} = \{i_1, \dots i_p\}$ is*

- *for $p \geq 2$,*

$$d_p(\mathscr{B}^{\mathscr{C}}_{\mathscr{V}}) = \sum_q (-)^{q+1} \sum_{\substack{\mathscr{B}'^{\mathscr{C}'}_{\mathscr{V}'} \in \mathfrak{B}^{p-1} \\ \mathscr{V}' \subset \mathscr{V} \; \mathscr{C}' = \mathscr{C} \setminus i_q}} \mathscr{B}'^{\mathscr{C}'}_{\mathscr{V}'}, \tag{17}$$

  which associates to a p-bubble the alternating sum of all $(p-1)$-bubbles formed by subsets of its vertices.
- *for $p = 1$, as the lines $\mathscr{B}^i_{v_1 v_2}$ connect a positive vertex (say v_1) to a negative one,*
  say v_2

$$d_1 \mathscr{B}^i_{v_1 v_2} = \mathscr{B}_{v_1} - \mathscr{B}_{v_2} \; . \tag{18}$$

- *for $p = 0, d_0 \mathscr{B}_v = 0$.*

These boundary operators give a well defined homology as

Lemma 2 *The boundary operators respect $d_{p-1} \circ d_p = 0$.*

Proof To check this consider the application of two consecutive boundary operators on a p-bubble

$$d_{p-1} d_p(\mathscr{B}^{\mathscr{C}}_{\mathscr{V}}) = \sum_q (-)^{q+1} \sum_{\substack{\mathscr{B}'^{\mathscr{C}'}_{\mathscr{V}'} \in \mathfrak{B}^{p-1} \\ \mathscr{V}' \subset \mathscr{V}\, \mathscr{C}' = \mathscr{C} \setminus i_q}} d_{p-1} \mathscr{B}'^{\mathscr{C}'}_{\mathscr{V}'} \tag{19}$$

$$= \sum_q (-)^{q+1} \sum_{\substack{\mathscr{B}'^{\mathscr{C}'}_{\mathscr{V}'} \in \mathfrak{B}^{p-1} \\ \mathscr{V}' \subset \mathscr{V}\, \mathscr{C}' = \mathscr{C} \setminus i_q}} \Big[\sum_{r<q} (-)^{r+1} \sum_{\substack{\mathscr{B}''^{\mathscr{C}''}_{\mathscr{V}''} \in \mathfrak{B}^{p-2} \\ \mathscr{V}'' \subset \mathscr{V}'\, \mathscr{C}'' = \mathscr{C}' \setminus i_r}} \mathscr{B}''^{\mathscr{C}''}_{\mathscr{V}''}$$

$$+ \sum_{r>q} (-)^{r} \sum_{\substack{\mathscr{B}''^{\mathscr{C}''}_{\mathscr{V}''} \in \mathfrak{B}^{p-2} \\ \mathscr{V}'' \subset \mathscr{V}'\, \mathscr{C}'' = \mathscr{C}' \setminus i_r}} \mathscr{B}''^{\mathscr{C}''}_{\mathscr{V}''} \Big], \tag{20}$$

where the sign of the second term changes, as i_r is the $r-1^{\text{th}}$ color of $\mathscr{C}' \setminus i_q$ if $q < r$. The two terms cancel if we exchange q and r in the second term. □

The operators d_p provide direct access to the attaching maps of p cells to $p-1$ cells. This allows one to define inductively a cellular complex associated to every CGFT graph. We present below the construction for colored GFT graphs in three dimensions.

Theorem 2 *A colored GFT graph $\mathscr{G}$ with four colors is a three dimensional cellular complex X^3, with cells identified by the p-bubbles.*

Proof Although straightforward this proof is somewhat abstract and technical. We will follow the classical reference (35) and employ the notations therein. The vertices of the graph are associated to 0 dimensional cells (denoted $e^0_{\mathscr{B}_v}$) and the lines are associated to 1 dimensional cells (denoted $e^1_{\mathscr{B}^i_{v_1 v_2}}$) attached to the vertices to form the 0-skeleton X^0 and the 1-skeleton X^1 of our complex (see Example 0.1 in (35)).

Consider now a 2 bubble $\mathscr{B}^{ij}_{\mathscr{V}}$. It is a closed circuit of lines with alternating colors. This circuit is isomorphic with a circle $\mathbb{S}^1$, hence there exists a bijection $\varphi_{\mathscr{B}^{ij}_{\mathscr{V}}} : \mathbb{S}^1 \to X^1$. The two cell associated to the 2-bubble is a topological cone over the circle $e^2_{\mathscr{B}^{ij}_{\mathscr{V}}} = C\mathbb{S}^1 = D^2$, hence a two dimensional disk. We attach the two cells $e^2_{\mathscr{B}^{ij}_{\mathscr{V}}}$ via the maps $\varphi_{\mathscr{B}^{ij}_{\mathscr{V}}}$ and obtain the 2-skeleton of our complex X^2. As a set, X^2 is the disjoint union of the one skeleton and the two cells $X^2 = X^1 \sqcup_{\mathscr{B}^{ij}_{\mathscr{V}}} e^2_{\mathscr{B}^{ij}_{\mathscr{V}}}$. Up to this stage our complex is just the classical CW complexes of ribbon graphs.

A 3-bubble $\mathscr{B}^{ijk}_{\mathscr{V}}$ is itself a colored graph with three colors and by the previous construction has an associated two dimensional cell complex $Y^2_{\mathscr{B}^{ijk}_{\mathscr{V}}}$, with zero, one and two cells denoted $f^0_{\mathscr{B}_v}, f^1_{\mathscr{B}^i_{v_1 v_2}}$ and $f^2_{\mathscr{B}^{ij}_{\mathscr{V}}}$. By Remarks 3 and 4, $Y^2_{\mathscr{B}^{ijk}_{\mathscr{V}}}$ is the cell complex of a link homeomorphic with an oriented closed connected surface of genus $g, M_g(\mathscr{B}^{ijk}_{\mathscr{V}})$.[4] The three cells of our complex are topological cones $e^3_{\mathscr{B}^{ijk}} = CM_g(\mathscr{B}^{ijk}_{\mathscr{V}})$ over the surfaces $M_g(\mathscr{B}^{ijk}_{\mathscr{V}})$.

The cells of $Y^2_{\mathscr{B}^{ijk}_{\mathscr{V}}}$ are in one to one correspondence with some 0, 1 or 2 bubble $\mathscr{B}_v, \mathscr{B}^i_{v_1v_2}, \mathscr{B}^{ij}_{\mathscr{V}}$, that is subgraphs of $\mathscr{B}^{ijk}_{\mathscr{V}}$. But, as $\mathscr{B}^{ijk}_{\mathscr{V}}$ is itself a subgraph $\mathscr{G}, \mathscr{B}_v, \mathscr{B}^i_{v_1v_2}, \mathscr{B}^{ij}_{\mathscr{V}}$ are also subgraphs of $\mathscr{G}$, hence one to one to some 0, 1, and 2 cell $e^0_{\mathscr{B}_v}, e^1_{\mathscr{B}^i_{\mathscr{V}}}, e^2_{\mathscr{B}^{ij}_{\mathscr{V}}}$ in the 2 skeleton of the graph X^2. Each $f^2_{\mathscr{B}^{ij}_{\mathscr{V}}}$ can then be mapped by some $\phi_{f^2_{\mathscr{B}^{ij}_{\mathscr{V}}}}$ onto the corresponding $e^2_{\mathscr{B}^{ij}_{\mathscr{V}}}$. We attach the three cell $\mathscr{B}^{ijk}_{\mathscr{V}}$ to X^2 by the map $\varphi_{\mathscr{B}^{ijk}_{\mathscr{V}}} : M_g(\mathscr{B}^{ijk}_{\mathscr{V}}) \to X^2$ defined as $\varphi_{\mathscr{B}^{ijk}_{\mathscr{V}}}|_{f^2_{\mathscr{B}^{ij}_{\mathscr{V}}}} = \phi_{f^2_{\mathscr{B}^{ij}_{\mathscr{V}}}}$. □

As a last remark note that, as the three cells are cones over arbitrary oriented surfaces, the complex X^3 is not a CW complex. It becomes one if and only if all the two complexes $Y^2_{\mathscr{B}^{ijk}_{\mathscr{V}}}$ are spheres $\mathbb{S}^2$ (as the cone over a spheres $\mathbb{S}^2$ is a three disk $\mathbb{D}^3$). In turn by Remark 1 and 5 this implies that the Euler characteristic $\chi(X^3) = 0$, hence X^3 a three dimensional closed connected manifold.

5 Minimal and Maximal Homology Groups

Before analyzing the Feynman amplitudes of CGFT graphs we will give some generic properties of the bubble homology induced by the boundary operator (17). Detailed examples of homology computations for graphs are presented in Appendix A.

First, by definition d_0 acting on 0-bubbles is zero. Thus, for any graph,

$$\ker(d_0) = \bigoplus_N \mathbb{Z}, \tag{21}$$

where $N = |\mathfrak{B}^0|$ is the number of vertices (0 bubbles) of the graph. Our first result concerns the minimal homology group of a colored graph.

Lemma 3 *For connected closed graphs $H_0 = \mathbb{Z}$.*

Proof The operators d_1 acting on a one chain is

$$d_1\alpha_1 = \sum_{\mathscr{B}^i_{v_1v_2} \in \mathfrak{B}^1} c^i_{v_1v_2} d_1 \mathscr{B}^i_{v_1v_2} \,. \tag{22}$$

The matrix of d_1 is then an incidence matrix of the oriented graph, where a line enters its positive end vertex and exists from its negative end vertex,

$$\Gamma_{\mathscr{B}_v\mathscr{B}^i_{v_1v_2}} = \begin{cases} 1, & \text{if } \mathscr{B}^i_{v_1v_2} \text{ enters } \mathscr{B}_v \\ -1, & \text{if } \mathscr{B}^i_{v_1v_2} \text{ exits } \mathscr{B}_v. \\ 0, & \text{else} \end{cases} \tag{23}$$

[4] That is M_0 is the two dimensional sphere $\mathbb{S}^2, M_1$ is the torus $\mathbb{S}^1 \times \mathbb{S}^1$, etc.

We will compute the $\mathrm{Im}(d_1)$ using a contraction procedure. We start by choosing a line $\mathscr{B}^i_{v_1 v_p}$ connecting the two distinct vertices v_1 and v_p. We collect all terms containing either $\mathscr{B}_{v_1}$ or $\mathscr{B}_{v_p}$ in Eq. (22) to get

$$\begin{aligned} d_1\alpha_1 = & \mathscr{B}_{v_1}\Gamma_{\mathscr{B}_{v_1}\mathscr{B}^i_{v_1v_p}}c^i_{v_1v_p} + \sum_{v_q\neq v_p}\mathscr{B}_{v_1}\Gamma_{\mathscr{B}_{v_1}\mathscr{B}^i_{v_1v_q}}c^i_{v_1v_q} \\ & + \mathscr{B}_{v_p}\Gamma_{\mathscr{B}_{v_p}\mathscr{B}^i_{v_1v_p}}c^i_{v_1v_p} + \sum_{v_r\neq v_1}\mathscr{B}_{v_p}\Gamma_{\mathscr{B}_{v_p}\mathscr{B}^i_{v_pv_r}}c^i_{v_pv_r} \\ & + \sum_{v\neq v_1,v_p}\mathscr{B}_v\Gamma_{\mathscr{B}_v\mathscr{B}^j_{vv'}}c^j_{vv'} \, . \end{aligned} \tag{24}$$

Using $\Gamma_{\mathscr{B}_{v_p}\mathscr{B}^i_{v_1v_p}} = -\Gamma_{\mathscr{B}_{v_1}\mathscr{B}^i_{v_1v_p}} = \pm 1$, we rewrite Eq. (24) as

$$\begin{aligned} d_1\alpha_1 = & (\mathscr{B}_{v_1}-\mathscr{B}_{v_p})\Gamma_{\mathscr{B}_{v_1}\mathscr{B}^i_{v_1v_p}}c^i_{v_1v_p} + \sum_{v_q\neq v_p}(\mathscr{B}_{v_1}-\mathscr{B}_{v_p})\Gamma_{\mathscr{B}_{v_1}\mathscr{B}^i_{v_1v_q}}c^i_{v_1v_q} \\ & + \sum_{v_q\neq v_p}\mathscr{B}_{v_p}\Gamma_{\mathscr{B}_{v_1}\mathscr{B}^i_{v_1v_q}}c^i_{v_1v_q} + \sum_{v_r\neq v_1}\mathscr{B}_{v_p}\Gamma_{\mathscr{B}_{v_p}\mathscr{B}^i_{v_pv_r}}c^i_{v_pv_r} \\ & + \sum_{v\neq v_1,v_p}\mathscr{B}_v\Gamma_{\mathscr{B}_v\mathscr{B}^j_{vv'}}c^j_{vv'} , \end{aligned} \tag{25}$$

and perform the change of basis in $C_0(\mathscr{G})$,

$$\mathscr{B}'_{v_1} = \mathscr{B}_{v_1} - \mathscr{B}_{v_p} , \quad \mathscr{B}'_{v_q} = \mathscr{B}_{v_q} , \; v_q \neq v_1 , \tag{26}$$

under which Equation (24) becomes

$$\begin{aligned} d_1\alpha_1 = & \mathscr{B}'_{v_1}\Gamma_{\mathscr{B}_{v_1}\mathscr{B}^i_{v_1v_p}}c^i_{v_1v_p} + \sum_{v_q\neq v_p}\mathscr{B}'_{v_1}\Gamma_{\mathscr{B}_{v_1}\mathscr{B}^i_{v_1v_q}}c^i_{v_1v_q} \\ & + \sum_{v_q\neq v_p}\mathscr{B}'_{v_p}\Gamma_{\mathscr{B}_{v_1}\mathscr{B}^i_{v_1v_q}}c^i_{v_1v_q} + \sum_{v_r\neq v_1}\mathscr{B}'_{v_p}\Gamma_{\mathscr{B}_{v_p}\mathscr{B}^i_{v_pv_r}}c^i_{v_pv_r} \\ & + \sum_{v\neq v_1,v_p}\mathscr{B}_v\Gamma_{\mathscr{B}_v\mathscr{B}^j_{vv'}}c^j_{vv'} . \end{aligned} \tag{27}$$

The first line of Eq. (27) is the only one involving $\mathscr{B}'_{v_1}$, which is linearly independent from all other $\mathscr{B}'_v$, therefore it spans a direction in $\mathrm{Im}(d_1)$. As $c^i_{v_1v_p} \in \mathbb{Z}$ and $\Gamma_{\mathscr{B}_{v_p}\mathscr{B}^i_{v_1v_p}} = \pm 1$, we have

$$\mathrm{Im}(d_1) = \mathbb{Z} \oplus \dots \tag{28}$$

The second and third lines of Eq. (27) corespond to the incidence matrix of a graph in which all vertices $\mathscr{B}_{v_q}$ who were connected by lines to $\mathscr{B}_{v_1}$ are now connected by lines with $\mathscr{B}_{v_p}$, that is the graph obtained from $\mathscr{G}$ by contracting the line $\mathscr{B}^i_{v_1v_p}$ and gluing the two vertices $\mathscr{B}_{v_1}$ and $\mathscr{B}_{v_p}$ into an unique vertex.

We iterate this contraction procedure for a spanning tree, that is $N-1$ times. Once such a tree is contracted, the final graph has only one vertex (rosette) v and all remaining lines are loop lines. For any remaining loop line, the coefficient in this final sum of $c^j_{vv'}$ will be $\Gamma_{\mathscr{B}_v\mathscr{B}^j_{vv'}} + \Gamma_{\mathscr{B}_{v'}\mathscr{B}^j_{vv'}} = 0$.

Therefore the image of d_1 is precisely

$$\mathrm{Im}(d_1) = \bigoplus_{N-1} \mathbb{Z} \Rightarrow H_0 = \frac{\ker(d_0)}{\mathrm{Im}(d_1)} = \mathbb{Z}, \tag{29}$$

and the kernel of d_1 is

$$\ker(d_1) = \bigoplus_{L-(N-1)} \mathbb{Z}. \tag{30}$$

□

For the maximal homology group of a graph $\mathscr{G}$ a similar result holds. In fact a contraction procedure for a spanning tree in the graph implied that the first homotopy group is $H_0 = \mathbb{Z}$, and a contraction procedure for a spanning tree in the gluing of simplices dual to a CGFT graph will in turn imply that $H_n = \mathbb{Z}$.

Note first that $\mathrm{Im}(d_{n+1}) = 0$. We have the lemma

Lemma 4 *For a connected closed nD graph $\mathscr{G}$,*

$$ker(d_n) = \mathbb{Z}, \quad Im(d_n) = \bigoplus_{|\mathfrak{B}^n|-1} \mathbb{Z}\,. \tag{31}$$

Proof Denote the set of all colors $\mathfrak{C} = \{0, 1, \dots, n\}$ and consider the boundary of an arbitrary n chain

$$\begin{aligned} d_n\Big[\sum_{\mathscr{B}^{\mathscr{C}}_{\mathscr{V}} \in \mathfrak{B}^n} c^{\mathscr{C}}_{\mathscr{V}} \mathscr{B}^{\mathscr{C}}_{\mathscr{V}}\Big] &= \sum_i \sum_{\substack{\mathscr{B}^{\mathscr{C}}_{\mathscr{V}} \in \mathfrak{B}^n \\ \mathscr{C} = \mathfrak{C}\setminus i}} c^{\mathscr{C}}_{\mathscr{V}} d_n \mathscr{B}^{\mathscr{C}}_{\mathscr{V}} \\ &= \sum_i \sum_{\substack{\mathscr{B}^{\mathscr{C}}_{\mathscr{V}} \in \mathfrak{B}^n \\ \mathscr{C} = \mathfrak{C}\setminus i}} c^{\mathscr{C}}_{\mathscr{V}} \sum_j \sum_{\substack{\mathscr{B}'^{\mathscr{C}'}_{\mathscr{V}'} \in \mathfrak{B}^{n-1} \\ \mathscr{C}' = \mathfrak{C}\setminus i \setminus j; \mathscr{V}' \subset \mathscr{V}}} \Big\{ (-)^{j+1} \mathscr{B}'^{\mathscr{C}'}_{\mathscr{V}'}|_{j<i} + (-)^j \mathscr{B}'^{\mathscr{C}'}_{\mathscr{V}'}|_{j>i} \Big\} \\ &= \sum_{i,j;j<i} \sum_{\substack{\mathscr{B}'^{\mathscr{C}'}_{\mathscr{V}'} \in \mathfrak{B}^{n-1} \\ \mathscr{C}' = \mathfrak{C}\setminus i \setminus j}} \mathscr{B}'^{\mathscr{C}'}_{\mathscr{V}'} \Big[(-)^{j+1} c^{\mathscr{C}}_{\mathscr{V}}|_{\substack{\mathscr{V} \supset \mathscr{V}' \\ \mathscr{C} = \mathfrak{C}\setminus i}} + (-)^i c^{\mathscr{C}}_{\mathscr{V}}|_{\substack{\mathscr{V} \supset \mathscr{V}' \\ \mathscr{C} = \mathfrak{C}\setminus j}} \Big]. \end{aligned} \tag{32}$$

Renaming the $c^{\mathscr{C}}_{\mathscr{V}} = (-)^i c^{\mathscr{C}}_{\mathscr{V}}$ if $\mathscr{C} = \mathfrak{C} \setminus i$ and Eq. (32) becomes

$$\sum_{i,j;j<i} \sum_{\substack{\mathscr{B}'^{\mathscr{C}'}_{\mathscr{V}'} \in \mathfrak{B}^{n-1} \\ \mathscr{C}' = \mathfrak{C}\setminus i \setminus j}} \mathscr{B}'^{\mathscr{C}'}_{\mathscr{V}'} (-)^{i+j+1} \Big[c^{\mathscr{C}}_{\mathscr{V}}|_{\substack{\mathscr{V} \supset \mathscr{V}' \\ \mathscr{C} = \mathfrak{C}\setminus i}} - c^{\mathscr{C}}_{\mathscr{V}}|_{\substack{\mathscr{V} \supset \mathscr{V}' \\ \mathscr{C} = \mathfrak{C}\setminus j}} \Big]\,. \tag{33}$$

First, note that Eq. (33) is zero if and only if all $c^{\mathscr{C}}_{\mathscr{V}}$ are equal, hence

$$\ker(d_n) = \mathbb{Z} \Rightarrow H_n = \mathbb{Z}. \tag{34}$$

To determine the image of d_n, consider the gluing Δ dual to the GFT graph $\mathscr{G}$. It has vertices dual to the n-bubbles of $\mathscr{G}$ and edges to the $(n-1)$-bubbles. We orient edges dual to $\mathscr{B}'^{\mathscr{C}'}_{\mathscr{V}'} \in \mathfrak{B}^{n-1}$ with colors $\mathscr{C}' = \mathfrak{C} \setminus i \setminus j$ and $j < i$ from the vertex dual to $\mathscr{B}^{\mathscr{C}}_{\mathscr{V}} \in \mathfrak{B}^n$ with $\mathscr{V}' \subset \mathscr{V}$ and colors $\mathscr{C} = \mathfrak{C} \setminus i$ to the vertex dual to $\mathscr{B}^{\mathscr{C}}_{\mathscr{V}} \in \mathfrak{B}^n$ with $\mathscr{V}' \subset \mathscr{V}$ and colors $\mathscr{C} = \mathfrak{C} \setminus j$. The matrix of the operator d_n is then the transposed of the incidence matrix of the Δ:

$$\Lambda_{\mathscr{B}'^{\mathscr{C}'}_{\mathscr{B}'}\mathscr{B}^{\mathscr{C}}_{\mathscr{V}}} = \begin{cases} 1, & \text{if the edge dual to } \mathscr{B}'^{\mathscr{C}'}_{\mathscr{V}'} \text{ enters the vertex dual to } \mathscr{B}^{\mathscr{C}}_{\mathscr{V}} \\ -1, & \text{if the edge dual to } \mathscr{B}'^{\mathscr{C}'}_{\mathscr{V}'} \text{ exits the vertex dual to } \mathscr{B}^{\mathscr{C}}_{\mathscr{V}}, \\ 0, & \text{else} \end{cases} \quad (35)$$

and redefining $\mathscr{B}'^{\mathscr{C}'}_{\mathscr{V}'} = \mathscr{B}'^{\mathscr{C}'}_{\mathscr{V}'}(-)^{i+j+1}$ Eq. (33) is rewritten:

$$d_n\alpha_n = \sum_{\mathscr{B}'^{\mathscr{C}'}_{\mathscr{V}'}} \mathscr{B}'^{\mathscr{C}'}_{\mathscr{V}'} \Lambda_{\mathscr{B}'^{\mathscr{C}'}_{\mathscr{V}'}\mathscr{B}^{\mathscr{C}}_{\mathscr{V}}} c^{\mathscr{C}}_{\mathscr{V}}. \quad (36)$$

In the dual gluing Δ, the $c^{\mathscr{C}}_{\mathscr{V}}$'s are associated to vertices and $\mathscr{B}'^{\mathscr{C}'}_{\mathscr{V}'}$ to lines. Therefore Eq. (36) has the same form as (24), with the $c^{\mathscr{C}}_{\mathscr{V}}$ and $\mathscr{B}'^{\mathscr{C}'}_{\mathscr{V}'}$ swapped!

Call $\tilde{\mathscr{T}}$ a rooted tree in the dual gluing Δ, and for all dual vertices $\mathscr{B}^{\mathscr{C}}_{\mathscr{V}}$ call $\tilde{\mathscr{B}}'^{\tilde{\mathscr{C}}'}_{\tilde{\mathscr{V}}'}$ the dual tree line touching this dual vertex and going towards the root. Under the change of variables parallel to (27), performed this time on the $c^{\mathscr{C}}_{\mathscr{V}}$, the quadratic form (36) becomes

$$d_n\alpha_n = \sum_{\mathscr{B}^{\mathscr{C}}_{\mathscr{V}} \in \mathfrak{B}^n} c^{\mathscr{C}}_{\mathscr{V}} \Big[\Lambda_{\tilde{\mathscr{B}}'^{\tilde{\mathscr{C}}'}_{\tilde{\mathscr{V}}'}\mathscr{B}^{\mathscr{C}}_{\mathscr{V}}} \tilde{\mathscr{B}}'^{\tilde{\mathscr{C}}'}_{\tilde{\mathscr{V}}'} + \sum_{\mathscr{B}'^{\mathscr{C}'}_{\mathscr{V}'} \notin \tilde{T}} \Lambda_{\mathscr{B}'^{\mathscr{C}'}_{\mathscr{B}'}\mathscr{B}^{\mathscr{C}}_{\mathscr{V}}} \mathscr{B}'^{\mathscr{C}'}_{\mathscr{B}'} \Big]. \quad (37)$$

The vectors in brackets are linearly independent as each of them contains a $\tilde{\mathscr{B}}'^{\tilde{\mathscr{C}}'}_{\tilde{\mathscr{V}}'}$ corresponding to a different tree line in $\tilde{\mathscr{T}}$. As $\Lambda_{\tilde{\mathscr{B}}'^{\tilde{\mathscr{C}}'}_{\tilde{\mathscr{V}}'}\mathscr{B}^{\mathscr{C}}_{\mathscr{V}}} = \pm 1, |\tilde{\mathscr{T}}| = |\mathfrak{B}^n| - 1$ and $c^{\mathscr{C}}_{\mathscr{V}} \in \mathbb{Z}$ we conclude that

$$\mathrm{Im}(d_n) = \bigoplus_{|\mathfrak{B}^n|-1} \mathbb{Z}. \quad (38)$$

□

The other homology groups $H_p, 0 < p < n$ depend of the particular CGFT graph one analyzes. Note that in three dimensions, in order to compute H_1 and H_2 one needs only to determine the kernel and image of the operator d_2. Several examples are presented in Appendix A.

6 Amplitudes and Homotopy

After the study of the homology groups of a GFT graph, the next natural step is to study its homotopy groups. We will only define here a homotopy equivalence for curves which will enable us to define the fundamental group of a GFT graph.

In strict parallel to triangulated polyhedra, we define an edge path as an ordered sequence of vertices $[v_n, \dots v_1]$ connected by lines (that is, $\forall i, \exists \mathscr{B}^l_{v_{i+1}v_i}$). An edge loop is a closed edge path, $v^n = v^1$. The paths

$$[v^{i+k}, v^{i+k-1}, \dots, v^{i+1}, v^i] \text{ and } [v^{i+k} v^{i+k+1}, \dots v^n, v^1, \dots v^{i-1}, v^i], \tag{39}$$

are homotopically equivalent if the set of vertices $v^n, \dots v^1$ span a 2 bubble.

The edge path group of a CGFT graph as follows. Start by associating group elements $g \in G$ to all lines of the graph. For all faces $\mathscr{B}^{ab}_{\mathscr{V}}$ of the graph, the set of vertices $\mathscr{V} = v_n, \dots v_1$ is a closed path. The *relations* defining the fundamental group are associated to the face of the graph and write

$$\mathscr{R}_{\mathscr{B}^{ab}_{\mathscr{V}}} = \prod_{\mathscr{B}^{j}_{v_{i+1}v_i} \in \mathscr{B}^{ab}_{\mathscr{V}}} g_{\mathscr{B}^{j}_{v_{i+1}v_i}}^{\sigma(\mathscr{B}^{i}_{v_{i+1}v_i})} = e, \tag{40}$$

where e is the unit element of G, and $\sigma(\mathscr{B}^{i}_{v_{i+1}v_i})$ is 1 (or -1) if the vertex v_i is positive (or negative). Finally, we set the group elements associated to a spanning tree $\mathscr{T}$ in the graph to e. The fundamental group of the graph is the group of words with generators $g_{\mathscr{B}^{j}_{v_{i+1}v_i}}, \mathscr{B}^{j}_{v_{i+1}v_i} \notin \mathscr{T}$ and relations (40).

On the other hand, as proved in (32) for a more general case, the amplitude of a colored GFT graph is

$$\mathscr{A}_{\mathscr{G}} = \int [dg] \prod_{\mathscr{B}^{ab}_{\mathscr{V}}} \delta(\mathscr{R}_{\mathscr{B}^{ab}_{\mathscr{V}}}) = \int_{g \notin \mathscr{T}} [dg] \prod_{\mathscr{B}^{ab}_{\mathscr{V}}} \delta(\mathscr{R}_{\mathscr{B}^{ab}_{\mathscr{V}}}), \tag{41}$$

for any tree $\mathscr{T} \in \mathscr{G}$. Therefore the Feynman amplitude of a graph is the volume of the relations defining the fundamental group π_1 of the graph over the base GFT group G![5] In this light the main result of (32) can be translated as follows (see (32) for the appropriate definitions): "type 1 graphs dual to manifold space-times are homotopically trivial". It is however difficult to give a complete characterization of homotopically trivial graphs. A first step in this direction is given by Lemma 5 below.

Lemma 5 *If a closed three dimensional graph is homotopically trivial, then all its 3-bubbles are planar.*

Proof If a graph is homotopically trivial then it is homologically trivial $H_1 = 0$. Equation (30) then implies

$$\mathrm{Im}(d_2) = \bigoplus_{L-(N-1)} \mathbb{Z}. \tag{42}$$

As d_2 is defined on $\mathfrak{B}^2$, with $|\mathfrak{B}^2| = F$, and $\ker(d_2) \supset \mathrm{Im}(d_3)$ we conclude that

$$F - [L - (N+1)] \geq B - 1, \tag{43}$$

where $B = |\mathfrak{B}^3|$.

On the other hand, Eq. 4, implies for the dual CGFT graph

$$N - L + F - B = - \sum_{\mathscr{B}^{\mathscr{C}}_{\mathscr{V}} \in \mathfrak{B}^3} g_{\mathscr{B}^{\mathscr{C}}_{\mathscr{V}}}, \tag{44}$$

where $g_{\mathscr{B}^{\mathscr{C}}_{\mathscr{V}}}$ is the genus of the 3-bubble $\mathscr{B}^{\mathscr{C}}_{\mathscr{V}}$. Equations (44) and (43) imply that $g_{\mathscr{B}^{\mathscr{C}}_{\mathscr{V}}} = 0, \forall \mathscr{B}^{\mathscr{C}}_{\mathscr{V}} \in \mathfrak{B}^3$, therefore all the 3-bubbles are planar. □

[5] In mathematical terms the volume of the representation variety $\pi_1 \to G$, (36).

This lemma shows that a homotopically trivial graph is dual to a manifold gluing whose associated complex is a CW complex. The reciprocal of Lemma 5 is not true. The first example in Appendix A is a non-homotopically trivial graph whose bubbles are all planar.

The amplitude of a CGFT graph is governed by the fundamental group π_1. As the first homology group H_1 we introduced in Sect. 4 is just the abelianization of π_1, it is reasonable to suppose that a rough power counting estimate should depend on H_1, whose evaluation for specific examples is much simpler than the one of π_1.

7 Conclusion

We started out work by an indepth study of singularities in the usual GFT models. We concluded that a certain type of topological singularities (baptized wrapping singularities) is generic for such theories always associated to graphs whose dual gluings are not pseudo-manifolds.

We then introduced a new (fermionic) group field theory whose combinatorics and topology is much more controlled. We were able to establish that all its graphs are dual to simplicial pseudo-manifolds and encode a combinatorial cellular complex. Using an appropriate boundary operator we detailed the homology of the CGFT graphs, introduced a homotopy transformation for paths on graphs and related the Feynman amplitude of graphs with the fundamental group.

A large amount of work should now be carried out in several directions. In a more quantum field theoretical approach one should not only continue the preliminary studies on the power counting of GFT's (5; 32) but also look for nontrivial fermionic instanton solutions corresponding to (37) and their possible relation with matter fields.

On the other hand our results could be used as a purely mathematical tool to further the understanding of three dimensional topological spaces. The encoding of the bubble complex in colored graphs provides a bridge between topological and combinatorial notions opening up the possibility to obtain, using the latter, new results on the former.

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A Homology Computations

In this appendix we compute the homology groups for three examples of four colored graphs.

A.1 First example

Consider the four colored graph of Fig. 13. The reader can check that all lines connect a positive and a negative vertex.

Fig. 13 A first example of 3D graph

The 3-bubbles of this graph are $\mathscr{B}^{012}_{v_1\dots v_8}, \mathscr{B}^{013}_{v_1\dots v_8}, \mathscr{B}^{023}_{v_1\dots v_8}$ and $\mathscr{B}^{123}_{v_1\dots v_8}$. The 2-bubbles (faces) are

Fig. 14 A second example of 3D graph

$$
\begin{aligned}
&\mathscr{B}^{01}_{v_1v_2v_8v_7},\quad \mathscr{B}^{01}_{v_3v_4v_6v_5},\quad \mathscr{B}^{02}_{v_1v_7v_3v_5},\quad \mathscr{B}^{02}_{v_4v_6v_2v_8},\\
&\mathscr{B}^{03}_{v_1v_7v_6v_4},\quad \mathscr{B}^{03}_{v_2v_8v_5v_3},\quad \mathscr{B}^{12}_{v_1v_2v_6v_5},\quad \mathscr{B}^{12}_{v_3v_4v_8v_7},\\
&\mathscr{B}^{13}_{v_1v_2v_3v_4},\quad \mathscr{B}^{13}_{v_5v_6v_7v_8},\quad \mathscr{B}^{23}_{v_1v_5v_8v_4},\quad \mathscr{B}^{23}_{v_2v_3v_7v_6}.
\end{aligned}
\tag{45}
$$

The 1-bubbles are

$$
\begin{aligned}
&\mathscr{B}^{0}_{v_1v_7},\ \mathscr{B}^{0}_{v_2v_8},\ \mathscr{B}^{0}_{v_3v_5},\ \mathscr{B}^{0}_{v_4v_6},\quad \mathscr{B}^{1}_{v_1v_2},\ \mathscr{B}^{1}_{v_3v_4},\ \mathscr{B}^{1}_{v_5v_6},\ \mathscr{B}^{1}_{v_7v_8},\\
&\mathscr{B}^{2}_{v_1v_5},\ \mathscr{B}^{2}_{v_2v_6},\ \mathscr{B}^{2}_{v_3v_7},\ \mathscr{B}^{2}_{v_4v_8},\quad \mathscr{B}^{3}_{v_1v_4},\ \mathscr{B}^{3}_{v_2v_3},\ \mathscr{B}^{3}_{v_5v_8},\ \mathscr{B}^{3}_{v_6v_7},
\end{aligned}
\tag{46}
$$

while the 0-bubbles are vertices. We need to analyze the kernel and the image of the operator d_2. Acting on a two chain d_2, we write

$$
\begin{aligned}
d_2\alpha_2 = {}& c^{01}_{1287}[\mathscr{B}^1_{v_1v_2}+\mathscr{B}^1_{v_7v_8}-\mathscr{B}^0_{v_1v_7}-\mathscr{B}^0_{v_2v_8}]+c^{01}_{3465}[\mathscr{B}^1_{v_3v_4}+\mathscr{B}^1_{v_5v_6}-\mathscr{B}^0_{v_3v_5}-\mathscr{B}^0_{v_4v_6}]\\
&+c^{02}_{1735}[\mathscr{B}^2_{v_1v_5}+\mathscr{B}^2_{v_3v_7}-\mathscr{B}^0_{v_1v_7}-\mathscr{B}^0_{v_3v_5}]+c^{02}_{4628}[\mathscr{B}^2_{v_2v_6}+\mathscr{B}^2_{v_4v_8}-\mathscr{B}^0_{v_4v_6}-\mathscr{B}^0_{v_2v_8}]\\
&+c^{03}_{1764}[\mathscr{B}^3_{v_1v_4}+\mathscr{B}^3_{v_6v_7}-\mathscr{B}^0_{v_1v_7}-\mathscr{B}^0_{v_4v_6}]+c^{03}_{2853}[\mathscr{B}^3_{v_2v_3}+\mathscr{B}^3_{v_5v_8}-\mathscr{B}^0_{v_2v_8}-\mathscr{B}^0_{v_3v_5}]\\
&+c^{12}_{1265}[\mathscr{B}^2_{v_1v_5}+\mathscr{B}^2_{v_2v_6}-\mathscr{B}^1_{v_1v_2}-\mathscr{B}^1_{v_5v_6}]+c^{12}_{3487}[\mathscr{B}^2_{v_3v_7}+\mathscr{B}^2_{v_4v_8}-\mathscr{B}^1_{v_3v_4}-\mathscr{B}^1_{v_7v_8}]\\
&+c^{13}_{1234}[\mathscr{B}^3_{v_1v_4}+\mathscr{B}^3_{v_2v_3}-\mathscr{B}^1_{v_1v_2}-\mathscr{B}^1_{v_3v_4}]+c^{13}_{5678}[\mathscr{B}^3_{v_5v_8}+\mathscr{B}^3_{v_6v_7}-\mathscr{B}^1_{v_5v_6}-\mathscr{B}^1_{v_7v_8}]\\
&+c^{23}_{1584}[\mathscr{B}^3_{v_5v_8}+\mathscr{B}^3_{v_1v_4}-\mathscr{B}^2_{v_1v_5}-\mathscr{B}^2_{v_4v_8}]+c^{23}_{2376}[\mathscr{B}^3_{v_2v_3}+\mathscr{B}^3_{v_6v_7}-\mathscr{B}^2_{v_2v_6}-\mathscr{B}^2_{v_3v_7}].
\end{aligned}
\tag{47}
$$

A lengthy but straightforward computation shows that

$$
\mathrm{Im}(d_2)=\bigoplus_8 \mathbb{Z}\oplus 2\mathbb{Z},\quad \ker d_2=\bigoplus_3 \mathbb{Z}. \tag{48}
$$

Using Lemma 3 we conclude that $\ker(d_1)=\bigoplus_{16-7}\mathbb{Z}$ and using Lemma 4 we conclude that $\mathrm{Im}(d_3)=\bigoplus_{3-1}\mathbb{Z}$. Therefore

$$
\begin{aligned}
H_0 &= \mathbb{Z},\\
H_1 &= \frac{\ker(d_1)}{\mathrm{Im}(d_2)}=\frac{\bigoplus_9\mathbb{Z}}{\bigoplus_8\mathbb{Z}\oplus 2\mathbb{Z}}=\mathbb{Z}_2,\\
H_2 &= \frac{\ker(d_2)}{\mathrm{Im}(d_3)}=\frac{\bigoplus_3\mathbb{Z}}{\bigoplus_3\mathbb{Z}}=0,\\
H_3 &= \mathbb{Z}.
\end{aligned}
\tag{49}
$$

Note first that these homology groups match those of $\mathbb{R}P^3$. Second, direct inspection shows that all the bubbles of this graph are planar. Indeed one can prove that the dual gluing is $\mathbb{R}P^3$, but this requires more effort than merely computing homology groups.

A.2 Second example

Consider now the graph of Fig. 14.

The 3-bubbles of this graph are $\mathscr{B}^{012}_{v_1v_2v_6v_5}$, $\mathscr{B}^{012}_{v_3v_4v_8v_7}$, $\mathscr{B}^{123}_{v_1v_5v_8v_4}$, $\mathscr{B}^{123}_{v_2v_3v_7v_6}$, $\mathscr{B}^{013}_{v_1\dots v_8}$ and $\mathscr{B}^{023}_{v_1\dots v_8}$. The 2-bubbles are

$$
\begin{aligned}
&\mathscr{B}^{01}_{v_1v_2v_5v_6},\ \mathscr{B}^{01}_{v_3v_4v_8v_7},\ \mathscr{B}^{02}_{v_1v_2v_5v_6},\ \mathscr{B}^{02}_{v_3v_4v_8v_7},\ \mathscr{B}^{03}_{v_1v_2v_3v_4},\ \mathscr{B}^{03}_{v_5v_6v_7v_8},\\
&\mathscr{B}^{12}_{v_1v_5},\ \mathscr{B}^{12}_{v_2v_6},\ \mathscr{B}^{12}_{v_4v_8},\ \mathscr{B}^{12}_{v_3v_7},\ \mathscr{B}^{13}_{v_1v_5v_8v_4},\ \mathscr{B}^{13}_{v_2v_3v_7v_6},\ \mathscr{B}^{23}_{v_1v_5v_8v_4},\ \mathscr{B}^{23}_{v_2v_3v_7v_6}.
\end{aligned}
\tag{50}
$$

Fig. 15 A third example of a graph

The 1-bubbles are

$$\mathscr{B}^0_{v_1v_2},\ \mathscr{B}^0_{v_5v_6},\ \mathscr{B}^0_{v_3v_4},\ \mathscr{B}^0_{v_7v_8},\ \mathscr{B}^1_{v_1v_5},\ \mathscr{B}^1_{v_2v_6},\ \mathscr{B}^1_{v_3v_7},\ \mathscr{B}^1_{v_4v_8},$$
$$\mathscr{B}^2_{v_1v_5},\ \mathscr{B}^2_{v_2v_6},\ \mathscr{B}^2_{v_3v_7},\ \mathscr{B}^2_{v_4v_8},\ \mathscr{B}^3_{v_1v_4},\ \mathscr{B}^3_{v_2v_3},\ \mathscr{B}^3_{v_5v_8},\ \mathscr{B}^3_{v_6v_7}. \tag{51}$$

Lemmas 3 and 4 give

$$H_0 = \mathbb{Z}, \quad \ker(d_1) = \bigoplus_9 \mathbb{Z}, \qquad H_3 = \mathbb{Z}, \quad \mathrm{Im}(d_3) = \bigoplus_5 \mathbb{Z}. \tag{52}$$

Manipulations similar to the ones for the first example give

$$\mathrm{Im}(d_2) = \bigoplus_9 \mathbb{Z} \to H_1 = 0, \quad \ker(d_2) = \bigoplus_5 \mathbb{Z} \to H_2 = 0. \tag{53}$$

These homology groups are consistent with those of $\mathbb{S}^3$. A direct computation of the Feynman amplitude shows that indeed this graph is homotopically trivial and, unsurprisingly, the dual gluing is indeed $\mathbb{S}^3$.

A.3 Third example

For the final example take the graph of Fig. 15.

The 3-bubbles of this graph are

$$\mathscr{B}^{012}_{v_1v_2v_3v_4v_5v_6},\ \mathscr{B}^{013}_{v_1v_2v_3v_4v_5v_6},\ \mathscr{B}^{023}_{v_1v_2v_3v_4v_5v_6},\ \mathscr{B}^{123}_{v_1v_2v_3v_4v_5v_6}\ . \tag{54}$$

Its 2-bubbles are

$$\mathscr{B}^{01}_{v_1v_2v_3v_4v_5v_6},\ \mathscr{B}^{02}_{v_1v_2v_3v_4v_5v_6},\ \mathscr{B}^{03}_{v_1v_2v_3v_4v_5v_6},\ \mathscr{B}^{12}_{v_1v_2v_3v_4v_5v_6},\ \mathscr{B}^{13}_{v_1v_2v_3v_4v_5v_6},$$
$$\mathscr{B}^{23}_{v_1v_2},\quad \mathscr{B}^{23}_{v_3v_4},\quad \mathscr{B}^{23}_{v_5v_6}, \tag{55}$$

and its 1-bubbles are

$$\mathscr{B}^0_{v_1v_6},\ \mathscr{B}^0_{v_2v_3},\ \mathscr{B}^0_{v_4v_5},\ \mathscr{B}^1_{v_1v_4},\ \mathscr{B}^1_{v_2v_5},\ \mathscr{B}^1_{v_3v_6},$$
$$\mathscr{B}^2_{v_1v_2},\ \mathscr{B}^2_{v_3v_4},\ \mathscr{B}^2_{v_5v_6},\ \mathscr{B}^3_{v_1v_2},\ \mathscr{B}^3_{v_3v_4},\ \mathscr{B}^3_{v_5v_6}. \tag{56}$$

Again from 3 and 4 we get

$$H_0 = \mathbb{Z}, \quad \ker(d_1) = \bigoplus_7 \mathbb{Z}, \qquad H_3 = \mathbb{Z}, \quad \mathrm{Im}(d_3) = \bigoplus_3 \mathbb{Z}, \tag{57}$$

and a direct computation gives

$$\mathrm{Im}(d_2) = \bigoplus_5 \mathbb{Z} \to H_1 = \mathbb{Z} \oplus \mathbb{Z}; \quad \ker(d_2) = \bigoplus_3 \mathbb{Z} \to H_2 = 0. \tag{58}$$

This graph represents a pseudo-manifold with two isolated singularities with links isomorphic with the torus.

B Euler Characteristic of Pseudo-manifolds in Three Dimensions

This result is classical but, for completeness, we present here a proof.

Lemma 6 *If a 3 dimensional gluing of tetrahedra with k vertices is a pseudo-manifold then*

$$\chi(X) = k - \frac{1}{2}\sum_i \chi(lkv_i). \tag{59}$$

Proof A pseudo-manifold X can be transformed into a manifold with boundary, M_X by removing its singular points by surgery. That is, as a set, X admits a decomposition $X = M_X \cup_i C[\partial M_X^{(i)}]$, with M_X a manifold with boundary $\partial M_X = \cup_i \partial M_X^{(i)}$ and $C[\partial M_X^{(i)}]$ is the topological cone over $\partial M_X^{(i)}$.

The first singular point has some neighborhood whose boundary is $\partial M_X^{(1)}$, a closed connected surface, and eliminating it by surgery from X we obtain $M_X^{(1)}$, a pseudo-manifold with one less singular point but with a boundary having a connected component. The initial pseudo-manifold is written $X = M_X^{(1)} \cup C[\partial M_X^{(1)}]$. Also note that $M_X^{(1)} \cap C[\partial M_X^{(1)}] = \partial M_X^{(1)}$, and, as X is locally compact, the Euler characteristic obeys an inclusion-exclusion relation

$$\chi(M_X^{(1)}) + \chi(C[\partial M_X^{(1)}]) = \chi(X) + \chi(\partial M_X^{(1)}). \tag{60}$$

The connected components $\partial M_X^{(i)}$ are two dimensional surfaces. The Euler characteristic of a cone over a surface is always 1. To prove this, let a simplicial complex on $\partial M_X^{(1)}$ with vertices v_i, lines l_j and triangles t_k. It lifts to a simplicial complex of the cone $C[\partial M_X^{(1)}]$ obtained by *coning* all simplices to an unique vertex a. The simplicial complex of the cone has one extra vertex, a itself, one extra edge av_i for each vertex v_i, one triangle al_j for each line l_j and one tetrahedron at_k for each triangle t_k. The Euler characteristic of the cone is written

$$\chi(C[\partial M_X^{(1)}]) = 1 + |v_i| - (|l_j| + |v_i|) + (|t_j| + |l_j|) - |t_j| = 1. \tag{61}$$

Hence

$$\chi(M_X^{(1)}) = \chi(X) - 1 + \chi(\partial M_X^{(1)}), \tag{62}$$

and iterating for all isolated singularities we conclude

$$\chi(M_X) = \chi(X) - C + \chi(\partial M_X), \tag{63}$$

where C is the number of connected components of the boundary of M_X (equal to the number of singular points of X). But for any manifold with boundary M_X in three dimensions, $\chi(M_X) = \frac{1}{2}\chi(\partial M_X)$, hence

$$\chi(X) = C - \frac{1}{2}\chi(\partial M_X). \tag{64}$$

One can continue to eliminate by surgery also regular points of M_X. Applying this procedure to *all* the internal points (singular or not) of a gluing of simplices proves the lemma. □

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