REGGE POLES AS CONSEQUENCES OF ANALYTICITY AND UNITARITY

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Conjectures regarding Regge poles in relativistic scattering amplitudes can be made on at least three levels:

(i) one can assume that the solution of the equations obtained from analyticity and elastic unitarity have the Regge asymptotic behaviour, so that his results may be applied in treating these equations;

(ii) one can assume that the exact scattering amplitude also has such a behaviour and compare its consequences with experiment, or

(iii) one can make the further conjecture of Chew and Frautschi that all particles correspond to Regge poles and none are more " elementary " than others.

I should like to report a proof of the first of these conjectures subject to certain conditions.

Progress in this direction has already been made by Barut and Zwanziger and by Bardakci who showed that, provided $A(s, t)$ did not tend to infinity faster than t for s fixed and negative, it had a Regge asymptotic behaviour with a background term which behaved like t. We have reduced the background term to $t^{-\gamma}(\gamma>0)$, and have also weakened their assumption which, though true in the exact scattering amplitude (the Froissart limit), is not necessarily true in this model. In other words, we prove that a cancellation will occur so that the asymptotic behaviour in t will depend on s, rather than assuming such a cancellation.

We shall assume that the double dispersion relation holds with a finite number of subtractions, and also that there exists an ϵ, greater than 0, such that

$$|A(s, t)-v(t)| < \frac{Mt^{1-\varepsilon}}{|s|} \qquad (1)$$

where $v(t)$ is the generalised potential for all t and $|s|$ sufficiently large. In other words, we assume that there is at most one subtraction in t for sufficiently large s. This condition has a bearing on the analyticity and unitarity equations quite apart from questions of Regge poles, as one observes when one investigates the number of subtractions required in the momentum transfer dispersion relations. If one cuts off the dispersion integrals in s at sufficiently high s, one can prove that only a finite number are necessary. The proof, however, makes *essential* use of the last term under the square-root sign in the kernel.

$$K(s\,;\, t, t', t'') = \left\{ s(s-4\mu^2)\left(t^2+t_1^2+t_2^2-2tt_1-2tt_2- -2t_1t_2-\frac{4tt_1t_2}{s-4\mu^2}\right)\right\}^{-\frac{1}{2}}.$$

Without the last term there would no longer be a finite number of subtractions. When we take the dispersion integral over all s, we must therefore ensure that the contributions from large s are sufficiently small; the condition for this turns out to be just the condition (1). We have not *proved* that a violation of (1) would lead to an infinite number of subtractions, but one would require delicate cancellations at infinite s to give a finite number of subtractions in that case, whereas this is automatic if (1) is satisfied.

It can be shown that (1) is in fact satisfied if the " generalised potential " is sufficiently small.

It is now possible to prove that the scattering amplitude is meromorphic in the l-plane to the right of the line $\mathrm{Re}(l)=\max(-\gamma, -\frac{1}{2})$, where γ is the largest value of ε for which (1) holds. While it is impossible to give the details of the proof in a 10 minute talk, we can outline the principles, which are not difficult. Froissart has pointed out that one can continue the partial waves $A(l, s)$ into the l-plane using the formula

$$A(l, s) = \frac{1}{s-4\mu^2}\int_{t_{\min}}^{\infty} dt A_3(s, t) Q_l\left(1+\frac{2t}{s-4\mu^2}\right), \qquad (2)$$

provided Re(l) is large enough for $A_3(s, t)t^{-l}$ to be bounded at infinity. However, one requires a continuation to smaller values of Re (l) before one can prove the Regge formula. One could define a continuation of $A(l, s)$ to a non-integral l by the formula

$$A'(l, s) = \frac{1}{s-4\mu^2} \int_{-s+4\mu^2}^{0} dt A(s, t) P_l\left(1+\frac{2t}{s-4\mu^2}\right), \quad (3)$$

and this definition has a meaning for all l. Except for integral l it is not the same as (2) and, as $A'(l, s)$ tends to infinity exponentially as $l \to i\infty$, it is useless for proving the Regge formula.

Suppose, however, we consider instead of (2) the function

$$B(l, s) = \frac{1}{s-4\mu^2} \int_{t\,\min}^{\infty} dt A_3(s, t) Q_l\left(-1-\frac{2t}{s-4\mu^2}\right). \quad (4)$$

Since $Q_l(-z) = e^{\pm i\pi l} Q_l(z)$, (4) contains the same information as (2). The function $B(l, s)$ will satisfy a dispersion relation in s, and it is not difficult to see that $f(l, s)$ the discontinuity of $B(l, s)$ across the left-hand cut, is given by

$$f(l, s) = \frac{1}{s-4\mu^2} \int_{t\,\min}^{-s+4\mu^2} dt A_3(s, t) P_l\left(-1-\frac{2t}{s-4\mu^2}\right). \quad (5)$$

We stress that (5) is a simple consequence of (4). The definition (5), however, has a meaning for all values of l. In other words the continuation (3), which is invalid for A, is valid for f, the discontinuity of B across the left hand cut, B being defined by (4).

We can now reconstruct $B(l, s)$ from $f(l, s)$ using the N/D method. Owing to the factor $e^{\pm i\pi l}$ connecting B and A we obtain an integral equation, not for B directly, but for $(s-4\mu^2)^{-l+n}B$, n being any integer. The value of n must be chosen so as to avoid singularities in the integral equation at $s = 4\mu^2$ and $s = \infty$, the condition turns out to be

$$n^{-3/2} < \text{Re } l < n+1 \quad n = 1, 2, 3, \ldots \quad (6)$$

To avoid a singularity at infinity, one has to make use of (1), and one can only avoid such a singularity provided that

$$\text{Re } l > -\gamma \quad (7)$$

As it is known that solutions of a non-singular linear integral equation are meromorphic functions of a parameter provided that the kernel is analytic in that parameter, it follows that $N(l, s)$ and $D(l, s)$ are meromorphic in l for Re$(l) > \max(-\gamma, -\frac{1}{2})$. Some care is required since one uses different integral equations for different ranges of l in order to satisfy (6), but it is not difficult to show that the solutions of these equations are analytic continuations of one another.

Although the functions $N(l, s)$ and $D(l, s)$ will have poles at those values of l for which the kernel of the integral equation has eigenfunctions, $B(l, s)$ is analytic at such values. However, $B(l, s)$ will have poles at the zeros of $D(l, s)$; these are the Regge poles. It has thus been proved that $B(l, s)$ is meromorphic for Re $(l) > \max(-\gamma, -\frac{1}{2})$.

In order to prove the Regge formula two further points must be established, namely that $A(l, s)$ tends to zero as l tends to $i\infty$, so that there will be a finite number of Regge poles, and that the "background term", representing the integral over Re$(l) = \max(-\gamma, -\frac{1}{2})$, has the same asymptotic behaviour in t as its integrand, namely $t^{-\gamma}$ or $t^{-\frac{1}{2}}$. These points can in fact be proved.

There is one essential difference between the present model and the potential model. In the latter case the n^{th}-order perturbation term falls off to zero with infinite s like $s^{-1/2n}$, whereas now all terms fall off to zero like s^{-1}. It follows that $l(\infty)$ the position of the Regge pole in the l-plane at infinite s, is no longer fixed, but depends on the generalised potential. If, as we increased the strength of the potential, Re $l(\infty)$ were ever to reach the value 1, the boundedness condition (1) would no longer be satisfied, the present proof would break down and the double dispersion relation would probably require an infinite number of subtractions. It is unlikely that a bound-state pole would move out to infinity in s. However, the theory may contain a " ghost ", i.e. a pole in partial-wave amplitudes which results from repulsive potentials, which has a negative residue and which moves *in* from infinity as the potential strength is increased. If such a ghost were to reach the S-wave one could neglect it provided it was sufficiently far out and, in any case, if one were using an S-wave subtraction, the question would not arise. Once it reaches the P-wave, however, (1) would be violated and the present proof would break down.

DISCUSSION

DOMOKOS: I would like to make one remark, partly of historical character, that a similar proof of the meromorphic character of the pion-pion scattering amplitude, as given by Professor Mandelstam, has been carried out in Dubna this spring, starting from the N/D equations and having shown that the left hand cut can be represented by the usual formula for non-integral l. However, and that is the question I would like to ask, we just started from the N/D equation for the proof and were able to show if you solve this N/D equation by a normal iteration method then in every step of the iteration you have a meromorphic function. But I just did not get the point of how Prof. Mandelstam makes the next step, that is to say knowing that every step in the iteration is meromorphic in l, then the whole solution itself is meromorphic. I mean, for example, that you may have a situation like this, that in every step of the iteration you have a cumulation of poles. The number of poles increases and they cumulate, for instance, to a cut, or is a behaviour like this impossible?

MANDELSTAM: It is impossible simply because one can use the Fredholm formula that N is equal to, say, N'/D' where N' and D' are not the N and D of the N/D method. Now N' and D' are both uniformly convergent series which involve integrals over the kernel itself, so that each term of both N' and D' is certainly analytic. As a uniformly convergent series of analytic functions is itself analytic, we have a ratio of two analytic functions which must be meromorphic.

PARTIAL WAVES WITH COMPLEX ANGULAR MOMENTA AND THEIR MOVING SINGULARITIES

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1. INTRODUCTION

Problems connected with the asymptotics of high energy scattering and the hypothesis of a dominant role of Regge poles [1] in these asymptotics have been discussed recently by many authors [2, 3, 4, 5, 6]. It appears that at this conference considerable attention will be paid to the same items. Therefore, I shall not dwell in detail on the structure of asymptotic behaviour rising from this hypothesis. I want only to note that while discussing the Regge pole hypothesis, as it seems to me, the fact that this hypothesis leads to a scattering essentially different from the usual diffraction, was not sufficiently emphasized. This scattering is analogous to the scattering on a system with radius and transparency increasing logarithmically with the energy [2]. The nature of this long-range action is not yet understood completely. If the position of Regge poles changes slowly with the energy, the radius of the system will increase slowly, its growth manifesting itself only at super-high energies. Nevertheless, the understanding of the nature of this long-range action is absolutely necessary, e.g. for the elucidation of the applicability of the hypothesis to the scattering on nuclei.

This report deals with the properties of the partial-wave amplitude as a function of angular momentum and energy.

I want to enumerate the results for which a proof will be given below.

1. The Mandelstam representation permits one to introduce uniquely the amplitudes with complex angular momenta $l f_l(t)$ which coincide with the usual partial waves at even and odd l, so that f_l has no singularities in the semiplane $\mathrm{Re}\, l > l_0$, is bounded in this semiplane, and satisfies the generalized unitarity condition.

2. The asymptotics of the high energy scattering is determined by singularities of $f_l(t)$ in the region $\mathrm{Re}\, l < l_0$.