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The $K_{\ell 3}$ scalar form factors in the standard model

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Abstract We discuss the predictions of the standard model for the scalar form factors of $K_{\ell 3}$ decays. Our analysis is based on the results of chiral perturbation theory, large N_c estimates of low-energy couplings and dispersive methods. It includes a discussion of isospin-violating effects of strong and electromagnetic origin.

1 Introduction

Recent high-statistics measurements of the $K_{\ell 3}$ form factor parameters λ'_+, λ''_+ and λ_0 are available from ISTRA+ [1], KTeV [2], NA48 [3] and KLOE [4]. Only ISTRA+ has analyzed the charged kaon decay $K^- \to \pi^0 \mu^- \nu$, whereas the results of the other three experiments are based on K_L decays. In particular for the scalar slope, the NA48 results are difficult to accommodate with those of the other experiments and the actual value of this quantity is not yet finally settled. A global analysis of the present experimental situation can be found in a recent compilation of the FLAVIAnet Kaon Working Group [5].

For a comparison with the experimental outcomes, we need a theoretical prediction, as precise as possible, for the behavior of the scalar form factors of $K^0_{\ell 3}$ and $K^+_{\ell 3}$ decays. We wish to address the following questions.

- Which of the values of λ_0 found by the different experimental groups are compatible with the standard model of particle physics?
- Which magnitude of isospin violation can be expected for the scalar form factors?

To this end we collect and extend the present theoretical information on the scalar $K^0_{\ell 3}$ and $K^+_{\ell 3}$ form factors. The principal theoretical tool of such an analysis is chiral perturbation theory [6–9], the low-energy effective theory of the standard model. It exploits the special rôle of the pseudoscalars π, η and K as Goldstone particles of spontaneous chiral symmetry breaking. This effective quantum field theory can be extended by the inclusion of photons [10–12] and leptons [13] as active degrees of freedom. As a complementary method, also dispersion techniques have been employed [14–19].

The outline of this paper is as follows. In Sect. 2 we recapitulate the basic definitions of the $K_{\ell 3}$ form factors, their current parametrizations and the Callan–Treiman theorem [20, 21]. We determine the quantity $F_K/F_\pi f_+^{K^0\pi^-}(0)$ (which is one of the basic input parameters of our analysis) by combining the relevant theoretical expressions with recent experimental data. This section is closed with a short review of the present theoretical and experimental status of the scalar slope parameter.

In Sect. 3 we display the formulae for the $K_{\ell 3}$ form factors at next-to-leading order (NLO) in the chiral expansion including strong isospin violation end electromagnetic contributions of the orders $(m_d - m_u)p^2$ and $e^2 p^2$ given in [22]. For later convenience, we also express the f_- form factors through the ratio F_K/F_π [13] instead of the low-energy coupling L_5^r. The corresponding expressions for the scalar form factors and slope parameters are shown in Sect. 4 and those relevant for the Callan–Treiman relation in Sect. 5. In Sect. 6 we update the numerical value of the parameter $\varepsilon^{(2)}$, which determines the size of strong isospin breaking. Our numerical results at NLO are then presented in Sect. 7. They show, for the first time, a quantitative comparison of strong isospin-breaking and electromagnetic contributions generated at this chiral order.

In the next step, we consider effects arising at NNLO. Our discussion in Sect. 8 starts with an analysis in the limit of isospin conservation. We combine the two-loop results obtained in [23] with a determination of the associated p^6 counterterms. The relevant low-energy couplings C_{12}^r and C_{34}^r can be estimated by using the $1/N_c$ expansion and truncating the hadronic spectrum to the lowest lying resonances [24]. We show that the uncertainty in the determi-

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nation of these coupling constants given in [24] can be reduced considerably by a careful consideration of the two-loop renormalization group equation [25] and by employing the analysis of [26] for the determination of the scalar and pseudoscalar resonance parameters. In this way, we obtain rather accurate theoretical predictions for the slope and curvature parameters in the isospin limit. The analogous analysis is made for the size of the scalar form factor at the Callan–Treiman point and at zero momentum transfer. In Sect. 9 we compare our findings for the slope and the curvature of the scalar form factor with the results obtained by dispersive methods [14–19].

In Sect. 10 we extend the results obtained at the order $(m_d - m_u)p^4$ [27] on the $K_{\ell 3}$ scalar form factors by an estimate of the associated local contributions relevant for the splitting $\lambda_0^{K^0\pi^+} - \lambda_0^{K^+\pi^0}$. Finally, we discuss the possible size of corrections to the Callan–Treiman relation induced by isospin violation at this chiral order.

Our conclusions are summarized in Sect. 11.

2 Basic facts

The $K_{\ell 3}$ decays

$$K^+(p_K) \to \pi^0(p_\pi)\ell^+(p_\ell)\nu_\ell(p_\nu), \tag{2.1}$$

$$K^0(p_K) \to \pi^-(p_\pi)\ell^+(p_\ell)\nu_\ell(p_\nu) \tag{2.2}$$

(and their charge conjugate modes) allow for the exploration of the hadronic matrix elements

$$\begin{aligned}&\langle \pi^0(p_\pi)\big|V_\mu^{4-i5}(0)\big|K^+(p_K)\rangle \\ &\quad= \frac{1}{\sqrt{2}}\big[f_+^{K^+\pi^0}(t)(p_K+p_\pi)_\mu + f_-^{K^+\pi^0}(t)(p_K-p_\pi)_\mu\big],\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}&\langle \pi^-(p_\pi)\big|V_\mu^{4-i5}(0)\big|K^0(p_K)\rangle \\ &\quad= f_+^{K^0\pi^-}(t)(p_K+p_\pi)_\mu + f_-^{K^0\pi^-}(t)(p_K-p_\pi)_\mu.\end{aligned} \tag{2.4}$$

The processes (2.1) and (2.2) thus involve the four $K_{\ell 3}$ form factors $f_\pm^{K^+\pi^0}(t)$, $f_\pm^{K^0\pi^-}(t)$, which depend on

$$t = (p_K - p_\pi)^2 = (p_\ell + p_\nu)^2, \tag{2.5}$$

the square of the four momentum transfer to the leptons. Only $K_{\mu 3}$ decays are sensitive to both $f_+^{K\pi}$ and $f_-^{K\pi}$. With K_{e3} decays only $f_+^{K\pi}$ can be tested, as in this case $f_-^{K\pi}$ enters together with the tiny quantity $m_e^2/M_K^2 \simeq 10^{-6}$ in the formula for the Dalitz plot density.

$f_+^{K\pi}$ is referred to as the vector form factor, because it specifies the P-wave projection of the crossed channel matrix elements $\langle 0|V_\mu^{4-i5}(0)|K\pi\rangle$. The S-wave projection is described by the scalar form factor

$$f_0^{K\pi}(t) = f_+^{K\pi}(t) + \frac{t}{M_K^2 - M_\pi^2} f_-^{K\pi}(t), \tag{2.6}$$

which implies

$$f_0^{K\pi}(0) = f_+^{K\pi}(0). \tag{2.7}$$

Older analyses of $K_{\ell 3}$ data usually used the linear parametrization

$$f_{+,0}^{K\pi}(t) = f_+^{K\pi}(0)\left(1 + \lambda_{+,0}\frac{t}{M_{\pi^+}^2}\right). \tag{2.8}$$

More recent high-statistics experiments also search for a quadratic term in the form factor expansion of $f_+^{K\pi}(t)$,

$$f_+^{K\pi}(t) = f_+^{K\pi}(0)\left[1 + \lambda'_+\frac{t}{M_{\pi^+}^2} + \frac{1}{2}\lambda''_+\left(\frac{t}{M_{\pi^+}^2}\right)^2\right]. \tag{2.9}$$

Alternatively, also a pole fit,

$$f_+^{K\pi}(t) = f_+^{K\pi}(0)\frac{M_V^2}{M_V^2 - t}, \tag{2.10}$$

$$f_0^{K\pi}(t) = f_+^{K\pi}(0)\frac{M_S^2}{M_S^2 - t}, \tag{2.11}$$

has been employed. Recently, a dispersive representation of the scalar form factor based on a twice subtracted dispersion relation was proposed in [17–19]:

$$\begin{aligned}&\frac{f_0^{K\pi}(t)}{f_+^{K\pi}(0)} = \exp\left[\frac{t}{\Delta_{K\pi}}\big(\ln C - G(t)\big)\right], \\ &G(t) = \frac{\Delta_{K\pi}(\Delta_{K\pi}-t)}{\pi}\int_{t_{K\pi}}^\infty \frac{ds}{s}\frac{\phi(s)}{(s-\Delta_{K\pi})(s-t-i\epsilon)}.\end{aligned} \tag{2.12}$$

The quantity $t_{K\pi}$ denotes the threshold of $K\pi$ scattering and $\Delta_{K\pi} = M_K^2 - M_\pi^2$.

From the theoretical point of view, the scalar $K_{\ell 3}$ form factor has a remarkable property: the low-energy theorem of Callan and Treiman [20, 21] predicts the size of $f_0^{K\pi}(t)$ at the (unphysical) momentum transfer $t = \Delta_{K\pi}$ to be

$$f_0^{K\pi}(\Delta_{K\pi}) = \frac{F_K}{F_\pi} + \Delta_{\rm CT}, \tag{2.13}$$

with a correction term $\Delta_{\rm CT}$ of $\mathcal{O}(m_u, m_d, e^2)$. In the isospin limit ($m_u = m_d$, $e = 0$), and at first non-leading order, the tiny value $\Delta_{\rm CT} = -3.5 \times 10^{-3}$ was worked out already some time ago [28].

Assuming for a moment a strict linear behavior of the scalar form factor in the range between $t=0$ and the Callan–Treiman point $t=\Delta_{K\pi}$, the slope parameter would be given by

$$\lambda_0 \simeq \frac{M_{\pi^+}^2}{\Delta_{K\pi}}\left(\frac{F_K}{F_\pi f_+^{K\pi}(0)} - 1\right) \tag{2.14}$$

as a consequence of (2.13). The ratio $F_K/F_\pi f_+^{K\pi}(0)$ appearing in (2.14) can be determined with remarkable precision from the experimental input, independent of $K_{\mu 3}$ data.

Because of its central importance for our subsequent analysis, we describe here the determination of

$$\frac{F_K}{F_\pi f_+^{K^0\pi^-}(0)} \tag{2.15}$$

in some detail. We want to point out that the decay constants used here always refer to the respective charged pseudoscalars ($F_\pi \equiv F_{\pi^0}$, $F_K \equiv F_{K^+}$). In the case of the pion, the distinction between charged and neutral decay constant amounts to a tiny effect of order $(m_d - m_u)^2$, whereas F_{K^+} differs from F_{K^0} by terms of order $m_d - m_u$ [9].

Including electromagnetic corrections [13, 29], the ratio of the (fully inclusive) $K_{\ell 2(\gamma)}$ and $\pi_{\ell 2(\gamma)}$ widths can be written as

$$\frac{\Gamma(K_{\ell 2(\gamma)})}{\Gamma(\pi_{\ell 2(\gamma)})} = \frac{|V_{us}|^2 F_K^2 M_{K^\pm}(1-z_{K\ell})^2}{|V_{ud}|^2 F_\pi^2 M_{\pi^\pm}(1-z_{\pi\ell})^2} \times \left\{1 + \frac{\alpha}{4\pi}\left[H(z_{K\ell}) - H(z_{\pi\ell}) + (3-Z)\ln\frac{M_K^2}{M_\pi^2} + \cdots\right]\right\}, \tag{2.16}$$

where $z_{P\ell} = m_\ell^2/M_P^2$. An explicit expression for the function $H(z)$ can be found in [13]. The chiral coupling [13] $Z \simeq 0.8$ arises from the electromagnetic mass difference of the pion,

$$M_{\pi^\pm}^2 - M_{\pi^0}^2 = 2e^2 Z F_0^2, \tag{2.17}$$

where F_0 denotes the pion decay constant in the chiral limit. The dots in (2.16) refer to contributions arising at $\mathcal{O}(e^2p^4)$. Inserting the measured widths [30]

$$\Gamma(K_{\mu 2(\gamma)}) = 0.5122(15) \times 10^8\ \mathrm{s}^{-1}, \tag{2.18}$$

$$\Gamma(\pi_{\mu 2(\gamma)}) = 0.38408(7) \times 10^8\ \mathrm{s}^{-1}, \tag{2.19}$$

we find

$$\frac{|V_{us}|F_K}{|V_{ud}|F_\pi} = 0.27567(40)(2)(29) = 0.27567(50). \tag{2.20}$$

The first two separated errors correspond to the experimental uncertainties of the $K_{\mu 2(\gamma)}$ and $\pi_{\mu 2(\gamma)}$ width, respectively. The third one is an estimate[1] of the unknown electromagnetic contributions of $\mathcal{O}(e^2p^4)$. Using (2.20), the quantity (2.15) we are interested in, can now be written as

$$\frac{F_K}{F_\pi f_+^{K^0\pi^-}(0)} = 0.27567(50) \times \frac{|V_{ud}|}{|V_{us}| f_+^{K^0\pi^-}(0)}. \tag{2.21}$$

For the determination of the product $|V_{us}| f_+^{K^0\pi^-}(0)$, we employ the frequently used formula [30]

$$\Gamma(K_{\ell 3(\gamma)}) = \frac{G_F^2 M_K^5}{192\pi^3} C_K^2 |V_{us}|^2 f_+^{K^0\pi^-}(0)^2 I_K^\ell \times S_{\mathrm{EW}}\left(1 + \delta_K^\ell + \delta_{\mathrm{SU}(2)}\right). \tag{2.22}$$

In order to avoid any bias from K_{e3}^+ (which would require additional theoretical input for the determination of $\delta_{\mathrm{SU}(2)}$) or $K_{\mu 3}$ data (involving also information about λ_0, the quantity we actually want to determine), we are exclusively using input from K_{Le3}^0 decays [2, 33–37] as given in [30]:

$$\Gamma\left(K_{Le3(\gamma)}^0\right) = 0.0792(4) \times 10^8\ \mathrm{s}^{-1}, \tag{2.23}$$

$$\lambda'_+ = 0.0249(13), \qquad \lambda''_+ = 0.0016(5), \qquad \rho_{\lambda',\lambda''} \simeq -0.95. \tag{2.24}$$

Taking into account the recently determined values [38] of the electromagnetic low-energy couplings X_i [13], we obtain

$$\delta_{K^0}^e = 0.0114(30) \tag{2.25}$$

as an update of our electromagnetic corrections presented in [39]. Putting everything together, we find

$$|V_{us}| f_+^{K^0\pi^-}(0) = 0.21616(68). \tag{2.26}$$

With [40]

$$|V_{ud}| = 0.97418(26), \tag{2.27}$$

extracted from superallowed nuclear Fermi transitions, we finally obtain[2]

$$\frac{F_K}{F_\pi f_+^{K^0\pi^-}(0)} = 1.2424(23)(39)(3) = 1.2424(45), \tag{2.28}$$

where the first error comes from (2.20), the second one from (2.26) and the third one from (2.27).

[1] See also [31, 32] for a recent calculation of $\mathcal{O}(e^2p^4)$ contributions to the ratio $R_{e/\mu}^{\pi,K}$.

[2] The small difference between our value and the one obtained in [16] within a similar approach is due to the slightly different input parameters.

Table 1 Experimental results for $\lambda_0^{K\pi} \times 10^3$

ISTRA+ ($K^+_{\mu 3}$)	KTeV ($K_{L\mu 3}$)	KTeV ($K_{L\mu 3}+K_{Le3}$)	NA48 ($K_{L\mu 3}$)	KLOE ($K_{L\mu 3}$)	KLOE ($K_{L\mu 3}+K_{Le3}$)
17.1 ± 2.2	12.8 ± 1.8	13.7 ± 1.3	9.5 ± 1.4	9.1 ± 6.5	15.4 ± 2.2

Inserting (2.28) in (2.14) gives $\lambda_0 \simeq 0.02$ as a rough estimate. In reality, as is also suggested by the pole parametrization (2.11), the second derivative of the scalar form factor is positive in the physical region and

$$\lambda_0^{K\pi} := \frac{M^2_{\pi^+}}{f_+^{K\pi}(0)} \frac{\mathrm{d} f_0^{K\pi}(t)}{\mathrm{d}t}\bigg|_{t=0} \tag{2.29}$$

becomes smaller than the value (2.14) obtained in the linear approximation. Indeed, an analysis performed in the isospin limit and at first non-leading chiral order gave [28] $\lambda_0^{K\pi} = 0.017(4)$. In the meantime, the chiral perturbation series of $f_0^{K\pi}(t)$ has been pushed forward to the two-loop level [23, 27, 41]. Combined with an estimate of the relevant $\mathcal{O}(p^6)$ low-energy couplings, the prediction [24] $\lambda_0^{K\pi} = 0.013(3)$ was obtained.

A further reduction of the size of the slope parameter (compared to the next-to-leading order result) is also supported by approaches using dispersive methods. Based on a detailed study of strangeness-changing scalar form factors by Oller, Jamin and Pich [14], they found [15] $\lambda_0^{K\pi} = 0.0157(10)$ and [16] $\lambda_0^{K\pi} = 0.0147(4)$, respectively. A similar result, $\lambda_0^{K\pi} = 0.01523(46) + 0.069\Delta_{\mathrm{CT}}$, based on the dispersive representation (2.12) was given in [18].

The present experimental situation is displayed in Table 1. Note that the numbers[3] shown here are those for which a quadratic parametrization (2.9) has been used for the simultaneous determination of the vector form factor $f_+^{K\pi}(t)$.

It is difficult to accommodate the result of NA48 with those of the other experiments.[4] It is also hard to see [42] how the outcome of NA48 could be reconciled with the results obtained in the dispersive analysis. Furthermore, if the numbers given by ISTRA+ (obtained from K^+ decays) and NA48 (extracted from K_L decays) were both true, this would signal an enormous isospin violation in the scalar form factors of $K_{\ell 3}$.

It is also an instructive exercise to estimate the scalar mass M_S of the pole parametrization (2.11) by imposing the Callan–Treiman relation (2.13). Inserting $f_0^{K\pi}(\Delta_{K\pi}) \simeq F_K/F_\pi$ in (2.11), one finds

$$M_S^2 \simeq \frac{\Delta_{K\pi}}{1 - F_\pi f_+^{K\pi}(0)/F_K} \simeq (1082\ \mathrm{MeV})^2, \tag{2.30}$$

this being in reasonable agreement with the result found by KTeV [2], $M_S = 1167(42)\ \mathrm{MeV/c^2}$, but rather far away from the value [3] $M_S = 1400(70)\ \mathrm{MeV/c^2}$ given by NA48.

3 $K_{\ell 3}$ form factors

In the following, we use the notation introduced in [22]. In particular,

$$\widetilde{f}_\pm^{K^+\pi^0}, \qquad \widetilde{f}_\pm^{K^0\pi^-} \tag{3.1}$$

denote the pure QCD contributions (in principle at any order in the chiral expansion) plus the electromagnetic contributions to the meson masses and π^0–η mixing. In fact,

$$f_+^{K^0\pi^-}(0) \to \widetilde{f}_+^{K^0\pi^-}(0) \tag{3.2}$$

has to be inserted in the master formula (2.22) for the analysis of $K_{\ell 3}$ decays. We also recall the (lowest-order) expressions for the pseudoscalar masses

$$\begin{aligned}
M^2_{\pi^\pm} &= 2B_0\widehat{m} + 2e^2 Z F_0{}^2, \\
M^2_{\pi^0} &= 2B_0\widehat{m}, \\
M^2_{K^\pm} &= B_0\left[(m_s+\widehat{m}) - \frac{2\varepsilon^{(2)}}{\sqrt{3}}(m_s-\widehat{m})\right] + 2e^2 Z F_0{}^2, \\
M^2_{\overset{(-)}{K}{}^0} &= B_0\left[(m_s+\widehat{m}) + \frac{2\varepsilon^{(2)}}{\sqrt{3}}(m_s-\widehat{m})\right], \\
M^2_\eta &= \frac{4}{3}B_0\left(m_s + \frac{\widehat{m}}{2}\right).
\end{aligned} \tag{3.3}$$

The mixing angle $\varepsilon^{(2)}$ is given by

$$\varepsilon^{(2)} = \frac{\sqrt{3}}{4}\frac{m_d - m_u}{m_s - \widehat{m}}, \tag{3.4}$$

the symbol $\widehat{m}$ stands for the mean value of the light quark masses,

$$\widehat{m} = \frac{1}{2}(m_u + m_d), \tag{3.5}$$

and B_0 is related to the vacuum condensate. Finally, M_π and M_K denote the isospin limits ($m_u = m_d$, $e = 0$) of the pion mass and the kaon mass, respectively:

$$M_\pi^2 = 2B_0\widehat{m}, \qquad M_K^2 = B_0(m_s + \widehat{m}). \tag{3.6}$$

[3] The ISTRA+ result [1] has been rescaled by $M^2_{\pi^+}/M^2_{\pi^0}$.

[4] See also the critical review of the present data in [5].

Whenever isospin-breaking effects are taken into account, the distinction of quantities like

$$\widetilde{f}_+^{K^+\pi^0} \neq \widetilde{f}_+^{K^0\pi^-}, \qquad \widetilde{f}_-^{K^+\pi^0} \neq \widetilde{f}_-^{K^0\pi^-},$$
$$M_\pi^\pm \neq M_{\pi^0} = M_\pi, \qquad M_{K^\pm} \neq M_{K^0} \neq M_K,$$

etc. has to be observed with meticulous care.

To order p^4, $(m_d - m_u)p^2$, e^2p^2, the f_+ form factors are given by [22, 28]

$$\begin{aligned}\widetilde{f}_+^{K^+\pi^0}(t) &= 1 + \sqrt{3}\big(\varepsilon^{(2)} + \varepsilon_S^{(4)} + \varepsilon_{EM}^{(4)}\big) \\ &\quad + \frac{1}{2}H_{K^+\pi^0}(t) + \frac{3}{2}H_{K^+\eta}(t) + H_{K^0\pi^-}(t) \\ &\quad + \sqrt{3}\varepsilon^{(2)}\left[\frac{5}{2}H_{K\pi}(t) + \frac{1}{2}H_{K\eta}(t)\right] \end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\widetilde{f}_+^{K^0\pi^-}(t) &= 1 + \frac{1}{2}H_{K^+\pi^0}(t) + \frac{3}{2}H_{K^+\eta}(t) + H_{K^0\pi^-}(t) \\ &\quad + \sqrt{3}\varepsilon^{(2)}\big[H_{K\pi}(t) - H_{K\eta}(t)\big]. \end{aligned} \tag{3.8}$$

The loop functions $H_{PQ}(t)$ can be found in [9, 28]. The quantity $\varepsilon_S^{(4)}$ is the strong contribution to the π^0–η mixing angle arising at first non-leading order [28] and $\varepsilon_{EM}^{(4)}$ is the corresponding term generated at $\mathcal{O}(e^2p^2)$ [11]. The explicit expressions for $\varepsilon_S^{(4)}$ and $\varepsilon_{EM}^{(4)}$ can be found in [22]. We note just in passing that (3.7) and (3.8) imply the relation

$$\widetilde{f}_+^{K^+\pi^0}(0) = \widetilde{f}_+^{K^0\pi^-}(0)\big[1 + \sqrt{3}\big(\varepsilon^{(2)} + \varepsilon_S^{(4)} + \varepsilon_{EM}^{(4)}\big)\big], \tag{3.9}$$

which defines

$$\delta_{SU(2)} = \begin{cases} 0 & \text{for } K_{\ell 3}^0, \\ 2\sqrt{3}(\varepsilon^{(2)} + \varepsilon_S^{(4)} + \varepsilon_{EM}^{(4)}) & \text{for } K_{\ell 3}^+ \end{cases} \tag{3.10}$$

to the order $(m_d - m_u)p^2$, e^2p^2.

The analogous expressions for the f_- form factors are given by [22]

$$\begin{aligned}\widetilde{f}_-^{K^+\pi^0}(t) &= \frac{4\Delta_{K\pi}}{F_0^2}\left(1 + \frac{\varepsilon^{(2)}}{\sqrt{3}}\right)\left[L_5^r(\mu) - \frac{3}{256\pi^2}\ln\frac{M_{K^\pm}^2}{\mu^2}\right] \\ &\quad - \frac{1}{128\pi^2F_0^2}\bigg[\big(3 + \sqrt{3}\varepsilon^{(2)}\big)M_\eta^2 \ln\frac{M_\eta^2}{M_{K^\pm}^2} \\ &\quad + 2\big(3 - \sqrt{3}\varepsilon^{(2)}\big)M_{K^0}^2\ln\frac{M_{K^0}^2}{M_{K^\pm}^2} \\ &\quad - 2\big(3 - \sqrt{3}\varepsilon^{(2)}\big)M_{\pi^\pm}^2\ln\frac{M_{\pi^\pm}^2}{M_{K^\pm}^2} \\ &\quad + \big(1 + 3\sqrt{3}\varepsilon^{(2)}\big)M_{\pi^0}^2\ln\frac{M_{\pi^0}^2}{M_{K^\pm}^2}\bigg] \\ &\quad + \sum_{PQ}\left\{\left[a_{PQ}(t) + \frac{\Delta_{PQ}}{2t}b_{PQ}\right]K_{PQ}(t) \right. \\ &\quad \left. + b_{PQ}\frac{F_0^2}{t}H_{PQ}(t)\right\}, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\widetilde{f}_-^{K^0\pi^-}(t) &= \frac{4\Delta_{K\pi}}{F_0^2}\left(1 + \frac{2\varepsilon^{(2)}}{\sqrt{3}}\right)\left[L_5^r(\mu) - \frac{3}{256\pi^2}\ln\frac{M_{\pi^\pm}^2}{\mu^2}\right] \\ &\quad - \frac{1}{128\pi^2F_0^2}\bigg[2M_{K^0}^2\ln\frac{M_{K^0}^2}{M_{\pi^\pm}^2} \\ &\quad + \big(3 + 2\sqrt{3}\varepsilon^{(2)}\big)M_\eta^2\ln\frac{M_\eta^2}{M_{\pi^\pm}^2} \\ &\quad - \big(3 + 2\sqrt{3}\varepsilon^{(2)}\big)M_{\pi^0}^2\ln\frac{M_{\pi^0}^2}{M_{\pi^\pm}^2}\bigg] \\ &\quad + \sum_{PQ}\left\{\left[c_{PQ}(t) + \frac{\Delta_{PQ}}{2t}d_{PQ}\right]K_{PQ}(t) \right. \\ &\quad \left. + d_{PQ}\frac{F_0^2}{t}H_{PQ}(t)\right\}. \end{aligned} \tag{3.12}$$

The sum runs over the meson pairs $K^+\pi^0$, $K^0\pi^+$ and $K^+\eta$ occurring in loop diagrams. The function $K_{PQ}(t)$ is given by

$$K_{PQ}(t) = \frac{\Delta_{PQ}}{2t}\bar{J}_{PQ}(t), \tag{3.13}$$

with the short-hand notation $\Delta_{PQ} = M_P^2 - M_Q^2$ and the loop function $\bar{J}_{PQ}(t)$ defined in [9]. The coefficients $a_{PQ}(t)$, b_{PQ} $c_{PQ}(t)$ and d_{PQ} are listed in the Appendix.

In the next step we are trading the low-energy constant $L_5^r(\mu)$ appearing in the expressions for the f_- form factors for the ratio F_K/F_π. To this end we employ the relation [13]

$$\begin{aligned}\frac{F_K}{F_\pi} &= 1 + \frac{4\Delta_{K\pi}}{F_0^2}L_5^r(\mu)\left(1 - \frac{2\varepsilon^{(2)}}{\sqrt{3}}\right) \\ &\quad - \frac{1}{8(4\pi)^2F_0^2}\bigg[3M_\eta^2\ln\frac{M_\eta^2}{\mu^2} + 2M_K^2\ln\frac{M_K^2}{\mu^2} \\ &\quad - 5M_\pi^2\ln\frac{M_\pi^2}{\mu^2}\bigg] \\ &\quad + \frac{\sqrt{3}\varepsilon^{(2)}}{4(4\pi)^2F_0^2}\bigg[M_\eta^2\ln\frac{M_\eta^2}{\mu^2} - M_\pi^2\ln\frac{M_\pi^2}{\mu^2} \\ &\quad + \frac{2}{3}\Delta_{K\pi}\left(\ln\frac{M_K^2}{\mu^2} + 1\right)\bigg] \end{aligned} \tag{3.14}$$

in (3.11) and (3.12). In this way we find

$$\begin{aligned}\widetilde{f}_-^{K^+\pi^0}(t) &= \left(\frac{F_K}{F_\pi}-1\right)\left(1+\sqrt{3}\varepsilon^{(2)}\right)\\ &\quad -\frac{\sqrt{3}\varepsilon^{(2)}}{(4\pi F_0)^2}\left(\Delta_{K\pi}-M_\pi^2\ln\frac{M_K^2}{M_\pi^2}\right)\\ &\quad +\frac{\Delta_{\pi^\pm\pi^0}}{4(4\pi F_0)^2}\left(5-3\ln\frac{M_K^2}{M_\pi^2}\right)\\ &\quad +\sum_{PQ}\left\{\left[a_{PQ}(t)+\frac{\Delta_{PQ}}{2t}b_{PQ}\right]K_{PQ}(t)\right.\\ &\quad \left. +b_{PQ}\frac{F_0^2}{t}H_{PQ}(t)\right\}\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\widetilde{f}_-^{K^0\pi^-}(t) &= \left(\frac{F_K}{F_\pi}-1\right)\left(1+\frac{4\varepsilon^{(2)}}{\sqrt{3}}\right)\\ &\quad -\frac{\varepsilon^{(2)}}{\sqrt{3}(4\pi F_0)^2}\left(\Delta_{K\pi}-M_\pi^2\ln\frac{M_K^2}{M_\pi^2}\right)\\ &\quad +\frac{\Delta_{\pi^\pm\pi^0}}{4(4\pi F_0)^2}\\ &\quad +\sum_{PQ}\left\{\left[c_{PQ}(t)+\frac{\Delta_{PQ}}{2t}d_{PQ}\right]K_{PQ}(t)\right.\\ &\quad \left. +d_{PQ}\frac{F_0^2}{t}H_{PQ}(t)\right\}.\end{aligned} \tag{3.16}$$

4 Scalar form factors

In the combination

$$\widetilde{f}_0^{K^+\pi^0}(t)=\widetilde{f}_+^{K^+\pi^0}(t)+\frac{t}{\Delta_{K^+\pi^0}}\widetilde{f}_-^{K^+\pi^0}(t) \tag{4.1}$$

the terms with $H_{PQ}(t)$ cancel completely because of the relation

$$\begin{aligned}&\frac{1}{2}H_{K^+\pi^0}(t)+\frac{3}{2}H_{K^+\eta}(t)+H_{K^0\pi^-}(t)\\ &\quad +\sqrt{3}\,\varepsilon^{(2)}\left[\frac{5}{2}H_{K\pi}(t)+\frac{1}{2}H_{K\eta}(t)\right]\\ &\quad +\sum_{PQ}b_{PQ}F_0^2H_{PQ}(t)/\Delta_{K^+\pi^0}=0.\end{aligned} \tag{4.2}$$

This is to be expected as $H_{PQ}(t)$ contains the low-energy coupling $L_9^r(\mu)$ being related to pure vector exchange.

Quite analogously, the terms with $H_{PQ}(t)$ disappear also in the combination

$$\widetilde{f}_0^{K^0\pi^-}(t)=\widetilde{f}_+^{K^0\pi^-}(t)+\frac{t}{\Delta_{K^0\pi^-}}\widetilde{f}_-^{K^0\pi^-}(t), \tag{4.3}$$

as a consequence of

$$\begin{aligned}&\frac{1}{2}H_{K^+\pi^0}(t)+\frac{3}{2}H_{K^+\eta}(t)+H_{K^0\pi^-}(t)\\ &\quad +\sqrt{3}\,\varepsilon^{(2)}\left[H_{K\pi}(t)-H_{K\eta}(t)\right]\\ &\quad +\sum_{PQ}d_{PQ}F_0^2H_{PQ}(t)/\Delta_{K^0\pi^-}=0.\end{aligned} \tag{4.4}$$

The scalar form factor of $K_{\ell 3}^+$ can now be written as

$$\begin{aligned}&\widetilde{f}_0^{K^+\pi^0}(t)\\ &\quad =\widetilde{f}_0^{K^+\pi^0}(0)\\ &\qquad +\frac{t}{\Delta_{K^+\pi^0}}\left\{\left(\frac{F_K}{F_\pi}-1\right)\left(1+\sqrt{3}\varepsilon^{(2)}\right)\right.\\ &\qquad -\frac{\sqrt{3}\varepsilon^{(2)}}{(4\pi F_0)^2}\left(\Delta_{K\pi}-M_\pi^2\ln\frac{M_K^2}{M_\pi^2}\right)\\ &\qquad +\frac{\Delta_{\pi^\pm\pi^0}}{4(4\pi F_0)^2}\left(5-3\ln\frac{M_K^2}{M_\pi^2}\right)\\ &\qquad \left.+\sum_{PQ}\left[\frac{1}{2}a_{PQ}(0)\Delta_{PQ}\bar{J}'_{PQ}(0)+\frac{1}{8}b_{PQ}\Delta_{PQ}^2\bar{J}''_{PQ}(0)\right]\right\}\\ &\qquad +\frac{1}{\Delta_{K^+\pi^0}}\sum_{PQ}\left\{\frac{1}{2}a'_{PQ}(0)\Delta_{PQ}t\bar{J}_{PQ}(t)\right.\\ &\qquad +\frac{1}{2}a_{PQ}(0)\Delta_{PQ}\left[\bar{J}_{PQ}(t)-t\bar{J}'_{PQ}(0)\right]\\ &\qquad \left.+\frac{1}{4}b_{PQ}\Delta_{PQ}^2\frac{\bar{J}_{PQ}(t)-t\bar{J}'_{PQ}(0)-t^2\bar{J}''_{PQ}(0)/2}{t}\right\}.\end{aligned} \tag{4.5}$$

The last term in the curly brackets is of order t^2 and the first derivative of the form factor can be read off directly from the term $\sim t$. The analogous expression for the $K_{\ell 3}^0$ scalar form factor is given by

$$\begin{aligned}&\widetilde{f}_0^{K^0\pi^-}(t)\\ &\quad =\widetilde{f}_0^{K^0\pi^-}(0)\\ &\qquad +\frac{t}{\Delta_{K^0\pi^-}}\left\{\left(\frac{F_K}{F_\pi}-1\right)\left(1+\frac{4\varepsilon^{(2)}}{\sqrt{3}}\right)\right.\\ &\qquad -\frac{\varepsilon^{(2)}}{\sqrt{3}(4\pi F_0)^2}\left(\Delta_{K\pi}-M_\pi^2\ln\frac{M_K^2}{M_\pi^2}\right)+\frac{\Delta_{\pi^\pm\pi^0}}{4(4\pi F_0)^2}\\ &\qquad \left.+\sum_{PQ}\left[\frac{1}{2}c_{PQ}(0)\Delta_{PQ}\bar{J}'_{PQ}(0)+\frac{1}{8}d_{PQ}\Delta_{PQ}^2\bar{J}''_{PQ}(0)\right]\right\}\\ &\qquad +\frac{1}{\Delta_{K^0\pi^-}}\sum_{PQ}\left\{\frac{1}{2}c'_{PQ}(0)\Delta_{PQ}t\bar{J}_{PQ}(t)\right.\end{aligned}$$

$$+\frac{1}{2}c_{PQ}(0)\Delta_{PQ}\left[\bar{J}_{PQ}(t)-t\bar{J}'_{PQ}(0)\right]$$
$$\left.+\frac{1}{4}d_{PQ}\Delta_{PQ}^2\frac{\bar{J}_{PQ}(t)-t\bar{J}'_{PQ}(0)-t^2\bar{J}''_{PQ}(0)/2}{t}\right\}. \tag{4.6}$$

In the further evaluation of the slope parameter

$$\lambda_0^{K^+\pi^0}:=\frac{M_{\pi^+}^2}{\widetilde{f}_+^{K^+\pi^0}(0)}\frac{\mathrm{d}\widetilde{f}_0^{K^+\pi^0}(t)}{\mathrm{d}t}\bigg|_{t=0} \tag{4.7}$$

from (4.5) we use (3.9) and arrive at

$$\begin{aligned}\lambda_0^{K^+\pi^0}&=\frac{M_{\pi^+}^2}{\Delta_{K^+\pi^0}}\\&\quad\times\left\{\frac{F_K}{F_\pi\widetilde{f}_+^{K^0\pi^-}(0)}-\frac{1}{\widetilde{f}_+^{K^0\pi^-}(0)}\right.\\&\quad-\frac{\sqrt{3}\varepsilon^{(2)}}{(4\pi F_0)^2}\left(\Delta_{K\pi}-M_\pi^2\ln\frac{M_K^2}{M_\pi^2}\right)\\&\quad+\frac{\Delta_{\pi^\pm\pi^0}}{4(4\pi F_0)^2}\left(5-3\ln\frac{M_K^2}{M_\pi^2}\right)+\left(1-\sqrt{3}\varepsilon^{(2)}\right)\\&\quad\times\sum_{PQ}\left[\frac{1}{2}a_{PQ}(0)\Delta_{PQ}\bar{J}'_{PQ}(0)\right.\\&\quad\left.\left.+\frac{1}{8}b_{PQ}\Delta_{PQ}^2\bar{J}''_{PQ}(0)\right]\right\}.\end{aligned} \tag{4.8}$$

For

$$\lambda_0^{K^0\pi^-}:=\frac{M_{\pi^+}^2}{\widetilde{f}_+^{K^0\pi^-}(0)}\frac{\mathrm{d}\widetilde{f}_0^{K^0\pi^-}(t)}{\mathrm{d}t}\bigg|_{t=0} \tag{4.9}$$

we obtain the result

$$\begin{aligned}\lambda_0^{K^0\pi^-}&=\frac{M_{\pi^+}^2}{\Delta_{K^0\pi^-}}\\&\quad\times\left\{\left(\frac{F_K}{F_\pi\widetilde{f}_+^{K^0\pi^-}(0)}-\frac{1}{\widetilde{f}_+^{K^0\pi^-}(0)}\right)\left(1+\frac{4\varepsilon^{(2)}}{\sqrt{3}}\right)\right.\\&\quad-\frac{\varepsilon^{(2)}/\sqrt{3}}{(4\pi F_0)^2}\left[\Delta_{K\pi}-M_\pi^2\ln\frac{M_K^2}{M_\pi^2}\right]+\frac{\Delta_{\pi^\pm\pi^0}}{4(4\pi F_0)^2}\\&\quad+\sum_{PQ}\left[\frac{1}{2}c_{PQ}(0)\Delta_{PQ}\bar{J}'_{PQ}(0)\right.\\&\quad\left.\left.+\frac{1}{8}d_{PQ}\Delta_{PQ}^2\bar{J}''_{PQ}(0)\right]\right\}.\end{aligned} \tag{4.10}$$

5 Callan–Treiman relations

For the investigation of the Callan–Treiman relations in the presence of isospin-breaking effects, it is convenient to consider the ratios

$$\frac{\widetilde{f}_0^{K^+\pi^0}(\Delta_{K^+\pi^0})}{\widetilde{f}_0^{K^+\pi^0}(0)},\qquad\frac{\widetilde{f}_0^{K^0\pi^-}(\Delta_{K^0\pi^-})}{\widetilde{f}_0^{K^0\pi^-}(0)}. \tag{5.1}$$

In the case of $K_{\ell 3}^+$ decays, we find

$$\begin{aligned}&\frac{\widetilde{f}_0^{K^+\pi^0}(\Delta_{K^+\pi^0})}{\widetilde{f}_0^{K^+\pi^0}(0)}\\&\quad=\frac{F_K}{F_\pi\widetilde{f}_+^{K^0\pi^-}(0)}\\&\qquad-\frac{\sqrt{3}\varepsilon^{(2)}}{(4\pi F_0)^2}\left(\Delta_{K\pi}-M_\pi^2\ln\frac{M_K^2}{M_\pi^2}\right)\\&\qquad+\frac{\Delta_{\pi^\pm\pi^0}}{4(4\pi F_0)^2}\left(5-3\ln\frac{M_K^2}{M_\pi^2}\right)+\left(1+\sqrt{3}\varepsilon^{(2)}\right)\\&\qquad\times\sum_{PQ}\left[a_{PQ}(\Delta_{K^+\pi^0})+\frac{\Delta_{PQ}b_{PQ}}{2\Delta_{K^+\pi^0}}\right]\\&\qquad\times K_{PQ}(\Delta_{K^+\pi^0}).\end{aligned} \tag{5.2}$$

A further evaluation of the coefficients $a_{PQ}(\Delta_{K^+\pi^0})$, b_{PQ} and of $K_{PQ}(\Delta_{K^+\pi^0})$ leads to the alternative form

$$\begin{aligned}&\frac{\widetilde{f}_0^{K^+\pi^0}(\Delta_{K^+\pi^0})}{\widetilde{f}_0^{K^+\pi^0}(0)}\\&\quad=\frac{F_K}{F_\pi\widetilde{f}_+^{K^0\pi^-}(0)}\\&\qquad-\frac{\sqrt{3}\varepsilon^{(2)}}{(4\pi F_0)^2}\left(\Delta_{K\pi}-M_\pi^2\ln\frac{M_K^2}{M_\pi^2}\right)\\&\qquad+\frac{\Delta_{\pi^\pm\pi^0}}{4(4\pi F_0)^2}\left(5-3\ln\frac{M_K^2}{M_\pi^2}\right)\\&\qquad+\frac{M_\pi^2}{2F_0^2}\left(1+\frac{12\varepsilon^{(2)}}{\sqrt{3}}-\frac{4\varepsilon^{(2)}M_K^2}{\sqrt{3}M_\pi^2}\right)\bar{J}_{K^+\pi^0}(\Delta_{K^+\pi^0})\\&\qquad-\frac{M_\pi^2}{F_0^2}\left(1+\frac{2\varepsilon^{(2)}}{\sqrt{3}}+\frac{4\varepsilon^{(2)}M_K^2}{\sqrt{3}M_\pi^2}-\frac{2\Delta_{\pi^\pm\pi^0}}{\Delta_{K\pi}}\right)\\&\qquad\times\bar{J}_{K^0\pi^-}(\Delta_{K^+\pi^0})\\&\qquad-\frac{M_\pi^2}{6F_0^2}\left(1+\frac{8\varepsilon^{(2)}}{\sqrt{3}}+\frac{4\varepsilon^{(2)}M_K^2}{\sqrt{3}M_\pi^2}-\frac{4\Delta_{\pi^\pm\pi^0}}{\Delta_{K\pi}}\right)\\&\qquad\times\bar{J}_{K^+\eta}(\Delta_{K^+\pi^0}).\end{aligned} \tag{5.3}$$

We remark that the specific combination of terms

$$\delta_{\rm CT}^{K^+\pi^0} := \frac{\widetilde{f}_0^{K^+\pi^0}(\Delta_{K^+\pi^0})}{\widetilde{f}_0^{K^+\pi^0}(0)} - \frac{F_K}{F_\pi \widetilde{f}_+^{K^0\pi^-}(0)} \tag{5.4}$$

vanishes at lowest order.[5] Contributions to this expression start only at NLO and constitute an appropriate measure for corrections to the Callan–Treiman relation in the presence of isospin violation. In contrast, the quantity

$$\Delta_{\rm CT}^{K^+\pi^0} = \widetilde{f}_0^{K^+\pi^0}(\Delta_{K^+\pi^0}) - \frac{F_K}{F_\pi} = \sqrt{3}\,\varepsilon^{(2)} + \cdots \tag{5.5}$$

receives a large (but trivial) contribution already at the tree level, making it less convenient for the discussion of deviations from the Callan–Treiman limit in the case $m_u \neq m_d$, $e \neq 0$. Note also that the NLO contribution to (5.4) can be expressed in terms of physical masses, the ratio $F_K/F_\pi \widetilde{f}_+^{K^0\pi^-}(0)$, $\varepsilon^{(2)}$ and the pion decay constant, whereas the computation of (5.5) at NLO requires additional information on $\varepsilon_{\rm S}^{(4)}$ and $\varepsilon_{\rm EM}^{(4)}$ entering in (3.9).

The analogous formulas in the case of $K^0_{\ell 3}$ are given by

$$\begin{aligned}
\frac{\widetilde{f}_0^{K^0\pi^-}(\Delta_{K^0\pi^-})}{\widetilde{f}_0^{K^0\pi^-}(0)}
&= \frac{F_K}{F_\pi \widetilde{f}_+^{K^0\pi^-}(0)} \\
&\quad + \frac{4\varepsilon^{(2)}}{\sqrt{3}}\left(\frac{F_K}{F_\pi \widetilde{f}_+^{K^0\pi^-}(0)} - \frac{1}{\widetilde{f}_+^{K^0\pi^-}(0)}\right) \\
&\quad - \frac{\varepsilon^{(2)}}{\sqrt{3}(4\pi F_0)^2}\left(\Delta_{K\pi} - M_\pi^2 \ln\frac{M_K^2}{M_\pi^2}\right) + \frac{\Delta_{\pi^\pm\pi^0}}{4(4\pi F_0)^2} \\
&\quad + \sum_{PQ}\left[c_{PQ}(\Delta_{K^0\pi^-}) + \frac{\Delta_{PQ} d_{PQ}}{2\Delta_{K^0\pi^-}}\right] \\
&\quad \times K_{PQ}(\Delta_{K^0\pi^-}),
\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}
\frac{\widetilde{f}_0^{K^0\pi^-}(\Delta_{K^0\pi^-})}{\widetilde{f}_0^{K^0\pi^-}(0)}
&= \frac{F_K}{F_\pi \widetilde{f}_+^{K^0\pi^-}(0)} \\
&\quad + \frac{4\varepsilon^{(2)}}{\sqrt{3}}\left(\frac{F_K}{F_\pi \widetilde{f}_+^{K^0\pi^-}(0)} - \frac{1}{\widetilde{f}_+^{K^0\pi^-}(0)}\right) \\
&\quad - \frac{\varepsilon^{(2)}}{\sqrt{3}(4\pi F_0)^2}\left(\Delta_{K\pi} - M_\pi^2 \ln\frac{M_K^2}{M_\pi^2}\right) + \frac{\Delta_{\pi^\pm\pi^0}}{4(4\pi F_0)^2} \\
&\quad - \frac{M_\pi^2}{2F_0^2}\left(1 - \frac{2\varepsilon^{(2)}}{\sqrt{3}} + \frac{2\Delta_{\pi^\pm\pi^0} M_K^2}{\Delta_{K\pi} M_\pi^2}\right)\bar{J}_{K^+\pi^0}(\Delta_{K^0\pi^-}) \\
&\quad - \frac{M_\pi^2}{6F_0^2}\left(1 + \frac{6\varepsilon^{(2)}}{\sqrt{3}} - \frac{2\Delta_{\pi^\pm\pi^0}}{\Delta_{K\pi}}\right)\bar{J}_{K^+\eta}(\Delta_{K^0\pi^-}),
\end{aligned} \tag{5.7}$$

respectively. In the following, the quantity

$$\delta_{\rm CT}^{K^0\pi^-} := \frac{\widetilde{f}_0^{K^0\pi^-}(\Delta_{K^0\pi^-})}{\widetilde{f}_0^{K^0\pi^-}(0)} - \frac{F_K}{F_\pi \widetilde{f}_+^{K^0\pi^-}(0)} \tag{5.8}$$

will be used to measure the size of corrections to the Callan–Treiman relation in the case of neutral kaon decays.

In the isospin limit, (5.3) as well as (5.7) reduce to the well known result [28]

$$\begin{aligned}
f_0^{K\pi}(\Delta_{K\pi}) &= \frac{F_K}{F_\pi} - \frac{M_\pi^2}{6F_0^2} \\
&\quad \times \left[3\bar{J}_{K\pi}(\Delta_{K\pi}) + \bar{J}_{K\eta}(\Delta_{K\pi})\right].
\end{aligned} \tag{5.9}$$

6 Size of isospin breaking

The size of strong isospin violation is determined by the mixing angle $\varepsilon^{(2)}$ defined in (3.4) or, equivalently, by the ratio of the quark mass differences

$$R := \frac{m_s - \widehat{m}}{m_d - m_u}. \tag{6.1}$$

Up to corrections of order m_q^2, the double ratio

$$Q^2 := \frac{m_s^2 - \widehat{m}^2}{m_d^2 - m_u^2} = R\frac{m_s/\widehat{m} + 1}{2} \tag{6.2}$$

is given by the meson masses and a purely electromagnetic contribution [9]:

$$Q^2 = \frac{\Delta_{K\pi} M_K^2 (1 + \mathcal{O}(m_q^2))}{M_\pi^2[\Delta_{K^0K^+} + \Delta_{\pi^+\pi^0} - (\Delta_{K^0K^+} + \Delta_{\pi^+\pi^0})_{\rm EM}]}. \tag{6.3}$$

As a consequence of Dashen's theorem [43], the electromagnetic term vanishes to lowest order $e^2 p^0$. It can be expressed through chiral logarithms and a certain combination of electromagnetic couplings [10, 11]:

$$\begin{aligned}
&(\Delta_{K^0K^+} + \Delta_{\pi^+\pi^0})_{\rm EM} \\
&\quad = e^2 M_K^2 \left[\frac{1}{4\pi^2}\left(3\ln\frac{M_K^2}{\mu^2} - 4 + 2\ln\frac{M_K^2}{\mu^2}\right)\right. \\
&\qquad + \frac{4}{3}(K_5 + K_6)^r(\mu) - 8(K_{10} + K_{11})^r(\mu) \\
&\qquad \left. + 16 Z L_5^r(\mu)\right] + \mathcal{O}(e^2 M_\pi^2).
\end{aligned} \tag{6.4}$$

[5] To lowest order, the second term is equal to 1, and the form factor itself is independent of the momentum transfer.

The numerical values of the electromagnetic coupling constants appearing in this expression have been determined by several authors [44–46]. Here we use the most recent result by Ananthanarayan and Moussallam [46]. They obtain a rather large deviation from Dashen's limit,

$$(\Delta_{K^0K^+} + \Delta_{\pi^+\pi^0})_{\rm EM} = -1.5\,\Delta_{\pi^+\pi^0}, \tag{6.5}$$

which corresponds to

$$Q = 20.7 \pm 1.2, \tag{6.6}$$

where we have added a rather generous error to account for higher order corrections.

For the determination of

$$R = \frac{2Q^2}{m_s/\widehat{m} + 1}, \tag{6.7}$$

we also need information about the quark mass ratio $m_s/\widehat{m}$ as our second input parameter. Employing different methods [47], typical values around $m_s/\widehat{m} \sim 24$ have been obtained in the literature. We want to corroborate this size of the quark mass ratio by a numerical update of the determination of $m_s/\widehat{m}$ with a method proposed by Leutwyler [48], using the decay widths of $\eta \to \gamma\gamma$ and $\eta' \to \gamma\gamma$. Defining the parameters c_η and $c_{\eta'}$ by [48]

$$\Gamma(P \to \gamma\gamma) = \frac{\alpha^2 M_P^3}{64\pi^3 F_\pi^2} c_P^2, \tag{6.8}$$

the experimental values for the decay widths given in [30] correspond to $c_\eta = 0.992 \pm 0.025$ and $c_{\eta'} = 1.245 \pm 0.022$. The quark mass ratio can be obtained from the system of equations [48–50]

$$F_\eta^8 c_\eta + F_{\eta'}^8 c_{\eta'} = F_\pi/\sqrt{3}, \tag{6.9}$$

$$\left(F_\eta^8\right)^2 + \left(F_{\eta'}^8\right)^2 = \frac{4F_K^2 - F_\pi^2}{3}, \tag{6.10}$$

$$\left(F_\eta^8\right)^2 M_\eta^2 + \left(F_{\eta'}^8\right)^2 M_{\eta'}^2 = \frac{8F_K^2 M_K^2 m_s/\widehat{m}}{3(m_s/\widehat{m}+1)} - \frac{F_\pi^2 M_\pi^2 (2m_s/\widehat{m} - 1)}{3}. \tag{6.11}$$

Equation (6.10) can be written in the form [48]

$$F_\eta^8 = F_8 \cos\vartheta_8, \qquad F_{\eta'}^8 = F_8 \sin\vartheta_8, \tag{6.12}$$

with

$$(F_8)^2 = \frac{4F_K^2 - F_\pi^2}{3}. \tag{6.13}$$

Using (6.9), the observed values of c_η and $c_{\eta'}$ require $\vartheta_8 = -22.0^\circ$. Inserting this in (6.11) yields the quark mass ratio

$$m_s/\widehat{m} = 24.7 \pm 1.0 \pm 0.3 \pm 0.1 = 24.7 \pm 1.1, \tag{6.14}$$

where the errors refer to the uncertainties of $\Gamma(\eta \to \gamma\gamma)$, $\Gamma(\eta' \to \gamma\gamma)$ and F_K/F_π. This value is perfectly consistent with $m_s/\widehat{m} = 24.4 \pm 1.5$, obtained in [47] based on different arguments.

Combining (6.6) and (6.14), (6.7) finally gives

$$R = 33.5 \pm 4.0 \pm 1.5 = 33.5 \pm 4.3. \tag{6.15}$$

A value for R of this size has been suggested in [51]. Note, however, that a recent analysis of $\eta \to 3\pi$ at the two-loop level [52] favors the values $R = 42.2$ and $Q = 23.2$.

The result in (6.15) corresponds to

$$\varepsilon^{(2)} = (1.29 \pm 0.17) \times 10^{-2} \tag{6.16}$$

and will be used in our subsequent numerical analysis. We also note that (6.16) leads to the numerical value

$$\delta_{\rm SU(2)} = 0.058(8) \tag{6.17}$$

for the parameter (3.10) in $K_{\ell 3}$ decays.

7 Numerics at $\mathcal{O}(p^4, (m_d - m_u)p^2, e^2p^2)$

For $M_{\pi^\pm}$, M_{π^0}, $M_{K^\pm}$ and M_{K^0} we take the PDG06 values [30], but for M_η we employ the Gell-Mann–Okubo relation [53, 54]:

$$3\Delta_{\eta K} = \Delta_{K\pi}. \tag{7.1}$$

As this relation has to be used to arrive at the theoretical formulas of Sects. 4 and 5, this is the only unambiguous choice for M_η at the considered chiral order.

With this set of masses, the numerical value for the ratio $F_K/F_\pi \tilde{f}_+^{K^0\pi^-}(0)$ given in (2.28), the size of strong isospin breaking (6.16) and $F_0 \simeq F_\pi = 92.2\,\text{MeV}$ [38] as our input parameters, we obtain the following results for the slope parameters at the chiral order p^4, $(m_d - m_u)p^2$ and e^2p^2:

$$\begin{aligned} \lambda_0^{K^0\pi^-} &= (\underbrace{16.64}_{m_u=m_d} + \underbrace{0.17}_{m_u\neq m_d} + \underbrace{0.14}_{\rm EM}) \times 10^{-3} \\ &= 16.95(40)(5) \times 10^{-3}, \end{aligned} \tag{7.2}$$

$$\begin{aligned} \lambda_0^{K^+\pi^0} &= (\underbrace{16.64}_{m_u=m_d} - \underbrace{0.12}_{m_u\neq m_d} - \underbrace{0.08}_{\rm EM}) \times 10^{-3} \\ &= 16.44(39)(4) \times 10^{-3}. \end{aligned} \tag{7.3}$$

The value in the limit of isospin conservation ($m_u = m_d$, $e = 0$) and the contributions generated by strong isospin breaking and electromagnetism are displayed separately. The latter two pieces turn out to be of the same size. In the total results, the first error refers to (2.28) and the second one to (6.16). We see that both sources of isospin violation generate only tiny shifts with respect to the result in the isospin limit, with a splitting of the two slope parameters given by

$$\Delta\lambda_0 := \lambda_0^{K^0\pi^-} - \lambda_0^{K^+\pi^0} = (5.1 \pm 0.9) \times 10^{-4}. \tag{7.4}$$

At the Callan–Treiman point we find

$$\delta_{\text{CT}}^{K^0\pi^-} = 1.7(1)(7) \times 10^{-3} \tag{7.5}$$

for the quantity defined in (5.8). The analogous measure (5.4) of the corrections to the Callan–Treiman relation for the charged kaon decay mode is given by

$$\delta_{\text{CT}}^{K^+\pi^0} = -10.4(0)(7) \times 10^{-3}. \tag{7.6}$$

In both cases, the first error is related to (2.28), and the second one to (6.16). The corresponding value in the isospin limit is given by

$$\frac{f_0^{K\pi}(\Delta_{K\pi})}{f_0^{K\pi}(0)} - \frac{F_K}{F_\pi f_+^{K\pi}(0)} = -3.6 \times 10^{-3}. \tag{7.7}$$

Switching off the electromagnetic contribution in (7.5) and (7.6), we obtain

$$\begin{aligned} \delta_{\text{CT}}^{K^0\pi^-}\big|_{e=0} &= 1.9 \times 10^{-3}, \\ \delta_{\text{CT}}^{K^+\pi^0}\big|_{e=0} &= -9.9 \times 10^{-3}. \end{aligned} \tag{7.8}$$

8 Analysis at $\mathcal{O}(p^6)$

In the isospin limit ($m_u = m_d$, $e = 0$), the NNLO result for scalar form factors has been given in the form [23]

$$\begin{aligned} &f_0^{K\pi}(t) + \frac{t}{\Delta_{K\pi}}\left(1 - \frac{F_K}{F_\pi}\right) \\ &\quad = 1 + \bar{\Delta}(t) + \Delta(0) \\ &\qquad - \frac{8\Delta_{K\pi}^2}{F_\pi^4}\left[C_{12}^r(M_\rho) + C_{34}^r(M_\rho)\right] \\ &\qquad + \frac{8t\Delta_{K\pi}}{F_\pi^4}\left[2C_{12}^r(M_\rho) + C_{34}^r(M_\rho)\right] \\ &\qquad + \frac{16tM_\pi^2}{F_\pi^4}\left[2C_{12}^r(M_\rho) + C_{34}^r(M_\rho)\right] \\ &\qquad - \frac{8t^2}{F_\pi^4}C_{12}^r(M_\rho). \end{aligned} \tag{8.1}$$

Following the strategy proposed in [24], we pull out the tree-level pieces $\sim L_i^r \times L_j^r$ from $\bar{\Delta}(t)$ and $\Delta(0)$ by defining[6]

$$D(0) = \Delta(0) - \frac{8\Delta_{K\pi}^2}{F_\pi^4} L_5^r(M_\rho)^2, \tag{8.2}$$

$$\bar{D}(t) = \bar{\Delta}(t) + \frac{8t\Delta_{K\pi}}{F_\pi^4} L_5^r(M_\rho)^2. \tag{8.3}$$

The scalar form factor can now be written as

$$\begin{aligned} f_0^{K\pi}(t) &= f_+^{K\pi}(0) + \frac{t}{\Delta_{K\pi}}\left(\frac{F_K}{F_\pi} - 1\right) \\ &\quad + \frac{8t\Delta_{K\pi}}{F_\pi^4}\left[2C_{12}^r(M_\rho) + C_{34}^r(M_\rho) - L_5^r(M_\rho)^2\right] \\ &\quad + \frac{16tM_\pi^2}{F_\pi^4}\left[2C_{12}^r(M_\rho) + C_{34}^r(M_\rho)\right] \\ &\quad - \frac{8t^2}{F_\pi^4}C_{12}^r(M_\rho) + \bar{D}(t), \end{aligned} \tag{8.4}$$

where

$$\begin{aligned} f_+^{K\pi}(0) &= 1 + D(0) \\ &\quad - \frac{8\Delta_{K\pi}^2}{F_\pi^4}\left[C_{12}^r(M_\rho) + C_{34}^r(M_\rho) - L_5^r(M_\rho)^2\right]. \end{aligned} \tag{8.5}$$

The expression for the normalized scalar form factor takes the form

$$\begin{aligned} \frac{f_0^{K\pi}(t)}{f_+^{K\pi}(0)} &= 1 + \frac{t}{\Delta_{K\pi}}\left(\frac{F_K}{F_\pi f_+^{K\pi}(0)} - \frac{1}{1 + D(0)}\right) \\ &\quad + \frac{8t(\Delta_{K\pi} - t)}{F_\pi^4}C_{12}^r(M_\rho) \\ &\quad + \frac{16tM_\pi^2}{F_\pi^4}\left[2C_{12}^r(M_\rho) + C_{34}^r(M_\rho)\right] \\ &\quad + \frac{\bar{D}(t)}{1 + D(0)}, \end{aligned} \tag{8.6}$$

allowing for the following conclusion.

Apart from the very small contribution

$$\frac{16tM_\pi^2}{F_\pi^4}\left[2C_{12}^r(M_\rho) + C_{34}^r(M_\rho)\right] = \Delta_{\text{CT}}^{\text{tree}, p^6}\frac{t}{\Delta_{K\pi}}, \tag{8.7}$$

which is suppressed by a factor M_π^2/M_K^2, the slope as well as the curvature of (8.6) depend only on the counterterm $C_{12}^r(M_\rho)$ if the loop functions $\bar{D}(t)$ and $D(0)$ are known and the quantity $F_K/F_\pi f_+^{K\pi}(0)$ is used as input parameter.

[6] Note that terms $\sim L_4^r \times L_5^r$, $L_5^r \times L_6^r$, $L_5^r \times L_8^r$, etc. cancel in the combination of terms entering in (8.1).

Taking $\Delta(0)$ and $\bar{\Delta}(t)$ ($K^0_{\ell 3}$) from [23] and $L^r_5(M_\rho)$ (fit 10) from [51], (8.2) and (8.3) assume the numerical values

$$D(0) = -0.0134 \pm 0.0005, \tag{8.8}$$

$$\bar{D}(t) = -0.23407\, t/\mathrm{GeV}^2 + 0.833045\, \left(t/\mathrm{GeV}^2\right)^2 + 1.25252\, \left(t/\mathrm{GeV}^2\right)^3. \tag{8.9}$$

The relevant p^6 counterterms have been determined by using the $1/N_c$ expansion and truncating the hadronic spectrum to the lowest lying resonances [24]. In this framework, the leading term in the large-N_c expansion of the relevant couplings can be expressed in terms of the scalar and pseudoscalar octet masses (M_S and M_P) and the pion decay constant [24]:

$$L_5^{\mathcal{SP}} = \frac{F_\pi^2}{4M_S^2}, \qquad C_{12}^{\mathcal{SP}} = -\frac{F_\pi^4}{8M_S^4},$$
$$C_{34}^{\mathcal{SP}} = \frac{3F_\pi^4}{16M_S^4} + \frac{F_\pi^4}{16M_S^4}\left(1 - \frac{M_S^2}{M_P^2}\right)^2. \tag{8.10}$$

One assumes that the expressions given in (8.10) determine the corresponding renormalized coupling constants at some typical hadronic matching scale μ:

$$C_i^r(\mu) = C_i^{\mathcal{SP}}. \tag{8.11}$$

The value of the coupling constant at our standard reference scale M_ρ can then be obtained by

$$C_i^r(M_\rho) = C_i^r(\mu) + \delta C_i(\mu, M_\rho), \tag{8.12}$$

where

$$\delta C_i(\mu, M_\rho) = \frac{1}{(4\pi)^2}\left\{\frac{\Gamma_i^{(2)}}{(4\pi)^2}\left(\ln\frac{\mu}{M_\rho}\right)^2 - \left[2\Gamma_i^{(1)} + \Gamma_i^{(L)}(M_\rho)\right]\ln\frac{\mu}{M_\rho}\right\} \tag{8.13}$$

is determined by the renormalization group equations [25]

$$\mu\frac{\partial C_i^r(\mu)}{\partial\mu} = \frac{1}{(4\pi)^2}\left[2\Gamma_i^{(1)} + \Gamma_i^{(L)}(\mu)\right],$$
$$\mu\frac{\partial \Gamma_i^{(L)}(\mu)}{\partial\mu} = -\frac{\Gamma_i^{(2)}}{8\pi^2}. \tag{8.14}$$

For our analysis we need [25]

$$\Gamma_{12}^{(2)} = \frac{19}{64}, \qquad \Gamma_{12}^{(1)} = -\frac{13}{768(4\pi)^2},$$
$$\Gamma_{12}^{(L)} = \frac{2}{3}L_1^r + \frac{4}{3}L_2^r + \frac{8}{9}L_3^r + \frac{3}{4}L_5^r \tag{8.15}$$

and

$$\Gamma_{34}^{(2)} = -\frac{13}{32}, \qquad \Gamma_{34}^{(1)} = -\frac{31}{2304(4\pi)^2},$$
$$\Gamma_{34}^{(L)} = -L_1^r - \frac{3}{2}L_2^r - \frac{11}{12}L_3^r + L_4^r - \frac{3}{2}L_5^r. \tag{8.16}$$

The analysis of [26] (scenario A) suggests the value $M_S = 1.48$ GeV for the lightest scalar nonet that survives the large-N_c limit. With this choice of the mass parameter one gets

$$L_5^{\mathcal{SP}} = 0.97 \times 10^{-3}, \tag{8.17}$$

which agrees exactly with the mean value of $L_5^r(M_\rho)$ obtained in fit 10 of [51]. For the pseudoscalar mass parameter, spectroscopy and chiral symmetry [26, 30] suggest the value $M_P = 1.3$ GeV. With this input one obtains[7]

$$C_{12}^r(M_\rho) = \left(-1.9^{+2.0}_{-0.4}\right) \times 10^{-6} \tag{8.18}$$

and

$$C_{34}^r(M_\rho) = \left(2.9^{+1.3}_{-5.0}\right) \times 10^{-6}. \tag{8.19}$$

The errors were estimated by calculating (8.13) using (8.15) and (8.16), respectively. The numerical values (together with their errors) of the $L_i^r(M_\rho)$ were taken from fit 10 of [51]. The matching scale μ was varied between M_η and 1 GeV, which provides us with an estimate of the intrinsic uncertainty due to subleading contributions in $1/N_c$. Note that the asymmetric errors in (8.18) and (8.19) originate from the quadratic term in (8.13) as a consequence of the two-loop renormalization group equation.

Expanding the scalar form factor as

$$\frac{f_0^{K\pi}(t)}{f_+^{K\pi}(0)} = 1 + \lambda_0^{K\pi}\frac{t}{M_{\pi^+}^2} + \frac{1}{2}c_0^{K\pi}\left(\frac{t}{M_{\pi^+}^2}\right)^2 + \cdots, \tag{8.20}$$

(8.6) implies that

$$\lambda_0^{K\pi} = M_{\pi^+}^2\left\{\frac{1}{\Delta_{K\pi}}\left(\frac{F_K}{F_\pi f_+^{K\pi}(0)} - \frac{1}{1+D(0)}\right) + \frac{8\Delta_{K\pi}}{F_\pi^4}C_{12}^r(M_\rho) + \frac{16M_\pi^2}{F_\pi^4}\left[2C_{12}^r(M_\rho) + C_{34}^r(M_\rho)\right] + \frac{\bar{D}'(0)}{1+D(0)}\right\} \tag{8.21}$$

[7]These values can be compared with the previous determinations in [15, 24, 55].

for the slope parameter. Using the two-loop results (8.8), (8.9) and estimating the pertinent combination of low-energy couplings in the way described above, we find

$$\lambda_0^{K\pi} = \left(13.9^{+1.3}_{-0.4} \pm 0.4\right) \times 10^{-3}. \tag{8.22}$$

The first error is related to the uncertainties in the determination of the C_i and the second one to those in $F_K/F_\pi f_+(0)$ and $D(0)$.

The expression for the curvature reads

$$c_0^{K\pi} = M_{\pi^+}^4 \left\{ -\frac{16}{F_\pi^4} C_{12}^r(M_\rho) + \frac{\bar{D}''(0)}{1 + D(0)} \right\}, \tag{8.23}$$

which leads to the prediction

$$c_0^{K\pi} = \left(8.0^{+0.3}_{-1.7}\right) \times 10^{-4}, \tag{8.24}$$

once (8.9) together with (8.18) have been inserted. Note that the naive pole parametrization (2.11) would predict

$$c_0^{K\pi}\big|_{\text{pole fit}} = 2\left(\lambda_0^{K\pi}\right)^2 \simeq 4 \times 10^{-4}, \tag{8.25}$$

where the numerical value was obtained by inserting (8.22). This exercise shows that the pole fit is unable to reproduce the curvature found in (8.24). It was already noticed in [17] that the pole parametrization underestimates the curvature derived from dispersion theory when realistic values of the slope and the pole mass are assumed (see also the discussion in the next section).

The combination of counterterms entering in (8.7) is given by

$$2C_{12}^r(M_\rho) + C_{34}^r(M_\rho) = \left(-0.9^{+3.8}_{-3.4}\right) \times 10^{-6}, \tag{8.26}$$

which translates into

$$\Delta_{\text{CT}}^{\text{tree},\, p^6} = \left(-0.8^{+3.5}_{-3.1}\right) \times 10^{-3}. \tag{8.27}$$

Combined with the two-loop result given in [27], the total p^6 result (in the isospin limit) reads

$$\Delta_{\text{CT}} = \left(-7.0^{+3.5}_{-3.1}\right) \times 10^{-3}. \tag{8.28}$$

For completeness, we also note

$$C_{12}^r(M_\rho) + C_{34}^r(M_\rho) - L_5^r(M_\rho)^2 = \left(0.1^{+1.1}_{-1.2}\right) \times 10^{-6}, \tag{8.29}$$

which determines the size of $f_+^{K\pi}(0)$ as

$$f_+^{K\pi}(0) = 0.986 \pm 0.007_{1/N_c} \pm 0.002_{M_S, M_P}. \tag{8.30}$$

Apart from varying the matching scale, we have also added a second error to account for the uncertainty in the choice of the resonance masses, as our central value for $f_{p^6}^{\text{tree}}$ given by

$$f_{p^6}^{\text{tree}} = -\frac{\Delta_{K\pi}^2}{2M_S^4}\left(1 - \frac{M_S^2}{M_P^2}\right)^2 \tag{8.31}$$

depends strongly on the (relative) size of the mass parameters. The value given in (8.30) is to be compared with the still currently used Leutwyler–Roos value, $f_+^{K\pi}(0) = 0.961 \pm 0.008$ [56].

9 Dispersive analysis

Employing the dispersive representation (2.12) of the scalar form factor used in [17–19], the slope parameter can be written as

$$\lambda_0^{K\pi} = \frac{M_{\pi^+}^2}{\Delta_{K\pi}}\left(\ln C - G(0)\right). \tag{9.1}$$

Evaluating (2.12) at $t = \Delta_{K\pi}$, one finds the relation

$$C = \frac{F_K}{F_\pi f_+^{K\pi}(0)} + \frac{\Delta_{\text{CT}}}{f_+^{K\pi}(0)}. \tag{9.2}$$

Using (2.28) and the estimate ± 0.01 for the uncertainty due to $\Delta_{\text{CT}}/f_+^{K\pi}(0)$, the parameter C assumes the value

$$C = 1.2424 \pm 0.0045 \pm 0.01, \tag{9.3}$$

or, equivalently,

$$\ln C = 0.2170 \pm 0.0036 \pm 0.0080. \tag{9.4}$$

Together with [17]

$$G(0) = 0.0398 \pm 0.0036 \pm 0.002, \tag{9.5}$$

the dispersive analysis gives

$$\lambda_0^{K\pi} = (15.1 \pm 0.8) \times 10^{-3} \tag{9.6}$$

for the slope parameter, this being consistent with the corresponding result (8.22) based on resonance saturation. This value is also in good agreement with other results using a dispersive approach [15, 16].

The expression for the curvature reads [17]

$$\begin{aligned} c_0^{K\pi} &= \left(\lambda_0^{K\pi}\right)^2 - \frac{2M_{\pi^+}^4 G'(0)}{\Delta_{K\pi}} \\ &= \left(\lambda_0^{K\pi}\right)^2 + (4.16 \pm 0.50) \times 10^{-4}. \end{aligned} \tag{9.7}$$

Inserting (9.6), the curvature is given by

$$c_0^{K\pi} = (6.4 \pm 0.6) \times 10^{-4}, \tag{9.8}$$

which is again consistent with the result (8.24) of the previous analysis.

Translated in terms of the size of the low-energy constant $C_{12}^r(M_\rho)$, the dispersive analysis seems to favor values at the upper end of the range suggested by (8.18). Explicitly, (9.8) corresponds to

$$C_{12}^r(M_\rho) = (0.0 \pm 0.7) \times 10^{-6}. \quad (9.9)$$

An analogous comparison can be made using the recent results [16] of the dispersive approach of Jamin, Oller and Pich [14]. Their method is based on a coupled-channel solution of the dispersion relation for the scalar form factor which includes also the $K\eta'$ channel. They find [16]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{f_0^{K\pi}(t)}{f_0^{K\pi}(0)}\bigg|_{t=0} = 0.773(21)\ \mathrm{GeV}^{-2}, \quad (9.10)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{f_0^{K\pi}(t)}{f_0^{K\pi}(0)}\bigg|_{t=0} = 1.599(52)\ \mathrm{GeV}^{-4}, \quad (9.11)$$

which corresponds to the values

$$\lambda_0^{K\pi} = (14.7 \pm 0.4) \times 10^{-3} \quad (9.12)$$

and

$$c_0^{K\pi} = (6.07 \pm 0.20) \times 10^{-4}. \quad (9.13)$$

10 Contributions of order $(m_d - m_u)p^4$

Recently, isospin breaking in the $K_{\ell 3}$ form factors has also been studied at the two-loop level [27]. The results for the scalar form factor of $K^0 \to \pi^- \ell^+ \nu_\ell$ with $C_i^r = 0$ turn out to be essentially the same as those in the isospin limit. From Fig. 13 of [27] one extracts

$$\Delta\lambda_0|_{C_i^r=e=0} \simeq 5 \times 10^{-4}. \quad (10.1)$$

The remaining contribution containing the p^6 low-energy couplings C_i^r is given by

$$\Delta\lambda_0|_{C_i^r} = \frac{32\varepsilon^{(2)}\Delta_{K\pi}M_{\pi^+}^2}{\sqrt{3}F_\pi^4} \times (2C_{12} + 6C_{17} + 6C_{18} + 3C_{34} + 3C_{35})^r(M_\rho). \quad (10.2)$$

This expression can be obtained by combining the pertinent terms given in Appendix B of [27]. Resonance estimates of the low-energy couplings appearing in (10.2) were given in [57]. Using these results, we find[8]

$$(2C_{12} + 6C_{17} + 6C_{18} + 3C_{34} + 3C_{35})^{\mathcal{SP}} = \frac{F_\pi^4}{4M_S^4}\left(1 - \frac{3M_S^2}{2M_P^2} - \frac{M_S^2}{M_{\eta'}^2} + 6\lambda_2^{\mathcal{SS}}\right). \quad (10.3)$$

[8] Additional contributions due to η' exchange [58] cancel in the combination $2C_{18} + C_{35}$.

Using our standard values for the resonance masses M_S and M_P and our usual determination of the uncertainty of the large N_c estimate, we find

$$(2C_{12} + 6C_{17} + 6C_{18} + 3C_{34} + 3C_{35})^r(M_\rho) = \left(-1.25 + 2.26\lambda_2^{\mathcal{SS}} \pm 0.7_{1/N_c}\right) \times 10^{-5}. \quad (10.4)$$

No estimates of the parameter $\lambda_2^{\mathcal{SS}}$ have become available so far, but in analogy to [57] $\lambda_1^{\mathcal{SP}} = -d_m/c_m$ one may expect [59]

$$|\lambda_2^{\mathcal{SS}}| \lesssim 1. \quad (10.5)$$

Varying $\lambda_2^{\mathcal{SS}}$ in this interval, adding (10.1) and the electromagnetic contribution discussed in Sect. 7, we expect the total value for the difference of the two slope parameters to lie within the rather small range

$$0 \lesssim \Delta\lambda_0 \lesssim 10^{-3}. \quad (10.6)$$

The two-loop contributions to the correction terms of the Callan–Treiman relation in the presence of isospin violation were also given in [27]. Translated in terms of the quantities defined in (5.8) and (5.4), they find

$$\delta_{\mathrm{CT}}^{K^0\pi^-}|_{C_i^r=e=0} = -5.6 \times 10^{-3} \quad (10.7)$$

and

$$\delta_{\mathrm{CT}}^{K^+\pi^0}|_{C_i^r=e=0} = -13.3 \times 10^{-3}, \quad (10.8)$$

respectively. These results should be supplemented by the associated local contributions arising at this order [27], which are, however, also plagued by partly undetermined low-energy couplings. We demonstrate this only for the purely isospin-violating combination

$$\left(\delta_{\mathrm{CT}}^{K^0\pi^-} - \delta_{\mathrm{CT}}^{K^+\pi^0}\right)|_{C_i^r} = \frac{32\varepsilon^{(2)}M_K^4}{\sqrt{3}F_\pi^4}(2C_{12} + 2C_{14} + 2C_{15} + 6C_{17} + 6C_{18} + 4C_{34} + 3C_{35})^r(M_\rho), \quad (10.9)$$

where terms $\sim \varepsilon^{(2)}M_\pi^2$ have been discarded. In addition to the undetermined parameter $\lambda_2^{\mathcal{SS}}$ already encountered in (10.3), the resonance estimate for the p^6 low-energy coupling C_{14} is still incomplete [57], preventing a reliable numerical determination of (10.9) (and even more so for the individual terms) for the time being.

Nevertheless, based on the numbers (7.5) and (7.6) found at NLO, the partial NNLO results shown in (10.7) and (10.8), our estimate of the isospin symmetric local p^6 contribution (8.27) and a rough order-of-magnitude estimate of not yet determined local terms of the order $(m_d - m_u)p^4$ (a typical term is shown in (10.9)), we expect numerically small corrections to the Callan–Treiman relation also in the presence of isospin violation with

$$\left|\delta_{\mathrm{CT}}^{K^0\pi^-}\right|, \left|\delta_{\mathrm{CT}}^{K^+\pi^0}\right| \lesssim 10^{-2}. \tag{10.10}$$

11 Summary and conclusions

In this work we have discussed the theoretical predictions for the scalar form factors of $K_{\ell 3}$ decays within the standard model.

The leading non-vanishing contribution to the scalar slope arises at order p^4 in the chiral expansion. The theoretical expression for the scalar form factor was worked out already more than twenty years ago [28] in the limit of isospin conservation. In this case, the slope parameter is uniquely determined by the pseudoscalar masses, the pion decay constant and the ratio

$$F_K/F_\pi \widetilde{f}_+^{K^0\pi^-}(0) = 1.2424(45). \tag{11.1}$$

The remarkably precise numerical value given here can be obtained by combining the latest experimental data on $K_{\mu 2(\gamma)}$, $\pi_{\mu 2(\gamma)}$, K_{Le3}^0 and V_{ud} with the corresponding theoretical expressions. Using this input, one finds

$$\lambda_0^{K\pi}\big|_{p^4} = (16.64 \pm 0.39) \times 10^{-3}. \tag{11.2}$$

The isospin-violating contributions of order $(m_d - m_u)p^2$ and e^2p^2 to the $K_{\ell 3}$ form factors were considered for the first time in [22]. The effects of strong isospin breaking are proportional to the mixing angle

$$\varepsilon^{(2)} = \frac{\sqrt{3}}{4}\,\frac{m_d - m_u}{m_s - \widehat{m}} = (1.29 \pm 0.17) \times 10^{-2}. \tag{11.3}$$

The numerical value shown here was obtained by using the corrections to Dashen's limit given in [46]. The electromagnetic contributions of order e^2p^2 entering the slope parameters $\lambda_0^{K^0\pi^-}$ and $\lambda_0^{K^+\pi^0}$ can be expressed through the electromagnetic pieces of the pseudoscalar masses as well as the coupling Z associated to the chiral Lagrangian of order e^2p^0, which can also be related to the pion mass difference (to the order considered). Both sources of isospin violation generate only a tiny shift of the two slope parameters (compared to the isospin symmetric limit) with a splitting $\Delta\lambda_0 = \lambda_0^{K^0\pi^-} - \lambda_0^{K^+\pi^0}$ given by

$$\Delta\lambda_0|_{(m_d-m_u)p^2, e^2p^2} = (5.1 \pm 0.9) \times 10^{-4} \tag{11.4}$$

at this chiral order.

The corrections arising at order p^6 (in the isospin limit) turn out to be quite sizeable. Combining the two-loop results of chiral perturbation theory [23] with an updated estimate of the necessary p^6 low-energy couplings, the numerical value of the slope parameter in the isospin symmetric limit is given by

$$\lambda_0^{K\pi} = \left(13.9^{+1.3}_{-0.4} \pm 0.4\right) \times 10^{-3}. \tag{11.5}$$

The main uncertainty in this result comes from a certain combination of p^6 low-energy couplings which has been determined by an updated analysis based on [24, 26].

Using the dispersive representation proposed in [17] with (2.28), we find

$$\lambda_0^{K\pi} = (15.1 \pm 0.8) \times 10^{-3}, \tag{11.6}$$

which is in good agreement with the value (11.5) obtained in chiral perturbation theory and also with other results [15, 16] using dispersion techniques.

The inclusion of isospin-violating contributions of order $(m_d - m_u)p^4$ does not change this picture substantially. We expect an additional uncertainty for the values of the slope parameters of at most $\pm 10^{-3}$, mainly due to low-energy couplings not yet fully determined. Combining the two-loop results given in [27] with an estimate of a further combination of low-energy couplings, the difference of the two slope parameters should be confined to the rather small range

$$0 \lesssim \Delta\lambda_0 \lesssim 10^{-3}. \tag{11.7}$$

In other words, if a difference of the size of the two slope parameters is detected at all, $\lambda_0^{K^0\pi^-}$ should be slightly larger than $\lambda_0^{K^+\pi^0}$.

At the Callan–Treiman point $t = \Delta_{K\pi}$, the size of the scalar form factor is predicted to be [20, 21]

$$f_0^{K\pi}(\Delta_{K\pi}) = \frac{F_K}{F_\pi} + \Delta_{\mathrm{CT}}, \tag{11.8}$$

where Δ_{CT} is of the order m_u, m_d, e. At order p^4 (in the isospin limit) the correction term $\Delta_{\mathrm{CT}} = -3.5 \times 10^{-3}$ was calculated in [28]. If isospin violation is included, it is advantageous to consider the quantities defined in (5.8) and (5.4). At the order p^4, $(m_d - m_u)p^2$ and e^2p^2, we find

$$\delta_{\mathrm{CT}}^{K^0\pi^-}\big|_{p^4,(m_d-m_u)p^2,e^2p^2} = (1.7 \pm 0.7) \times 10^{-3} \tag{11.9}$$

and

$$\delta_{\mathrm{CT}}^{K^+\pi^0}\big|_{p^4,(m_d-m_u)p^2,e^2p^2} = (-10.4 \pm 0.7) \times 10^{-3}. \tag{11.10}$$

In spite of the large corrections to the correction term itself, the Callan–Treiman relation still holds with excellent precision also if isospin-violating contributions are taken into account.

Corrections to Δ_{CT} arising at NNLO are also (potentially) large. At the same time, the uncertainty of the theoretical result is increased by the presence of p^6 low-energy couplings. Combining the two-loop result given in [27] with our estimate for $2C_{12}^r + C_{34}^r$, we find (in the isospin limit)

$$\Delta_{\text{CT}} = \left(-7.0^{+3.5}_{-3.1}\right) \times 10^{-3}. \tag{11.11}$$

The loop contributions of order $(m_d - m_u)p^4$ were considered in [27]. The associated counterterm contributions depend on partly undetermined low-energy couplings. In spite of these theoretical uncertainties, we expect only small corrections to the Callan–Treiman relation with

$$\left|\delta_{\text{CT}}^{K^0\pi^-}\right|, \left|\delta_{\text{CT}}^{K^+\pi^0}\right| \lesssim 10^{-2}. \tag{11.12}$$

The experimental results for the scalar slope parameter found by ISTRA+, KTeV and KLOE are in agreement with the predictions of the standard model. On the other hand, the value found by NA48 can hardly be reconciled with our theoretical results. Furthermore, an isospin violation in $\Delta\lambda_0$ as would be suggested by the simultaneous validity of the results of ISTRA+ and NA48 is definitely ruled out within the standard model.

The naive pole parametrization of the scalar form factor should be avoided. It contains the implicit assumption of the existence of a relation between slope and curvature which is not fulfilled in the standard model.

At the present theoretical and experimental level of precision, the correct treatment of electromagnetic corrections in $K_{\mu 3}$ decays is mandatory for the extraction of form factor parameters from the experimental data. The appropriate procedure was described in [22, 60].

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Appendix

In this section we list the coefficients $a_{PQ}(t)$, b_{PQ}, $c_{PQ}(t)$, and d_{PQ} given in [22]. We have

$$\begin{aligned}
a_{K^+\pi^0}(t) &= \frac{2M_K^2 + 2M_\pi^2 - t}{4F_0^2} \\
&\quad + \left(\frac{\varepsilon^{(2)}}{\sqrt{3}}\right)\frac{-2M_K^2 + 22M_\pi^2 - 9t}{4F_0^2} + 4\pi\alpha Z, \\
a_{K^0\pi^-}(t) &= \frac{-2M_K^2 - 2M_\pi^2 + 3t}{2F_0^2} \\
&\quad + \left(\frac{\varepsilon^{(2)}}{\sqrt{3}}\right)\frac{-2M_K^2 + 6M_\pi^2 - 3t}{2F_0^2} - 16\pi\alpha Z, \\
a_{K^+\eta}(t) &= \frac{2M_K^2 + 2M_\pi^2 - 3t}{4F_0^2} \\
&\quad + \left(\frac{\varepsilon^{(2)}}{\sqrt{3}}\right)\frac{6M_K^2 - 2M_\pi^2 - 3t}{4F_0^2} + 12\pi\alpha Z,
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
b_{K^+\pi^0} &= -\frac{\Delta_{K\pi}}{2F_0^2} - \left(\frac{7\varepsilon^{(2)}}{2\sqrt{3}}\right)\frac{\Delta_{K\pi}}{F_0^2} - 4\pi\alpha Z, \\
b_{K^0\pi^-} &= -\frac{\Delta_{K\pi}}{F_0^2} - \left(\frac{\varepsilon^{(2)}}{\sqrt{3}}\right)\frac{\Delta_{K\pi}}{F_0^2} - 8\pi\alpha Z, \\
b_{K^+\eta} &= -\frac{3\Delta_{K\pi}}{2F_0^2} + \left(\frac{\sqrt{3}\varepsilon^{(2)}}{2}\right)\frac{\Delta_{K\pi}}{F_0^2} - 12\pi\alpha Z,
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
c_{K^+\pi^0}(t) &= -\frac{2M_K^2 + 2M_\pi^2 - 3t}{4F_0^2} + \left(\frac{\varepsilon^{(2)}}{\sqrt{3}}\right)\frac{-4M_K^2 + 3t}{2F_0^2} \\
&\quad - 8\pi\alpha Z, \\
c_{K^0\pi^-}(t) &= \frac{t}{2F_0^2}, \\
c_{K^+\eta}(t) &= \frac{2M_K^2 + 2M_\pi^2 - 3t}{4F_0^2} + \left(\frac{\varepsilon^{(2)}}{\sqrt{3}}\right)\frac{4M_K^2 - 3t}{2F_0^2},
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
d_{K^+\pi^0} &= -\frac{\Delta_{K\pi}}{2F_0^2} - \left(\frac{4\varepsilon^{(2)}}{\sqrt{3}}\right)\frac{\Delta_{K\pi}}{F_0^2} + 4\pi\alpha Z, \\
d_{K^0\pi^-} &= -\frac{\Delta_{K\pi}}{F_0^2} - \left(\frac{2\varepsilon^{(2)}}{\sqrt{3}}\right)\frac{\Delta_{K\pi}}{F_0^2} + 8\pi\alpha Z, \\
d_{K^+\eta} &= -\frac{3\Delta_{K\pi}}{2F_0^2} + 12\pi\alpha Z.
\end{aligned} \tag{A.4}$$

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