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Free-fall rainbow BTZ black hole

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ABSTRACT

In this paper, we investigate properties of rainbow BTZ black holes in the scenario with the free-fall (FF) orthonormal frame. After the FF rainbow BTZ metric is obtained, the Hawking temperature is calculated via the Hamilton-Jacobi method, and the thermodynamic properties are discussed. For radiated particles with a subluminal modified dispersion relation, we find that the effects of rainbow gravity tend to increase the Hawking temperature but decrease the black hole entropy. However, it shows that the FF rainbow BTZ black hole possesses an infinite effective Hawking temperature for radiated particles with a superluminal modified dispersion relation.

1. Introduction

Shortly after Einstein proposed general relativity in 1915, Schwarzschild derived the first black hole solution (Schwarzschild metric) from Einstein's equations [1]. From then on, the study of black holes has become an important field of modern physics. Due to the strong gravitational pull from a black hole, the classical theory of a black hole predicts that anything, including light, cannot escape from the black hole. However, considering quantum effects, Stephen Hawking demonstrated that black holes radiate a thermal flux of quantum particles [2].

After this discovery, it was realized that there is the trans-Planckian problem in Hawking's calculations [3]. Hawking radiation appears to originate from a mode with a large initial frequency, far beyond the Planck mass m_p, which experiences an exponentially high gravitational redshift near the horizon. Therefore, Hawking's prediction relies on the validity of quantum field theory for arbitrary high energy in curved spacetime. On the other hand, quantum field theory is considered to be more of an effective field theory whose nature remains unknown. This observation raises the question whether any unknown physics at the Planck scale could strongly influence Hawking radiation. It is widely believed that trans-Planckian physics can be expressed in certain modifications of existing models.

Although a full theory of quantum gravity is not yet available, various theories of quantum gravity, such as loop quantum gravity, string theory and quantum geometry, predict that the transformation laws of special relativity are modified at very high energies. In particular, the Deformed Special Relativity (DSR) theory makes the Planck length a new invariant scale and guarantees the nonlinear Lorentz transformation in the momentum spacetime [4–7]. Specifically, the modified energy-momentum dispersion relation of a particle of energy E and momentum p in DSR can take the form of

$$E^2 f^2\left(E/m_p\right) - p^2 g^2\left(E/m_p\right) = m^2, \tag{1}$$

where m_p is the Planck mass, and $f(x)$ and $g(x)$ are two general functions with the following properties:

$$\lim_{x\to 0} f(x) = 1 \text{ and } \lim_{x\to 0} g(x) = 1. \tag{2}$$

The modified dispersion relation (MDR) might play an important role in astronomical and cosmological observations, such as the threshold anomalies of ultra high energy cosmic rays and TeV photons [8–13]. In phenomenological physics, ground observations and astrophysical experiments have tested the predictions of MDR theory [14–17]. One of the most popular choices for the functions $f(x)$ and $g(x)$ has been proposed by Amelino-Camelia et al.[18, 19], which gives

$$f(x) = 1 \text{ and } g(x) = \sqrt{1-\eta x^n}. \tag{3}$$

As shown in [19], this formula is compatible with some of the results obtained in the Loop-Quantum-Gravity approach and reflects the results obtained in κ-Minkowski and other noncommutative spacetimes. Phenomenological implications of this "Amelino-Camelia (AC) dispersion relation" are also reviewed in [19].

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To incorporate DSR into the framework of general relativity, Magueijo and Smolin [20] proposed the "Gravity's rainbow", where the spacetime background felt by a test particle would depend on its energy. Consequently, the energy of the test particle deforms the background geometry and hence the dispersion relation. As regards the metric, it would be replaced by a one-parameter family of metrics which depends on the energy of the test particle, forming a "rainbow metric". Specifically, for a BTZ black hole, the corresponding "rainbow metric" solution to the rainbow Einstein's Field equations in a stationary orthonormal frame is given in [20]:

$$ds^2 = \frac{-h\left(r\right)}{g^2\left(E/m_p\right)}dt^2 + \frac{\frac{dr^2}{h(r)} + r^2\left[N_\phi\left(r\right)dt + d\phi\right]^2}{g^2\left(E/m_p\right)}. \tag{4}$$

Later, Alsaleh specifically studied the thermodynamic properties of the rainbow BTZ black hole in a stationary orthonormal frame [21]. Since the rainbow metric is the metric that the radiated particles "see", a more natural orthonormal frame is the one anchored to the particles. Actually, in section 3 we will show that the rainbow BTZ black hole in the free-fall orthonormal frame is given by

$$\begin{aligned} ds^2 = &\left[\frac{1}{g^2\left(E/m_p\right)} - \frac{1}{f^2\left(E/m_p\right)}\right]h\left(r\right)dt^2 \\ &+ \frac{-h(r)dt^2 + \frac{dr^2}{h(r)} + r^2\left[N_\phi\left(r\right)dt + d\phi\right]^2}{g^2\left(E/m_p\right)}. \end{aligned} \tag{5}$$

In the remainder of this paper, we dub the rainbow BTZ black holes (4) and (5) as stationary frame (SF) and free-fall frame (FF) rainbow BTZ black holes, respectively. In this paper, we aim to explore the thermodynamic properties of the FF Rainbow BTZ black hole.

There are some methods proposed to obtain Hawking radiation [22–27]. Recently, a semi-classical method of modeling Hawking radiation as a tunneling process has been developed and attracted much attention. This method was first proposed by Kraus and Wilczek [28,29], which is known as the null geodesic method. Later, the tunneling behaviors of particles were investigated using the Hamilton-Jacobi method [30–32]. Furthermore, taking the effects of quantum gravity into account, the Hamilton-Jacobi equation was modified, and the modified Hawking temperature was derived [33–38]. These motivate us to use the Hamilton-Jacobi method to study the gravity rainbow effect of Hawking radiation [39,40]. The cases with the stationary orthonormal frame have been extensively studied by many authors [41–59].

The BTZ black hole is a solution of Einstein field equations in three dimensional curved space, which describes a rotating AdS geometry [60]. It plays an important role in quantum field theory and string theory. In this paper, we will study the quantum gravity effect on BTZ black hole in the framework of gravity rainbow theory with the free-fall orthonormal frame. The remainder of our paper is organized as follows. In section 2, the thermodynamic properties of BTZ black holes are briefly reviewed. In section 3, the metric of a FF rainbow BTZ black hole is derived, and its Hawking temperature is obtained using the Hamilton-Jacobi method. The temperature and entropy of a FF rainbow BTZ black hole are computed. Finally, the thermodynamic properties of the FF static rainbow BTZ black hole will be studied. Section 4 is devoted to our discussion and conclusion. Throughout the paper we take geometrized units $c = 8G = k_b = 1$, where the Planck constant $\hbar$ is square of the Planck mass m_p.

2. BTZ black hole

The BTZ black hole is an important solution to Einstein field equation in a $(2+1)$ dimensional space with a negative cosmological constant. The action is

$$S = \frac{1}{2\pi}\int\sqrt{-g}\left[R + 2\Lambda\right], \tag{6}$$

where $\Lambda = -l^2$ is the cosmological constant, and l is AdS radius. The BTZ black hole solution to the action (6) is [60]

$$ds^2 = -h\left(r\right)dt^2 + \frac{1}{h\left(r\right)}dr^2 + r^2\left[N_\phi\left(r\right)dt + d\phi\right]^2, \tag{7}$$

where

$$\begin{aligned} h\left(r\right) &= -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \\ N_\phi\left(r\right) &= -\frac{J}{2r^2}. \end{aligned} \tag{8}$$

The parameters M and J can be interpreted as the mass and the angular momentum of the BTZ black hole. The BTZ black hole has two horizons located at $r = r_\pm$, which are determined by $h\left(r_\pm\right) = 0$:

$$r_\pm^2 = \frac{Ml^2 \pm \sqrt{M^2l^4 - l^2J^2}}{2}. \tag{9}$$

In terms of $r_\pm$, we can rewrite $h\left(r\right)$, M and J as

$$h\left(r\right) = \frac{1}{l^2r^2}\left(r^2 - r_+^2\right)\left(r^2 - r_-^2\right), M = \frac{r_+^2 + r_-^2}{l^2} \text{ and } J = \frac{2r_+r_-}{l}. \tag{10}$$

The surface gravity κ of the BTZ black hole at the outer horizon $r = r_+$ is

$$\kappa = \frac{r_+^2 - r_-^2}{l^2r_+}. \tag{11}$$

So the Hawking temperature T_h of the BTZ black hole is [61–64]

$$T_h = \frac{\hbar\kappa}{2\pi} = \frac{\hbar\left(r_+^2 - r_-^2\right)}{2\pi l^2 r_+}. \tag{12}$$

The first law of thermodynamics for the BTZ black hole was obtained in [65], which reads

$$dM = T_h dS + \Omega_H dJ, \tag{13}$$

where S and $\Omega_H = \frac{J}{2r_+^2} = \frac{r_-}{lr_+}$ are the entropy and the angular velocity of the BTZ black hole, respectively. The eqn. (13) can be rewritten in the form:

$$dS = \frac{dM - \Omega_H dJ}{T_h} = \frac{4\pi dr_+}{\hbar}, \tag{14}$$

which, by integration, leads to

$$S = \frac{4\pi r_+}{\hbar}. \tag{15}$$

Note that the BTZ black hole entropy S was also computed using the Euclidean action method [60,62,64,66,67]. The heat capacity of the BTZ black hole is

$$C_J = T_h\left(\frac{\partial S}{\partial T_h}\right)_J = \frac{4\pi r_+\Delta}{2 - \Delta}, \tag{16}$$

where $\Delta = \sqrt{1 - \frac{l^2J^2}{M^2l^4}}$. Since $0 \le \Delta \le 1$, the BTZ black hole always has a positive heat capacity, which implies the thermodynamic system for the BTZ black hole is stable.

3. Free-fall rainbow BTZ black hole

In this section, we obtain the Free-fall (FF) rainbow BTZ black hole and then calculate its Hawking temperature via the Hamilton-Jacobi method. Finally, we discuss the rainbow corrections to the entropy of the black hole.

3.1. Free-fall rainbow BTZ metric

First, we generalize the energy-independent BTZ metric (7) to the energy-dependent BTZ rainbow metric. Generally, the energy-independent metric is given by

$$ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu, \tag{17}$$

can be rewritten in terms of a set of energy-independent orthonormal frame fields e_a:

$$ds^2 = \eta^{ab} e_a \otimes e_b, \tag{18}$$

where $a = (0, i)$ and i is the spatial index. For the MDR (1), the energy-dependent rainbow counterpart for the energy-independent metric (17) can be obtained using equivalence principle [20], which gives

$$\begin{aligned} d\tilde{s}^2 &= \tilde{g}_{\mu\nu} dx^\mu \otimes dx^\nu = \eta^{ab} \tilde{e}_a \otimes \tilde{e}_b \\ &= \left(\frac{1}{g^2(E/m_p)} - \frac{1}{f^2(E/m_p)} \right) e_0 \otimes e_0 + \frac{ds^2}{g^2(E/m_p)}, \end{aligned} \tag{19}$$

where the energy-dependent orthonormal frame fields are

$$\tilde{e}_0 = \frac{e_0}{f(E/m_p)} \text{ and } \tilde{e}_i = \frac{e_i}{g(E/m_p)}. \tag{20}$$

Note that a tilde is used for an energy-dependent quantity.

Obviously, different choices of the orthonormal frame can give different rainbow metrics. In fact, the form of the rainbow metric crucially depends on the time component of the orthonormal frame, e_0. For the BTZ metric (7), e_0 in the literature [21,68,69] is usually chosen to be

$$e_0 = \sqrt{h(r)} dt, \tag{21}$$

where this orthonormal frame basis is hovering above the black hole. On the other hand, the particles radiated from the black hole travel along the geodesics. Since the rainbow metric is the metric that the radiated particles "see", a more natural orthonormal frame is the one anchored to the particles.

For massive particles moving along the geodesics, $(e_0)^\mu$ is just the 3-velocity vector u^μ of the geodesics. To compute the 3-velocity vector, we consider p_t and p_ϕ, which are conserved along geodesics (since the metric does not depend explicitly on t and ϕ). It leads immediately to the first integrals of the t- and ϕ- equations, which are given by

$$\begin{aligned} u_t &= g_{tt} u^t + g_{t\phi} u^\phi = \left[-h(r) + r^2 N_\phi^2(r) \right] \dot{t} + r^2 N_\phi \dot{\phi} = -E/m, \\ u_\phi &= g_{\phi t} u^t + g_{\phi\phi} u^\phi = r^2 N_\phi \dot{t} + r^2 \dot{\phi} = L/m. \end{aligned} \tag{22}$$

Here, m, E and L are the mass, the energy and the angular momentum of the particles. Solving eqns. (22) for $\dot{t}$ and $\dot{\phi}$ gives

$$\begin{aligned} u^t &= \dot{t} = \frac{N_\phi(r) L + E}{mh(r)}, \\ u^\phi &= \dot{\phi} = \frac{L}{mr^2} - \frac{N_\phi(r) \left[N_\phi(r) L + E \right]}{mh(r)}. \end{aligned} \tag{23}$$

We then use $g^{\mu\nu} u_\mu u_\nu = -1$ to find u^r. So the 3-velocity vector of the radiated particle is

$$\begin{aligned} u^\mu = \Bigg(& \frac{N_\phi(r) L + E}{mh(r)}, \sqrt{\frac{\left[E + N_\phi(r) L \right]^2}{m^2} - h(r) \left(1 + \frac{L^2}{m^2 r^2} \right)}, \\ & \frac{L}{mr^2} - \frac{N_\phi(r) \left[N_\phi(r) L + E \right]}{mh(r)} \Bigg). \end{aligned} \tag{24}$$

Therefore, the time component of the orthonormal frame anchored to the radiated particles is then given by

$$\begin{aligned} e_0 = u_\mu dx^\mu = & -\frac{E + N_\phi(r) L}{m} dt \\ & + \frac{\sqrt{\frac{\left[E + N_\phi(r) L \right]^2}{m^2} - h(r) \left(1 + \frac{L^2}{m^2 r^2} \right)}}{h(r)} dr. \end{aligned} \tag{25}$$

From eqn. (19), the rainbow BTZ metric is

$$\begin{aligned} ds^2 = & \left[\frac{1}{g^2(E/m_p)} - \frac{1}{f^2(E/m_p)} \right] e_0 \otimes e_0 \\ & + \frac{-h(r) dt^2 + \frac{dr^2}{h(r)} + r^2 \left[N_\phi(r) dt + d\phi \right]^2}{g^2(E/m_p)}, \end{aligned} \tag{26}$$

where e_0 is given by eqn. (25). The rainbow BTZ metric (26) is dubbed as "Free-fall (FF) rainbow BTZ black hole."

3.2. Effective Hawking temperature

We now use the Hamilton-Jacobi method to calculate the Hawking temperature of the FF rainbow black hole (26). In the Hamilton-Jacobi method, one ignores the self-gravitation of emitted particles and assumes that their action satisfies the relativistic Hamilton-Jacobi equation. The tunneling probability for the classically forbidden trajectory from inside to outside the horizon is obtained by using the Hamilton-Jacobi equation to calculate the imaginary part of the action for the tunneling process. In [39], it showed that, in the rainbow metric $ds^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu$, the Hamilton-Jacobi equations for massive particles can simply be written as

$$\tilde{g}_{\mu\nu} \partial^\mu I \partial^\nu I = m^2, \tag{27}$$

where I is the radiated particle's classical action. Since the FF rainbow black hole (26) is independent of t and ϕ, we can employ the following ansatz for the action I

$$I = -Et + W(r) + L\phi, \tag{28}$$

where, as above defined, E and L are the radiated particle's energy and angular momentum, respectively. Defining $p_r(r) \equiv W'(r)$, we can use the FF rainbow black hole (26) to rewrite the Hamilton-Jacobi equation (27) as the equation in terms of p_r, which is too long to put here. Solving the Hamilton-Jacobi equation for p_r, we obtain

$$p_r^\pm(r) = \frac{A_r^\pm(r)}{h(r) \tilde{H}(r)} \tag{29}$$

where $+/-$ denotes the outgoing/ingoing solutions. Here, $A_r^\pm(r)$ are regular functions of r without poles, the detailed forms of which are rather complicated and not relevant. In the denominator of $p_r^\pm(r)$, we define

$$\tilde{H}(r) \equiv h(r) - \epsilon\left(E/m_p\right) \frac{\left[E + LN_\phi(r)\right]^2}{m^2},$$
$$\epsilon\left(E/m_p\right) \equiv 1 - \frac{g^2\left(E/m_p\right)}{f^2\left(E/m_p\right)}. \tag{30}$$

The corresponding action is

$$I_\pm = -Et + \int p_r^\pm(r)\,dr + L\phi, \tag{31}$$

where the imaginary part of $I_\pm$ comes from the integral of $p_r^\pm(r)$.

According to the Hamilton-Jacobi method, the residues of $p_r^\pm(r)$ lead to the Hawking temperature of the radiation. So the poles of $p_r^\pm(r)$ correspond to the locations of the horizons of the FF rainbow black hole (26). Eqn. (29) shows that the poles of $p_r^\pm(r)$ are at the location where $h(r) = 0$ or $\tilde{H}(r) = 0$. For $h(r) = 0$, one simply has $r = r_\pm$. Moreover, it can show that there are also two solutions to $\tilde{H}(r) = 0$, namely $r = \tilde{r}_\pm$ with $\tilde{r}_+ \geq \tilde{r}_-$. The ordering of $r_\pm$ and $\tilde{r}_\pm$ depends on the sign of $\epsilon\left(E/m_p\right)$. Since $h'(r) > 0$ for $r \geq r_+$ and $h'(r) < 0$ for $r \leq r_-$, we then have $\tilde{r}_+ \geq r_+ \geq r_- \geq \tilde{r}_-$ when $\epsilon\left(E/m_p\right) > 0$. In this case, $\tilde{r}_+$ is the radius of the outermost horizon. Similarly, in the case with $\epsilon\left(E/m_p\right) < 0$, the radius of the outermost horizon is r_+.

To find the Hawking temperature on the outermost horizon, we need to calculate the imaginary part of $I_\pm$ by integrating $p_r^\pm(r)$ along the semicircle around the outermost horizon. As shown in [70,71], the probability of a particle tunneling from inside to outside the horizon is

$$P_{emit} \propto \exp\left[-\frac{2}{\hbar}\left(\mathrm{Im}\, I_+ - \mathrm{Im}\, I_-\right)\right]. \tag{32}$$

For a particle of energy E and angular momentum L residing in a system with temperature T and angular velocity ω, the Maxwell–Boltzmann distribution is [72]

$$P \propto \exp\left[-\frac{E - \omega L}{T}\right]. \tag{33}$$

From eqn. (33), the effective Hawking temperature can be read off from the Boltzmann factor in P_{emit}:

$$\tilde{T}_h = \frac{\hbar\left[E + N_\phi(r_h) L\right]}{2\left(\mathrm{Im}\, I_+ - \mathrm{Im}\, I_-\right)}, \tag{34}$$

where $-N_\phi(r_h) = -\tilde{g}_{t\phi}/\tilde{g}_{\phi\phi} = -g_{t\phi}/g_{\phi\phi}$ is the angular velocity of the FF rainbow BTZ black hole, and $r_h = r_+$ or $\tilde{r}_+$ is the outermost horizon. For the $\epsilon\left(E/m_p\right) < 0$ and $\epsilon\left(E/m_p\right) > 0$ cases, we find

- $\epsilon\left(E/m_p\right) < 0$: The outermost horizon is at $r = r_+$. Using the residue theory for the semi circle around $r = r_+$, we get

$$\mathrm{Im}\, I_+ = \mathrm{Im}\, I_- = \frac{\pi\left[E + N_\phi(r_+) L\right]}{h'(r_+)}, \tag{35}$$

which gives the effective Hawking temperature

$$\tilde{T}_h = \infty. \tag{36}$$

- $\epsilon\left(E/m_p\right) > 0$: The outermost horizon is at $r = \tilde{r}_+$. Using the residue theory for the semi circle around $r = \tilde{r}_+$, we get

$$\mathrm{Im}\, I_+ = 0 \tag{37}$$

$$\mathrm{Im}\, I_- = -\frac{2\pi\left[E + N_\phi(r_+) L\right]}{\tilde{H}'(\tilde{r}_+)} \times \sqrt{\frac{g^2\left(E/m_p\right)\left(1 + \frac{L^2}{m^2\tilde{r}_+^2}\right)}{f^2\left(E/m_p\right)} - \frac{L^2}{m^2\tilde{r}_+^2}}, \tag{38}$$

which gives the effective Hawking temperature

$$\tilde{T}_h = \frac{\hbar\tilde{H}'(\tilde{r}_+)}{4\pi\sqrt{1 - \epsilon\left(E/m_p\right)\left(1 + \frac{L^2}{m^2\tilde{r}_+^2}\right)}}. \tag{39}$$

When $g\left(E/m_p\right) = f\left(E/m_p\right) = 1$, one has $\epsilon\left(E/m_p\right) = 0$, $\tilde{r}_+ = r_+$ and $\tilde{H}(r) = h(r)$, which means that, as expected, $\tilde{T}_h$ of the FF rainbow BTZ black hole would reduce to T_h of the BTZ black hole, given in eqn. (12). To express $\tilde{T}_h$ in terms of E and L, we need to solve $\tilde{H}(r) = 0$ for $\tilde{r}_+$, whose expression is quite complicated. However for $\epsilon\left(E/m_p\right) \ll 1$, one has that, to $\mathcal{O}\left(\epsilon\left(E/m_p\right)\right)$,

$$\tilde{T}_h \approx T_h\left\{1 + \frac{\epsilon\left(E/m_p\right)}{2}\left[1 + \frac{L^2}{m^2 r_+^2} + \frac{\hbar^2\left(E - \frac{JL}{2r_+^2}\right)^2\left(\frac{2}{l^2} + \frac{3J^2}{2r_+^4}\right)}{8\pi^2 T_h^2 m^2} - \frac{\hbar\left(E - \frac{JL}{2r_+^2}\right)LJ}{\pi T_h m^2 r_+^3}\right]\right\}. \tag{40}$$

Since the Hawking radiation spectrum is dominated by low angular momentum modes [73], we can set $L = 0$ for simplicity. In this case, the effective Hawking temperature becomes

$$\tilde{T}_h \approx T_h\left\{1 + \frac{\epsilon\left(E/m_p\right)}{2}\left[1 + \frac{\hbar^2 E^2\left(\frac{2}{l^2} + \frac{3J^2}{2r_+^4}\right)}{8\pi^2 T_h^2 m^2}\right]\right\}, \tag{41}$$

which shows that the rainbow gravity correction tends to increase the Hawking temperature of the BTZ black hole.

The effective Hawking temperature of an ordinary (SF) rainbow BTZ black hole (4) is given by [21,39,48,49]

$$\tilde{T}_h = T_h\frac{g\left(E/m_p\right)}{f\left(E/m_p\right)} = T_h\sqrt{1 - \epsilon\left(E/m_p\right)}, \tag{42}$$

where has a simpler expression and is quite different from that of a FF rainbow BTZ black hole. While the effective Hawking temperature in the FF case depends on the energy and angular momentum of the raidatated particles, eqn. (42) shows that the effective Hawking temperature in the SF case only depends on the energy of the raidatated particles. Note that the effective Hawking temperature of the SF rainbow BTZ black hole is always finite even when $\epsilon\left(E/m_p\right) < 0$.

3.3. Thermodynamics of FF static rainbow BTZ black hole

For simplicity, we estimate the rainbow corrected temperature and entropy for a FF static rainbow BTZ black hole, which has $J = 0$. First, we can use the Heisenberg uncertainty principle to estimate the momentum p of an emitted particle [74,75]:

$$p \sim \delta p \sim \hbar/\delta x \sim \hbar/\tilde{r}_+. \tag{43}$$

Using $\tilde{H}(\tilde{r}_+) = 0$ and eqns. (43) and (1), we can express the energy E in terms of the black hole mass M. The effective Hawking temperature (39) then can be written as a function of M, $T(M)$, which can be interpreted as the rainbow corrected temperature of

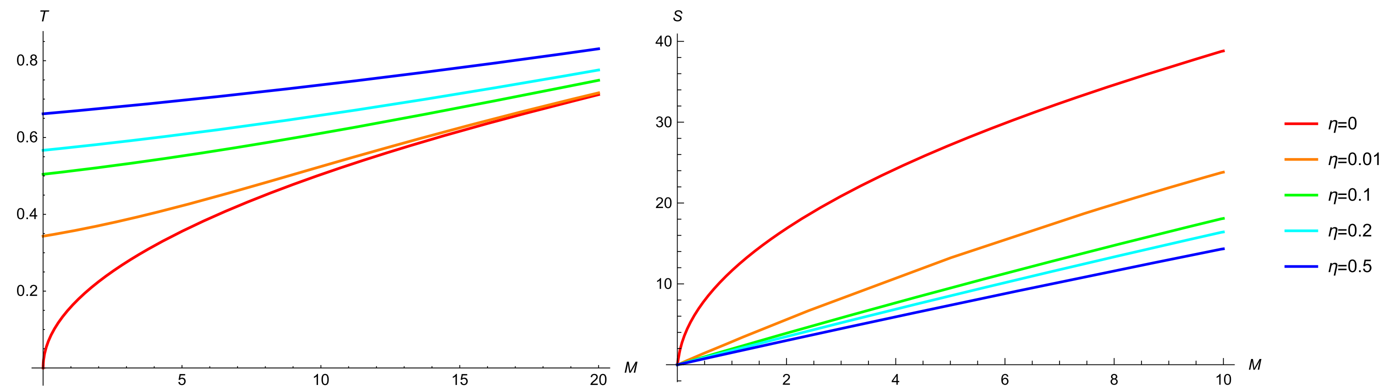

Fig. 1. Plots of the corrected temperature $T(M)$ and entropy $S(M)$ of a FF rainbow BTZ black hole for various values of η.

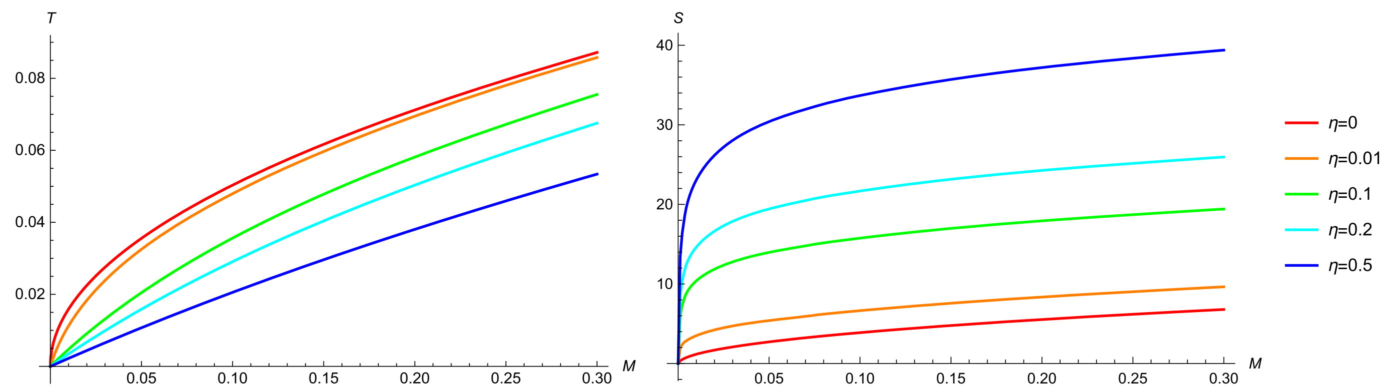

Fig. 2. Plots of the corrected temperature $T(M)$ and entropy $S(M)$ of a SF rainbow BTZ black hole for various values of η.

the black hole. Using the first law of black hole thermodynamics, we find that the entropy of the black hole is

$$S(M) = \int \frac{dM}{T(M)}. \tag{44}$$

When $M \gg \frac{\hbar^2}{m^2 l^2}$, the corrected Hawking temperature and entropy are estimated as

$$T(M) \sim \frac{\hbar\sqrt{M}}{2\pi l}\left[1+\frac{\epsilon\left(m/m_p\right)(M+1)}{2M}\right],$$
$$S(M) \sim \frac{\hbar M^{\frac{3}{2}}}{3l\pi}\left[1+\frac{\epsilon\left(m/m_p\right)(M+3)}{2M}\right], \tag{45}$$

respectively.

To numerically investigate $T(M)$ and $S(M)$, we focus on the Amelino-Camelia dispersion relation (3) with $n=2$ and $\eta>0$. We plot $T(M)$ and $S(M)$ for various values of η in Fig. 1, where we take $m=0.01$, $l=1$ and $\hbar=1$. The left panel of Fig. 1 shows that the black hole temperature increases with increasing η, which implies that the rainbow effects would speed up the evaporation of the black hole. Moreover, the terminal temperature is zero when $\eta=0$ while it is greater than zero when $\eta>0$, which means the rainbow effects would lead to a more violent death of the black hole. On the other hand, the right panel of Fig. 1 shows the black hole entropy decreases with increasing η. Therefore, the black hole tends to store less information when the rainbow effects are turned on.

Similarly, we can estimate the rainbow corrected temperature and entropy of a SF rainbow BTZ black hole with $J=0$. For the Amelino-Camelia dispersion relation (3) with $n=2$ and $\eta>0$, eqns. (1), (3) and (43) give the rainbow modified temperature of the static SF rainbow BTZ black hole

$$T(M) = \frac{\hbar\sqrt{M}}{2\pi l}\sqrt{1-\eta\frac{1+m^2 M}{M+\eta}}. \tag{46}$$

The entropy of the SF static rainbow BTZ black hole is also given by

$$S(M) = \int \frac{dM}{T(M)}. \tag{47}$$

We plot $T(M)$ and $S(M)$ for various values of η in the SF rainbow BTZ case in Fig. 2, where we take $m=0.01$, $l=1$ and $\hbar=1$. Contrary to the FF rainbow BTZ case (the left panel of Fig. 1), the left panel of Fig. 2 shows that, in the SF rainbow BTZ case, the black hole temperature decreases with increasing η, and hence the rainbow effects would slow down the evaporation of the black hole. Additionally, the terminal temperature is always zero even if $\eta>0$. The right panel of Fig. 2 shows that the black hole entropy increases with increasing η in the SF rainbow BTZ case, while the right panel of Fig. 1 shows that the black hole entropy decreases with increasing η in the FF rainbow BTZ case.

4. Discussion and conclusion

In this paper, we considered a FF rainbow BTZ black hole and analyzed the effects of rainbow gravity on the temperature and entropy. The metric of the FF rainbow BTZ black hole was first derived. Then, we used the Hamilton-Jacobi method to obtain the effective Hawking temperature of the rainbow BTZ black hole, which was shown to depend on the energy and angular momentum of radiated particles. Note that, for a massless particle, one has

Table 1
Results and implications for the Hawking temperature and the black hole entropy for the FF and SF rainbow BTZ black holes in the subluminal case.

	FF rainbow BTZ black hole	SF rainbow BTZ black hole
Temperature	The rainbow effects increase the temperature, which leads to a more violent death with a nonzero terminal temperature.	The rainbow effects decrease the temperature, which leads to a more peaceful death with a zero terminal temperature.
Entropy	The rainbow effects decrease the entropy, which means the black hole can store less information.	The rainbow effects increase the entropy, which means the black hole can store more information.

$$\frac{E}{p} = \sqrt{1 - \epsilon\left(E/m_p\right)}, \tag{48}$$

which means that $\epsilon\left(E/m_p\right) > 0$ corresponds to the subluminal case that has $E/p < 1$, and that $\epsilon\left(E/m_p\right) < 0$ to the superluminal case that has $E/p > 1$. It was found that the radiated particles experience an infinite effective Hawking temperature in the superluminal case, which might shed light on the black hole firewall paradox. However, the effective Hawking temperature of a SF rainbow BTZ black hole is always finite. In the subluminal case, the effective Hawking temperature was given in eqn. (39), and the corrected temperature and entropy of the black hole were estimated using the uncertainty principle. Focusing on the Amelino-Camelia dispersion relation, we numerically studied the temperature and entropy of the SF and FF rainbow BTZ black holes. The corresponding results and implications are summarized in Table 1.

We have discussed rainbow Schwarzschild black holes in the SF [39] and FF [40] cases. For FF and SF rainbow Schwarzschild black holes, we found that the effects of rainbow gravity play a similar role, e.g., they tend to decrease the Hawking temperature and increase the black hole entropy in the subluminal case. However, our results showed that the Hawking temperature and entropy of SF and FF rainbow BTZ black holes behave rather differently. It seems that the effects of rainbow gravity are quite model-dependent. So it would be interesting to study the thermodynamic properties of various black holes of different theories of gravity in different rainbow models, which may help us explore the effects of quantum gravity.

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References

[1] K. Schwarzschild, On the gravitational field of a mass point according to Einstein's theory, Sitz.ber. Preuss. Akad. Wiss. Berl. Math. Phys. 1916 (1916) 189, arXiv:physics/9905030.
[2] S.W. Hawking, Particle creation by black holes, Commun. Math. Phys. 43 (1975) 199, https://doi.org/10.1007/BF02345020, Erratum: Commun. Math. Phys. 46 (1976) 206, https://doi.org/10.1007/BF01608497.
[3] W.G. Unruh, Notes on black hole evaporation, Phys. Rev. D 14 (1976) 870, https://doi.org/10.1103/PhysRevD.14.870.
[4] G. Amelino-Camelia, Testable scenario for relativity with minimum length, Phys. Lett. B 510 (2001) 255, https://doi.org/10.1016/S0370-2693(01)00506-8, arXiv:hep-th/0012238.
[5] G. Amelino-Camelia, Relativity in space-times with short distance structure governed by an observer independent (Planckian) length scale, Int. J. Mod. Phys. D 11 (2002) 35, https://doi.org/10.1142/S0218271802001330, arXiv:gr-qc/0012051.
[6] J. Magueijo, L. Smolin, Lorentz invariance with an invariant energy scale, Phys. Rev. Lett. 88 (2002) 190403, https://doi.org/10.1103/PhysRevLett.88.190403, arXiv:hep-th/0112090.
[7] J. Magueijo, L. Smolin, Generalized Lorentz invariance with an invariant energy scale, Phys. Rev. D 67 (2003) 044017, https://doi.org/10.1103/PhysRevD.67.044017, arXiv:gr-qc/0207085.
[8] G. Amelino-Camelia, J.R. Ellis, N.E. Mavromatos, D.V. Nanopoulos, S. Sarkar, Tests of quantum gravity from observations of gamma-ray bursts, Nature 393 (1998) 763, https://doi.org/10.1038/31647, arXiv:astro-ph/9712103.
[9] D. Colladay, V.A. Kostelecky, Lorentz violating extension of the standard model, Phys. Rev. D 58 (1998) 116002, arXiv:hep-ph/9809521.
[10] S.R. Coleman, S.L. Glashow, High-energy tests of Lorentz invariance, Phys. Rev. D 59 (1999) 116008, arXiv:hep-ph/9812418.
[11] G. Amelino-Camelia, T. Piran, Planck scale deformation of Lorentz symmetry as a solution to the UHECR and the TeV gamma paradoxes, Phys. Rev. D 64 (2001) 036005, arXiv:astro-ph/0008107.
[12] T. Jacobson, S. Liberati, D. Mattingly, TeV astrophysics constraints on Planck scale Lorentz violation, Phys. Rev. D 66 (2002) 081302, arXiv:hep-ph/0112207.
[13] T.A. Jacobson, S. Liberati, D. Mattingly, F.W. Stecker, New limits on Planck scale Lorentz violation in QED, Phys. Rev. Lett. 93 (2004) 021101, arXiv:astro-ph/0309681.
[14] R.K. Mittleman, I.I. Ioannou, H.G. Dehmelt, N. Russell, Bound on CPT and Lorentz symmetry with a trapped electron, Phys. Rev. Lett. 83 (1999) 2116, https://doi.org/10.1103/PhysRevLett.83.2116.
[15] F. Cane, D. Bear, D.F. Phillips, M.S. Rosen, C.L. Smallwood, R.E. Stoner, R.L. Walsworth, V.A. Kostelecky, Bound on Lorentz and CPT violating boost effects for the neutron, Phys. Rev. Lett. 93 (2004) 230801, https://doi.org/10.1103/PhysRevLett.93.230801, arXiv:physics/0309070.
[16] L. Shao, B.Q. Ma, Lorentz violation induced vacuum birefringence and its astrophysical consequences, Phys. Rev. D 83 (2011) 127702, https://doi.org/10.1103/PhysRevD.83.127702, arXiv:1104.4438 [astro-ph.HE].
[17] D. Petry, MAGIC Telescope Collaboration, The MAGIC telescope – prospects for GRB research, Astron. Astrophys. Suppl. Ser. 138 (3) (1999) 601, https://doi.org/10.1051/aas:1999369, arXiv:astro-ph/9904178.
[18] G. Amelino-Camelia, J.R. Ellis, N.E. Mavromatos, D.V. Nanopoulos, Distance measurement and wave dispersion in a Liouville string approach to quantum gravity, Int. J. Mod. Phys. A 12 (1997) 607, arXiv:hep-th/9605211.
[19] G. Amelino-Camelia, Quantum-spacetime phenomenology, Living Rev. Relativ. 16 (2013) 5, arXiv:0806.0339 [gr-qc].
[20] J. Magueijo, L. Smolin, Gravity's rainbow, Class. Quantum Gravity 21 (2004) 1725, arXiv:gr-qc/0305055.
[21] S. Alsaleh, Thermodynamics of BTZ black holes in gravity's rainbow, Int. J. Mod. Phys. A 32 (15) (2017) 1750076, https://doi.org/10.1142/S0217751X17500762, arXiv:1703.09617 [physics.gen-ph].
[22] S. Hemming, E. Keski-Vakkuri, The spectrum of strings on BTZ black holes and spectral flow in the SL(2,R) WZW model, Nucl. Phys. B 626 (2002) 363, https://doi.org/10.1016/S0550-3213(02)00021-4, arXiv:hep-th/0110252.
[23] A.J.M. Medved, Radiation via tunneling from a de Sitter cosmological horizon, Phys. Rev. D 66 (2002) 124009, https://doi.org/10.1103/PhysRevD.66.124009, arXiv:hep-th/0207247.
[24] E.C. Vagenas, Semiclassical corrections to the Bekenstein-Hawking entropy of the BTZ black hole via selfgravitation, Phys. Lett. B 533 (2002) 302, https://doi.org/10.1016/S0370-2693(02)01695-7, arXiv:hep-th/0109108.
[25] M. Arzano, A.J.M. Medved, E.C. Vagenas, Hawking radiation as tunneling through the quantum horizon, J. High Energy Phys. 0509 (2005) 037, https://doi.org/10.1088/1126-6708/2005/09/037, arXiv:hep-th/0505266.
[26] S.Q. Wu, Q.Q. Jiang, Remarks on Hawking radiation as tunneling from the BTZ black holes, J. High Energy Phys. 0603 (2006) 079, https://doi.org/10.1088/1126-6708/2006/03/079, arXiv:hep-th/0602033.
[27] G.Q. Li, Hawking radiation and entropy of a black hole in Lovelock-Born-Infeld gravity from the quantum tunneling approach, Chin. Phys. C 41 (4) (2017) 045103, https://doi.org/10.1088/1674-1137/41/4/045103.
[28] P. Kraus, F. Wilczek, Selfinteraction correction to black hole radiance, Nucl. Phys. B 433 (1995) 403, arXiv:gr-qc/9408003.
[29] P. Kraus, F. Wilczek, Effect of selfinteraction on charged black hole radiance, Nucl. Phys. B 437 (1995) 231, arXiv:hep-th/9411219.
[30] K. Srinivasan, T. Padmanabhan, Particle production and complex path analysis, Phys. Rev. D 60 (1999) 024007, arXiv:gr-qc/9812028.

[31] M. Angheben, M. Nadalini, L. Vanzo, S. Zerbini, Hawking radiation as tunneling for extremal and rotating black holes, J. High Energy Phys. 0505 (2005) 014, arXiv:hep-th/0503081.
[32] R. Kerner, R.B. Mann, Tunnelling, temperature and Taub-NUT black holes, Phys. Rev. D 73 (2006) 104010, arXiv:gr-qc/0603019.
[33] D. Chen, H. Wu, H. Yang, Fermion's tunnelling with effects of quantum gravity, Adv. High Energy Phys. 2013 (2013) 432412, https://doi.org/10.1155/2013/432412, arXiv:1305.7104 [gr-qc].
[34] D. Chen, H. Wu, H. Yang, Observing remnants by fermions' tunneling, J. Cosmol. Astropart. Phys. 1403 (2014) 036, https://doi.org/10.1088/1475-7516/2014/03/036, arXiv:1307.0172 [gr-qc].
[35] D.Y. Chen, Q.Q. Jiang, P. Wang, H. Yang, Remnants, fermions' tunnelling and effects of quantum gravity, J. High Energy Phys. 1311 (2013) 176, https://doi.org/10.1007/JHEP11(2013)176, arXiv:1312.3781 [hep-th].
[36] D. Chen, Z. Li, Remarks on remnants by fermions' tunnelling from black strings, Adv. High Energy Phys. 2014 (2014) 620157, https://doi.org/10.1155/2014/620157, arXiv:1404.6375 [hep-th].
[37] D. Chen, H. Wu, H. Yang, S. Yang, Effects of quantum gravity on black holes, Int. J. Mod. Phys. A 29 (26) (2014) 1430054, https://doi.org/10.1142/S0217751X14300543, arXiv:1410.5071 [gr-qc].
[38] B. Mu, P. Wang, H. Yang, Minimal length effects on tunnelling from spherically symmetric black holes, Adv. High Energy Phys. 2015 (2015) 898916, https://doi.org/10.1155/2015/898916, arXiv:1501.06025 [gr-qc].
[39] B. Mu, P. Wang, H. Yang, Thermodynamics and luminosities of rainbow black holes, J. Cosmol. Astropart. Phys. 1511 (11) (2015) 045, https://doi.org/10.1088/1475-7516/2015/11/045, arXiv:1507.03768 [gr-qc].
[40] J. Tao, P. Wang, H. Yang, Free-fall frame black hole in gravity's rainbow, Phys. Rev. D 94 (6) (2016) 064068, https://doi.org/10.1103/PhysRevD.94.064068, arXiv:1602.08686 [gr-qc].
[41] Y. Ling, Rainbow universe, J. Cosmol. Astropart. Phys. 0708 (2007) 017, https://doi.org/10.1088/1475-7516/2007/08/017, arXiv:gr-qc/0609129.
[42] Y. Ling, Q. Wu, The big bounce in rainbow universe, Phys. Lett. B 687 (2010) 103, https://doi.org/10.1016/j.physletb.2010.03.028, arXiv:0811.2615 [gr-qc].
[43] R. Garattini, M. Sakellariadou, Does gravity's rainbow induce inflation without an inflaton?, Phys. Rev. D 90 (4) (2014) 043521, https://doi.org/10.1103/PhysRevD.90.043521, arXiv:1212.4987 [gr-qc].
[44] A. Awad, A.F. Ali, B. Majumder, Nonsingular rainbow universes, J. Cosmol. Astropart. Phys. 1310 (2013) 052, https://doi.org/10.1088/1475-7516/2013/10/052, arXiv:1308.4343 [gr-qc].
[45] Y. Ling, X. Li, H.b. Zhang, Thermodynamics of modified black holes from gravity's rainbow, Mod. Phys. Lett. A 22 (2007) 2749, https://doi.org/10.1142/S0217732307022931, arXiv:gr-qc/0512084.
[46] P. Galan, G.A.Mena Marugan, Entropy and temperature of black holes in a gravity's rainbow, Phys. Rev. D 74 (2006) 044035, https://doi.org/10.1103/PhysRevD.74.044035, arXiv:gr-qc/0608061.
[47] H. Li, Y. Ling, X. Han, Modified (A)dS Schwarzschild black holes in rainbow spacetime, Class. Quantum Gravity 26 (2009) 065004, https://doi.org/10.1088/0264-9381/26/6/065004, arXiv:0809.4819 [gr-qc].
[48] A.F. Ali, Black hole remnant from gravity's rainbow, Phys. Rev. D 89 (10) (2014) 104040, https://doi.org/10.1103/PhysRevD.89.104040, arXiv:1402.5320 [hep-th].
[49] A.F. Ali, M. Faizal, M.M. Khalil, Remnant for all black objects due to gravity's rainbow, Nucl. Phys. B 894 (2015) 341, https://doi.org/10.1016/j.nuclphysb.2015.03.014, arXiv:1410.5706 [hep-th].
[50] Y. Gim, W. Kim, Thermodynamic phase transition in the rainbow Schwarzschild black hole, J. Cosmol. Astropart. Phys. 1410 (2014) 003, https://doi.org/10.1088/1475-7516/2014/10/003, arXiv:1406.6475 [gr-qc].
[51] S.H. Hendi, M. Faizal, Black holes in Gauss-Bonnet gravity's rainbow, Phys. Rev. D 92 (4) (2015) 044027, https://doi.org/10.1103/PhysRevD.92.044027, arXiv:1506.08062 [gr-qc].
[52] A.F. Ali, M. Faizal, M.M. Khalil, Remnants of black rings from gravity's rainbow, J. High Energy Phys. 1412 (2014) 159, https://doi.org/10.1007/JHEP12(2014)159, arXiv:1409.5745 [hep-th].
[53] A.F. Ali, M. Faizal, M.M. Khalil, Absence of black holes at LHC due to gravity's rainbow, Phys. Lett. B 743 (2015) 295, https://doi.org/10.1016/j.physletb.2015.02.065, arXiv:1410.4765 [hep-th].
[54] A. Ashour, M. Faizal, A.F. Ali, F. Hammad, Branes in gravity's rainbow, Eur. Phys. J. C 76 (5) (2016) 264, https://doi.org/10.1140/epjc/s10052-016-4124-7, arXiv:1602.04926 [hep-th].
[55] A.F. Ali, M. Faizal, B. Majumder, Absence of an effective horizon for black holes in gravity's rainbow, Europhys. Lett. 109 (2) (2015) 20001, https://doi.org/10.1209/0295-5075/109/20001, arXiv:1406.1980 [gr-qc].
[56] Y. Heydarzade, P. Rudra, F. Darabi, A.F. Ali, M. Faizal, Vaidya spacetime in massive gravity's rainbow, Phys. Lett. B 774 (2017) 46, https://doi.org/10.1016/j.physletb.2017.09.049, arXiv:1710.00673 [gr-qc].
[57] S.H. Hendi, M. Momennia, B. Eslam Panah, M. Faizal, Nonsingular universes in Gauss–Bonnet gravity's rainbow, Astrophys. J. 827 (2) (2016) 153, https://doi.org/10.3847/0004-637X/827/2/153, arXiv:1703.00480 [gr-qc].
[58] S.H. Hendi, A. Dehghani, M. Faizal, Black hole thermodynamics in Lovelock gravity's rainbow with (A)dS asymptote, Nucl. Phys. B 914 (2017) 117, https://doi.org/10.1016/j.nuclphysb.2016.10.022, arXiv:1702.02431 [hep-th].
[59] S.H. Hendi, S. Panahiyan, B. Eslam Panah, M. Faizal, M. Momennia, Critical behavior of charged black holes in Gauss-Bonnet gravity's rainbow, Phys. Rev. D 94 (2) (2016) 024028, https://doi.org/10.1103/PhysRevD.94.024028, arXiv:1607.06663 [gr-qc].
[60] M. Banados, C. Teitelboim, J. Zanelli, The black hole in three-dimensional spacetime, Phys. Rev. Lett. 69 (1992) 1849, https://doi.org/10.1103/PhysRevLett.69.1849, arXiv:hep-th/9204099.
[61] B. Chen, S.x. Liu, J.j. Zhang, Thermodynamics of black hole horizons and Kerr/CFT correspondence, J. High Energy Phys. 1211 (2012) 017, https://doi.org/10.1007/JHEP11(2012)017, arXiv:1206.2015 [hep-th].
[62] R.G. Cai, Z.J. Lu, Y.Z. Zhang, Critical behavior in (2+1)-dimensional black holes, Phys. Rev. D 55 (1997) 853, https://doi.org/10.1103/PhysRevD.55.853, arXiv:gr-qc/9702032.
[63] R.G. Cai, J.B. Griffiths, Null particle solutions in three-dimensional (anti-)de Sitter spaces, J. Math. Phys. 40 (1999) 3465, https://doi.org/10.1063/1.532900, arXiv:gr-qc/9905011.
[64] R.G. Cai, J.H. Cho, Thermodynamic curvature of the BTZ black hole, Phys. Rev. D 60 (1999) 067502, https://doi.org/10.1103/PhysRevD.60.067502, arXiv:hep-th/9803261.
[65] J.M. Bardeen, B. Carter, S.W. Hawking, The four laws of black hole mechanics, Commun. Math. Phys. 31 (1973) 161, https://doi.org/10.1007/BF01645742.
[66] S.W. Kim, W.T. Kim, Y.J. Park, H. Shin, Entropy of the BTZ black hole in (2+1)-dimensions, Phys. Lett. B 392 (1997) 311, https://doi.org/10.1016/S0370-2693(96)01572-9, arXiv:hep-th/9603043.
[67] R.B. Mann, S.N. Solodukhin, Quantum scalar field on three-dimensional (BTZ) black hole instanton: heat kernel, effective action and thermodynamics, Phys. Rev. D 55 (1997) 3622, https://doi.org/10.1103/PhysRevD.55.3622, arXiv:hep-th/9609085.
[68] S.H. Hendi, Asymptotically charged BTZ black holes in gravity's rainbow, Gen. Relativ. Gravit. 48 (4) (2016) 50, https://doi.org/10.1007/s10714-016-2044-3, arXiv:1507.04733 [gr-qc].
[69] M. Dehghani, Thermodynamics of charged dilatonic BTZ black holes in rainbow gravity, Phys. Lett. B 777 (2018) 351, https://doi.org/10.1016/j.physletb.2017.12.048.
[70] J. Tao, P. Wang, H. Yang, Black hole radiation with modified dispersion relation in tunneling paradigm: static frame, Nucl. Phys. B 922 (2017) 346, https://doi.org/10.1016/j.nuclphysb.2017.06.022, arXiv:1505.03045 [gr-qc].
[71] P. Wang, H. Yang, S. Ying, Black hole radiation with modified dispersion relation in tunneling paradigm: free-fall frame, Eur. Phys. J. C 76 (1) (2016) 27, https://doi.org/10.1140/epjc/s10052-015-3858-y, arXiv:1505.04568 [gr-qc].
[72] L.D. Landau, E.M. Lifshitz, Linda E. Reichl, Statistical physics, part 1 (third edition), Phys. Today 34 (1) (1981) 74, https://doi.org/10.1063/1.2889978.
[73] E. Mc Caughey, Hawking radiation screening and Penrose process shielding in the Kerr black hole, Eur. Phys. J. C 76 (4) (2016) 179, https://doi.org/10.1140/epjc/s10052-016-4028-6, arXiv:1603.08774 [gr-qc].
[74] J.D. Bekenstein, Black holes and entropy, Phys. Rev. D 7 (1973) 2333.
[75] R.J. Adler, P. Chen, D.I. Santiago, The generalized uncertainty principle and black hole remnants, Gen. Relativ. Gravit. 33 (2001) 2101, arXiv:gr-qc/0106080.