Proceedings of Institute of Mathematics of NAS of Ukraine 2002, Vol. 43, Part 2, 432–438

The Lipkin–Meshkov–Glick Model and its Deformations through Polynomial Algebras

Nathalie DEBERGH and Florica STANCU

Fundamental Theoretical Physics, University of Liège, Institute of Physics, B.5, Sart Tilman, B-4000 Liège, Belgium
E-mail: *Nathalie.Debergh@ulg.ac.be, fstancu@ulg.ac.be*

We search for solutions of the many-particle Hamiltonian of Lipkin, Meshkov and Glick in the context of the $sl(2,\mathbb{R})$ deformed polynomial algebra. The reducibility of the original model is proved according to the representations of this algebra. A new symmetry is uncovered, which further splits any matrix of a given j multiplet into two submatrices. In this way the diagonalization of the Hamiltonian matrix is simplified and the entire spectrum of the many-particle Hamiltonian is easily recovered. Supplementary eigenvalues stemming from the deformed algebra approach are also introduced. We indicate how they can lead to a new class of deformed-type models.

1 Introduction

Quantum mechanical equations with analytic solutions are rare. Only some interactions like e.g. the harmonic oscillator or the Coulomb potential give rise to a class of equations which are called exactly solvable. But sometimes one can weaken the condition of exact solvability by asking for an exact knowledge of a *finite* number of solutions only. This leads to what is referred to in [1] as quasi-exact solvability. Quasi-exactly solvable models have been essentially developed in a nonrelativistic context. They are characterized by the fact that, up to a change of variables as well as a transformation at the level of the wavefunctions, their Hamiltonians can be expressed as at most a quadratic function of the generators of a Lie algebra, namely $sl(2,\mathbb{R})$ for algebras of rank one. These generators stabilize a finite-dimensional space and so do the Hamiltonians which can be easily diagonalized within this space.

In physical examples at most a quadratic function of generators of $sl(2,\mathbb{R})$ is a consequence of the assumption of a two-body interaction between particles. One of the well-known quasi-exactly solvable models is that proposed by Lipkin, Meshkov and Glick [2], developed for treating many particle systems. Another one is the spin Van der Waals model used in statistical mechanics [3]. Interestingly enough, it has been shown that these two models are equivalent and represent particular cases of a more general Hamiltonian [4]. In these two cases the $sl(2,\mathbb{R})$ generators are called quasi-spin or pseudo-spin operators.

Here we refer to the work of Lipkin, Meshkov and Glick (LMG), who constructed a two N-fold degenerate level Hamiltonian where N is the number of fermions in the system. The two levels are separated by an energy ϵ. The simplified version of the LMG Hamiltonian, which we consider here, contains only terms which mix particle-hole configurations. The corresponding Hamiltonian reads

$$H_{\rm LMG} = \epsilon j_0 + \frac{\delta\epsilon}{2N}\left(j_+^2 + j_-^2\right), \tag{1}$$

where δ is the interaction strength, while the $sl(2,\mathbb{R})$ generators j_0, $j_\pm$ are realized as

$$j_0 = -\frac{N}{2} + \frac{1}{2}\sum_{m=1}^{N}\left(\alpha_m^\dagger\alpha_m + \beta_m^\dagger\beta_m\right), \tag{2}$$

$$j_+ = \sum_{m=1}^{N} \alpha_m^\dagger \beta_m^\dagger, \tag{3}$$

$$j_- = \sum_{m=1}^{N} \alpha_m \beta_m, \tag{4}$$

and satisfy

$$[j_0, j_\pm] = \pm j_\pm, \qquad [j_+, j_-] = 2j_0. \tag{5}$$

In the definitions (2)–(4) the fermion operators $\beta_m^\dagger$, β_m create and annihilate holes in the lower level, while $\alpha_m^\dagger$, α_m create and annihilate particles in the upper level. These operators are such that

$$\{\alpha_m, \alpha_n^\dagger\} = \{\beta_m, \beta_n^\dagger\} = \delta_{mn},$$
$$[\alpha_m, \beta_n] = [\alpha_m, \beta_n^\dagger] = [\beta_m, \alpha_n^\dagger] = [\alpha_m^\dagger, \beta_n^\dagger] = 0.$$

The Casimir operator of the $sl(2,\mathbb{R})$ algebra

$$C_1 = \frac{1}{2}\{j_+, j_-\} + j_0^2 \tag{6}$$

evidently commutes with the Hamiltonian (1). Hence the Hamiltonian matrix splits into submatrices each associated with a given value of j and of order $2j+1$. Each state in a j multiplet has a different number of excited particle-hole pairs. The interaction part of (1) mixes states within the same j multiplet but cannot mix states having different eigenvalues of C_1. It can only excite or de-excite two particle-hole pairs or in other words it can only change the eigenvalue of j_0 by two units. From the definition (2), it follows that the eigenvalues of j_0 are given by half the difference between the number of particles in the upper level and the number of particles in the lower level. Then the maximum eigenvalue of j_0 and of j is $\frac{N}{2}$. The largest matrix to be diagonalized in (1) is thus of dimension $N+1 = 2j+1$.

The main purpose of this paper is to revisit the LMG Hamiltonian (1) in the context of the $sl(2,\mathbb{R})$ deformed polynomial algebra. In such a context, we show that the largest matrix associated to a given N can be split into two submatrices of dimensions $\frac{N}{2}+1$ and $\frac{N}{2}$ for N even and two submatrices, both of dimensions $\frac{N+1}{2}$ for N odd. This is due to the presence, apart from (6), of an additional invariant, i.e. the Casimir operator of the deformed algebra. Moreover, the polynomial deformation technique leads to new representations corresponding to new eigenvalues appropriate to a deformed LMG model.

2 The deformed polynomial algebra approach

Instead of (1) in this section we propose to consider the following Hamiltonian [5]

$$H = \epsilon(2J_0 + \delta(J_+ + J_-)) \tag{7}$$

containing the operators J_0, $J_\pm$, which satisfy the following polynomial algebra (as compared with (5))

$$[J_0, J_\pm] = \pm J_\pm, \tag{8}$$

$$[J_+, J_-] = -\frac{16}{N^2} J_0^3 + \frac{2}{N^2}\left(2j^2 + 2j - 1\right) J_0, \tag{9}$$

where j is an eigenvalue of the operator C_1 as defined by (6). Such a choice is justified by the fact that the particular realization of the algebra (8)–(9)

$$J_0 = \frac{1}{2} j_0, \qquad J_\pm = \frac{1}{2N} j_\pm^2, \tag{10}$$

makes the Hamiltonian (7) to coincide with (1). However realizations other than (10) can be produced in general, leading to new eigenvalues, different from those of (1), as shown below.

Indeed the Casimir operator of the $sl(2,\mathbb{R})$ deformed polynomial algebra (8)–(9) is

$$C_2 = J_+ J_- - \frac{4}{N^2} J_0^4 + \frac{8}{N^2} J_0^3 + \frac{2j^2+2j-5}{N^2} J_0^2 - \frac{2j^2+2j-1}{N^2} J_0 \tag{11}$$

and two types of finite-dimensional representations arise. The first ones are defined according to

$$\begin{aligned} &J_0|J,M\rangle = (M+c)|J,M\rangle, \\ &J_+|J,M\rangle = f(M)|J,M+1\rangle, \qquad J_-|J,M\rangle = g(M)|J,M-1\rangle, \end{aligned} \tag{12}$$

with $M = -J, -J+1, \ldots, J-1, J,\ J = 0, \frac{1}{2}, 1, \ldots$ and

$$\begin{aligned} f(M-1)g(M) = {} & \frac{1}{N^2}(J-M+1)(J+M) \\ & \times \left(2j^2+2j-1-4J^2-4J-4M^2+4M+8(1-2M)c-24c^2\right). \end{aligned}$$

The real number c can take three distinct values [6] given by

$$c = 0 \qquad \text{and} \qquad c = \pm\sqrt{\frac{1}{4}j(j+1) - \frac{1}{8} - J(J+1)}.$$

The second type of representations are characterized by the following equations

$$\begin{aligned} &J_0|J',M'\rangle = \left(\frac{M'}{2}\right)|J',M'\rangle, \\ &J_+|J',M'\rangle = f'\left(M'\right)|J',M'+2\rangle, \qquad J_-|J',M'\rangle = g'\left(M'\right)|J',M'-2\rangle, \end{aligned} \tag{13}$$

where $J' = 0, 1, 2, \ldots$ and

$$\begin{aligned} f'\left(M'-2\right)g'\left(M'\right) = {} & \frac{1}{4N^2}\left(J'-M'+2\right)\left(J'+M'\right) \\ & \times \left(2j^2+2j-1-J'^2-2J'-M'^2+2M'\right) \end{aligned} \tag{14}$$

if $M' = -J', -J'+2, \ldots, J'-2, J'$ and

$$f'\left(M'-2\right)g'\left(M'\right) = \frac{1}{4N^2}\left(J'-M'+1\right)\left(J'+M'-1\right)\left(2j^2+2j-J'^2-M'^2+2M'\right) \tag{15}$$

if $M' = -J'+1, -J'+3, \ldots, J'-3, J'-1$. In the cases where $J' = \frac{1}{2}, \frac{3}{2}, \ldots$, J' must be equal to j (M' to m) and

$$f'(m-2)g'(m) = \frac{1}{4N^2}(j+m)(j+m-1)(j-m+1)(j-m+2).$$

It is important to note that the polynomial algebra provides a new "quantum number" c, as introduced above. It helps to distinguish between the eigenvalues of (7) corresponding to even and odd N. For N even one has $c = 0$ and for N odd $c = \pm 1/4$.

In the following we shall drop the representation (13) due to the fact that it is reducible. Indeed evaluating the eigenvalue of the Casimir operator C_2 of (11) within the invariant subspace of the representation (13) we obtain two distinct values which implies that the invariant subspace splits into the direct sum

$$\left(J' = n, c = 0\right)_{(13)} = \left(J = \frac{n}{2}, c = 0\right)_{(12)} \oplus \left(J = \frac{n-1}{2}, c = 0\right)_{(12)}, \tag{16}$$

where the left hand side refers to the representation space of (13) and each bracket in the right hand side designates an invariant subspace of (12). A similar decomposition holds for half-integer j

$$\left(J' = j = n + \frac{1}{2}, c = 0\right)_{(13)} = \left(J = \frac{n}{2}, c = \frac{1}{4}\right)_{(12)} \oplus \left(J = \frac{n}{2}, c = -\frac{1}{4}\right)_{(12)} \tag{17}$$

for any integer n. The original LMG model defined by (1) or equivalently by (7) with the realization (10) is clearly connected to the representations (13) with $J' = j$ (J' being an integer or a half integer). We can conclude that the LMG Hamiltonian matrix is reducible. More precisely, according to equations (16) and (17), a matrix Hamiltonian of dimension $2n+1$ can be split into a direct sum of two submatrices of dimensions $n+1$ and n for j even and a matrix of dimension $2n+2$ can be split into two matrices, each of dimension $n+1$, for j half integer. We thus obtain the result mentioned in the Introduction with $N = 2n$ and $N = 2n+1$ respectively, such result being significant for a large number of particles. Then searching for the spectrum of the Hamiltonian (7) amounts to the diagonalization of the matrix $\langle H \rangle$ given by

$$\begin{pmatrix} 2J+2c & \delta f(J-1) & 0 & 0 & \cdot & \cdot & 0 \\ \delta g(J) & 2J-2+2c & \delta f(J-2) & 0 & \cdot & \cdot & 0 \\ 0 & \delta g(J-1) & 2J-4+2c & \delta f(J-3) & \cdot & \cdot & 0 \\ 0 & 0 & \delta g(J-2) & 2J-6+2c & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -2J+2+2c & \delta f(-j) \\ 0 & 0 & 0 & 0 & \cdot & \delta g(-J+1) & -2J+2c \end{pmatrix}. \tag{18}$$

obtained in the invariant subspace defined by (12). In the following section we are going to illustrate these findings on specific examples.

3 Examples

3.1 The $N = 2$ case

We first consider the simplest $N = 2$ case in order to easily illustrate our results. The complete LMG matrix is of dimension 4, corresponding to the four possible states of two particles occupying two levels (the two particles can be on the lower level, or on the upper one, or one particle can be on the lower while the other can be on the upper level or vice-versa). Following the original LMG Hamiltonian (1), the matrix of dimension 4 splits into $3+1$ while the matrix of (7) in the invariant space of the representation (12) splits into $2+1+1$ (corresponding to $J = \frac{1}{2}$ and $J = 0$ twice). The eigenvalues E (in units of ϵ) are obtained from the diagonalization of three matrices of type (18) of dimensions 2, 1 and 1 respectively. The eigenvalues are summarized in the following table

j	J	E
0	0	0
1	0	0
	$\frac{1}{2}$	$\pm\sqrt{1+\frac{1}{4}\delta^2}$

3.2 The $N = 8$ case

For $N = 8$, there are $2^8 = 256$ states. The largest original LMG matrix corresponds to $j = \frac{N}{2} = 4$, the others being associated to $j = 3$ (7 times), $j = 2$ (20 times), $j = 1$ (28 times) and $j = 0$ (14 times). Following the decompositions (16)–(17) the polynomial algebra leads to other representations: $J = 2$ (1 time), $J = \frac{3}{2}$ (8 times), $J = 1$ (27 times), $J = \frac{1}{2}$ (48 times) and $J = 0$ (42 times). The corresponding eigenvalues come from the diagonalization of matrices of type (18) and are given in units of ϵ in the following table

j	J	E
0	0	0
1	0	0
	$\frac{1}{2}$	$\pm\sqrt{1+\frac{1}{64}\delta^2}$
2	$\frac{1}{2}$	$\pm\sqrt{1+\frac{9}{64}\delta^2}$
	1	$0, \pm\sqrt{4+\frac{3}{16}\delta^2}$
3	1	$0, \pm\sqrt{4+\frac{15}{16}\delta^2}$
	$\frac{3}{2}$	$\pm\sqrt{5+\frac{33}{64}\delta^2 \pm \sqrt{16+\frac{3}{2}\delta^2+\frac{27}{128}\delta^4}}$
4	$\frac{3}{2}$	$\pm\sqrt{5+\frac{113}{64}\delta^2 \pm \sqrt{16+\frac{19}{2}\delta^2+\frac{275}{128}\delta^4}}$
	2	$0, \pm\sqrt{10+\frac{59}{32}\delta^2 \pm \sqrt{36-\frac{9}{8}\delta^2+\frac{2025}{1024}\delta^4}}$

4 Supplementary eigenvalues

In the previous section the tables contain the eigenvalues of the Hamiltonian (1) only. They were obtained through the polynomial algebra technique. However the polynomial algebra is richer than the usual $sl(2,\mathbb{R})$ algebra, associated with the quasi-spin formalism in the LMG model. As seen above, its representations have three labels (J, c, j) instead of one (j) for $sl(2,\mathbb{R})$. Thus the number of representations is larger. This is particularly clear from the table corresponding to $N = 8$. Indeed when $j = 2$ for example, we can see that the eigenvalues of the LMG Hamiltonian are recovered when $J = \frac{1}{2}$ and $J = 1$ while the case $J = 0$ is missing and must correspond to another model. The same situation holds for $j = 3$, when $J = 0$ or $J = \frac{1}{2}$ and $j = 4$ when $J = 0$, $J = \frac{1}{2}$ and $J = 1$. These new possibilities are excluded by the Hamiltonian (1) but not by (7). They lead to supplementary eigenvalues as summarized in the following table

j	J	E
2	0	0
3	0	0
	$\frac{1}{2}$	$\pm\sqrt{1+\frac{21}{64}\delta^2}$
4	0	0
	$\frac{1}{2}$	$\pm\sqrt{1+\frac{37}{64}\delta^2}$
	1	$0, \pm\sqrt{4+\frac{31}{16}\delta^2}$

Taking for example the maximal value of j, i.e. $j = 4$ associated to $N = 8$ we can see that these supplementary eigenvalues are surprisingly close to some of the original LMG Hamiltonian. Indeed when $\delta = 1$, we have

j	J	E
4	0	0
	$\frac{1}{2}$	± 1.256
	1	$0, \pm 2.437$
	$\frac{3}{2}$	$\pm 1.228, \pm 3.467$
	2	$0, \pm 2.402, \pm 4.232$

i.e. very close to the numerical values shown in [5]. The same kind of results hold for any number of particles. In order to fix the ideas, for an even number $N = 2n$ of particles, the largest matrix corresponds to $j = n$, the values $J = \frac{n-1}{2}$, $\frac{n}{2}$ give rise to the LMG eigenvalues while the cases $J = 0, \frac{1}{2}, 1, \ldots, \frac{n}{2} - 1$ lead to supplementary solutions, close and larger than the LMG ones for a fixed j. Moreover the closeness is better realized for δ smaller, as it can be seen from the analytic expressions.

A natural question then arises: to what kind of model do correspond these supplementary eigenvalues? In order to answer this question, let us once again concentrate on the case of $N = 8$ particles and, this time, on the representations (13). We have five different values as far as J' is concerned, i.e. $J' = 0, 1, 2, 3, 4$. In fact, according to (13)–(15) we can generalize the realization (10) to

$$J_{\pm} = \frac{1}{16} M(J') j_{\pm}^2, \tag{19}$$

where $M(J')$ is a diagonal matrix depending on J' and of dimension $2J' + 1$. In principle this matrix should be different for each value of J'. It is interesting to note that this diagonal matrix reduces to the identity I for $J' = J'_{\max} = \frac{N}{2} = 4$ only, in agreement with (10). With this generalization the Hamiltonian (7) becomes

$$H = \epsilon j_0 + \frac{\delta \epsilon}{2N} \left(M(J') j_+^2 + j_-^2 M(J') \right) \tag{20}$$

with $J' = 0, 2, 4, \ldots, \frac{N}{2}$ and $M(\frac{N}{2}) = I$. In general the operators (19) can also be written as

$$J_+ = \frac{1}{2N} M(J') j_+^2 \equiv \frac{1}{2N} (j'_+)^2, \qquad J_- = \frac{1}{2N} j_-^2 M(J') \equiv \frac{1}{2N} (j'_-)^2$$

with

$$[j_0, j'_{\pm}] = \pm j'_{\pm}, \tag{21}$$

$$[j'_+, j'_-] = \sum_{k=0}^{J'-1} c_k j_0^{2k+1}, \tag{22}$$

where c_k are coefficients being fixed according to N and J'. The relations (21)–(22) are those of a polynomial deformation of $sl(2, \mathbb{R})$ except when $J' = 1$ and $J' = \frac{N}{2}$ where it is equivalent to $sl(2, \mathbb{R})$ ($J' = 0$ leading to trivial results). We can then conclude that our model (7) or equivalently (20) represents the usual LMG model (corresponding to $J' = \frac{N}{2}$) plus $\frac{N}{2}$ deformed LMG models (corresponding to $J' = 0, 1, \ldots, \frac{N}{2} - 1$ and $M(J') \neq I$), the deformed models giving rise to supplementary eigenvalues as discussed in this section. In all cases $M(J')$ is uniquely defined by the deformed algebra.

5 Summary

We have presented a derivation of the entire spectrum of the many-particle Hamiltonian of Lipkin, Meshkov and Glick in the context of the $sl(2, \mathbb{R})$ deformed polynomial algebra. For any given number N of particles the spectrum first splits into j multiplets of the $sl(2,\mathbb{R})$ algebra. The eigenvalues associated with the largest j are non-degenerate except for $E = 0$. We have shown that the Hamiltonian matrix of each j further splits into two submatrices corresponding to two distinct irreducible representations of the deformed polynomial algebra. In order to illustrate the method we have derived explicit analytic expressions for the eigenvalues of the LMG Hamiltonian for $N = 2$ and 8. Our method can evidently be extended to any N.

Furthermore we have shown that the deformed polynomial algebra related to the LMG model implies a larger spectrum than that of the model itself. Some of the new eigenvalues present characteristics similar to those of the LMG model and actually correspond to a superposition of specific deformed LMG models where, once again, the deformed polynomial algebra $sl(2,\mathbb{R})$ plays a prominent role.

[1] Turbiner A.V., Quasi-exact solvable problems and $sl(2)$ algebra, *Comm. Math. Phys.*, 1988, V.118, 467–474; Ushveridze A.G., Quasi-exactly solvable models in quantum mechanics, I.O.P., 1994.

[2] Lipkin H.J., Meshkov N. and Glick A.J., Validity of many-body approximation methods for a solvable model, (I) Exact solutions and perturbation theory, *Nucl. Phys.*, 1965, V.62, 188–198; (II) Linearization procedures, ibid., 199–210; (III) Diagram summations, ibid., 211–224;
Klein A. and Marshalek E.R., Boson realizations of Lie algebras with applications to nuclear physics, *Rev. Mod. Phys.*, 1995, V.63, 375–558;
Lipkin H.J., An extension of a two-shell solvable model to include two collective excitations, *Nucl. Phys. A*, 1999, V.658, 240–245.

[3] Dekeyser R. and Lee M.H., Nonequilibrium statistical mechanics of the spin 1/2 Van der Waals model. I. Time evolution of a single spin, *Phys. Rev. B*, 1991, V.43, 8123–8130.

[4] Gilmore R., Critical properties of pseudospin Hamiltonians, *J. Math. Phys.*, 1984, V.25, 2336–2338.

[5] Debergh N and Stancu Fl., On the exact solutions of the Lipkin–Meshkov–Glick model, *J. Phys. A*, 2001, V.34, 3265–3276; ibid., 4267 (corrigenda).

[6] Debergh N., The relation between polynomial deformations of $sl(2,\mathbb{R})$ and quasi-exact solvability, *J. Phys. A*, 2000, V.33, 7109–7121.