FERMION MASS GENERATION IN CHIRAL-SYMMETRIC GAUGE THEORIES

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abstract

The possibility of generating fermion masses in chiral-symmetric gauge theories has been examined with the help of renormalization group equations.

1. Renormalization Group Equations

In discussing dynamical generation of fermion masses the most useful tool is the renormalization group (RG) method, and we shall briefly recapitulate essential features of this method.

Green's functions in gauge theories with exponentiated mass-insertion[1)] are defined by

$$G^{(n,m)}(x,...,y,...,z,...:K) \\ =< 0|T[\psi(x)...\overline{\psi}(y)...\phi_\lambda(z)...\exp\left(iKm\int d^4uS(u)\right)]|0>, \tag{1.1}$$

where the scalar density S is bilinear in the fermion fields and is normalized by

$$< p|S|p >= \overline{u}(p)u(p). \tag{1.2}$$

$u(p)$ and $\overline{u}(p)$ denote the Dirac spinors of the fermion fields corresponding to a single fermion state $|p>$ of momentum p. Also, for a given local operator $A(w)$ we can define Green's functions of the form:

$$A^{(n,m)}(w;x,...,y,...,z,...:K) \\ =< 0|T\left[A(w)\psi(x)...\overline{\psi}(y)...\phi_\lambda(z)...\exp(iKm\int d^4uS(u))\right]|0> . \tag{1.3}$$

These Green's functions satisfy homogeneous Callan-Symanzik (CS) equations of the form:[1)]

$$(\mathcal{D} + n\gamma_F + m\gamma_V)G^{(n,m)}(...; g, m, \alpha : K) = 0, \tag{1.4}$$

where n and m denote the numbers of the fermion fields and of the gauge fields in the T-product, respectively, and

$$\mathcal{D} = m\frac{\partial}{\partial m} + \beta\frac{\partial}{\partial g} - 2\alpha\gamma_V\frac{\partial}{\partial \alpha} - \bigl(1 + (1-\gamma_S)K\bigr)\frac{\partial}{\partial K}. \tag{1.5}$$

γ_F, γ_V and γ_S denote, respectively, the anomalous dimensions of the operators ψ, ϕ_λ and S.

For practical purposes it is convenient to change the set of parameters from g, m, α and K to g, m, α and m_R, where m_R is called the effective mass and is defined by[2)]

$$m_R = m\bigl(1 + KB^{-1}(g)\bigr), \tag{1.6}$$

where $B(g)$ is characterized by the following equation:

$$\beta\frac{dB}{dg} + (1-\gamma_S)B = 1. \tag{1.7}$$

In terms of the new set of parameters the differential operator $\mathcal{D}$ assumes the following form:[3)]

$$\mathcal{D} = m\frac{\partial}{\partial m} + \beta\frac{\partial}{\partial g} - 2\alpha\gamma_V\frac{\partial}{\partial \alpha} - \gamma_\theta m_R\frac{\partial}{\partial m_R}, \tag{1.8}$$

where γ_θ is related to B through

$$1 + \gamma_\theta = B^{-1}. \tag{1.9}$$

2. Chiral Symmetry

In classical gauge theories the concept of chiral symmetry is equivalent to the vanishing of the bare fermion mass m_0. The classical Dirac equation for the fermion

$$\gamma_\mu(\partial_\mu - ig\frac{\lambda^a}{2}\phi^a_\mu)\psi + m_0\psi = 0 \tag{2.1}$$

is invariant under the chiral transformation

$$\psi \to \exp(i\alpha\gamma_5)\cdot\psi, \tag{2.2}$$

provided that the bare mass m_0 vanishes,

$$m_0 = 0. \tag{2.3}$$

The equivalence ceases to be valid in quantum theory, however, because of the divergent character of the theory. Indeed, in quantum field theory we often encounter a tricky relation:

$$0 \times \infty = \text{finite}, \tag{2.4}$$

which is reminiscent of anomalies characteristic of quantum field theory. We shall elucidate on this point in QCD.

Let us decompose the quark propagator as

$$S_F(x) = -(\gamma\cdot\partial)\Delta_1(x) + m\Delta_2(x), \tag{2.5}$$

and let us assume a spectral representation of $\Delta_i(x)$ as

$$\Delta_i(x) = \int d\mu^2 \rho_i(\mu^2)\Delta_F(x,\mu^2), \quad (i = 1,2) \tag{2.6}$$

where $\Delta_F(x,\mu^2)$ denotes the free propagator for mass μ. Then the physical mass m is related to the bare mass m_0 through[2)]

$$m_0\int d\mu^2\rho_1(\mu^2) = m\int d\mu^2\rho_2(\mu^2), \tag{2.7}$$

provided that $<\phi^a_\mu> = 0$. In the Landau gauge the RG analysis leads us to the following relationships[2)] in QCD:

$$\int d\mu^2\rho_1(\mu^2) = Z_2^{-1}(g), \quad \text{finite}, \tag{2.8}$$

$$\int d\mu^2 \rho_2(\mu^2) = 0. \tag{2.9}$$

Then we can deduce on the basis of Eqs.(2.7), (2.8) and (2.9) that

$$m_0/m = 0. \tag{2.10}$$

This result poses a serious doubt on the equivalence between chiral symmetry and the vanishing bare mass. Suppose that the physical mass m is finite, then Eq.(2.10) implies $m_0 = 0$, and consequently the resulting theory should be chiral-symmetric no matter how we choose the physical mass if we should insist on the classical equivalence. This sounds very unlikely, however. We must admit, therefore, that the classical equivalence should be broken by quantum corrections, and we must look for a proper definition of chiral symmetry expressed in terms of renormalized quantities alone.

3. Dynamical Breakdown of Chiral Symmetry

In an attempt to define chiral symmetry we propose to define it by the existence of an axial-vector current X_λ satisfying the following two conditions:

$$\partial_\lambda X_\lambda = 0, \tag{3.1}$$

and the equal-time commutation relations (ETCR):

$$\delta(x_0 - y_0)[X_0(x), \psi(y)] = -\gamma_5 \psi(y) \delta^4(x - y), \tag{3.2a}$$

$$\delta(x_0 - y_0)[X_0(x), \overline{\psi}(y)] = -\overline{\psi}(y)\gamma_5 \delta^4(x - y). \tag{3.2b}$$

In what follows we shall look for the condition for the existence of such a current in QCD. For this purpose we introduce some unrenormalized expressions first.

$$\begin{aligned} A_\lambda^{(0)} &= i\overline{\psi}^{(0)}\gamma_\lambda\gamma_5\psi^{(0)}, \\ P^{(0)} &= i\overline{\psi}^{(0)}\gamma_5\psi^{(0)}, \\ S^{(0)} &= \overline{\psi}^{(0)}\psi^{(0)}. \end{aligned} \tag{3.3}$$

Then we have

$$\partial_\lambda X_\lambda^{(0)} = 2m_0 P^{(0)}. \tag{3.4}$$

A flavor-changing current $X_\lambda^{(0)}$ can be identified with $A_\lambda^{(0)}$, but for a flavor-conserving current as (3.3) we have to identify $X_\lambda^{(0)}$ with the following combination:[4]

$$X_\lambda^{(0)} = A_\lambda^{(0)} - C_\lambda^{(0)}, \tag{3.5}$$

where $C_\lambda^{(0)}$ denotes the Chern-Simons term whose explicit form is irrelevant in what follows. The current satisfies the ETCR (3.2) so that we may assume non-renormalization of $X_\lambda^{(0)}$, and it may be identified with the renormalized one:

$$X_\lambda = X_\lambda^{(0)}. \tag{3.6}$$

The scalar density is renormalized as in Eq.(1.2), but the pseudoscalar density will be renormalized by

$$m_0 P^{(0)} = mP, \tag{3.7}$$

in conformity with the renormalization prescription adopted by Adler and Bardeen.[5] Then we have

$$\partial_\lambda X_\lambda = 2mP. \tag{3.8}$$

Here we have assumed that all flavors of quarks carry the same physical mass for simplicity. This is not the only way of renormalizing $P^{(0)}$, and we shall introduce an alternative prescription in what follows.

The unrenormalized currents satisfy the following ETCR:

$$\delta(x_0 - y_0)[X_0(x), S^{(0)}(y)] = 2iP^{(0)}(y)\delta^4(x-y), \tag{3.9}$$

so that its renormalized version is given by

$$\delta(x_0 - y_0)[X_0(x), S(y)] = 2ibP(y)\delta^4(x-y), \tag{3.10}$$

where

$$P^{(0)} = Z_P P, \quad S^{(0)} = Z_S S, \quad b = Z_P Z_S^{-1}. \tag{3.11}$$

Now we introduce $\tilde{P}$ by

$$\tilde{P} = bP, \quad P^{(0)} = Z_S \tilde{P}. \tag{3.12}$$

The RG equation for b in the Landau gauge reads as

$$\Big(\beta \frac{d}{dg} + \gamma_S - \gamma_P\Big) b = 0, \tag{3.13}$$

and the renormalization prescription (3.7) gives[4)]

$$b = 1 - \gamma_P. \tag{3.14}$$

We can easily find that this b is related to B, with the help of Eq.(1.7), through

$$B = b^{-1}, \tag{3.15}$$

and

$$\gamma_\theta = -\gamma_P. \tag{3.16}$$

Combining these relationships we finally arrive at

$$\partial_\lambda X_\lambda = 2mB(g)\tilde{P}. \tag{3.17}$$

A relationship indicating the anomalous character of the theory is illustrated by

$$mB(g) = m_0 Z_S. \tag{3.18}$$

The bare mass m_0 is zero and Z_S is divergent, whereas the *l.h.s.* is finite in general, so that this relationship is an avatar of the anomalous relation (2.4).

Thus, in order for the theory to be chiral-symmetric Eq.(3.1) must be satisfied. This in turn implies

$$mB(g) = 0. \tag{3.19}$$

There are two ways of satisfying Eq.(3.19), namely, either $m = 0$ or $B(g) = 0$. In the former case the fermion is massless and the theory is trivially chiral-symmetric, but in the latter case chiral symmetry is dynamically broken thereby generating the NG boson.

Next, by starting from Eq.(3.8) and the ETCR (3.2) we can derive the Ward-Takahashi identity in an obvious notation:

$$i(p-q)_\lambda \Gamma_{5\lambda}(p,q) = -2m\Gamma_5(p,q) + S_F^{-1}(p)\gamma_5 + \gamma_5 S_F^{-1}(q), \tag{3.20}$$

where p and q are outgoing and incoming momenta, respectively. In the lowest order perturbation theory, $\Gamma_{5\lambda}(p,q), \Gamma_5(p,q)$ and $S_F^{-1}(p)$ reduce to $\gamma_\lambda\gamma_5$, γ_5 and $(ip\cdot\gamma + m)$, respectively.

Let us assume that $m \neq 0$ and $B(g) = 0$, corresponding to Eq.(3.1), so that $\Gamma_5 = 0$ in Eq.(3.20), and then let us take the limit $g \to p$ to find

$$\lim_{q\to p} i(p-q)_\lambda \Gamma_{5\lambda}(p,q) = \{\gamma_5, S_F^{-1}(p)\} \tag{3.21}$$

The non-vanishing of the *r.h.s.* for $m \neq 0$ implies the existence of a massless pole, in the vertex $\Gamma_{5\lambda}(p,q)$, of the form

$$\gamma_5(p-q)_\lambda/(p-q)^2, \tag{3.22}$$

indicating generation of a massless NG boson.

4. The Schwinger-Dyson Equation

When $mB(g) \neq 0$, the theory is *not* chiral-symmetric. First, we shall study the Schwinger-Dyson (SD) equation in this case. Then the massless NG boson is absent, and

Eq.(3.20) reduces, in the limit $q \to p$, to

$$\{\gamma_5, S_F^{-1}(p)\} = 2m\Gamma_5(p,p). \tag{4.1}$$

The vertex function $\Gamma_5(p,p)$ satisfies a Bethe-Salpeter (BS) equation of the following form:

$$\Gamma_5(p,p) = Z_2 Z_P^{-1}\gamma_5 + \int d^4p' K(p,p')\Gamma_5(p',p'), \tag{4.2}$$

where spinor indices have been suppressed. By combining Eqs.(4.1) and (4.2) we also have a BS equation for the *l.h.s* of Eq.(4.1):

$$\{\gamma_5, S_F^{-1}(p)\} = 2m_0 Z_2\gamma_5 + \int d^4p' K(p,p')\{\gamma_5, S_F^{-1}(p')\}. \tag{4.3}$$

Because of the divergent character of the theory we find[2)]

$$Z_P^{-1} = 0, \quad m_0 = 0, \quad Z_2 = \text{finite}, \tag{4.4}$$

and both Eqs.(4.2) and (4.3) reduce to homogeneous ones.

The so-called SD equation is then given by

$$\{\gamma_5, S_F^{-1}(p)\} = \int d^4p' K(p,p')\{\gamma_5, S_F^{-1}(p')\}. \tag{4.5}$$

Although m_0 has been put equal to zero, the system described by this equation is *not* chiral-symmetric. In what follows we shall study the behavior of $\{\gamma_5, S_F^{-1}(p)\}$ for large values of p^2 with the help of the RG equation:

$$(\mathcal{D} + \gamma_P - 2\gamma_F)\Gamma_5(p,p;g,m,m_R) = 0. \tag{4.6}$$

The anomalous dimensions are given in QCD by

$$\begin{aligned}
\beta(g) &= -\frac{b}{2}g^3 + \ldots \qquad &&; \; b = \frac{1}{24\pi^2}(33 - 2N_f),\\
\gamma_S(g) &= -cg^2 + \ldots &&; \; c = 1/2\pi^2,\\
\gamma_\theta(g) &= -\gamma_P(g),\\
\gamma_F(g) &\sim O(g^4),
\end{aligned} \tag{4.7}$$

in the Landau gauge. By solving Eq.(4.6) we find the asymptotic form of Γ_5 for large values of p^2 as

$$\Gamma_5(p,p;g,m,m_R) \sim \gamma_5 B(g) Z_2(g) C(g) \Big(\ell n \ \frac{p^2}{m^2}\Big)^{-c/b}, \tag{4.8}$$

and consequently

$$\{\gamma_5, S_F^{-1}(p)\} \sim 2m\gamma_5 B(g) Z_2(g) C(g) \Big(\ell n \ \frac{p^2}{m^2}\Big)^{-c/b}. \tag{4.9}$$

This corresponds to Lane's G_+ solution.[6]

Next we shall study what will become of the NG boson when $mB(g) \neq 0$. The operator $\tilde{P}$ is BRS invariant, and we expect

$$< 0|\tilde{P}(x)\tilde{P}(y)|0 > \neq 0. \tag{4.10}$$

When color confinement is realized, composite hadron states saturate the intermediate states.[7] In particular, we pick out a single particle state $|\pi>$ satisfying

$$< 0|\tilde{P}(x)|\pi > \neq 0. \tag{4.11}$$

Then Eq.(3.17) implies $< 0|X_\lambda(x)|\pi > \neq 0$ and we may put

$$< 0|X_\lambda(x)|\pi > = \frac{2}{M(g)}\partial_\lambda < 0|\tilde{P}(x)|\pi >, \tag{4.12}$$

which defines the proportionality constant $M(g)$. Combination of Eqs.(3.17) and (4.12) yields

$$(\Box - \mu^2) < 0|\tilde{P}(x)|\pi > = 0, \tag{4.13}$$

where

$$\mu^2 = mM(g)B(g). \tag{4.14}$$

Thus, in general, we have a massive pseudoscalar bound state instead of the massless NG boson unless $B(g) = 0$. We shall study the high p^2 behavior of the BS amplitude

$$< 0|T\Big[\psi_\alpha\Big(\frac{x}{2}\Big)\overline{\psi}_\beta\Big(-\frac{x}{2}\Big)\Big]|\pi> . \tag{4.15}$$

For this purpose we introduce the operator product expansion and assume that it is dominated by the pseudoscalar term:

$$T\Big[\psi_\alpha\Big(\frac{x}{2}\Big)\overline{\psi}_\beta\Big(-\frac{x}{2}\Big)\Big] \sim f(x)(\gamma_5)_{\alpha\beta}\tilde{P}(0) + ..., \tag{4.16}$$

so that we have

$$< 0|T\Big[\psi_\alpha\Big(\frac{x}{2}\Big)\overline{\psi}_\beta\Big(-\frac{x}{2}\Big)\Big]|\pi> \sim f(x)(\gamma_5)_{\alpha\beta} < 0|\tilde{P}(0)|\pi> . \tag{4.17}$$

The RG equation for $f(x)$ is given by

$$(\mathcal{D} + 2\gamma_F - \gamma_S)f(x) = 0, \tag{4.18}$$

and the high p^2 behavior of $f(p)$, Fourier transform of $f(x)$, is given by

$$(p^2)^2 f(p) \sim \Big(\ell n\ \frac{p^2}{m^2}\Big)^{c/b} \tag{4.19}$$

The amputated BS amplitude $\tau(p)$ for $p^2 \to \infty$ is related to $f(p)$ through

$$\gamma_5 f(p) \sim S_F(p)\gamma_5\tau(p)S_F(p), \tag{4.20}$$

so that we have for $p^2 \to \infty$ the asymptotic form of $\tau(p)$ as

$$\tau(p) \sim \frac{1}{p^2}\Big(\ell n\ \frac{p^2}{m^2}\Big)^{c/b} \tag{4.21}$$

This corresponds to Lane's G_- solution.[6]

So far we have assumed $mB(g) \neq 0$ and have obtained massive pseudoscalar bound state, but what will happen when $m \neq 0$ and $B(g) = 0$? Let us assume that $B(g)$ vanishes for $g = g_X$,

$$B(g_X) = 0, \tag{4.22}$$

then Eq.(4.14) indicates that the massive pseudoscalar boson reduces to the massless NG boson. Causality implies

$$\mu^2 = mB(g)M(g) \geq 0. \tag{4.23}$$

Since $B(g)$ changes its sign at $g = g_X$, so does $M(g)$, too, by causality. Namely, we have

$$M(g_X) = 0, \tag{4.24}$$

and hence Eq.(4.12) leads us to

$$< 0|\tilde{P}(x)|\pi >= 0, \quad \text{for} \quad g = g_X. \tag{4.25}$$

In this case we are aware that the *r.h.s.* of Eq.(4.17) vanishes, and Eq.(4.17) must be modified as

$$< 0|T\Big[\psi_\alpha\Big(\frac{x}{2}\Big)\overline{\psi}_\beta\Big(-\frac{x}{2}\Big)\Big]|\pi >\sim h(x)(\gamma_\lambda\gamma_5)_{\alpha\beta}, < 0|X_\lambda(0)|\pi > . \tag{4.26}$$

Non-renormalization of X_λ, expressed by Eq.(3.6), implies

$$\gamma_X = 0, \tag{4.27}$$

and the RG equation for $h(x)$ is given by

$$(\mathcal{D} + 2\gamma_F)h(x) = 0. \tag{4.28}$$

The high p^2 behavior of $h(p)$ is then given by

$$(p^2)^2 h(p) \sim \text{const}. \tag{4.29}$$

The amputated BS amplitude $\sigma(p)$ defined by an equation similar to Eq.(4.20) behaves for large p^2 as

$$\sigma(p) \sim \frac{\text{const}}{p^2}. \tag{4.30}$$

$\sigma(p)$ is proportional to Eq.(3.21) so that we find

$$\{\gamma_5, S_F^{-1}(p)\} \sim \frac{\text{const}}{p^2}, \quad \text{for} \quad g = g_X. \tag{4.31}$$

This is quite distinct from Eq.(4.9).

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