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To cite this article: Luigi Accardi 2017 *J. Phys.: Conf. Ser.* **819** 012001

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# Quantum Probability, Orthogonal Polynomials and Quantum Field Theory

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**Abstract.** The main thesis of the present paper is that: Quantum Probability is not a generalization of classical probability, but it is a deeper level of it. Classical random variables have an intrinsic (microscopic) non-commutative structure that generalize usual quantum theory. The study of this generalization is the core of the non-linear quantization program.

## 1. Quick outline of the historical development

The notion of **Interacting Fock space** (IFS in the following) emerged naturally from the stochastic limit of Quantum Electro-dynamics (QED) obtained in the paper [3]: this was the first paper where an exact although asymptotical result was obtained in QED **without introducing dipole-type approximations**.

The complexity of the emerging mathematical structures suggested to introduce an axiomatic approach in order to isolate the basic properties underlying these structures. This led to the first axiomatization of the notion of IFS (see section 18 of the paper [2]).

At that time the connection with orthogonal polynomials had not yet been realized, but two important properties of this new mathematical structure were already clear and were emphasized in the paper namely:

- (i) That IFS are natural candidates for models of self-interacting quantum fields (from which the name: **Interacting Fock Space**).
- (ii) That IFS provide a new and very general, easy to handle and constructive, technique to produce an infinite multiplicity of **new examples of quantum statistical independence**. The notion of IFS provided a unifying framework for a multiplicity of such examples that, up to that moment had been studied by many authors from an empirical point of view.

One year after the paper [2] the **identification of the category of 1-mode IFS's with the category of orthogonal polynomials in 1 real variable** was discovered in the paper [1]. This result was extended to finitely many variables in the paper [4] where the an intrinsic formulation of the multi-dimensional Jacobi relations was produced, apparently for the first time in the literature. This result was later extended to infinitely many variables in the paper [5] (see also the survey [7]). An important step in the understanding of the structure of multi-dimensional orthogonal polynomials was done in the paper [6] where a characterization theorem of states on the  $\ast$ -algebra  $\mathcal{P}(R^d)$ , of all polynomials in  $d$  real variables, was given in terms of commutators. The importance of this theorem lies in the fact that, in the symmetric case, this

reduces the classification problem to a classification of **square matrices**, which is much easier to handle than classification of rectangular matrices.

However the above mentioned commutator theorem still left open the problem of individuating the exact multi-dimensional analogue of the Jacobi sequences  $(\omega_n)$  and  $(\alpha_n)$ , that contain the **minimal information** gained in passing from the knowledge of all moments of order  $n$  of a given state on  $\mathcal{P}(R^d)$  to the knowledge of all moments of order  $n + 1$ . This **minimality**, in the 1-dimensional case allows a reconstruction theorem, i. e. Favard Lemma, that can be considered as the converse statement of the Jacobi relations. The final step, in the multi-dimensional reconstruction theorem, was done in the paper [8] where the multi-dimensional analogues of the Jacobi sequences  $(\omega_n)$  and  $(\alpha_n)$  was first deduced.

Even in the 1-dimensional case, the quantum probabilistic approach to the theory of orthogonal polynomials allowed to solve some problems in classical probability that were open since many decades even for the standard Gaussian measure on  $\mathbb{R}$ . In particular the solution of the problem of expressing the Jacobi sequences and the orthogonal polynomials of the square of a random variable in terms of the corresponding sequences of the random variable itself was known only in the symmetric case and was extended to the non-symmetric case in [12]. The same problem for arbitrary powers was solved in [?].

Each classical random variable, scalar or vector valued, was shown to be canonically associated to a family of commutation relations generalizing the usual Heisenberg commutation relations and this leads to a purely algebraic classification of classical probability measures based on the associated  $*$ -Lie algebras [11]. As a consequence of this fact, one proves that, to every classical probability measure with all moments, one can canonically associate an extension of usual quantum theory [9]. This gives a unified framework and at the same time a generalization of the program of **non-linear quantization** [10].

The same algebraic approach to the classification of classical probability measures led to the introduction of an information complexity index for classical probability measures on  $\mathbb{R}$  which in particular allowed a new characterization of the Meixner classes [13].

In the present paper we discuss in what sense the theory of orthogonal polynomials in infinitely many variables leads to non-trivial extensions of quantum field theory just as the finite dimensional case leads to non-trivial extensions of usual quantum theory canonically associated to classical probability measures [14].

## 2. Classical probability, notations

Let  $X$  be real valued random variable (always with all moments in the following) and let  $\mu$  be the probability distribution of  $X$ .  $\mu$  is a probability measure on  $\mathbb{R}$  and from now on we identify  $X \equiv \mu$ . Denote by  $e_1$  a basis of  $\mathbb{R}$  and:

$$\mathcal{P}(\mathbb{R}, e_1) \equiv \mathcal{P} := \{\text{algebra of polynomial functions } (\mathbb{R}, e_1) \rightarrow \mathbb{C}\}$$

$$\mu(P) := \int_{\mathbb{R}} P(x) \mu(dx) := \text{The state on } \mathcal{P} \text{ induced by } \mu$$

The measure  $\mu$  induces on  $\mathcal{P}$  the pre-scalar product:

$$\langle P, Q \rangle := \mu(P^* Q)$$

**Theorem 1** *Pre-scalar products on  $\mathcal{P}$ , induced by a state are characterized by the property*

$$X = X^* \quad (\text{modulo zero-norm vectors})$$

$\langle \cdot, \cdot \rangle$ -orthogonalizing the monomials  $(X^n)$  one obtains the **monic  $\mu$ -orthogonal polynomials**, denoted  $(\Phi_n)$ .

**Theorem 2 (Carl Gustav Jacob Jacobi (150 years ago))** *The monic 3-diagonal relations hold:*

$$x\Phi_n(x) = \Phi_{n+1}(x) + \alpha_n\Phi_n(x) + \omega_n\Phi_{n-1}(x) \\ (\omega_n) \quad ; \quad (\alpha_n) \quad ; \quad (\text{monic Jacobi sequences})$$

**Operator interpretation:** identity of vectors in the pre-Hilbert space  $(\mathcal{P}, \mu)$ :

$$X\Phi_n = a^+\Phi_n + a^0\Phi_n + a^-\Phi_n$$

Since  $(\Phi_n)$  is an orthogonal basis this implies an identity of operators on  $(\mathcal{P}, \mu)$ :

$$X = a^+ + a^0 + a^-$$

This is the **quantum decomposition of the classical random variable**  $X$ .  $a^+, a^-, a^0$  are called the **CAP operators** (Creation, Annihilation, Preservation).

**Remark 1** *For 150 years the Jacobi relation have been studied by analysts. In QP one looks at the same object from a different point of view, namely: The classical random variable  $X$  has a **microscopic structure** defined by the operators  $a^+, a^-, a^0$ . This new point of view has deep implications. Let us see some of them.*

### 3. Commutation relations canonically associated to the classical random variable $X$

Start from the quantum decomposition of the classical random variable

$$X = a^+ + a^0 + a^-$$

and recall the definition of the monic CAP operators:

$$a^+\Phi_n = \Phi_{n+1} \quad ; \quad a^-\Phi_n = \omega_n\Phi_{n-1} \\ \Rightarrow a^-a^+\Phi_n = a^-\Phi_{n+1} = \omega_{n+1}\Phi_n \\ a^+a^-\Phi_n = \omega_na^+\Phi_{n-1} = \omega_n\Phi_n$$

Subtracting, one finds the **commutation relations**

$$(a^-a^+ - a^+a^-)\Phi_n = (\omega_{n+1} - \omega_n)\Phi_n \\ \Longleftrightarrow [a^-, a^+]\Phi_n = (\omega_{n+1} - \omega_n)\Phi_n$$

Introducing the **number operator**:

$$\Lambda\Phi_n := n\Phi_n$$

the  $\omega$ -commutation relations

$$[a^-, a^+]\Phi_n = (\omega_{n+1} - \omega_n)\Phi_n \quad ; \quad \forall n \in \mathbb{N}$$

become

$$[a^-, a^+] = \omega_{\Lambda+1} - \omega_\Lambda =: \partial\omega_\Lambda$$

Notice that these are **operator valued commutation relations (CR)** and that they depend only on:  $[X] :=$  equivalence class of  $X$  for the equivalence relation:

$$X \sim Y \Longleftrightarrow \omega_{X,n} = \omega_{Y,n} \quad , \quad \forall n$$

**Theorem 3** *There exists exactly one equivalence class of probability measures on  $\mathbb{R}$  whose associated canonical commutation relations*

$$[a^-, a^+] = \omega_{\Lambda+1} - \omega_{\Lambda} =: \partial\omega_{\Lambda}$$

*coincide (up to the value of the Planck constant) with the Heisenberg commutation relations*

$$[a^-, a^+] = \hbar \cdot 1$$

This is: the **equivalence class of Gaussian measures**, that coincides with the **equivalence class of Poisson measures**.

**Example (1):** The  $q$ -deformed 1-mode IFS.

**Lemma 1** *For any  $q \geq -1$  and any 1-mode IFS, the following two statements are equivalent ( $\omega_0 := 1$ ):*

$$\omega_n = \begin{cases} \sum_{k=0}^{n-1} q^k, & \text{if } q > -1 \\ 1, & \text{if } q = -1 \text{ and } n \leq 1; \\ 0, & \text{if } q = -1 \text{ and } n \geq 2 \end{cases} \quad (1)$$

$$aa^+ - qa^+a = 1 \quad (2)$$

**Example (2):** The Parthasarathy–Schürmann  $q$ -deformed 1-mode IFS (Azema martingale).

**Lemma 2** *For any  $x \in \mathbb{C} \setminus \{0\}$ , and any 1-mode IFS, the following statements are equivalent ( $q := |x|^2$ ):*

$$\omega_n := n \cdot |x|^{2(n-1)} =: n \cdot q^{(n-1)}; \quad \forall n \geq 1 \quad (3)$$

$$aa^+ - |x|^{2\Lambda} a^+a \iff aa^+ - qa^+a = q^{\Lambda} \quad (4)$$

There are many other interesting examples. But the most important thing is the general method. This allows to look at usual quantum mechanics from a new, insightful point of view. In the following we briefly describe this new point of view.

#### 4. Extensions of quantum mechanics

We have seen that **usual quantum mechanics corresponds to the Gauss–Poisson classes of measures**. The natural question: **What happens with the other measures?** leads to several inequivalent probabilistic extensions of quantum mechanics. More precisely, to every classical random variable  $X$  one associates an  $[X]$ -dependent quantum mechanics in the following way. To the quantum decomposition of  $X$ :

$$X = a^+ + a^0 + a^- = a_X^+ + a_X^0 + a_X^-$$

one associates **the free  $X$ -Hamiltonian**

$$H := ca_X^+ a_X^- = ca^+ a^- \quad ; \quad c > 0$$

The corresponding **Schrödinger equation** is:

$$\partial_t \psi_t = -iH\psi_t \quad (\hbar = 1) \quad (5)$$

and the **Heisenberg evolution** for  $a^{\pm}$  is

$$a^{\pm}(t) := u_t(a^{\pm}) := e^{itH} a^{\pm} e^{-itH} \quad ; \quad a^{\pm}(0) = a^{\pm}$$

Its unique solution is given by:

$$a^-(t) = e^{-itc(\omega_\Lambda - \omega_{\Lambda-1})} a^-$$

$$a^+(t) = a^+ e^{itc(\omega_\Lambda - \omega_{\Lambda-1})}$$

**Theorem 4** *The map*

$$t \in \mathbb{R} \mapsto a^\pm(t) = \begin{cases} a^+ e^{it(\omega_\Lambda - \omega_{\Lambda-1})} \\ e^{-it(\omega_\Lambda - \omega_{\Lambda-1})} a^- \end{cases}$$

*can be uniquely extended to a  $*$ -algebra automorphism  $(u_t^{(0)})$ , called the **generalized free evolution**, of the **quantum polynomial algebra***

$$\mathcal{P}(a^+, a^-, a^0, 1) := \text{algebraic span of } \{a^+, a^-, a^0, 1\}$$

Since  $\Phi$  is a cyclic unit vector for  $\mathcal{P}(X)$ , it is also cyclic for the **quantum polynomial algebra**:

$$\mathcal{P}(a^+, a^-, a^0, 1) \supseteq \mathcal{P}(X)$$

On the quantum polynomial algebra  $\mathcal{P}(a^+, a^-, a^0, 1)$ , there is a **semi-finite trace** given by

$$z \in \mathcal{P}(a^+, a^-, a^0, 1) \mapsto \tau(z) := \sum_n \langle \tilde{\Phi}_n, z \tilde{\Phi}_n \rangle$$

$$\tilde{\Phi}_n := \text{normalized orthogonal polynomials}$$

Recall the expression of the generalized free evolution

$$u_t(X) = e^{it(\omega_\Lambda - \omega_{\Lambda-1})} X e^{-it(\omega_\Lambda - \omega_{\Lambda-1})}$$

Fix  $\beta > 0$  and a positive operator  $W \in \mathcal{P}(a^+, a^-, a^0, 1)$ . The  $((u_t), \beta)$ -KMS equilibrium condition, for  $W$ , is:

$$\text{Tr}(W u_{t-i\beta}(X) Y) = \text{Tr}(W Y u_t(X))$$

Since  $X, Y \in \mathcal{P}(a^+, a^-, a^0, 1)$  are arbitrary, this is equivalent to

$$W = e^{-\beta(\omega_\Lambda - \omega_{\Lambda-1})}$$

Usually  $W$  will not be trace-class, so  $\tau(W \cdot)$  is a  $((u_t), \beta)$ -**KMS weight**. If it is trace-class, we normalize and get the  $\omega$ -Gibbs state:

$$\tau(W \cdot) = \tau \left( \frac{e^{-\beta(\omega_\Lambda - \omega_{\Lambda-1})}}{Z_\beta} \cdot \right)$$

## Conclusions

(i) Non-commutativity arises in QP

**not artificially put by hands**, but **naturally deduced** from a 150-year-old result in analysis.

(ii) Every equivalence class of probability measures on  $\mathbb{R}$ , for the equivalence relation

$$\mu \sim \nu \iff (\omega_{\mu,n}) = (\omega_{\nu,n}) \quad \forall n$$

is canonically associated an **extension of usual quantum mechanics**. Usual quantum theory is covered exactly by the Gauss-Poisson class.

#### 4.1. Non-trivial feed-back for usual quantum mechanics

Why these extension of usual quantum mechanics are important?

The fact is that  $\omega$ -commutation relations can be bosonized [16]. So one can look at the boson form of the  $\omega$ -Schrödinger equation. This gives an analogy with integrable systems and action-angle variables that is now under investigation.

### 5. Developments in the multi-dimensional case

The **main structural result** in the theory of 1-d orthogonal polynomials is **Favard Lemma**.

**Theorem 5 (Favard Lemma 1930)** *For every state  $\varphi$ , on  $\mathcal{P}$ , there exists a sequence of real numbers*

$$\alpha : \mathbb{N} \rightarrow \alpha_n \in \mathbb{R}$$

*and a sequences of positive real numbers*

$$\omega : \mathbb{N} \rightarrow \omega_n \in \mathbb{R}_+$$

*satisfying*

$$\omega_k = 0 \Rightarrow \omega_n = 0, \forall n \geq k$$

*with the following properties. Denoting  $\{\Phi_n\}$  the sequence of monic orthogonal polynomials associated to  $\mu$ , normalized so that*

$$\Phi_{-1} = 0 ; \Phi_0 = 1 ; \Phi_1(x) = x - \alpha_1$$

*the following (monic Jacobi) relations hold for each  $n \in \mathbb{N}$ :*

$$x\Phi_n(x) = \Phi_{n+1}(x) + \alpha_n\Phi_n(x) + \omega_n\Phi_{n-1}(x)$$

**Conversely**, *given two sequences of real numbers*

$$(\alpha_n) \quad , \quad (\omega_n)$$

*satisfying the above conditions, there exists a unique state  $\mu$ , on  $\mathcal{P}$  which satisfies the Jacobi induction relation with respect to these two sequences.*

The connection with probability theory is given by the Hamburger theorem according to which every state  $\mu$  on  $\mathcal{P}$  defines at least one probability measure on  $\mathbb{R}$ . Such a clear implication is missing in dimensions  $d \geq 2$  (see the survey paper [15] for references on the failure of Hamburger theorem in dimensions  $\geq 2$ . This is related to a variant of Hilbert's 17-th problem.

### 6. Multi-dimensional orthogonal polynomials: Notations

Let  $d \in \mathbb{N}$ ,  $D := \{1, \dots, d\}$  – polynomial  $*$ -algebra in  $d$  commuting indeterminates:

$$\mathcal{P} := \mathbb{C}[(X_j)_{j \in D}]$$

– algebraic generators: **coordinate functions** in a given linear basis  $e = (e_j)$  of  $\mathbb{R}^d$ .

$$X_j = X_j^*$$

– linear generators: monomials

$$M = X_1^{n_1} \cdots X_d^{n_d} \quad ; \quad n_j \in \mathbb{N}$$

– degree of  $M$

$$\deg(M) := \sum_{j \in D} n_j = n$$

$$\mathcal{P}_n^0 := \begin{cases} \text{the vector subspace of } \mathcal{P} \text{ generated} \\ \text{by the monomials of degree } n \end{cases} \quad (6)$$

This gives the, purely algebraic, vector space (not orthogonal) **monomial gradation**

$$\mathcal{P} = \sum_{n \in \mathbb{N}} \mathcal{P}_n^0$$

This gradation is purely algebraic: it does not depend on any measure.

### 6.1. Emergence of the symmetric tensor algebra

**Theorem 6** Let  $(e_j)_{j \in D}$  be the canonical linear basis of  $\mathbb{C}^d$ . The map

$$e_j \mapsto X_j, \quad j \in D, \quad 1_{\mathcal{T}_{sym}(\mathbb{C}^d)} \mapsto 1_{\mathcal{P}} \quad (7)$$

extends uniquely to a gradation preserving isomorphism of commutative  $*$ -algebras:

$$\text{Ten}_{sym}(\mathbb{C}^d) := \sum_{n \in \mathbb{N}} (\mathbb{C}^d)^{\hat{\otimes} n} \rightarrow \sum_{n \in \mathbb{N}} \mathcal{P}_n^0 \equiv \mathcal{P} \quad (8)$$

### 6.2. Probability measures on $\mathbb{R}^d$ and states and pre-scalar products on $\mathcal{P}$

**Theorem 7** For a pre-scalar product  $\langle \cdot, \cdot \rangle$  on  $\mathcal{P}$  the following statements are **equivalent**:

(i) There exists a state  $\varphi$  on  $\mathcal{P}$  such that:

$$\varphi(f^*g) = \langle f, g \rangle \quad ; \quad f, g \in \mathcal{P}$$

(ii) The pre-scalar product  $\langle \cdot, \cdot \rangle$  satisfies

$$\langle 1_{\mathcal{P}}, 1_{\mathcal{P}} \rangle_{\varphi} = 1 \quad (9)$$

and, for each  $j \in D$ , multiplication by the **coordinate**  $X_j$  is a **symmetric linear operator** on  $\mathcal{P}$  with respect to  $\langle \cdot, \cdot \rangle$ :

$$\langle X_j f, g \rangle = \langle f, X_j g \rangle \quad (10)$$

### 6.3. The degree filtration and the orthogonal gradation

Define

$$\begin{aligned} \mathcal{P}_{n]} &:= \text{lin-span}\{\text{monomials } M : \deg(M) \leq n\} = \\ &= \text{polynomials of degree } \leq n \end{aligned}$$

$$P_{n]} : \mathcal{P} \rightarrow \mathcal{P}_{n]} =$$

= **pre-Hilbert** space orthogonal projector defined by the monomial basis.

$$m \leq n \Rightarrow P_{m]} P_{n]} = P_{m]} \quad (\text{increasing filtration})$$

$$P_n := P_{n]} - P_{n-1]} : \mathcal{P} \rightarrow \mathcal{P}_{n-1]}^{\perp} \cap \mathcal{P}_{n]} = \mathcal{P}_n$$



$(P_n)$  is a **partition of 1**, i.e.:

$$m \neq n \rightarrow P_m P_n = 0 \quad ; \quad \sum_{n \in \mathbb{N}} P_n = 1$$

Therefore  $(P_n)$  defines the following **orthogonal gradation** of  $\mathcal{P}$ :

$$\mathcal{P} = \bigoplus_{n \in \mathbb{N}} \mathcal{P}_n = \bigoplus_{n \in \mathbb{N}} P_n(\mathcal{P})$$

**Theorem 8** *The Orthogonal gradation is linearly isomorphic to Monomial gradation*

$$\mathcal{P}_n \sim \mathcal{P}_n^0 \sim \mathbb{C}^{d_n} \sim (\mathbb{C}^d)^{\widehat{\otimes} n} \quad (11)$$

where  $\widehat{\otimes}$  denotes symmetric tensor product.

$$\mathcal{P} = \bigoplus_{n \in \mathbb{N}} \mathcal{P}_n = \bigoplus_{n \in \mathbb{N}} \left( (\mathbb{C}^d)^{\widehat{\otimes} n}, \langle \cdot, \cdot \rangle_{\mathcal{P}_n} \right)$$

But we know from quantum mechanics that

$$\bigoplus_{n \in \mathbb{N}} \left( (\mathbb{C}^d)^{\widehat{\otimes} n}, \langle \cdot, \cdot \rangle_{\widehat{\otimes} n} \right) = \Gamma \left( \mathbb{C}^d, \langle \cdot, \cdot \rangle_{\text{Eucl}} \right)$$

**This suggests that the theory of OP is a IFS-generalization of Boson quantization.**  
We will see that this intuition is correct.

#### 6.4. The intrinsic formulation of the Symmetric Jacobi Relations

**Theorem 9 (Symmetric Jacobi Relations)** *In the above notations one has for each  $n$ :*

$$X_j P_n = P_{n+1} X_j P_n + P_n X_j P_n + P_{n-1} X_j P_n$$

This is the intrinsic formulation of the Symmetric Jacobi Relations.

#### 6.5. The CAP operators of a state on $\mathcal{P}$

Here CAP stands for: Creation–Annihilation–Preservation. The symmetric Jacobi Relation

$$X_j P_n = P_{n+1} X_j P_n + P_n X_j P_n + P_{n-1} X_j P_n$$

suggests to define, for each  $n \in \mathbb{N}$ , the linear operators

$$a_{j;n}^+ := P_{n+1} X_j P_n : \mathcal{P}_n \rightarrow \mathcal{P}_{n+1}$$

$$a_{j;n}^0 := P_n X_j P_n : \mathcal{P}_n \rightarrow \mathcal{P}_n$$

$$a_{j;n}^- := P_{n-1} X_j P_n : \mathcal{P}_n \rightarrow \mathcal{P}_{n-1}$$

$$a_{j;n}^\pm \equiv \text{rectangular matrices}$$

$$a_{j;n}^0 \equiv \text{square matrix}$$

**Theorem 10** *Defining the real-linear operators*

$$\begin{aligned} a_j^+ &:= \sum_n P_{n+1} X_j P_n = \sum_n a_{j;n}^+ \text{ (creation)} \\ a_j^- &:= \sum_n P_{n-1} X_j P_n = \sum_n a_{j;n}^- \text{ (annihilation)} \\ a_j^0 &:= \sum_n P_n X_j P_n = \sum_n a_{j;n}^0 \text{ (preservation)} \end{aligned}$$

One has, for each  $j \in \{1, \dots, d\}$ :

$$\begin{aligned} (a_j^+)^* &= a_j^- & ; & & (a_j^0)^* &= a_j^0 \\ X_j &= a_j^+ + a_j^0 + a_j^- & ; & & j &= 1, \dots, d \end{aligned} \quad (12)$$

The decomposition (12) is called **the quantum decomposition of  $X_j$** .

#### 6.6. Commutation relations

Remember that, in the 1-d case, the commutation relations were consequence only of the fact that

$$a^+ a^- , a^- a^+ , a^0$$

are gradation preserving operators. The basic new ingredient with respect to the 1-d case is that: **the  $X_j$  commute!**

**Theorem 11** *The decomposition*

$$X_j := a_j^+ + a_j^0 + a_j^- , \quad j \in \{1, \dots, d\}. \quad (13)$$

is unique and the operators  $X_j$  commute. Conversely, if the operators  $X_j$  ( $j \in \{1, \dots, d\}$ ) are defined by (17), with

$$a_j^0 = (a_j^0)^* ; a_j^- = (a_j^+)^* ; j \in \{1, \dots, d\}.$$

then they commute if and only if the operators  $a_j^+ , a_j^0 , a_j^-$  satisfy the following commutation relations: for all  $j, k \in \{1, \dots, d\}$  such that  $j < k$

$$[a_j^+ , a_k^+] = 0 \quad (14)$$

$$[a_j^+ , a_k^-] + [a_j^0 , a_k^0] + [a_j^- , a_k^+] = 0 \quad (15)$$

$$[a_j^+ , a_k^0] + [a_j^0 , a_k^+] = 0 \quad (16)$$

#### 6.7. Abstract essence of the multi-dimensional Favard problem

**Problem.** Let be given:

- a pre-Hilbert space  $H$ ;
- an orthogonal gradation of  $H$ :

$$H = \bigoplus_{n \in \mathbb{N}} H_n ;$$

- adjointable operators

$$a_j^+ : H_n \rightarrow H_{n+1}$$

with adjoint  $a_j^-$ . Can one find  $d$  gradation preserving operators  $a_j^j : H_n \rightarrow H_n$  ( $\forall n$ ) such that the operators

$$X_j := a_j^+ + a_j^0 + a_j^- , \quad j \in \{1, \dots, d\}. \quad (17)$$

are commutative?

From Theorem 11 we know the problem is equivalent to find  $d$  gradation preserving operators  $a_j^0 : H_n \rightarrow H_n$  ( $\forall n$ ) which satisfy the commutation relations given above, i.e.

$$[a_j^+, a_k^+] = 0 \quad (18)$$

$$[a_j^+, a_k^-] + [a_j^0, a_k^0] + [a_j^-, a_k^+] = 0 \quad (19)$$

$$[a_j^+, a_k^0] + [a_j^0, a_k^+] = 0 \quad (20)$$

If there exist operators

$$a_j^+ : H_n \rightarrow H_{n+1} \quad ; \quad (a_j^+)^* = a_j^- : H_{n-1} \rightarrow H_n$$

$$a_j^j : H_n \rightarrow H_n \quad ; \quad \forall n$$

that satisfy the above commutation relations, the gradation

$$H = \bigoplus_{n \in \mathbb{N}} H_n;$$

is said to have a 3-diagonal structure.

So the Favard problem can be reformulated as: **parametrize the gradations on  $\mathcal{P}(\mathbb{R}^d)$  that admit a 3-diagonal structure.**

### 6.8. Product states

**Definition 1** A state  $\mu$  on  $\mathcal{P}(\mathbb{R}^d)$  is called a product state if, for any  $n \in \mathbb{N}$ , any  $f_1, \dots, f_d \in \mathcal{P}(\mathbb{R})$ , one has:

$$\mu(f_1(X_1) \cdots f_d(X_d)) = \prod_{j \in D} \mu_j(f_j(X_j)) \quad (21)$$

where  $\mu_j$  denotes the restriction of  $\mu$  on  $\mathcal{P}(X_j)$ . In this case we write  $\mu =: \bigotimes_{j \in D} \mu_j$ .

Recall that  $\Phi_0 := 1_{\mathcal{P}} \in \mathcal{P}$  and that the embedding  $f \in \mathcal{P}(\mathbb{R}) \mapsto f(X_j) \in \mathcal{P}(X_j)$  is identity preserving. Therefore one has the identification

$$\Phi_{j,0} \equiv \Phi_0 \quad ; \quad \forall j \in \{1, \dots, d\} \quad (22)$$

**Theorem 12** Let  $\mu$  be a state on  $\mathcal{P}$  and let

$$X_j = a_j^+ + a_j^0 + a_j^- \quad ; \quad j \in D \quad (23)$$

be the quantum decomposition of  $X_j$  with respect to  $\mu$ . Then:

(i)  $\mu$  is a **product state** if and only if, for all  $j \neq k \in D$ , one has

$$[a_j^-, a_k^+] = [a_j^-, a_k^0] = [a_j^0, a_k^0] = 0 \quad (24)$$

(ii) For  $n \in \mathbb{N}$  and  $j \in D$ , denote  $\tilde{\Phi}_{j,n}$  the

normalized orthogonal polynomial of degree  $n$  of  $\mu_j$ . Then the orthogonal polynomials of degree  $n$  of  $\mu$  are

$$\left\{ \tilde{\Phi}_{1,m_1}(X_1) \cdots \tilde{\Phi}_{d,m_d}(X_d) \cdot \Phi_0 : (m_1, \dots, m_d) \in \{0, 1, \dots, n\}_n^d \right\} \quad (25)$$

where  $\tilde{\Phi}_{j,m_j}(X_j)$  denotes the multiplication operator by  $\tilde{\Phi}_{j,m_j}$  on  $\mathcal{P}(\mathbb{R}^d) \cdot \Phi_0$  and

$$\{0, 1, \dots, n\}_n^d := \left\{ \bar{m} := (m_1, \dots, m_d) \in \{0, 1, \dots, n\}^d : \sum_{j \in D} m_j = n \right\} \quad (26)$$

### 6.9. Other consequences of the commutation relations

**Theorem 13** *The set of vectors*

$$a_{j_n}^+ \cdots a_{j_1}^+ \Phi_0 \quad (27)$$

( $\Phi_0$  – the constant function = 1) **are total in**  $\mathcal{P} \cdot \Phi_0$ .

In quantum mechanics the vectors (27) are called **number (or finite particle) vectors**. Define the sequence of positive definite (PD) kernels by  $\tilde{\Omega}_0 = 1 \in \mathbb{C}$  and

$$\tilde{\Omega}_{n+1}(e_j, e_k) := (a_{j|n}^+)^* a_{k|n}^+ ; \quad \forall n \in \mathbb{N}, \quad \forall j, k \in D \quad (28)$$

$$\begin{aligned} \langle a_{|n}^+ \xi_n, a_{|n}^+ \eta_n \rangle_{n+1} &= \langle \xi_n, (a_{|n}^+)^* a_{|n}^+ \eta_n \rangle_n \\ &= \langle \xi_n, \tilde{\Omega}_{n+1}(e_j, e_k) \eta_n \rangle_n \end{aligned}$$

Clearly

$$\tilde{\Omega}_{n+1}(e_j, e_k) : H_n \rightarrow H_n$$

**Lemma 3** *The commutation relations*

$$[a_j^+, a_k^-] + [a_j^0, a_k^0] + [a_j^-, a_k^+] = 0 \quad (29)$$

are equivalent to

$$\tilde{\Omega}_1(e_j, e_k) = \tilde{\Omega}_1(e_k, e_j) \in \mathbb{R} \quad (30)$$

$$Im(\tilde{\Omega}_{n+1}(e_j, e_k)) = Im(a_{k|n-1}^+ (a_{j|n-1}^+)^*) + Im((a_{k|n}^0)^* a_{j|n}^0) ; \quad \forall n \geq 1 \quad (31)$$

for all  $j, k \in D$  such that  $j < k$  and all  $n \in \mathbb{N}$ .

**Lemma 4** *The commutation relations*

$$[a_j^+, a_k^0] + [a_j^0, a_k^+] = 0 \quad (32)$$

are equivalent to

$$a_{j|n+1}^0 a_{k|n}^+ - a_{k|n+1}^0 a_{j|n}^+ = a_{k|n}^+ a_{j|n}^0 - a_{j|n}^+ a_{k|n}^0 \quad (33)$$

for all  $j, k \in D$  such that  $j < k$  and all  $n \in \mathbb{N}$ .

This result gives an inductive linear constraint on the  $a_{j|n+1}^0$  given the  $a_{j|n}^0$  and the  $a_{j|n}^+$ .

**Lemma 5** *The commutativity of creators is a consequence of Lemma 4.*

**Remark 2** *The orthogonal polynomial gradations define a very special sub-class of symmetric interacting Fock spaces.*

## 7. Acknowledgements

The author acknowledges support by the RSF grant 14-11-00687, Steklov Mathematical Institute.

- [1] Accardi L and Bożejko M 1998 *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **1** 663-670
- [2] Accardi L, Lu Y G and Volovich I 1997 *Publications of IAS* 1997-008
- [3] Accardi L and Lu Y G 1996 *Comm. Math. Phys.* **180** 605-632
- [4] Accardi L and Nhani M 2002 In book: *The Crossroad of Non-Commutativity, Infinite-Dimensionality* (Singapore: World Scientific) pp 192-205
- [5] Accardi L, Kuo H H and Stan A 2004 *Inf. Dim. Anal. Quant. Prob. Rel. Top.* **7** 485-505
- [6] Accardi L, Kuo H H and Stan A 2007 *Infin. Dim. Anal. Quantum Probab. Rel. Topics* **10** 591-612

- [7] Accardi L, Kuo H H and Stan A 2005 In book: *Quantum Probability and Infinite Dimensional Analysis: From Foundations to Applications* (Singapore: World Scientific) pp 1-11.
- [8] Accardi L, Barhoumi A and Dhahri A *Infin. Dim. Anal. Quantum Probab. Rel. Topics* (in press)
- [9] Accardi L In book: *Workshop on IDAQP and their Applications, 3 - 7 March 2014, Singapore* (Singapore: World Scientific) (in press)
- [10] Accardi L and Boukas A 2015 *Prob. Math Stat.* **35** 41–72
- [11] Accardi L, Barhoumi A, Lu Y G and Rhaima M 2016 In book: *Proceedings of the International Workshop "Lie Theory and Its Applications in Physics", (LT-11), 15–21 June, 2015 Varna, "Springer Proceedings in Mathematics and Statistics* Vol. 191 pp 3–21
- [12] Accardi L, Barhoumi A and Rhaima M 2016 *Inter. J. Probab. Stochast. Process.* 1-15
- [13] Accardi L, Barhoumi A and Rhaima M 2016 *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **19** 1650015
- [14] Accardi L, Dhahri A, Gibilisco P 2017 *Milan Journal of Mathematics* (2017) (submitted)
- [15] Alpay D, Jorgensen P E T and Kimsey D P 2015 *Infin. Dim. Anal. Quantum Probab. Rel. Topics* **18** 1550024
- [16] Bhat B VR and Parthasarathy K R *Lect. Notes in Mathematics* **1485**