Calabi-Yau Threefolds and Heterotic String Compactification



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A thesis submitted for the degree of

 $Doctor \ of \ Philosophy$

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This thesis is concerned with Calabi-Yau threefolds and vector bundles upon them, which are the basic mathematical objects at the centre of smooth supersymmetric compactifications of heterotic string theory. We begin by explaining how these objects arise in physics, and give a brief review of the techniques of algebraic geometry which are used to construct and study them. We then turn to studying multiply-connected Calabi-Yau threefolds, which are of particular importance for realistic string compactifications. We construct a large number of new examples via free group actions on complete intersection Calabi-Yau manifolds (CICY's). For special values of the parameters, these group actions develop fixed points, and we show that, on the quotient spaces, this leads to a particular class of singularities, which are quotients of the conifold. We demonstrate that, in many cases at least, such a singularity can be resolved to yield another smooth Calabi-Yau threefold, with different Hodge numbers and fundamental group. This is a new example of the interconnectedness of the moduli spaces of distinct Calabi-Yau threefolds.

In the second part of the thesis we turn to a study of two new 'three-generation' manifolds, constructed as quotients of a particular CICY, which can also be represented as a hypersurface in $dP_6 \times dP_6$, where dP_6 is the del Pezzo surface of degree six. After describing the geometry of this manifold, and especially its non-Abelian quotient, in detail, we show how to construct on the quotient manifolds vector bundles which lead to four-dimensional heterotic models with the standard model gauge group and three generations of particles. The example described in detail has the spectrum of the minimal supersymmetric standard model plus a single vector-like pair of colour triplets.

Acknowledgements

I have been lucky to benefit from the help and support of many people before and during my time at Oxford, and I am grateful to all of them, but I would particularly like to acknowledge the following people.

- Most of all I thank my wife Emma, whose endless love and encouragement motivates and strengthens me in all pursuits.
- A special thank-you goes to my parents Pam and Owen, who have always supported and encouraged me, even though my choices have taken me half a world away from them.
- I thank my supervisors and collaborators Andre Lukas, Xenia de la Ossa, Philip Candelas, Volker Braun and Ron Donagi. They have taught me a great deal, and this thesis would not have come to be without their help. I also thank Mark Gross, Balázs Szendrői and Duco van Straten for sharing with me some of their expertise in algebraic geometry.
- I would like to thank everybody with whom I have talked informally about physics and mathematics during my D.Phil., but especially Matt McCullough. Such discussions, especially over cups of coffee or pints of ale, have been an invaluable and immensely enjoyable part of the whole experience.
- Finally, I would like to acknowledge the financial support of the Sir Arthur Sims Travelling Scholarship, from the University of Melbourne, and the Oxford-Australia University College Old Members' Scholarship.

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1. Introduction

Superstring theory has many remarkable features which set it apart from any other attempt to describe fundamental physics. The consistent quantisation of the theory requires that the dimension of spacetime is ten — no other theory makes such a prediction. The spectrum of excitations of a string contains gravitons, Yang-Mills gauge fields, and charged fermions, all of which we know are required for a sensible description of our universe. Furthermore, string theory provides field equations for the vacuum configurations of these fields, which reduce at large distances to general relativity coupled to a Yang-Mills gauge theory. It also appears that the theory is finite, and therefore makes sense at all energy scales. This striking structure makes string theory a very appealing candidate for the correct description of physics at the most fundamental level. Nevertheless, there is still a lot of work to be done to properly understand the theory itself. Various non-perturbative features have been understood, but a complete nonperturbative definition of the theory is yet to be found.

At least as important as understanding the foundations of string theory is attempting to connect it directly to experiment. The big problem for this pursuit, referred to broadly as 'string phenomenology', is that at low energies, string theory is typically indistinguishable from quantum field theory; quintessentially 'stringy' effects only show up around the string scale. Since this is basically the same as the fundamental scale of the gravitational interactions, it is likely to be well out of reach of current, and foreseeable future, experiments (the speculative 'large extra dimensions' scenario gets around this conclusion, but we will not discuss this possibility here). For now, the best we can do is to try to construct solutions of string theory which, at low energies, resemble the standard model of particle physics (in fact, usually its supersymmetric extension, the minimal supersymmetric standard model, or MSSM). If this can be achieved, then we can ask what other features these models have, and what their consequences are for next-generation experiments.

In this thesis, the focus will be on the $E_8 \times E_8$ heterotic string (first described in [1, 2]). In flat space, at low energies, this theory reduces to ten-dimensional $\mathcal{N} = 1$ supergravity coupled to $E_8 \times E_8$ super-Yang-Mills theory. Its massless bosonic fields are therefore the graviton, the gauge field A, a scalar field ϕ , called the dilaton, and an anti-symmetric rank-two tensor B, the field strength of which is given by

$$H = dB + \frac{\alpha'}{4} \left(\omega_L - \omega_Y\right) \tag{1.1}$$

where ω_L and ω_Y are the Chern-Simons three-forms corresponding to the tangent bundle and the gauge bundle, respectively, and α' is related to the string tension T by $\alpha' = 1/2\pi T$. This modified definition of the field strength plays an important role in heterotic string theory.

The gauge group and fermion content of the standard model are naturally contained in E_8 super-Yang-Mills theory, which makes the $E_8 \times E_8$ heterotic theory a compelling starting point for

constructing realistic string models of particle physics. For this, it is necessary to 'compactify' the theory, so that it is effectively four-dimensional at low energies. We will see that compactification involves specifying a non-vanishing background configuration for the gauge fields, which has the dual effect of partially breaking the gauge symmetry and generating a chiral spectrum in four dimensions.

One condition which is usually imposed on such string compactifications is that they leave supersymmetry unbroken at the compactification scale. This is partly a pragmatic choice, since it makes finding solutions much easier, but it is also consistent with the widespread view that lowenergy supersymmetry will play an important role in beyond-the-standard-model physics. We must therefore understand solutions of the heterotic string equations of motion which preserve $\mathcal{N} = 1$ supersymmetry in four dimensions. The first detailed analysis of this condition was performed in [3]; here we give only a heuristic justification of the results.

Suppose spacetime is of the form $\mathcal{M}_4 \times X$, where \mathcal{M}_4 is four-dimensional Minkowski space, and X is a compact six-dimensional manifold. We want to know what restrictions are placed on X by the requirement of unbroken supersymmetry. Fermion fields are taken to vanish in the background, so the supersymmetry variations of the bosonic fields automatically vanish. The condition we require is therefore that there exists some spinor for which the supersymmetry variation of each fermionic field vanishes.¹ Under the simplifying assumption that $H = d\phi = 0$, the variation of the dilatino vanishes identically, and that of the 'internal' polarisations of the gravitino is simply

$$\delta\psi_m = \nabla_m \epsilon$$

where m is an index labelling the coordinates of the compact space X, and ∇ is its spin connection. So supersymmetry requires that X supports a covariantly constant spinor. To see what this implies, recall that $\text{Spin}(6) \cong SU(4)$, and the two inequivalent Weyl spinor representations are $\mathbf{4}$ and $\overline{\mathbf{4}}$. The stabiliser of some vector in the $\mathbf{4}$ is $SU(3) \subset SU(4)$, so we conclude that the holonomy of X must be contained in SU(3). A real 2*n*-fold with holonomy contained in U(n)is a Kähler manifold [4], so X is necessarily Kähler. In the decomposition of the connection according to $\mathfrak{u}(n) \cong \mathfrak{su}(n) \oplus \mathfrak{u}(1)$, the $\mathfrak{u}(1)$ factor corresponds to the connection on the canonical bundle ω_X of X, so the holonomy of X is contained in SU(n) if and only if the holonomy of ω_X is trivial. Since X is Kähler, the curvature of ω_X has the same components as the Ricci tensor, so we learn that X is Ricci flat.

So we require X to be a Ricci-flat Kähler manifold.² Such metrics are notoriously difficult to construct, but it is a famous conjecture of Calabi, later proven by Yau, that if X is a Kähler manifold with vanishing first Chern class, $c_1(X) = 0$, then there exists a unique Ricci-flat Kähler metric on X in each Kähler class. Such spaces have become known as *Calabi-Yau manifolds*,

¹Unbroken supersymmetry automatically ensures that the field equations are solved, so we do not need to consider them separately.

²In fact this only guarantees that the *restricted* holonomy of ω_X is trivial. We will see as we go along that this is sufficient for threefolds, in which we will be interested.

and since [3], Calabi-Yau threefolds have played a central role in string theory compactification. Thanks to the simplification offered by the Calabi-Yau theorem, a huge number have now been constructed; the best-known classes are the complete intersections in products of projective spaces (CICY's) [5], and hypersurfaces in weighted \mathbb{P}^4 or more general toric varieties [6, 7].

Now that we have determined the conditions on the background geometry, we turn to the gauge fields. As well as the vanishing of the gaugino variation, we must satisfy equation (1.1). Since we have already made the assumption that H = 0, this is now very restrictive. There is, however, an obvious solution, which is to set the gauge connection equal to the spin connection, so that $\omega_Y = \omega_L$, and dB = 0. Happily, this also causes the gaugino variation (with respect to the covariantly constant spinor) to vanish, so we have our supersymmetric background.

We should elaborate on the meaning of setting the gauge fields equal to the connection on $\mathcal{T}X$. Of the two copies of E_8 provided by the heterotic string, we will focus on only one. The structure group of $\mathcal{T}X$ is SU(3), and E_8 contains a maximal subgroup³ $SU(3) \times E_6$. When we say that the gauge field is equal to the connection on $\mathcal{T}X$, we mean that we construct an E_8 connection from the spin connection on $\mathcal{T}X$, via the embedding $SU(3) \hookrightarrow SU(3) \times E_6 \hookrightarrow E_8$. This scenario is often referred to as the 'standard embedding'.

An obvious question to ask now is what the unbroken gauge group is for the solution(s) we have found. Mathematically, the object of interest is the holonomy group \mathcal{H} of the gauge connection; the unbroken gauge group in four dimensions is the centraliser⁴ of \mathcal{H} in E_8 . So for the standard embedding, we obtain a model with E_6 gauge symmetry in four dimensions.

Obviously E_6 gauge symmetry at low energies is unrealistic, so we must find a way to further reduce the gauge group. An obvious way to proceed is to let the background gauge field take values in a larger subgroup of E_8 , so that fewer generators commute with the background configuration. Other maximal subgroups of E_8 include $SU(4) \times SO(10)$, and $SU(5) \times SU(5)$, so if we could find solutions in which the gauge connection has holonomy SU(4) or SU(5), we would obtain models with SO(10) or SU(5) gauge symmetry in four dimensions. It has been argued by Witten and Witten in [10] that such solutions can be constructed by starting with a Calabi-Yau manifold X, and a gauge field which satisfies the 'slope-zero Hermitian-Yang-Mills equations'. If we let i, j, \ldots and $\bar{i}, \bar{j} \ldots$ denote holomorphic and anti-holomorphic indices on X, then these equations can be written as

$$F_{ij} = F_{\bar{\imath}\bar{\jmath}} = 0 , \quad g^{i\bar{\jmath}}F_{i\bar{\jmath}} = 0$$
 (1.2)

where F is the gauge field strength, and g is the metric on X. The first two equations here simply require that the gauge field be a connection on a holomorphic vector bundle V, but the third equation is much more difficult to solve. Fortunately, there is a theorem due to Donaldson

³Strictly speaking, the Lie algebra $\mathfrak{su}(3) \oplus \mathfrak{e}_6$ is a maximal sub-algebra of \mathfrak{e}_8 , but the correct statement at the Lie group level is that E_8 contains as a maximal subgroup $(SU(3) \times E_6)/\mathbb{Z}_3$. Such global issues will not be important for us, and we will generally ignore them.

⁴There is a caveat here: if \mathcal{H} contains a U(1) factor, then this will also be a factor in its centraliser, however the associated gauge boson obtains a large mass via the Green-Schwarz mechanism. See [8, 9] for a detailed discussion of this case.

in two complex dimensions [11], and Uhlenbeck and Yau in arbitrary dimensions [12], which says that on a Kähler manifold, (1.2) admits a solution if and only if V is *polystable*,⁵ and that such a solution is unique. This makes the problem more tractable, and is analogous to the Calabi-Yau theorem, which allows us to find Ricci-flat Kähler manifolds somewhat indirectly.

Given a Calabi-Yau manifold X, and a gauge field which is a solution of (1.2), we would like to be able to analyse the resulting low-energy physics. Perhaps the simplest feature to try to understand is the number of massless fields of various types. There are always gravitational modes, including the various geometric moduli, but these will not interest us here. Instead, we focus on massless fields descending from the super-Yang-Mills fields. The ten-dimensional vectors give rise to both vectors and scalars in four dimensions, depending on their polarisation, and since supersymmetry is unbroken, these necessarily pair up with fermions to give, respectively, vector and chiral multiplets of $\mathcal{N} = 1$ SUSY.

We will take as an example the case where the holonomy group \mathcal{H} of the gauge connection is SU(4). The gauge fields take values in the adjoint representation **248** of E_8 , and under $SU(4) \times SO(10)$ this decomposes as:

$$E_8 \supset SU(4) \times SO(10)$$
248 = (15, 1) \oplus (1, 45) \oplus (4, 16) \oplus ($\overline{4}, \overline{16}$) \oplus (6, 10)

In obvious notation, denote these representations by $(\mathbf{r}_{\alpha}, \mathbf{R}_{\alpha})$. Then if x^{μ} and y^{m} are coordinates on the external space and the Calabi-Yau manifold respectively, the Kaluza-Klein expansion of the gauge field takes the form

$$A = \sum_{\alpha,s} A^{\alpha,s}_{\mu}(x) \, dx^{\mu} f^{\alpha,s}(y) \, + \sum_{\alpha,t} \phi^{\alpha,t}(x) \, \eta^{\alpha,t}_m(y) dy^m \tag{1.3}$$

where s and t label the Kaluza-Klein modes of four dimensional vectors and scalars, respectively. Massless four-dimensional fields A_{μ} or ϕ then come from harmonic f or η , and the number of these depends on the representation \mathbf{r}_{α} of SU(4), since this determines the covariant derivative.

We will let V denote the rank-four holomorphic vector bundle associated to our SU(4)connection by the fundamental **4** representation. The $(\mathbf{4}, \mathbf{16})$ component of the gauge field therefore takes values in V, while the $(\overline{\mathbf{4}}, \overline{\mathbf{16}})$ takes values in the dual bundle V^{\sim} . The **6** of SU(4) is the rank-two anti-symmetric tensor, so the $(\mathbf{6}, \mathbf{10})$ fields correspond to $\wedge^2 V$. Similarly, the $(\mathbf{1}, \mathbf{45})$ component takes values in \mathcal{O}_X – the trivial line bundle on X – and the $(\mathbf{15}, \mathbf{1})$ takes values in $(V \otimes V^{\sim})/\mathcal{O}_X$ – the bundle of traceless endomorphisms of V. Here \mathcal{O}_X is identified with the SU(4)-invariant sub-bundle spanned, in terms of a local frame $\{v_a\}$ for V, by $\sum_a v_a v_a^*$. So determining the spectrum of the four-dimensional theory amounts to finding the number of harmonic 0- and 1-forms taking values in these bundles. For the first part we have the following important fact: stability of V implies that, of the bundles above, only \mathcal{O}_X admits a non-trivial harmonic 0-form, which is just a constant function. This means that the massless gauge bosons are precisely those corresponding to the adjoint **45** of SO(10), justifying our statement that this

⁵Stability is a straightforward, but somewhat technical condition; it is discussed later in $\S6.2$, and in [12, 13].

is the unbroken gauge group.

The next thing to calculate is how many zero modes we obtain from the second term in (1.3). The gauge field is real, so let $\eta = \eta_m dy^m$ be a real one-form on X, valued in some holomorphic vector bundle V' (not necessarily related to V from above). We can, instead, expand η in terms of complex coordinates on X:

$$\eta = \eta_i \, dz^i + \eta_{\bar{\imath}} \, d\bar{z}^i = \eta^{(1,0)} + \eta^{(0,1)}$$

where $\eta_{\bar{\iota}} = \overline{\eta_{\bar{\iota}}}$, since η is real. Now we invoke some very useful properties of Kähler manifolds. If X is Kähler, its Laplacian preserves the 'type' of a form – the number of holomorphic and anti-holomorphic indices – so η is harmonic if and only if its (1,0) and (0,1) parts are separately harmonic. Hodge decomposition then gives a one-to-one correspondence between the space of V'-valued harmonic (p,q) forms and the Dolbeault cohomology group $H^{p,q}_{\overline{\partial}}(X,V')$. Finally, Dolbeault's theorem states that this is isomorphic to the sheaf cohomology group $H^q(X,\Omega^pV')$, where Ω^pV' is the sheaf of holomorphic V'-valued (p,0)-forms. So calculating the number of real harmonic one-forms with values in V' boils down to calculating the sheaf cohomology group $H^1(X,V')$. We will see how to do such calculations later.

So solutions of (1.2) can give us models with unbroken gauge symmetry SO(10) or SU(5) in four dimensions, and we can, at least in principle, calculate their spectrum as described above. But we still need to find a way to get the gauge group down to that of the standard model, and the Higgs fields used for this purpose in traditional grand unified theories (GUTs) are not present in these models. It is true that the centraliser of $SU(5) \times U(1)$ in E_8 is exactly the standard model gauge group $G_{\rm SM} := SU(3) \times SU(2) \times U(1)$, but the U(1) here is the hypercharge gauge group $U(1)_Y$ itself, and as already mentioned, its gauge boson obtains a large mass if the structure group of V includes $U(1)_Y$ as a factor.

Fortunately, in theories with extra dimensions there is another way to reduce the unbroken gauge symmetry in four dimensions. If the internal manifold X has non-trivial fundamental group, then there exist topologically non-trivial gauge fields with vanishing field strength. In other words, we may turn on Wilson lines around homotopically non-trivial paths, taking values in the otherwise unbroken gauge group G, without affecting our solution of equation (1.2). This is equivalent to specifying a map $\varphi : \pi_1(X) \to G$, and consistency requires that this be a homomorphism of groups. The result is that the unbroken gauge group is the subgroup of G which commutes with the image of φ . In recent years, several examples of stable bundles on multiply-connected Calabi-Yau threefolds have been constructed which give rise to realistic light spectra [14-18].

From the preceding discussion, it is clear that multiply-connected Calabi-Yau threefolds are very important objects for heterotic string model building, and it is therefore of interest to construct large numbers of such manifolds and study their properties. It is a basic fact of topology that any multiply-connected space X can be constructed as a quotient of its universal cover by a freely-acting group isomorphic to $\pi_1(X)$. Furthermore, it is clear that the universal cover of a Calabi-Yau manifold is again a Calabi-Yau manifold, so the task of constructing multiply-connected Calabi-Yau threefolds requires us to find simply-connected ones which admit free actions by discrete groups of isometries. There is, however, a subtlety here: we do not know, *a priori*, that a free quotient of a Calabi-Yau is again Calabi-Yau. To understand this, recall that X having SU(3) holonomy is equivalent to the canonical bundle ω_X being trivial. This in turn is equivalent to the existence of a nowhere-vanishing global holomorphic section, Ω , usually referred to simply as 'the holomorphic three-form' on X. This generates the cohomology group $H^{3,0}(X)$, and in fact for any Calabi-Yau threefold, we have [19]

$$h^{p,0}(X) = \begin{cases} 1 , & p = 0,3 \\ 0 , & p = 1,2 \end{cases}$$

where $h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X)$. If a group G acts via free isometries on X, then the quotient X/G will be Kähler, but it will have SU(3) holonomy only if each element of G preserves Ω . Any $g \in G$ induces a map on cohomology, $g^* : H^*(X) \to H^*(X)$, and since g has no fixed points, a special case of the Lefschetz fixed point theorem [19, 20] gives us

$$0 = \sum_{p=0}^{3} (-1)^{p} \operatorname{Tr} \left(g^{*} \big|_{H^{p,0}(X)} \right)$$

Here g^* necessarily acts on $H^{0,0}(X)$ as the identity, since this group is just generated by a constant, non-zero function on X. The only other non-vanishing group is $H^{3,0}(X)$, generated by Ω , so we conclude that $g^*(\Omega) = \Omega$.

The argument above is completely general, and guarantees that the quotient of any Calabi-Yau threefold by a group of freely-acting symmetries is again Calabi-Yau. Several manifolds were constructed this way in the early days of string phenomenology [21, 22], and we will see many examples later.

In Chapter 2 we give a brief introduction to some aspects of algebraic geometry which are particularly important for later parts of this work.⁶ In Chapter 3 we find a large number of new multiply-connected Calabi-Yau threefolds by following known free group actions through conifold transitions. Chapter 4 raises and answers the question of what happens to the geometry when these group actions are allowed to develop fixed points: the quotient spaces develop isolated singularities which are quotients of the conifold. We demonstrate that in many cases, these singularities have resolutions which give rise to new topological transitions between Calabi-Yau manifolds. In Chapter 5 we describe a particularly interesting manifold with two distinct quotients, by groups of order twelve, with Euler number -6, which therefore give three net generations of fermions via the standard embedding. Finally, in Chapter 6 we describe how to modify the standard embedding to obtain $G_{\rm SM}$ instead of E_6 as the unbroken gauge group, and present a model whose spectrum is that of the MSSM plus a vector-like pair of colour triplets.

⁶Due to lack of space, the necessary background material on the *differential* geometry of complex and Kähler manifolds is not covered here (although many concepts and results have already been used). Good introductions include [23, 24].

2. Aspects of Algebraic Geometry

An algebraic variety is, loosely speaking, a space which can be specified locally by the vanishing of a set of polynomials in some finite number of variables. Most of the Calabi-Yau manifolds which we construct are actually algebraic varieties, and the algebraic point of view turns out to be quite powerful at answering certain questions about their geometry.¹ For this reason we turn now to a (very) brief summary of some important concepts and results of algebraic geometry. There are several good textbooks which cover a much wider range of material than can be presented here, e.g. [20, 25]. We will always work over the complex numbers, and eschew rigour in favour of pragmatism.

Before turning to specific topics, we review a number of foundational ideas and definitions:

 Let 𝒴 ⊂ ℂ[x₁,...,x_n] be an ideal in the ring of polynomials in n complex variables. Then we define Z(𝒴) to be the common vanishing locus, in ℂⁿ, of polynomials in the ideal:

$$Z(\mathscr{I}) = \{ p \in \mathbb{C}^n \mid f(p) = 0 \ \forall f \in \mathscr{I} \}$$

An affine algebraic variety is any set of the form $Z(\mathscr{I})$ for some ideal \mathscr{I} .

• For an affine variety $X \subset \mathbb{C}^n$, we can define \mathscr{I}_X to be the ideal in $\mathbb{C}[x_1, \ldots, x_n]$ consisting of polynomials vanishing identically on X. The *coordinate ring* of X is, roughly speaking, the ring of polynomials functions on X; its definition is

$$\mathscr{A}(X) = \frac{\mathbb{C}[x_1, \dots, x_n]}{\mathscr{I}_X}$$

This is intuitively clear: elements of \mathscr{I}_X are precisely those polynomials vanishing on X, so are identified with zero to obtain the ring of polynomials on X.

We state as a fact that there is a one-to-one correspondence between affine algebraic varieties and ideals satisfying² $\mathscr{I} = \sqrt{\mathscr{I}}$, given by $X \to \mathscr{I}_X$ and $\mathscr{I} \to Z(\mathscr{I})$.

- An affine variety X is said to be *irreducible* if \mathscr{I}_X is a prime ideal. This is the same as saying that X cannot be expressed as a non-trivial union of two sub-varieties.
- A general *algebraic variety* is constructed by gluing together affine varieties in such a way that polynomials on one are identified with polynomials on the other (so that 'algebraic' quantities coincide).
- An ideal $\mathscr{I} \subset \mathbb{C}[z_0, z_1, \ldots, z_N]$ is called *homogeneous* if it is generated by homogeneous polynomials. Note that the vanishing locus of a homogeneous polynomial is invariant under rescaling of the coordinates, and can therefore be defined in \mathbb{P}^N .

 $^{^{1}}$ It is possible to take almost a purely algebraic approach, but we will not go to this extreme. Frequently, the algebraic and analytical approaches are complementary.

²Recall that the *radical* is defined by $\sqrt{\mathscr{I}} := \{f \mid f^r \in \mathscr{I} \text{ for some } r > 0\}.$

A projective variety is the common vanishing locus, in \mathbb{P}^N , of the polynomials in some homogeneous ideal.³

The Fubini-Study metric on projective space is Kähler, which means that smooth projective varieties are Kähler manifolds. It is for this reason that the projective property is important to us in the study of Calabi-Yau manifolds.

• \mathcal{O}_X is the *structure sheaf* of X. Its local sections are holomorphic rational functions of the local complex coordinates; \mathcal{O}_X is therefore a sheaf of rings. More precisely, if X is affine and $U \subset X$ is open, then the sections of \mathcal{O}_X over U are

$$\Gamma(U, \mathcal{O}_X) = \{ f/g \mid f, g \in \mathscr{A}(X), g(p) \neq 0 \ \forall p \in U \}$$

 \mathcal{O}_X^* is the subsheaf of nowhere-vanishing functions. This is a sheaf of multiplicative groups.

- \mathcal{K}_X is the sheaf of meromorphic rational functions on X (so its local sections are the same as those of \mathcal{O}_X above, but without the restriction that g be everywhere non-vanishing); like \mathcal{O}_X , it is a sheaf of rings. Its global sections, $\Gamma(X, \mathcal{K}_X)$, constitute the *function field* of X. \mathcal{K}_X^* is the subsheaf of not-identically-zero functions. Like \mathcal{O}_X^* , this is a sheaf of multiplicative groups.
- The *Segré embedding* is very important to us because it establishes that the product of projective spaces is a projective variety. It is defined by the following map:

 $\phi: \mathbb{P}^N \times \mathbb{P}^M \to \mathbb{P}^{(N+1)(M+1)-1}$

 $(z_0, \ldots, z_N) \times (w_0, \ldots, w_M) \mapsto (z_0 \, w_0, z_0 \, w_1, \ldots, z_0 \, w_M, z_1 \, w_0, \ldots, z_N \, w_M)$

It can be checked that this is indeed an embedding. If we take homogeneous coordinates $Z_{i,\alpha}$, where i = 0, ..., N and $\alpha = 0, ..., M$, ordered so that ϕ is given by $Z_{i,\alpha} = z_i w_{\alpha}$, then the image of ϕ is the zero set of the polynomials $Z_{i,\alpha}Z_{j,\beta} - Z_{i,\beta}Z_{j,\alpha}$. It therefore satisfies the criterion given above for a projective variety, which shows that $\mathbb{P}^N \times \mathbb{P}^M$ is projective.

2.1 Divisors and line bundles

Loosely speaking, the divisors on a variety X are the codimension-one objects. There are two ways to define them, one geometric, and the other algebraic. In the case that X is smooth, these coincide in a well-defined way, but in general they do not. Throughout this section, we will take $\{\mathcal{U}_{\alpha}\}$ to be a fixed open affine cover of X, and define $\mathcal{U}_{\alpha\beta} := \mathcal{U}_{\alpha} \cap \mathcal{U}_{\beta}$.

2.1.1 Divisors

A prime divisor on X is an irreducible codimension-one subvariety of X. A general Weil divisor is a formal finite⁴ linear combination, with integer coefficients, of prime divisors. If

 $^{^{3}}$ Complex analytic varieties can be defined by replacing the polynomial rings of this section with rings of analytic functions. *Chow's theorem* states that all projective analytic varieties are algebraic, so we do not 'miss' anything in the projective case by restricting to algebraic varieties.

⁴If X is non-compact, this is replaced by *locally finite*, but we will ignore this complication.

all the coefficients are non-negative, the divisor is called *effective*. The Weil divisors form an additive group, the *divisor group* Div(X).

If $f \in \Gamma(X, \mathcal{K}_X^*)$ is a global meromorphic function on X, we associate a divisor to it as follows. If $V \subset X$ is an irreducible hypersurface, choose a smooth point (defined in §2.3) $p \in V$, and local coordinates x_1, \ldots, x_n on X such that V is given locally by $x_1 = 0$. Then there is a unique integer k for which $f = x_1^k g$, where g is holomorphic and non-zero at $x_1 = 0$. k is called the order of f along V, $\operatorname{ord}_V(f)$, and we define a divisor

$$(f) = \sum_{V} \operatorname{ord}_{V}(f) V \tag{2.1}$$

It should be clear from context when parentheses are intended to indicate a divisor in this way. Two divisors D_1, D_2 are said to be *linearly equivalent*, denoted $D_1 \sim D_2$, if there exists a global meromorphic function f such that $D_1 = D_2 + (f)$. The quotient of Div(X) by this equivalence relation is the *divisor class group* Cl(X) of X.

A Cartier divisor D on X is a global section of $\mathcal{K}_X^*/\mathcal{O}_X^*$. That is, D can be specified by local meromorphic functions $h_{\alpha} \in \Gamma(\mathcal{U}_{\alpha}, \mathcal{K}_X^*)$ such that on each $\mathcal{U}_{\alpha\beta}, h_{\alpha}/h_{\beta} \in \Gamma(\mathcal{U}_{\alpha\beta}, \mathcal{O}_X^*)$. A Cartier divisor is called *effective* if each h_{α} is actually holomorphic (has no poles).

There is a natural map from the group of Cartier divisors to Div(X), given on an open patch \mathcal{U}_{α} by $D \to (h_{\alpha})$, as defined in equation (2.1); it is easy to see that this patches together into a well-defined map. This map is injective, and is a homomorphism, since $(h^1h^2) = (h^1) + (h^2)$. If X is smooth, this map is also surjective, giving an isomorphism between the two notions of divisor. Note also that effective Cartier divisors get mapped to effective Weil divisors. For the rest of this section, we will only consider the case where X is smooth, so the two definitions of divisor coincide as above.

2.1.2 Holomorphic line bundles

A holomorphic line bundle is a complex line bundle, \mathcal{L} , with holomorphic transition functions $g_{\alpha\beta} \in \Gamma(\mathcal{U}_{\alpha\beta}, \mathcal{O}_X^*)$, *i.e.* its sections satisfy $\sigma_{\alpha} = g_{\alpha\beta}\sigma_{\beta}$, where σ_{α} is a section of \mathcal{L} over \mathcal{U}_{α} . There is a one-to-one correspondence between elements of the divisor class group and holomorphic line bundles (up to isomorphism), which we will now describe.

A divisor D gives rise to a holomorphic line bundle $\mathcal{L}(D)$ in a straightforward way: the transition functions for $\mathcal{L}(D)$ are given by $g_{\alpha\beta} = h_{\alpha} h_{\beta}^{-1}$, where h_{α} is a local defining equation for D on \mathcal{U}_{α} . Note that, trivially, the h_{α} define a meromorphic section h of $\mathcal{L}(D)$, since we have $h_{\alpha} = (h_{\alpha} h_{\beta}^{-1})h_{\beta}$. Now let $D' = (\sigma)$, where σ is any meromorphic section of $\mathcal{L}(D)$; therefore it is given by local sections satisfying $\sigma_{\alpha} = (h_{\alpha} h_{\beta}^{-1})\sigma_{\beta}$. Then clearly $\sigma_{\alpha}/h_{\alpha} = \sigma_{\beta}/h_{\beta}$ on $\mathcal{U}_{\alpha\beta}$, so this defines a global meromorphic function $f = \sigma/h$, and D' = D + (f), so $D' \sim D$. The converse also holds: any divisor linearly equivalent to D is given by a meromorphic section of $\mathcal{L}(D)$. Furthermore, the transition functions for $\mathcal{L}(D')$ are $g'_{\alpha\beta} = \sigma_{\alpha} \sigma_{\beta}^{-1} = h_{\alpha} h_{\beta}^{-1} = g_{\alpha\beta}$, so $\mathcal{L}(D') \cong \mathcal{L}(D)$. In this way, linear equivalence of divisors corresponds to isomorphism of line bundles.⁵

The group of isomorphism classes of holomorphic line bundles (or equivalence classes of divisors), on X is called its *Picard group*, Pic(X). Such a line bundle is specified by its transition functions, $g_{\alpha\beta} \in \Gamma(\mathcal{U}_{\alpha\beta}, \mathcal{O}_X^*)$, which must satisfy the consistency condition $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$. This says that the $g_{\alpha\beta}$ must be a Čech 1-cocycle for the sheaf \mathcal{O}_X^* . The line bundle is trivial if and only if for each α, β , we have $g_{\alpha\beta} = f_{\alpha} f_{\beta}^{-1}$ for some $f_{\alpha} \in \Gamma(\mathcal{U}_{\alpha}, \mathcal{O}_X^*), f_{\beta} \in \Gamma(\mathcal{U}_{\beta}, \mathcal{O}_X^*)$, which is the statement that the $g_{\alpha\beta}$ are a Čech coboundary. So in fact, the Picard group is the same as the first Čech cohomology group of the sheaf \mathcal{O}_X^* :

$$\operatorname{Pic}(X) \cong \check{H}^1(X, \mathcal{O}_X^*)$$

2.2 Holomorphic vector bundles and their cohomology

More generally, a complex vector bundle V on a complex manifold X is called *holomorphic* if its transition functions can be taken to be holomorphic functions of the local complex coordinates. This allows us to define holomorphic sections of V, since holomorphicity will be preserved across patches.

As discussed in Chapter 1, the low-energy spectrum of a heterotic string compactification is determined by certain Dolbeault cohomology groups of holomorphic vector bundles. If we replace such a bundle V by the sheaf of its holomorphic sections, then there are two other important cohomology theories we can consider — Grothendieck's sheaf cohomology, and Čech cohomology. Happily, all these theories give the same cohomology groups, so we may use any we wish.⁶ The easiest to calculate is invariably Čech cohomology, so we use this for explicit calculations.

A convenient way to calculate the cohomology of some vector bundle is, if possible, to fit it into a short exact sequence with other bundles, the cohomology of which is already known. For, if we have a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

on an n-fold X, then the cohomology groups fit into a long exact sequence

and all higher cohomology groups vanish. The maps across each row are the obvious ones induced on Čech cohomology by the maps in the short exact sequence, whereas the definition of δ^* is slightly complicated, but is described in any book on homological algebra, *e.g.* [26]. One

0

⁵Changing the local trivialisations of the line bundle is the same as $h_{\alpha} \to f_{\alpha}h_{\alpha}$ for local non-vanishing holomorphic functions f_{α} . This clearly doesn't change the divisor.

⁶The different cohomology theories have different utility. De Rham cohomology comes up most naturally in physics (and, on a Kähler manifold, is refined by Dolbeault cohomology), sheaf cohomology is convenient for proving theorems via homological algebra, and Čech cohomology lends itself well to explicit calculation.

very useful fact is that if we take the tensor product of each term in an exact sequence with some other bundle, we obtain a new exact sequence.

There is one further simplification to note, which we have already used implicitly above by always restricting to rational functions rather than arbitrary analytic functions. All the manifolds we deal with in this work are projective, and in this case we can appeal to the powerful results of Serre's 'GAGA' paper [27], which roughly let us replace all complex analytic quantities with algebraic ones. More precisely, the transition functions of a holomorphic vector bundle V can always be chosen to be made up of rational functions, allowing us to define the sheaf of 'algebraic sections' of V, locally isomorphic to $\mathcal{O}_X^{\oplus \operatorname{rk}(V)}$. Furthermore, the cohomology of this sheaf is the same as the cohomology of the sheaf of all holomorphic sections of V.

2.2.1 Bundles on hypersurfaces

In this subsection we will take X to be a smooth hypersurface in a manifold M, and let $\{\mathcal{U}_{\alpha}\}$ be an open cover of M (which of course restricts to an open cover of X), with local coordinates $x_{\alpha} : \mathcal{U}_{\alpha} \to \mathbb{C}^{n+1}$. X can be thought of as an effective divisor on M, and is given by the vanishing of a holomorphic section h of the line bundle $\mathcal{L}(X)$, which we will take to have transition functions $g_{\alpha\beta}$, so that $h_{\alpha} = g_{\alpha\beta}h_{\beta}$.

Recall that the normal bundle to X in M, denoted $\mathcal{N}_{X|M}$, is defined to be $\mathcal{T}M|_X/\mathcal{T}X$. Since $h_{\alpha} = 0$ is a local defining equation for X on \mathcal{U}_{α} , at a point $p \in X \cap \mathcal{U}_{\alpha}$, $\mathcal{T}_pX \subset \mathcal{T}_pM$ is defined by

$$\mathcal{T}_p X = \{ v \in \mathcal{T}_p M \mid dh_\alpha(v) = 0 \}$$

We can therefore define a local coordinate function on the fibres of $\mathcal{N}_{X|M}$ by $v \to dh_{\alpha}(v)$. This lets us determine the transition functions of $\mathcal{N}_{X|M}$; on $\mathcal{U}_{\alpha} \cap \mathcal{U}_{\beta}$, we have

$$dh_{\alpha}(v) = \sum_{i} v_{\alpha i} \frac{\partial h_{\alpha}}{\partial x_{\alpha i}} = \sum_{i} v_{\beta i} \frac{\partial (g_{\alpha \beta} h_{\beta})}{\partial x_{\beta i}}$$
$$= \sum_{i} \left(h_{\beta} v_{\beta i} \frac{\partial g_{\alpha \beta}}{\partial x_{\beta i}} + g_{\alpha \beta} v_{\beta i} \frac{\partial h_{\beta}}{\partial x_{\beta i}} \right) = g_{\alpha \beta} dh_{\beta}(v)$$

where we have used the fact that $h_{\beta} \equiv 0$, because we are working on X. We see that the transition functions for $\mathcal{N}_{X|M}$ are exactly those of $\mathcal{L}(X)$, restricted to X, so we have derived the isomorphism

$$\mathcal{N}_{X|M} \cong \mathcal{L}(X)|_X$$

The preceding paragraph can be summarised by the following short exact sequence on X:

$$0 \longrightarrow \mathcal{T}X \longrightarrow \mathcal{T}M|_X \xrightarrow{dh} \mathcal{L}(X)|_X \longrightarrow 0$$

This allows us to relate the tangent bundle of X to bundles restricted from M to X. This is typically the first step in calculating the cohomology of some bundle on X, as we will wish to do later. The next is to 'lift' the calculations to M itself. To do so, we need an exact sequence relating functions on X to those on M. This follows from our earlier discussion of affine coordinate rings – local functions on X are identified with local functions on M, modulo the equivalence relation $h \sim 0$. So we want to set all multiples of h to zero; by definition, multiples of h constitute the *ideal sheaf* \mathscr{I}_X of X. The short exact sequence is therefore

$$0 \longrightarrow \mathscr{I}_X \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_X \longrightarrow 0$$

The sheaf \mathscr{I}_X is a line bundle, since it is locally generated by the single function h_{α} . We will now find explicit transition functions for \mathscr{I}_X .

As for any line bundle, a local section of \mathscr{I}_X is a single holomorphic function σ_{α} on M, with the map $\mathscr{I}_X \to \mathcal{O}_M$ given by $\sigma_{\alpha} \to \sigma_{\alpha} h_{\alpha}$. Since the transition functions for \mathcal{O}_M are trivial, on $\mathcal{U}_{\alpha} \cap \mathcal{U}_{\beta}$ we have

$$\sigma_{\alpha} h_{\alpha} = \sigma_{\beta} h_{\beta}$$
$$\implies \qquad \sigma_{\alpha} = h_{\alpha}^{-1} h_{\beta} \sigma_{\beta} = g_{\alpha\beta}^{-1} \sigma_{\beta}$$

so we see that $\mathscr{I}_X \cong \mathcal{L}(X)^{-1} \cong \mathcal{L}(-X)$. The short exact sequence becomes

$$0 \longrightarrow \mathcal{L}(-X) \stackrel{h}{\longrightarrow} \mathcal{O}_M \longrightarrow \mathcal{O}_X \longrightarrow 0$$

Given a short exact sequence such as this, we can take the tensor product of each term with any vector bundle to obtain another short exact sequence. This allows us to lift cohomology calculations from X to M, as we will need to do repeatedly later.

One bundle of particular interest, especially since we will be studying Calabi-Yau manifolds, is the canonical bundle. If X is a smooth hypersurface in a manifold M, then its canonical bundle is given by the *adjunction formula* (see *e.g.* [25])

$$\omega_X = \omega_M \big|_X \otimes \mathcal{N}_{X|M} \tag{2.2}$$

We see then that X has trivial canonical bundle if and only if it is given by a section of the anti-canonical bundle of M, so that $\mathcal{N}_{X|M} \cong \mathcal{L}(X)|_X \cong \omega_M^{-1}|_X$.

2.2.2 Projective space

In this thesis, we will always calculate cohomology groups by relating them to the cohomology of line bundles on projective space, so we need an explicit description of such cohomology groups. Take homogeneous coordinates (z_0, \ldots, z_N) on \mathbb{P}^N , an open affine cover $\{\mathcal{U}_i\}_{i=0}^N$ given by $\mathcal{U}_i = \{z_i \neq 0\}$, and affine coordinates $x_{i,j} = z_j/z_i$ on \mathcal{U}_i .

A hyperplane in \mathbb{P}^N is given by the vanishing of some linear polynomial in the homogeneous coordinates. The ratio of two such polynomials is a well-defined global meromorphic function on \mathbb{P}^N (since it has homogeneity degree zero), so any two hyperplanes are linearly equivalent as divisors. The corresponding line bundle is denoted by $\mathcal{O}_{\mathbb{P}^N}(1)$, and generates the Picard group of \mathbb{P}^N . Its k^{th} tensor power is denoted by $\mathcal{O}_{\mathbb{P}^N}(k)$.

The cohomology of these line bundles is particularly simple. Recalling that small h's indicate the dimension of a cohomology group (as a \mathbb{C} vector space), we have

$$h^{0}(\mathbb{P}^{N}, \mathcal{O}_{\mathbb{P}^{N}}(k)) = \begin{cases} \frac{(N+k)!}{N!k!} , & k \ge 0 \\ 0 , & k < 0 \end{cases} \quad h^{N}(\mathbb{P}^{N}, \mathcal{O}_{\mathbb{P}^{N}}(k)) = \begin{cases} 0 , & k \ge -N \\ \frac{(-k-1)!}{N!(-N-k-1)!} , & k < -N \end{cases}$$
$$h^{i}(\mathbb{P}^{N}, \mathcal{O}_{\mathbb{P}^{N}}(k)) = 0 , & 1 \le i \le N-1 \end{cases}$$

The non-vanishing cohomology groups also have a simple representation in terms of homogeneous polynomials. A generic monomial is

$$\prod_{i=0}^{N} z_i^{l_i} \dots z_N^{l_N} \quad , \quad l_i \ge 0 \ \forall \ i$$

For $k \geq 0$, $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k))$ is simply spanned by the degree k monomials. We identify these with Čech 0-cycles by dividing by z_i^k on \mathcal{U}_i , to get well-defined local holomorphic functions. For k < -N, $H^N(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k))$ is spanned by the *inverses* of degree (-k) monomials with $l_i \geq 1 \forall i$. These are identified with Čech N-cycles on $\bigcap_{i=0}^N \mathcal{U}_i$ by taking, say, $\{x_{0,i} = z_i/z_0\}$ as local coordinates, and multiplying by z_0^{-k} .

We will take this explicit description of the cohomology of line bundles on \mathbb{P}^N as the basis for the calculation of bundle cohomology on projective varieties.

The Euler sequence

The only other bundle on \mathbb{P}^N which we will discuss here is the holomorphic tangent bundle, which has a simple description in terms of line bundles. To see this, let $\pi : \mathbb{C}^{N+1} \setminus \mathbf{0} \to \mathbb{P}^N$ be the natural projection map, with differential $d\pi$. Interpret the homogeneous coordinates $\{z_0, \ldots, z_N\}$ as coordinates on \mathbb{C}^{N+1} , and consider a general vector field

$$v(z) = \sum_{i} v_i(z) \frac{\partial}{\partial z_i}$$
(2.3)

This will descend to a well-defined vector field on \mathbb{P}^N if and only if the image of v(z) under $d\pi$ is the same as that of $v(\lambda z)$ for any $\lambda \in \mathbb{C}^*$, since z and λz project to the same point of \mathbb{P}^N . To study this condition, we work in the affine patch⁷ where $z_0 \neq 0$, with local coordinates

$$(x_1, \ldots, x_N) := \pi(z_0, \ldots, z_N) = \left(\frac{z_1}{z_0}, \ldots, \frac{z_N}{z_0}\right)$$

From this we easily obtain the differential

$$d\pi \left(\frac{\partial}{\partial z_0}\right) = -\sum_{i=1}^N \frac{z_i}{z_0^2} \frac{\partial}{\partial x_i} \quad , \quad d\pi \left(\frac{\partial}{\partial z_i}\right) = \frac{1}{z_0} \frac{\partial}{\partial x_i} \quad i \neq 0$$

By inspection, then, our vector field v defined in equation (2.3) will descend to \mathbb{P}^N if and only if $v_i(\lambda z) = \lambda v_i(z)$ for all i. In other words, each of the N + 1 components of v must be linear in the homogeneous coordinates, so we can interpret v as a section of $\mathcal{O}_{\mathbb{P}^N}(1)^{\oplus N+1}$. It is also clear that $d\pi$ is surjective, since $\frac{\partial}{\partial x_i} = d\pi \left(z_0 \frac{\partial}{\partial z_i} \right)$, so we have an exact sequence

$$\mathcal{O}_{\mathbb{P}^N}(1)^{\oplus N+1} \xrightarrow{d\pi} \mathcal{T}\mathbb{P}^N \to 0$$

To complete this to a short exact sequence, we need the kernel of $d\pi$. It follows easily from the expressions above that it is generated by the *Euler vector field*, which we define to be

$$e := \sum_{i=0}^{N} z_i \frac{\partial}{\partial z_i}$$

This can clearly be interpreted as a section of $\mathcal{O}_{\mathbb{P}^N}(1)^{\oplus N+1}$, and since it does not vanish anywhere, it defines a trivial sub-bundle, isomorphic to $\mathcal{O}_{\mathbb{P}^N}$. The short exact sequence we have

⁷It is left to the reader to check that everything we say remains true when changing to a different affine patch.

just derived is known as the *Euler sequence*:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^N} \longrightarrow \mathcal{O}_{\mathbb{P}^N}(1)^{\oplus N+1} \xrightarrow{d\pi} \mathcal{T}\mathbb{P}^N \longrightarrow 0$$
(2.4)

2.3 Singularities and resolutions

Suppose we have an affine algebraic variety, X, which is the common vanishing locus of polynomials f_1, \ldots, f_k in \mathbb{C}^{k+n} , and take an arbitrary point $p \in X$. X is said to be *smooth at* p if $df_1 \wedge \ldots \wedge df_k \neq 0$ at p. Otherwise X is said to be *singular at* p. If X is smooth at every point, then it has the structure of a complex manifold.

Sometimes, given a singular variety X, it is useful to 'resolve' the singularities by finding a smooth variety which is "nearly isomorphic" to X. Before we make this idea precise, we will describe the process of blowing up a sub-variety, which is the main technique used.

2.3.1 Blowing up

One very important operation on algebraic varieties is that of *blowing up* along some subvariety. To avoid complication, we will define blowing up in a limited context, but it will be sufficient for our purposes.

First we define the blow-up of projective space \mathbb{P}^N along some smooth sub-variety $V \subset \mathbb{P}^N$. Let V be of dimension N - k, given by k polynomial equations $f_0 = f_1 = \ldots = f_{k-1} = 0$ (this is the statement that V is a *complete intersection*). Introduce another projective space \mathbb{P}^{k-1} , with homogeneous coordinates (z_0, \ldots, z_{k-1}) . Then the blow-up of \mathbb{P}^N along V is defined in $\mathbb{P}^N \times \mathbb{P}^{k-1}$ by

$$f_i z_j - f_j z_i = 0$$
, $i, j = 0, \dots, k - 1$ (2.5)

To see what this new variety looks like, consider the map $\pi : \mathbb{P}^N \times \mathbb{P}^{k-1} \to \mathbb{P}^N$ given by projection onto the first factor. Take some point $p \in \mathbb{P}^N \setminus V$. Then at least one polynomial f_i is non-zero at p; we take $f_0(p) \neq 0$, without loss of generality. Then the equations (2.5) simply say that $z_i/z_0 = f_i/f_0$, so $\pi^{-1}(p)$ is a unique point. On the other hand, if $p \in V$, then $f_i(p) = 0 \forall i$, and (2.5) is satisfied for any values of the z_i , so that $\pi^{-1}(p) \cong \mathbb{P}^{k-1}$.

So, topologically, the blow-up of \mathbb{P}^N along the codimension-k variety V is simply \mathbb{P}^N with each point on V replaced by a copy of \mathbb{P}^{k-1} . We should think of this \mathbb{P}^{k-1} as being the set of normal directions to V in \mathbb{P}^N (we will see why below). $\pi^{-1}(V)$ is called the *exceptional divisor* of the blow-up.

Now let $X \subset \mathbb{P}^N$ be some other variety, and let $Y = X \cap V$. Then the blow-up of X along Y, which we will denote by \widetilde{X} , is defined to be

$$\widetilde{X} = \overline{\pi^{-1}(X \setminus Y)} \tag{2.6}$$

where an overline denotes the topological closure. Note that, thanks to the Segré embedding, the blow-up of a projective variety is again projective. This will be important to us later.

It is easiest to understand the definition of a blow-up via an example. We will blow up \mathbb{C}^2 at the origin, $(x_1, x_2) = (0, 0)$ (this fits into the above discussion by compactifying \mathbb{C}^2 to \mathbb{P}^2), and ask what the blow-up of some smooth curve C, passing through the origin, looks like. As described above, the blown-up plane is constructed by introducing a \mathbb{P}^1 , with homogeneous coordinates (z_1, z_2) , and imposing, in $\mathbb{C}^2 \times \mathbb{P}^1$, the equation

$$x_2 \, z_1 - x_1 \, z_2 = 0$$

Away from the origin of \mathbb{C}^2 , this is equivalent to $(z_1, z_2) \propto (x_1, x_2)$, but at the origin, it is satisfied identically, so we get an entire copy of \mathbb{P}^1 , which is the exceptional divisor E of the blow-up.

Let C be a smooth curve embedded by some function $f : \mathbb{C} \to \mathbb{C}^2$, $f(t) = (at, bt) + \mathcal{O}(t^2)$, where a and b are constants. Then as $t \to 0$, we have $x_2/x_1 \to b/a$, so taking the closure as per equation (2.6), we see that \widetilde{C} intersects the exceptional curve $E \cong \mathbb{P}^1$ at the point $(z_1, z_2) = (a, b)$, corresponding to the tangent direction to C at the origin.

So now we see the geometric meaning of blowing up — sub-varieties which intersect the blown-up sub-variety get separated according to the direction from which they approach it. We will see more examples as we go along.

2.3.2 Resolution of singularities

Suppose X is singular along some set S. A resolution of X is a smooth variety \hat{X} along with a surjective morphism $\pi : \hat{X} \to X$, such that π is an isomorphism on $\hat{X} \setminus \pi^{-1}(S)$. In practice, resolutions are typically achieved by blowing up X along the singular set, or some larger sub-variety which contains the singular set.

Example

As a simple example, let us take X to be the hypersurface S given by $uv - w^2 = 0$ in \mathbb{C}^3 . It is clear that this is singular at the origin, and in fact, this variety is $\mathbb{C}^2/\mathbb{Z}_2$, where the \mathbb{Z}_2 action is $(x, y) \to (-x, -y)$. To see this, note that the coordinate ring of $\mathbb{C}^2/\mathbb{Z}_2$ is just the \mathbb{Z}_2 -invariant part of $\mathbb{C}[x, y]$. It is easy to see that this is generated by 1 and the following invariants:

$$u=x^2\ ,\ v=y^2\ ,\ w=xy$$

These satisfy the identity $uv - w^2 = 0$, so we have

$$\mathscr{A}(\mathbb{C}^2/\mathbb{Z}_2) \cong \frac{\mathbb{C}[u, v, w]}{(uv - w^2)} = \mathscr{A}(S)$$
(2.7)

where parenthese around a ring element denote the principal ideal it generates. By the correspondence between ideals and affine varieties, this establishes that $S \cong \mathbb{C}^2/\mathbb{Z}_2$.

We will now construct a resolution of S by blowing up the singular point. The blow-up of the ambient \mathbb{C}^3 at the origin is given in $\mathbb{C}^3 \times \mathbb{P}^2$ by

$$u z_1 - v z_0 = u z_2 - w z_0 = v z_2 - w z_1 = 0$$

where (z_0, z_1, z_2) are homogeneous coordinates on the \mathbb{P}^2 . Away from the origin, these equations amount to $(z_0, z_1, z_2) \propto (u, v, w)$, so to study the blow-up \tilde{S} away from the origin of \mathbb{C}^3 , we can make this replacement in the equation defining S. As per equation (2.6), \tilde{S} is obtained by taking the closure of the resulting solution set inside $\mathbb{C}^3 \times \mathbb{P}^2$, so \widetilde{S} is simply given by

$$u z_1 - v z_0 = 0$$
, $u z_2 - w z_0 = 0$
 $v z_2 - w z_1 = 0$, $z_0 z_1 - z_2^2 = 0$

It is easy to check that these equations define a smooth variety, which is isomorphic to Saway from (u, v, w) = (0, 0, 0). There is also an obvious projection $\tilde{S} \to S$, inherited from the projection of $\mathbb{C}^3 \times \mathbb{P}^2$ onto its first factor. The exceptional divisor – the pre-image of the singular point in S – is given by $z_0 z_1 - z_2^2 = 0$ in \mathbb{P}^2 . This curve is isomorphic to \mathbb{P}^1 , which can be seen explicitly by defining the map $\phi : \mathbb{P}^1 \to \mathbb{P}^2$, $(t_1, t_2) \mapsto (t_1^2, t_2^2, t_1 t_2)$, and checking that it is an embedding, with image given exactly by the equation above.

The singularity $\mathbb{C}^2/\mathbb{Z}_2$ is also known as the A_1 surface singularity. For any n > 0, the A_n singularity is the quotient given by $(x, y) \sim (\zeta x, \zeta^n y)$, where $\zeta = \exp(2\pi i/(n+1))$. These all have crepant resolutions (*i.e.* the resolution has trivial canonical class), where the exceptional set is a chain of $n \mathbb{P}^1$'s, labelled E_i , with intersections $E_i \cdot E_j = -2\delta_{ij} + \delta_{i,j+1} + \delta_{i,j-1}$.

2.3.3 The conifold, and conifold transitions

In this thesis we are primarily interested in Calabi-Yau threefolds. Possibly the simplest singularity that any variety can have is called an ordinary double point, or node. In the threefold case, the local model of such a singularity is called the *conifold*, which we will denote by C, and is a hypersurface in \mathbb{C}^4 given by the equation

$$y_1 y_4 - y_2 y_3 = 0 \tag{2.8}$$

It is easy to check that this is smooth everywhere except at the origin of \mathbb{C}^4 , where it has a singularity.

The conifold can be thought of as a limiting case of a family of smooth hypersurfaces, usually called *deformed* or *smoothed* conifolds, given by

$$y_1 y_4 - y_2 y_3 - \epsilon = 0 \tag{2.9}$$

where ϵ is a constant. It is easy to show that topologically, taking $\epsilon \to 0$ causes an embedded S^3 to shrink to zero size [28]. The deformed conifold is in fact a non-compact Calabi-Yau threefold, and conifold singularities can therefore occur in Calabi-Yau spaces.

A good question to ask is how we might resolve the conifold. One way to do so is to blow up its singular point, as we did in the surface example above, but this introduces an exceptional divisor which contributes to the canonical class of the resolved manifold (see §4.3.1). We therefore obtain a manifold which is *not* Calabi-Yau, so this is of little use to us. There is, however, another way to resolve the singularity, for which the resolution is also Calabi-Yau. Ccontains a Weil divisor, D, specified by the two equations⁸

$$y_1 = 0 , \quad y_2 = 0$$
 (2.10)

⁸We could alternatively consider the divisor D' given by $y_1 = y_3 = 0$; the resulting resolution differs by a 'flop' from the one presented here.

These are easily seen to give an irreducible two-dimensional subvariety of C, since y_3 and y_4 can take any values, but it is impossible to specify D in a neighbourhood of the singular point by a single equation in C, so it does not correspond to a Cartier divisor.

Consider blowing up along D. To do so, we add a \mathbb{P}^1 factor, with homogeneous coordinates t_0, t_1 , to the ambient space, and add the equation $y_1t_0 - y_2t_1 = 0$. This has the effect of setting $(t_0, t_1) \propto (y_2, y_1)$ away from D, so the blow up of \mathcal{C} along D is given in $\mathbb{C}^4 \times \mathbb{P}^1$ by

$$y_1 t_0 - y_2 t_1 = 0$$

$$-y_3 t_0 + y_4 t_1 = 0 \qquad \Leftrightarrow \qquad \begin{pmatrix} y_1 & -y_2 \\ -y_3 & y_4 \end{pmatrix} \begin{pmatrix} t_0 \\ t_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (2.11)$$

Since t_0 and t_1 cannot simultaneously vanish, these equations only have a solution where the determinant of the matrix vanishes. The determinant is just the polynomial defining the conifold C, so there is an obvious projection from the blown-up variety to C, inherited from the projection of $\mathbb{C}^4 \times \mathbb{P}^1$ onto its first factor. This map is one-to-one when any y_i is non-zero, but at the origin in \mathbb{C}^4 , the t's are completely undetermined, so we get an entire copy of \mathbb{P}^1 . Finally, it is easy to check that the equations (2.11) define a smooth variety, so indeed this gives a resolution of C; the exceptional set is just the \mathbb{P}^1 lying over the origin. Therefore there is no exceptional divisor; for this reason, it is known as the *small resolution* of the conifold. This means there can be no extra contribution to the canonical class, *i.e.* the resolution is crepant, and the resolved manifold is Calabi-Yau. If we take into account the different Kähler metrics on the resolved manifold, we can think about the resolution as being a continuous process, increasing the size of the exceptional \mathbb{P}^1 from zero.

So we have seen that we can pass continuously from the deformed conifold to the resolved conifold. This process is known as a conifold transition, and is shown schematically in Figure 2.1. It is possible under certain circumstances for a compact Calabi-Yau manifold to develop several



Figure 2.1: A graphical representation of a conifold transition. This and many other graphics herein were produced with Mathematica [29].

nodes and undergo a conifold transition, yielding a topologically distinct Calabi-Yau manifold.⁹ This is central to the work presented in Chapter 3, and we will defer further details to there.

2.4 Toric geometry

Toric geometry represents a wide-ranging generalisation of the following familiar example. Start with the multiplicative group \mathbb{C}^* , which we can also think of as an algebraic variety with

⁹This is not possible in the case when only a single node occurs, see [28, 30].

coordinate ring $\mathbb{C}[x, x^{-1}]$. Adding a single point ('the origin'), we obtain the complex plane \mathbb{C} . We can then add the point at infinity, giving us the Riemann sphere, or \mathbb{P}^1 .

Notice that the group \mathbb{C}^* with which we started acts on the resulting space. For $\lambda \in \mathbb{C}^*$, the action is $z \to \lambda z$ for $z \in \mathbb{C}^*$, and the two points we added are fixed points.

With the example above in mind, we say that an *n*-dimensional algebraic variety X is a toric variety if it contains the algebraic torus $\mathbb{T}^n := (\mathbb{C}^*)^n$ as a dense subset, and the group action of \mathbb{T}^n on itself extends to an action on X. Toric varieties can therefore be thought of as (partial) compactifications of algebraic tori. This definition seems very restrictive, but we will see that toric geometry encompasses a wide range of interesting examples, and offers a powerful computational framework.

Here we will consider a number of different ways of looking at toric varieties, all of which are equivalent (although it will not be proven). There are also several reviews of toric geometry to be found in the physics literature [31-34], and a substantial text by Fulton [35].

2.4.1 Affine toric varieties

To specify an *n*-dimensional affine toric variety, we begin with a lattice $N \cong \mathbb{Z}^n$, and its dual $M := \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, with pairing \langle , \rangle . We also define associated vector spaces, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, and similarly for $M_{\mathbb{R}}$. The pairing \langle , \rangle naturally extends to these vector spaces.

A strongly convex rational polyhedral cone σ in $N_{\mathbb{R}}$ is a cone, with its tip at the origin, generated by a finite number of lattice vectors, and satisfying $\sigma \cap -\sigma = \mathbf{0}$. From now on, 'cone' will refer to this special type of cone. Given a cone σ , its dual σ is defined as follows

$$\sigma^{*} := \{ u \in M_{\mathbb{R}} \mid \langle u, v \rangle \ge 0 \ \forall \ v \in \sigma \}$$

It is also useful to define the semi-group $S_{\sigma} = \sigma \cap M$. We state some important properties of these objects, proofs of which can be found in [35]:

- (σ) = σ
- S_{σ} is finitely generated.
- σ generates $M_{\mathbb{R}}$ over \mathbb{R} i.e. it is *n*-dimensional.

The last point obviously follows from the requirement that σ is strongly convex.

Given any Abelian semi-group S, one can construct a commutative ring $\mathbb{C}[S]$ by taking the free \mathbb{C} -algebra generated by the set $\{\chi^u \mid u \in S\}$ and then imposing the relations $\chi^u \chi^{u'} = \chi^{u+u'}$. For a cone σ , we therefore obtain an affine variety¹⁰

$$U_{\sigma} := \operatorname{Spec}(\mathbb{C}[S_{\sigma}])$$

This gives us an algebraic variety, because S_{σ} is finitely generated. To see that it is actually a toric variety, we observe that $\mathbb{T}^n = \operatorname{Spec}(\mathbb{C}[M]) = \operatorname{Spec}(\mathbb{C}[S_{\{0\}}])$, so the injection $S_{\sigma} \hookrightarrow M$

¹⁰For our purposes, Spec(R), where R is a ring, is just the affine variety with coordinate ring R. For the proper definition of Spec(R) as a *scheme*, see [25].

induces a morphism $\mathbb{T}^n \to U_{\sigma}$. In fact this is an embedding, with \mathbb{T}^n being specified inside U_{σ} by $\prod_{u \in S_{\sigma}} \chi^u \neq 0$.

A cone τ is called a *face* of σ if there is some $u \in S_{\sigma}$ such that

$$\tau = \{ v \in \sigma \mid \langle u, v \rangle = 0 \}$$

In this case, the semi-group S_{τ} is very simply related to S_{σ} :

$$S_{\tau} = S_{\sigma} + \mathbb{Z}_{>0}(-u)$$

If we translate this into geometry, we find that U_{τ} is a principle open subset of U_{σ} :

$$U_{\tau} = \{ x \in U_{\sigma} \mid \chi^u(x) \neq 0 \}$$

2.4.2 Fans and toric varieties

A general toric variety is constructed by gluing together affine toric varieties in a special way. This is most easily described by the 'fan' of the toric variety, which we now introduce. A fan Σ is a collection of cones (as described above) satisfying the following two conditions:

- If $\sigma \in \Sigma$, and τ is a face of σ , then $\tau \in \Sigma$.
- If $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \cap \sigma_2$ is a face of each.

We saw above how to obtain an affine variety U_{σ} from a cone σ . Given a fan Σ , we construct a toric variety X_{Σ} as

$$X_{\Sigma} := \coprod_{\sigma \in \Sigma} U_{\sigma} \big/ \sim$$

where we have to define the equivalence relation. Suppose τ is the common face of σ_1 and σ_2 . Then we have seen that U_{τ} is a principal open subset in both U_{σ_1} and U_{σ_2} , and we use this to glue them together. Let ι_i be the map embedding U_{τ} in U_{σ_i} ; the identifications are then

$$(x_1 \in U_{\sigma_1}) \sim (x_2 \in U_{\sigma_2}) \iff \exists y \in U_{\tau} \text{ such that } \iota_1(y) = x_1, \ \iota_2(y) = x_2$$

We now turn to the inverse process of constructing a fan from a toric variety X. Take coordinates (t_1, \ldots, t_n) on $\mathbb{T}^n \subset X$, and define the action of an element $\lambda := (\lambda_1, \ldots, \lambda_n) \in \mathbb{T}^n$ by $\lambda \cdot (t_1, \ldots, t_n) = (\lambda_1 t_1, \ldots, \lambda_n t_n)$. The requirement that this extends to an action of \mathbb{T}^n on X is not merely a technical assumption; it is crucial for studying the structure of X. In fact, like any space on which a group acts, X can be decomposed into disjoint orbits of the group action, and it is this which underlies the construction of the fan.

To see how this works, we study one-parameter subgroups of the torus, given by morphisms

$$\phi_{\mathbf{v}}: \mathbb{C}^* \hookrightarrow \mathbb{T}^n, \quad \phi_{\mathbf{v}}(\tau) = (\tau^{v_1}, \dots, \tau^{v_n})$$

where \mathbf{v} is a *n*-tuple of integers, (v_1, \ldots, v_n) ; such morphisms are therefore parametrised by a lattice $N \cong \mathbb{Z}^n$. We now take the limit $\tau \to 0$, which doesn't exist in \mathbb{T}^n (unless $\mathbf{v} = \mathbf{0}$), but which might exist in X. If the limit exists, we can define the following closed sub-variety of X:

$$V_{\mathbf{v}} := \overline{\{ \lim_{\tau \to 0} \lambda \cdot \phi_{\mathbf{v}}(\tau) \mid \lambda \in \mathbb{T}^n \}} \subset X$$

The convex hull, in $\mathbb{N}_{\mathbb{R}}$, of the origin and the set of points which in this way determine the same V, is a strongly convex rational polyhedral cone, and we write $V(\sigma)$ for the sub-variety corresponding to the cone σ . The set of all cones found in this way forms a fan $\Sigma(X)$, and there is a one-to-one order-reversing correspondence between cones in $\Sigma(X)$ and the closed \mathbb{T}^n -invariant sub-varieties of X, where the ordering on both cones and sub-varieties is given by inclusion. This correspondence between cones and \mathbb{T}^n -invariant sub-varieties becomes particularly clear when we introduce homogeneous coordinates for toric varieties, in §2.4.3.

This discussion also suggests a very simple criterion for the compactness of a toric variety, which is in fact true: X is compact if and only if $\Sigma(X)$ fills the vector space $N_{\mathbb{R}}$.

The construction just described is inverse to the one discussed above, in the sense that $X_{\Sigma(X)} = X$, and $\Sigma(X_{\Sigma}) = \Sigma$.

2.4.3 Homogeneous coordinates

We will now briefly review the construction of a toric variety in terms of homogeneous coordinates, as described by Cox [36]. Given a fan Σ , let $\{v_{\rho}\}_{\rho=1...d}$ be the set of minimal lattice vectors generating the one-dimensional cones in Σ , and associate a complex variable z_{ρ} with each v_{ρ} ; these will be our homogeneous coordinates. Our first step is to delete a subset of the space \mathbb{C}^d spanned by these variables: if some set $\{v_{\rho_1}, \ldots, v_{\rho_l}\}$ does *not* span a cone in the fan, we remove the set $\{z_{\rho_1} = \ldots = z_{\rho_l} = 0\}$. Denote by \mathcal{S} the union of all such sets. What Cox showed in [36] is that X may be constructed as a quotient of $\mathbb{C}^d \setminus \mathcal{S}$; we will now describe the quotient group G, and its action on $\mathbb{C}^d \setminus \mathcal{S}$.

Denote by $A_{n-1}(X)$ the group of equivalence classes of Weil divisors on X. This group is generated by the toric divisors, and two toric divisors are equivalent if and only if they differ by $(\chi^u) = \sum_{\rho} \langle u, v_{\rho} \rangle D_{\rho}$ for some $u \in M$. We can express this in the following short exact sequence of Abelian groups

$$0 \longrightarrow M \xrightarrow{\phi} \mathbb{Z}^d \longrightarrow A_{n-1}(X) \longrightarrow 0$$

where \mathbb{Z}^d is the free Abelian group generated by $\{D_\rho\}$. We now apply the contravariant functor $\operatorname{Hom}(-, \mathbb{C}^*)$ to this, to obtain

$$0 \longrightarrow \operatorname{Hom}(A_{n-1}(X), \mathbb{C}^*) \longrightarrow \operatorname{Hom}(\mathbb{Z}^d, \mathbb{C}^*) \xrightarrow{\phi^*} \operatorname{Hom}(M, \mathbb{C}^*) \longrightarrow 0$$

where $\phi^*(g) = g \circ \phi$. The group G is defined to be ker ϕ^* , and the action of $g \in G$ on $\mathbb{C}^d \setminus S$ is

$$g \cdot (z_1, \ldots, z_d) = (g_1 z_1, \ldots, g_d z_d)$$

where $g_{\rho} := g(D_{\rho})$. The above is a very concrete prescription, and in practice it is easy to determine the group G explicitly. An arbitrary morphism $g \in \operatorname{Hom}(\mathbb{Z}^d, \mathbb{C}^*)$ is given by $g(\sum_{\rho} a_{\rho} D_{\rho}) = \prod_{\rho} g_{\rho}^{a_{\rho}}$, where $g_{\rho} \in \mathbb{C}^*$. For $u \in M$, we therefore have

$$\phi^*(g)(u) = g(\phi(u)) = \prod_{\rho} g_{\rho}^{\langle u, v_{\rho} \rangle}$$

Finding G is therefore a matter of finding all sets $\{g_{\rho}\}$ such that the above evaluates to unity for all $u \in M$. It is sufficient to check this condition for the standard basis (e_1^*, \ldots, e_n^*) of M. The description in terms of homogeneous coordinates makes particularly simple the correspondence between fans and toric sub-varieties, mentioned in the last section. If σ is a cone spanned by $v_{\rho_1}, \ldots, v_{\rho_k}$, then the corresponding sub-variety is given by $z_{\rho_1} = \ldots = z_{\rho_k} = 0$.

Example

In order to illustrate the procedure described above, let us consider an example which will arise in Chapter 4. Let σ be the three-dimensional cone generated in \mathbb{Z}^3 by the following four vectors:

$$v_1 = (1, 1, 0) , v_2 = (1, 2, 3)$$

 $v_3 = (1, 0, 2) , v_4 = (1, 3, 1)$
(2.12)

The two-dimensional faces are spanned by the pairs $\langle v_1, v_3 \rangle$, $\langle v_1, v_4 \rangle$, $\langle v_2, v_3 \rangle$, and $\langle v_2, v_4 \rangle$. Note that all four vectors lie on a hyperplane, which is equivalent to the corresponding affine toric variety U_{σ} having trivial canonical class (see the next subsection), and allows us to represent the fan by drawing its intersection with the hyperplane, as in Figure 2.2.



Figure 2.2: The toric diagram of a particular \mathbb{Z}_5 quotient of the conifold, discussed in the text.

We first introduce variables z_1, z_2, z_3, z_4 on \mathbb{C}^4 , and remove the following set:

$$\mathcal{S} = \{z_1 = z_2 = 0, (z_3, z_4) \neq (0, 0)\} \cup \{z_3 = z_4 = 0, (z_1, z_2) \neq (0, 0)\}$$

Next we need to determine the quotient group G. Let $g = (g_1, g_2, g_3, g_4)$ be an arbitrary element of Hom $(\mathbb{Z}^4, \mathbb{C}^*)$. Evaluating $\phi^*(g)$ on the standard basis (e_1^*, e_2^*, e_2^*) of M gives us the equations we need to solve:

$$\phi^*(g) = (g_1 \, g_2 \, g_3 \, g_4 \, , \, g_1 \, g_2^2 \, g_4^3 \, , \, g_2^3 \, g_3^2 \, g_4) = (1, 1, 1)$$

These can quickly be re-arranged to obtain

$$g_1 = g_2^7 g_3^6$$
, $g_4 = g_2^{-3} g_3^{-2}$, $g_2^5 g_3^5 = 1$

The solution is then easy:

$$G = \{ (\lambda, \zeta^2 \lambda, \zeta \lambda^{-1}, \zeta^2 \lambda^{-1}) \mid \lambda \in \mathbb{C}^*, \ \zeta^5 = 1 \} \cong \mathbb{C}^* \times \mathbb{Z}_5$$

We can see that U_{σ} is in fact a \mathbb{Z}_5 quotient of the conifold by introducing the following coordinates invariant under $\mathbb{C}^* \subset G$:

$$y_1 = z_1 z_3$$
, $y_2 = z_1 z_4$, $y_3 = z_2 z_3$, $y_4 = z_2 z_4$

These identically satisfy $y_1y_4 - y_2y_3 = 0$, which is the equation for the conifold, and are subject

$$(y_1, y_2, y_3, y_4) \sim (\zeta y_1, \zeta^2 y_2, \zeta^3 y_3, \zeta^4 y_4)$$

2.4.4 Toric geometry and Calabi-Yau manifolds

Given that our main interest is in Calabi-Yau manifolds, an obvious question at this point is whether these can be constructed as toric varieties. Although we haven't discussed the conditions under which a toric variety X admits a Kähler metric, it is true that many smooth toric varieties are indeed Kähler, but what about the condition $c_1(X) = 0$?

It can be shown that the canonical divisor class is given by minus the sum of the toric divisors [35]: $K_X \sim -\sum_{\rho} D_{\rho}$. So the Calabi-Yau condition is easy to state in terms of the toric data:

$$c_1(X) = 0 \iff \sum_{\rho} D_{\rho} \sim 0$$

We said previously that a toric divisor is linearly equivalent to zero if and only if it can be expressed as $\sum_{\rho} \langle u, v_{\rho} \rangle D_{\rho}$ for some $u \in M$, so the above condition is equivalent to the existence of a u such that

$$\langle u, v_{\rho} \rangle = 1 \quad \forall \quad \rho$$

In words, this says that all the v_{ρ} lie on some hyperplane. If this is the case, then it is clear that $\Sigma(X)$ does not fill $N_{\mathbb{R}}$, so X is non-compact.

To summarise, toric Calabi-Yau varieties do exist, but they are necessarily non-compact. This doesn't mean that toric geometry is of no use in constructing compact Calabi-Yau manifolds, as we will now see.

Reflexive polyhedra and mirror manifolds

Batyrev described a general method for constructing mirror pairs of Calabi-Yau manifolds as hypersurfaces in toric varieties [37]. Let ∇ be a polyhedron given by the convex hull in $N_{\mathbb{R}}$ of some finite set $\{v_{\rho}\}$ of lattice vectors. ∇ is said to be *reflexive* if it contains the origin, and the distance between the origin and the hyperplane defined by any face of ∇ is 1. If we take any polyhedral sub-division of the faces of ∇ , then the cones over these polyhedra form a fan, and in this way we obtain a toric variety.

The polyhedron $\Delta \subset M$ dual to ∇ is then defined by

$$\Delta = \{ u \in M \mid \langle u, v \rangle \ge -1 \ \forall \ v \in \nabla \}$$

Points in the dual lattice M give rise to holomorphic functions on $\mathbb{T}^n \subset X$ via

$$u \mapsto \prod_{i} t_{i}^{u_{i}} \tag{2.13}$$

where $\{u_i\}$ are the components of u relative to the standard basis for N. When the polyhedra are reflexive, the integral points of the dual polyhedron Δ correspond in this way to global sections of the anticanonical bundle of X, so the closure of the vanishing locus of a linear combination of these monomials is a Calabi-Yau variety. We can also dualise this construction: taking cones over the faces of Δ gives the fan of another toric variety X^* , and points in ∇ correspond to sections of its anti-canonical bundle. In this way we obtain the mirror family.

3. New Calabi-Yau Manifolds with Small Hodge Numbers

It is known that the moduli spaces of many families of Calabi-Yau manifolds form a connected web. The question of whether all Calabi–Yau manifolds form a single web depends on the degree of singularity that is permitted for the varieties that connect the distinct families of smooth manifolds. If only conifolds are allowed then, since shrinking two-spheres and three-spheres to points cannot affect the fundamental group, manifolds with different fundamental groups will form disconnected webs. In this chapter, based on [38], we examine webs of multiply-connected manifolds, which tend to lie near tip of the distribution of Calabi–Yau manifolds, where the Hodge numbers $(h^{1,1}, h^{2,1})$ are both small. Beginning from a small number of previously-known quotients, we generate a number of new manifolds via conifold transitions.

3.1 Overview: CICY's and quotient manifolds

In [39], attention was drawn to the fact that there is an interesting region in the distribution of Calabi–Yau manifolds where both the Hodge numbers $(h^{1,1}, h^{2,1})$ are small. This region contains at least two manifolds that are interesting from the perspective of elementary particle phenomenology. These are quotients of two manifolds which can be represented as:

$$X^{14,23} = \begin{array}{c} \mathbb{P}^3 \begin{bmatrix} 1 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}^{14,23} , \quad X^{19,19} = \begin{array}{c} \mathbb{P}^1 \begin{bmatrix} 1 & 1 \\ 3 & 0 \\ \mathbb{P}^2 \begin{bmatrix} 0 & 3 \end{bmatrix}^{19,19} \end{array}$$
(3.1)

where the superscripts denote the Hodge numbers $(h^{1,1}, h^{2,1})$. These manifolds belong to a class known as complete intersection Calabi–Yau manifolds (CICY's), which can be presented as the complete intersection of polynomials in a product of projective spaces; the notation means that we start with a product of the projective spaces, and define the manifold by the simultaneous vanishing of several homogeneous polynomials, one for each column of the matrix. Each column in the matrix then represents the multi-degree of the corresponding polynomial. The condition that a configuration corresponds to a Calabi–Yau manifold, that is has $c_1 = 0$, is the condition that each row of the matrix sum to one more than the dimension of the corresponding projective space. A list of the almost 8,000 CICY's was compiled in [5]; these have Euler numbers in the range $-200 \le \chi \le 0$, are all simply connected, and have $h^{1,1} + h^{2,1} \ge 30$.

The first manifold in (3.1) admits a free \mathbb{Z}_3 action, first described in [21, 40], and the resulting quotient, referred to here as Yau's manifold, was used for the first attempt at constructing a realistic string theory model of particle physics [41, 42]. The second, called the 'split bicubic' when represented this way, admits a free $\mathbb{Z}_3 \times \mathbb{Z}_3$ action, the quotient by which has been used as the basis for the so-called "heterotic standard models" investigated in [14, 16, 43, 44]. Yau's construction is the prototypical example of the procedure we will implement here, so we describe it in some detail. One obtains a smooth manifold by taking the defining polynomials to be

$$f = a_0 w_0 z_0 + a_1 \sum_j w_j z_j + a_2 \sum_j w_j z_{j+1} + a_3 \sum_j w_{j+1} z_j + a_4 w_0 \sum_j z_j + a_5 (\sum_j w_j) z_0$$

$$g = w_0^3 - w_1 w_2 w_3 + b_1 \sum_j w_j^3 + b_2 w_0 \sum_j w_j w_{j+1}$$

$$h = z_0^3 - z_1 z_2 z_3 + c_1 \sum_j z_j^3 + c_2 z_0 \sum_j z_j z_{j+1}$$

where the (w_0, w_j) and (z_0, z_j) , j = 1, 2, 3, are homogeneous coordinates for the two \mathbb{P}^3 's and a_α , b_α and c_α are coefficients. The separate treatment of the zeroth coordinate anticipates the action of a \mathbb{Z}_3 symmetry group, with generator S that simultaneously permutes the coordinates w_j and z_k :

$$S: (w_0, w_j) \times (z_0, z_k) \rightarrow (w_0, w_{j+1}) \times (z_0, z_{k+1})$$

where the indices j and k are understood mod 3. It is easy to check that the action of S is fixed point free when restricted to X, so the quotient $X^{14,23}/\mathbb{Z}_3$ is smooth and has in fact $(h^{1,1}, h^{2,1}) = (6,9)$ and hence $\chi = -6$, where χ denotes the Euler number.

A review of constructions of Calabi–Yau manifolds is given in [39], where it is observed that finding manifolds with small Hodge numbers, that is with say $h^{1,1} + h^{2,1} \leq 22$, is largely synonymous with finding quotients by freely acting groups. Our aim in this chapter is to find such manifolds by finding CICY's that admit freely acting symmetries and then taking the quotient in a manner analogous to that which leads to Yau's manifold.

Manifolds that admit a freely acting symmetry seem to be genuinely rare so our strategy is to try to trace the symmetries through the web of CICY manifolds.¹ To explain this we first digress to explain how conifold transitions between CICY's can be realised by processes on their configuration matrices, called *splitting* and *contraction*. This is old knowledge, and a more detailed account than we shall give here, together with many matters pertaining to CICY's, may be found in [19]. For a recent interesting reference in which some of the manifolds that are important to us here appear in a different context, see [46].

3.1.1 Splitting and Contraction

Consider the bicubic CICY, with configuration matrix

$$X^{2,83} = \frac{\mathbb{P}^2}{\mathbb{P}^2} \begin{bmatrix} 3\\ 3 \end{bmatrix}$$

We can vary the polynomial f defining such a manifold until it takes the special form

$$f_0 \stackrel{\text{def}}{=} U(\xi)V(\eta) - W(\xi)Z(\eta) = 0$$
(3.2)

Note that this bears a close resemblance to equation (2.8), which defines the conifold. In fact, f_0 necessarily defines a singular bicubic, since f_0 and all its derivatives vanish at the $3^4 = 81$ points where U = V = W = Z = 0. If we assume we are in the generic case where $dU \wedge dW$ and

¹After the work in this chapter was published in [38], Braun developed and implemented an algorithm to systematically find all free group actions on CICY's [45].



Figure 3.1: The tip of the distribution of Calabi–Yau manifolds, as it stood after the work in this chapter was completed. The Euler number $\chi = 2(h^{1,1}-h^{2,1})$ is plotted horizontally, $h^{1,1}+h^{2,1}$ is plotted vertically and the oblique axes bound the region $h^{1,1} \ge 0$, $h^{2,1} \ge 0$. Manifolds with $h^{1,1}+h^{2,1} \le 22$ are identified in Table 3.8.

 $dV \wedge dZ$ are both non-zero at these points, then each is actually just a node, since U, V, W, Z then act as local coordinates on $\mathbb{P}^2 \times \mathbb{P}^2$. Referring back to §2.3.3, we see that we can simultaneously resolve all 81 nodes by replacing f_0 by the matrix equation

$$\begin{pmatrix} U(\xi) & W(\xi) \\ Z(\eta) & V(\eta) \end{pmatrix} \begin{pmatrix} t_0 \\ t_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
(3.3)

We can see by comparing to (3.1) that the process of resolving the nodes has taken us to the family of $X^{19,19}$ manifolds, so we learn that $X^{2,83}$ and $X^{19,19}$ are connected by a conifold transition. The corresponding path in moduli space is of finite length with respect to the physical metric [47, 48]; it is for this reason that conifolds are of physical relevance. The process just described is realised on the configuration matrix by splitting a column²

$$\mathbb{P}^{2} \begin{bmatrix} 3 \\ 3 \end{bmatrix} \xrightarrow{\mathbb{P}^{1}} \mathbb{P}^{1} \begin{bmatrix} 1 & 1 \\ 3 & 0 \\ \mathbb{P}^{2} \begin{bmatrix} 0 & 3 \end{bmatrix}$$

More generally, a conifold transition between CICY's may be implemented by splitting with a \mathbb{P}^N in place of the \mathbb{P}^1

$$\mathcal{P}[\mathbf{c}, M] \to \frac{\mathbb{P}^N}{\mathcal{P}} \begin{bmatrix} 1 & 1 & \cdots & 1 & \mathbf{0} \\ \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_{N+1} & M \end{bmatrix}$$

where $\mathcal{P} = \mathbb{P}^{N_1} \times \ldots \times \mathbb{P}^{N_k}$ is any product of projective spaces, $\mathbf{c} = \sum_{j=1}^{N+1} \mathbf{c_j}$ is a vector which we decompose as the sum of N + 1 vectors with nonnegative components, and M is a matrix. On the right we have a configuration with N + 1 equations that are linear in the N + 1 coordinates of the \mathbb{P}^N . Since these coordinates cannot all vanish, the determinant of the matrix of coefficients, which has multidegree \mathbf{c} , must vanish. Therefore, given any manifold belonging to the configuration on the right, it projects down to a variety belonging to the one on the left, such that the polynomial corresponding to the column \mathbf{c} is actually a determinant. Denote a generic such variety by X_0 (these will frequently be singular).

Let A denote the matrix of coefficients referred to above. Since its determinant vanishes on X_0 , A cannot have rank N + 1. If it has rank N, which is the generic case, then it determines a unique point in \mathbb{P}^N . Now we ask: of what dimension is the set on which the rank drops to N - 1? To answer this, we use the fact that the space of $(N + 1) \times (N + 1)$ matrices of rank at most k has dimension k(2(N + 1) - k) (see [49] for a nice derivation of a more general result). The difference between the cases k = N and k = N - 1 is therefore

$$N(N+2) - (N-1)(N+3) = 3$$

The locus on which the rank of A drops to N - 1 is therefore of codimension three, so it intersects X_0 in discrete points. Over each such point the equations determine a line, \mathbb{P}^1 , in \mathbb{P}^N . In general, A cannot have rank less than N - 1 on X_0 , since this would occur on a set of codimension greater than three.

For the configuration on the left one sees that there is a smoothed equation $F_s = \det A + s K$, and that the effect of the limit $s \to 0$ is to shrink a finite number of S^3 's to nodes. The burden of these comments is that we have the same situation here as previously: we may proceed from a smooth manifold, X, corresponding to the configuration on the left, shrink a certain number of S^3 's to nodes to arrive at a singular variety X_0 and then resolve the nodes with \mathbb{P}^1 's to arrive at a smooth manifold corresponding to the configuration on the right, which we denote by \hat{X} . The Euler number of S^3 is zero, and the Euler number of \mathbb{P}^1 is two, so the Euler numbers of X

$$\mathbb{P}^2_{\mathbb{P}^2} \begin{bmatrix} 3\\ 3 \end{bmatrix} \to \mathbb{P}^2_{\mathbb{P}^2} \begin{bmatrix} 1 & 1\\ 2 & 1\\ 1 & 2 \end{bmatrix}$$

 $^{^{2}}$ This is not a unique process since, in general, a column can be split in different ways. Another way to split the bicubic is

and \hat{X} are related by

$$\chi(\widehat{X}) = \chi(X) + 2\nu$$

where ν is the number of nodes of X_0 . Thus $\chi(\hat{X}) \geq \chi(X)$, and there is equality if and only if the manifold corresponding to det A = 0 has, in fact, no nodes. When this is the case, $\hat{X} = X$, so we conclude that $\hat{X} = X$ if and only if their Euler numbers are equal. This is a useful criterion which we shall use frequently in the following. We refer to the process that we have denoted by \rightarrow as a *splitting* and we refer to the reversed process as a *contraction*.

3.1.2 Configurations and diagrams

Configurations that differ merely by a permutation of their rows or columns determine the same manifold. A more intrinsic representation is given by a diagram that expresses the combinatorics of the degrees of the polynomials in the coordinates of each of the ambient spaces. The individual projective spaces are represented by open disks and the polynomials by filled disks. The degree of the polynomial with respect to a given ambient space is encoded by the number of lines connecting the corresponding disks. The diagrams for the quintic, bicubic and split bicubic are as follows:



The transpose of a configuration matrix also defines a CICY (by choosing the dimensions of the ambient spaces appropriately). It is perhaps surprising, but simple to check, that this space is always another threefold. Transposition interchanges spaces and polynomials, so as an operation on the diagram it amounts to simply interchanging the two sorts of disk. The transposition of CICY's remains a somewhat mysterious process owing to the fact that two different matrices, M_1 and M_2 say, can represent the same family of manifolds — this is the case, for example, if the matrices differ by an ineffective split. The manifolds corresponding to the transposes M_1^T and M_2^T , however, will often be different. We will come across several examples of this presently. Although not well understood, transposition seems to play a role in the webs of manifolds that admit free group actions, as we discuss below.

As already mentioned, it sometimes happens that splitting doesn't actually change the manifold. We use two instances of this phenomenon frequently in the following. The first reduces redundancy and is based on the identity

This is simply the statement that a bilinear equation in $\mathbb{P}^1 \times \mathbb{P}^1$ is equivalent to $s_0 t_1 = s_1 t_0$, and this is equivalent to $(s_0, s_1) = (t_0, t_1)$. This identity prevents us from splitting indefinitely since the splits eventually become ineffective. A second identity, that arises often in relation to manifolds with a \mathbb{Z}_3 symmetry, is

The configuration on the left is a redundant split of the configuration on the right. Both configurations correspond to the del Pezzo surface dP_6 , given by a \mathbb{P}^2 blown up in three generic points;³ this surface will be very important to us in later chapters. It can happen (as in the CICY's of equation (3.1), for example) that $h^{1,1}$ can exceed the number of ambient spaces. In suitable cases, use of the above identity, read from right to left, can be used to increase the number of ambient spaces and so give explict representation to more of the Kähler parameters.

3.1.3 Calculation of the Euler characteristic and Hodge numbers

The topological data of the CICY's has been known for a long time; a technique for calculating the Euler number $\chi = 2(h^{1,1} - h^{2,1})$ was given in the original paper [5], and the individual Hodge numbers were calculated in [50]. The list of CICY's, with the respective Hodge numbers appended, exists as a computer file. Finding the Hodge numbers corresponding to a given matrix is therefore, now, just a matter of looking up the relevant matrix in the list. There are two complications to doing this. The first owes to the fact that an attempt was made to eliminate redundant splits in the compilation of the list, so not all matrices are included. For the cases considered here this is not a significant problem, since in all cases where the matrix does not occur in the list it is related to a space that is listed by redundant splittings and contractions. A second problem is that there is no canonical way to write the CICY matrices, so if a given configuration exists in the list it will very likely appear with its rows and columns permuted.

Determining the Hodge numbers of the quotient manifolds is slightly more challenging. The easiest quantity to compute is the Euler number, since this simply divides by the order of the freely acting group. In principle then, we need only calculate one of the individual Hodge numbers, since the other can then be determined from $\chi = 2(h^{1,1} - h^{2,1})$, however in most cases we calculate both numbers as a check on our results.

The easiest way to determine $h^{1,1}$ is to find a representation of the CICY in which all nontrivial (1, 1)-forms on the original manifold arise as the pullbacks of the hyperplane classes of the embedding spaces. The group action on the second cohomology of the CICY is then determined by the action on the ambient space. In all cases except one (see §3.3.7), each ambient space is either part of an orbit of spaces which are mapped to each other by the symmetry, or the symmetry acts linearly on the homogeneous coordinates within a given \mathbb{P}^N . In the first case, each orbit contributes 1 to the count for the quotient manifold, and in the second case the space still contributes 1 by itself, since a holomorphic linear action maps hyperplanes biholomorphically to hyperplanes.

³Note that we follow the mathematical tradition and label the del Pezzo surfaces dP_d by their degree d. The blow-up of \mathbb{P}^2 at n generic points is a del Pezzo surface of degree d = 9 - n, for $n = 0, 1, \ldots, 8$.

To calculate $h^{2,1}$, we count independent parameters in the polynomials defining the symmetric CICY. There exist cases of CICY's where this does not work, but a theorem from [51] assures us of the effectiveness of the technique in the majority of cases. Paraphrased, the theorem states that if the diagram for a Calabi-Yau 3-fold X cannot be disconnected by cutting a single leg, and the counting of parameters for the polynomials agrees with $h^{2,1}$, then the independent parameters in the polynomials act as a basis for $H^{2,1}(X)$. This means that $h^{2,1}$ for the quotient manifold will agree with the number of independent parameters in the polynomials after imposing the symmetry. Examples of how to count the independent parameters can be found later in this chapter, and this is guaranteed by the above theorem to work in all cases presented here, except for the very last one, in §3.4.4.

In some examples we find group actions which are not free, but still yield new Calabi-Yau manifolds if we resolve the resulting orbifold singularities. Since the Euler number is topological, we can in these cases calculate it by treating the resolution as a surgery. Suppose a group G acts non-freely on a CICY X, and denote by Σ the set of points on X which are fixed by some element(s) of G. We remove some neighbourhood U_{Σ} of Σ , leaving a manifold-with-boundary X' on which G acts freely. The Euler number is simply

$$\chi(X') = \chi(X) - \chi(U_{\Sigma})$$

Taking the quotient gives another manifold X'/G with boundary $\partial(X'/G) = (\partial \overline{U}_{\Sigma})/G$. We then 'glue in' a neighbourhood \mathcal{M} of the exceptional set of the resolution, to give again a smooth Calabi-Yau threefold \hat{X} . The resulting Euler number is

$$\chi(\widehat{X}) = \frac{\chi(X) - \chi(U_{\Sigma})}{|G|} + \chi(\mathcal{M})$$

Calculating the individual Hodge numbers depends on the details of the fixed set Σ , and we will leave the discussion to the relevant parts of the text. Note that Σ and thus \mathcal{M} will generally have multiple connected components.

3.1.4 Checking transversality of the defining equations

A complete intersection threefold is, locally, the vanishing locus of N polynomials in an N+3 complex dimensional space. The condition that the resulting variety be of three dimensions and smooth is that the form $dp_1 \wedge \cdots \wedge dp_N$ be non-vanishing at all points of intersection. When this is so, the polynomials are said to be *transverse*. This condition amounts to the following. For each coordinate patch, with coordinates x_m , we check that the $(N+3) \times N$ Jacobian matrix

$$H = \left(\frac{\partial p_i}{\partial x_m}\right)$$

has rank N on the locus $p_i = 0$. This, in turn, requires checking that the equations $p_i = 0$ taken together with the vanishing of all the $N \times N$ minors of H have no simultaneous solution for general choice of coefficients in the polynomials. It is enough to know that there is no solution for a particular choice of parameters since we then know that there is no solution for a general choice. For this it suffices to assign suitable integer values to the coefficients and to

perform a Gröbner basis calculation. Such a calculation is frequently only practical in finite characteristic, since in the process of generating the Gröbner basis the coefficients of the basis polynomials, if taken over \mathbb{R} say, grow very large and, for the computations that we perform here, the computation of the basis will fail to complete. If we choose integer values for the parameters of the defining polynomials of the manifold then the derivatives and determinants, that we take in constructing the ideal, preserve the fact that the coefficients are integers. If there is a simultaneous solution of the equations then there is also a simultaneous solution in characteristic p. Such a solution may not exist in \mathbb{F}_p but it will exist in a finite extension; that is in \mathbb{F}_{p^n} for sufficiently large n. This is the same as saying that the solution will be given by a consistent set of equations with coefficients in \mathbb{F}_p . The Gröbner basis calculation finds these if they exist. If, on the other hand, the polynomials do not reduce to a consistent set over \mathbb{F}_p , then they cannot have been consistent to start with. The upshot is that if the defining polynomials are transverse over \mathbb{F}_p , for some prime p and choice of integral coefficients, then they are transverse over $\mathbb C$ for generic coefficients. There can however be an 'accidental' solution p even if there was no solution to the original equations (although there will only be a finite number of these 'bad primes'). An example where the variety is singular over \mathbb{F}_p but smooth over \mathbb{C} is provided by the quintic threefold with equation $\sum_{j=1}^{5} x_j^5 = 0$. This is smooth over \mathbb{C} but is singular over \mathbb{F}_5 since all the partial derivatives vanish identically.

We implemented this procedure directly in Mathematica 6.0 and also in SINGULAR 3.0.4 [52], which we run from within Mathematica by means of the STRINGVACUA package [53]. The SINGULAR implementation of the Gröbner basis calculation is significantly faster and this is of practical importance since for large matrices the number of minors of the Jacobian matrix grows rapidly, and there are also many coordinates and coordinate patches. The number of minors and coordinate patches can be in the hundreds, and for these cases the Mathematica implementation seems to be impractical. The number of coordinate patches is not, in itself, as big a problem as it might seem. Consider, as a trivial example, the problem of checking the transversality of \mathbb{P}^4 [5] over the five standard coordinate patches, $\mathcal{U}_j = \{x_j \neq 0\}$, of \mathbb{P}^4 . Having first checked transversality over \mathcal{U}_0 , we then check transversality over \mathcal{U}_1 but, since we already know that the polynomial is transverse when $x_0 \neq 0$, we may now set $x_0 = 0$. Similarly when we come to checking transversality over \mathcal{U}_2 we may set $x_0 = x_1 = 0$, and so on. The complexity of the algorithms grows very rapidly with the number of variables so this simplification, which reduces the number of variables, leads to a very significant increase in speed.

3.1.5 Webs of CICY's with freely acting symmetries

There is a simple and elegant argument [19, 54] that shows that it is possible to pass from any CICY configuration to any other by a sequence of splittings and contractions. Thus the parameter space of CICY's forms a web, connected by conifold loci. The CICY's themselves are all simply connected, while the quotient of a CICY by a freely acting group G has fundamental


Table 3.1: A web of CICY's that admit a freely acting \mathbb{Z}_5 symmetry. The groups that are appended to the configurations are the largest for which we have found a free action of the group.



Table 3.2: Webs derived from Table 3.1 with fundamental groups \mathbb{Z}_5 and $\mathbb{Z}_5 \times \mathbb{Z}_2$.

group G. In general, splitting will not commute with taking the quotient by a freely acting group, since the subvariety of symmetric manifolds, in the parameter space, need not intersect the conifold locus. On the other hand, by suitable choice of splitting, we will sometimes be able to achieve this. When this happens we can move along a web of parameter spaces corresponding to manifolds with fundamental group G. We stress again that, since a conifold transition cannot alter the fundamental group of a manifold, there will be disconnected webs corresponding to different fundamental groups.

We began this investigation with the hypothesis that Calabi–Yau manifolds admitting a

freely acting group are very rare, and although we have found a number of new examples of such manifolds, our experience is consistent with a paucity of these. Manifolds admitting free actions by larger groups are seemingly particularly rare. There are manifolds with free actions of groups G of order 64, the largest known [45, 55-57]. To our knowledge, the resulting quotients all lie at the very tip of the distribution, with $(h^{1,1}, h^{2,1}) = (2, 2)$. At order 49 there is a manifold known with a free action of $\mathbb{Z}_7 \times \mathbb{Z}_7$ [55, 58] but, to our knowledge, it is the only one such and this also has $(h^{1,1}, h^{2,1}) = (2, 2)$. Only one manifold is known which admits a group of order 36. This has a free action of $\mathbb{Z}_6 \times \mathbb{Z}_6$ and $(h^{1,1}, h^{2,1}) = (6, 6)$. What is more, all the cases we have listed thus far belong to the remarkable class of manifolds investigated by Gross and Popescu [55] that are fibred by Abelian surfaces and have Hodge numbers $(h^{1,1}, h^{2,1}) = (n, n)$ for n = 2, 4, 6, 8, 10. There are several free actions of groups of order 32; the resulting quotients have $(h^{1,1}, h^{2,1}) = (1,3)$ or $(h^{1,1}, h^{2,1}) = (2,2)$. At order 25 there is a short web consisting of the $\mathbb{Z}_5 \times \mathbb{Z}_5$ quotients of the quintic threefold $\mathbb{P}^4[5]$ and a resolution of the Horrocks-Mumford quintic, which is a highly nodal form of $\mathbb{P}^4[5]$. The resolution has $(h^{1,1}, h^{2,1}) = (4, 4)$ and is again one of the Gross-Popescu manifolds.

In [38], we found webs for the groups \mathbb{Z}_5 , $\mathbb{Z}_5 \times \mathbb{Z}_2$, \mathbb{Z}_3 , $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_2$, and \mathbb{H} , with \mathbb{H} the order-8 quaternion group. We should explain how \mathbb{H} arises in this story. Our investigation began with a search for symmetry generators of order 3 and order 5, and so we find groups containing \mathbb{Z}_3 and \mathbb{Z}_5 as subgroups. We did not attempt a search for generators of order 2, since we suspected that there are an unfeasibly large number of these to deal with by hand, and this was confirmed by the results of [45]. Having sought generators of order 3 and 5, it seems natural to seek also generators of order 4, particularly in relation to the important configuration $\mathbb{P}^7[2\ 2\ 2\ 2]$ and its close relatives. The diagrams for a selection of these are given below and show a \mathbb{Z}_4 rotational symmetry however, apart from $\mathbb{P}^7[2\ 2\ 2\ 2]$, this symmetry does not act freely. It turns out that there is in fact a freely acting \mathbb{Z}_4 symmetry for each of these manifolds, but it is contained in \mathbb{H} as a subgroup, and does not act simply as a rotation of the diagram.



The process of exploring the webs starts by looking for manifolds with free actions by \mathbb{Z}_5 , \mathbb{Z}_3 and \mathbb{H} . The web with group \mathbb{Z}_5 is much smaller than that for \mathbb{Z}_3 , and is shown in Table 3.1. From this table we can form a web of \mathbb{Z}_5 quotients and also a yet smaller web of $\mathbb{Z}_5 \times \mathbb{Z}_2$ quotients; these



Table 3.3: The ineffective splits that show that the configuration with Hodge numbers (7, 27) from Table 3.1 corresponds to the same manifold as its transpose.

are shown in Table 3.2. In a similar way we obtain from the \mathbb{Z}_3 web smaller webs corresponding to $\mathbb{Z}_6 \cong \mathbb{Z}_3 \times \mathbb{Z}_2$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$.

A first remark is that the \mathbb{Z}_5 , \mathbb{Z}_3 and \mathbb{H} webs show a striking property with respect to transposition of the matrices. For each configuration that appears in these tables, the transposed configuration does also. In Table 3.1, for example, this is clear apart from the configuration with $(h^{1,1}, h^{2,1}) = (7, 27)$. This configuration is however equal to its transpose as we see by performing a further ineffective splitting followed by an ineffective contraction, as illustrated in Table 3.3. All three configurations have Euler number -40. It is straightforward to see that by contracting the first five \mathbb{P}^1 rows of the large configuration we return to the configuration with $(h^{1,1}, h^{2,1}) = (7, 27)$ whereas if, on the other hand, we contract the rows of the large configuration corresponding to the two \mathbb{P}^4 's then we pass to the transpose of the (7,27) configuration. For the \mathbb{Z}_3 -web there are a number of identities analogous to this last one that ensure that the transpose of each configuration is also a configuration of the web. As an illustration of this consider the relation between the (6, 24) configuration and the (15, 15). This was written as it is to emphasise that $X^{15,15}$ is a split of $X^{6,24}$. The six \mathbb{P}^2 rows of $X^{15,15}$ can however be contracted without changing the Euler number, and when this is done we see that this configuration is also given by the transpose of $X^{6,24}$. It cannot, however, be simply the case that every web is invariant under transposition, since there are many examples of CICY's for which a manifold and its transpose admit different freely acting symmetries. The curious property that characterises the \mathbb{Z}_3 , \mathbb{Z}_5

and \mathbb{H} webs is that if a manifold of the \mathbb{Z}_3 -web, say, admits a freely acting group $G \supset \mathbb{Z}_3$ then, for all the cases studied herein, the transpose admits a freely acting group G' that also contains \mathbb{Z}_3 as a subgroup; a similar statement is true for the \mathbb{Z}_5 and \mathbb{H} webs.

In this work we have not attempted an automated search for CICY's that admit freely acting groups, but rather have examined only splits of manifolds that seem likely to admit the desired symmetry. Recently, Braun has developed and implemented an algorithm to systematically discover free group actions of the type described here; the algorithm and results are presented in [45].

The new manifolds discovered via our techniques are shown in Figure 3.1 and listed in Table 3.8. Among these are some that seem especially interesting. It is apparent from Table 3.4 that the manifold with Hodge numbers (15, 15) is very special, and we find \mathbb{Z}_2 , \mathbb{Z}_3 and $\mathbb{Z}_3 \times \mathbb{Z}_2$ free quotients with Hodge numbers (9,9), (7,7) and (3,3). This manifold is an elliptic fibration and it would seem to be analogous in a number of ways to $X^{19,19}$ which is at the heart of the heterotic models developed at the University of Pennsylvania [14, 15, 17, 43]. The manifold $X^{9,27}$ is also special. It admits a \mathbb{Z}_3 symmetry generator S that acts freely, so that the quotient $X^{9,27}/S$ has $\chi = -12$ and fundamental group \mathbb{Z}_3 , but also admits a \mathbb{Z}_2 generator U which does not act freely, instead fixing two elliptic curves. On taking the further quotient by U and resolving the singularities, we find an analogue of the Tian-Yau manifold in that it has Euler number -6and fundamental group \mathbb{Z}_3 , but now has Hodge numbers (5,8) instead of (6,9). We find also a number of interesting manifolds with Hodge numbers near the tip of the distribution as is evident from Figure 3.1.

Although our main aim is to take quotients by freely acting groups, we come upon several cases where there are also symmetry generators with non-trivial fixed point sets, which are often elliptic curves as above. Taking quotients by non-freely acting generators and resolving singularities provides a number of examples of new manifolds. The fundamental groups of the resolved orbifolds which appear in this way can be determined quite simply. If X is simply-connected and we take its quotient by the group G and resolve fixed points, the fundamental group of the resulting manifold is G/H, where H is the normal subgroup generated by elements of G which act with fixed points. This is intuitive because the spaces we glue in to repair orbifold points are all simply connected, so any loop encircling such a point is homotopically trivial in the resolved manifold. In the physics literature this result goes back to [59]; it can be proven by a straightforward application of van Kampen's theorem [60].

In [38], the webs are traced through, and each new manifold is described in detail. Here we will simply present a representative sample, consisting of some of the more interesting cases.



Table 3.4: The web of CICY's that admit a freely acting \mathbb{Z}_3 symmetry. Not all splits are shown and $Y^{6,33}$ of [38] is omitted for lack of space. The two configurations with Hodge numbers (19, 19) are the same manifold.



Table 3.5: The webs of CICY's obtained, from Table 3.4 as quotients by freely acting \mathbb{Z}_3 and $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetries. In the \mathbb{Z}_3 table there are two occurrences of $X^{3,48}/\mathbb{Z}_3$ owing to the fact that there are distinct quotients by the \mathbb{Z}_3 -generators R and S, see §3.3.1.



Figure 3.2: The webs of CICY's with fundamental group \mathbb{Z}_3 , $\mathbb{Z}_3 \times \mathbb{Z}_2$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$.



Table 3.6: The web of the six CICY's that we consider here that admit a freely acting \mathbb{H} symmetry, together with some related manifolds that we have met previously.



Table 3.7: The web derived from Table 3.6 of manifolds with fundamental group \mathbb{H} .

Table 3.8: The Manifolds of the Tip

The manifolds with $y = h^{1,1} + h^{2,1} \leq 22$ from Figure 3.1. In the 'Manifold' column $X^{19,19}$ denotes the split bicubic and multiple quotient groups indicates different quotients with the same Hodge numbers. We denote by \mathbb{H} the quaternion group and the notation $\mathbb{P}^7[2,2,2,2]^{\sharp}$ and $\mathbb{P}^7[2,2,2,2]^{\sharp\sharp}$ denote two different singularizations. The vectors appended to the symmetries of the two weighted CICY's indicate how the generators act. The generator ($\mathbb{Z}_3 : 1, 2, 1, 2, 0, 0, 0, 0$), for example, acts by multiplying the first coordinate by ω , the second by ω^2 , etc., with ω a nontrivial cube root of unity. In the entry corresponding to $(h^{1,1}, h^{2,1}) = (6, 6)$ we write $\mathbb{P}^{5}[3,3]^{\sharp}$ to denote a resolution of a conifold of $\mathbb{P}^{5}[3,3]$ and 1 denotes the trivial group. We also write $\widehat{X^{a,b}/G}$ for the desingularisation of a quotient of $X^{a,b}$ by a non-freely acting group G. The column labelled by π_1 gives the fundamental group. We only state this explicitly for resolutions of singular quotients; for smooth quotients, π_1 is simply the quotient group. For each manifold with $\chi < 0$ there is a mirror which we do not list explicitly.

(χ, y)	$(h^{1,1}, h^{2,1})$	Manifold	π_1	Reference	
(-40,22)	(1,21)	$\mathbb{P}^4[5]/\mathbb{Z}_5$		_	
(-36,22)	(2,20)	$ \mathbb{P}^{2} \begin{bmatrix} 1 \ 1 \ 1 \ 0 \\ \mathbb{P}^{5} \begin{bmatrix} 1 \ 1 \ 1 \ 0 \\ 1 \ 1 \ 1 \ 3 \end{bmatrix}_{/\mathbb{Z}_{3}} $		[38]	
(-20,22)	(6,16)	$ \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^4 \\ \mathbb{P}^4 \\ \mathbb{P}^4 \\ \mathbb{P}^4 \\ \mathbb{P}^0 \\ \mathbb{P}^1 \\ \mathbb{P}$		[38]	
(-12,22)	(8,14)	$\mathbb{P}\begin{pmatrix}42221100\\00001120\\10000001\end{pmatrix}\begin{bmatrix}8&4\\0&4\\2&0\end{bmatrix}/(\mathbb{Z}_{2}:1,1,0,0,1,0,1,0)$		[61]	
(0,22)	(11,11)	$ \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{array} \begin{bmatrix} 1 & 1 \\ 3 & 0 \\ 0 & 3 \end{bmatrix}_{/\mathbb{Z}_2} $		[62]	
(-30,21)	(3,18)	$ \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^5 \\ \mathbb{P}$		[38]	
(-30,21)	(3,18)	$\mathbb{P}^2egin{pmatrix} 1 \ $		[38]	
(-36,20)	(2,18)	$ \begin{array}{c} \mathbb{P}^1 \begin{bmatrix} 2 \\ 2 \\ \mathbb{P}^1 \end{bmatrix} \\ \mathbb{P}^1 \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}_{\mathbb{Z}_4} \end{array} $		[38]	
(-24,20)	(4,16)	$\mathbb{P}\begin{pmatrix}1111110\\1100001\end{pmatrix}\begin{bmatrix}3&3\\3&0\end{bmatrix}_{/(\mathbb{Z}_3:1,2,1,2,0,0,0,0)}$		[61]	
Continued on the following page					

Table 3.8 – Continued	from	previous	page
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(χ, y)	$(h^{1,1}, h^{2,1})$	Manifold	π_1	Reference	
(-24,20)	(4,16)	$ \begin{array}{c} \mathbb{P}^{1} \begin{bmatrix} 0 \ 1 \ 1 \\ \mathbb{P}^{1} \\ 0 \ 1 \ 1 \\ \mathbb{P}^{1} \\ \mathbb{P}^{1} \end{bmatrix} \begin{bmatrix} 0 \ 1 \ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ \mathbb{P}^{1} \\ \mathbb{P}^{1} \end{bmatrix}_{ \begin{array}{c} 1 \\ 1 \\ 1 \end{bmatrix}_{ \begin{array}{c} 1 \\ 1 \\ 2 \end{bmatrix}_{ \begin{array}{c} \mathcal{I} \\ \mathbb{Z}_{3} \end{array}} } $		[38]	
(-20,20)	(5,15)	$ \begin{array}{c} \mathbb{P}^1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{bmatrix}_{\mathcal{I} = 1} \mathbb{P}^1 \\ \mathbb{P}^1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 \end{bmatrix}_{\mathbb{Z}_2 \times \mathbb{Z}_2} $		[38]	
(0,20)	(10,10)	$\mathbb{P}^7[\widehat{2,2,2,2}]^{\sharp\sharp}$		[55, 57]	
(-24,18)	(3,15)	$\mathbb{P}^{5}[\widehat{3,3]}_{/\mathbb{Z}_{3}} imes \mathbb{Z}_{2}$	\mathbb{Z}_3	[38]	
(-24,18)	(3,15)	$ \begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^5 \\ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \\ \mathbb{Z}_3 \end{array} $		[38]	
(0,18)	(9,9)	$ \begin{array}{c} \mathbb{P}^1 & \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ \mathbb{P}^1 & 0 & 1 & 0 & 0 \\ \mathbb{P}^1 & 0 & 0 & 1 & 0 & 0 \\ \mathbb{P}^1 & 1 & 0 & 0 & 1 & 0 \\ \mathbb{P}^1 & 1 & 0 & 0 & 1 & 0 \\ \mathbb{P}^1 & 0 & 0 & 1 & 0 \\ \mathbb{P}^1 & 1 & 0 & 0 & 0 & 1 \\ \mathbb{P}^1 & 1 & 0 & 0 & 0 & 1 \\ \mathbb{P}^1 & \mathbb{P}^1 & 0 & 1 & 0 & 0 \\ \mathbb{P}^1 & 0 & 0 & 0 & 1 \\ \end{array} \right]_{\mathbb{Z}_2} $		[38]	
(-30,17)	(1,16)	$\mathbb{P}^2 egin{pmatrix} 1 \$		[38]	
(-18,17)	(4,13)	$ \begin{array}{c} \mathbb{P}^2 \begin{bmatrix} 3 & 0 \\ \mathbb{P}^3 \begin{bmatrix} 1 & 3 \end{bmatrix}_{/\mathbb{Z}_3} \end{array} $		[63]	
(-18,17)	(4,13)	$ \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^5 \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ \end{bmatrix}_{/\mathbb{Z}_3} $		[38]	
(-18,17)	(4,13)	$ \begin{array}{c} \mathbb{P}^1 & \begin{bmatrix} 1 \ 1 \ 0 \ 0 \\ \mathbb{P}^1 \\ 1 \ 0 \ 1 \ 0 \\ \mathbb{P}^1 & 1 \ 0 \ 0 \ 1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 & \begin{bmatrix} 0 \ 1 \ 1 \ 1 \\ 0 \ 1 \ 1 \\ \mathbb{P}_3 \\ \mathbb{P}_3 \end{array} \right _{\mathbb{Z}_3} $		[38]	
(-24,16)	(2,14)	$ \begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ \mathbb{P}^2 \end{bmatrix}_{\mathbb{Z}_3} \end{array} $		[38]	
Continued on the following page					

(χ, y)	$(h^{1,1}, h^{2,1})$	Manifold	π_1	Reference	
(-12,16)	(5,11)	$ \begin{array}{c} \mathbb{P}^1 \begin{bmatrix} 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \\ \mathbb{P}^1 \end{bmatrix} \begin{bmatrix} 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \\ 0 \end{bmatrix} \\ \mathbb{P}^1 \end{bmatrix} \begin{bmatrix} 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \\ \mathbb{P}^1 \end{bmatrix} \begin{bmatrix} 0 \ 1 \ 0 \ 0 \ 0 \ 0 \end{bmatrix} \\ \begin{bmatrix} 1 \ 0 \ 0 \end{bmatrix} \\ \mathbb{P}^1 \end{bmatrix} \begin{bmatrix} 0 \ 1 \ 0 \ 0 \ 0 \ 0 \end{bmatrix} \\ \begin{bmatrix} 1 \ 0 \ 0 \ 0 \ 0 \ 0 \end{bmatrix} \\ \mathbb{P}^1 \end{bmatrix} \begin{bmatrix} 1 \ 0 \ 0 \ 0 \ 0 \ 0 \end{bmatrix} \\ \begin{bmatrix} 1 \ 0 \ 0 \ 0 \ 0 \ 0 \end{bmatrix} \\ \mathbb{P}^1 \end{bmatrix} \\ \begin{bmatrix} 1 \ 0 \ 0 \ 0 \ 0 \ 0 \end{bmatrix} \\ \mathbb{P}^1 \end{bmatrix} \\ \begin{bmatrix} 1 \ 0 \ 0 \ 0 \ 0 \ 0 \end{bmatrix} \\ \mathbb{P}^1 \end{bmatrix} \\ \begin{bmatrix} 1 \ 0 \ 0 \ 0 \ 0 \ 0 \end{bmatrix} \\ \mathbb{P}^1 \end{bmatrix} \\ \begin{bmatrix} 1 \ 0 \ 0 \ 0 \ 0 \ 0 \end{bmatrix} \\ \mathbb{P}^1 \end{bmatrix} \\ \begin{bmatrix} 1 \ 0 \ 0 \ 0 \ 0 \ 0 \end{bmatrix} \\ \mathbb{P}^1 \end{bmatrix} \\ \begin{bmatrix} 1 \ 0 \ 0 \ 0 \ 0 \ 0 \end{bmatrix} \\ \mathbb{P}^1 \end{bmatrix} \\ \mathbb{P}^5 \end{bmatrix} \\ \begin{bmatrix} 1 \ 1 \ 1 \ 1 \end{bmatrix} \\ \begin{bmatrix} 1 \ 1 \ 1 \end{bmatrix} \\ \\ \begin{bmatrix} 1 \ 1 \ 1 \end{bmatrix} \\ \\ \begin{bmatrix} 1 \ 1 \ 1 \end{bmatrix} \\ \\ \begin{bmatrix} 1 \ 1 \ 1 \end{bmatrix} \\ \\ \begin{bmatrix} 1 \ 1 \ 1 \end{bmatrix} \\ \\ \begin{bmatrix} 1 \ 1 \ 1 \end{bmatrix} \\ \\ \begin{bmatrix} 1 \ 1 \ 1 \end{bmatrix} \\ \\ \begin{bmatrix} 1 \ 1 \ 1 \end{bmatrix} \\ \\ \end{bmatrix} \\ \\ \begin{bmatrix} 1 \ 1 \ 1 \ 1 \end{bmatrix} \\ \\ \\ \begin{bmatrix} 1 \ 1 \ 1 \ 1 \end{bmatrix} \\ \\ \\ \begin{bmatrix} 1 \ 1 \ 1 \ 1 \end{bmatrix} \\ \\ \\ \begin{bmatrix} 1 \ 1 \ 1 \ 1 \end{bmatrix} \\ \\ \\ \begin{bmatrix} 1 \ 1 \ 1 \ 1 \end{bmatrix} \end{bmatrix} \\ \\ \\ \begin{bmatrix} 1 \ 1 \ 1 \ 1 \ 1 \end{bmatrix} \end{bmatrix} \\ \\ \\ \begin{bmatrix} 1 \ 1 \ 1 \ 1 \ 1 \end{bmatrix} \end{bmatrix} \\ \\ \\ \begin{bmatrix} 1 \ 1 \ 1 \ 1 \ 1 \end{bmatrix} \end{bmatrix} \\ \\ \\ \begin{bmatrix} 1 \ 1 \ 1 \ 1 \ 1 \end{bmatrix} \end{bmatrix} \\ \\ \\ \begin{bmatrix} 1 \ 1 \ 1 \ 1 \ 1 \ 1 \end{bmatrix} \end{bmatrix} \\ \\ \\ \begin{bmatrix} 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \end{bmatrix} \end{bmatrix} \\ \\ \\ \begin{bmatrix} 1 \ $		[38]	
(-8,16)	(6,10)	$ \mathbb{P}^{4} \begin{bmatrix} 1 \ 2 \ 2 \ 0 \ 0 \\ \mathbb{P}^{4} \begin{bmatrix} 1 \ 0 \ 0 \ 2 \ 2 \end{bmatrix}_{\mathbb{Z}_{2} \times \mathbb{Z}_{2}} $		[38]	
(0,16)	(8,8)	$\widehat{X^{19,19}_{/\mathbb{Z}_2 \times \mathbb{Z}_2}}$	\mathbb{Z}_2	[38]	
(0,16)	(8,8)	$\mathbb{P}(1\widehat{1114})[8]^{\sharp}$		[55]	
(4,16)	(9,7)	$X^{\widehat{5,59}}_{/\mathbb{Z}_3 imes \mathbb{Z}_3}$	\mathbb{Z}_3	[38]	
(8,16)	(10,6)	$X^{\widehat{8,44}}_{/\mathbb{Z}_3 imes \mathbb{Z}_3}$	\mathbb{Z}_3	[38]	
(-18,15)	(3,12)	$ \begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^2 \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ \mathbb{P}^2 \end{bmatrix}_{/\mathbb{Z}_3} $		[38]	
(-10,15)	(5,10)	$X^{\widehat{3,48}}_{/\mathbb{Z}_3 imes \mathbb{Z}_3}$	1	[38]	
(-6,15)	(6,9)	$ \begin{array}{c} \mathbb{P}^3 \begin{bmatrix} 1 & 3 & 0 \\ \mathbb{P}^3 \begin{bmatrix} 1 & 0 & 3 \end{bmatrix}_{/\mathbb{Z}_3} \end{array} $		[40, 21]	
(10,15)	(10,5)	$\widehat{X^{6,33}_{/\mathbb{Z}_3\times\mathbb{Z}_3}}$	\mathbb{Z}_3	[38]	
(-20,14)	(2,12)	$\mathbb{P}^4 egin{bmatrix} 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 $		[38]	
(-16,14)	(3,11)	$ \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ 0 1 0 0 0 1 0 0 \\ \mathbb{P}^1 \\ 0 0 0 1 0 0 0 1 0 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^7 \\ \mathbb{1} 1 1 1 1 1 1 1 \mathbf$		[38]	
(-16,14)	(3,11)	$ \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \\ \mathbb{1} 1 1 1 \\ \mathbb{P}_4 \\ \mathbb{P}_4 \end{array} \right) \\ \mathbb{P}_4 \\ \mathbb{P}$		[38]	
Continued on the following page					

Table 3.8 – Continued from previous page

 $(h^{1,1}, h^{2,1})$ (χ, y) Manifold π_1 Reference \mathbb{P}^2 [1 0 0 1 0 0 1 0 0] $\mathbb{P}^2 | 0 \mathbf{1} 0 0 \mathbf{1} 0 0 \mathbf{1} 0 0 \mathbf{1} 0 | 0$ $\mathbb{P}^2 \mid 0 \ 0 \ \mathbf{1} \ 0 \ 0 \ \mathbf{1} \ 0 \ 0 \ \mathbf{1}$ (-12, 14)[38](4,10) \mathbb{P}^2 | 1 1 1 0 0 0 0 0 0 $\mathbb{P}^{2} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ \mathbb{Z}_3 $\widehat{X^{3,39}_{/\mathbb{Z}_3 \times \mathbb{Z}_2}}$ (-12, 14)(4,10) \mathbb{Z}_3 [38] $\mathbb{P}^1 \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix}$ $\mathbb{P}^1 | \mathbf{1} 0 0 \mathbf{1} 0$ $\mathbb{P}^1 | \mathbf{1} 0 0 0 \mathbf{1}$ $\begin{array}{c}
\mathbb{P}^{1} \\
\mathbb{P$ (-8, 14)[38](5,9) $\mathbb{P}^{2} \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix}_{\mathbb{Z}_{3}}$ $\widehat{X^{6,24}_{/\mathbb{Z}_3 \times \mathbb{Z}_3}}$ 1 (-4, 14)(6,8)[38] $\mathbb{P}^1 \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \\
 \mathbb{P}^2 \begin{bmatrix} 3 & 0 \end{bmatrix}$ (0,14)(7,7)[62] $\overset{\mathfrak{a}}{\mathbb{P}^2} \begin{bmatrix} \check{0} & 3 \end{bmatrix}_{\{\mathbb{Z}_3, \ \mathbb{Z}_2 \times \mathbb{Z}_2\}}$ $\mathbb{P}^1 \begin{bmatrix} \mathbf{1} & 0 & \mathbf{0} & \mathbf{1} & 0 & \mathbf{0} \end{bmatrix}$ $\mathbb{P}^1 \mid 0 \ \mathbf{1} \ 0 \ \mathbf{1} \ 0 \ \mathbf{0}$ $\mathbb{P}^1 \mid 0 \ 0 \ \mathbf{1} \ \mathbf{1} \ 0 \ 0$ $\mathbb{P}^1 | \mathbf{1} 0 0 0 \mathbf{1} 0$ (0, 14) $\mathbb{P}^1 \mid 0 \ \mathbf{1} \ 0 \ \mathbf{1} \ 0 \ \mathbf{1} \ 0$ (7,7)[38] $\mathbb{P}^1 \mid 0 \ 0 \ \mathbf{1} \ 0 \ \mathbf{1} \ 0$ $\mathbb{P}^1 \mid \mathbf{1} \mid \mathbf{0} \mid \mathbf{0} \mid \mathbf{0} \mid \mathbf{0} \mid \mathbf{1}$ $\mathbb{P}^1 \mid 0 \ \mathbf{1} \ 0 \ 0 \ 0 \ \mathbf{1}$ $\mathbb{P}^{1}\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}_{\mathbb{Z}_{3}}$ $\widehat{X^{15,15}_{/\mathbb{Z}_3 \times \mathbb{Z}_3}}$ 1 [38](0,14)(7,7) $\mathbb{P}^{2} \begin{bmatrix} 3 \\ 3 \end{bmatrix}_{\mathbb{Z}_{3} \times \mathbb{Z}_{3}}$ [39](-18, 13)(2,11) $\widehat{X^{9,27}_{/\mathbb{Z}_3 \times \mathbb{Z}_2}}$ (-6, 13)(5,8)[38] \mathbb{Z}_3 $\begin{array}{c} \mathbb{P}^1 & \begin{bmatrix} 2 \\ 2 \\ \mathbb{P}^1 & 2 \\ 2 \\ \mathbb{P}^1 & 2 \\ \mathbb{P}^1 & 2 \end{bmatrix}$ (-16, 12)[38](2,10) $\mathbb{Z}_4 \times \mathbb{Z}_2$ $\begin{array}{c} \mathbb{P}^{4} \begin{bmatrix} 1 & 2 & 2 & 0 & 0 \\ \mathbb{P}^{4} \begin{bmatrix} 1 & 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 2 & 2 \end{bmatrix}_{/\mathbb{Z}_{4}}$ [38](-8, 12)(4,8) $\widehat{X^{19,19}_{/\mathbb{Z}_3 \times \mathbb{Z}_2}}$ (0,12)(6,6)[38] \mathbb{Z}_3 $\widehat{\mathbb{P}^5[3,3]}^{\sharp}/G \ , \ \ G \subset \mathbb{Z}_6 \times \mathbb{Z}_6$ (0,12)(6,6)[55]Continued on the following page

Table 3.8 - Continued from previous page

		Table 3.8 - Continued from previous p	page	
(χ,y)	$(h^{1,1}, h^{2,1})$	Manifold	π_1	Reference
(-16,10)	(1,9)	$ \begin{array}{c} \mathbb{P}^1 \begin{bmatrix} 1 \ 1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{bmatrix}_{\mathbb{Z}_5}^{\mathbb{Z}_5} $		[38]
(-16,10)	(1,9)	$\mathbb{P}_5[3,3]_{/\mathbb{Z}_3 imes\mathbb{Z}_3}$		[38]
(-16,10)	(1,9)	$\mathbb{P}^{7}[2,2,2,2]/\{\mathbb{H},\mathbb{Z}_{8},\mathbb{Z}_{4}\times\mathbb{Z}_{2},\mathbb{Z}_{2}\times\mathbb{Z}_{2}\times\mathbb{Z}_{2}\}$		[22, 57, 64, 65]
(-16,10)	(1,9)	$ \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \right]_{/\mathbb{H}} $		[38]
(-12,10)	(2,8)	$ \begin{array}{c} \mathbb{P}^1 & \begin{bmatrix} 1 \ 1 \ 0 \ 0 \ 0 \ 0 \\ \mathbb{P}^1 & 1 \ 0 \ 1 \ 0 \ 0 \ 0 \\ \mathbb{P}^1 & 1 \ 0 \ 1 \ 0 \ 0 \\ \mathbb{P}^1 & 1 \ 0 \ 0 \ 1 \ 0 \\ \mathbb{P}^1 & 1 \ 0 \ 0 \ 0 \ 1 \\ \mathbb{P}^4 & \begin{bmatrix} 1 \ 1 \ 0 \ 0 \ 0 \ 0 \\ 0 \ 1 \ 1 \ 1 \\ \mathbb{P}^4 \end{bmatrix}_{\mathbb{Z}_5} $		[38]
(-12,10)	(2,8)	$ \mathbb{P}^{2} \begin{bmatrix} 1 \ 1 \ 1 \ 0 \\ \mathbb{P}^{5} \begin{bmatrix} 1 \ 1 \ 1 \ 0 \\ 1 \ 1 \ 1 \ 3 \end{bmatrix}_{/\mathbb{Z}_{3} \times \mathbb{Z}_{3}} $		[38]
(-12,10)	(2,8)	$ \begin{array}{c} \mathbb{P}^{1} \\ \mathbb{P}^{2} \\ \mathbb{Z}_{3} \times \mathbb{Z}_{2} \end{array} $		[38]

(-8,10)	(3,7)	$ \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}$		[38]	
(-8,10)	(3,7)	$\mathbb{P}^{5}[3,3]_{/\mathbb{Z}_{3}\times\mathbb{Z}_{3}\times\mathbb{Z}_{2}}$	$\mathbb{Z}_3 \times \mathbb{Z}_3$	[38]	
(-8,10)	(3,7)	$ \mathbb{P}^{2} \begin{bmatrix} 1 \ 1 \ 1 \ 0 \ 0 \ 0 \\ \mathbb{P}^{2} \\ \mathbb{P}^{5} \end{bmatrix} \mathbb{P}^{5} \begin{bmatrix} 1 \ 1 \ 1 \ 0 \ 0 \ 0 \\ 1 \ 1 \ 1 \end{bmatrix} _{\mathbb{Z}_{3} \times \mathbb{Z}_{3} } $		[38]	
(0,10)	(5,5)	$ \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{array} \begin{bmatrix} 1 & 1 \\ 3 & 0 \\ 0 & 3 \end{bmatrix}_{/\mathbb{Z}_4} $		[62], [38]	
(0,10)	(5,5)	$\widehat{X^{15,15}_{/\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2}}$	\mathbb{Z}_2	[38]	
(-14,9)	(1,8)	{Resoln. of a Pfaffian CY manifold}/ \mathbb{Z}_7		[58]	
Continued on the following page					

(χ, y)	$(h^{1,1}, h^{2,1})$	Manifold	π_1	Reference	
(0,8)	(4,4)	$\widehat{X^{19,19}_{/\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2}}$	$\mathbb{Z}_3 \times \mathbb{Z}_2$	[38]	
(0,8)	(4,4)	Resoln. of a Horrocks-Mumford quintic		$[55, \S{3.2}]$	
(-8,8)	(2,6)	$ \begin{array}{c} \mathbb{P}^1 & \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ \mathbb{P}^1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ \mathbb{P}^1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ \mathbb{P}^1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ \mathbb{P}^7 & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \right) / \mathbb{H} $		[38]	
(-8,8)	(2,6)	$ \begin{array}{c} \mathbb{P}^{1} \\ \mathbb{P}^{1} \\ \mathbb{P}^{1} \\ 0 & 2 & 0 & 0 \\ \mathbb{P}^{1} \\ 0 & 0 & 2 & 0 \\ \mathbb{P}^{1} \\ \mathbb{P}^{3} \\ \mathbb{P}^{3} \\ 1 & 1 & 1 \\ \end{array} \right)_{/\mathbb{H}} $		[38]	
(-10,7)	(1,6)	$\frac{\mathbb{P}^4 \begin{bmatrix} \textbf{1} \ \textbf{1} \ \textbf{1} \ \textbf{1} \ \textbf{1} \\ \mathbb{P}^4 \begin{bmatrix} \textbf{1} \ \textbf{1} \ \textbf{1} \ \textbf{1} \\ \textbf{1} \ \textbf{1} \ \textbf{1} \\ \textbf{1} \end{bmatrix}}{/\mathbb{Z}_5 \times \mathbb{Z}_2}$		[38]	
(-8,6)	(1,5)	$\mathbb{P}^4[5]/\mathbb{Z}_5{ imes}\mathbb{Z}_5$		_	
(-8,6)	(1,5)	$ \begin{array}{c} \mathbb{P}^1 & \begin{bmatrix} 1 & 1 \\ \mathbb{P}^1 & \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^1 & \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ \mathbb{P}^1 & \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ \mathbb{P}^2 & \mathbb{Z}_2 \end{array} $		[38]	
(-8,6)	(1,5)	$\mathbb{P}^{7}[2,2,2,2]/G$, $ G = 16$		[22, 56, 57]	
(-8,6)	(1,5)	$ \begin{array}{c} \mathbb{P}^{1} \begin{bmatrix} 2\\ 2\\ 2\\ \mathbb{P}^{1} \end{bmatrix}_{2} \\ \mathbb{P}^{1} \begin{bmatrix} 2\\ 2\\ 2\\ \end{bmatrix}_{/\mathbb{H} \times \mathbb{Z}_{2}} $		[38]	
(-4,6)	(2,4)	$ \begin{array}{c} \mathbb{P}^{1} \\ \mathbb{P}^{4} \\ \mathbb{P}^{4} \\ \mathbb{P}^{4} \\ \mathbb{P}^{4} \\ \mathbb{P}^{0} \\ \mathbb{P}^{1} \\ P$		[38]	
(-4,6)	(2,4)	$ \mathbb{P}^{4} \begin{bmatrix} 1 \ 2 \ 2 \ 0 \ 0 \\ \mathbb{P}^{4} \begin{bmatrix} 1 \ 0 \ 0 \ 2 \ 2 \end{bmatrix}_{/\mathbb{H}} $		[38]	
(0,6)	(3,3)	$ \mathbb{P}^{1}_{\mathbb{P}^{2}} \begin{bmatrix} 1 & 1 \\ 3 & 0 \\ 0 & 3 \end{bmatrix}_{\mathbb{Z}^{3} \times \mathbb{Z}_{3}, \mathbb{H}, \mathbb{Z}_{4} \times \mathbb{Z}_{2}, \mathbb{Z}_{3} \times \mathbb{Z}_{2}, \mathbb{Z}_{5} } $		[62], [38]	
Continued on the following page					

Table 3.8 – Continued from previous page

(χ,y)	$(h^{1,1}, h^{2,1})$	Manifold 7		Reference
(0,6)	(3,3)	$ \begin{array}{c} \mathbb{P}^1 \begin{bmatrix} 1 \ 0 \ 0 \ 1 \ 0 \ 0 \\ \mathbb{P}^1 \\ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \\ \mathbb{P}^1 \\ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \\ \mathbb{P}^1 \\ \mathbb{P}^2 $		[38]
(-4,4)	(1,3)	$ \begin{array}{c} \mathbb{P}^1 \begin{bmatrix} 1 & 1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{bmatrix}_{\mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2} \\ / \mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \end{array} $		[38]
(-4,4)	(1,3)	$\mathbb{P}^{7}[2,2,2,2]/G$, $ G = 32$		[22, 56, 57]
(0,4)	(2,2)	$\mathbb{P}^{7}[\widehat{2,2,2,2}]^{\sharp}/G$, $ G $ divides 64		[55-57]
(0,4)	(2,2)	Resoln. of Pfaffian CY w. 49 nodes		[58]

Table 3.8 – Continued from previous page

3.2 Some examples of free actions by \mathbb{Z}_5

3.2.1 $X^{2,52}$; a split of the quintic threefold

A family of quintics $\mathbb{P}^4[5]$ admit a free action by the group \mathbb{Z}_5 . Seeking manifolds that are related to $\mathbb{P}^4[5]$ and which maintain the symmetry, it is natural to consider the split configuration

$$X^{2,52} = \mathbb{P}^{4} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ \end{bmatrix}_{-100}^{2,52} \tag{3.5}$$

This will serve as a simple first example of the required techniques. We denote the coordinates of the two \mathbb{P}^4 's by x_j and y_k and the five polynomials by p_i . The polynomials are bilinear in the coordinates of the two \mathbb{P}^4 's, so we may write

$$p_i(x,y) = \sum_{j,k} A_{ijk} x_j y_k \tag{3.6}$$

>

where the A_{ijk} are arbitrary complex coefficients (this will be true for all polynomials unless otherwise noted). In the present context it is convenient to understand the indices as taking values in \mathbb{Z}_5 . Let us choose to make the equations covariant under a generator S, acting as

$$S: x_i \to x_{i+1}, \quad y_i \to y_{i+1}; \quad p_i \to p_{i+1}$$

This requires

$$A_{ijk} = A_{i-1, j-1, k-1}$$
 hence $A_{ijk} = A_{0, j-i, k-i}$

If we write a_{jk} in place of A_{0jk} and change indices of summation, our polynomials take the form

$$p_i(x,y) = \sum_{j,k} a_{jk} x_{j+i} y_{k+i}$$
(3.7)

The coordinates of the points fixed by S in $\mathbb{P}^4 \times \mathbb{P}^4$ are of the form $x_j = \zeta^j$ and $y_j = \tilde{\zeta}^j$ with $\zeta^5 = \tilde{\zeta}^5 = 1$, and it is easy to see that, for general choice of the coefficients a_{jk} , none of the p_i vanish at these points. We have also checked that the polynomials are transverse, and thus the quotient variety $X^{2,52}/S$ is smooth. The Hodge numbers for the quotient follow from the fact that S acts separately on each \mathbb{P}^4 , so taking the quotient does not change $h^{1,1}$, while the new Euler number is -100/5 = -20.

If the coefficient matrix a_{jk} is taken to be symmetric then the polynomials equation (3.7) are also invariant under a \mathbb{Z}_2 generator

$$U: x_j \leftrightarrow y_j; p_i \rightarrow p_i$$

The fixed points of U are such that $y_k = x_k$ and $p_i(x, x) = 0$. The latter equations are 5 quadratic equations acting in a \mathbb{P}^4 , which we expect to have no common solution for a generic symmetric matrix a. It is easy to check that this is, in fact, the case by means of a Gröbner basis calculation. For the U-quotients $X^{2,52}/U$ and $X^{2,52}/S \times U$ we have $h^{1,1} = 1$, since U identifies the two \mathbb{P}^4 's. The value of $h^{2,1}$ then follows from the fact that the Euler number divides by the order of the group. Thus we have shown the existence of the quotients of the following table:

$(h^{1,1}, h^{2,1}) (X^{2,52}/G)$	(1, 6)	(2, 12)	(1, 26)
G	$\mathbb{Z}_5 imes \mathbb{Z}_2$	\mathbb{Z}_5	\mathbb{Z}_2

Table 3.9: The Hodge numbers of smooth quotients of $X^{2,52}$.

3.2.2 $X^{7,27}$; a further split of the quintic

We can perform a further split of the quintic by splitting $X^{2,52}$ to obtain the configuration



The structure of the matrix and diagram suggests the possibility of a free $\mathbb{Z}_5 \times \mathbb{Z}_2$ action, and we will show that such a symmetry does in fact exist. We again take coordinates $(x_i, y_j), i, j \in \mathbb{Z}_5$ for $\mathbb{P}^4 \times \mathbb{P}^4$, along with coordinates t_{ia} , $i \in \mathbb{Z}_5$, $a \in \mathbb{Z}_2$ for the five \mathbb{P}^1 's, so that the defining polynomials are

$$p_{i} = \sum_{j} x_{j} (A_{ij} t_{i0} + B_{ij} t_{i1})$$
$$q_{i} = \sum_{j} y_{j} (C_{ij} t_{i0} + D_{ij} t_{i1})$$

We extend the $\mathbb{Z}_5 \times \mathbb{Z}_2$ action of the last section to the new ambient space, so that it is now generated by

$$S: x_i \to x_{i+1}, y_i \to y_{i+1}, t_{ia} \to t_{i+1,a}; p_i \to p_{i+1}, q_i \to q_{i+1}$$

$$U: x_i \leftrightarrow y_i, t_{i0} \leftrightarrow t_{i1}; p_i \leftrightarrow q_i$$

The most general form of the polynomials covariant under these transformations is

$$p_{i} = \sum_{j} x_{j} (A_{i-j} t_{i0} + B_{i-j} t_{i1})$$
$$q_{i} = \sum_{j} y_{j} (B_{i-j} t_{i0} + A_{i-j} t_{i1})$$

In order to show that the action is free it suffices to check that both S and U act without fixed points. A fixed point of S is of the form $x_i = \zeta^i$, $y_i = \tilde{\zeta}^i$, where $\zeta^5 = \tilde{\zeta}^5 = 1$, and $t_{ia} = t_a$. At these points, the polynomials are given by

$$p_i = \zeta^i \sum_k \zeta^{-k} (A_k t_0 + B_k t_1)$$
$$q_i = \tilde{\zeta}^i \sum_k \tilde{\zeta}^{-k} (B_k t_0 + A_k t_1)$$

This system reduces to a pair of linear equations for $(t_0, t_1) \in \mathbb{P}^1$, and for general coefficients will have no solution.

A fixed point of U is given by $y_i = x_i$ and $(t_{i0}, t_{i1}) = (1, \pm 1)$. The polynomials then become

$$p_i = \sum_j (A_{i-j} \pm B_{i-j}) x_j t_{i0}, \quad q_i = \pm p_i$$

For general coefficients, regardless of the choice of the signs, the equations $p_i = 0$ place five independent linear constraints on the x_j , and therefore have no non-trivial solutions. We have established that $\mathbb{Z}_5 \times \mathbb{Z}_2$ acts on the manifold without fixed points. We can use our freedom to change coordinates to isolate the independent coefficients in the polynomials. The actions of Sand U are preserved by any coordinate change of the form

$$x_i \to \sum_k \gamma_k \, x_{i+k} \,, \quad y_i \to \sum_k \gamma_k \, y_{i+k}$$
$$(t_{i0}, \, t_{i1}) \to (\alpha \, t_{i0} + \beta \, t_{i1}, \beta \, t_{i0} + \alpha \, t_{i1})$$

We can use a transformation of the x's and y's to set $A_i = A_0 \delta_{i0}$. Then we can transform the t's to enforce $A_0 = B_0$, and absorb B_0 into the normalisation of the polynomials. This leaves us with

$$p_{i} = x_{i} t_{i0} + \sum_{j \neq i} B_{i-j} x_{j} t_{i1}$$
$$q_{i} = \sum_{j \neq i} B_{i-j} y_{j} t_{i0} + y_{i} t_{i1}$$

We have checked that these polynomials are transverse. The Euler number of the quotient manifold is $\chi = -40/10 = -4$, and the group action identifies the five \mathbb{P}^1 's, as well as the two \mathbb{P}^4 's, leaving $h^{1,1} = 2$. Thus $h^{2,1} = 4$, which agrees with the number of parameters in the polynomials.

We have confirmed the existence of a smooth $\mathbb{Z}_{10} \cong \mathbb{Z}_5 \times \mathbb{Z}_2$ quotient, and therefore also of the two- and five-fold covers obtained by taking the separate \mathbb{Z}_5 and \mathbb{Z}_2 quotients. We record the Hodge numbers in the following short table.

$(h^{1,1}, h^{2,1}) (X^{7,27}/G)$	(2, 4)	(3,7)	(6, 16)
G	$\mathbb{Z}_5 \times \mathbb{Z}_2$	\mathbb{Z}_5	\mathbb{Z}_2

Table 3.10: Hodge numbers of quotients of $X^{7,27}$.

3.3 Some manifolds admitting free actions by \mathbb{Z}_3

3.3.1 $X^{3,48}$; a \mathbb{P}^2 split of the bicubic

The bicubic $X^{2,83}$, which was discussed in §3.1, admits a free action by $\mathbb{Z}_3 \times \mathbb{Z}_3$, so is a promising starting point for a web of \mathbb{Z}_3 -symmetric families. A very symmetrical-looking split of the bicubic is given by splitting with a single \mathbb{P}^2 :

$$X^{3,48} = \begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ 1 & 1 & 1 \\ \mathbb{P}^2 \\ \end{bmatrix}_{-90}^{3,48}$$

There are two nice ways to draw the corresponding diagram. The first suggests a \mathbb{Z}_3 symmetry while the second makes it apparent, in the diagram, that the matrix is identical to its transpose



Let us denote the coordinates of the three \mathbb{P}^2 's by x_{ij} where *i* labels the space and *j* its coordinates; we understand the labels to take values in \mathbb{Z}_3 . The polynomials take the form

$$p_i = \sum_{jkl} A_{ijkl} \, x_{0,j} \, x_{1,k} \, x_{2,l} \tag{3.8}$$

The counting of parameters, before we impose any symmetries, is that there are 3^4 coefficients. There are 3×8 degrees of freedom corresponding to redefinitions of the coordinates, up to scale, and a further 9 degrees of freedom corresponding to redefinitions of the polynomials

$$x_{ij} \rightarrow \sum_{k} \alpha_{ijk} x_{ik} ; p_i \rightarrow \sum_{k} \beta_{ik} p_k$$

The count is therefore that there are 81 - 24 - 9 = 48 effective parameters in the polynomials, which agrees with the value of $h^{2,1}$.

We impose first an internal symmetry

$$S: x_{i,j} \to x_{i,j+1}; \quad p_i \to p_{i+1}$$

which requires the coefficients to satisfy $A_{ijkl} = A_{0,j-i,k-i,l-i}$. So we set $C_{ijk} = A_{0ijk}$, to obtain

$$p_i = \sum_{jkl} C_{jkl} x_{0,j+i} x_{1,k+i} x_{2,l+i}$$
(3.9)

Now there are 27 coefficients and we are allowed redefinitions

$$x_{i,j} \rightarrow \sum_{k} \alpha_{ik} x_{i,j+k} \text{ and } p_i \rightarrow \sum_{k} \beta_k p_{i+k}$$

with 6 degrees of freedom, up to scale, originating in the redefinition of the coordinates and 3 in the redefinition of the polynomials. The count is therefore that there are now 27 - 6 - 3 = 18 free parameters in the equations.

Fixed points of S, in the embedding space, are of the form $x_{ij}^S = \xi_i^j$, with the ξ_i cube roots of unity. Evaluating on the fixed points we have

$$p_i(x^S) = (\xi_0 \xi_1 \xi_2)^i p_0(x^S)$$

and $p_0(x^S) \neq 0$ for generic choices of coefficients. The polynomials are also transverse for generic coefficients though this is easiest to show for the more symmetric polynomials we shall consider shortly. The quotient $X^{3,48}/S$ will have $\chi = -90/3 = -30$, and $h^{1,1} = 3$ since S only acts internally on each \mathbb{P}^2 . This implies that $(h^{1,1}, h^{2,1}) = (3, 18)$ so we have agreement between $h^{2,1}$ and the number of parameters in the polynomials.

We now return to equation (3.8) and impose instead the symmetry

$$R: x_{i,j} \to x_{i+1,j}; p_i \to p_i$$

This requires A_{ijkl} to be invariant under cyclic permutation of the last three indices. A tensor B_{jkl} that is invariant under cyclic permutation of its labels has 11 degrees of freedom so our coefficients A_{ijkl} have 33. Now there are a total of 8 degrees of freedom in the allowed redefinition of coordinates and 9 in the redefinition of the polynomials,

$$x_{i,j} \rightarrow \sum_{k} \alpha_{jk} x_{i,k} , p_i \rightarrow \sum_{k} \beta_{ik} p_k$$

so the count is that there are 33 - 8 - 9 = 16 degrees of freedom in the polynomials. The generator R identifies the three \mathbb{P}^{2} 's, so $h^{1,1} = 1$ for the quotient. Since the Euler number again divides we find $(h^{1,1}, h^{2,1}) = (1, 16)$ and the value we find for $h^{2,1}$ confirms our parameter count. Fixed points of R are of the form

$$x_{ij}^R = w_j$$

The w_j parametrise a \mathbb{P}^2 so the three equations $p_i(w) = 0$ will, generically, have no solution. This can be checked explicitly, together with the fact that the polynomials are transverse, for the simple polynomials given below.

If we impose invariance under both R and S then the polynomials are as in equation (3.9) but with the coefficient tensor invariant under cyclic permutation of its indices. A counting of parameters analogous to that above reveals that there are 11 - 2 - 3 = 6 free parameters in the polynomials. A choice that exhibits these is

$$p_{i} = x_{0,i} x_{1,i} x_{2,i} + \sum_{s=\pm 1} \left\{ E_{s} x_{0,i+s} x_{1,i+s} x_{2,i+s} + F_{s} \sum_{k=0}^{2} x_{k,i} x_{k+1,i+s} x_{k+2,i+s} + G_{s} \sum_{k=0}^{2} x_{k,0} x_{k+s,1} x_{k+2s,2} \right\}$$
(3.10)

and it is straightforward to check that these are transverse, for generic choice of the coefficients.

We have seen that R and S act without fixed points. The diagonal generators, however, do have fixed points. We have $RS x_{i,j} = x_{i+1,j+1}$ and this action has fixed points that satisfy

$$x_{i,j}^{RS} = x_{i-1,j-1}^{RS}$$
 hence $x_{i,j}^{RS} = x_{0,j-i}^{RS}$

If we regress to writing the polynomials as in equation (3.9) with cyclically invariant C_{jkl} then we see that, when evaluated on the fixed points,

$$p_{i+1} = \sum_{jkl} C_{jkl} x_{1,j+i}^{RS} x_{2,j+i-1}^{RS} x_{0,j+i-2}^{RS} = p_i$$

Thus the three polynomials p_i are all equivalent to p_0 , and this is a cubic in $x_{0,k}^{RS}$, so the fixed point set is an elliptic curve that we denote by \mathcal{E}^{RS} . In a similar way we see that R^2S has fixed points

$$x_{i,j}^{R^2S} = x_{0,j+i}^{R^2S}$$

and that these points make up a second elliptic curve \mathcal{E}^{R^2S} . A point of intersection of the two elliptic curves would be a simultaneous fixed point of RS and R^2S and hence also of R and S. We have seen above that R and S act without fixed points, so the two elliptic curves cannot intersect.

The singular set of $X^{3,48}/R \times S$ therefore consists of the A_2 surface singularity $\mathbb{C}^2/\mathbb{Z}_3$, fibred over two disjoint elliptic curves. It is well-known that blowing up the singular point of the A_2 surface gives a crepant resolution, and that the exceptional divisor consists of two irreducible components. The resolution can be fibred over the elliptic curves (which is the same as blowing up along these curves) to resolve the threefold singularity, and this introduces two divisor classes for each curve. For the resolved manifold we therefore have $h^{1,1} = 1+2\times 2 = 5$. An elliptic curve has Euler number 0, and so does any bundle over it, so the resolution does not change the Euler number, which implies that $h^{2,1} = 6 + 2 \times 2 = 10$. Since RS and R^2S both have fixed points, and these two elements generate the whole quotient group, the resolved manifold obtained this way is simply connected. The results of this section are summarised in the following table

Hodge numbers	(1, 16)	(3, 18)	(5, 10)
Manifold	$X^{3,48} / \mathbb{Z}_3$	$X^{3,48}/\mathbb{Z}_3$	$\widehat{X^{3,48}/\mathbb{Z}_3} \times \mathbb{Z}_3$
Fundamental group	\mathbb{Z}_3	\mathbb{Z}_3	1

Table 3.11: Hodge numbers of smooth quotients, and the resolution of the singular quotient, of $X^{3,48}$.

3.3.2 $X^{4,40}$; a second split of the bicubic

We can introduce a fourth \mathbb{P}^2 and split one column of the above configuration to obtain

$$X^{4,40} = \begin{array}{c} \mathbb{P}^{2} \begin{bmatrix} \mathbf{1} & 0 & 0 & \mathbf{1} & \mathbf{1} \\ \mathbb{P}^{2} \\ \mathbb{P}^{2} \\ \mathbb{P}^{2} \\ \mathbb{P}^{2} \end{bmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix}_{-72}^{4,40}$$

with diagrams



We will now show that this manifold admits a free action by \mathbb{Z}_3 , and in doing so we will come across a new subtlety in counting the number of meaningful parameters in a set of polynomials.

Take coordinates x_{ij} on the first three spaces as in the previous subsection, and coordinates u_i on the last space. Denote by q_i the first three polynomials, and by p_1, p_2 the last two. We can then extend the definition of the \mathbb{Z}_3 generator R, from the last section, to

$$R: x_{i,j} \rightarrow x_{i+1,j}, \quad u_i \rightarrow u_{i+1}; \quad q_i \rightarrow q_{i+1}$$

The most general polynomials covariant under this action are

$$q_{i} = \sum_{j,k} A_{jk} x_{ij} u_{i+k}$$

$$p_{1} = \sum_{i,j,k} B_{ijk} x_{0,i} x_{1,j} x_{2,k}$$

$$p_{2} = \sum_{i,j,k} C_{ijk} x_{0,i} x_{1,j} x_{2,k}$$
(3.11)

where B_{ijk} and C_{ijk} are cyclic in their indices. We have checked that these polynomials are transverse. The fixed points of R in the ambient space occur when $x_{ij} = x_j^*$, and $u_i = \zeta$, where $\zeta^3 = 1$. At these points, the equations $q_0 = p_1 = p_2 = 0$ impose three independent homogeneous constraints in the \mathbb{P}^2 parametrised by x_j^* , so there are no solutions, in general. Therefore the quotient is smooth.

Since R identifies three of the four ambient spaces, $h^{1,1}$ will be reduced to 2 for the quotient. The Euler number will be -72/3 = -24, and this implies that $h^{2,1} = 14$. We can confirm this with a parameter count. We start with the general non-symmetric case, in which there are 81 terms in the polynomials. There are $4 \times 8 = 32$ parameters in coordinate changes, and 3 parameters in rescaling the q_i . Next we use an observation first made in [51] in the context of the manifold $X^{2,56}$. For a solution to $q_i = 0$ to exist for all i, we must have $0 = \det(\partial q_i/\partial u_j) \equiv D$. But D is a homogeneous trilinear polynomial in the first three spaces, so the most general redefinition of the polynomials p_1, p_2 is

$$p_1 \to \kappa_{11} p_1 + \kappa_{12} p_2 + K_1 D$$
, $p_2 \to \kappa_{21} p_1 + \kappa_{22} p_2 + K_2 D$

which contains 6 more parameters. Therefore the number of meaningful parameters in the defining polynomials is 81 - (32 + 3 + 6) = 40, which agrees with $h^{2,1}$. Now impose covariance under R. Demanding that C_{ijk} is cyclic leaves 11 independent components, so the polynomials now contain $9 + (2 \times 11) = 31$ coefficients. The coordinate changes compatible with the action of R are $x_{ij} \rightarrow \sum_k \alpha_{jk} x_{ik}$ and $u_i \rightarrow \sum_j \beta_{ij} u_j$, which contain 8+2 = 10 parameters, up to irrelevant scaling. Finally, we can rescale the q_i by a common factor, and redefine p_1 and p_2 exactly the same way as in the non-symmetric case, so altogether this gives 7 more parameters, leaving 31 - (10+7) = 14 independent coefficients. This agrees with our previous determination of $h^{2,1}$, and we have found a quotient manifold with fundamental group \mathbb{Z}_3 and $(h^{1,1}, h^{2,1}) = (2, 14)$.

3.3.3 $\mathbb{P}^{5}[3,3]^{1,73}$

The fact that $\mathbb{P}^{5}[3,3]$ admits a freely acting $\mathbb{Z}_{3} \times \mathbb{Z}_{3}$ symmetry has been known for some time. We recall this in order to be able to take a further quotient by a \mathbb{Z}_{2} generator, V, that does not act freely. This generator reappears in splits of $\mathbb{P}^{5}[3,3]$, including in a configuration that we study in §3.3.5, the quotient of which by V leads, after resolution, to a manifold with $\chi = -6$.

We take coordinates $(x_0, x_1, x_2, y_0, y_1, y_2)$ for \mathbb{P}^5 and impose two cubics, p_1 and p_2 , that we take to be invariant under the symmetries generated by

$$S: x_i \to x_{i+1}, y_i \to y_{i+1} \qquad T: x_i \to \zeta^i x_i, y_i \to \zeta^i y_i$$

with ζ a non-trivial cube root of unity. To write suitable invariant equations define the following invariant cubics

$$\mathcal{X}_{0} = \sum_{i} x_{i}^{3} , \quad \mathcal{X}_{1} = \sum_{i} x_{i} y_{i}^{2} , \quad \mathcal{X}_{2} = \sum_{i} x_{i} y_{i+1} y_{i+2} , \quad \mathcal{X}_{3} = x_{0} x_{1} x_{2}$$

$$\mathcal{Y}_{0} = \sum_{i} y_{i}^{3} , \quad \mathcal{Y}_{1} = \sum_{i} x_{i}^{2} y_{i} , \quad \mathcal{Y}_{2} = \sum_{i} x_{i} x_{i+1} y_{i+2} , \quad \mathcal{Y}_{0} = y_{0} y_{1} y_{2}$$

In terms of these we form invariant polynomials

$$p_1 = \sum_{\alpha=0}^3 \left(A_\alpha \,\mathcal{X}_\alpha + B_\alpha \,\mathcal{Y}_\alpha \right) \,, \quad p_2 = \sum_{\alpha=0}^3 \left(C_\alpha \,\mathcal{X}_\alpha + D_\alpha \,\mathcal{Y}_\alpha \right)$$

and one can check that these polynomials are transverse for generic values of the coefficients, though this is easier to check for the more symmetric polynomials that we shall write below. To show that the group generated by S and T acts without fixed points, it is sufficient to show that ST^k acts without fixed points for k = 0, 1, 2, and this is straightforward. There are 16 coefficients in p_1 and p_2 and, up to scale, a three-parameter freedom to redefine the coordinates in a way that preserves the symmetry

$$x_i \rightarrow \alpha x_i + \beta y_i, \quad y_i \rightarrow \gamma x_i + \delta y_i$$

There is also a four-parameter freedom to redefine p_1 and p_2

$$p_1 \rightarrow a p_1 + b p_2, \quad p_2 \rightarrow c p_1 + d p_2$$

Thus there are 16 - 3 - 4 = 9 effective parameters in the polynomials. For the quotient it is still the case that $h^{1,1} = 1$, and $\chi = -144/9 = -16$ so $h^{2,1} = 9$, confirming our counting of parameters.

Consider now the specialisation of the defining equations with $C_{\alpha} = B_{\alpha}$ and $D_{\alpha} = A_{\alpha}$

$$p_1 = \sum_{\alpha=0}^{3} \left(A_{\alpha} \mathcal{X}_{\alpha} + B_{\alpha} \mathcal{Y}_{\alpha} \right), \quad p_2 = \sum_{\alpha=0}^{3} \left(B_{\alpha} \mathcal{X}_{\alpha} + A_{\alpha} \mathcal{Y}_{\alpha} \right)$$
(3.12)

These equations are covariant under the \mathbb{Z}_2 -generator

 $V: x_i \leftrightarrow y_i; p_1 \leftrightarrow p_2$

We have checked that the polynomials (3.12) are transverse. Now the counting is that there are 8 coefficients, a one-parameter family of coordinate redefinitions $x_i \to \alpha x_i + \beta y_i$, $y_i \to \beta x_i + \alpha y_i$ and a two-parameter family of redefinitions of the defining equations $p_1 \to ap_1 + bp_2$, $p_2 \to bp_1 + ap_2$. Thus there are now 8 - 1 - 2 = 5 parameters in the defining equations. To check for fixed points, we note that if $S^k T^l V$ has fixed points then so has V. The fixed points of V, in the embedding space, are of the form (x_i, x_j) and $(x_i, -x_j)$ and these correspond to two disjoint elliptic curves

$$\mathcal{E}_{\pm}$$
 : $\sum_{\alpha} (A_{\alpha} \pm B_{\alpha}) \mathcal{X}_{\alpha}(x, x) = 0$

The resolution of the resulting singularities is almost identical to that in the last section, except that here we have the A_1 surface singularity, $\mathbb{C}^2/\mathbb{Z}_2$. In this case, the exceptional divisor has only a single component, so we get only one new divisor class from each curve. Once again, the Euler number does not change, so $h^{1,1} = 1 + 2 = 3$ and $h^{2,1} = 5 + 2 = 7$.

We summarise with a table the manifolds that we have obtained from quotients of the manifold $X^{1,73} = \mathbb{P}^5[3,3]$.

Hodge numbers	(1,9)	(1, 25)	(3,7)	(3, 15)	(3, 39)
Manifold	$X^{1,73}_{/\mathbb{Z}_3 \times \mathbb{Z}_3}$	$X^{1,73}_{/\mathbb{Z}_3}$	$\widehat{X^{1,73}_{/\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2}}$	$\widehat{X^{1,73}_{/\mathbb{Z}_3 \times \mathbb{Z}_2}}$	$\widehat{X^{1,73}_{/\mathbb{Z}_2}}$
Fundamental group	$\mathbb{Z}_3{ imes}\mathbb{Z}_3$	\mathbb{Z}_3	$\mathbb{Z}_3{ imes}\mathbb{Z}_3$	\mathbb{Z}_3	1

Table 3.12: Hodge numbers for quotients of $X^{1,73}$.

3.3.4 $X^{3,39}$; splitting the two cubics

The next manifold we consider is a split of both $\mathbb{P}^{5}[3,3]$ and its transpose.



Clearly, contracting the two \mathbb{P}^2 -rows brings us back to $\mathbb{P}^5[3,3]$ while contracting the \mathbb{P}^5 -row takes us to the transpose. Note also a parallel with our first split of the quintic: as in that case, the Euler number of the split manifold is half the Euler number of the original manifold. Let us take coordinates u_j and v_k for the two \mathbb{P}^2 's and write $(x_0, x_1, x_2, y_0, y_1, y_2)$ for the coordinates of the \mathbb{P}^5 . We denote by p_i the first three polynomials and by q_i the remaining three. With these conventions the polynomials have the form

$$p_i = \sum_{j,k} (A_{ijk} \, x_k + B_{ijk} \, y_k) \, u_j \, , \quad q_i = \sum_{j,k} (C_{ijk} \, x_k + D_{ijk} \, y_k) \, v_j$$

where the indices i, j, k are understood to take values in \mathbb{Z}_3 . We take the equations to be covariant under the action of a generator

$$S: (x_i, y_j) \to (x_{i+1}, y_{j+1}), (u_i, v_j) \to (u_{i+1}, v_{j+1}); (p_i, q_j) \to (p_{i+1}, q_{j+1})$$

Covariance of the polynomials requires the relations $A_{ijk} = A_{i-k,j-k,0}$ with analogous relations for B, C and D. By setting k = 2(i + j + m) and writing $a_{mn} = A_{-m+n,m+n,0}$ we bring the polynomials to the form

$$p_{i} = \sum_{j,m} (a_{i-j,m} x_{2(i+j+m)} + b_{i-j,m} y_{2(i+j+m)}) u_{j}$$

$$q_{i} = \sum_{j,m} (c_{i-j,m} x_{2(i+j+m)} + d_{i-j,m} y_{2(i+j+m)}) v_{j}$$
(3.13)

The parameter count is that there $4 \times 3 \times 3 = 36$ coefficients and from this we need to subtract the number of degrees of freedom in making changes of coordinates that preserve the action of S, and the number of degrees of freedom in redefining the polynomials in a way that preserves the symmetry. Neglecting an overall change of scale, there is a two parameter freedom in changing $u_j \to \sum_k g_k u_{j+k}$ and similarly for v. Again neglecting an overall scale, there is an 11 parameter freedom to redefine the coordinates x and y

$$x_i \rightarrow \sum_k (\alpha_k x_{i+k} + \beta_k y_{i+k}) , \quad y_i \rightarrow \sum_k (\gamma_k x_{i+k} + \delta_k y_{i+k})$$

(in this context the δ_k denote parameters, rather than the Kronecker symbol). Finally, there is a three-parameter freedom to redefine $p_i \to \sum_k \kappa_k p_{i+k}$, and similarly for the q_i , where we do permit a scaling of the polynomials. The count is that we have a total of 36 - 21 = 15 free parameters.

It is straightforward to show that the polynomials (3.13) are fixed point free and transverse. Hence we have checked the existence of the quotient

$$\begin{array}{cccccc} \mathbb{P}^2 & \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbb{P}^2 & \\ \mathbb{P}^5 & \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix} \right]_{\mathbb{Z}_3}^{3,15}$$

For the quotient, $h^{1,1} = 3$ and $\chi = -24$, so $h^{2,1} = 15$, confirming our parameter count.

If we now seek to impose the second symmetry

$$T: (x_i, y_j) \rightarrow (\zeta^i x_i, \zeta^j y_j), (u_i, v_j) \rightarrow (\zeta^i u_i, \zeta^j v_j); (p_i, q_j) \rightarrow (\zeta^{2i} p_i, \zeta^{2j} q_j)$$

where again ζ is a non-trivial cube root of unity, then we must take $a_{l,m} = 0$ for $m \neq 0$ and

similarly for b, c and d. Dropping the second index on the coefficients, we have

$$p_{i} = \sum_{j} (a_{i-j} x_{2(i+j)} + b_{i-j} y_{2(i+j)}) u_{j}$$

$$q_{i} = \sum_{j} (c_{i-j} x_{2(i+j)} + d_{i-j} y_{2(i+j)}) v_{j}$$
(3.14)

In contradistinction to the quintic case, these polynomials are fixed point free and transverse for general values of the coefficients. Now there are 12 coefficients, a three-parameter family of coordinate redefinitions up to scaling

$$x_i \rightarrow \alpha x_i + \beta y_i \quad , \quad y_i \rightarrow \gamma x_i + \delta y_i$$

and two rescalings, $p_i \to \tilde{\alpha} p_i$ and $q_i \to \tilde{\beta} q_i$ of the polynomials. Thus the polynomials (3.14) contain 12-3-2=7 effective parameters. The Euler number is now -24/3=-8 and $h^{1,1}=3$, as previously, so $h^{2,1}=7$, in agreement with our counting of parameters. A convenient way to represent the 7 parameters is to demand that $c_j = b_j$ and that $a_0 = 1$ and $b_0 = 0$, leaving the two free components of a_j , the two free components of b_j and the three d_j .

We have thus checked the existence of the $\mathbb{Z}_3 \times \mathbb{Z}_3$ quotient

$$\mathbb{P}^{2} \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 \\ \mathbb{P}^{2} & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbb{P}^{5} \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix} \Big|_{\mathbb{Z}_{3} \times \mathbb{Z}}$$

Another possibility is to impose, instead of the second \mathbb{Z}_3 symmetry, a \mathbb{Z}_2 symmetry, generated by

$$U: u \leftrightarrow v, x_i \leftrightarrow y_i$$

Clearly this must act on the polynomials as $p_i \leftrightarrow q_i$, which forces them to take the form

$$p_{i} = \sum_{j,m} \left(a_{i-j,m} \, x_{2(i+j+m)} + b_{i-j,m} \, y_{2(i+j+m)} \right) \, u_{j}$$

$$q_{i} = \sum_{j,m} \left(b_{i-j,m} \, x_{2(i+j+m)} + a_{i-j,m} \, y_{2(i+j+m)} \right) \, v_{j}$$
(3.15)

Again S acts without fixed points, but U fixes two curves in the manifold. To see this, note that fixed points of U in the ambient space are those for which $y_i = \lambda x_i$, with $\lambda = \pm 1$, and $v_i = u_i$. At these points, we have $q_i = \lambda p_i$, and

$$p_i = \sum_{j,m} (a_{i-j,m} + \lambda b_{i-j,m}) x_{2(i+j+m)} u_j$$

We see then that the two fixed curves \mathcal{E}_\pm are in fact CICY's corresponding to

$$\mathbb{P}^2 \begin{bmatrix} 1 & 1 & 1 \\ \mathbb{P}^2 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and thus elliptic curves. We already know how to resolve such singularities, and calculate the Hodge numbers of the new space. The generator U identifies the two ambient \mathbb{P}^{2} 's, so the embedding spaces now contribute only 2 to $h^{1,1}$. The blow-up of each curve gives another divisor class, and once again, the resolution process does not change the Euler number. The resolved orbifold therefore has fundamental group \mathbb{Z}_3 , $\chi = -12$, and Hodge numbers (4, 10).

We can also obtain $h^{2,1} = 10$ more directly. There are 18 parameters in the polynomials, but

we are free to make the following coordinate changes consistent with the action of the group:

$$\begin{aligned} x_i \to \sum_j (\alpha_j \, x_{i+j} + \beta_j \, y_{i+j}) \,, \ y_i \to \sum_j (\beta_j \, x_{i+j} + \alpha_j \, y_{i+j}) \\ u_i \to \sum_j \gamma_j \, u_{i+j} \,, \ v_i \to \sum_j \gamma_j \, v_{i+j} \end{aligned}$$

We neglect overall scaling, so this is a 7 parameter freedom. We can also redefine the polynomials, in a way consistent with the symmetries:

$$p_i \to \sum_j \kappa_j \, p_{i+j} \;, \;\; q_i \to \sum_j \kappa_j \, q_{i+j}$$

Here the overall scale is significant, so we have another 3 parameters, giving us 10 in total. Subtracting this from the 18 parameters in the original equations leaves 8. Since the resolution process does not change the Euler number, it must introduce two new (2, 1) forms, to cancel the contribution of the new (1, 1) forms, so this gives an independent determination of $h^{2,1} = 10$.

We summarise the manifolds found in this section in the following table:

Hodge numbers	(3,7)	(3, 15)	(4, 10)
Manifold	$X^{3,39}_{/\mathbb{Z}_3 imes\mathbb{Z}_3}$	$X^{3,39}_{/\mathbb{Z}_3}$	$\widehat{X^{3,39}_{/\mathbb{Z}_3 \times \mathbb{Z}_2}}$
Fundamental group	$\mathbb{Z}_3{ imes}\mathbb{Z}_3$	\mathbb{Z}_3	\mathbb{Z}_3

Table 3.13: Hodge numbers of smooth quotients of $X^{3,39}$.

3.3.5 $X^{9,27}$; a second split of both cubics

Starting with $X^{3,39}$, we can split each bilinear polynomial by introducing a \mathbb{P}^1 , and then contract the two \mathbb{P}^2 's, to arrive at



We take coordinates s_{ia} for the first three \mathbb{P}^1 's, t_{ia} for the remaining three, and (x_i, y_j) for \mathbb{P}^5 with $i, j \in \mathbb{Z}_3$ and $a \in \mathbb{Z}_2$. We first impose the \mathbb{Z}_3 symmetry generated by

$$S: \ s_{ia} \to s_{i+1,a} \ , \ \ t_{ia} \to t_{i+1,a} \ , \ \ x_i \to x_{i+1} \ , \ \ y_i \to y_{i+1} \ ; \ \ p_i \to p_{i+1} \ , \ \ q_i \to q_{i+1}$$

where the p_i and q_i are the bilinear polynomials, and the two trilinear polynomials, which we will label P and Q respectively, are invariant. By appropriate choice of coordinates, we can bring the polynomials to the form

$$P = \frac{1}{3}(m_{000} + m_{111}) , \quad Q = \frac{1}{3}(n_{000} + n_{111})$$

$$p_i = x_i s_{i0} + \sum_j (A_{i-j} x_j + B_{i-j} y_j) s_{i1} , \quad q_i = y_i t_{i0} + \sum_j (C_{i-j} x_j + D_{i-j} y_j) t_{i1}$$
(3.17)

where $m_{abc} = \sum_{i} s_{ia} s_{i+1,b} s_{i+2,c}$ and $n_{abc} = \sum_{i} t_{ia} t_{i+1,b} t_{i+2,c}$. There is a scaling $(x_i, y_j) \to (x_i, \mu y_j)$, $q_i \to \mu q_i$ that preserves the form of the polynomials. The effect of the scaling is to change the coefficients $B_k \to \mu B_k$ and $C_k \to \mu^{-1} C_k$, with A_k and D_k remaining unchanged. This freedom can be used to set $B_0 = C_0$, for example, so there are 11 free parameters in the equations. For generic coefficients the polynomials are transverse, though this is easier to check for the \mathbb{Z}_6 -invariant subfamily that we will come to shortly.

The solution set has no fixed points under S. The fixed point analysis goes as follows. Fixed points of S are of the form

$$s_{ia} = s_a , \ t_{ia} = t_a , \ (x_i, y_j) = (\lambda \zeta^i, \mu \zeta^j)$$

where $\zeta^3 = 1$, and (λ, μ) parametrise a \mathbb{P}^1 . The equations P = 0 and Q = 0 become cubics in s_a and t_a respectively and restrict s_a and t_a to discrete values. The equations $p_i = q_i = 0$ then become two independent equations in the variables (λ, μ) , which in general will have no solution for (λ, μ) a point of a \mathbb{P}^1 .

As in the previous example we may pass to an extended representation for which the hyperplane sections of the embedding spaces generate the second cohomology, and this enables us to calculate the Hodge numbers of the quotient. To this end consider the representation

corresponding to the diagram



We extend the action of S to the new \mathbb{P}^2 's

 $S: u_i \rightarrow u_{i+1}, v_i \rightarrow v_{i+1}$

By applying twice the process of the previous subsection we may, without loss of generality, take equations for the extended manifold to be of the form

$$P_{i} = u_{i} s_{i0} + u_{i+1} s_{i1} , \quad Q_{i} = v_{i} t_{i0} + v_{i+1} t_{i1}$$

$$p_{i} = x_{i} s_{i0} + \sum_{j} (A_{i-j} x_{j} + B_{i-j} y_{j}) s_{i1} , \quad q_{i} = y_{i} t_{i0} + \sum_{j} (C_{i-j} x_{j} + D_{i-j} y_{j}) t_{i1}$$
(3.19)

where we again take $B_0 = C_0$. These polynomials are also transverse and fixed point free, though, again, it is easiest to check the transversality for the \mathbb{Z}_6 invariant subfamily that follows. The six \mathbb{P}^1 's form two orbits under the action of S so it is now clear that, for the quotient, $h^{1,1} = 5$. For the Euler number we have $\chi = -12$ hence $h^{2,1} = 11$, which agrees with our parameter count. Thus we have found a quotient

Let us now impose more symmetry, and consider the subfamily of equation (3.20) that is invariant under the \mathbb{Z}_2 generator

$$U: x_i \leftrightarrow y_i, s_{ia} \leftrightarrow t_{ia}, u_i \leftrightarrow v_i$$

The polynomials are now

$$P_{i} = u_{i} s_{i0} + u_{i+1} s_{i1} , \quad Q_{i} = v_{i} t_{i0} + v_{i+1} t_{i1}$$

$$p_{i} = x_{i} s_{i0} + \sum_{j} (A_{i-j} x_{j} + B_{i-j} y_{j}) s_{i1} , \quad q_{i} = y_{i} t_{i0} + \sum_{j} (B_{i-j} x_{j} + A_{i-j} y_{j}) t_{i1}$$
(3.20)

which are transverse for generic choices of coefficients. The constraint between B_0 and C_0 is now irrelevant, so the equations contain 6 parameters. The generator U does not act freely: the fixed point set consists of two disjoint curves, \mathcal{E}_{\pm} , for which $y_i = \lambda x_i$, with $\lambda = \pm 1$, $t_{ia} = s_{ia}$ and $v_i = u_i$, and with the coordinates subject to the independent constraints $p_i = P_i = 0$. The curves \mathcal{E}_{\pm} correspond to the one dimensional CICY's

In this way we see that \mathcal{E}_{\pm} are again elliptic curves, and can be resolved as in §3.3.3. Since the Euler numbers of the \mathcal{E}_{\pm} are zero the Euler number of the resolved quotient is simply -36/6 = -6. The generators S and U, between them, identify the six \mathbb{P}^1 's and the two \mathbb{P}^2 's, so the embedding spaces contribute 3 to $h^{1,1}$. Two additional (1, 1)-forms and two additional (2, 1)-forms arise from the resolution of the fixed curves. In this way we find a manifold with fundamental group \mathbb{Z}_3 , $\chi = -6$ and $(h^{1,1}, h^{2,1}) = (5, 8)$. We summarise the new manifolds found in this section in a short table

Hodge numbers	(5, 11)	(5, 8)
Manifold	$X^{9,27}_{/\mathbb{Z}_3}$	$\widehat{X^{9,27}_{/\mathbb{Z}_3 \times \mathbb{Z}_2}}$
Fundamental group	\mathbb{Z}_3	\mathbb{Z}_3

Table 3.14: Hodge numbers of manifolds constructed from quotients of $X^{9,27}$, including a new 'three generation' manifold, with Euler number -6.

3.3.6 $X^{5,50}$; a one-sided split of $X^{1,73}$

In this section we started with the manifold $X^{1,73}$, defined by two cubic polynomials in \mathbb{P}^5 , and so far we have considered only splits which treat the two polynomials symmetrically. If, instead, we leave one cubic intact and split the other one twice (then perform an ineffective \mathbb{P}^2 contraction, as in the last subsection), we obtain



We take coordinates (x_i, y_j) for the \mathbb{P}^5 and $t_{i,a}$ for the \mathbb{P}^1 's, where $a \in \mathbb{Z}_2$, and define a \mathbb{Z}_3 action generated by

$$S: x_i \to x_{i+1} , y_j \to y_{j+1} , t_{i,a} \to t_{i+1,a}$$

We will denote the trilinear equation by r, the three bilinear equations by p_i , and the cubic by q. We demand that q and r be invariant under the action of S, and that the p_i transform as $p_i \rightarrow p_{i+1}$. If, as previously, we define the S-invariant quantities $m_{abc} = \sum_i t_{i,a} t_{i+1,b} t_{i+2,c}$, then the polynomials take the form

$$p_{i} = \sum_{j} \left((A_{j} x_{i+j} + B_{j} y_{i+j}) t_{i,0} + (C_{j} x_{i+j} + D_{j} y_{i+j}) t_{i,1} \right)$$
$$r = \sum_{ijk} \left(G_{jk} x_{i} x_{i+j} x_{i+k} + H_{jk} x_{i} x_{i+j} y_{i+k} + J_{jk} x_{i} y_{i+j} y_{i+k} + K_{jk} y_{i} y_{i+j} y_{i+k} \right)$$
(3.21)

$$q = E_0 m_{000} + E_1 m_{100} + E_2 m_{110} + E_3 m_{111}$$

We have checked that these polynomials are transverse, and that the action of S is fixed-pointfree. To find the Hodge numbers of the quotient, we consider the extended representation in which all the (1, 1)-forms are all pullbacks from the ambient spaces:



Here we take coordinates u_i on the \mathbb{P}^2 , and extend the \mathbb{Z}_3 action to include $u_i \to u_{i+1}$. Thus the action identifies the three \mathbb{P}^1 's, but only acts internally on the other spaces, leaving the quotient with $h^{1,1} = 3$. Since the Euler number will be $\chi = -90/3 = -30$, it must be that $h^{2,1} = 18$. We can as usual check the value of $h^{2,1}$ by counting coefficients, which we do first for the manifold before imposing the symmetry.

Consider the first configuration. The most general polynomial q has $2^3 = 8$ coefficients, each p_i has $2 \times 6 = 12$, and r is characterised by a symmetric rank three SO(6) tensor, which has $6 \times 7 \times 8/1 \times 2 \times 3 = 56$ components, giving altogether 100 coefficients. A general change of coordinates has $3 \times 3 + 35 = 44$ parameters, neglecting an overall scaling in each space, and we can also rescale each of the five polynomials. Finally, there is one rather non-trivial redefinition

of the polynomials. The general form of the p_i is

$$p_i = \mathcal{A}_i(x, y) t_{i0} + \mathcal{B}_i(x, y) t_{i1}$$

where \mathcal{A} and \mathcal{B} are linear, so $p_i = 0$ then gives

$$t_{i1} = -\frac{\mathcal{A}_i(x,y)}{\mathcal{B}_i(x,y)} t_{i0}$$

If we substitute this into q = 0 (and multiply by $\mathcal{B}_0 \mathcal{B}_1 \mathcal{B}_2$) we obtain a cubic constraint on x, y, which can be used to eliminate a single coefficient from the polynomial r. Our count of independent coefficients is therefore 100 - 50 = 50, which is indeed the value of $h^{2,1}$.

Repeating the above analysis for the \mathbb{Z}_3 -symmetric case has only one subtlety: G_{jk} and K_{jk} have only four independent components each, while H_{jk} and J_{jk} have six. This gives a total of 36 coefficients in (3.21). Coordinate changes consistent with the symmetry are

$$x_{i} \to \sum_{j} (\alpha_{j} x_{i+j} + \beta_{j} y_{i+j}) , \quad y_{i} \to \sum_{j} (\gamma_{j} x_{i+j} + \delta_{j} y_{i+j})$$

$$(t_{i0}, t_{i1}) \to (\mu t_{i0} + \nu t_{i1}, \rho t_{i0} + \sigma t_{i1})$$
(3.22)

Neglecting overall scaling there are 14 parameters here. We can also rescale all the p_i by the same factor, and rescale r and q, giving three more. The complicated redefinition discussed above eliminates one more coefficient, leaving 36 - 18 = 18, which agrees with our previously-determined value of $h^{2,1}$. In summary, the quotient manifold has fundamental group \mathbb{Z}_3 and Hodge numbers $(h^{1,1}, h^{2,1}) = (3, 18)$.

3.3.7 $X^{8,44}$

We now turn to a manifold which will interest us greatly in Chapter 5. Here it provides, as well as several new free quotients, another example of resolving the fixed points of a group action, as well as our only example of a freely-acting group which acts non-linearly on the ambient space.



Again take coordinates s_{ia} for the first three \mathbb{P}^1 's and t_{ia} for the remaining three, with $i \in \mathbb{Z}_3$ and $a \in \mathbb{Z}_2$. We define an action of \mathbb{Z}_3 generated by

$$S: s_{ia} \to s_{i+1,a}, \quad t_{ia} \to t_{i+1,a}$$

In order to write polynomials invariant under S it is again useful to consider the invariant quantities

$$m_{abc} = \sum_{j} s_{ja} \, s_{j+1,b} \, s_{j+2,c} \quad , \quad n_{abc} = \sum_{j} t_{ja} \, t_{j+1,b} \, t_{j+2,c}$$

and

$$l_{abcdef} = \sum_{j} s_{ja} \, s_{j+1,b} \, s_{j+2,c} \, t_{jd} \, t_{j+1,e} \, t_{j+2,f}$$

The S-invariant polynomials can then be written as

$$p = \sum_{abc} A_{abc} m_{abc} , \quad q = \sum_{abc} B_{abc} n_{abc} , \quad r = \sum_{abcdef} C_{abcdef} l_{abcdef}$$
(3.24)

Note that the individual terms obey the symmetries $m_{abc} = m_{cab}$, $n_{abc} = n_{cab}$, $l_{abcdef} = l_{cabfde}$, and therefore we may require the same symmetry in the coefficients. The polynomials are transverse and the action of S is fixed point free, but this is easier to check for the more symmetric polynomials which follow.

We now impose also a \mathbb{Z}_2 symmetry generated by

$$U: s_{ia} \rightarrow (-1)^a s_{ia}, t_{ia} \rightarrow (-1)^a t_{ia}$$

The polynomials (3.24) are also invariant under U if the indices of each term sum to 0 in \mathbb{Z}_2 . By choosing coordinates suitably we may take the first two polynomials to be

$$p = \frac{1}{3}m_{000} + m_{110}$$
 and $q = \frac{1}{3}n_{000} + n_{110}$ (3.25)

We shall want to discuss the general form of the third polynomial r in some detail, but a simple choice suffices to show that the polynomials can be chosen to be transverse and fixed point free. Such a choice is

$$r = \frac{1}{9}m_{000}n_{000} + A_1m_{100}n_{100} + A_2m_{111}n_{100} + A_3m_{100}n_{111} + A_4m_{111}n_{111}$$
(3.26)

To show that the polynomials are fixed point free, it suffices to check the fixed points of S and of U. A fixed point of S is such that $s_{ja} = s_a$ and $t_{ja} = t_a$, independent of j. The polynomials p and q impose the conditions

$$s_0(s_0^2 + 3s_1^2) = 0$$
 and $t_0(t_0^2 + 3t_1^2) = 0$

and for generic values of the coefficients, (3.26) does not vanish at the nine solutions to these equations. Fixed points of U, in the embedding space, consist of the 64 points with s_{ja} and t_{ja} given by independent choices of $\{(0, 1), (1, 0)\}$. For each of the 8 choices of fixed points for s_{ja} there is precisely one of the polynomials m_{abc} that is nonzero and similarly there is precisely one of the polynomials n_{abc} that is nonzero for the fixed points of t_{ja} . Our three polynomials then cannot vanish if all the coefficients shown are nonzero.

We have shown that the $\mathbb{Z}_3 \times \mathbb{Z}_2$ quotient exists. In this representation of the space we see only six of the eight independent cohomology classes of H^2 among the embedding spaces. As in previous examples we may here also pass to a representation for which all of H^2 is generated by the hyperplane classes of the ambient spaces. Such a representation is given by the transpose of the configuration in (3.16)

$$X^{8,44} = \begin{bmatrix} \mathbb{P}^{1} \\ \mathbb{P}^{2} \\ \mathbb{P}^{2$$



The polynomials p and q are each replaced here by three equations involving the coordinates of the additional \mathbb{P}^2 's

$$p_{i} = s_{i0}(u_{i+1} + u_{i+2}) + s_{i1}(u_{i+1} - u_{i+2})$$

$$q_{i} = t_{i0}(u_{i+1} + u_{i+2}) + t_{i1}(u_{i+1} - u_{i+2})$$
(3.27)

$$q_i = t_{i0}(v_{i+1} + v_{i+2}) + t_{i1}(v_{i+1} - v_{i+2})$$

It is convenient to think of these as matrix equations $p_i = \sum_j M_{ij}(s)u_j$ and $q_i = \sum_j M_{ij}(t)v_j$ with

$$M_{ij}(s) = \begin{pmatrix} 0 & s_{00} + s_{01} & s_{00} - s_{01} \\ s_{10} - s_{11} & 0 & s_{10} + s_{11} \\ s_{20} + s_{21} & s_{20} - s_{21} & 0 \end{pmatrix}$$
(3.28)

Eliminating the new coordinates from equation (3.27) yields det $M(s) = \det M(t) = 0$ and returns us to equation (3.25). The fact that the Euler numbers for the two configurations are the same shows that eliminating the new coordinates does not introduce any nodes, so that the manifolds are indeed isomorphic. We have taken the equations p_i and q_i above to be covariant under S and it is straightforward to take the S-quotient since this just identifies the \mathbb{P}^1 's in two groups of three while leaving the \mathbb{P}^2 's invariant. It follows that $X^{8,44}/S$ has Hodge numbers $(h^{1,1}, h^{2,1}) = (4, 16)$.

It is, at first sight, puzzling how to describe the action of U in the extended representation. For while the polynomials p, q and r are invariant under U it is easy to see that there is no *linear* transformation of the coordinates u_i and v_i that renders the polynomials p_i and q_i covariant.

Consider the equations $p_i = \sum_j M_{ij}(s)u_j = 0$. Clearly the matrix $M_{ij}(s)$ cannot have rank three on X. It is immediate, from the explicit form of (3.28), that $M_{ij}(s)$ can never have rank one. Thus the matrix always has rank two. Hence, given three points (s_{i0}, s_{i1}) , the equations $p_i = 0$ determine a unique point $u_i \in \mathbb{P}^2$. Conversely, given a point $u_i \in \mathbb{P}^2$, that is not one of the three special points $u_i = (1, 0, 0), (0, 1, 0), (0, 0, 1)$, and the general form of M_{ij} from (3.28), we determine a unique point $(s_{i0}, s_{i1}) \in (\mathbb{P}^1)^3$. For the special points, however, an entire \mathbb{P}^1 is left undetermined by this process. For $u_i = (1, 0, 0)$, for example (s_{00}, s_{01}) is undetermined. Thus the surface determined by the equations $p_i = 0$ is the del Pezzo surface dP_6 , given by a \mathbb{P}_2 blown up at the three points we have specified.

We know that, given the values of (s_{i0}, s_{i1}) for all *i*, there is a unique solution for u_i up to scale, so, since we know the action of U on the s_{ia} , we may deduce the action on the u_i . To this end we write down an explicit solution and act on it with U:

$$\begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -(s_{10} + s_{11})(s_{20} - s_{21}) \\ (s_{10} + s_{11})(s_{20} + s_{21}) \\ -(s_{10} - s_{11})(s_{20} - s_{21}) \end{pmatrix} \xrightarrow{U} \begin{pmatrix} -(s_{10} - s_{11})(s_{20} + s_{21}) \\ (s_{10} - s_{11})(s_{20} - s_{21}) \\ -(s_{10} + s_{11})(s_{20} + s_{21}) \end{pmatrix}$$

$$= \frac{1}{(s_{10} + s_{11})(s_{20} - s_{21})} \begin{pmatrix} u_1 u_2 \\ u_0 u_2 \\ u_0 u_1 \end{pmatrix}$$

$$(3.29)$$

Since an overall scale is irrelevant, we see that the action of U is related to a Cremona transformation of \mathbb{P}^2 , given by $u_i \to u_{i+1}u_{i+2}$, or equivalently $u_i \to 1/u_i$. This is defined everywhere except at the three special points, and squares to the identity away from the special lines $u_0 = 0$, $u_1 = 0$, $u_2 = 0$. Furthermore it maps the special lines to the special points. There is a nice way to understand this in terms of the del Pezzo surface. Since this is just the blow up of \mathbb{P}^2 at the three special points, it contains six distinguished lines – the three special lines $u_i = 0$, just described, and the three exceptional lines arising as the blow ups – which form a hexagon. The actions of S and U can be thought of as rotations of this hexagon of order 3 and 2 respectively. The reason the action of U cannot be realised as an isomorphism of \mathbb{P}^2 is that the projection blows down three sides of the hexagon to points. The reader may worry that the



Figure 3.3: The surface dP_6 is obtained by blowing up three points in \mathbb{P}^2 . The three exceptional lines together with the three lines that passed through pairs of the points that were blown up form a hexagon. The effect of U in dP_6 is to make a rotation of order 2 of the hexagon. The effect of U in \mathbb{P}^2 is to blow up the special points and blow down the special lines and so 'maps' the triangle on the left to the triangle on the right, and cannot be realised as an isomorphism of \mathbb{P}^2 .

above map is ill-defined on our manifold just as on \mathbb{P}^2 , but this is not the case. Consider the point $(1,0,0) \in \mathbb{P}^2$, at which the Cremona transformation is not defined. Inspection of (3.28) shows that this point occurs when $s_{11} = s_{10}$ and $s_{21} = -s_{20}$. The value of s_{01}/s_{00} is however undetermined, as we have just seen. The second line of equation (3.29) shows that this \mathbb{P}^1 is mapped bijectively to the line $u_1 = 0$. The same analysis applies to the other two special points. The action of U is thus well-defined on the del Pezzo surface, and hence also on our manifold, which is embedded in the Cartesian product of two copies of this surface.

To determine $h^{1,1}$ for the quotient manifolds we need to calculate the action of U on the pullback of the hyperplane class of \mathbb{P}^2 . We will see how to do this in Chapter 5, but we can alternatively determine $h^{2,1}$ for the quotients involving U by a careful counting of parameters in the polynomials, and deduce $h^{1,1}$ from the Euler number. Let us return to the polynomials (3.24) that define $X^{8,44}$. We may choose coordinates s_{ja} and t_{ja} such that the polynomials P and Q take the form shown in equation (3.25). Fixing the form of these polynomials does not completely exhaust the freedom to make redefinitions. For the s_{ja} we may consider the $SL(2, \mathbb{C})$ transformations

$$\begin{pmatrix} s_{j0} \\ s_{j1} \end{pmatrix} \rightarrow \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \begin{pmatrix} s_{j0} \\ s_{j1} \end{pmatrix}$$

There are a total of 9 parameters in these transformations. There are 8 monomials $s_{1a} s_{2b} s_{3c}$,

and an overall scale is irrelevant, so fixing the form of P imposes 7 constraints. We are left with a two parameter freedom to make redefinitions of the s_{ja} (this corresponds to the fact that dP_6 is toric, which we will discuss later). If we restrict to $SL(2, \mathbb{C})$ transformations that are infinitesimally close to the identity then we can check this quite explicitly. The transformations that preserve P take the form

$$\begin{pmatrix} s_{j0} \\ s_{j1} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \epsilon\beta_j \\ \epsilon\beta_j & 1 \end{pmatrix} \begin{pmatrix} s_{j0} \\ s_{j1} \end{pmatrix}$$
(3.30)

with the β_j subject to the constraint $\sum_j \beta_j = 0$.

Consider now the monomials

$$s_{1a} \, s_{2b} \, s_{3c} \, t_{1d} \, t_{2e} \, t_{3f}$$

that can be included in r. There are 64 of these, of which 32 are even under U and 32 odd. Terms of the form

$$m_{011} t_{1d} t_{2e} t_{3f}$$
 and $s_{1a} s_{2b} s_{3c} n_{011}$

of which there are 7 that are even and 8 that are odd, may be eliminated from r through the equations P = 0 and Q = 0. There is also the freedom to redefine r by an overall scale which we use to, say, set the coefficient of the monomial $m_{000}n_{000}$ to 1/9. We still need to dispose of a two parameter freedom to redefine the s_{ja} and another two parameter freedom to redefine the t_{ja} . Consider the effect of making the redefinition equation (3.30) in the leading monomial $m_{000}n_{000}$

$$m_{000} n_{000} \rightarrow m_{000} n_{000} + \epsilon (\beta_0 s_{01} s_{10} s_{20} + \beta_1 s_{00} s_{11} s_{20} + \beta_2 s_{00} s_{10} s_{21}) n_{000}$$

We may use this freedom, together with the corresponding freedom for the t_{ja} , to eliminate the 4 odd monomials

 $s_{01} s_{10} s_{20} n_{000}$, $s_{00} s_{11} s_{20} n_{000}$, $m_{000} t_{00} t_{10} t_{21}$, $m_{000} t_{00} t_{11} t_{20}$

The counting is that we have 32-7-1 = 24 free parameters in r associated with even monomials and 32-8-4 = 20 associated with odd monomials. The total count is 44 which agrees with the value of $h^{2,1}$. Since we have a complete description of the parameter space we know that the even parameters are the parameters of the quotient $X^{8,44}/U$ which therefore has Hodge numbers $(h^{1,1}, h^{2,1}) = (6, 24)$. A more formal argument proceeds via the Lefschetz fixed point theorem with the same result.

We turn now to the S-quotient, for which we start with the following complete list of the labels for the polynomials l_{abcdef} .

Even polynomials	Odd polynomials		
(000 000) (001 001) (011 000) (111 001) (000 011) (001 010) (011 011) (111 111) (001 100) (011 101) (001 111) (011 110)	$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$		

We take r to have leading term $\frac{1}{3}l_{000000} = \frac{1}{9}m_{000}n_{000}$, which removes one parameter, and we

should also remove 3 even parameters owing to the freedom to eliminate terms of the form

 $m_{011} n_{000} = 3l_{011000}$, $m_{000} n_{011} = 3l_{000011}$ and $m_{011} n_{011} = l_{011011} + l_{011101} + l_{011110}$

by means of P and Q. Elimination of terms of the form

 $m_{011} n_{001}$, $m_{011} n_{111}$, $m_{001} n_{011}$, $m_{111} n_{011}$

removes 4 odd parameters. Thus we are left with 12 - 3 - 1 = 8 even parameters and 12 - 4 = 8 odd parameters. The total number is 16 which is indeed the value of $h^{2,1}$ for $X^{8,44}/S$. We learn also that $h^{2,1} = 8$ for the quotient $X^{8,44}/U \times S$ and hence $h^{1,1} = 2$. Again we could deduce this more formally by applying the Lefschetz fixed point theorem.

Instead of the \mathbb{Z}_2 we can try to impose a second \mathbb{Z}_3 generated by

$$T: s_{ia} \to \zeta^a s_{ia} ; t_{ia} \to \zeta^a t_{ia}$$

with $\zeta = \exp 2\pi i/3$. The most general polynomials invariant under both S and T are given by

$$p = A_0 m_{000} + A_1 m_{111}$$
, $q = B_0 n_{000} + B_1 n_{112}$

$$+ C_7 l_{111000} + C_8 l_{000111} + C_9 l_{111111}$$

The only coordinate changes which are allowed by the group action are $(s_{i0}, s_{i1}) \rightarrow (\alpha_s s_{i0}, \beta_s s_{i1})$ and $(t_{i0}, t_{i1}) \rightarrow (\alpha_t t_{i0}, \beta_t t_{i1})$. We can use this along with the freedom to rescale p and q to set $A_a = B_a = \frac{1}{3}$, for a = 0, 1. We can also use the freedom to redefine r by multiples of p and q to eliminate C_7, C_8 and C_9 in favour of C_0 , and then rescale r so that $C_0 = \frac{1}{3}$. This leaves us with

$$p = \frac{1}{3}(m_{000} + m_{111})$$
, $q = \frac{1}{3}(m_{000} + m_{111})$

$$r = \frac{1}{3}l_{000000} + C_1 \, l_{100110} + C_2 \, l_{100101} + C_3 \, l_{100011} + C_4 \, l_{110100} + C_5 \, l_{110010} + C_6 \, l_{110001}$$

These are transverse, but the action of T has fixed points. The elements ST and S^2T , however, act freely, which is straightforward to show [38]. The fixed points in the ambient space are given by $(s_{i0}, s_{i1}), (t_{i0}, t_{i1}) \in \{(0, 1), (1, 0)\}$ for all i, corresponding to $8 \times 8 = 64$ points. However, if the same choice is made for all values of i for either s or t, the polynomials are easily seen not to vanish, and the remaining choices for (say) s are all equivalent under S to either $\{(1,0), (0,1), (0,1)\}$ or $\{(1,0), (1,0), (0,1)\}$. Therefore there are actually only $2 \times 6 = 12$ points to consider. At each of these points we have p = q = 0, but a moment's thought shows that in exactly half the cases, $r \neq 0$. Therefore the action of T on $X^{8,44}/S$ has six isolated fixed points. This leads to local singularities of the form $\mathbb{C}^3/\mathbb{Z}_3$ on $X^{8,44}/S \times T$, where the \mathbb{Z}_3 action is $(x_1, x_2, x_3) \rightarrow (\zeta x_1, \zeta x_2, \zeta x_3)$. This is a well-known space, given by blowing down the zero section of the bundle $\mathcal{O}_{\mathbb{P}^2}(-3)$, and we can therefore resolve the singular points simply by blowing them up.⁴ Each exceptional divisor is a copy of \mathbb{P}^2 , with Euler number 3, so the Euler number of the resulting manifold is $(-24-6)/3+6\times 3=8$, and it will have fundamental group \mathbb{Z}_3 . We expect $h^{2,1} = 6$, as that is the number of free parameters in the above polynomials, and

⁴Note that $\omega_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$, so the adjunction formula confirms that the blown-up space is Calabi-Yau.

since the T acts only internally on each ambient space we will get 4 (1, 1)-forms from these plus 6 from the blow ups of the fixed points, giving $h^{1,1} = 10$. Indeed this gives the correct Euler number.

Hodge numbers	(2, 8)	(4, 16)	(6, 24)	(10, 6)
Manifold	$X^{8,44}_{/\mathbb{Z}_3 \times \mathbb{Z}_2}$	$X^{8,44}_{/\mathbb{Z}_3}$	$X^{8,44}_{/\mathbb{Z}_2}$	$X^{\widehat{8,44}}_{/\mathbb{Z}_3 \times \mathbb{Z}_3}$
Fundamental group				\mathbb{Z}_3

We summarise with a table the manifolds that we have obtained from quotients of $X^{8,44}$.

Table 3.15: Hodge numbers for quotients of $X^{8,44}$.

3.3.8 The manifold $X^{19,19}$

Starting with the configuration matrix in (3.23), we can introduce one more \mathbb{P}^1 , and split the sextic polynomial, to obtain

$$X^{19,19} = \begin{array}{c} \mathbb{P}^{1} \begin{bmatrix} \mathbf{1} \ \mathbf{1} \ \mathbf{0} \ \mathbf{0} \\ \mathbb{P}^{1} \\ \mathbb{P}^{1}$$

10.10

We have already met the manifold $X^{19,19}$, in the guise of the split bicubic.⁵ Although we will not discuss it in detail here, we pause to note that it is rather remarkable, in that it admits free actions by many groups (on different loci in its moduli space, of course). A number of Abelian examples are described in [62]; later in this chapter we will present a free action by the quaternion group, and in Chapter 5 we will find a free action by a non-Abelian group of order 12.

3.4 Some manifolds with quaternionic symmetry

We will now present several of the manifolds discovered in [38] to admit free actions by the quaternion group, denoted here by \mathbb{H} .

3.4.1 The manifold $\mathbb{P}^7[2 \ 2 \ 2 \ 2]$

Consider first the configuration

$$X^{1,65} = \mathbb{P}^7[2 \ 2 \ 2 \ 2]$$

The manifold $X^{1,65} = \mathbb{P}^7[2\ 2\ 2\ 2]$ admits free actions by groups, both Abelian and non-Abelian, of order 32 [22, 56, 57], see also [39] for a brief review. Here we will be interested only in the

⁵To show that it is the same manifold, split each of the trilinear polynomials with a \mathbb{P}^2 , then contract the six original \mathbb{P}^1 's, and observe that the Euler number does not change.
order-8 group \mathbb{H} , the 'quaternion group':

$$\mathbb{H} = \left\{ \, 1, \, \, i, \, \, j, \, \, k, \, -1, \, -i, \, -j, \, -k \, \right\}$$

Denote the homogeneous coordinates of \mathbb{P}^7 by x_{α} , where the index α takes values in the group \mathbb{H} . We take the four quadrics to be of the form

$$p_{\alpha} = x_{\alpha}^{2} + x_{-\alpha}^{2} + a x_{\alpha j} x_{-\alpha j} + b (x_{\alpha i} x_{-\alpha k} + x_{-\alpha i} x_{\alpha k}) + c (x_{\alpha i} x_{\alpha k} + x_{-\alpha i} x_{-\alpha k})$$

and we see that $p_{-\alpha} = p_{\alpha}$. The symmetries $U_{\gamma}, \gamma \in \mathbb{H}$ act by

$$U_{\gamma}: x_{\alpha} \rightarrow x_{\gamma\alpha}; p_{\alpha} \rightarrow p_{\gamma\alpha}$$

We leave it unproven here that the given polynomials are transverse, and that the U_{γ} all act freely on the corresponding manifold.

3.4.2 $X^{5,37}$, a symmetrical split of $\mathbb{P}^{7}[2 \ 2 \ 2 \ 2]$

We can split the configuration above in a symmetrical-looking way to obtain another manifold that admits the quaternionic group as a freely acting symmetry. The split matrix and its diagram are given by





It is useful to denote by \mathbb{H}_+ the set of 'positive' unit quaternions

$$\mathbb{H}_{+} = \{1, i, j, k\}$$

and use these elements to label the four \mathbb{P}^1 's. The coordinates of the \mathbb{P}^1 labelled by $\sigma \in \mathbb{H}_+$ are taken to be $(s_{\sigma}, s_{-\sigma})$ and the polynomials 'connected' to this \mathbb{P}^1 are denoted by p_{σ} and q_{σ} . We also take coordinates x_{α} , $\alpha \in \mathbb{H}$ for the \mathbb{P}^7 as above.

Consider the polynomials

$$p_{\alpha} = s_{\alpha} x_{\alpha} + s_{-\alpha} x_{-\alpha}$$

$$q_{\alpha} = \sum_{\beta \in \mathbb{H}} a_{\beta} \left(s_{\alpha} x_{\alpha\beta} - s_{-\alpha} x_{-\alpha\beta} \right)$$
(3.32)

where the index α runs over \mathbb{H} . This is a harmless extension of the indexing on the polynomials, since $p_{-\alpha} = p_{\alpha}$ and $q_{-\alpha} = -q_{\alpha}$. For $\gamma \in \mathbb{H}$, these equations are covariant under the action

$$U_{\gamma}: x_{\alpha} \rightarrow x_{\gamma \alpha} , s_{\alpha} \rightarrow s_{\gamma \alpha}; p_{\alpha} \rightarrow p_{\gamma \alpha} , q_{\alpha} \rightarrow q_{\gamma \alpha}$$

To check that the action is fixed point free it is sufficient to check that U_{-1} acts without fixed points, since the elements $\pm i, \pm j, \pm k$ all square to -1. A fixed point of U_{-1} has the form

$$x_{-\alpha}^* = \lambda x_{\alpha}^*, \ (s_{\sigma}^*, s_{-\sigma}^*) = (1, s_{-\sigma}^*) \text{ with } \lambda^2 = (s_{-\sigma}^*)^2 = 1$$

with the x_{α}^* not all zero. Imposing the constraints $p_{\sigma} = q_{\sigma} = 0$ requires

$$(1 + \lambda s^*_{-\sigma}) x^*_{\sigma} = 0$$
 and $(1 - \lambda s^*_{-\sigma}) \sum_{\beta \in \mathbb{H}} a_{\beta} x^*_{\sigma\beta} = 0$

and these require, for generic choice of the coefficients a_{β} , that the x_{σ}^* all vanish.

Generic invariant polynomials with the parity $p_{-\alpha} = p_{\alpha}$ and $q_{-\alpha} = -q_{\alpha}$ can be brought to the form of equation (3.32) by suitable redefinition of the coordinates x_{α} . This does not completely fix the coordinates, since there remains a one-parameter freedom to make the redefinition

$$x_{\alpha} \rightarrow \lambda x_{\alpha} + \mu x_{-\alpha}, \quad s_{\alpha} \rightarrow \lambda s_{\alpha} - \mu s_{-\alpha}; \quad \lambda^2 - \mu^2 = 1$$

which preserves the form of the p_{α} . Such a redefinition changes the coefficients in the q_{α}

$$a_{\beta} \rightarrow (\lambda^2 + \mu^2) a_{\beta} + 2\lambda \mu a_{-\beta}$$

and we may use this freedom to require $a_{-1} = 0$, for example. In this way we see that there are 6 free parameters in the polynomials (3.32). We have checked that these polynomials are generically transverse.

The Euler number for the quotient is -64/8 = -8, and $h^{1,1} = 2$, since the four \mathbb{P}^1 's are identified under the action of \mathbb{H} , so the Hodge numbers for the quotient are $(h^{1,1}, h^{2,1}) = (2, 6)$, and $h^{2,1}$ agrees with our counting of parameters. The group \mathbb{H} has subgroups \mathbb{Z}_4 and \mathbb{Z}_2 generated by (say) *i* and -1 respectively, and we give the Hodge numbers of the corresponding quotients in the table below.

$(h^{1,1}, h^{2,1}) (X^{5,37}/G)$	(2, 6)	(3, 11)	(5, 21)
G	H	\mathbb{Z}_4	\mathbb{Z}_2

Table 3.16: The Hodge numbers of smooth quotients of $X^{5,37}$.

3.4.3 $X^{4,68}$ —the tetraquadric

Contracting the \mathbb{P}^7 of the configuration above brings us to the tetraquadric, which is the transpose of $\mathbb{P}^7[2,2,2,2]$.

For this manifold it is possible to write a defining polynomial that is transverse and also invariant and fixed point free under the group $\mathbb{H} \times \mathbb{Z}_2$. We again choose coordinates $(s_{\sigma}, s_{-\sigma}), \sigma \in \mathbb{H}_+$ for the four \mathbb{P}^1 's and define symmetry generators $U_{\gamma}, \gamma \in \mathbb{H}$, as before, together with a new generator, W,

 $U_{\gamma}: s_{\alpha} \to s_{\gamma\alpha} \text{ for } \alpha \in \mathbb{H}, W: (s_{\sigma}, s_{-\sigma}) \to (s_{\sigma}, -s_{-\sigma}) \text{ for } \sigma \in \mathbb{H}_{+}$

There are $3^4 = 81$ tetraquadric monomials in the s_{α} . One of these is the fundamental monomial, $\prod_{\alpha \in \mathbb{H}} s_{\alpha}$, that is invariant under the full group. Of the other 80 monomials, 40 are even under W and 40 odd. The 40 even monomials fall into 5 orbits of length 8 under the action of \mathbb{H} . Thus there is a 5 parameter family of invariant polynomials. The symmetry $\mathbb{H}\times\mathbb{Z}_2$ does not permit any redefinition of the coordinates, so the number of parameters in the polynomials is also the number of parameters of the manifold. For the quotient, $h^{1,1} = 1$, since the four \mathbb{P}^1 's are identified, and the Euler number is -128/16 = -8. Hence $(h^{1,1}, h^{2,1}) = (1,5)$, confirming the parameter count.

It is straightforward to check that the generic member of this family is transverse and fixed point free. In order to check that the group action is fixed point free it suffices to check that U_{-1} , W and $U_{-1}W$ act without fixed points. Each of these symmetries has a set of 16 fixed points in the embedding space, and it is simple to check that the generic polynomial does not vanish on any of these points.

$(h^{1,1}, h^{2,1}) (X^{4,68}/G)$	(1, 5)	(1, 9)	(2, 10)	(2,18)	(4, 20)	(4, 36)
G	$\mathbb{H}{\times}\mathbb{Z}_2$	H	$\mathbb{Z}_4{ imes}\mathbb{Z}_2$	\mathbb{Z}_4	$\mathbb{Z}_2{ imes}\mathbb{Z}_2$	\mathbb{Z}_2

Table 3.17: The Hodge numbers of smooth quotients of $X^{4,68}$.

3.4.4 An \mathbb{H} quotient of the manifold with $(h^{1,1}, h^{2,1}) = (19, 19)$

We can split the tetraquadric, and obtain the CICY matrix below, corresponding to Euler number $\chi = 0$. Another way to obtain this configuration is to contract two \mathbb{P}^{1} 's in (3.31), and since the Euler number doesn't change, we find that this is another presentation of the manifold $X^{19,19}$.



We will see that, presented as the split tetraquadric, the manifold admits a free action by \mathbb{H} . We start by labelling four of the \mathbb{P}^1 's by the positive unit quaternions as indicated in the diagram. The coordinates of these spaces are taken to be $(s_{\sigma}, s_{-\sigma}), \sigma \in \mathbb{H}_+$. The \mathbb{P}^1 introduced by the splitting is taken to have coordinates (t_1, t_i) . Generators $U_{\gamma}, \gamma \in \mathbb{H}$ act on these coordinates and on polynomials p_{α} , to be given below, as follows

$$U_{\gamma}: s_{\alpha} \rightarrow s_{\gamma\alpha}, t_{\alpha} \rightarrow t_{\gamma\alpha}; p_{\alpha} \rightarrow p_{\gamma\alpha}$$

where we will understand the t_{α} and the p_{α} as subject to the identifications

$$t_{\alpha} = t_{j\alpha}$$
 and $p_{\alpha} = p_{j\alpha}$

It follows from these identifications that the only independent values for the coordinates t_{α} are t_1 and t_i , and similarly for the polynomials p_{α} . Consider the following three polynomials, which

we denote by m_a , a = 1, 2, 3

$$(s_1^2 + s_{-1}^2)(s_j^2 + s_{-j}^2)$$
, $(s_1^2 + s_{-1}^2)s_js_{-j} + s_1s_{-1}(s_j^2 + s_{-j}^2)$, $s_1s_{-1}s_js_{-j}$

These are biquadratic in the variables (s_1, s_{-1}) and (s_j, s_{-j}) , and invariant under U_j . We form linear combinations

$$f_1 = \sum_{a=1}^{3} B_a m_a$$
 and $g_i = \sum_{a=1}^{3} C_a m_a$

and define

$$f_{\alpha} = U_{\alpha} f_1$$
 and $g_{\alpha} = U_{-\alpha i} g_i$

Note that, by construction, $f_{\alpha} = f_{\alpha j} = f_{-\alpha} = f_{j\alpha}$, and similarly for g_{α} . Thus f_{α} and g_{α} each take only two independent values as α ranges over \mathbb{H} , and these can be taken to be the values corresponding to $\alpha = 1$ and $\alpha = i$. The defining polynomials can be written in terms of the f_{α} and g_{α} as

$$p_{\alpha} = t_{\alpha} f_{\alpha} + t_{\alpha i} g_{\alpha i}$$

To check that the action of \mathbb{H} is fixed point free it is, again, only necessary to check that U_{-1} acts freely. In the embedding space, a fixed point of U_{-1} has $(s_{\sigma}^*, s_{-\sigma}^*) = (1, \pm 1)$ for each $\sigma \in \mathbb{H}_+$. For each of the 16 fixed points, the independent polynomials p_1 and p_i give two equations for (t_1, t_i) and these have no solution apart from $t_1 = t_i = 0$ for a general choice of coefficients. It is straightforward to check that the polynomials p_1 and p_i are transverse.

The parameter count is that there are 6 free coefficients in the definition of f_1 and g_i . There is a two-parameter freedom to redefine coordinates $s_{\alpha} \rightarrow \lambda s_{\alpha} + \mu s_{-\alpha}$ and $t_{\alpha} \rightarrow \nu t_{\alpha} + \rho t_{\alpha i}$, and there is a one-parameter freedom to rescale the polynomials $p_{\alpha} \rightarrow \tau p_{\alpha}$. This suggests that the manifold has 6 - 3 = 3 parameters. We do not have a presentation of the manifold that accounts for all of H_2 in terms of the embedding spaces, and we shall simply assume that our count of parameters is correct in this case. The Hodge numbers for $X^{19,19}/\mathbb{H}$ are, subject to this assumption, $(h^{1,1}, h^{2,1}) = (3, 3)$.

4. Hyperconifold Singularities

In Chapter 3, families of multiply-connected Calabi-Yau threefolds were constructed by finding free actions of discrete groups on simply-connected manifolds. We saw that in such cases, the group action is free for a generic choice of the complex structure of the symmetric manifold, but may develop fixed points for special choices, usually corresponding to a codimension-one locus in complex structure moduli space (i.e. a single condition on the coefficients of the polynomials). In this chapter, based on [66], we will find that on such loci the corresponding variety develops a certain type of hyperquotient singularity,¹ rather than an orbifold singularity as one might expect. These singularities have local descriptions as discrete quotients of the conifold, and are referred to here as hyperconifolds. We show that in many cases, such a singularity can be resolved to yield a distinct compact Calabi-Yau manifold.

We have already discussed why smooth multiply-connected Calabi-Yau threefolds are important for heterotic string model building, but smooth Calabi-Yau manifolds are not the only ones relevant to string theory. The moduli spaces of families of such manifolds have bound-aries corresponding to singular spaces, and some of these (such as conifold loci) are moreover at a finite distance as measured by the moduli space metric [48]. String theory is neverthe-less well-defined throughout the moduli space, as demonstrated by Strominger [68]. Even more remarkable is the fact, utilised in Chapter 3, that the moduli spaces of topologically distinct families can meet along such singular loci, and in fact it has been speculated that the moduli space of all Calabi-Yau threefolds is connected in this way [69]. A series of beautiful papers in the 90's established that type II string theories can pass smoothly through some such singular points, realising spacetime topology change via flops and conifold transitions [31, 70]. Conifold transitions can also be used as a tool for finding new Calabi-Yau manifolds, as in Chapter 3 and [71].

The most generic singularities which occur in threefolds are the nodes, or conifold singularities, with which we are already familiar. In this chapter we will show that for multiplyconnected threefolds, there are worse singularities which are just as generic, in that they also occur on codimension-one loci in moduli space. Specifically, if the moduli are chosen such that the (generically free) group action on the covering space actually has fixed points, these turn out always to be singular points, generically nodes. The singularities on the quotient are therefore certain quotients of the conifold, and as we will see, these also have local descriptions as toric varieties. Standard techniques from toric geometry, as reviewed in §2.4, are therefore utilised throughout this chapter.

The group actions on the conifold which will arise here all have the singular point as their unique fixed point. We will refer to the resulting quotients as 'hyperconifolds', or sometimes

¹This term was coined by Reid in [67], for a discrete quotient of a hypersurface singularity.

G-hyperconifolds to be explicit about the quotient group G. Although the toric formalism allows us to find local crepant resolutions (i.e. resolutions with trivial canonical bundle) of these singularities in each case, the important question is whether these preserve the Calabi-Yau conditions when embedded in the compact varieties of interest. In particular, the existence of a Kähler form is a global question. For many of the examples we can argue that a Calabi-Yau resolution does indeed exist for all varieties containing these singularities, and furthermore we can calculate the Hodge numbers of such a resolution in terms of those of the original smooth manifold. This therefore gives a systematic way of constructing new Calabi-Yau manifolds from known multiply-connected ones. By analogy with conifold transitions, the process by which we pass from the original smooth Calabi-Yau through the singular variety to its resolution will be dubbed a 'hyperconifold transition'. Like a conifold transition, the Hodge numbers of the new manifold are different, but unlike flops or conifold transitions, the fundamental group also changes.

Quotients of the conifold have been considered previously in the physics literature, mostly in the context of D3-branes at singularities (*e.g.* [72-76]), although the most-studied group actions in this context have fixed-point sets of positive dimension. The most simple example in this paper, the \mathbb{Z}_2 -hyperconifold, has however appeared in numerous papers (e.g. [77-80], and recently in the context of heterotic theories with flux [81]), while the \mathbb{Z}_3 case appears in an appendix in [82]. To the best of my knowledge, hyperconifold singularities have not before been explicitly embedded in compact varieties. The work herein provides a general method to find compactifications of string/brane models based on these singularities.

The layout of the chapter is as follows. In §4.1 the \mathbb{Z}_5 quotient of the quintic is presented as an example of a compact Calabi-Yau threefold which develops a hyperconifold singularity. The toric description of such singularities is also introduced here. §4.2 contains the main mathematical result of this chapter. It is demonstrated that if one starts with a family of threefolds which generically admit a free group action, then specialises to a sub-family for which the action instead develops a fixed point, then this point is necessarily a singularity (generically a node). The quotient variety therefore develops a hyperconifold singularity; the toric descriptions of these are given, and their topology described. In §4.3, Calabi-Yau resolutions are shown to exist for many of these singular varieties, and the Hodge numbers of these resolutions are calculated. In §4.4 a few initial observations are made relating to the possibility of hyperconifold transitions being realised in string theory.

The notation for the various varieties appearing in this chapter will be:

- X is a generic member of a family of smooth Calabi-Yau threefolds which admit a free holomorphic action of the group G. The tilde therefore indicates a covering space here, rather than a blow-up as in Chapter 2.
- X is the (smooth, Calabi-Yau) quotient \widetilde{X}/G .

- \widetilde{X}_0 is a (*G*-invariant) deformation of \widetilde{X} such that the action of *G* is no longer free.
- X_0 is the singular quotient space \widetilde{X}_0/G . This can be thought of as living on the boundary of the moduli space of smooth manifolds X.
- \widehat{X} will denote a resolution of X_0 , with projection $\pi : \widehat{X} \to X_0$. We will denote by E the exceptional set of this resolution.

4.1 A \mathbb{Z}_5 example

We will begin with a simple example to illustrate the idea. Consider a quintic hypersurface in \mathbb{P}^4 , and denote such a variety by \widetilde{X} . Take homogeneous coordinates z_i for the ambient space, with $i \in \mathbb{Z}_5$, so that such a hypersurface is given by f = 0, where

$$f = \sum A_{ijklm} z_i z_j z_k z_l z_m \tag{4.1}$$

If we denote by g_5 the generator of the cyclic group $G \cong \mathbb{Z}_5$, we can define an action of this group on the ambient \mathbb{P}^4 as follows:

$$g_5: z_i \to \zeta^i z_i$$
 where $\zeta = e^{2\pi i/5}$

The hypersurface \widetilde{X} will be invariant under this action if A_{ijklm} is zero except for those indices satisfying $i + j + k + l + m \equiv 0 \mod 5$. It is easy to check that for a general such choice of coefficients, \tilde{X} is smooth and the \mathbb{Z}_5 action has no fixed points, so the quotient variety, denoted X, is also smooth. For special choices of complex structure though, the hypersurface given by f = 0 will contain fixed points, and it is this case which will interest us here.

Consider the fixed point² $[1,0,0,0,0] \in \mathbb{P}^4$, and take local affine coordinates around this point, given by $y_a = z_a/z_0$, where a = 1, 2, 3, 4. Then the \mathbb{Z}_5 action is given locally by

 $y_a \to \zeta^a y_a$

and an invariant polynomial must locally be of the form

$$f = \alpha_0 + y_1 y_4 - y_2 y_3 + \text{higher-order terms}$$

where $\alpha_0 := A_{00000}$ is one of the constant coefficients in (4.1), and we have chosen the coefficients of the quadratic terms by rescaling the coordinates.³ If we make the special choice $\alpha_0 = 0$ (which corresponds to a codimension one locus in the moduli space of invariant hypersurfaces), we obtain a variety X_0 on which the action is no longer free.

But now we see what turns out to be a general feature of this sort of situation: when $\alpha_0 = 0$ we actually have f = df = 0 at the fixed point, meaning it is a node, or conifold, singularity on \widetilde{X}_0 . This means that on its quotient X_0 we get a particular type of hyperquotient singularity. We will now study this singularity by the methods of toric geometry.

²The analysis is the same for any of the five fixed points of the \mathbb{Z}_5 action. ³The quadratic terms correspond to some quadratic form η on \mathbb{C}^4 . Assuming that η is non-degenerate, it will always take the given form in appropriate coordinates. For general choices of coefficients in (4.1), η will indeed by non-degenerate.

4.1.1 The conifold and \mathbb{Z}_5 -hyperconifold as toric varieties

We will, as before, take the conifold \mathcal{C} to be described in \mathbb{C}^4 by the equation

$$y_1 y_4 - y_2 y_3 = 0 \tag{4.2}$$

This is a toric variety whose fan consists of a single cone (and its faces), spanned by the vectors

$$v_1 = (1, 0, 0) , v_2 = (1, 1, 1)$$

 $v_3 = (1, 1, 0) , v_4 = (1, 0, 1)$

$$(4.3)$$

We can see that the four vertices lie on a hyperplane; this is equivalent to the statement that the conifold is a non-compact Calabi-Yau variety. The intersection of its fan with the hyperplane is shown in Figure 4.1.



Figure 4.1: The toric diagram for the conifold.

We can also give homogeneous coordinates for the conifold, following the prescription described in §2.4.3. Explicitly, we have $C = (\mathbb{C}^4 \setminus S) / \sim$, where the excluded set is given by $S = \{z_1 = z_2 = 0, (z_3, z_4) \neq (0, 0)\} \cup \{z_3 = z_4 = 0, (z_1, z_2) \neq (0, 0)\}$, and the equivalence relation is

$$(z_1, z_2, z_3, z_4) \sim (\lambda z_1, \lambda z_2, \lambda^{-1} z_3, \lambda^{-1} z_4) \text{ for } \lambda \in \mathbb{C}^*$$

$$(4.4)$$

The explicit isomorphism between this representation and the hypersurface defined by (4.2) is given by the same relations which appeared in §2.4.3

$$y_1 = z_1 z_3 , \quad y_2 = z_1 z_4 , \quad y_3 = z_2 z_3 , \quad y_4 = z_2 z_4$$

$$(4.5)$$

We saw that the \mathbb{Z}_5 -hyperconifold singularity is obtained by imposing the equivalence relation $(y_1, y_2, y_3, y_4) \sim (\zeta y_1, \zeta^2 y_2, \zeta^3 y_3, \zeta^4 y_4)$, where $\zeta = e^{2\pi i/5}$. Using the above equations we can express this in terms of the z coordinates as

$$(z_1, z_2, z_3, z_4) \sim (z_1, \zeta^2 z_2, \zeta z_3, \zeta^2 z_4)$$

This equivalence relation must be imposed in addition to the earlier one,⁴ and we see that we obtain the same variety as described in the example in $\S2.4.3$. It is again a toric Calabi-Yau variety; the intersection of its fan with the hyperplane on which the vertices lie was drawn in Figure 2.2.

⁴Note that the power of ζ multiplying z_1 can always be chosen to be trivial by simultaneously applying a rescaling from (4.4).

The singularity could easily be resolved by sub-dividing the fan, but we will postpone a discussion of resolution of singularities until later. First we want to prove that the example presented here is far from unique.

4.2 Local hyperconifold singularities in general

The preceding discussion can be generalised to many other families of multiply-connected Calabi-Yau threefolds. To this end, consider such a threefold \tilde{X} which, for appropriately chosen complex structure, admits a free holomorphic action by some discrete group G. Then there exists a smooth quotient $X = \tilde{X}/G$, the deformations of which correspond to G-invariant deformations of \tilde{X} . For simplicity, we will herein consider only the case in which G is cyclic, $G \cong \mathbb{Z}_N$. This is not a great restriction, since there seem to be very few free actions of non-Abelian groups on Calabi-Yau manifolds, and in any case, every non-Abelian group has Abelian subgroups, to which the following discussion applies.

As we have seen for the \mathbb{Z}_5 -symmetric quintic, it may be that for special choices of the complex structure of X (generally on a codimension-one locus in its moduli space) the action of \mathbb{Z}_N on \widetilde{X} will no longer be free. One might expect the resulting singularities on X to be simple orbifold singularities, locally of the form $\mathbb{C}^3/\mathbb{Z}_N$. In the case of the quintic though, we actually obtained a quotient of the conifold. We now demonstrate that this is a general phenomenon.

4.2.1 Analysis of fixed points

Let g_N denote the generator of \mathbb{Z}_N , suppose that \widetilde{X} is locally determined by k equations $f_1 = \ldots = f_k = 0$ in \mathbb{C}^{k+3} , on which some \mathbb{Z}_N action is given, and let $P_0 \in \mathbb{C}^{k+3}$ be a fixed point of this action. Then we can choose local coordinates x_1, \ldots, x_{k+3} at P_0 such that the action of g_N is given by $x_i \to \zeta^{q_i} x_i$, where $\zeta = e^{2\pi i/N}$ and $q_i \in \{0, \ldots, N-1\}$. Let I be the set of fixed points of this action, and order the coordinates such that I is given locally by $x_{\dim I+1} = \ldots = x_{k+3} = 0$. This is equivalent to $q_1 = \ldots = q_{\dim I} = 0$ and $q_i \neq 0$ for $i > \dim I$.

By taking linear combinations of the polynomials if necessary, we can assume that $g_N \cdot f_a = \zeta^{Q_a} f_a$. What we mean by this is that $f_a(g_N \cdot P) = \zeta^{Q_a} f_a(P)$ for $P \in \mathbb{C}^{k+3}$. This immediately implies that if $Q_a \neq 0$, then we must have $f_a|_I \equiv 0$. But since by assumption \widetilde{X} does not generically intersect I, at least dim I + 1 of the polynomials must be non-trivial when restricted to I, so that they have no common zeros. We conclude that at least dim I+1 of the polynomials must be invariant under the group action.

Now suppose that we choose special polynomials such that the corresponding variety X_0 intersects I at a point, and identify this point with P_0 above: $I \cap \widetilde{X}_0 = \{P_0\}$. The expansion of an invariant polynomial f_a (i.e. $Q_a = 0$) around P_0 is then

$$f_a = \sum_{i=1}^{\dim I} C_{a,i} x_i + \mathcal{O}(x^2)$$

Now we can see what goes wrong. At P_0 we have

$$\frac{\partial f_a}{\partial x_i} = \begin{cases} C_{a,i} & , & i \le \dim I \\ 0 & , & i > \dim I \end{cases}$$

so the matrix $\partial f_a/\partial x_i$, for f_a ranging over invariant polynomials, has maximal rank dim *I*. But since, as argued above, there are at least dim *I* + 1 invariant polynomials, at the point P_0 we get $f_a = 0$ for all *a* and

$$df_1 \wedge \ldots \wedge df_{\dim I+1} = 0$$
 and hence $df_1 \wedge \ldots \wedge df_k = 0$

So the variety \widetilde{X}_0 is singular at this point, and in fact generically it will have a node, or conifold singularity. This means that the quotient variety X_0 develops a worse local singularity: a quotient of the conifold by a \mathbb{Z}_N action fixing only the singular point. This is what we will now call a \mathbb{Z}_N -hyperconifold.

It should be noted that there is no reason for any other singularities to occur on X_0 , and indeed it can be checked in specific cases that only one singular point develops.

4.2.2 The hyperconifolds torically

We now want to give explicit descriptions of the types of singularities whose existence in compact Calabi-Yau varieties we demonstrated above. There are known Calabi-Yau threefolds with fundamental group \mathbb{Z}_N for N = 2, 3, 4, 5, 6, 7, 8, 10, 12, and all cases except N = 7 occur as quotients of CICYs [22, 38, 55, 83]. For these we can perform analyses similar to that presented earlier for the \mathbb{Z}_5 quotient of the quintic, and obtain singular varieties containing isolated hyperconifold singularities. Each of these has a local toric description, which will be presented below. Since these are all toric Calabi-Yau varieties, the vectors generating the onedimensional cones of their fans lie on a hyperplane; Figure 4.2 and Figure 4.3 are collections of diagrams showing the intersections of the fans with this hyperplane. It should be noted that from the diagrams it is obvious that each singularity admits at least one toric crepant resolution. However we are only interested in those which give a Calabi-Yau resolution of the compact variety in which the singularity resides. Determining whether such a resolution exists requires more work, which we defer to §4.3.

\mathbb{Z}_2 quotient

Note that, as discussed above, the only point on the conifold fixed by the group actions we are considering will be the singular point itself. As such, there is only a single possible action of \mathbb{Z}_2 :

$$(y_1, y_2, y_3, y_4) \to (-y_1, -y_2, -y_3, -y_4)$$

In terms of the homogeneous coordinates this gives the additional equivalence relation

$$(z_1, z_2, z_3, z_4) \sim (z_1, z_2, -z_3, -z_4)$$

The resulting singularity is one which *has* appeared in the physics literature, as mentioned earlier. The difference here is that we have given a prescription for embedding this singularity in a compact Calabi-Yau variety, in such a way that it admits both a smooth deformation and, as we will see later, a crepant, Kähler resolution.

\mathbb{Z}_3 quotient

Similarly to the \mathbb{Z}_2 case, there is only a single action of \mathbb{Z}_3 on the conifold with an isolated fixed point:

$$(y_1, y_2, y_3, y_4) \to (\zeta y_1, \zeta y_2, \zeta^2 y_3, \zeta^2 y_4)$$

where $\zeta = e^{2\pi i/3}$. In terms of the homogeneous coordinates this leads to

$$(z_1, z_2, z_3, z_4) \sim (z_1, \zeta z_2, \zeta z_3, \zeta z_4)$$



Figure 4.2: The toric diagrams for the \mathbb{Z}_N -hyperconifolds, where N = 2, 3, 4, 5, 6, 8.

\mathbb{Z}_4 quotient

The group \mathbb{Z}_4 has a \mathbb{Z}_2 subgroup which must also act non-trivially on each coordinate y_a , so again there is a unique action consistent with this:

$$(y_1, y_2, y_3, y_4) \rightarrow (i y_1, i y_2, -i y_3, -i y_4)$$

In terms of the homogeneous coordinates this implies

$$(z_1, z_2, z_3, z_4) \sim (z_1, -z_2, i z_3, i z_4)$$

\mathbb{Z}_5 quotient

This is the first case where there are two actions of the group on the conifold which fix only the origin. This is true for \mathbb{Z}_5 and several of the larger cyclic groups discussed below, but in each case only one of the actions actually occurs in known examples. For \mathbb{Z}_5 it is

$$(y_1, y_2, y_3, y_4) \to (\zeta y_1, \zeta^2 y_2, \zeta^3 y_3, \zeta^4 y_4)$$
 (4.6)

where $\zeta = e^{2\pi i/5}$. We have already seen this in our original example of the quintic. In terms of the homogeneous coordinates the new equivalence relation is

$$(z_1, z_2, z_3, z_4) \sim (z_1, \zeta^2 z_2, \zeta z_3, \zeta^2 z_4)$$

\mathbb{Z}_6 quotient

For \mathbb{Z}_6 we can once again find the action by general arguments. If we require all elements of the group to act with only a single fixed point, there is only one possibility:

$$(y_1, y_2, y_3, y_4) \to (\zeta y_1, \zeta y_2, \zeta^5 y_3, \zeta^5 y_4)$$

where $\zeta = e^{\pi i/3}$. The identification on the homogeneous coordinates is therefore

$$(z_1, z_2, z_3, z_4) \sim (z_1, \zeta^4 z_2, \zeta z_3, \zeta z_4)$$

\mathbb{Z}_8 quotient

As in the \mathbb{Z}_5 case, there are multiple actions of \mathbb{Z}_8 on the conifold which fix only the origin, but only one is realised in the present context. The only free \mathbb{Z}_8 actions on compact Calabi-Yau threefolds known to the author are the one described in [22] and those related to it by conifold transitions [55, 56]. These can be deformed to obtain a local conifold singularity with the following quotient group action

$$(y_1, y_2, y_3, y_4) \to (\zeta y_1, \zeta^3 y_2, \zeta^5 y_3, \zeta^7 y_4)$$

where $\zeta = e^{\pi i/4}$. The equivalence relation on the homogeneous coordinates is therefore

$$(z_1, z_2, z_3, z_4) \sim (z_1, \zeta^4 z_2, \zeta z_3, \zeta^3 z_4)$$

\mathbb{Z}_{10} quotient

Several free actions of \mathbb{Z}_{10} on Calabi-Yau manifolds were described in Chapter 3. If we allow one of these to develop a fixed point, the resulting action on the conifold is

$$(y_1, y_2, y_3, y_4) \to (\zeta y_1, \zeta^3 y_2, \zeta^7 y_3, \zeta^9 y_4)$$

where $\zeta = e^{\pi i/5}$. The corresponding equivalence relation on the homogeneous coordinates is

$$(z_1, z_2, z_3, z_4) \sim (z_1, \zeta^6 z_2, \zeta z_3, \zeta^3 z_4)$$

\mathbb{Z}_{12} quotient

The largest cyclic group known to act freely on a Calabi-Yau manifold is \mathbb{Z}_{12} , and this was discovered only recently [83]. The resulting action on the conifold is

$$(y_1, y_2, y_3, y_4) \to (\zeta y_1, \zeta^5 y_2, \zeta^7 y_3, \zeta^{11} y_4)$$

where $\zeta = e^{\pi i/6}$. The corresponding equivalence relation on the homogeneous coordinates is

 $(z_1, z_2, z_3, z_4) \sim (z_1, \zeta^6 z_2, \zeta z_3, \zeta^5 z_4)$



Figure 4.3: The toric diagrams for the \mathbb{Z}_{10} - and \mathbb{Z}_{12} -hyperconifold singularities.

4.2.3 Topology of the singularities

Topologically, the conifold is a cone over $S^3 \times S^2$. It would be nice to relate the group actions described herein to this topology. Evslin and Kuperstein have provided a convenient parametrisation of the base of the conifold for just this sort of purpose [84], which we will use here. Parametrise the conifold as the set of degenerate 2×2 complex matrices

$$W = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} , \quad \det W = 0$$
(4.7)

and identify the base with the subset satisfying $Tr(W^{\dagger}W) = 1$. Now identify S^3 with the underlying topological space of the group SU(2), and S^2 with the space of unit two-vectors modulo phases. Then we map the point $(X, v) \in S^3 \times S^2$ to

$$W = X v v^{\dagger} \tag{4.8}$$

This is shown to be a homeomorphism in [84]. The actions of $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$, described in §4.2.2, are all realised in this description by

$$W \to \begin{pmatrix} \zeta & 0\\ 0 & \zeta^{-1} \end{pmatrix} W \tag{4.9}$$

where $\zeta = e^{2\pi i/N}$, N = 2, 3, 4, 6 respectively. We see from (4.8) that this can be considered as an action purely on the S^3 factor of the base, the quotient by which is the lens space L(N, 1). Topologically then, the singularity is locally a cone over $L(N, 1) \times S^2$. In fact these same spaces were considered in [85].

The more complicated cases of \mathbb{Z}_5 , \mathbb{Z}_8 , \mathbb{Z}_{10} and \mathbb{Z}_{12} quotients do not have such a straightforward topological description, but could be analysed along the lines of [84].

4.3 Global resolutions

In the preceding section, we described the local structure of the hyperconifolds using toric geometry; now we want to address the question of their resolution. Certainly, if we consider each case as a non-compact variety, they can all easily be resolved using toric methods. The interesting question is whether or not the compact varieties containing these singularities can be resolved to yield new Calabi-Yau manifolds.

4.3.1 Blowing up a node

It will be useful to first consider blowing up an ordinary node, and only then turn to its quotients. Again we take the conifold C to be given in \mathbb{C}^4 by (4.2):

$$y_1 y_4 - y_2 y_3 = 0$$

The singularity lies at the origin, and we can resolve it by blowing up this point. To do so we introduce a \mathbb{P}^3 with homogeneous coordinates (t_1, t_2, t_3, t_4) , and consider the usual equations $y_i t_j - y_j t_i = 0$ in $\mathbb{C}^4 \times \mathbb{P}^3$. This has the effect of setting $(t_1, t_2, t_3, t_4) \propto (y_1, y_2, y_3, y_4)$ when at least one y_i is non-zero, but leaving the t's completely undetermined at the origin. In this way we 'blow up' a single point to an entire copy of \mathbb{P}^3 , and have a natural projection map π which blows it down again. The blow up of the conifold is then defined to be the closure of the

pre-image of its smooth points:

$$\widehat{\mathcal{C}} = \overline{\pi^{-1}(\mathcal{C} \setminus \{\mathbf{0}\})}$$

Therefore $\widehat{\mathcal{C}}$ is isomorphic to \mathcal{C} away from the node, but the node itself is replaced by the surface in \mathbb{P}^3 given by

$$t_1 t_4 - t_2 t_3 = 0$$

which is in fact just $\mathbb{P}^1 \times \mathbb{P}^1$. This is the exceptional divisor of the blow-up, and we will denote it by E. Another important piece of information is the normal bundle $\mathcal{N}_{E|\hat{\mathcal{C}}}$ to E inside $\hat{\mathcal{C}}$. If $\mathcal{O}(n,m)$ denotes the line bundle which restricts to the n^{th} (resp. m^{th}) power of the hyperplane bundle on the first (resp. second) \mathbb{P}^1 , then in this case the normal bundle is $\mathcal{O}(-1,-1)$. This can be verified by taking an affine cover and writing down transition functions, but the toric formalism, to which we turn shortly, will let us see this much more easily. In any case, with this information we can demonstrate that $\hat{\mathcal{C}}$ is *not* Calabi-Yau. To see why, recall the adjunction formula (2.2), which gives the canonical bundle of the hypersurface E in terms of that of $\hat{\mathcal{C}}$

$$\omega_E = \omega_{\widehat{\mathcal{C}}} \Big|_E \otimes \mathcal{N}_{E|\widehat{\mathcal{C}}} \tag{4.10}$$

Therefore if $\omega_{\widehat{\mathcal{C}}}$ were trivial, we would have $\omega_E \cong \mathcal{N}_{E|\widehat{\mathcal{C}}} \cong \mathcal{O}(-1, -1)$, but it is a well-known fact that $\omega_E = \mathcal{O}(-2, -2)$, so we conclude that $\widehat{\mathcal{C}}$ has a non-trivial canonical bundle. This is why the blow up of a node does not generally feature in discussions of Calabi-Yau manifolds. We will see soon why it becomes relevant once we want to consider quotients.

The final important point is that the blowing up procedure automatically gives us another projective variety (i.e. a Kähler manifold if it is smooth), as we discussed in Chapter 2.

4.3.2 The toric picture, and the \mathbb{Z}_2 -hyperconifold

We can also blow up the node on the conifold using toric geometry. Recall that the fan for C consists of a single cone spanned by the four vectors given in (4.3), plus its faces. To this set of vectors we want to add $v_5 = v_1 + v_2 = v_3 + v_4$, and sub-divide the fan accordingly. The result is shown in Figure 4.4. We now have five homogeneous coordinates, and two independent rescalings (we will not explicitly describe the set to be removed before taking the quotient — this can be read off from the fan).

$$(z_1, z_2, z_3, z_4, z_5) \sim (\lambda \, z_1, \lambda \, z_2, \mu \, z_3, \mu \, z_4, \lambda^{-1} \mu^{-1} z_5) \quad \lambda, \mu \in \mathbb{C}^*$$
(4.11)

From this data we can easily see that (z_1, z_2) parametrise a \mathbb{P}^1 , as do (z_3, z_4) , and z_5 is a coordinate on the fibre of $\mathcal{O}(-1, -1)$. When $z_5 \neq 0$, we can choose $\mu = \lambda^{-1} z_5$, and we obtain the isomorphism to $\mathcal{C} \setminus \{0\}$. The remaining points are on the toric divisor given by $z_5 = 0$, and it is clear that this is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. So this toric variety is indeed the blow up of \mathcal{C} at the origin. The toric formalism makes it clear that the resolution is not crepant, since the new vector does not lie on the same hyperplane as the others.

We now turn our attention to the \mathbb{Z}_2 -hyperconifold, which was described in §4.2.2, and which we will denote by \mathcal{C}_2 . In this case the blow up of the singular point is obtained by adding a vector



Figure 4.4: The fans for the blow-up of the conifold at its singular point, and its \mathbb{Z}_2 quotient.

which lies on the same hyperplane as the first four, meaning that the resulting resolution \hat{C}_2 is also Calabi-Yau. The relations are now $2v_5 = v_1 + v_2 = v_3 + v_4$, and the five vectors can be taken to be

$$v_1 = (1, -1, 0)$$
, $v_2 = (1, 1, 0)$
 $v_3 = (1, 0, -1)$, $v_4 = (1, 0, 1)$
 $v_5 = (1, 0, 0)$

The resulting equivalence relations on the homogeneous coordinates are

$$(w_1, w_2, w_3, w_4, w_5) \sim (\lambda \, w_1, \lambda \, w_2, \mu \, w_3, \mu \, w_4, \lambda^{-2} \mu^{-2} w_5) \quad \lambda, \mu \in \mathbb{C}^*$$
(4.12)

This is very similar to (4.11), but in this case w_5 is seen to be a coordinate on $\mathcal{O}(-2, -2)$ rather than $\mathcal{O}(-1, -1)$. As such, the adjunction formula (4.10) says that the canonical bundle of $\widehat{\mathcal{C}}_2$ restricts to be trivial on the exceptional divisor, consistent with $\widehat{\mathcal{C}}_2$ being Calabi-Yau.

There is another nice way to think about the resolution of C_2 . We begin by noticing that the blown-up conifold \widehat{C} is actually a ramified double cover of \widehat{C}_2 , with the explicit map being given by

$$w_i = z_i \text{ for } i = 1, 2, 3, 4 \qquad w_5 = (z_5)^2$$

$$(4.13)$$

This deserves some elaboration. It is clear from (4.13) that the map is two-to-one everywhere except along the exceptional divisor given by $z_5 = 0$. In fact it can be thought of as an identification $z_5 \sim -z_5$ on the fibres of $\mathcal{O}(-1, -1)$ over $\mathbb{P}^1 \times \mathbb{P}^1$. Since the fixed point set of



Figure 4.5: The toric diagram for the resolution of the \mathbb{Z}_2 -hyperconifold.

the involution $z_5 \to -z_5$ is of complex codimension one, taking the quotient actually does not introduce any singularity (which is clear here since $\widehat{\mathcal{C}}_2$ is manifestly smooth). So we can think of the resolution of \mathcal{C}_2 in two ways: either we blow up the singular point of \mathcal{C}_2 , or we blow up the node on the covering space, and then take the \mathbb{Z}_2 quotient.

Note that the procedure described above is completely local (we blew up a point), and therefore can be performed inside any compact Calabi-Yau variety X_0 in which the singularity C_2 occurs, to yield a new compact Calabi-Yau manifold. This should be compared to the small resolution of the conifold, which involves blowing up a sub-variety which extends 'to infinity' in C (in fact the variety given by $y_1 = y_2 = 0$), so that the existence of the Calabi-Yau resolution depends on the global structure.

4.3.3 The \mathbb{Z}_{2M} -hyperconifolds

Having demonstrated the existence of crepant projective resolutions (i.e. Calabi-Yau resolutions) for Calabi-Yau varieties containing the singularity C_2 , we can easily do the same for the quotients of the conifold by all cyclic groups of even order. This is achieved by breaking the process down into several steps.⁵

The unique \mathbb{Z}_2 subgroup of \mathbb{Z}_{2M} fixes exactly the singular point, and we can blow this up by adding the ray through the point $v_5 = \frac{1}{4}(v_1 + v_2 + v_3 + v_4)$ and sub-dividing the fan accordingly. Alternatively we can think of this as first taking the quotient by $\mathbb{Z}_2 \subset \mathbb{Z}_{2M}$, blowing up the resulting \mathcal{C}_2 singularity, and then taking the quotient by the induced action of $\mathbb{Z}_{2M}/\mathbb{Z}_2 \cong \mathbb{Z}_M$. Either way, we obtain a variety with only \mathbb{Z}_M orbifold singularities. There are then two cases:

- If *M* is odd, it turns out that there is a unique way to further sub-divide the fan to obtain a smooth variety. In [86] it is shown that for a projective threefold with only orbifold singularities, one obtains a global projective crepant resolution by choosing an appropriate crepant resolution on each affine patch. If there is a unique choice for each, we therefore automatically obtain the projective resolution, so we are done.
- If M is even, then \mathbb{Z}_M contains a unique \mathbb{Z}_2 subgroup, and the fixed point set of this ⁵The following argument is partly due to Balázs Szendrői.

subgroup is a pair of disjoint curves which are toric orbits (this follows from inspecting the diagrams case-by-case). These are given by two-cones, spanned by v_i, v_j , and in each case the vector $\frac{1}{2}(v_i + v_j)$ is integral, so can be added to the fan to blow up the fixed curve (in fact this is just the well-known resolution of the A_1 surface singularity, fibred over the curves). We iterate this process until we are left with $\mathbb{Z}_{M'}$ orbifold singularities for M'odd, and the fan then has a unique smooth subdivision.

Note that we are not claiming that the resolutions obtained are the unique Kähler ones. Several of the hyperconifolds admit multiple resolutions differing by flops, and it is possible that more than one of these corresponds to a projective resolution.

The preceding prescription is easy to understand in particular cases, as we will now illustrate with the complicated \mathbb{Z}_{12} -hyperconifold. We begin with the fan in Figure 4.3, and blow up the singular point, which adds a ray through the geometric centre of the top-dimensional cone, and divides it into four (see Figure 4.6). The result is a fan for a variety containing a chain of four genus zero curves meeting in points. Two of these are curves of $\mathbb{C}_2/\mathbb{Z}_2$ orbifold singularities, and the other two of $\mathbb{C}_2/\mathbb{Z}_3$ singularities. The four points of intersection are locally $\mathbb{C}_3/\mathbb{Z}_6$ orbifold singularities. We can blow up the (disjoint) \mathbb{Z}_2 curves by bisecting the corresponding two-cones and sub-dividing the fan accordingly. This leaves us with eight top-dimensional simplicial cones, each of which has a unique crepant sub-division, giving us the final smooth, crepant, Kähler resolution of the singularity.

We can perform the same analysis for each \mathbb{Z}_{2M} -hyperconifold, obtaining the fans in Figure 4.7. The reader may find it amusing to follow the steps in each case, and verify the resulting fans. At present there is no argument that varieties containing the \mathbb{Z}_3 - and \mathbb{Z}_5 -hyperconifolds also admit Kähler crepant resolutions, but one is naturally drawn to conjecture that this is the case. The following comments would then apply to these cases too.

It is easy to obtain certain topological data about these resolutions. From the toric diagrams we see that in each case the exceptional set E is simply connected, which is the case for any toric variety whose fan contains a top-dimensional cone. Therefore the resolution of X_0 is simply-connected, even though the smooth Calabi-Yau X, of which X_0 is a deformation, had fundamental group \mathbb{Z}_N . This contrasts with the case of a conifold transition, where the fundamental group does not change.

We can also simply read off the diagram that the exceptional set of the resolution of the \mathbb{Z}_N hyperconifold has Euler characteristic $\chi(E) = 2N$, since χ is just the number of top-dimensional cones in the fan. We can therefore calculate $\chi(\hat{X})$ quite easily. Topologically, \tilde{X}_0 is obtained from \tilde{X} by shrinking an S^3 to a point P_0 , so $\chi(\tilde{X}_0) = \chi(\tilde{X}) + 1$. We delete P_0 , quotient by \mathbb{Z}_N , then glue in E, so

$$\chi(\hat{X}) = \chi(\hat{X})/N + \chi(E) = \chi(X) + 2N$$
(4.14)

Finally, the resolution of the \mathbb{Z}_N -hyperconifold introduces N-1 new divisor classes, so we can

actually calculate all the Betti numbers of \hat{X} in terms of those of X:

$$b_1(\hat{X}) = b_5(\hat{X}) = 0$$
, $b_2(\hat{X}) = b_4(\hat{X}) = b_2(X) + N - 1$, $b_3(\hat{X}) = b_3(X) - 2$ (4.15)

The Hodge numbers of the new manifold are therefore

$$h^{1,1}(\widehat{X}) = h^{1,1}(X) + N - 1$$
, $h^{2,1}(\widehat{X}) = h^{2,1}(X) - 1$ (4.16)

4.4 Hyperconifold transitions in string theory?

A natural question to ask is whether the hyperconifold transitions described in this chapter can be realised in string theory, as their cousins flops and conifold transitions can. At this stage we will merely make some suggestive observations.

Consider type IIB string theory on a Calabi-Yau manifold X with fundamental group \mathbb{Z}_N , and vary the complex structure moduli until we approach a singular variety X_0 . We have seen (in the simplest cases) that topologically this looks like shrinking a three-cycle L(N, 1) to a point. Therefore, just as in the conifold case, there will be D3-brane states becoming massless [68]. These manifest in the low-energy theory as a hypermultiplet charged under a U(1) gauge group coming from the R-R sector, and although it becomes massless there is still a D-term potential preventing its scalars from developing a VEV. However, these D-brane states are not necessarily the only ones becoming massless at the hyperconifold point — there are twisted sectors coming from strings wrapping non-trivial loops on L(N, 1), and these strings attain zero length at the singular point. It is therefore conceivable that these twisted sectors give rise to a new branch of the low-energy moduli space, and that moving onto this branch corresponds to resolving the singularity of the internal space. Since there are N - 1 new divisors/Kähler parameters on the resolution, there must be N - 1 new flat directions.

The conjecture then is that in the low-energy field theory at the hyperconifold point, there is a new (N-1)-dimensional branch of moduli space coming from the twisted sectors in the string theory. The new flat directions correspond to Kähler parameters on the resolution of the singular variety, and giving them VEVs resolves the singularity. It would be interesting to try to verify this picture.



Figure 4.6: The three steps involved in resolving the \mathbb{Z}_{12} -hyperconifold singularity. First we blow up the singular point, then the two curves fixed by $\mathbb{Z}_2 \subset \mathbb{Z}_6$, and finally we perform the unique maximal subdivision of the resulting fan.



Figure 4.7: The fans for the Kähler crepant resolutions of the \mathbb{Z}_{2M} hyperconifold singularities, where M = 2, 3, 4, 5.

5. New Three-Generation Manifolds

In this chapter, based on [83], we study further the CICY $X^{8,44}$, of §3.3.7, which has Euler number -72. We found a free action of \mathbb{Z}_6 on this manifold, but in [45], Braun discovered that this can be realised as a subgroup of two distinct groups of order 12, each of which can be made to act freely on $X := X^{8,44}$. These are the cyclic group \mathbb{Z}_{12} and a non-Abelian group \mathfrak{G} , isomorphic to the dicyclic group Dic₃. The quotient manifolds have $\chi = -6$ and Hodge numbers $(h^{1,1}, h^{2,1}) = (1, 4)$. Our main focus here is the non-Abelian quotient, due mainly to the relative novelty of having a large non-Abelian fundamental group. We detail the structure of the group $\mathfrak{G} \cong \text{Dic}_3$ and its representations, and calculate its action on the (co)homology of X. We also analyse the possible discrete symmetries which act on the quotient manifold $X_{\mathfrak{G}} := X/\mathfrak{G}$, and obtain the interesting result that for generic values of the complex structure parameters the symmetry is \mathbb{Z}_2 .

The final interesting feature of the geometry of $X_{\mathfrak{G}}$ is that there exists a limit in which it develops three conifold points, and we demonstrate how these may be resolved to give another Calabi-Yau manifold with $(h^{1,1}, h^{2,1}) = (2,2)$, which is yet another quotient of the (19,19) manifold first discussed in §3.1. We also find the probable mirror manifold of $X_{\mathfrak{G}}$ by adapting the toric construction of Batyrev.

5.1 A manifold with a $\chi = -6$ quotient

The manifold X can be represented as a CICY in several ways, one of which we studied earlier. There are two such representations for which the group actions are represented linearly on the homogeneous coordinates of the ambient space; the corresponding configuration matrices are

These configurations give us a very explicit description of the manifold X in terms of polynomials. We have already seen the second configuration in §3.3.7.

At this stage we should note that soon after it was put together, the CICY list was searched [87, 88] for manifolds that admit a freely acting symmetry such that the quotient has $\chi = -6$, since this gives three generations via the standard embedding. In these searches, the manifold presented here, which occurs 3 times in the list of [5] (two of these presentations are the configurations displayed above, the third is a hybrid of the two), was wrongly rejected. In [87] one of the presentations of this manifold was recognised as admitting a symmetry group \mathbb{Z}_3 , and we extended this to \mathbb{Z}_6 in Chapter 3. The groups of order 12 finally came to light during Braun's recent project to classify all the freely acting symmetries for the manifolds of the CICY list [45].

It transpires that, apart from the covering manifold of Yau's three-generation manifold, the manifold presented here is the only one to admit a smooth quotient with $\chi = -6$.

If one thinks carefully about the configurations (5.1), another picture of X emerges. The manifold can be realised as a hypersurface in a space $S \times S$, where S is the twofold defined by two bilinear equations in $\mathbb{P}^2 \times \mathbb{P}^2$ or a single trilinear equation in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. The surface specified in this way is the del Pezzo surface dP₆.

The manifold X is then an anti-canonical hypersurface inside the fourfold $S \times S$. The surface $S = dP_6$ is toric so we find ourselves within the general framework of toric hypersurfaces and reflexive polyhedra [37]. This will prove useful later.

5.1.1 The quotient manifold

The new configuration for X is



The manifold is defined by 5 polynomials, that we denote by p^1 , p^2 , q^1 , q^2 , and r, defined on $(\mathbb{P}^2)^4$. We take coordinates $x_{\alpha j}$ for the four \mathbb{P}^2 , where the indices α and j take values in \mathbb{Z}_4 and \mathbb{Z}_3 , respectively. The columns of the matrix correspond to the degrees of the polynomials in the coordinates of each space, in the order $\{p^1, q^1, r, q^2, p^2\}$. The diagram on the right encodes the same information and shows how the polynomials, represented by the blue dots, depend on the variables of the four \mathbb{P}^2 's, which correspond to the open red dots. The fact that the dots are all connected by single lines in the diagram corresponds to the fact that the polynomials are all multilinear.

We begin by seeking equations that are covariant under the cyclic permutation of the four \mathbb{P}^{2} 's,

$$g_4: x_{\alpha j} \rightarrow x_{\alpha+1,j}; p^1 \leftrightarrow p^2, q^1 \leftrightarrow q^2, r \rightarrow r$$

For w_j and z_k homogeneous coordinates on $\mathbb{P}^2 \times \mathbb{P}^2$, define bilinear polynomials P and Q by

$$P(w,z) = \sum_{jk} A_{jk} w_j z_k , \qquad Q(w,z) = \sum_{jk} B_{jk} w_j z_k$$

where the coefficient matrices A_{jk} and B_{jk} are symmetric. Define also g_4 -invariant polynomials

$$m_{ijkl} = \frac{1}{4} \sum_{\alpha} x_{\alpha,i} x_{\alpha+1,j} x_{\alpha+2,k} x_{\alpha+3,l}$$

In terms of these quantities we may take defining polynomials, for X, of the form

$$p^{1} = P(x_{1}, x_{3}) , \qquad p^{2} = P(x_{2}, x_{4})$$
$$q^{1} = Q(x_{1}, x_{3}) , \qquad q^{2} = Q(x_{2}, x_{4})$$
$$r = \sum_{ijkl} C_{ijkl} m_{ijkl}$$

Note that the quantities m_{ijkl} are cyclically symmetric in their indices so, in the definition of r, we take the sum over indices i, j, k, l to run over combinations that are identified up to cyclic permutation, and the coefficients C_{ijkl} to be cyclically symmetric. Consider now a second symmetry

$$g_3: x_{\alpha j} \to \zeta^{(-1)^{\alpha} j} x_{\alpha j}; p^1 \to p^1, p^2 \to p^2, q^1 \to \zeta q^1, q^2 \to \zeta^2 q^2$$

where ζ is a nontrivial cube root of unity. Covariance under g_3 restricts the coefficients that can appear in the defining polynomials. We see that

 $A_{jk} = 0 \text{ unless } j+k \equiv 0 \mod 3$ $B_{jk} = 0 \text{ unless } j+k \equiv 2 \mod 3$ $C_{ijkl} = 0 \text{ unless } i+k \equiv j+l \mod 3$

Thus, removing overall scales, P and Q are of the form

$$P(w,z) = w_0 z_0 + a (w_1 z_2 + w_2 z_1) , \quad Q(w,z) = w_1 z_1 + b (w_0 z_2 + w_2 z_0)$$

while r is a linear combination of 9 of the m_{ijkl}

$$r = c_0 m_{0000} + c_1 m_{1111} + c_2 m_{2222} + c_3 m_{0011} + c_4 m_{0212} + c_5 m_{0022} + c_6 m_{1122} + c_7 m_{0102} + c_8 m_{0121}$$
(5.2)

The freedom to redefine the coordinates $x_{\alpha j}$ is restricted by the action of g_3 and g_4 . The remaining freedom allows only a two-parameter redefinition

$$(x_{\alpha 0}, x_{\alpha 1}, x_{\alpha 2}) \rightarrow (x_{\alpha 0}, \beta x_{\alpha 1}, \gamma x_{\alpha 2})$$

$$(5.3)$$

which we may use to set the constants a and b that appear in p and q to unity. We may also redefine r by multiples of the other polynomials. Let \tilde{P} and \tilde{Q} be generic polynomials with the same degrees and covariance properties as P and Q, that is

$$\tilde{P}(w,z) = a_1 w_0 z_0 + a_2 (w_1 z_2 + w_2 z_1) , \quad \tilde{Q}(w,z) = b_1 w_1 z_1 + b_2 (w_0 z_2 + w_2 z_0)$$

These provide a four-parameter freedom to redefine r:

$$r \rightarrow r + \left(\tilde{P}(x_1, x_3) P(x_2, x_4) + P(x_1, x_3) \tilde{P}(x_2, x_4)\right) + \left(\tilde{Q}(x_1, x_3) Q(x_2, x_4) + Q(x_1, x_3) \tilde{Q}(x_2, x_4)\right)$$

The coefficients c_2 , c_3 and c_4 are unaffected by this process but we may use this freedom to eliminate the terms in equation (5.2) with coefficients c_5 , c_6 , c_7 and c_8 , say. An overall scale is of no consequence so the resulting polynomial has four parameters. We summarise the final form of the polynomials as

$$p^{1} = x_{10} x_{30} + x_{11} x_{32} + x_{12} x_{31} , \qquad q^{1} = x_{11} x_{31} + x_{10} x_{32} + x_{12} x_{30}$$
$$p^{2} = x_{20} x_{40} + x_{21} x_{42} + x_{22} x_{41} , \qquad q^{2} = x_{21} x_{41} + x_{20} x_{42} + x_{22} x_{40}$$
(5.4)

$$r = c_0 m_{0000} + c_1 m_{1111} + c_2 m_{2222} + c_3 m_{0011} + c_4 m_{0212}$$

Consider again the coordinate transformations given in (5.3). We see that the polynomials p^1 , p^2 , q^1 and q^2 are invariant provided that $\beta\gamma = 1$ and $\gamma^3 = 1$. We take, therefore, $\beta = \zeta$ and $\gamma = \zeta^2$. The polynomial r, however, is not invariant, since under the given transformation, $m_{ijkl} \rightarrow \zeta^{i+j+k+l} m_{ijkl}$. The effect is equivalent to changing the coefficients. In this way we see that there is a \mathbb{Z}_3 -action on the coefficients, and that we should identify

$$(c_0, c_1, c_2, c_3, c_4) \sim (c_0, \zeta c_1, \zeta^2 c_2, \zeta^2 c_3, \zeta^2 c_4)$$
 (5.5)

Return to the symmetries g_4 and g_3 : these generate a group that we shall denote by \mathfrak{G} . Note that

$$g_4 g_3 = g_3^2 g_4$$

so the group is non-Abelian. This relation, however, permits the enumeration of its elements as $g_3^m g_4^n$, $0 \le m \le 2$, $0 \le n \le 3$. It may also be expressed as $g_4 g_3 g_4^{-1} = g_3^2$, so the group contains \mathbb{Z}_3 as a normal subgroup, and can be thought of as a semi-direct product $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$. Thus \mathfrak{G} has order 12 and is in fact isomorphic to the dicyclic group Dic₃. A fact that will be useful shortly is that the element $g_6 = g_3^2 g_4^2$ generates a \mathbb{Z}_6 subgroup of \mathfrak{G} and that the elements of \mathfrak{G} may also be enumerated as $g_6^m g_4^n$ with $0 \le m \le 5$ and n = 0 or 1.

$g_3^m g_4^n$	1	g_4	g_{4}^{2}	g_4^3	g_3	$g_3 g_4$	$g_3 g_4^2$	$g_3 g_4^3$	g_3^2	$g_3^2 g_4$	$g_3^2 g_4^2$	$g_3^2 g_4^3$
$g_6^m g_4^n$	1	g_4	g_{6}^{3}	$g_6^3 g_4$	g_{6}^{2}	$g_{6}^{2} g_{4}$	g_6^5	$g_6^5 g_4$	g_6^4	$g_{6}^{4} g_{4}$	g_6	$g_6 g_4$
Order	1	4	2	4	3	4	6	4	3	4	6	4

Table 5.1: The elements of the group Dic₃ presented in the form $g_3^m g_4^n$, for $0 \le m \le 2$, $0 \le n \le 3$, and $g_6^m g_4^n$, for $0 \le m \le 5$, n = 0, 1, together with the order of each element.

In order to check for fixed points of \mathfrak{G} , note that if an element h has a fixed point then so has h^m for each $m \ge 1$. The order of an element of \mathfrak{G} , that is not the identity, must divide the order of the group so can be 2, 3, 4, or 6. If an element h has order 4 then h^2 has order 2 and if it has order 6 then h^3 has order 2. Hence it is enough to check the elements of order 2 and 3 for fixed points. The only elements of order 3 are g_3 and g_3^2 and if g_3^2 has a fixed point then so has $g_3^4 = g_3$. Thus it suffices to check g_3 and g_4^2 , the latter being the unique element of order 2. A fixed point of g_3 is such that $x_{\alpha j}$ takes one of the values $\{(1,0,0), (0,1,0), (0,0,1)\}$, for each α . Thus there are 3^4 fixed points in the embedding space $(\mathbb{P}^2)^4$. It is an easy check to see that these points do not coincide with simultaneous zeros of the defining polynomials provided none of the coefficients c_2 , c_3 , and c_4 vanish. A fixed point of g_4^2 is such that $x_{1j} = x_{3j} = w_j$ and $x_{2j} = x_{4j} = z_j$ for some w_j and z_j that satisfy the equations

$$egin{array}{rll} P(w,w) &=& 0 \;, & P(z,z) \;=\; 0 \ Q(w,w) \;=\; 0 \;, & Q(z,z) \;=\; 0 \ r(w,z,w,z) \;=\; 0 \end{array}$$

and it is easily checked that these five equations do not have a solution in $\mathbb{P}^2 \times \mathbb{P}^2$ for generic values of the parameters. We pause to do this explicitly since the values of the parameters for which there *are* fixed points will be of interest later. The equations P(w, w) = Q(w, w) = 0have four solutions for w. These are w = (0, 0, 1) and $w = (1, \omega, -\omega^2/2)$, for $\omega^3 = 1$, and the solutions for z are the same, giving rise to 16 points in $\mathbb{P}^2 \times \mathbb{P}^2$. The polynomial r(w, z, w, z)does not vanish on any of these points unless at least one of the quantities

$$c_2 , c_2 + 2c_4 , c_0^3 + c_1^3 + d_2^3 - 3c_0c_1d_2$$
 (5.6)

where $16d_2 = c_2 + 16c_3 + 4c_4$, vanishes. We have checked that the polynomials (5.4) are transverse following the methods of Chapter 3. We conclude that the quotient $X_{\mathfrak{G}}$ is a smooth Calabi-Yau manifold. Now we may regard the manifold X as a hypersurface in $\mathcal{S} \times \mathcal{S}$, where \mathcal{S} is the surface

$$S = \frac{\mathbb{P}^2}{\mathbb{P}^2} \begin{bmatrix} 1 & 1\\ 1 & 1 \end{bmatrix}$$
(5.7)

which has Euler number 6 and ample anti-canonical bundle, and is therefore the del Pezzo surface dP_6 , obtained by blowing up three points of \mathbb{P}^2 that are in general position. It is instructive to verify this explicitly and to locate the three blown up points by considering the defining equations (5.4). The polynomials p^1 and q^1 , that define the first copy of \mathcal{S} , are $p^1 = x_1^T A x_3 = 0$ and $q^1 = x_1^T B x_3 = 0$ where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Given $x_1 \in \mathbb{P}^2$, consider the corresponding values of x_3 which solve these equations. For generic values, x_3 is determined up to scale, hence uniquely as a point of \mathbb{P}^2 , as the vector orthogonal to $x_1^T A$ and $x_1^T B$. At the three points coming from left eigenvectors of AB^{-1} , however, we have $x_1^T A \propto x_1^T B$, and there is therefore a whole \mathbb{P}^1 of solutions for x_3 . These three points are $x_1 = (1, 1, 1), (1, \zeta, \zeta^2)$ and $(1, \zeta^2, \zeta)$, with ζ again a non-trivial cube root of unity. The corresponding \mathbb{P}^1 's are the exceptional curves E_1, E_2, E_3 . Also important to us are the three lines L_{ij} in S that correspond to the lines in the \mathbb{P}^2 that join the points that are blown up to E_i and E_j . The three L_{ij} together with the three E_i form a hexagon in S, as sketched in Figure 5.1. We saw earlier, and will confirm presently, that the order 6 symmetry g_6 acts on this hexagon by rotation.



Figure 5.1: A sketch of dP_6 , in the centre, showing the hexagon formed by the six (-1)lines. This surface may be realised in $\mathbb{P}^2 \times \mathbb{P}^2$ as the locus p(w, z) = q(w, z) = 0 defined by two bilinear polynomials in the coordinates w_j and z_j of the two \mathbb{P}^2 's. If we project to the first \mathbb{P}^2 then the image is as in the sketch on the left, in which the three lines E_i project to points. If, instead, we project to the second \mathbb{P}^2 then the image is as in the sketch on the right, in which the three lines L_{ij} have been projected to points.

5.1.2 The group representations of Dic₃

Before proceeding, we pause to describe the representations of the group $\mathfrak{G} \cong \text{Dic}_3$. There are four one-dimensional representations, in which g_3 acts trivially and g_4 is multiplication by one of the fourth roots of unity. We will denote these, in an obvious notation, by R_1, R_i, R_{-1} and R_{-i} . These are the homomorphisms of Dic₃ to its Abelianisation \mathbb{Z}_4 . There are also two distinct two-dimensional representations, distinguished in a coordinate-invariant way by $\text{Tr}(g_4^2) = \pm 2$. These we will denote by $R_{\pm}^{(2)}$. For completeness we display the corresponding matrices in a basis for which g_3 is diagonal:

$$R_{+}^{(2)}: \quad g_{3} \to \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{2} \end{pmatrix}, \quad g_{4} \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$R_{-}^{(2)}: \quad g_{3} \to \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{2} \end{pmatrix}, \quad g_{4} \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The non-obvious tensor products of representations are as follows

$$R_{1} \otimes R_{\pm}^{(2)} = R_{-1} \otimes R_{\pm}^{(2)} = R_{\pm}^{(2)}, \qquad R_{i} \otimes R_{\pm}^{(2)} = R_{-i} \otimes R_{\pm}^{(2)} = R_{\mp}^{(2)}$$

$$R_{+}^{(2)} \otimes R_{+}^{(2)} = R_{1} \oplus R_{-1} \oplus R_{+}^{(2)}, \qquad R_{-}^{(2)} \otimes R_{-}^{(2)} = R_{1} \oplus R_{-1} \oplus R_{+}^{(2)}$$

$$R_{+}^{(2)} \otimes R_{-}^{(2)} = R_{i} \oplus R_{-i} \oplus R_{-}^{(2)}$$

5.1.3 Group action on homology

The cohomology group $H^2(X^{8,44})$ descends from that of $H^2(S \times S) = H^2(S) \oplus H^2(S)$. For each S we have $h^{1,1}(S) = 4$, and the cohomology group is spanned by (the duals of) the hyperplane class, H, and the three E_i . The intersection numbers of the four classes $\{H, E_1, E_2, E_3\}$ can be expressed as a matrix η . The self-intersection of the hyperplane class is just 1, and we can always choose a hyperplane which misses the three blown-up points. The exceptional curves arise as blow-ups of distinct smooth points, so are disjoint and each have self-intersection -1, as usual. The intersection numbers are therefore

$$H \cdot H = 1, \quad H \cdot E_i = 0, \quad E_i \cdot E_j = -\delta_{ij} \quad \text{so} \quad \eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

By considering the intersection numbers of L_{ij} with H and the E_k one sees that

$$L_{ij} = H - E_i - E_j$$

and that the L_{ij} are also (-1)-lines, that is $L_{ij} \cdot L_{ij} = -1$. Of course, everything said above also applies on the second copy of S. To distinguish the cohomology classes coming from this copy, we denote them by $\tilde{H}, \tilde{E}_1, \tilde{E}_2, \tilde{E}_3$.

(-1)-line	w	z
E_1	(1, 1, 1)	$z_0 + z_1 + z_2 = 0$
L_{12}	$w_0 + \zeta w_1 + \zeta^2 w_2 = 0$	$(1,\zeta^2,\zeta)$
E_2	$(1,\zeta,\zeta^2)$	$z_0+\zeta^2 z_1+\zeta z_2=0$
L_{23}	$w_0 + w_1 + w_2 = 0$	(1, 1, 1)
E_3	$(1,\zeta^2,\zeta)$	$z_0 + \zeta z_1 + \zeta^2 z_2 = 0$
L_{31}	$w_0 + \zeta^2 w_1 + \zeta w_2 = 0$	$(1,\zeta,\zeta^2)$

Table 5.2: The equations that define the six (-1)-lines in coordinates (w_i, z_j) on $\mathbb{P}^2 \times \mathbb{P}^2$.

We know the action of the group generators on the spaces, and this allows us to calculate the induced action on $H^2(S \times S)$. Choosing the ordered basis $\{H, E_1, E_2, E_3, \tilde{H}, \tilde{E}_1, \tilde{E}_2, \tilde{E}_3\}$, we can write 8×8 matrices U(g) representing the action of $g \in \mathfrak{G}$. It is clear that g_3 preserves the hyperplane classes and rotates the exceptional curves into each other, so that

$$U(g_3) = \begin{pmatrix} \mathsf{G}_3 & 0\\ 0 & \mathsf{G}_3^2 \end{pmatrix} \text{ with } \mathsf{G}_3 = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The action of g_4 is slightly more complicated to read off. The coordinates for the two copies of S are (x_{1j}, x_{3k}) and (x_{2j}, x_{4k}) and g_4 acts by mapping

 $(x_1, x_3) \rightarrow (x_2, x_4)$, while $(x_2, x_4) \rightarrow (x_3, x_1)$

We can think of this action as being an exchange of the two copies of S followed by the involution $x_1 \leftrightarrow x_3$ on the first copy. We need to calculate the action of this involution on $H^2(S)$. To this end, choose one of the exceptional curves, say E_1 , which lies above (1,1,1) in $\mathbb{P}^2_{x_1}$. Then it is described in $\mathbb{P}^2_{x_3}$ by the line $x_{30} + x_{31} + x_{32} = 0$. On the coordinates, the involution acts as $x_1 \leftrightarrow x_3$, so it maps E_1 to the curve described by $x_{10}+x_{11}+x_{12}=0$ and $(x_{30}, x_{31}, x_{32}) = (1,1,1)$. Since this line passes through both $(1,\zeta,\zeta^2)$ and $(1,\zeta^2,\zeta)$ in the x_1 plane, which are the other points that are blown up, we see that it is actually the line referred to earlier as L_{23} . Thus

the action of the involution is $E_1 \leftrightarrow L_{23}$, or more generally $E_i \leftrightarrow L_{i+1,i+2}$. This is enough information to work out the action of g_4 on $H^2(\mathcal{S} \times \mathcal{S})$, with respect to the basis given above. The result is

$$U(g_4) = \begin{pmatrix} 0 & \mathsf{G}_2 \\ 1 & 0 \end{pmatrix} \text{ with } \mathsf{G}_2 = \begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

It is a quick check that $G_2^2 = G_3^3 = 1$ and that G_2 and G_3 commute and preserve the intersection matrix η . It is useful also to express the transformations in terms of the (-1)-lines of the two copies of S, eliminating the explicit reference to H. Denote by D_a and \tilde{D}_b , $a, b \in \mathbb{Z}_6$, the six (-1)-lines on the two copies of S, with ordering

$$D_a = (E_1, L_{12}, E_2, L_{23}, E_3, L_{31})$$
 and $\tilde{D}_b = (\tilde{E}_1, \tilde{L}_{12}, \tilde{E}_2, \tilde{L}_{23}, \tilde{E}_3, \tilde{L}_{31})$ (5.8)

In terms of these, the action of the generators g_6 and g_4 is

$$g_6: D_a \times \widetilde{D}_b \to D_{a-1} \times \widetilde{D}_{b+1} \text{ and } g_4: D_a \times \widetilde{D}_b \to D_{b+3} \times \widetilde{D}_a$$
 (5.9)

If we change our basis for $H^2(\mathcal{S} \times \mathcal{S})$ such that $U(g_3)$ becomes diagonal, we can compare with §5.1.2 and see that the eight-dimensional representation decomposes into the sum

$$R_1 \oplus R_{-1} \oplus R_{\mathrm{i}} \oplus R_{-\mathrm{i}} \oplus R_+^{(2)} \oplus R_+^{(2)}$$

In particular, there is only a single invariant, implying $h^{1,1} = 1$ for the quotient. This invariant corresponds to the canonical class of the ambient space, as it must, which we can see explicitly as follows. The group element $g_6 = g_3^2 g_4^2$ generates a \mathbb{Z}_6 subgroup, as noted previously. We have

$$U(g_6) = \begin{pmatrix} \mathsf{G}_6 & 0\\ 0 & \mathsf{G}_6^{-1} \end{pmatrix} \quad \text{where} \quad \mathsf{G}_6 = \mathsf{G}_2 \,\mathsf{G}_3^2 = \begin{pmatrix} 2 & 1 & 1 & 1\\ -1 & -1 & -1 & 0\\ -1 & 0 & -1 & -1\\ -1 & -1 & 0 & -1 \end{pmatrix}$$

which acts separately on the two surfaces. The two canonical classes

$$K_{\mathcal{S}} = 3H - E_1 - E_2 - E_3$$
 and $K_{\widetilde{\mathcal{S}}} = 3\widetilde{H} - \widetilde{E}_1 - \widetilde{E}_2 - \widetilde{E}_3$

are invariant under g_6 , and since the eigenvalues of G_6 are $\{1, -1, \zeta, \zeta^2\}$ we see that these are the only two invariant homology classes. A class that is invariant under g_6 is invariant under \mathfrak{G} if it is also invariant under g_4 , and we immediately see that the only such invariant combination is $K_S + \widetilde{K}_S$. The fact that $h^{1,1} = 1$ for the quotient variety implies $h^{1,1} = 1$ for the hypersurface $X_{\mathfrak{G}}$. Since the Euler number divides by the order of the group, we find that $\chi = -72/12 = -6$ and it follows that $h^{2,1} = 4$, in agreement with our count of parameters in the defining polynomials.

5.2 Symmetries of X and $X_{\mathfrak{G}}$

The study of possible symmetries of the quotient manifold is necessarily related to a study of the parameter space of the manifold since such symmetries will exist only for special values of the parameters. For the manifold $X^{8,44}$ and its quotient the symmetries originate in the symmetries of $S \cong dP_6$, so we start by discussing these.

5.2.1 Symmetries of $dP_6 \times dP_6$

We summarise the symmetries of $dP_6 \times dP_6$ following [89]. The dP_6 surface contains, as remarked previously, six (-1)-curves that intersect in a hexagon. It turns out that the entire symmetry group of the hexagon, that is, the dihedral group

$$\operatorname{Dih}_6 = \mathbb{Z}_6 \rtimes \mathbb{Z}'_2 \cong (\mathbb{Z}_2 \times \mathbb{Z}_3) \rtimes \mathbb{Z}'_2$$

acts on dP₆. In addition, the dP₆ surface is toric, corresponding to the fan over the polygon shown later in Figure 5.4, and so is acted on by the torus $(\mathbb{C}^*)^2$. As previously, we realise the surface as the complete intersection in $\mathbb{P}^2 \times \mathbb{P}^2$ of the polynomials

$$P = w_0 z_0 + w_1 z_2 + w_2 z_1 , \quad Q = w_1 z_1 + w_0 z_2 + w_2 z_0$$

In these coordinates, the symmetry group acts as follows:

• \mathbb{Z}_2 acts via the coordinate exchange

$$(w_0, w_1, w_2) \leftrightarrow (z_0, z_1, z_2)$$
 (5.10)

• \mathbb{Z}_3 acts as the phase rotation

$$\left((w_0, w_1, w_2), (z_0, z_1, z_2)\right) \mapsto \left((w_0, \zeta w_1, \zeta^2 w_2), (z_0, \zeta z_1, \zeta^2 z_2)\right)$$
(5.11)

• \mathbb{Z}'_2 acts via the coordinate exchange

$$\left((w_0, w_1, w_2), (z_0, z_1, z_2)\right) \mapsto \left((w_1, w_0, w_2), (z_1, z_0, z_2)\right)$$
(5.12)

• The toric action will be described in §5.4, where we discuss the toric point of view in detail.

5.2.2 Symmetries of the quotient

If we refer to the group above as Aut(S), then the symmetries of $S \times S$ are

$$\operatorname{Aut}(\mathcal{S} \times \mathcal{S}) = (\operatorname{Aut}(\mathcal{S}) \times \operatorname{Aut}(\mathcal{S})) \rtimes \mathbb{Z}_2''$$

where the \mathbb{Z}_2'' here exchanges the two factors. Not all of these symmetries descend to symmetries of the quotient $X_{\mathfrak{G}}$, since an element of $\operatorname{Aut}(\mathcal{S} \times \mathcal{S})$ can (i) fail to be a symmetry of the covering manifold X owing to the fact that it does not preserve the hypersurface r = 0, or (ii) it may fail to commute appropriately with \mathfrak{G} . We shall see presently that the symmetry group of the quotient $X_{\mathfrak{G}}$ is reduced to a subgroup of (a single copy of) the dihedral group Dih_6 . This subgroup is \mathbb{Z}_2 for generic values of the parameters, $\mathbb{Z}_2 \times \mathbb{Z}_2$ on a certain 3 parameter family of manifolds and the full group Dih_6 for an interesting 2 parameter family of singular varieties. The condition for a symmetry of the covering space to descend to a symmetry of the quotient is quite restrictive, so we postpone testing the invariance of the hypersurface r = 0 and begin with a discussion parallel to that of [90]. If we let π denote the projection onto the quotient by \mathfrak{G} , then the condition that a symmetry h be a symmetry of the quotient is that $h \pi(x) = \pi(hx)$, which is the condition that for each $g \in \mathfrak{G}$ there is an element $g' \in \mathfrak{G}$ such that gh = hg'. For the group \mathfrak{G} , this is the condition that for each g we have

$$hgh^{-1} = g_3^m g_4^n$$

for some integers $0 \le m \le 2$, $0 \le n \le 3$. It suffices to apply this condition to the generators g_3 and g_4 themselves. The form of the right hand side of this relation is restricted by the fact that hg_3h^{-1} must be an element of \mathfrak{G} that is of order 3 and hg_4h^{-1} must be of order 4. There are just two elements of order 3, which are g_3 and g_3^2 , and six of order 4, which are the elements $g_3^m g_4^m$ and $g_3^m g_4^3$ for $0 \le m \le 2$. Thus we may write

$$hg_3h^{-1} = g_3^{1+k}$$
, $hg_4h^{-1} = g_3^m g_4^{1+2n}$; $k = 0, 1$, $0 \le m \le 2$, $n = 0, 1$ (5.13)

Consider first the case that k, m and n all vanish; these are the symmetries h that commute with \mathfrak{G} . A symmetry that commutes with both g_3 and g_4 must be of the form h_3^l or $g_4^2 h_3^l$, for some $l \in \{0, 1, 2\}$, where

$$h_3: x_{\alpha j} \to \zeta^j x_{\alpha j}$$

This is the \mathbb{Z}_3 symmetry of (5.11), understood as applying symmetrically to the two copies of \mathcal{S} , and also a symmetry operation that we first saw in §5.1.1. Consider now the symmetry operation of (5.10), which we take to act on the first copy of \mathcal{S} only, since acting equally on both copies is equivalent to the operation g_4^2 ,

$$h_2: x_{1,j} \leftrightarrow x_{3,j}, x_{2,j} \rightarrow x_{2,j}, x_{4,j} \rightarrow x_{4,j}$$

It is an easy check that h_2 satisfies equation (5.13) with (k, m, n) = (0, 0, 1). Next we take a symmetry h'_2 corresponding to (5.12), which we take to act symmetrically on the two copies of S

$$h'_2: x_{\alpha,j} \to x_{\alpha,1-j}$$

This satisfies equation (5.13) with (k, m, n) = (1, 0, 0). We note also that $h = g_3^2$ satisfies equation (5.13) with (k, m, n) = (0, 1, 0). In terms of these symmetries we can give a transformation

$$h = (h_2')^n g_3^{2m} h_2^k$$

that satisfies equation (5.13) for general values of the integers k, m and n. Furthermore, this solution to the conditions is uniquely determined modulo h_3 and g_4^2 , since if h and \tilde{h} both satisfy equation (5.13) for given k, m and n, then $\tilde{h}h^{-1}$ commutes with \mathfrak{G} . We have shown that the symmetry group of $X_{\mathfrak{G}}$ is a subgroup of $\mathfrak{H} = \langle h_2, h'_2, h_3 \rangle$. One sees that

$$h_2h_3 = h_3h_2$$
, $h_2h'_2 = h'_2h_2$ and $h'_2h_3 = h_3^2h'_2$

We set $h_6 = h_2 h_3^2$ so that $h_2 = h_6^3$ and $h_3 = h_6^2$. Thus \mathfrak{H} is generated by h_6 and h_2' and we note that

$$h_2' h_6 = h_6^5 h_2'$$

We see that $\mathfrak{H} \cong \text{Dih}_6$ and we recover, in this way, the dihedral group.

We have met a number of symmetry operations in the course of this discussion, and we gather these together in Table 5.3 for reference. We now examine which of the symmetries of \mathfrak{H} preserve the hypersurface r = 0. The transformation h_2 affects only the coordinates x_{1j} and x_{3k} . The effect on the polynomials m_{ijkl} is $m_{ijkl} \to m_{kjil}$ and this clearly preserves the polynomials m_{iiii}

Symmetry	Relation	$x_{lpha j}$	D_a	\widetilde{D}_b
g_3	g_6^2	$\zeta^{(-1)^{lpha}j} x_{lpha j}$	D_{a-2}	\widetilde{D}_{b+2}
g_4		$x_{\alpha+1,j}$	\widetilde{D}_a	D_{b+3}
g_6	$g_3^2 g_4^2$	$\zeta^{(-1)^{lpha+1}j} x_{lpha+2,j}$	D_{a-1}	\widetilde{D}_{b+1}
h_2	h_6^3	$x_{1j} \leftrightarrow x_{3j}, \ x_{2j} \rightarrow x_{2j}, \ x_{4j} \rightarrow x_{4j}$	D_{a+3}	\widetilde{D}_b
h_2'		$x_{lpha, 1-j}$	D_{6-a}	\widetilde{D}_{6-b}
h_3	h_6^2	$\zeta^j x_{lpha j}$	D_{a+2}	\widetilde{D}_{b+2}
h_6	$h_{2}h_{3}^{2}$	$x_{1j} \rightarrow \zeta^{2j} x_{3j}, \ x_{2j} \rightarrow \zeta^{2j} x_{2j},$ $x_{3j} \rightarrow \zeta^{2j} x_{1j}, \ x_{4j} \rightarrow \zeta^{2j} x_{4j}$	D_{a+1}	\widetilde{D}_{b-2}

Table 5.3: A table of the various symmetry operations that we have met with their actions on the coordinates and the (-1)-lines D_a and \tilde{D}_b .

and it also preserves m_{0011} and m_{0212} since these are transformed to polynomials m_{ijkl} whose indices are related to the original ones by cyclic permutation. Thus $X_{\mathfrak{G}}$ is invariant under h_2 for all values of the parameters. The transformation h'_2 transforms $m_{ijkl} \rightarrow m_{1-i,1-j,1-k,1-l}$. The effect is to interchange the polynomials m_{0000} and m_{1111} , the other terms in r being invariant since the indices transform by a cyclic permutation. Thus r is invariant if $c_0 = c_1$. For generic coefficients satisfying this condition, the quotient variety is smooth and is invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action generated by h'_2 and h_2 . The transformation h_3 has been discussed in §5.1.1 and can be understood as inducing the transformation $(c_0, c_1, c_2, c_3, c_4) \rightarrow (c_0, \zeta c_1, \zeta^2 c_2, \zeta^2 c_3, \zeta^2 c_4)$ on the parameters. If we make the choice $c_0 = c_1 = 0$ then r also transforms homogeneously, $r \rightarrow \zeta^2 r$. Thus there is a \mathbb{P}^2 within the parameter space corresponding to parameters $(0, 0, c_2, c_3, c_4)$ for which each quotient $X_{\mathfrak{G}}$ has a group Dih_6 of automorphisms. These varieties are, however, all singular and, for generic values of c_2 , c_3 and c_4 , have 3 nodes. These nodal varieties will be studied in detail in §5.3, where we demonstrate that the nodes can be resolved to obtain a new family of Calabi-Yau manifolds. There are also two isolated points given by $c_j = (1, 0, 0, 0, 0)$ and $c_j = (0, 1, 0, 0, 0)$ that correspond to Dih₆-invariant varieties that are very singular. Although we have not yet described the torus action, we note here that for the special parameter choice $c_i = (0, 0, 1, 4, 4)$, the hypersurface r = 0 is invariant under the complete $(\mathbb{C}^*)^4$ action. This is a point corresponding to a very singular variety, which we will see later is the union of the 12 irreducible toric divisors.

5.3 Conifold transition to a manifold with $(h^{1,1}, h^{2,1}) = (2,2)$

We have noted that when $c_0 = c_1 = 0$ there is a two-parameter family of Dih₆ invariant varieties $X_{\mathfrak{G}}$ and that these are all singular, the generic member having 3 nodes. These arise as 36 nodes on the covering manifold X, which form three \mathfrak{G} -orbits, or a single $\mathfrak{G} \times \mathbb{Z}_3$ orbit where the \mathbb{Z}_3 is generated by h_3 . The fact that there are 36 singularities for generic (c_2, c_3, c_4) is best checked by a Gröbner basis calculation. The location of the singularities will be given presently, and once it is known that these are the only singularities then it is easy to check that they are nodes by expanding the equations in a neighbourhood of a singular point. Owing to the $\mathfrak{G} \rtimes \mathfrak{H}$ action it is sufficient to examine any one of the singularities locally. We describe the resolution of the three nodes on $X_{\mathfrak{G}}$ in two steps. First we demonstrate that the nodal varieties X admit Kähler small resolutions, by identifying smooth divisors which intersect the nodes in an appropriate way, and blowing up along these divisors. We then show that such a resolution is \mathfrak{G} -equivariant, and therefore yields a resolution of $X_{\mathfrak{G}}$.

5.3.1 The parameter space of 3-nodal quotients $X_{\mathfrak{G}}$

Before studying the question of singularity resolution, we will describe the parameter space Γ of 3-nodal, Dih₆-invariant quotients $X_{\mathfrak{G}}$. The generic quotient with $c_0 = c_1 = 0$ has three nodes, however on a certain locus within this space there are more severe degenerations. We find this locus to consist of the components listed in Table 5.4, which arise in the Gröbner basis calculation that finds the nodes. We have already seen most of these conditions at some point. For example, several of the linear conditions are those for elements of \mathfrak{G} to have fixed points on X. When this occurs, the covering space X necessarily has a node at the fixed point, as we saw in Chapter 4, so $X_{\mathfrak{G}}$ develops a hyperconifold singularity. The remaining linear conditions lead to varieties with additional nodes and/or quotients of nodes. We shall see the significance of the initial, quadratic, condition shortly. A sketch indicating the intriguing manner in which these components intersect is given in Figure 5.2. The curves listed in Table 5.4 show an unexpected symmetry under the \mathbb{Z}_2 -automorphism

$$c_2 \rightarrow 2 c_3 , \quad c_3 \rightarrow \frac{1}{2} c_2 , \quad c_4 \rightarrow -c_4$$

This operation fixes the curves $\Gamma^{(o)}$, $\Gamma^{(v)}$ and $\Gamma^{(x)}$ and interchanges the curves

 $\Gamma^{(i)} \ \leftrightarrow \ \Gamma^{(ii)} \ , \quad \Gamma^{(iii)} \ \leftrightarrow \ \Gamma^{(iv)} \ , \quad \Gamma^{(vi)} \ \leftrightarrow \ \Gamma^{(vii)} \ , \quad \Gamma^{(viii)} \ \leftrightarrow \ \Gamma^{(ix)}$

It is unclear whether this is a genuine symmetry of the geometry of the parameter space. It is not a symmetry at the same level as the \mathbb{Z}_3 -symmetry (5.5) which arises from a coordinate transformation that preserves the form of the defining polynomials of X. We can see this by noting that the transformation interchanges, for example, varieties on curve $\Gamma^{(i)}$, which have 6 nodes, with varieties on curve $\Gamma^{(ii)}$ which have singularities of a different type. The curve $\Gamma^{(x)}$ is not a curve of varieties that have more severe singularities, since the number of nodes here is still generically three. We include it in the discriminant as a curve that, being fixed under the

Component	Equation	Type of Singularity
Г(о)	$4c_2c_3 - c_4^2 = 0$	3 higher order singularities
$\Gamma^{(i)}$	$4c_2 + c_3 - 2c_4 = 0$	6 nodes
Γ ⁽ⁱⁱ⁾	$c_2 + 16 c_3 + 4 c_4 = 0$	3 nodes, 1 g_4^2 -node, 1 g_4 -node
$\Gamma^{(iii)}$	$c_2 = 0$	3 nodes, 1 G -node
$\Gamma^{(iv)}$	$c_3 = 0$	$3 \text{ nodes}, 1 g_3$ -node
$\Gamma^{(v)}$	$c_4 = 0$	$3 \text{ nodes}, 1 g_3$ -node
$\Gamma^{(vi)}$	$8 c_3 + c_4 = 0$	4 nodes
$\Gamma^{(\mathrm{vii})}$	$4c_2 - c_4 = 0$	4 nodes
$\Gamma^{(\mathrm{viii})}$	$c_3 - c_4 = 0$	5 nodes
$\Gamma^{(ix)}$	$c_2 + 2 c_4 = 0$	3 nodes, 1 g_4^2 -node
$\Gamma^{(x)}$	$c_2 + 2 c_3 = 0$	3 nodes

Table 5.4: The generic member of Γ is a variety with 3 nodes but along these curves the varieties develop more severe singularities. Some of these extra singularities are orbifolds of nodes as indicated. A g_3 -node, for example, is a node fixed by the symmetry g_3 .

automorphism, may require special consideration.

5.3.2 The nodes and their resolution

To describe the location of the singularities, we refer to the (-1)-lines on the two copies of S labelled and ordered as in equation (5.8). On each D_a we specify a unique point, $\operatorname{pt}(D_a)$, by the condition $x_{12}x_{32} = 0$. Thus the point $\operatorname{pt}(E_1)$, for example, is specified by the coordinate values $x_{1j} = (1, 1, 1)$ and $x_{3j} = (1, -1, 0)$, while the point $\operatorname{pt}(L_{12})$ is given by $x_{1j} = (1, -\zeta^2, 0)$ and $x_{3j} = (1, \zeta^2, \zeta)$. In an analogous way, we define on each \tilde{D}_a a point, $\operatorname{pt}(\tilde{D}_a)$, by the condition $x_{22}x_{42} = 0$. For brevity, we will often abbreviate $\operatorname{pt}(D_a) = \operatorname{pt}_a$ and $\operatorname{pt}(\tilde{D}_a) = \widetilde{\operatorname{pt}}_a$ respectively. The 36 nodes are the points

$$\operatorname{pt}(D_a) \times \operatorname{pt}(D_b)$$
, $a, b \in \mathbb{Z}_6$

The action of the symmetries $\mathfrak{G} \rtimes \mathfrak{H}$ on the nodes follows from the action of the symmetries on the (-1)-lines. For g_6 and g_4 these are given by equation (5.9) and the action of h_3 is easily read off from Table 5.2. We have

$$g_{6}: \operatorname{pt}_{a} \times \widetilde{\operatorname{pt}}_{b} \to \operatorname{pt}_{a-1} \times \widetilde{\operatorname{pt}}_{b+1}, \quad g_{4}: \operatorname{pt}_{a} \times \widetilde{\operatorname{pt}}_{b} \to \operatorname{pt}_{b+3} \times \widetilde{\operatorname{pt}}_{a}$$
$$h_{6}: \operatorname{pt}_{a} \times \widetilde{\operatorname{pt}}_{b} \to \operatorname{pt}_{a+1} \times \widetilde{\operatorname{pt}}_{b-2}, \quad h_{2}': \operatorname{pt}_{a} \times \widetilde{\operatorname{pt}}_{b} \to \operatorname{pt}_{6-a} \times \widetilde{\operatorname{pt}}_{6-b}$$
(5.14)

Notice that the generators of \mathfrak{G} preserve the sum $a + b \mod 3$ so this sum distinguishes the



Figure 5.2: Two sketches of the surface Γ , the locus of Dih₆-invariant varieties showing the discriminant of the space of 3-nodal varieties. The components of the discriminant locus are labeled according to Table 5.4. For the resolved manifold with Hodge numbers $(h^{1,1}, h^{2,1}) = (2, 2)$ this is the space of complex structures. The second sketch zooms out to show how the components intersect. The four intersections of the pairs of blue and purple lines lie on the dashed line $\Gamma^{(x)}$.

three \mathfrak{G} -orbits. The transformation h_3 permutes these orbits so the 36 nodes form a single orbit under $\mathfrak{G} \rtimes \mathfrak{H}$. The points $\operatorname{pt}_a \in S$ have the property that if the coordinates x_{1j} and x_{3j} are restricted to these values then the equation r = 0 is identically satisfied for all $(x_{2j}, x_{4k}) \in \widetilde{S}$. Thus $\mathcal{D}_a = \operatorname{pt}_a \times \widetilde{S}$ is a Weil divisor in X, as is $\widetilde{\mathcal{D}}_b = \mathcal{S} \times \widetilde{\operatorname{pt}}_b$. The six divisors \mathcal{D}_a are mutually disjoint, and each contains six nodes. The same applies to the six divisors $\widetilde{\mathcal{D}}_b$, and the two collections intersect precisely in the 36 nodes $\operatorname{pt}_a \times \widetilde{\operatorname{pt}}_b$. The configuration of the divisors and nodes is sketched in Figure 5.3. As we will see below, the given divisors are non-Cartier in a neighbourhood of each node, and we can blow up X along such a divisor to obtain a small resolution of each node it contains. We may therefore resolve all 36 nodes by blowing up each of the 'horizontal' divisors $\mathcal{S} \times \widetilde{\operatorname{pt}}_b$. In this way we obtain a Kähler manifold \widehat{X} that has vanishing first Chern class and $\chi = 0$. Alternatively, we can blow up each of the 'vertical' divisors $\operatorname{pt}_a \times \widetilde{S}$, but as we will see, this gives the same manifold.

We may examine the singularities and their resolutions locally since, as remarked previously, all the singularities are related by the group $\mathfrak{G} \rtimes \mathfrak{H}$. We expand about a singularity by taking coordinates $x_{\alpha 0} = 1$ and $x_{\alpha j} = x_{\alpha j}^{\sharp} + \epsilon_{\alpha j}$ for j = 1, 2, where the $x_{\alpha j}^{\sharp}$ are the coordinates of a singularity. We have 8 coordinates $\epsilon_{\alpha j}$ and we may solve the four equations $p^1 = p^2 = q^1 = q^2 = 0$ for the $\epsilon_{\alpha 2}$ as functions of the $\epsilon_{\alpha 1}$. We are left with the constraint r = 0 which, for the particular point $x^{\sharp} = (1, 1, 1) \times (1, -1, 0) \times (1, 1, 1) \times (1, -1, 0)$, becomes

$$\epsilon_1(A\,\epsilon_2 - B\,\epsilon_4) - \epsilon_3(B\,\epsilon_2 - C\,\epsilon_4) = 0 \tag{5.15}$$

where

$$A = 2(c_2 + c_3 + c_4)$$
, $B = 4c_2 - 2c_3 + c_4$, $C = 2(4c_2 + c_3 - 2c_4)$

We see that the singularity is indeed a node provided that the determinant of the matrix associated to the quadratic form does not vanish. This determinant is proportional to the quantity $(AC-B^2)^2 = 3^4(c_4 - 4c_2c_3)^2$, so this requirement provides an understanding of the quadratic condition in Table 5.4. We now see that the 'vertical' divisor $\mathcal{D}_0 = \text{pt}_0 \times \tilde{\mathcal{S}}$ is non-Cartier in a neighbourhood of the node, being given by the two local equations $\epsilon_1 = \epsilon_3 = 0$. We may blow up X along this divisor by introducing a \mathbb{P}^1 with coordinates (s_0, s_1) , and considering the following equation in $X \times \mathbb{P}^1$:

$$\epsilon_1 s_0 + \epsilon_3 s_1 = 0 \tag{5.16}$$

If $\pi: X \times \mathbb{P}^1 \to X$ is projection onto the first factor, then the blow up of X is given by

$$\widehat{X} = \overline{\pi^{-1}(X \setminus \mathcal{D}_0)}$$

that is, \widehat{X} is the closure of the preimage of the smooth points of X. As usual, \widehat{X} defined in this way is indeed just X with each node on \mathcal{D}_0 replaced by a \mathbb{P}^1 . Alternatively we may blow up along the 'horizontal' divisor $\widetilde{\mathcal{D}}_0 = \mathcal{S} \times \widetilde{\mathrm{pt}}_0$; the discussion is the same, but instead of (5.16) we take

$$(B\epsilon_2 - C\epsilon_4)s_0 + (A\epsilon_2 - B\epsilon_4)s_1 = 0$$
(5.17)



Figure 5.3: The divisors $\operatorname{pt}_a \times \widetilde{\mathcal{S}}$ and $\mathcal{S} \times \widetilde{\operatorname{pt}}_b$. These intersect in the 36 nodes which form three \mathfrak{G} -orbits that are distinguished by colour.
The vanishing of the combinations $B \epsilon_2 - C \epsilon_4$ and $A \epsilon_2 - B \epsilon_4$ is equivalent to the vanishing of ϵ_2 and ϵ_4 , provided $AC - B^2 \neq 0$. We can see that the two resolutions are identical by observing that they are each given by the following matrix equation in $\mathbb{C}^4 \times \mathbb{P}^1$, where $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \in \mathbb{C}^4$:

$$\begin{pmatrix} \epsilon_1 & \epsilon_3 \\ B \epsilon_2 - C \epsilon_4 & A \epsilon_2 - B \epsilon_4 \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \end{pmatrix} = 0$$

We can now argue that our resolution of the 36 nodes on X is \mathfrak{G} -equivariant, and therefore gives a resolution of the 3 nodes on $X_{\mathfrak{G}}$. Suppose we blow up the six divisors \mathcal{D}_a . These are just permuted by g_6 , so the resolution is manifestly \mathbb{Z}_6 invariant. The element g_4 on the other hand interchanges the six \mathcal{D}_a for the six $\widetilde{\mathcal{D}}_a$, but we have shown above that at each node $\operatorname{pt}_a \times \widetilde{\operatorname{pt}}_b$ we obtain the same resolution whether we blow up \mathcal{D}_a or $\widetilde{\mathcal{D}}_b$. The resolution is therefore also equivariant under the g_4 action, and thus under the action of the whole group \mathfrak{G} . Finally we can ask about the Hodge numbers of $\widehat{X}_{\mathfrak{G}}$. The space of complex structures is two-dimensional, so $h^{2,1}(\widehat{X}_{\mathfrak{G}}) = 2$. To obtain $h^{1,1}$, note that we blow up a single divisor on X/\mathbb{Z}_6 , and this resolution happens also to be g_4 -covariant, so we simply have $h^{1,1}(\widehat{X}_{\mathfrak{G}}) = h^{1,1}(X_{\mathfrak{G}}) + 1 = 2$. This is consistent, because we obtain $\widehat{X}_{\mathfrak{G}}$ by resolving 3 nodes, which gives $\chi(\widehat{X}_{\mathfrak{G}}) = \chi(X_{\mathfrak{G}}) + 6 = 0$. The considerations above suggest, along the lines of [39], that there may be 3-generation heterotic models on $\widehat{X}_{\mathfrak{G}}$ that derive from the model we present on $X_{\mathfrak{G}}$ is at least Dih₆ at all points in its moduli space, so any such theory will feature this quite large discrete symmetry group.

5.3.3 Identifying the (2,2) manifold with a quotient of $X^{19,19}$

We can in fact give a very explicit description of the resolved manifold, which turns out to be yet another quotient of $X^{19,19}$. To see this, we work on the covering space, and note that the divisor we blow up, given by the union of the six \mathcal{D}_a , can be specified globally by the equations

$$x_{10}x_{31} + x_{11}x_{30} = x_{12}x_{32} = 0 (5.18)$$

as well as the vanishing of the polynomials (5.4) defining X. This corresponds to the fact that, when we set $c_0 = c_1 = 0$, the polynomial r simplifies to

$$r = x_{12}x_{32} \left(\frac{1}{4} c_4 x_{21} x_{40} + \frac{1}{4} c_4 x_{20} x_{41} + c_2 x_{22} x_{42} \right) + (x_{10} x_{31} + x_{11} x_{30}) \left(\frac{1}{4} c_3 x_{21} x_{40} + \frac{1}{4} c_3 x_{20} x_{41} + \frac{1}{4} c_4 x_{22} x_{42} \right)$$
(5.19)

We see now that we are in the very familiar situation of splitting a CICY configuration. Explicitly, we blow up the six \mathcal{D}_a by introducing a \mathbb{P}^1 , with homogeneous coordinates (s_0, s_1) , and taking the new polynomials

$$p^{1} = x_{10} x_{30} + x_{11} x_{32} + x_{12} x_{31} , \quad q^{1} = x_{11} x_{31} + x_{10} x_{32} + x_{12} x_{30}$$
$$p^{2} = x_{20} x_{40} + x_{21} x_{42} + x_{22} x_{41} , \quad q^{2} = x_{21} x_{41} + x_{20} x_{42} + x_{22} x_{40}$$

$$r_1 = s_0(x_{10}x_{31} + x_{11}x_{30}) - s_1x_{12}x_{32}$$

$$r_{2} = s_{0} \left(\frac{1}{4} c_{4} x_{21} x_{40} + \frac{1}{4} c_{4} x_{20} x_{41} + c_{2} x_{22} x_{42} \right) + s_{1} \left(\frac{1}{4} c_{3} x_{21} x_{40} + \frac{1}{4} c_{3} x_{20} x_{41} + \frac{1}{4} c_{4} x_{22} x_{42} \right)$$

These correspond to the configuration matrix

$$X^{19,19} = \begin{array}{ccccc} \mathbb{P}^{1} \begin{bmatrix} 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ \mathbb{P}^{2} & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \\ \mathbb{P}^{2} & \mathbb{P}^{2} & \begin{bmatrix} \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \end{bmatrix}_{0}^{19,19}$$

This is not how we have presented the manifold $X^{19,19}$ previously, but we confirm that it is indeed the same manifold as follows. It can be checked that the configuration above gives Euler number $\chi = 0$. If we contract (say) the first and second \mathbb{P}^2 's of the above configuration, we arrive at a familiar configuration — that of the split bicubic:

$$\mathbb{P}^{1} \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ \mathbb{P}^{2} \begin{bmatrix} 3 & 0 \end{bmatrix}_{0}^{19}$$

The fact that the contraction does not change the Euler number means that the two manifolds are actually identical, so we conclude that the resolved manifold is $X^{19,19}$.

We have argued above that the resolution is \mathfrak{G} -equivariant, which implies that there is a free action of \mathfrak{G} on $X^{19,19}$. We omit the details, but it is straightforward to see that this is true, by extending the group action to the new coordinates (s_0, s_1) , and checking that the polynomials given above are fixed point free. We already know that they are transverse, as we resolved all the singularities of $X_{\mathfrak{G}}$, but this can also be confirmed explicitly as a check.

5.4 Toric considerations and the mirror manifold

5.4.1 The Newton polyhedron and its dual

It is a felicitous fact that the del-Pezzo surface dP_6 is toric and has a fan with six onedimensional cones $\{v_a\}$ as shown in Figure 5.4. These correspond to the toric divisors, which



Figure 5.4: The fan and polygon for dP_6 . The one-dimensional cones correspond to the toric divisors as indicated.

are exactly the D_a . Let Σ denote this collection of one-dimensional cones

$$\Sigma = \{v_a\} = \{(1,1), (0,1), (-1,0), (-1,-1), (0,-1), (1,0)\}$$

We can also think of Σ as the polygon over the fan, which in the present case is a hexagon. The dual polygon is naïvely Σ rotated by 90°, but by appropriate choice of coordinates in M we can simply identify the two. We have previously been specifying $S \cong dP_6$ as a complete intersection in $\mathbb{P}^2 \times \mathbb{P}^2$, as described in §5.1. In order to describe the toric action it is convenient to first make a change of coordinates to

$$y_{\alpha j} = \sum_{k} V_{jk} x_{\alpha k}$$
 where $V = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta^2 & \zeta \\ 1 & \zeta & \zeta^2 \end{pmatrix}$ (5.20)

This brings the polynomials p^1 and q^1 to the form

$$p^1 = y_{10} y_{30} + y_{11} y_{31} + y_{12} y_{32} , \quad q^1 = y_{10} y_{30} + \zeta^2 y_{11} y_{31} + \zeta y_{12} y_{32}$$

In these coordinates the blown-up points in the y_1 plane are (1, 0, 0), (0, 1, 0) and (0, 0, 1). It is easy to see that for any $(\mu, \lambda) \in (\mathbb{C}^*)^2$, the polynomials are invariant under

$$(y_{10}, y_{11}, y_{12}) \times (y_{30}, y_{31}, y_{32}) \rightarrow (y_{10}, \lambda y_{11}, \mu y_{12}) \times (y_{30}, \lambda^{-1} y_{31}, \mu^{-1} y_{32})$$

The torus $(\mathbb{C}^*)^2$ can then be explicitly embedded in dP₆ as follows:

$$(t_1, t_2) \rightarrow (1, t_1 \zeta, t_2 \zeta^2) \times (1, t_1^{-1} \zeta, t_2^{-1} \zeta^2) \subset \mathbb{P}^2 \times \mathbb{P}^2$$
 (5.21)

We can do the same for the second copy $\widetilde{\mathcal{S}}$, and call the extra toric variables t_3, t_4 . The polyhedron, ∇ , over the fan for $\mathcal{S} \times \widetilde{\mathcal{S}}$ is obtained as the convex hull of the union of Σ with a second orthogonal copy $\widetilde{\Sigma}$, corresponding to $\widetilde{\mathcal{S}}$, so its vertices are

$$\operatorname{vert}(\nabla) = \{ (v_a, 0) \mid v_a \in \Sigma \} \cup \{ (0, v_b) \mid v_b \in \widetilde{\Sigma} \}$$

$$(5.22)$$

The Newton polyhedron, Δ , is dual to ∇ , in the sense of §5.4.1, and is given by the Minkowski sum of Σ and $\tilde{\Sigma}$, with vertices

$$\operatorname{vert}(\Delta) = \{ (v_a, v_b) \mid v_a \in \Sigma, v_b \in \widetilde{\Sigma} \}$$

$$(5.23)$$

The polyhedron ∇ has 12 vertices, which are the points given explicitly by equation (5.22), and no other integral points apart from the origin. It has 36 three-faces that are tetrahedra. The polyhedron Δ has 36 vertices and 12 three-faces that are all hexagonal prisms; these each have two hexagonal two-faces and six rectangular two-faces. The 36 points given explicitly by equation (5.23) are the vertices. The polyhedron also contains the origin and 12 additional integral points, one interior to each of the hexagonal two-faces. These 12 additional points are in fact the vertices of ∇ , so we have $\nabla \subset \Delta$. These facts are most quickly established by having recourse to a programme such as POLYHEDRON or PALP [91] that analyse reflexive polyhedra. We can, with benefit of hindsight, get useful insight into this structure and understand Δ and ∇ rather simply in terms of the divisors D_a and \tilde{D}_b . We will not rederive the structure of the polyhedra but will content ourselves with giving a description using only what we know about these divisors. It is easy to see from the relation (5.23) that there is a one-to-one correspondence between vertices of Δ and pairs of divisors

$$\nu_{a\,b} \leftrightarrow (D_a, D_b) , \quad a, b \in \mathbb{Z}_6$$

The integral points that are interior to the six blue (resp. pink) hexagonal two-faces in Figure 5.5 each correspond to a divisor \tilde{D}_b (resp. D_a) and will be labelled $\tilde{\iota}_b$ (resp. ι_a); the vertices of this hexagonal face are the ν_{ab} as a (resp. b) varies. We see that the same vertices arise in both the blue and pink three-faces, and this gives the correspondence between the blue and pink two-faces in Figure 5.5. The rectangular two-faces contain the vertices $\{\nu_{a,b}, \nu_{a+1,b}, \nu_{a+1,b+1}, \nu_{a,b+1}\}$. Having associated the points of Δ with the divisors D_a and \tilde{D}_b we see that there is a natural $\mathfrak{G} \rtimes \mathfrak{H}$ action on the points. Let ρ and σ denote the matrices

$$\rho = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and note that $\rho^6 = \mathbf{1}$ and $\rho^3 = -\mathbf{1}$. It is an immediate check that these act as $\rho v_a = v_{a-1}$ and $\sigma v_a = v_{6-a}$. In virtue of the polyhedron of Figure 5.4 it comes as no surprise that ρ and σ furnish a representation of Dih₆. We know the action of $\mathfrak{G} \rtimes \mathfrak{H}$ on the D_a and \widetilde{D}_b from equation (5.14) and in this way we see that the action of the generators on the points of Δ is given by a linear action on the lattice M, with matrices

$$g_6 = \begin{pmatrix} \rho & 0\\ 0 & \rho^{-1} \end{pmatrix}, \qquad g_4 = \begin{pmatrix} 0 & -1\\ 1 & 0 \end{pmatrix}$$
$$h_6 = \begin{pmatrix} \rho^{-1} & 0\\ 0 & -\rho \end{pmatrix}, \qquad h'_2 = \begin{pmatrix} \sigma & 0\\ 0 & \sigma \end{pmatrix}$$



Figure 5.5: The three-faces of the polyhedron Δ . Six hexagonal three-faces are stacked to make the blue prism on the left and the remaining six are stacked to make the pink prism. The top and bottom faces of these prisms are identified to make the two intersecting rings shown on the right. The faces that are depicted as exposed are identified between the two rings with the result that there is in fact no boundary.

The method we have used relies on the fact that the polygon Σ is self-dual; a more general approach would be to use (5.21) to find the action of $\mathfrak{G} \rtimes \mathfrak{H}$ on the toric coordinates, and translate this to an action on M by utilising its identification with rational monomials on the torus. So far we have described Δ . The polyhedron ∇ is simpler. The 12 vertices of ∇ are the points ι_a and $\tilde{\iota}_b$. Thus ∇ is contained in Δ and the vertices of ∇ are the points interior to the two-faces of Δ . The three-faces are the 36 tetrahedra with vertices { ι_a , ι_{a+1} , $\tilde{\iota}_b$, $\tilde{\iota}_{b+1}$ }. Owing to the fact that ∇ is contained in Δ , the group acts on the points of ∇ in the same representation as the action on Δ .



Figure 5.6: The vertices of the three-faces of the polyhedron ∇ are the interior points to the two-faces of Δ . The three-faces are tetrahedra, as shown in the first figure. Note however a hazard of projecting from four dimensions to three. Four of the vertices of Δ project onto the faces of the tetrahedron, but they do not actually lie on the tetrahedron, as they appear to in the figure. Six of these tetrahedra fit together to form the polyhedron in the center. Six of these polyhedra, in turn, fit together to form the star-shaped polyhedron on the right, with the exposed faces identified in pairs.

5.4.2 Triangulations

The mirror of $X^{8,44}$ is realised as the resolution of a hypersurface in the toric variety defined by the fan over the faces of Δ . The toric variety defined by Δ is singular, and since the singularities will intersect a hypersurface, so is the hypersurface. The singularities of the hypersurface are resolved by resolving the singularities of the embedding space [37]. This is done by sub-dividing the cones to refine the fan. The cones are sub-divided by sub-dividing the three-faces of Δ into smaller polyhedra, and the ambient variety becomes smooth if the faces of Δ are divided into polyhedra of minimal volume (which must then be tetrahedra of minimal volume). This process of sub-dividing the top-dimensional faces is known as triangulation. We start with an $\mathfrak{G} \rtimes \mathfrak{H}$ invariant triangulation of Δ by dividing the 3-faces into wedges as shown in Figure 5.6. We may denote the blue and pink wedges that contain the two-face { $\nu_{a,b}$, $\nu_{a+1,b}$, $\nu_{a+1,b+1}$, $\nu_{a,b+1}$ } by $W_{a,b}$ and $\widetilde{W}_{a,b}$ respectively,

$$W_{ab} = \{\nu_{a,b}, \nu_{a+1,b}, \nu_{a+1,b+1}, \nu_{a,b+1}, \tilde{\iota}_b, \tilde{\iota}_{b+1}\}$$
$$\widetilde{W}_{ab} = \{\nu_{a,b}, \nu_{a+1,b}, \nu_{a+1,b+1}, \nu_{a,b+1}, \iota_a, \iota_{a+1}\}$$

The group $\mathfrak{G} \rtimes \mathfrak{H}$ acts on the wedges as expected: $g_6 W_{ab} = W_{a-1,b+1}$ and $g_4 W_{ab} = \widetilde{W}_{b+3,a}$, and so on.

The triangulation of Δ into the wedges yields a toric variety with singularities along curves, and therefore a hypersurface with point singularities. Each wedge can be cut into 3 tetrahedra of minimal lattice volume. This further subdivision will yield a smooth toric variety. For reasons that will become clear shortly, we only enforce the \mathfrak{G} -symmetry at this point. Therefore, a fundamental region for the group action is a three-face of Δ (one-sixth of the blue or pink prism in Figure 5.5), consisting of 6 wedges. Each wedge can be triangulated in 6 different ways. These 6 possibilities can be distinguished by how they bisect the three rectangular two-faces of the wedge,¹ as shown in Figure 5.7. We will label these choices as $(uUu), \ldots, (dDu)$ corresponding



Figure 5.7: The 6 different triangulations of a wedge inside a three-face of Δ , which is a fundamental region for the triangulation.

to whether the three lines go up or down. When assembling the triangulated wedges into the hexagonal fundamental region, we must ensure that the triangulations match along the two-faces of the wedges. Therefore, the whole triangulation can be written as a cyclic string of six wedge-triangulations such that only two *u*'s or two *d*'s meet, that is, $(\cdots u)(u \cdots)$ or $(\cdots d)(d \cdots)$. This ensures a consistent triangulation in the interior of the fundamental region. The boundary of the fundamental region will intersect the boundaries of some of its \mathfrak{G} -images and compatibility of the triangulations along the outward-facing two-faces requires that the *i*-th and the (i+3)-rd wedge have the rectangular two-face cut in the same way, both 'U' or both 'D'. Up to symmetries² of the fundamental region, there are 6 distinct triangulations of a three-face. These are shown in

 $^{^{1}}$ A wedge has three rectangular faces and each of these can be bisected in two ways so there are eight ways to bisect the faces. Two of these ways, however, do not correspond to triangulations of the wedge.

 $^{^{2}}$ The symmetries of the fundamental region are Dih₆ transformations together with a reflection in a 'horizontal' plane.

Triangulation	Regular
$\{(dDu), (uUd), (dDu), (uDu), (uUd), (dDd)\}$	No
$\{(dDu), (uUd), (dDd), (dDu), (uUd), (dDd)\}$	Yes
$\{(dDu), (uUd), (dDd), (dDd), (uUu), (dDd)\}$	No
$\{(dDu), (uDu), (uUd), (dDd), (dDd), (uUu)\}$	No
$\{(dDd), (dDd), (dDd), (dDd), (dDd), (dDd)\}$	No
$\{(dDd), (dDd), (uUu), (dDd), (dDd), (uUu)\}$	No

Table 5.5: Distinct choices for the triangulation of one hexagonal three-face, the fundamental region for the \mathfrak{G} -action. The last column states whether the ensuing \mathfrak{G} -invariant triangulation of Δ is regular.

Table 5.5. It is interesting that precisely one of these yields a regular \mathfrak{G} -invariant triangulation of Δ . Recall that a regular triangulation is one that is induced by the 'crease lines' of the graph of a convex support function, a condition that is equivalent to the toric variety being Kähler. We always use the regular triangulation in the following. It is interesting also that the regular triangulation is not the \mathfrak{H} -invariant triangulation $(dDd)(dDd)(dDd)\dots(dDd)$. Thus we learn that the mirror manifold is not Dih₆-invariant. Note, however, that the the regular triangulation (dDu)(uUd)(dDd)(dDu)(uUd)(dDd) does repeat with period 3. The upper-case characters that correspond to the triangulations of the two-faces of Δ are required to repeat with period three, in virtue of our observations above, but the lower-case characters are not constrained by this requirement. We see that the mirror remains invariant under the symmetry h_2 .

Note that Δ has 49 points, none of which lie interior to a three-face. These yield $h^{1,1} = 44$ divisor classes after deleting the origin and modding out the 4 linear relations between the points. We have seen that there is a \mathfrak{G} -action on Δ , and there is a corresponding \mathfrak{G} -action on the divisors. Since the group action on the toric hypersurface is free, the invariant combinations of divisors form a basis for the divisors on the quotient. In other words, one has to identify the divisors on the covering space with their \mathfrak{G} -images. There are 4 orbits, and, therefore, $4 (= h^{1,1}(X_{\mathfrak{G}}^*))$ linearly independent divisors on the quotient. Consider further the \mathfrak{G} -action on the points of Δ . These fall into five orbits which we denote by Δ_i , $0 \le i \le 4$. The origin of Δ , which forms an orbit of length one, is Δ_0 . Of the four remaining orbits, one, which we choose to be Δ_1 , consists of the 12 points of Δ that are internal to two-faces. These are the points of $\nabla - \iota_a = (v_a, 0)$ and $\tilde{\iota}_b = (0, v_b)$. The remaining three orbits consist of the vertices $\nu_{ab} = (v_a, v_b)$ which fall into orbits according to the value of $a + b \mod 3$. We take Δ_2 , Δ_3 and Δ_4 to consist of the vertices ν_{ab} such that $a + b \mod 3$ takes the values 0, 1 and 2, respectively. We abuse notation by identifying divisors on the quotient with the corresponding orbits of vertices. As divisors we have a relation

$$\Delta_0 = -\sum_{i=1}^4 \Delta_i$$

The calculation that finds the convex piecewise linear function that determines the regularity of the triangulation yields also the generators of the the Mori cone (the dual to the Kähler cone). These are given by

	Δ_0	Δ_1	Δ_2	Δ_3	Δ_4
$l_1 = (-$	-1,	0,	0, -	-1,	2)
$l_2 = ($	0, -	-1,	0,	1,	0)
$l_3 = ($	0,	2, -	-1, -1	-1,	0)
$l_4 = ($	0, -	-1,	2,	0, -	-1)

5.4.3 The mirror manifold

We have previously observed that ∇ is contained in Δ . As a result, the mirror family is defined by a polynomial which is a specialisation of r, given by constraints on the coefficients. We will denote this specialisation by r^* . The equation $r^* = 0$ defines a singular variety which, generically, has 72 nodes, and the mirror X^* is obtained as its resolution. In §5.3, in order to construct the (2, 2) manifold, we resolved the 36-nodal varieties by finding a suitable set of divisors and blowing up along these. In this way we demonstrated the existence of a \mathfrak{G} -invariant Calabi-Yau resolution. In the present case we do not know of suitable divisors, however we may now avail ourselves of the techniques of toric geometry. We have just seen that there exists a maximal triangulation of Δ which is regular and \mathfrak{G} -invariant, and this provides the \mathfrak{G} -invariant, Calabi-Yau resolution of the 72-nodal varieties corresponding to $r^* = 0$.

Let us therefore consider the form of the polynomial r. The integral points of Δ correspond to monomials on the embedded torus, and r, restricted to the torus, is a linear combination of these. A four-parameter family of \mathfrak{G} -invariant Laurent polynomials is obtained by writing

$$f = \sum_{i=0}^{4} \gamma_i f_i$$
 where $f_i = \sum_{u \in \Delta_i} t^u$

Being invariant, this family of Laurent polynomials must be equivalent to the family from (5.4) and the γ_i must be another system of coordinates on the parameter space and so expressible in terms of the c_i . To determine the relations,³ we set r = f on the embedded torus given in (5.21). In this way we find the following correspondence

$$\begin{aligned} \gamma_0 &= c_2 + c_3 + c_4 & c_0 &= 3(\zeta \gamma_2 + \zeta^2 \gamma_3 + \gamma_4) \\ \gamma_1 &= \frac{1}{12} (4c_2 - 2c_3 + c_4) & c_1 &= 3(\zeta^2 \gamma_2 + \zeta \gamma_3 + \gamma_4) \\ \gamma_2 &= \frac{1}{36} (4\zeta^2 c_0 + 4\zeta c_1 + (4c_2 + c_3 - 2c_4)) & c_2 &= \frac{1}{9} (\gamma_0 + 12\gamma_1 + 12(\gamma_2 + \gamma_3 + \gamma_4)) & (5.24) \\ \gamma_3 &= \frac{1}{36} (4\zeta c_0 + 4\zeta^2 c_1 + (4c_2 + c_3 - 2c_4)) & c_3 &= \frac{4}{9} (\gamma_0 - 6\gamma_1 + 3(\gamma_2 + \gamma_3 + \gamma_4)) \\ \gamma_4 &= \frac{1}{36} (4c_0 + 4c_1 + (4c_2 + c_3 - 2c_4)) & c_4 &= \frac{4}{9} (\gamma_0 + 3\gamma_1 - 6(\gamma_2 + \gamma_3 + \gamma_4)) \end{aligned}$$

We have seen that ∇ is obtained by deleting the vertices of Δ . Thus $\nabla = \Delta_0 \cup \Delta_1$ and the

³Of course there is a scaling ambiguity in r, so in fact the relations between the γ_i and c_i are only determined up to scale.

polynomial r^* corresponds to setting $\gamma_2 = \gamma_3 = \gamma_4 = 0$. In virtue of the relations above we see that this is equivalent to the conditions

$$c_0 = c_1 = 0$$
 and $4c_2 + c_3 - 2c_4 = 0$ (5.25)

We learn that the parameter space of the mirror is contained as a curve in the parameter space of $X_{\mathfrak{G}}$ and, moreover, that this curve lies in Γ and is the curve $\Gamma^{(i)}$ of Table 5.4. It is worth pursuing the forms of f and r^* a little further. From the relation

$$f = \gamma_0 + \gamma_1 f_1$$
 with $f_1 = t_1 + \frac{1}{t_1} + t_2 + \frac{1}{t_2} + t_3 + \frac{1}{t_3} + t_4 + \frac{1}{t_4} + \frac{t_1}{t_2} + \frac{t_2}{t_1} + \frac{t_3}{t_4} + \frac{t_4}{t_3}$

it is compelling that the point corresponding to the large complex structure limit is the point $\gamma_1 = 0$, which in terms of the c_j is $c_j = (0, 0, 1, 4, 4)$. As may be seen from Figure 5.2, this is a point where the components of the discriminant locus have a high order contact so it may well be necessary to blow up this point, as in Figure 5.8, in order to discuss the monodromies about the large complex structure limit adequately. In any event, we have come rather rapidly to an identification of this limit.

Returning to the polynomial r^* we have, on imposing the conditions in equation (5.25)

$$r^* = \frac{1}{9}\gamma_0 \left(m_{2222} + 4m_{0011} + 4m_{0212} \right) + \frac{4}{3}\gamma_1 \left(m_{2222} - 2m_{0011} + m_{0212} \right)$$
(5.26)

The polynomial varying with γ_0 factorises

 $m_{2222} + 4m_{0011} + 4m_{0212} = s(x_1, x_3) s(x_2, x_4)$ with $s(w, z) = w_0 z_1 + w_1 z_0 + w_2 z_2$

The part of r that varies with γ_1 also simplifies

$$m_{2222} - 2m_{0011} + m_{0212} = -\frac{1}{2}s(x_1, x_3)s(x_2, x_4) + \frac{3}{4}(s(x_1, x_3)x_{22}x_{42} + x_{12}x_{32}s(x_2, x_4))$$

The polynomial s is a natural analogue of p and q, in the sense that the monomials in p have indices which sum to 0 mod 3, and the indices for the monomials of q sum to 2 mod 3. For s the indices sum to 1 mod 3. Consider now the locus s = 0 in dP₆, which corresponds to the locus p = q = s = 0 in $\mathbb{P}^2 \times \mathbb{P}^2$. At first sight one might be tempted to identify this as a torus, since it is given by three bilinear equations in $\mathbb{P}^2 \times \mathbb{P}^2$. This however is a false conclusion owing to the fact that the intersection p = q = s = 0 is not transverse. A little thought, and



Figure 5.8: The resolution of the point (1, 4, 4) of Figure 5.2 requires a sequence of two blow ups which introduce the two exceptional divisors E_0 and E_1 .

consultation with Table 5.2, reveals that all six divisors D_a lie in the locus s = 0 and that this locus is precisely the hexagon formed by the D_a . We may think of this as a degenerate elliptic curve which has become a chain of six \mathbb{P}^1 's. The hexagon less its vertices consists of the six one-dimensional orbits of the torus action and the vertices are the zero-dimensional orbits. Therefore when $\gamma_1 = 0$, so that $r^* = \frac{1}{9}s(x_1, x_3)s(x_2, x_4)$, we obtain a very singular variety which is invariant under the whole torus action.

The reader may be worried about an apparent contradiction between the fact that f_0 appears to be equal to unity, and (5.26), where $f_0 = \frac{1}{9}s(x_1, x_3) s(x_2, x_4)$. This is resolved by noticing that in writing r = f only on the torus (5.21), it is implicit that $y_{\alpha 0} = 1$. Writing s in terms of the torus coordinates, we find that $s(x_1, x_3) = s(x_2, x_4) = 3$. This is a result of the choice $y_{\alpha 0} = 1$. It does not contradict the fact that s vanishes on the hexagon since no point of the hexagon lies on the torus. With s = 3, we get $f_0 = \frac{1}{9}s(x_1, x_3)s(x_2, x_4) = 1$, so there is, in fact, no contradiction.

For generic γ_1/γ_0 , the variety described by the family of polynomials r^* has 72 nodes which form six \mathfrak{G} -orbits. These comprise the 36 nodes that we have met previously and 36 nodes that are new and that are located at the points $\operatorname{pt}'_a \times \widetilde{\operatorname{pt}}'_b$, where pt'_a denotes the point on the hexagon corresponding to the intersection of the divisors D_a and D_{a+1} , with $\widetilde{\operatorname{pt}}'_b$ understood analogously. The polyhedron Δ , with its \mathfrak{G} -invariant triangulation, provides a Calabi–Yau resolution of these nodes. The resolution of 6 nodes on the quotient $X_{\mathfrak{G}}$ gives a manifold with $\chi = +6$ and Hodge numbers $(h^{1,1}, h^{2,1}) = (4, 1)$, which we identify with the mirror, $X^*_{\mathfrak{G}}$, of $X_{\mathfrak{G}}$. In Table 5.6 we list the values of the parameter for which the variety, whose resolution is $X^*_{\mathfrak{G}}$, develops extra singularities. This occurs at the values of γ_0/γ_1 for which $\Gamma^{(i)}$ intersects the other components of the discriminant. In each case where $\Gamma^{(i)}$ intersects another component Γ' , the generic singularities of each. So the singularities associated with such an intersection are the 3 generic nodes, together with the 3 extra nodes of $\Gamma^{(i)}$ and the extra singularities of Γ' .

Curve	(o)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)
γ_0/γ_1	∞	4	-12	6	-3	5	∞	∞	-4	4
Sing.		$6, 1 g_4, 1 g_4^2$	$6, 1 \mathfrak{G}$	$6, 1 g_3$	$6, 1 g_3$	7			$6,1g_{4}^{2}$	6

Table 5.6: The values of the parameter γ_0/γ_1 for which the mirror manifold is singular together with the type of singularity. The values given for γ_0/γ_1 are those corresponding to the intersection of $\Gamma^{(i)}$ with the other components of the discriminant locus that were listed in Table 5.4. The entry $(6, 1 g_4, 1 g_4^2)$, for example means 6 nodes, 1 g_4 -node and 1 g_4^2 -node. Where the intersection is at the large complex structure limit, for which $\gamma_0/\gamma_1 = \infty$, the singularity is not listed.

5.5 The Abelian quotient

The manifold X also admits a free quotient by the Abelian group \mathbb{Z}_{12} . With notation similar to that used in §5.1, the action of the group generator is given by

$$g_{12}: x_{\alpha j} \rightarrow \zeta^j x_{\alpha+1,j} , \quad q^1 \rightarrow \zeta^2 q^2 , \quad q^2 \rightarrow \zeta^2 q^1 , \quad s^1 \rightarrow \zeta s^2 , \quad s^2 \rightarrow \zeta s^1 , \quad r \rightarrow r$$

and the covariant polynomials are

$$q^{1} = x_{10} x_{32} + x_{12} x_{30} + x_{11} x_{31} , \qquad s^{1} = x_{10} x_{31} + x_{11} x_{30} + x_{12} x_{32}$$
$$q^{2} = x_{20} x_{42} + x_{22} x_{40} + x_{21} x_{41} , \qquad s^{2} = x_{20} x_{41} + x_{21} x_{40} + x_{22} x_{42}$$

 $r = C_0 m_{0000} + C_1 m_{2211} + C_2 m_{2121} + C_3 m_{2010} + C_4 m_{1110}$

We can check that the corresponding variety is smooth, and that the induced action of \mathbb{Z}_{12} is free. So we obtain another smooth quotient of X, this one with fundamental group \mathbb{Z}_{12} .

5.5.1 Group action on homology

The representation theory of \mathbb{Z}_{12} is very straightforward: there are exactly 12 distinct onedimensional representations, in which the generator of \mathbb{Z}_{12} corresponds to multiplication by one of the twelfth roots of unity. We will denote by R_k the representation in which the generator acts as multiplication by $\exp(2\pi i k/12)$. Then, repeating the type of argument used in §5.1.3, we find that \mathbb{Z}_{12} acts on $H^2(X)$ through the representation

$$R_0 \oplus R_2 \oplus R_3 \oplus R_4 \oplus R_6 \oplus R_8 \oplus R_9 \oplus R_{10}$$

There is again a single invariant, corresponding to the canonical class of the ambient space, so the Hodge numbers of the quotient are once more $(h^{1,1}, h^{2,1}) = (1, 4)$.

5.6 Alternative representations

We have seen that the manifold $X^{8,44}$ can be viewed as an anti-canonical hypersurface in $S \times S$, or as a CICY in different ways. This is because there are alternative ways of representing $S \cong dP_6$ as a complete intersection in projective spaces. In addition to the representation in equation (5.7) the following representation is useful

$$\mathcal{S} = \begin{bmatrix} \mathbb{P}^1 & 1\\ \mathbb{P}^1 & 1\\ \mathbb{P}^1 & 1 \end{bmatrix}$$
(5.27)

The identification follows from the fact that the configuration on the right has Euler number 6 and ample anti-canonical class. In this way we arrive at the original CICY configuration for X



This configuration has already been described, together with the action of the common \mathbb{Z}_6 subgroup of Dic₃ and \mathbb{Z}_{12} , in §3.3.7. We can therefore give a brief account here that concentrates on the action of the enlarged groups. We take coordinates s_{ja} on the first three \mathbb{P}^{1} 's and t_{ja} on the last three, where j = 0, 1, 2, a = 0, 1, and polynomials p, q, and r, as indicated by the diagram. As before, we will construct the polynomials from the quantities

$$m_{abc} = \sum_{i=0}^{2} s_{i,a} \, s_{i+1,b} \, s_{i+2,c} \,, \qquad n_{abc} = \sum_{i=0}^{2} t_{i,a} \, t_{i+1,b} \, t_{i+2,c}$$

$$l_{abcdef} = \sum_{i=0}^{2} s_{i,a} \, s_{i+1,b} \, s_{i+2,c} \, t_{i,d} \, t_{i+1,e} \, t_{i+2,f}$$
(5.28)

5.6.1 The non-Abelian quotient

First consider the group Dic₃. We can take the action of the two generators g_3, g_4 to be given by

Note that the two generators S and U of §3.3.7 correspond to the symmetries g_3 and g_4^2 . Then, with an appropriate choice of coordinates, the most general polynomials transforming as above are

$$p = \frac{1}{3}m_{000} + m_{110} , \qquad q = \frac{1}{3}n_{000} + n_{110}$$
$$r = \frac{1}{3}\tilde{c}_0 l_{111111} + \tilde{c}_1 l_{001001} + \tilde{c}_2 l_{001010} + \tilde{c}_3 l_{001100} + \tilde{c}_4 (l_{111001} + l_{001111})$$

Note that a further invariant term $l_{000011} - l_{011000}$ is excluded from r, since

$$l_{000011} - l_{011000} = m_{000} n_{011} - m_{011} n_{000} = p n_{011} - m_{011} q$$

and so it corresponds merely to a redefinition of r by multiples of the polynomials p and q. This being so, we see that there is a 4 parameter family of polynomials r. It is straightforward to check that for generic values of the four undetermined coefficients, the resulting variety is smooth, and the group action is free.

5.6.2 The \mathbb{Z}_{12} quotient

Now we turn to the quotient by \mathbb{Z}_{12} . The group generator acts as

$$g_{12}: s_{j,a} \to (-1)^{a+1} t_{j+1,a} , t_{j,a} \to s_{j+1,a} ; p \to -q , q \to p , r \to r$$

It is easy to see that this is related to the generators of Dic₃, given above, by $g_{12}^2 = g_3^2 g_4^2$. We can take the same polynomials p and q as above, and a slightly different r:

$$p = \frac{1}{3}m_{000} + m_{110} , \qquad q = \frac{1}{3}n_{000} + n_{110}$$
$$r = \frac{1}{3}C_0 l_{111111} + C_1 l_{001001} + C_2 (l_{001010} + l_{001100}) + C_3 (l_{011110} - l_{110011}) + C_4 (l_{111001} + l_{001111})$$

We leave out the term $l_{000011} - l_{011000}$ for the same reasons as above, although it is also invariant under \mathbb{Z}_{12} . The variety defined by the above equations is smooth for a generic choice of coefficients.

6. Semi-Realistic Heterotic Models

We turn in this chapter to the study of the $E_8 \times E_8$ heterotic string compactified on X, and in particular on its quotients, the geometry of which we described in great detail in the last chapter. The standard embedding of the spin connection in the gauge group gives rise to an E_6 gauge theory with 3 chiral generations of particles. As discussed already, E_6 models are of limited interest, so we construct stable rank-five bundles as deformations of $\mathcal{T}X \oplus \mathcal{O}_X^{\oplus 2}$, which descend to the quotient and therefore lead to three-generation SU(5) models. This SU(5) can then be broken by Wilson lines to the standard model gauge group, and we calculate the spectrum of the resulting models. An initial attempt at model building on $X_{\mathfrak{G}}$ was made in [83], but this has now been superceded thanks to a better understanding of the mathematics involved, and the new work is described here instead.

In the following we make extensive use of standard Lie group theory, for which the comprehensive reference is the review by Slansky [92].

Compactification of $E_8 \times E_8$ heterotic string theory on the manifold $X_{\mathfrak{G}}$, with the standard embedding, leads to an effective theory with unbroken 'visible' gauge group E_6 , with four chiral multiplets in the **27** and one in the **27**. There are also singlet fields corresponding to bundle moduli; these come from the cohomology group $H^1(X_{\mathfrak{G}}, \Omega^1 X_{\mathfrak{G}} \otimes \mathcal{T} X_{\mathfrak{G}})$, which is the \mathfrak{G} -invariant part of $H^1(X, \Omega^1 X \otimes \mathcal{T} X)$; this latter cohomology group is calculated in Appendix A.2.2.

We wish to break the E_6 gauge symmetry further, to achieve the gauge group of the standard model, $G_{\rm SM} = SU(3) \times SU(2) \times U(1)$, without destroying the appealing feature of having three net chiral generations of particles. Given that the unbroken gauge group is the centraliser in E_8 of the holonomy group of the gauge connection, there are two related mechanisms at our disposal to achieve this. The first is to continuously deform the internal gauge field, which corresponds to the Higgs mechanism in four dimensions. This amounts to taking the vector bundle corresponding to the background gauge field to be an irreducible deformation of $\mathcal{T}X_{\mathfrak{G}} \oplus \mathcal{O}_{X_{\mathfrak{G}}}^{\oplus r}$, where $\mathcal{T}X_{\mathfrak{G}}$ is the tangent bundle of $X_{\mathfrak{G}}$, $\mathcal{O}_{X_{\mathfrak{G}}}$ is the trivial line bundle on $X_{\mathfrak{G}}$, and r = 1, 2. In this way, the structure group of the bundle becomes SU(4) or SU(5). The centralisers of these groups in E_8 are Spin(10) and SU(5), respectively, which are more attractive groups for phenomenology than E_6 (taking r > 2 would break E_8 to a group of rank < 4, which precludes it from containing $G_{\rm SM}$). For a specific manifold, however, it requires checking that stable irreducible deformations of the relevant bundles exist, in order to achieve the desired symmetry breaking. We will discuss below why, on $X_{\mathfrak{G}}$, it is possible to obtain Spin(10) but not SU(5) in this way. In any case, obtaining the standard model gauge group directly by this method is impossible.

The second possibility, on a multiply connected manifold such as $X_{\mathfrak{G}}$, is to give non-zero values to Wilson lines around homotopically non-trivial paths, as already discussed in Chapter 1. This amounts to taking the gauge bundle to be $\mathcal{T}X_{\mathfrak{G}} \oplus \mathcal{W}$, where \mathcal{W} is a non-trivial flat bundle.

 \mathcal{W} is specified by choosing a group homomorphism $\Phi : \pi_1(X_{\mathfrak{G}}) \cong \mathfrak{G} \to E_6$, and letting the holonomy of the connection on \mathcal{W} around a path γ be given by $\Phi([\gamma])$. The holonomy group therefore acquires an extra discrete factor $\Phi(\mathfrak{G})$, and the unbroken gauge group is the centraliser of this in E_6 . However, it is a result of McInnes that it is impossible to break either E_6 or Spin(10) to $G_{\rm SM}$ by turning on discrete Wilson lines [93]. In fact, the smallest unbroken gauge group which can be obtained this way, while still containing $G_{\rm SM}$, is $SU(3) \times SU(2) \times U(1) \times U(1)$.

So to obtain G_{SM} as the low-energy gauge group, it is necessary to combine both discrete Wilson lines and continuous bundle deformations. What we will end up doing is constructing, on the covering space X, an irreducible stable rank-five bundle V by taking a \mathfrak{G} -equivariant deformation of $\mathcal{T}X \oplus \mathcal{O}_X \oplus \mathcal{O}_X$, where $\mathcal{O}_X \oplus \mathcal{O}_X$ is equipped with a non-trivial equivariant structure (corresponding to a non-trivial Wilson line on the quotient space). This leaves SU(5)as the unbroken gauge group on the covering space, but when we pass to the quotient, we can break this to G_{SM} via an Abelian Wilson line. Describing this procedure will occupy most of this chapter.

6.1 Bundle deformations and the low-energy theory

Although, strictly speaking, we will not need it here, we will first repeat the argument of [83] that, on $X_{\mathfrak{G}}$, it is impossible to break E_6 to SU(5) via a deformation of the gauge bundle. This has the advantage of highlighting the relationship between the mathematical arguments to follow, and standard results of supersymmetric field theories in four dimensions.

We obtain unbroken E_6 gauge symmetry by choosing the non-trivial part of the gauge connection to be equal to that on the tangent bundle $\mathcal{T}X_{\mathfrak{G}}$ of the manifold, with structure group SU(3). We will now consider, as discussed above, taking instead a non-trivial deformation of $\mathcal{T}X_{\mathfrak{G}} \oplus \mathcal{O}_{X_{\mathfrak{G}}}$ or $\mathcal{T}X_{\mathfrak{G}} \oplus \mathcal{O}_{X_{\mathfrak{G}}} \oplus \mathcal{O}_{X_{\mathfrak{G}}}$, with structure group SU(4) or SU(5), respectively. Since deformation is a continuous process, it must correspond to the Higgs mechanism in the low energy theory, whereby the vector multiplets lying outside the unbroken sub-algebra gain mass by eating chiral multiplets with the same charges. The gauge bosons transform in the adjoint representation of E_6 , while the families and antifamilies transform in the **27** and **27**, respectively. These representations decompose under the Spin(10) subgroup as:

$$78 = 45 \oplus 16 \oplus \overline{16} \oplus 1 \;, \qquad 27 = 16 \oplus 10 \oplus 1 \;, \qquad \overline{27} = \overline{16} \oplus 10 \oplus 1 \;$$

So E_6 can be Higgsed to Spin(10) by giving a VEV to the Spin(10)-singlets appearing in the decomposition of the **27** and $\overline{\mathbf{27}}$. The vector multiplets in $\mathbf{16} \oplus \overline{\mathbf{16}} \oplus \mathbf{1}$ will then eat corresponding chiral multiplets, leaving massless chiral multiplets in the following representation of Spin(10):

$$(3 \times \mathbf{16}) \oplus (5 \times \mathbf{10}) \oplus (4 \times \mathbf{1})$$

Some of the fields in the **10** representation will also typically gain mass from the 27^3 and $\overline{27}^3$ Yukawa couplings, but we can't say more without knowing the details of these couplings. For the current argument, the only important point to notice is that all the anti-generations are gone after this step.

Now suppose we want to go further, and deform to an SU(5) bundle, in order to break Spin(10) to SU(5). Once again, this would be a continuous process, and so would correspond to the Higgs mechanism in the low energy theory, and we can repeat the analysis above. The relevant representations decompose as follows:

$$45 = 24 \oplus 10 \oplus \overline{10} \oplus 1$$
, $16 = 10 \oplus \overline{5} \oplus 1$, $10 = 5 \oplus \overline{5}$.

So, repeating the argument above, in this case there would have to be chiral multiplets transforming as $\mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{1}$ to be eaten by the corresponding vector fields. But there are no chiral multiplets transforming as $\overline{\mathbf{10}}$, so we conclude that this cannot happen.¹ Notice that this is a consequence of having only one anti-generation of E_6 to begin with, so the same argument holds on any manifold with $h^{1,1} = 1$.

The simple analysis above relies on general features of supersymmetric physics in four dimensions, and tells us that on certain manifolds we *cannot* deform $\mathcal{T} \oplus \mathcal{O}^{\oplus r}$ to an irreducible stable bundle, but we would like to know when we *can* find such deformations. A complete mathematical answer to this question was given in [94], and will be reviewed in the next section. For now we just mention that it requires r linearly independent (1,1) cohomology classes, so again we find that there is no stable deformation for r = 2, because $h^{1,1}(X_{\mathfrak{G}}) = 1$. This is a simple example of the beautiful interplay between mathematical features of the compactification and low energy supersymmetric field theory; for a recent thorough analysis of vector bundle stability in this spirit, see [95, 96].

6.2 The deformed bundle

We have a Calabi-Yau manifold X, and its quotient $X_{\mathfrak{G}}$, with Euler number -6. Our objective is to take the rank-five bundle $\mathcal{T}X \oplus \mathcal{O}_X \oplus \mathcal{O}_X$, and deform it to an irreducible stable bundle, which is equivariant under the action of the quotient group \mathfrak{G} . To do so, we will adopt the techniques introduced by Li and Yau in [94] (see also [97]), and incorporate equivariance as we go along.

We will consider X represented as a CICY in the ambient space $P \equiv (\mathbb{P}^1)^6$, as before

$$X = \begin{bmatrix} \mathbb{P}^{1} & 1 & 0 & 1 \\ \mathbb{P}^{1} & 1 & 0 & 1 \\ \mathbb{P}^{1} & 1 & 0 & 1 \\ \mathbb{P}^{1} & 0 & 1 & 1 \\ \mathbb{P}^{1} & 0 & 1 & 1 \\ \mathbb{P}^{1} & 0 & 1 & 1 \end{bmatrix}$$

The tangent bundle, $\mathcal{T}P$, of the ambient space fits into a short exact sequence, arising as a direct sum of the Euler sequences for each of the \mathbb{P}^1 's. Restricting this sequence to X gives an exact sequence

$$0 \longrightarrow \mathcal{O}_X^{\oplus 6} \longrightarrow \mathcal{O}_X(100000)^{\oplus 2} \oplus \ldots \oplus \mathcal{O}_X(000001)^{\oplus 2} \xrightarrow{\psi_1} \mathcal{T}P|_X \longrightarrow 0$$
(6.1)

¹We could alternatively note that, in the absence of any $\overline{16}$ fields, there is a *D*-term potential preventing a VEV for the **16** fields.

The other important short exact sequence is the one defining the tangent bundle of X in terms of that of P,

$$0 \longrightarrow \mathcal{T}X \longrightarrow \mathcal{T}P|_X \xrightarrow{\psi_2} \mathcal{O}_X(111000) \oplus \mathcal{O}_X(000111) \oplus \mathcal{O}_X(111111) \longrightarrow 0$$
(6.2)

The morphism τ is given by the derivatives of the three polynomials defining X.

At this point, let us introduce some space-saving notation:

$$G := \mathcal{O}_X(100000)^{\oplus 2} \oplus \ldots \oplus \mathcal{O}_X(000001)^{\oplus 2}$$

$$\mathcal{N} := \mathcal{O}_X(111000) \oplus \mathcal{O}_X(000111) \oplus \mathcal{O}_X(111111)$$

(6.3)

From (6.1) and (6.2), we can construct a surjective morphism $\Phi_0 = \psi_2 \circ \psi_1$. It is then easy to check that the bundle $\tilde{F}_0 := \ker \Phi_0$ fits into the following commutative diagram, with exact rows and columns



We see in this way that \tilde{F}_0 is an extension of $\mathcal{T}X$ by $\mathcal{O}_X^{\oplus 6}$. Of course, this makes \tilde{F}_0 a rank-nine bundle, whereas we are interested in rank-five bundles. To rectify this, we choose a sub-bundle $\mathcal{O}_X^{\oplus 4} \stackrel{\iota}{\hookrightarrow} \mathcal{O}_X^{\oplus 6}$, and define a rank-five bundle $F_0 = \tilde{F}_0/\mathrm{im}\,\iota$. This can also be expressed in the following short exact sequence

$$0 \longrightarrow \mathcal{O}_X^{\oplus 4} \stackrel{\iota}{\longrightarrow} \widetilde{F}_0 \longrightarrow F_0 \longrightarrow 0$$

We will be more specific about the map ι when we discuss equivariance under the quotient group $\mathfrak{G} \cong \text{Dic}_3$. For now we just note that F_0 is an extension of $\mathcal{T}X$ by $\mathcal{O}_X^{\oplus 6}/\mathcal{O}_X^{\oplus 4} \cong \mathcal{O}_X^{\oplus 2}$. The bundle F_0 is not suitable for string compactification, but to see why we must discuss vector bundle stability.

6.2.1 Bundle stability and deformations

Let V be a holomorphic vector bundle on an n-dimensional Kähler manifold X, with Kähler form ω . The *slope* of V, denoted $\mu(V)$, is defined to be

$$\mu(V) = \frac{1}{\operatorname{rk}(V)} \int_X c_1(V) \wedge \omega^{n-1}$$

The bundle V is stable if, for every coherent sub-sheaf² $\mathscr{F} \subset V$, with $\operatorname{rk}(\mathscr{F}) < \operatorname{rk}(V)$, we have $\mu(\mathscr{F}) < \mu(V)$; V is polystable if it is a direct sum of stable bundles, all with the same slope. The Donaldson-Uhlenbeck-Yau theorem says that polystability is equivalent to V admitting a Hermitian-Yang-Mills connection. In our case, $c_1(V) = 0$, and hence $\mu(V) = 0$.

Here we will be interested only in the stability of bundles which are obtained by deformation of $\mathcal{T}X \oplus \mathcal{O}_X^{\oplus r}$. The infinitesimal deformations of any holomorphic vector bundle V are given by the cohomology group³ $H^1(X, \operatorname{End}(V)) = H^1(X, V \otimes V)$. In the case of interest, using the fact that $H^1(X, \mathcal{O}_X) = 0$ because X is Calabi-Yau, this group is

$$H^{1}(X, \Omega^{1}X \otimes \mathcal{T}X) \oplus H^{1}(X, \Omega^{1}X)^{\oplus r} \oplus H^{1}(X, \mathcal{T}X)^{\oplus r}$$

$$(6.5)$$

where $\Omega^1 X$ is the holomorphic cotangent bundle. The first term here corresponds to deformations of $\mathcal{T}X$, and so is relatively uninteresting for our purposes. The group $H^1(X, \mathcal{T}X)$ is naturally isomorphic to⁴ Ext¹($\mathcal{O}_X, \mathcal{T}X$), which parametrises extensions of \mathcal{O}_X by $\mathcal{T}X$, *i.e.* bundles F coming from exact sequences of the form

$$0 \longrightarrow \mathcal{T}X \longrightarrow F \longrightarrow \mathcal{O}_X \longrightarrow 0$$

Similarly, we can see by dualising that $H^1(X, \Omega^1 X)$ is isomorphic to $\text{Ext}^1(\mathcal{T}X, \mathcal{O}_X)$, corresponding to extensions of the opposite type:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow F \longrightarrow \mathcal{T}X \longrightarrow 0$$

It is easy to see that a bundle constructed as an extension in either way cannot be stable. Indeed, in the first case we have an injection $\mathcal{T}X \hookrightarrow F$, and $\mu(\mathcal{T}X) = 0$ since X is Calabi-Yau, and in the second case we have $\mathcal{O}_X \hookrightarrow F$, and clearly $\mu(\mathcal{O}_X) = 0$. So any such extension bundles are unstable. However, if the deformation class of F has pieces in each of the last two terms of (6.5), then F has no such simple interpretation, and might be stable. Indeed, it was shown by Li and Yau in [94] that if F is a deformation of $\mathcal{T}X \oplus \mathcal{O}_X^{\oplus r}$, then F is stable if and only if the projections of its deformation class onto each of the last two terms in (6.5) consist of r linearly independent (1, 1) and (2, 1) cohomology classes, respectively.

So it is now clear what we can do to construct a stable rank five bundle. Prior to our discussion of stability, we had a bundle F_0 which was an extension of $\mathcal{T}X$ by $\mathcal{O}_X^{\oplus 2}$, and therefore unstable. Our approach, following [94], will be to deform the map Φ_0 , given by the derivatives of the polynomials p, q, r, to a more general map Φ , with kernel \tilde{F} . If we demand that we still have $\operatorname{im} \iota \subset \tilde{F}$, then we can form the rank-five bundle $F = \tilde{F}/\operatorname{im} \iota$. Assuming that general enough deformations of Φ exist, the generic F defined this way will be stable. It is of course essential that F also be equivariant under the action of \mathfrak{G} , so that it passes to a bundle on the quotient manifold $X_{\mathfrak{G}}$. For this reason we now examine the equivariant structures on the various

²We have not defined coherent sheaves, but may pretend here that they are just holomorphic vector bundles. ³Roughly, we are changing the transition functions of V by some small amount, so want functions mapping $V \to V$ on intersections of two affine patches. These must be a Čech cocyle in order to still satisfy the consistency conditions for transition functions, but if they are a Čech coboundary, they correspond merely to changing the local trivialisations of V. For a detailed discussion of deformation theory and cohomology, see [98].

⁴This is immediate from the definition of H^* and Ext^* as derived functors.

bundles appearing in (6.4).

6.2.2 Equivariant structures

We need to find an explicit description of the \mathfrak{G} -equivariant structures on the bundles $\mathcal{O}^{\oplus 6}$, G and \mathcal{N} . In fact they each admit multiple \mathfrak{G} -equivariant structures, but unique choices are singled out by the requirement that all the maps in (6.4) be equivariant.

Equivariant structures on $\mathcal{O}_X^{\oplus r}$ are the same thing as flat bundles on $X_{\mathfrak{G}}$. These are classified by the representations of \mathfrak{G} , because we construct a flat, rank-r bundle by equipping $\mathcal{O}_X^{\oplus r}$ with some r-dimensional representation of \mathfrak{G} . To clarify what is meant by this, note that \mathcal{O}_X is generated by its global sections, which are just constant functions on X. A \mathfrak{G} -equivariant structure on $\mathcal{O}_X^{\oplus r}$ is therefore induced by any action of \mathfrak{G} on the r-dimensional vector space of global sections. We will denote by $\mathcal{L}_{R^{(1)}}$ the flat line bundle on $X_{\mathfrak{G}}$ associated to a onedimensional representation $R^{(1)}$ of \mathfrak{G} on \mathcal{O}_X , and by $\mathcal{W}_{R^{(2)}}$ the flat rank-two bundle on $X_{\mathfrak{G}}$ associated to a two-dimensional representation $R^{(2)}$ on $\mathcal{O}_X \oplus \mathcal{O}_X$.

Let us consider again the generalised Euler sequence, (6.1), restricted to X

$$0 \longrightarrow \mathcal{O}_X^{\oplus 6} \longrightarrow \mathcal{O}_X(100000)^{\oplus 2} \oplus \ldots \oplus \mathcal{O}_X(000001)^{\oplus 2} \longrightarrow \mathcal{T}P|_X \longrightarrow 0$$

The bundle $\mathcal{T}P$ carries an equivariant structure under any group action, given by the differentials of the maps which represent the group; we must endow the first two terms with compatible equivariant structures, so that this sequence descends to $X_{\mathfrak{G}}$. For the middle sheaf this is achieved by identifying its sections with vector fields on $(\mathbb{C}^2)^6$, which are linear in the appropriate variables (this was discussed in detail in §2.2.2). The first map is then given (on global sections) by

$$(\lambda_0, \lambda_1, \lambda_2, \mu_0, \mu_1, \mu_2) \to \lambda_0 \left(s^{00} \frac{\partial}{\partial s^{00}} + s^{01} \frac{\partial}{\partial s^{01}} \right) + \ldots + \mu_2 \left(t^{20} \frac{\partial}{\partial t^{20}} + t^{21} \frac{\partial}{\partial t^{21}} \right)$$

For this map to commute with the action of \mathfrak{G} , we must let \mathfrak{G} act on the first term of (6.1) as follows:

$$g_4(\lambda_0, \lambda_1, \lambda_2, \mu_0, \mu_1, \mu_2) = (\mu_0, \mu_2, \mu_1, \lambda_0, \lambda_2, \lambda_1)$$

$$g_3(\lambda_0, \lambda_1, \lambda_2, \mu_0, \mu_1, \mu_2) = (\lambda_1, \lambda_2, \lambda_0, \mu_1, \mu_2, \mu_0)$$

It is straightforward to express this as a sum of irreducible representations of \mathfrak{G}

$$H^0(X, \mathcal{O}_X^{\oplus 6}) \sim R_1 \oplus R_{-1} \oplus R_+^{(2)} \oplus R_+^{(2)}$$
 (6.6)

so upon taking the quotient, $\mathcal{O}_X^{\oplus 6}$ descends to $\mathcal{L}_1 \oplus \mathcal{L}_{-1} \oplus \mathcal{W}_{R^{(2)}_{\perp}} \oplus \mathcal{W}_{R^{(2)}_{\perp}}$ on $X_{\mathfrak{G}}$.

We now need to determine how the group acts on the normal bundle to X in P, which is $\mathcal{N} = \mathcal{O}_X(111000) \oplus \mathcal{O}_X(000111) \oplus \mathcal{O}_X(111111)$. Consider the map $\Phi_0 = (dp, dq, dr)$, appearing in (6.4). When we act on the coordinates with $g_4 \in \mathfrak{G}$, it induces $p \to -q$, $q \to p$, $r \to r$, and therefore $dp \to -dq$, $dq \to dp$, $dr \to dr$. This then determines, by equivariance of Φ_0 , the action of \mathfrak{G} on \mathcal{N} , and in particular, we see that an extra minus sign appears relative to the 'naïve' equivariant structure. Explicitly, the difference appears in

$$(s_{0,a}s_{1,b}s_{2,c} , 0 , 0) \xrightarrow{g_4} (0 , -(-1)^{a+b+c+1} t_{0,a}t_{1,c}t_{2,b} , 0)$$

$$= (0 , (-1)^{a+b+c} t_{0,a}t_{1,c}t_{2,b} , 0)$$

$$(6.7)$$

In particular, notice that when considered as sections of \mathcal{N} (over the ambient space, not X, where they vanish identically) with this equivariant structure, we have $g_4: p \to q$.

6.2.3 Constructing bundle deformations

 $\hat{r} =$

Having discussed the form of the bundle deformations we require, we now turn to finding them explicitly (this is important, since they are not guaranteed to exist).

Let the deformed map $\Phi : G \to \mathcal{N}$ be given by $(\hat{p}, \hat{q}, \hat{r})$. Since sections of G are represented by vector fields, and those of \mathcal{N} simply by homogeneous polynomials, Φ can be written in terms of differentials of the homogeneous coordinates:

$$\hat{p} = \sum_{i,a,b,c} A^{i}_{abc} \, ds_{i,a} s_{i+1,b} s_{i+2,c} \quad , \quad \hat{q} = \sum_{i,a,b,c} B^{i}_{abc} \, t_{i,a} t_{i+1,b} dt_{i+2,c}$$

$$\sum_{i,a,b,c} \left(C^{i}_{abcdef} \, ds_{i,a} s_{i+1,b} s_{i+2,c} t_{i,d} t_{i+1,e} t_{i+2,f} + D^{i}_{abcdef} \, s_{i,a} s_{i+1,b} s_{i+2,c} t_{i,d} t_{i+1,e} dt_{i+2,f} \right)$$

We know that $\Phi_0 = (dp, dq, dr)$, so the general Φ must transform the same way; all components must be invariant under g_3 , while $g_4 \cdot (\hat{p}, \hat{q}, \hat{r}) = (-\hat{q}, \hat{p}, \hat{r})$.

The way we have parametrised things, g_3 -invariance simply amounts to all coefficients being independent of their 'i' index; we define $A_{abc} \equiv A^i_{abc}$, and similarly for the other coefficients. It is only marginally more complicated to determine the conditions for g_4 -equivariance; they are

$$A_{abc} = (-1)^{a+b+c} A_{abc} , \quad B_{abc} = A_{cba}$$

$$C_{abcdef} = (-1)^{a+b+c+d+e+f} C_{abcdef} \ , \ \ D_{abcdef} = (-1)^{a+b+c+1} C_{fedcbe} \ , \ \ D_{abcdef} = (-1)^{$$

So the *B* and *D* coefficients are determined in terms of the *A*'s and *C*'s, and these must be even, in the sense that they vanish unless their indices add to zero in \mathbb{Z}_2 . So at this stage we have four free parameters in \hat{p} and \hat{q} , which are $A_{000}, A_{011}, A_{101}$ and A_{110} , and 32 in \hat{r} , which are $C_{abcde,a+b+c+d+e}$ for all choices of a, b, c, d, e.

It is not true that each choice of coefficients satisfying the above conditions corresponds to a distinct bundle $\tilde{F} = \ker \Phi$. Rather trivially, we can always rescale \hat{p} and \hat{q} equally, and \hat{r} independently, without changing \tilde{F} . For \hat{p} and \hat{q} , this rescaling is the only redundancy in the parameters given, but there are two other distinct ways to change the map \hat{r} , as defined above, without actually deforming the bundle \tilde{F} .

First note that, if a + b + c = 0, then the following term may be added to \hat{r} :

$$\sum_{i} \left(ds_{i,a} s_{i+1,b} s_{i+2,c} q - p t_{i,c} t_{i+1,b} dt_{i+2,a} \right)$$

Of course, p and q vanish identically on X, so this has no effect. Inspecting the form of the polynomials p and q, we see that we can use this to eliminate all parameters $C_{abc\,000}$ in favour of $C_{abc\,011}, C_{abc\,101}$, and $C_{abc\,110}$.

Next we observe that, since \tilde{F} is defined to consist of everything annihilated by \hat{p}, \hat{q} and \hat{r} , adding multiples of \hat{p} and \hat{q} to \hat{r} does not change anything. To be invariant under \mathfrak{G} , such terms must be of the form $\hat{p}\tilde{q} - \tilde{p}\hat{q}$, where \tilde{p} and \tilde{q} are of the same degrees as p and q, and also transform like them under \mathfrak{G} . The most general such polynomials are

$$\tilde{p} = \alpha m_{000} + \beta m_{011}$$
, $\tilde{q} = \alpha n_{000} + \beta n_{011}$

where m_{abc} and n_{abc} are defined in equation (5.28). It may seem like this allows us to remove two more parameters, but we must be careful. If we choose $\alpha = \beta$, then \tilde{p} and \tilde{q} are just multiples of p and q, and we have already accounted for these. In fact, the right thing to do is to enforce $\alpha = 0$; one way to see this is that $\alpha \neq 0$ would contribute to C_{000000} , which we already set to zero in the previous step. By choosing β appropriately, we can, say, remove the parameter C_{000011} .

Above we have found the effective, \mathfrak{G} -equivariant deformations of the map Φ . These correspond to equivariant deformations of the rank-nine bundle \tilde{F} . However, we are interested in deformations of the tangent bundle $\mathcal{T}X = \tilde{F}/\mathcal{O}_X^{\oplus 6}$, or of the rank-five bundle $F = \tilde{F}/\mathcal{O}_X^{\oplus 4}$, on which we want to compactify the heterotic string. These correspond to demanding, respectively, that $\mathcal{O}_X^{\oplus 6}$ or $\mathcal{O}_X^{\oplus 4}$ is in the kernel of Φ , so we can still form the quotient.

We can think of $\mathcal{O}_X^{\oplus 6}$ as being generated by the Euler vectors on the six ambient spaces, which are defined as

$$e_i = \sum_a s_{i,a} \frac{\partial}{\partial s_{i,a}} , \quad \tilde{e}_i = \sum_a t_{i,a} \frac{\partial}{\partial t_{i,a}}$$

Via this identification, we will think of sections of $\mathcal{O}_X^{\oplus 6}$ as vector fields on $(\mathbb{C}^2)^6$ throughout.

Deforming the tangent bundle

Deformations of the tangent bundle come from maps Φ which annihilate all six Euler vectors. Before moving on, we need to highlight the important point that this only needs to be true on X, where $p = q = r \equiv 0$. What this means is that if we let \mathscr{I} be the polynomial ideal (p, q, r), then we require only that, for example $\hat{p}(e_i) \sim 0$ modulo \mathscr{I} . For tangent bundle deformations, we will find it easy enough to work this out by hand, but in some cases it is far easier to appeal to a computer algebra program such as Mathematica for these calculations. The procedure is to first find a Gröbner basis for \mathscr{I} , and then use the built-in Mathematica function PolynomialReduce to express polynomials uniquely as an element of \mathscr{I} plus a remainder [29].

First consider the equation $\hat{p}(e_i) = 0$ on X, for some fixed i. We have

$$\hat{p}(e_i) = \sum_{a,b,c} A_{abc} \, s_{i,a} s_{i+1,b} s_{i+2,c}$$

where the sum runs over only those indices satisfying a + b + c = 0. A polynomial of this degree vanishes modulo \mathscr{I} if and only if it is a multiple of p. It is easy to see that, independently of i, this requires that all the A_{abc} are equal,⁵ so that $\hat{p} \propto dp$. In other words, there are no possible deformations here. Since \hat{q} is determined from \hat{p} , and the Euler vector \tilde{e}_i from e_{-i} , by the action of g_4 , it is not necessary to consider $\hat{q}(\tilde{e}_i) = 0$ separately.

⁵It is to be expected that each value of *i* gives the same condition, since \hat{p} is invariant under g_3 , and $dg_3(e_i) = e_{i-1}$. The same reasoning also applies to subsequent calculations.

We now turn to finding the most general \hat{r} annihilating all the Euler vectors. By \mathfrak{G} -invariance, we need only consider, say, $\hat{r}(e_0) = 0$. Substituting the \mathfrak{G} -invariant expression for \hat{r} , we get

$$\hat{r}(e_0) = \sum_{a,b,c,d,e,f} C_{abcdef} \, s_{0,a} s_{1,b} s_{2,c} t_{0,d} t_{1,e} t_{2,f} \tag{6.8}$$

where the sum is restricted by a + b + c + d + e + f = 0. We want to know when this vanishes modulo \mathscr{I} . We first recall some useful terminology from §3.3.7: we call *even* those terms for which a + b + c = 0, and *odd* those for which a + b + c = 1. This distinction is helpful because, inspecting the forms of p, q and r, we see that p and q consist only of even terms, and r only of odd terms, so we can consider these separately.

Consider first the even terms in equation (6.8); we require that they can be expressed as a multiple of p plus a multiple of q. But recall that we have set $C_{abc\,000} = C_{000011} = 0$ by redefinitions, and any non-zero multiple of q contributes to these terms. So we require the even terms to be a multiple of p. It helps to write out p explicitly:

$$p = s_{0,0}s_{1,0}s_{2,0} + s_{0,0}s_{1,1}s_{2,1} + s_{0,1}s_{1,0}s_{2,1} + s_{0,1}s_{1,1}s_{2,0}$$

From this and equation (6.8), we see that the requirement that the even terms give a multiple of p is simply

$$C_{abcdef} = C_{000\,def} \ \forall \ d, e, f$$

There are therefore only two free even parameters, which can be taken to be C_{000101} and C_{000110} .

The odd terms are even easier. Each odd term appears exactly once in r itself, so demanding that the odd terms in equation (6.8) are a multiple of r fixes all odd parameters to be proportional to a single one, which we can generically take to be C_{111111} . The constants of proportionality will depend on the coefficients of r.

In summary, there are three free parameters in \hat{r} , which correspond to a two-dimensional space of bundles, since the overall scale of \hat{r} is irrelevant. Note that if we specialise the coefficients to $C_{000101} = C_{000110} = 0$, then \hat{r} is simply a multiple of dr, so indeed the bundles we have constructed are deformations of $\mathcal{T}X$.

Deforming the rank-five bundle

To study deformations of the rank-five bundle $F_0 = \tilde{F}_0/\operatorname{im} \iota$, we need only impose the weaker condition $\operatorname{im} \iota \subset \tilde{F}$, as discussed in the first part of §6.2. Since we are free to choose the map ι , this amounts to requiring that some four-dimensional sub-bundle of $\mathcal{O}_X^{\oplus 6}$ is annihilated by Φ . The bundle $\mathcal{O}_X^{\oplus 6}$, which is generated by the Euler vectors, transforms under \mathfrak{G} according to (6.6), so to ensure that F is equivariant we must choose a four-dimensional sub-representation of that in (6.6) to be annihilated by Φ . We will see in the next section that to obtain an acceptable spectrum we must take it to be $R_+^{(2)} \oplus R_+^{(2)}$.

It is easy to check that the two copies of $R^{(2)}_+$ in $\mathcal{O}^{\oplus 6}_X$ are spanned, respectively, by (l_1, m_1)

and (l_2, m_2) , where

$$l_1 = e_0 + \zeta e_1 + \zeta^2 e_2 \quad , \quad m_1 = \tilde{e}_0 + \zeta^2 \tilde{e}_1 + \zeta \tilde{e}_2$$
$$l_2 = e_0 + \zeta^2 e_1 + \zeta e_2 \quad , \quad m_2 = \tilde{e}_0 + \zeta \tilde{e}_1 + \zeta^2 \tilde{e}_2$$

so we must ensure that Φ annihilates each of these.

First we demand that \hat{p} annihilates l_1 ; substituting the expressions for each gives

$$0 = \hat{p}(l_1) = A_{000} \sum_i \zeta^i s_{i,0} s_{i+1,0} s_{i+2,0} + A_{011} \sum_i \zeta^i s_{i,0} s_{i+1,1} s_{i+2,1} + A_{101} \sum_i \zeta^i s_{i,1} s_{i+1,0} s_{i+2,1} + A_{110} \sum_i \zeta^i s_{i,1} s_{i+1,1} s_{i+2,0} = (A_{011} + \zeta^2 A_{101} + \zeta A_{110}) \sum_i \zeta^i s_{i,0} s_{i+1,1} s_{i+2,1}$$

A polynomial of this degree vanishes on X only if it is proportional to p itself. There is no dependence on A_{000} , so this parameter is unconstrained. The remaining polynomial cannot be a non-zero multiple of p, so its coefficient must vanish, giving a linear constraint on the other coefficients in \hat{p} . The expression for $\hat{p}(l_2)$ is very similar:

$$0 = \hat{p}(l_2) = (A_{011} + \zeta A_{101} + \zeta^2 A_{110}) \sum_{i} \zeta^{2i} s_{i,0} s_{i+1,1} s_{i+2,1}$$

Combining the two equations, we still get no condition on A_{000} , but the other coefficients in \hat{p} must satisfy

$$A_{011} + \zeta^2 A_{101} + \zeta A_{110} = A_{011} + \zeta A_{101} + \zeta^2 A_{110} = 0$$

This is equivalent to $A_{110} = A_{101} = A_{011}$. So overall, there are two free parameters in \hat{p} (and recall that \hat{q} is obtained from \hat{p} by acting with g_4 , so is not independent); we have, in fact

$$\hat{p} = \frac{1}{3} A_{000} \, d(m_{000}) + A_{011} \, d(m_{011})$$

where the m_{abc} were defined in equation (5.28). This corresponds to a one-parameter family of deformations, since the overall scale of \hat{p} is irrelevant. Note also that for $A_{011} = A_{000}$, \hat{p} is just a multiple of dp, so indeed this is a deformation of Φ_0 .

Next we need to determine the constraints on \hat{r} . First of all, notice that the restricted form of \hat{p} and \hat{q} just derived still lets us eliminate from \hat{r} the same terms that we chose earlier. Apart from this small step, the calculation of \hat{r} is much more complicated than that for \hat{p} and \hat{q} , and we resort to using computer algebra, as described earlier. The solution has 11 free parameters; its explicit form is not very enlightening, and is omitted.

We can now argue that the generic F constructed above is stable. First, we show that the bundle $\tilde{F}_0 = \ker \Phi_0$ is a completely *non-split* extension of $\mathcal{T}X$ by $\mathcal{O}_X^{\oplus 6}$. Suppose not, so that $\tilde{F}_0 = F'_0 \oplus \mathcal{O}_X$. Then there is a projection onto the second factor, corresponding to a non-trivial element of $\operatorname{Hom}(\tilde{F}_0, \mathcal{O}_X)$. But we know \tilde{F}_0 fits into the short exact sequence $0 \to \tilde{F}_0 \to G \to \mathcal{N} \to 0$, as in (6.4), so $\operatorname{Hom}(\tilde{F}_0, \mathcal{O}_X)$ injects into $\operatorname{Hom}(G, \mathcal{O}_X)$, since Hom is left-exact. But $\operatorname{Hom}(G, \mathcal{O}_X) = H^0(X, G^{\check{}}) \cong H^3(X, G) = 0$, where the non-trivial isomorphism follows from Serre duality, and the last equality comes from calculations in the next section. So we conclude that $\operatorname{Hom}(\widetilde{F}_0, \mathcal{O}_X) = 0$, and the extension does not split.

So the deformation class of $F_0 = \tilde{F}_0/\text{im }\iota$ consists of two linearly independent (1, 1) cohomology classes. We then deform it to a bundle F which is no longer an extension by $\mathcal{O}_X^{\oplus 2}$ or \mathcal{O}_X . This implies that the (2, 1) parts of the deformation class are non-zero and linearly independent (otherwise F would be an extension by $\mathcal{O}_X^{\oplus 2}$ of some deformation of $\mathcal{T}X$, or an extension by \mathcal{O}_X of some deformation of $\mathcal{T}X \oplus \mathcal{O}_X$). By continuity, the (1, 1) parts of the deformation class are generically still linearly independent, so F is stable by the criterion of Li and Yau.

6.3 Calculating the cohomology/spectrum

We have constructed a family of stable bundles, the generic member of which we are denoting by F, with structure group SU(5). Embedding this SU(5) in E_8 gives us a heterotic model with unbroken gauge group⁶ $SU(5)_{GUT}$. The decomposition of the adjoint of E_8 is

$$E_8 \supset SU(5) \times SU(5)_{\rm GUT}$$

$$oxed{248} = (oldsymbol{1}, oldsymbol{24}) \oplus (oldsymbol{24}, oldsymbol{1}) \oplus (oldsymbol{5}, oldsymbol{10}) \oplus (oldsymbol{\overline{5}}, oldsymbol{\overline{10}}) \oplus (oldsymbol{\overline{5}}, oldsymbol{\overline{10}}) \oplus (oldsymbol{\overline{10}}, oldsymbol{5})$$

The left-handed standard model fermions appear in the $\mathbf{10} \oplus \mathbf{\overline{5}}$ representation of $SU(5)_{\text{GUT}}$, and therefore come from the fundamental **5** and the rank-two anti-symmetric **10** of the structure group of F. The number of massless chiral superfields in each representation is given by the following numbers

$$n_{10} = h^1(X, F) \quad n_{\overline{5}} = h^1(X, \wedge^2 F)$$
$$n_{\overline{10}} = h^1(X, F^{\tilde{}}) \quad n_{5} = h^1(X, \wedge^2 F^{\tilde{}})$$

where $h^i(X, F) := \dim_{\mathbb{C}} H^i(X, F)$, and similarly for other bundles. So it is these cohomology groups we wish to calculate. In this section we just present the results; an outline of the calculations is presented in Appendix A.

6.3.1 Calculating n_{10} and $n_{\overline{10}}$

To summarise the previous section, the bundle, F, in which we are interested, is given by the cohomology of the following complex (which is *not* an exact sequence)

$$0 \longrightarrow \mathcal{O}_X^{\oplus 4} \xrightarrow{\iota} G \xrightarrow{\Phi} \mathcal{N} \longrightarrow 0 \tag{6.9}$$

where $\Phi \circ \iota = 0$, and G and \mathcal{N} are defined in (6.3). Introducing $\tilde{F} := \ker \Phi$ therefore allows us to split this into two short exact sequences, from which we can then calculate the cohomology of F

$$0 \longrightarrow \widetilde{F} \longrightarrow G \xrightarrow{\Phi} \mathcal{N} \longrightarrow 0 \tag{6.10}$$

$$0 \longrightarrow \mathcal{O}_X^{\oplus 4} \stackrel{\iota}{\longrightarrow} \widetilde{F} \longrightarrow F \longrightarrow 0 \tag{6.11}$$

The first thing to do is calculate the cohomology of G and \mathcal{N} , which in each case is a direct sum of the cohomology of the direct summands. The result is

$$h^{i}(X,G) = \begin{cases} 24 & , i = 0\\ 0 & , i = 1,2,3 \end{cases}$$
(6.12)

⁶Here we borrow some notation from [99].

$$h^{i}(X, \mathcal{N}) = \begin{cases} 62 & , i = 0\\ 2 & , i = 1\\ 0 & , i = 2, 3 \end{cases}$$
(6.13)

The long exact sequence in cohomology, following from (6.10), is therefore

At this stage we must assume that Φ is sufficiently generic, in the following sense. We have demanded that $\mathcal{O}_X^{\oplus 4}$ injects into \tilde{F} , which implies that $h^0(X, \tilde{F}) \geq 4$. We will assume from now on that this inequality is saturated,⁷ in which case we can now completely determine the cohomology of \tilde{F} :

$$h^{i}(X, \widetilde{F}) = \begin{cases} 4 & , i = 0 \\ 42 & , i = 1 \\ 2 & , i = 2 \\ 0 & , i = 3 \end{cases}$$
(6.15)

We note here that these numbers are independent of the map Φ , up to the genericity conditions already mentioned. There is also a single non-trivial check we can make on our calculation: since \tilde{F} is obtained by deforming the direct sum of $\mathcal{T}X$ with a flat bundle, its index must be the same as that of $\mathcal{T}X$. Indeed, we see that

$$\operatorname{ind}(\widetilde{F}) = h^0(\widetilde{F}) - h^1(\widetilde{F}) + h^2(\widetilde{F}) - h^3(\widetilde{F}) = -36$$

To get the cohomology of F from the cohomology of \tilde{F} and the sequence (6.11) note that, since X is Calabi-Yau, we have

$$h^{i}(X, \mathcal{O}_{X}) = \begin{cases} 1 & , i = 0, 3 \\ 0 & , i = 1, 2 \end{cases}$$

From this and equation (6.11) we immediately obtain

$$h^{1}(X,F) = h^{1}(X,\tilde{F}) = 42$$

$$h^{2}(X,F) = h^{2}(X,\tilde{F}) - h^{3}(X,\tilde{F}) + 4 = 6$$
(6.16)

6.3.2 Calculating n_5 and $n_{\overline{5}}$

Calculating the number of massless fields in the **5** and $\overline{\mathbf{5}}$ representations of $SU(5)_{\text{GUT}}$ is much harder, since it is given by the cohomology of the bundle $\wedge^2 F$. Below we will see how this can, in principle, be calculated. In practice, some of the calculations became too difficult to do by hand, and also would not finish in Mathematica. Volker Braun completed the calculation of the cohomology groups using different techniques; the final results are those that he obtained.

It will be easier to calculate the cohomology of $\wedge^2 F^{\check{}}$, and use Serre duality. The fundamental fact we require is that, given any short exact sequence $0 \to A \to B \to C \to 0$, there exists another exact sequence

$$0 \longrightarrow \wedge^2 A \longrightarrow \wedge^2 B \longrightarrow B \otimes C \longrightarrow S^2 C \longrightarrow 0$$
(6.17)

⁷We can see from (6.11) that $h^0(X, F) = h^0(X, \tilde{F}) - 4$, so actually this condition is necessary for F to be stable. We can also verify it directly, given the explicit maps Φ we constructed.

where S^2C is the second symmetric tensor power of C. The morphisms here are naturally induced by the morphisms in the original sequence.

The bundle $F\,\check{}$ fits into the exact sequence

$$0 \longrightarrow F^{\check{}} \longrightarrow \widetilde{F}^{\check{}} \longrightarrow \mathcal{O}_X^{\oplus 4} \longrightarrow 0$$

Referring to (6.17), we see that we get an exact sequence

$$0 \longrightarrow \wedge^2 F^{\text{\tiny{\circ}}} \longrightarrow \wedge^2 \widetilde{F}^{\text{\tiny{\circ}}} \longrightarrow \widetilde{F}^{\text{\tiny{\circ}} \oplus 4} \longrightarrow \mathcal{O}_X^{\oplus 10} \longrightarrow 0$$

If we introduce K_1 as the kernel of the final map, this splits into two short exact sequences

$$0 \longrightarrow \wedge^2 F^{\tilde{}} \longrightarrow \wedge^2 \tilde{F}^{\tilde{}} \longrightarrow K_1 \longrightarrow 0$$

$$0 \longrightarrow K_1 \longrightarrow \tilde{F}^{\tilde{}} \oplus 4 \longrightarrow \mathcal{O}_X^{\oplus 10} \longrightarrow 0$$
(6.18)

We can immediately obtain the cohomology of K_1 from the second sequence and the calculations we have already done:

$$h^{i}(X, K_{1}) = \begin{cases} 0 & , i = 0 \\ 18 & , i = 1 \\ 168 & , i = 2 \\ 6 & , i = 3 \end{cases}$$

The much more difficult calculation is to find the cohomology of $\wedge^2 \tilde{F}^{\check{}}$. Since stability of the bundle F implies that $h^3(X, \wedge^2 F^{\check{}}) = h^0(X, \wedge^2 F^{\check{}}) = 0$, the first sequence in (6.18) tells us that we get $h^0(X, \wedge^2 \tilde{F}^{\check{}}) = 0$ and $h^3(X, \wedge^2 \tilde{F}^{\check{}}) = 6$. For the other degrees, it will prove slightly easier to again appeal to Serre duality, and calculate instead the cohomology of $\wedge^2 \tilde{F}$. From (6.10), we can see that this fits into the exact sequence

$$0 \longrightarrow \wedge^2 \widetilde{F} \longrightarrow \wedge^2 G \longrightarrow G \otimes \mathcal{N} \longrightarrow S^2 \mathcal{N} \longrightarrow 0$$

Once again, we introduce a kernel, K_2 , so that we get two short exact sequences

$$0 \longrightarrow \wedge^2 \widetilde{F} \longrightarrow \wedge^2 G \longrightarrow K_2 \longrightarrow 0$$

$$0 \longrightarrow K_2 \longrightarrow G \otimes \mathcal{N} \longrightarrow S^2 \mathcal{N} \longrightarrow 0$$
(6.19)

The cohomology groups of $\wedge^2 G$, $G \otimes \mathcal{N}$ and $S^2 \mathcal{N}$ are given (and the techniques required to calculate them are outlined) in Appendix A.2, so we can write down the resulting long exact sequences in cohomology. The first is

while the second is

Clearly then, $H^3(X, K_2) = 0$. It is easy enough to show that the map from $H^1(X, G \otimes \mathcal{N})$ to $H^1(X, S^2\mathcal{N})$ (\mathbb{C}^{24} to \mathbb{C}^{14} in the above) is surjective, so $H^2(X, K_2) = 0$. To find the other two cohomology groups of K_2 , we would need to find the kernel of the map between the groups $H^0(X, G \otimes \mathcal{N})$ and $H^0(X, S^2\mathcal{N})$ (which are \mathbb{C}^{1104} and \mathbb{C}^{650} above). It would then be possible to work 'backwards' through the various long exact sequences, and end up with the cohomology of $\wedge^2 F$, and therefore of $\wedge^2 F$. Unfortunately, at this point our techniques failed to be practical, and we could not carry out these last steps. Volker Braun completed the calculation by other methods, and obtained

$$h^{i}(X, \wedge^{2}F) = \begin{cases} 0 & , i = 0\\ 42 & , i = 1\\ 6 & , i = 2\\ 0 & , i = 3 \end{cases}$$

6.3.3 Group action on cohomology, and the low-energy spectrum

Our ultimate goal is to try to find a model which has just the spectrum of the MSSM at low energies. This requires projecting out all zero modes coming from the $\overline{10}$ of $SU(5)_{GUT}$, and all except one electroweak doublet from the **5**, which is identified with the up-type Higgs.

The charged fields carry gauge indices as well as an index labelling the cohomology group from which they arise. The quotient group \mathfrak{G} naturally acts on the cohomology of an equivariant bundle, and if we choose non-zero Wilson lines in order to break $SU(5)_{\rm GUT}$, this amounts to letting the group act on the gauge indices. The zero modes on $X_{\mathfrak{G}}$ are therefore obtained by taking the tensor product of these two representations of \mathfrak{G} and picking out the invariants. Our procedure will be to first determine which choices of Wilson lines achieve the required symmetry breaking and result in no $\overline{10}$ zero modes. We can then calculate the resulting $\mathbf{5}$ zero modes. Note that discrete Wilson lines do not change the curvature of the bundle, and therefore the index remains unchanged. So we are guaranteed by the index theorem to always have three net generations of particles in each representation, and this means that we do not have to calculate separately the number of, say, $\mathbf{10}$ and $\overline{\mathbf{10}}$ zero modes.

There are very few choices of \mathfrak{G} Wilson lines which break $SU(5)_{\text{GUT}}$ to G_{SM} . The centraliser of G_{SM} is just the Abelian hypercharge group $U(1)_Y$, so we must choose a one-dimensional representation $\mathfrak{G} \to U(1)_Y$, and these all factor through the Abelianisation, \mathbb{Z}_4 , of \mathfrak{G} , obtained by setting $g_3 = \mathbf{1}$. There are therefore only three possibilities, given by

$$g_4 \mapsto \operatorname{diag}(1, 1, 1, -1, -1), \quad g_4 \mapsto \operatorname{diag}(-1, -1, -1, i, i)$$

$$g_4 \mapsto \operatorname{diag}(-1, -1, -1, -i, -i)$$
(6.20)

We need to know the transformation properties of the massless charged fields in the representations 10 and $\overline{10}$, corresponding to the cohomology groups $H^1(X, F)$ and $H^1(X, F^{\circ})$. But first we should make a comment about Serre duality. On a Calabi-Yau threefold, we frequently use this in the form $H^i(X, \mathscr{F}) \cong H^{3-i}(X, \mathscr{F}^{\circ})$, but in fact, the natural isomorphism is $H^i(X, \mathscr{F}) \cong H^{3-i}(X, \mathscr{F}^{\circ} \otimes \omega_X)'$, where a prime denotes the dual vector space. For a Calabi-Yau, of course, $\omega_X \cong \mathcal{O}_X$, and a freely-acting group acts trivially on this, as discussed in Chapter 1. Therefore, when applying Serre duality, we must simply take the contragredient representation of \mathfrak{G} , corresponding to the action on the dual space.

From now on, we will take the isomorphism symbol, \cong , to mean that two objects are isomorphic *and* that they are equipped with the same action of the group \mathfrak{G} . Note that, since our bundle maps are all \mathfrak{G} -equivariant, long exact sequences split into separate sequences for sums of each irreducible representation of \mathfrak{G} .

Considering first the 10 fields, we find from our long exact sequences in cohomology that

$$H^{1}(X,F) \cong H^{1}(X,\widetilde{F}) \cong \frac{H^{0}(X,\mathcal{N})}{\left(H^{0}(X,G)/H^{0}(X,\mathcal{O}_{X}^{\oplus 4})\right)}$$
(6.21)

Of the three vector spaces appearing on the right, we can choose the action of \mathfrak{G} on $H^0(X, \mathcal{O}_X^{\oplus 4})$ as any four-dimensional sub-representation of (6.6), but the representations acting on $H^0(X, \mathcal{N})$ and $H^0(X, G)$ are fixed, and must be calculated. It is straightforward to obtain

$$H^{0}(X,G) \sim 2 \operatorname{Reg}_{\mathfrak{G}} = 2 \left(R_{1} \oplus R_{-1} \oplus R_{i} \oplus R_{-i} \oplus 2 R_{+}^{(2)} \oplus 2 R_{-}^{(2)} \right)$$
(6.22)

We can consider sections of \mathcal{N} on the ambient space P first, where we find

$$H^{0}(P, \mathcal{N}) = H^{0}(P, \mathcal{O}(111000) \oplus \mathcal{O}(000111)) \oplus H^{0}(P, \mathcal{O}(111111))$$

= $(R_{1} \oplus R_{-1} \oplus R_{i} \oplus R_{-i} \oplus \operatorname{Reg}_{\mathfrak{G}}) \oplus (R_{1} \oplus 4\operatorname{Reg}_{\mathfrak{G}})$ (6.23)

On X, the sections p, q and r vanish identically, so we delete the corresponding representations (bearing in mind the 'non-trivial' equivariant structure on \mathcal{N} , discussed in the last section):

$$H^{0}(X, \mathcal{N}) = R_{i} \oplus R_{-i} \oplus 5 \operatorname{Reg}_{\mathfrak{G}}$$

$$(6.24)$$

So now if we make the choice $H^0(X, \mathcal{O}_X^{\oplus 4}) \sim 2 R_+^{(2)}$, we get, from equation (6.21),

$$H^{1}(X,F) \sim R_{i} \oplus R_{-i} \oplus 2 R^{(2)}_{+} \oplus 3 \operatorname{Reg}_{\mathfrak{G}}$$

$$(6.25)$$

In particular, we see that the representations R_1 and R_{-1} appear exactly three times each (inside $3 \operatorname{Reg}_{\mathfrak{G}}$). Therefore if we choose our Wilson lines, breaking $SU(5)_{GUT}$, such that only these representations act on the gauge indices, we will get precisely three copies of the **10** of $SU(5)_{GUT}$, with no exotics. It is easy to see that the only choice which works is the first listed in (6.20), given by $g_4 \rightarrow \operatorname{diag}(1, 1, 1, -1, -1)$.

Notice that we had to choose the map ι appropriately. If instead its image carried the representation $R_1 \oplus R_{-1} \oplus R_+^{(2)}$, we would always be left with unwanted zero modes. In fact, this is the scenario which was advocated in [83]; the non-Abelian Wilson lines chosen there correspond to the bundle $\mathcal{T}X_{\mathfrak{G}} \oplus \mathcal{W}_{R_+^{(2)}}$. We see from the present work that the appropriate symmetry breaking could indeed be achieved in that model, but the resulting spectrum is not realistic.

We have seen above that what we want to do is deform $\mathcal{T}X_{\mathfrak{G}} \oplus \mathcal{L}_1 \oplus \mathcal{L}_{-1}$. But there seems to be a problem — the determinant bundle of this is not trivial, so it is not an SU(5) bundle, and nor will its deformations be. The way out is easy to see: we must twist it by \mathcal{L}_{-1} , so that the bundle we actually deform is $(\mathcal{T}X_{\mathfrak{G}} \otimes \mathcal{L}_{-1}) \oplus \mathcal{L}_{-1} \oplus \mathcal{L}_1$. Equivalently, we simply take $F_{\mathfrak{G}} \otimes \mathcal{L}_{-1}$, where $F_{\mathfrak{G}}$ is the bundle F defined above, pushed forward to $X_{\mathfrak{G}}$. Luckily, none of our conclusions change, since the representation of \mathfrak{G} on the cohomology of F is invariant under tensor product with R_{-1} . The index theorem tells us that with the choices above, we will get no zero modes of the $\overline{\mathbf{10}}$ fields, but it is nice to see this explicitly, by calculating the group action on $H^1(X, F^{\tilde{}})$. From the duals of (6.10) and (6.11), we quickly find that

$$H^{1}(X, F^{\sim}) \cong H^{0}(X, \mathcal{O}_{X}^{\oplus 4}) \oplus H^{1}(X, \widetilde{F}^{\sim})$$

$$H^{1}(X, \widetilde{F}^{\sim}) \cong H^{2}(X, \mathcal{N}^{\sim})$$

(6.26)

We have already decided to take $H^0(X, \mathcal{O}_X^{\oplus 4}) \sim 2 R_+^{(2)}$, which is unaffected by dualisation since the representation $R_+^{(2)}$ is its own contragredient. We therefore need only to work out the group action on $H^2(X, \mathcal{N}) \cong \mathbb{C}^2$. This group comes from the first two factors in \mathcal{N} :

$$H^{2}(X, \mathcal{N}) = H^{2}(X, \mathcal{O}_{X}(-1, -1, -1, 0, 0, 0) \oplus \mathcal{O}_{X}(0, 0, 0, -1, -1, -1))$$
(6.27)

Chasing through the long exact cohomology sequences, and using the well-known cohomology of line bundles on projective spaces, reviewed in Chapter 2, we find that this group is generated, in Čech cohomology, by

$$\frac{p}{s^{00}s^{01}s^{10}s^{11}s^{20}s^{21}}\ ,\ \frac{q}{t^{00}t^{01}t^{10}t^{11}t^{20}t^{21}}$$

If we take into account the extra '-1' factor in the equivariant structure, these indeed transform as $R_i \oplus R_{-i}$. So overall, $H^1(X, F) \sim R_i \oplus R_{-i} \oplus 2R_+^{(2)}$, and we see explicitly that our choice of Wilson lines above projects out all zero modes coming from $\overline{10}$ fields.

Having determined that there is a unique choice of Wilson lines which removes all exotics coming from the $\overline{10}$ of $SU(5)_{GUT}$, we must find the resulting massless spectrum arising from the **5** and $\overline{5}$. By similar techniques to those used above, we find

$$H^1(X, \wedge^2 F) \sim R_1 \oplus R_{-1} \oplus 2 R_-^{(2)} \oplus 3 \operatorname{Reg}_{\mathfrak{G}}$$

Our choice of Wilson lines, forced on us by the requirement that there are no zero modes coming from the $\overline{10}$, is such that the colour triplet and electroweak doublet in the $\overline{5}$ transform as R_1 and R_{-1} respectively. Each of these representations appears four times in the above, so as well as a pair of Higgs doublets, the model also contains a vector-like pair of colour triplets, with the quantum numbers of the right-handed down quark. Although such fields would probably obtain electroweak-scale masses after supersymmetry breaking, and thus avoid direct detection limits, they will ruin gauge unification, and therefore unfortunately make this model look fairly unappealing.

6.4 Models on the \mathbb{Z}_{12} quotient

So it seems that we can't obtain the MSSM spectrum by beginning with the standard embedding on $X_{\mathfrak{G}}$, but we can use what we have learned above to perform a quick preliminary analysis of the possible models on the \mathbb{Z}_{12} quotient manifold. Let $\mathscr{G} = \mathbb{Z}_{12}$, and denote the quotient by $X_{\mathscr{G}}$. The way to proceed is to find all deformation classes of bundles on $X_{\mathscr{G}}$ which descend from deformations of $\mathcal{T}X \oplus \mathcal{O}_X \oplus \mathcal{O}_X$, and determine which of them allow a choice of Wilson lines which projects out all the $\overline{\mathbf{10}}$ zero modes on X. Having found all choices of bundle and Wilson line which allow this, we must then search for an example in which all colour triplets coming from **5** modes are projected out.

6.4.1 Wilson line breaking of $SU(5)_{GUT}$

The first thing we will do is determine the ways that \mathbb{Z}_{12} -valued Wilson lines can break $SU(5)_{\text{GUT}}$ to G_{SM} . Their values must lie in the hypercharge subgroup, so if g_{12} generates \mathbb{Z}_{12} , we choose its image to be

$$g_{12} \to \exp\left[\frac{2\pi \mathrm{i}}{12}\operatorname{diag}(x, x, x, y, y)\right]$$
 (6.28)

where $3x + 2y \equiv 0 \pmod{12}$. It is straightforward to solve this for the possible values of x and y (we exclude (x, y) = (0, 0), since this is no breaking):

$$\begin{array}{rcl} (x,y) &=& (0,6) & (2,3) & (2,9) & (4,0) & (4,6) & (6,3) \\ & & (6,9) & (8,0) & (8,6) & (10,3) & (10,9) \end{array}$$

We should also ask what the ' \mathbb{Z}_{12} charges' of the various fields in the **10** and $\overline{\mathbf{5}}$ representations of $SU(5)_{\text{GUT}}$ are in terms of x and y. For the $\overline{\mathbf{5}}$, the doublets have charge -y, and the triplets charge -x. The **10** is the rank-two anti-symmetric tensor, so the charges of the components are x + y, 2x and 2y. These values, for each choice of (x, y) above, are listed in Table 6.1.

(x, y)	x+y	2x	2y
(0, 6)	6	0	0
(2,3)	5	4	6
(2,9)	11	4	6
(4,0)	4	8	0
(4, 6)	10	8	0
(6,3)	9	0	6
(6,9)	3	0	6
(8,0)	8	4	0
(8, 6)	2	4	0
(10,3)	1	8	6
(10,9)	7	8	6

Table 6.1: The possible choices for (x, y) in equation (6.28), and the resulting Z_{12} charges of the different components of the **10** of $SU(5)_{GUT}$.

6.4.2 Bundle deformations and the spectrum

Let R_r be the \mathbb{Z}_{12} representation corresponding to charge r. The representation of \mathbb{Z}_{12} on the cohomology group $H^{1,1}(X)$ was given in Chapter 5:

$$H^{1,1}(X) \sim R_0 \oplus R_2 \oplus R_3 \oplus R_4 \oplus R_6 \oplus R_8 \oplus R_9 \oplus R_{10}$$
 (6.30)

We want to deform $\mathcal{T}X \oplus \mathcal{O}_X \oplus \mathcal{O}_X$ to a \mathbb{Z}_{12} -equivariant irreducible bundle. On each \mathcal{O}_X , we can choose an equivariant structure corresponding to one of the R_r ; we will denote by \mathcal{L}_r the resulting flat line bundle on the quotient. Irreducible deformations of $\mathcal{T}X \oplus \mathcal{L}_{r_1} \oplus \mathcal{L}_{r_2}$ exist only if both R_{r_1} and R_{r_2} occur in (6.30). We must then ensure that we have an SU(5) bundle by twisting, if necessary, with an appropriate flat bundle. In other words, we really want a deformation of

$$\mathcal{L}_{\tilde{r}} \otimes (\mathcal{T} X_{\mathscr{G}} \oplus \mathcal{L}_{r_1} \oplus \mathcal{L}_{r_2}) \tag{6.31}$$

where $5\tilde{r} + r_1 + r_2 \equiv 0 \pmod{12}$. We can always find an \tilde{r} to solve this, because 5 is invertible in \mathbb{Z}_{12} , so there are 28 distinct deformation classes of bundles on the quotient, corresponding to the choice of two representations from (6.30). We should note that it is not possible to find deformations of all these bundles by the technique of §6.2. The six (1,1) forms coming from the ambient space transform as $R_0 \oplus R_2 \oplus R_4 \oplus R_6 \oplus R_8 \oplus R_{10}$, so we can only construct deformations when $r_1, r_2 \in \{0, 2, 4, 6, 8, 10\}$. However, all (1,1) forms are represented by toric divisors, so it would be possible to construct deformations using the toric description in the other cases.

Let F be the irreducible rank-five bundle obtained by deforming (6.31). The first thing to calculate is $H^1(X, F^{\check{}})$, and its transformation under \mathbb{Z}_{12} . The recipe for this is clear from our earlier work on the Dic₃ quotient: if we deform $\mathcal{T}X_{\mathscr{G}} \oplus \mathcal{L}_{r_1} \oplus \mathcal{L}_{r_2}$, h^1 of the bundle decreases by two, and we delete $R_{r_1} \oplus R_{r_2}$ from (6.30). The effect of then tensoring the bundle with $\mathcal{L}_{\tilde{r}}$ is to tensor the representation on the cohomology with $R_{\tilde{r}}$. The 28 choices for $(r_1, r_2; \tilde{r})$, and the corresponding representations on $H^1(X, F^{\check{}})$, are listed in Table 6.2.

$(r_1,r_2; ilde{r})$	\mathbb{Z}_{12} charges of $H^1(X, F^{\sim})$
(0,2;2)	0, 5, 6, 8, 10, 11
(0,3;9)	1, 3, 5, 6, 7, 11
(0,4;4)	0, 1, 2, 6, 7, 10
(0,6;6)	2, 3, 4, 8, 9, 10
(0,8;8)	0, 2, 5, 6, 10, 11
(0,9;3)	$1, 5, 6, 7, 9\ 11$
(0, 10; 10)	0, 1, 2, 4, 6, 7
(2,3;11)	3, 5, 7, 8, 9, 11
(2,4;6)	0, 2, 3, 4, 6, 9
(2,6;8)	0, 4, 5, 6, 8, 11
(2,8;10)	1, 2, 4, 7, 8, 10
(2,9;5)	1, 3, 5, 8, 9, 11
(2, 10; 0)	0, 3, 4, 6, 8, 9
(3,4;1)	1, 3, 7, 9, 10, 11
(3,6;3)	0, 1, 3, 5, 7, 11
(3,8;5)	2, 3, 5, 7, 9, 11
(3,9;0)	0, 2, 4, 6, 8, 10
(3,10;7)	1, 3, 4, 7, 9, 11
(4,6;10)	0, 1, 6, 7, 8, 10
(4,8;0)	0, 2, 3, 6, 9, 10
(4,9;7)	1, 3, 5, 7, 9, 10
(4, 10; 2)	$2, 4, \overline{5}, 8, 10, 11$
(6,8;2)	0, 2, 4, 5, 6, 11
$(\overline{6,9;9)}$	$0, 1, \overline{5}, 7, 9, 11$
(6,10;4)	0, 1, 4, 6, 7, 8
(8,9;11)	$1, 2, \overline{3, 5, 9, 11}$
(8,10;6)	$0, 3, \overline{6, 8, 9, 10}$
(9, 10; 1)	$1, 3, \overline{4, 5, 7, 9}$

Table 6.2: The bundle F is a deformation of $\mathcal{L}_{\tilde{r}} \otimes (\mathcal{T}X_{\mathscr{G}} \oplus \mathcal{L}_{r_1} \oplus \mathcal{L}_{r_2})$, and we calculate the action on its cohomology as described in the text. 'Charge' r stands for the representation R_r .

Recall that to identify zero modes on $X_{\mathscr{G}}$, we take the representation of \mathscr{G} on the relevant cohomology group, tensor it with the action defined by the choice of Wilson line, and pick out the invariants. So to find the combinations of bundle F and Wilson line which result in no $\overline{10}$ zero modes surviving on $X_{\mathscr{G}}$, we need only compare Table 6.1 to Table 6.2, and choose pairs of rows, one from each, with no common charges. The cases which pass this test are compiled in Table 6.3, which summarises the results of this section.

$(r_1, r_2; \tilde{r})$			(x, y)		
(0,3;9)	(4, 0)	(4, 6)	(8, 0)	(8,6)	
(0,6;6)	(0, 6)				
(0,9;3)	(4, 0)	(4, 6)	(8, 0)	(8,6)	
(2,3;11)	(0, 6)	(8, 6)			
(2,8;10)	(0, 6)	(6, 3)	(6, 9)		
(2,9;5)	(0, 6)	(8, 6)			
(3,4;1)	(0, 6)	(2, 3)	(4, 0)	(8, 0)	(8, 6)
(3,8;5)	(0, 6)	(4, 0)	(4, 6)	(8, 0)	(10, 3)
(3, 10; 7)	(0, 6)	(4, 6)			
(4,9;7)	(0, 6)	(2, 9)	(4, 0)	(8, 0)	(8, 6)
(4, 10; 2)	(0, 6)	(6, 3)	(6, 9)		
(8,9;11)	(0, 6)	(4, 0)	(4, 6)	(8, 0)	(10, 9)
(9, 10; 1)	(0, 6)	(4, 6)			

Table 6.3: An enumeration of the bundles on $X_{\mathscr{G}}$ which lead to no exotic massless fields descending from the $\overline{\mathbf{10}}$ of $SU(5)_{\text{GUT}}$; there are a total of 43 choices which pass this simple test. The rank-five bundle F is a deformation of $\mathcal{L}_{\tilde{r}} \otimes (\mathcal{T}X_{\mathscr{G}} \oplus \mathcal{L}_{r_1} \oplus \mathcal{L}_{r_2})$, and Z_{12} Wilson lines are embedded in $SU(5)_{\text{GUT}}$ via the map $g_{12} \to \exp\left[\frac{2\pi i}{12}\operatorname{diag}(x, x, x, y, y)\right]$.

6.5 Moduli stabilisation

Until now, we have completely ignored the presence of the massless moduli fields in the models above, which must eventually be given reasonably large masses in any realistic theory. The possibility of doing this for the examples herein has not been investigated, and we will not discuss it in detail, but it seems necessary to comment on the points made in [100].

The main observation of [100] is that if one tries to stabilise the geometric moduli in a model with the standard embedding, then any vector-like matter fields will obtain masses of the same order as the moduli, because they are related by the fact that the gauge bundle is the tangent bundle. This would be a problem for our models, as the bundle deformations we have described correspond to vector-like flat directions in the low-energy theory, so we do not want these lifted. However, [100] did not take into account the effects of discrete Wilson lines. To see why this is important, note that the moduli of the quotient manifold are precisely the group-invariant moduli of X. In the presence of Wilson lines, though, the massless matter fields generically come from harmonic forms on X which are *not* group invariant, but rather transform in the representation dual to that on the gauge degrees of freedom.

In the \mathbb{Z}_{12} models discussed above, the flat directions correspond to the representations R_{r_1} and R_{r_2} , so as long as both of these are non-trivial, the results of [100] do not preclude us from stabilising the moduli while retaining the necessary low-energy flat directions.

6.6 Summary

This chapter has been quite long and technical, and it seems prudent to finish with an overview of what we have done. We began with a quotient manifold $X_{\mathfrak{G}}$ with $\chi(X_{\mathfrak{G}}) = -6$, which therefore gives us, via the standard embedding, a supersymmetric E_6 GUT with three net generations of fermions. However, it is a group-theoretical fact that E_6 cannot be broken to $G_{\rm SM}$ by any choice of discrete Wilson lines. In fact, the only way to obtain $G_{\rm SM}$ as the low energy gauge group is to start with an irreducible rank five bundle, and then break the remaining group $SU(5)_{\rm GUT}$ with a discrete Abelian Wilson line.

On the covering space X, presented as a CICY in $(\mathbb{P}^1)^6$, we saw that there is a natural way to construct a rank-nine bundle \tilde{F}_0 , which is a non-trivial extension of $\mathcal{T}X$ by $\mathcal{O}_X^{\oplus 6}$, as the kernel of a bundle morphism Φ_0 . Since $\mathcal{O}_X^{\oplus 6}$ injects into \tilde{F}_0 , we are free to choose some sub-bundle $\mathcal{O}_X^{\oplus 4}$, and quotient by it to obtain a rank-five bundle $F_0 = \tilde{F}_0/\mathcal{O}_X^{\oplus 4}$, which is therefore an extension of $\mathcal{T}X$ by $\mathcal{O}_X^{\oplus 2}$. Such a bundle cannot be stable, but using a theorem of Li and Yau, we can deform it to a stable bundle by deforming the map Φ_0 to a more general one Φ . In doing so, we must ensure that we still have $\mathcal{O}_X^{\oplus 4} \subset \tilde{F} \equiv \ker \Phi$, in order to be able to form the stable rank-five quotient bundle $F = \tilde{F}/\mathcal{O}_X^{\oplus 4}$.

In order for the bundle F to descend to the quotient manifold $X_{\mathfrak{G}}$, it must be equivariant under the group action. We ensure this by imposing equivariance under \mathfrak{G} at each step of the above process. The bundle $\mathcal{O}_X^{\oplus 6}$ occurs with a natural equivariant structure, corresponding to a particular six-dimensional representation of \mathfrak{G} , and we choose the sub-bundle corresponding to some rank-four sub-representation to be divided out to obtain F. The interpretation of this is quite interesting, and a new feature of our work. A choice of discrete Wilson lines corresponds to some flat rank-r bundle on $X_{\mathfrak{G}}$, which in turn is equivalent to a non-trivial \mathfrak{G} -equivariant structure on $\mathcal{O}_X^{\oplus r}$. The initial bundle F_0 is, on the quotient manifold, an extension of $\mathcal{T}X_{\mathfrak{G}}$ by a rank-two flat bundle, corresponding to the action of \mathfrak{G} on $\mathcal{O}_X^{\oplus 2}$ on the covering space. We can therefore interpret our construction as adding discrete Wilson lines to the standard embedding, and *then* deforming the bundle, corresponding to the Higgs mechanism in the low-energy field theory. This cannot change the net number of generations, so we obtain a model with gauge group $G_{\rm SM}$ and three generations of chiral fermions. Mathematically this is because on the covering space, F is a deformation of $\mathcal{T}X \oplus \mathcal{O}_X^{\oplus 2}$, and therefore has the same index.

We showed that the model based on the bundle F has the exact spectrum of the MSSM, plus a single vector-like pair of colour triplets. It is therefore probably ruled out by the requirement of gauge unification. However, we also discussed how to perform an analogous construction on the quotient of X by \mathbb{Z}_{12} , in which case there are many more choices for Wilson lines. We found 43 distinct ways to obtain $G_{\rm SM}$ as the unbroken gauge group, with no exotic fields coming from the $\overline{10}$ of $SU(5)_{\rm GUT}$, but have not yet calculated the spectrum of massless fields originating in the 5 and $\overline{5}$ representations. There could well be viable models amongst these 43 candidates.

A. Cohomology Calculations

A.1 Cohomology on dP_6

The manifold X of Chapter 5 can be represented as a hypersurface in $dP_6 \times dP_6$, and this provides a convenient way to calculate the cohomology of many bundles on X, as long as we can calculate cohomology on dP_6 .

We will here express $S \equiv dP_6$ as a hypersurface in $M := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, defined by a single trilinear equation. If we need to be specific, we will take this equation to be¹

$$p = s_{0,0}s_{1,0}s_{2,0} + s_{0,0}s_{1,1}s_{2,1} + s_{0,1}s_{1,0}s_{2,1} + s_{0,1}s_{1,1}s_{2,0}$$
(A.1)

where $s_{i,j}$ is the *j*-th homogeneous coordinate on the *i*-th \mathbb{P}^1 . The ideal sheaf of $S \subset M$ is therefore isomorphic to $\mathcal{O}_M(-1, -1, -1)$, and fits into the short exact sequence

$$0 \longrightarrow \mathcal{O}_M(-1, -1, -1) \xrightarrow{p} \mathcal{O}_M \longrightarrow \mathcal{O}_S \longrightarrow 0$$
(A.2)

This allows us to relate the cohomology of sheaves on M to the cohomology of the same sheaves restricted to S. For example, for any line bundle $\mathcal{O}_M(a, b, c)$ on M, we have

$$0 \longrightarrow \mathcal{O}_M(a-1,b-1,c-1) \xrightarrow{p} \mathcal{O}_M(a,b,c) \longrightarrow \mathcal{O}_S(a,b,c) \longrightarrow 0$$
(A.3)

Since we can easily calculate the cohomology of such sheaves on M, this lets us calculate their cohomology on S. There are a few simple cases which are worth listing explicitly, since they come up so often (note that a, b, c may be permuted arbitrarily without affecting the cohomology):

$$\underline{a, b, c \ge 0}$$

$$h^{0}(S, \mathcal{O}_{S}(a, b, c)) = ab + ac + bc + a + b + c + 1 , \text{ others zero}$$
(A.4)

 $\underline{a} = 0$

$$h^{i}(S, \mathcal{O}_{S}(0, b, c)) = h^{i}(M, \mathcal{O}_{M}(0, b, c))$$
 (A.5)

a = -1

$$h^{i}(S, \mathcal{O}_{S}(-1, b, c)) = h^{i+1}(M, \mathcal{O}_{M}(-2, b-1, c-1))$$
(A.6)

Note that, by adjunction, the canonical bundle of S is in fact $\omega_S = \mathcal{O}_S(-1, -1, -1)$. One consequence is that the case a, b, c < 0 is Serre dual to $a, b, c \ge 0$, and can therefore be read off from the above. Other cases are slightly more complicated, but all are easily calculable from (A.3), and knowledge of line bundle cohomology on M.

A.1.1 The tangent bundle and related bundles

We also have the short exact sequence defining the normal bundle to S in M, which is isomorphic to $\mathcal{O}_S(1,1,1)$, in terms of the tangent bundles of S and M

$$0 \longrightarrow \mathcal{T}S \longrightarrow \mathcal{T}M|_S \xrightarrow{dp} \mathcal{O}_S(1,1,1) \longrightarrow 0$$
(A.7)

¹All non-degenerate choices are equivalent.

At this point we note that, on \mathbb{P}^1 , we have the identity $\mathcal{T}\mathbb{P}^1 \cong \mathcal{O}_{\mathbb{P}^1}(2)$. Since M is a direct product of three \mathbb{P}^1 's, we get

$$\mathcal{T}M = \mathcal{O}_M(2,0,0) \oplus \mathcal{O}_M(0,2,0) \oplus \mathcal{O}_M(0,0,2) \tag{A.8}$$

This identity proves very useful in simplifying some calculations. Another follows from the fact that, for any rank *n* holomorphic vector bundle \mathcal{F} , there are isomorphisms $\wedge^r \mathcal{F} \cong \wedge^{n-r} \mathcal{F} \otimes \wedge^n \mathcal{F}$. Taking $\mathcal{F} = \Omega^1 S$, we get $\Omega^1 S \cong \mathcal{T} S \otimes \omega_S$.

We can get the cohomology of $\mathcal{T}S$ directly from (A.7), which gives the exact sequences

$$0 \longrightarrow H^0(S, \mathcal{T}S) \longrightarrow \mathbb{C}^9 \xrightarrow{dp} \mathbb{C}^7 \longrightarrow H^1(S, \mathcal{T}S) \longrightarrow 0$$
$$0 \longrightarrow H^2(S, \mathcal{T}S) \longrightarrow 0$$

We can see that we need to calculate the kernel and/or cokernel of dp acting on global sections of the bundle $\mathcal{T}M|_S$. This sort of calculation comes up frequently, so we will work this one through explicitly as an example. The space $H^0(S, \mathcal{O}_S(1, 1, 1))$ is the space of polynomials linear in the homogeneous coordinates of each \mathbb{P}^1 , divided by the subspace generated by p, since $p \equiv 0$ on S. The space $H^0(S, \mathcal{T}M|_S)$ is best described as the space of linear vector fields on $(\mathbb{C}^2)^3$, as in §2.2.2, divided by the subspace generated by the Euler vectors $e_i = \sum_a s_{i,a} \partial/\partial s_{i,a}$. It is then fairly easy to show that dp is surjective, since each trilinear monomial is in its image; for example

$$s_{0,0}s_{1,1}s_{2,1} = dp\left(s_{0,0}\frac{\partial}{\partial s_{0,0}} + \frac{1}{2}\left[s_{0,1}\frac{\partial}{\partial s_{0,1}} - s_{1,0}\frac{\partial}{\partial s_{1,0}} - s_{2,0}\frac{\partial}{\partial s_{2,0}}\right]\right)$$

Every monomial can be obtained in a similar way, so dp is surjective, and we get

$$h^{i}(S, \mathcal{T}S) = \begin{cases} 2 & , i = 0 \\ 0 & , i = 1, 2 \end{cases}$$

To get the cohomology of $\Omega^1 S$, dualise (A.7) to obtain the short exact sequence

$$0 \longrightarrow \mathcal{O}_S(-1, -1, -1) \longrightarrow \Omega^1 M|_S \longrightarrow \Omega^1 S \longrightarrow 0$$
(A.9)

The resulting long exact sequence in cohomology splits up into

$$0 \longrightarrow \mathbb{C}^3 \longrightarrow H^1(S, \Omega^1 S) \longrightarrow \mathbb{C} \longrightarrow 0$$
$$0 \longrightarrow H^i(S, \Omega^1 S) \longrightarrow 0 \qquad i = 0, 2$$

We therefore find

$$h^{i}(S, \Omega^{1}S) = \begin{cases} 4 & , & i = 1 \\ 0 & , & i = 0, 2 \end{cases}$$

We would now like to find the cohomology of the slightly trickier beast $\Omega^1 S \otimes \mathcal{T}S$. To do so, we tensor (A.9) by $\mathcal{T}S$ to obtain a new short exact sequence

$$0 \longrightarrow \mathcal{T}S(-1, -1, -1) \longrightarrow \mathcal{T}S \otimes \Omega^1 M|_S \longrightarrow \Omega^1 S \otimes \mathcal{T}S \longrightarrow 0$$
 (A.10)

The first term here could also be written as $\mathcal{T}S \otimes \omega_S \cong \Omega^1 S$, so we already know its cohomology. To obtain that of the middle term, note that from equation (A.8), we have

$$\mathcal{T}S \otimes \Omega^1 M|_S \cong \mathcal{T}S(-2,0,0) \oplus \mathcal{T}S(0,-2,0) \oplus \mathcal{T}S(0,0,-2)$$

Its cohomology is therefore the direct sum of three identical pieces. We can obtain one of these by tensoring (A.7) by $\mathcal{O}_S(-2,0,0)$ to get

$$0 \longrightarrow \mathcal{T}S(-2,0,0) \longrightarrow \mathcal{O}_S \oplus \mathcal{O}_S(-2,2,0) \oplus \mathcal{O}_S(-2,0,2) \longrightarrow \mathcal{O}_S(-1,1,1) \longrightarrow 0$$

The resulting long exact cohomology sequence is

$$0 \longrightarrow H^0(S, \mathcal{T}S(-2, 0, 0)) \longrightarrow \mathbb{C} \longrightarrow \mathbb{C} \longrightarrow H^1(S, \mathcal{T}S(-2, 0, 0)) \longrightarrow \mathbb{C}^6 \longrightarrow 0$$

It is not too hard to show that the map between the two copies of \mathbb{C} is surjective, which implies that $h^0(S, \mathcal{T}S(-2, 0, 0)) = 0$ and $h^1(S, \mathcal{T}S(-2, 0, 0)) = 6$. Then the long exact sequence coming from equation (A.10) reduces to

$$0 \longrightarrow H^0(S, \Omega^1 S \otimes \mathcal{T}S) \longrightarrow \mathbb{C}^4 \longrightarrow \mathbb{C}^{18} \longrightarrow H^1(S, \Omega^1 S \otimes \mathcal{T}S) \longrightarrow 0$$

The kernel of the middle map here is one-dimensional, so we conclude that $h^0(S, \Omega^1 S \otimes \mathcal{T} S) = 1$ and $h^1(S, \Omega^1 S \otimes \mathcal{T} S) = 15$. In summary

$$h^{i}(S, \Omega^{1}S \otimes \mathcal{T}S) = \begin{cases} 1 & , i = 0 \\ 15 & , i = 1 \\ 0 & , i = 2 \end{cases}$$

A.2 Cohomology on $X^{8,44}$

We are really interested in the cohomology of bundles on the Calabi-Yau threefold $X^{8,44}$, which can be realised as an anti-canonical hypersurface in $Z \equiv dP_6 \times dP_6$. It will be useful to distinguish the two copies of dP_6 , so we will denote this ambient space by $Z = S_1 \times S_2$. We therefore have two projections, $\pi_i : Z \to S_i$, and bundles such as $\mathcal{T}Z$ and ω_Z can be expressed in terms of pullbacks from the two surfaces:

$$\mathcal{T}Z = \pi_1^* \mathcal{T}S_1 \oplus \pi_2^* \mathcal{T}S_2 \ , \ \ \omega_Z = \pi_1^* \omega_{S_1} \otimes \pi_2^* \omega_{S_2}$$

To avoid cluttering the notation, we will usually drop the explicit reference to π_i^* , and write for example $\mathcal{T}Z = \mathcal{T}S_1 \oplus \mathcal{T}S_2$. It should be clear from the context when this is done.

Calculating the cohomology of most interesting bundles on Z is greatly simplified by the fact that it is a product of two surfaces, which means we have the Künneth formula for cohomology over \mathbb{C} :

$$H^{i}(Z, \pi_{1}^{*}E_{1} \otimes \pi_{2}^{*}E_{2}) \cong \bigoplus_{j \leq i} H^{j}(S_{1}, E_{1}) \otimes H^{i-j}(S_{2}, E_{2})$$
 (A.11)

where E_i is a bundle on S_i .

The 'restriction' sequence and normal bundle sequence for $X \subset Z$ are given by

$$0 \longrightarrow \omega_Z \xrightarrow{r} \mathcal{O}_Z \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{A.12}$$

$$0 \longrightarrow \mathcal{T}X \longrightarrow \mathcal{T}Z|_X \xrightarrow{dr} \omega_Z^{-1}|_X \longrightarrow 0$$
(A.13)

where we have denoted by r the section of ω_Z^{-1} which defines X. Note that, in terms of bundles on $(\mathbb{P}^1)^6$, we have $\omega_Z = \mathcal{O}_Z(-1, -1, -1, -1, -1, -1)$.

Example: $\mathcal{O}_X(2, 2, 2, 0, 0, 0)$

As part of our calculations in Chapter 6, we required the cohomology on X of bundles such as $G, G \otimes \mathcal{N}$ etc. These are all direct sums of line bundles which are pulled back from the ambient space $(\mathbb{P}^1)^6$, so we must understand the cohomology of such line bundles. Here we will demonstrate just one calculation, but they are all the same in form.

We first write down the sequence which expresses the restriction of $\mathcal{O}_Z(2,2,2,0,0,0)$ to X:

$$0 \to \mathcal{O}_Z(1, 1, 1, -1, -1, -1) \xrightarrow{r} \mathcal{O}_Z(2, 2, 2, 0, 0, 0) \to \mathcal{O}_X(2, 2, 2, 0, 0, 0) \to 0$$
(A.14)

We then use the results of Appendix A.1 and equation (A.11) to calculate the cohomology of the first two terms here:

$$h^{i}(Z, \mathcal{O}_{Z}(1, 1, 1, -1, -1, -1)) = \begin{cases} 7 & , i = 2 \\ 0 & , \text{ otherwise} \end{cases}$$
 (A.15)

$$h^{i}(Z, \mathcal{O}_{Z}(2, 2, 2, 0, 0, 0)) = \begin{cases} 19 & , i = 0 \\ 0 & , \text{ otherwise} \end{cases}$$
 (A.16)

The cohomology on X then follows very simply from the long exact sequence arising from equation (A.14), without even having to worry about explicit maps:

$$h^{i}(X, \mathcal{O}_{X}(2, 2, 2, 0, 0, 0)) = \begin{cases} 19 & , i = 0 \\ 7 & , i = 1 \\ 0 & , i = 2, 3 \end{cases}$$
(A.17)

A.2.1 Some cohomologies

Using the technique exemplified in the previous example, we can calculate the cohomology of bundles involved in finding the spectrum of our model in Chapter 6. In particular, the following bundles appear in certain exact sequences of interest:

$$\wedge^{2}G = \mathcal{O}_{X}(2,0,0,0,0,0) \oplus \ldots \oplus \mathcal{O}_{X}(0,0,0,0,0,2)$$

$$\oplus 4 \Big[\mathcal{O}_{X}(1,1,0,0,0,0) \oplus \mathcal{O}_{X}(1,0,1,0,0,0) \oplus \ldots \oplus \mathcal{O}_{X}(0,0,0,0,1,1) \Big]$$
(A.18)

$$G \otimes \mathcal{N} = 2 \Big[\mathcal{O}_X(2, 1, 1, 0, 0, 0) \oplus \mathcal{O}_X(1, 2, 1, 0, 0, 0) \oplus \mathcal{O}_X(1, 1, 2, 0, 0, 0) \\ \oplus \mathcal{O}_X(1, 1, 1, 1, 0, 0) \oplus \mathcal{O}_X(1, 1, 1, 0, 1, 0) \oplus \mathcal{O}_X(1, 1, 1, 0, 0, 1) \\ \oplus \mathcal{O}_X(1, 0, 0, 1, 1, 1) \oplus \mathcal{O}_X(0, 1, 0, 1, 1, 1) \oplus \mathcal{O}_X(0, 0, 1, 1, 1, 1) \\ \oplus \mathcal{O}_X(0, 0, 0, 2, 1, 1) \oplus \mathcal{O}_X(0, 0, 0, 1, 2, 1) \oplus \mathcal{O}_X(0, 0, 0, 1, 1, 2) \\ \oplus \mathcal{O}_X(2, 1, 1, 1, 1, 1) \oplus \ldots \oplus \mathcal{O}_X(1, 1, 1, 1, 1, 2) \Big]$$
(A.19)

$$S^{2}\mathcal{N} = \mathcal{O}_{X}(2,2,2,0,0,0) \oplus \mathcal{O}_{X}(1,1,1,1,1,1) \oplus \mathcal{O}_{X}(2,2,2,1,1,1)$$
$$\oplus \mathcal{O}_{X}(0,0,0,2,2,2) \oplus \mathcal{O}_{X}(1,1,1,2,2,2) \oplus \mathcal{O}_{X}(2,2,2,2,2,2,2)$$
(A.20)
Calculating the cohomology of these bundles is straightforward, and the results are

$$h^{i}(X, \wedge^{2}G) = \begin{cases} 258 , i = 0 \\ 6 , i = 2 \\ 0 , i = 1, 3 \end{cases}$$
(A.21)

$$h^{i}(X, G \otimes \mathcal{N}) = \begin{cases} 1104 & , i = 0 \\ 24 & , i = 1 \\ 0 & , i = 2, 3 \end{cases}$$
(A.22)

$$h^{i}(X, S^{2}\mathcal{N}) = \begin{cases} 650 , i = 0\\ 14 , i = 1\\ 0 , i = 2, 3 \end{cases}$$
(A.23)

A.2.2 Deformations of $\mathcal{T}X^{8,44}$

We can apply the techniques outlined here to calculate the number of infinitesimal deformations of $\mathcal{T}X$, which is the number of massless gauge singlets in the heterotic model arising from the standard embedding on X. This number is $h^1(X, \text{End }\mathcal{T}X) = h^1(X, \Omega^1 X \otimes \mathcal{T}X)$.

Start by tensoring (A.13) with $\Omega^1 X$, to get the short exact sequence

$$0 \to \Omega^1 X \otimes \mathcal{T} X \to \Omega^1 X \otimes \mathcal{T} Z|_X \to \Omega^1 X \otimes \omega_Z^{-1}|_X \to 0$$
(A.24)

We see that we need the cohomology of the last two terms. The last is easiest to obtain; simply dualise (A.13) and tensor with ω_Z^{-1} to obtain

$$0 \to \mathcal{O}_X \to (\Omega^1 Z \otimes \omega_Z^{-1})|_X \to \Omega^1 X \otimes \omega_Z^{-1}|_X \to 0$$
(A.25)

The cohomology of the first term here follows from the fact that X is Calabi-Yau (we can also calculate it explicitly, as a check). Using identities $\Omega^1 Z = \Omega^1 S_1 \oplus \Omega^1 S_2$, $\omega_z = \omega_{S_1} \otimes \omega_{S_2}$, and $\Omega^1 S_i \otimes \omega_{S_i}^{-1} \cong \mathcal{T}S_i$, the middle term becomes $\mathcal{T}S_1 \otimes \omega_{S_2}^{-1} \oplus \mathcal{T}S_2 \otimes \omega_{S_1}^{-1}$, and we can calculate its cohomology, before restriction to X, from the results of the last section:

$$h^{i}(Z, \Omega^{1}Z \otimes \omega_{Z}^{-1}) = \begin{cases} 28 & , i = 0\\ 0 & , \text{ otherwise} \end{cases}$$
(A.26)

To restrict to X, we tensor (A.12) with $\Omega^1 Z \otimes \omega_Z^{-1}$, to get

$$0 \to \Omega^1 Z \to \Omega^1 Z \otimes \omega_Z^{-1} \to (\Omega^1 Z \otimes \omega_Z^{-1})|_X \to 0$$

The resulting long exact sequence is

$$0 \to \mathbb{C}^{28} \to H^0(X, (\Omega^1 Z \otimes \omega_Z^{-1})|_X) \to \mathbb{C}^8 \to 0$$
$$0 \to H^i(X, (\Omega^1 Z \otimes \omega_Z^{-1})|_X) \to 0 , \quad i = 1, 2, 3$$

so we have

$$h^{i}(X, (\Omega^{1}Z \otimes \omega_{Z}^{-1})|_{X}) = \begin{cases} 36 & , i = 0\\ 0 & , \text{ otherwise} \end{cases}$$
(A.27)

Going back to (A.25), we can now write out the associated long exact sequence in cohomology:

$$\begin{split} 0 &\to \mathbb{C} \to \mathbb{C}^{36} \to H^0(X, \Omega^1 X \otimes \omega_Z^{-1}|_X) \to 0 \\ 0 &\to H^2(X, \Omega^1 X \otimes \omega_Z^{-1}|_X) \to \mathbb{C} \to 0 \\ 0 &\to H^i(X, \Omega^1 X \otimes \omega_Z^{-1}|_X) \to 0 \qquad i = 1,3 \end{split}$$

So we can simply read off the cohomology of $\Omega^1 X \otimes \omega_Z^{-1}|_X$:

$$h^{i}(X, \Omega^{1}X \otimes \omega_{Z}^{-1}|_{X}) = \begin{cases} 35 & , i = 0\\ 1 & , i = 2\\ 0 & , i = 1, 3 \end{cases}$$
(A.28)

Obtaining the cohomology of $\Omega^1 X \otimes \mathcal{T} Z|_X$ is not much more difficult given the work we have already done in Appendix A.1. First we dualise (A.13) and tensor it by $\mathcal{T} Z = \mathcal{T} S_1 \oplus \mathcal{T} S_2$, to obtain

$$0 \to \bigoplus_{i=1,2} \left(\mathcal{T}S_i \otimes \omega_{S_{3-i}} \right)|_X \to \bigoplus_{i=1,2} \left(\mathcal{T}S_i \otimes \Omega^1 S_i \oplus \mathcal{T}S_i \otimes \Omega^1 S_{3-i} \right)|_X \to \Omega^1 X \otimes \mathcal{T}Z|_X \to 0$$

The cohomology groups of the first two terms follow from the results of Appendix A.1 and the Künneth formula, and the resulting long exact cohomology sequence gives

$$h^{i}(X, \Omega^{1}X \otimes \mathcal{T}Z|_{X}) = \begin{cases} 2 & , i = 0\\ 46 & , i = 1 \end{cases}$$

The cohomology at degrees 2 and 3 is harder to calculate, but we will see that we do not need it.

The first part of the long exact sequence following from (A.24) is therefore

$$0 \to H^0(X, \Omega^1 X \otimes \mathcal{T} X) \to \mathbb{C}^2 \to \mathbb{C}^{35} \to H^1(X, \Omega^1 X \otimes \mathcal{T} X) \to \mathbb{C}^{46} \to 0$$

We know that $\mathcal{T}X$ is stable, so $H^0(X, \Omega^1 X \otimes \mathcal{T}X) \cong \mathbb{C}$. Since $\Omega^1 X \otimes \mathcal{T}X$ is self-dual, Serre duality also gives us $H^i(X, \Omega^1 X \otimes \mathcal{T}X) \cong H^{3-i}(X, \Omega^1 X \otimes \mathcal{T}X)$, so we get

$$h^{i}(X, \Omega^{1}X \otimes \mathcal{T}X) = \begin{cases} 1 & , i = 0, 3 \\ 80 & , i = 1, 2 \end{cases}$$
(A.29)

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