

2-Forms Gravity: An Introduction

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(Dedicated to Prof. Jerzy Plebański on the occasion of his 65th birthday)

1. Introduction

The purpose of this contribution is to provide a pedagogical introduction to a formulation of (complex) general relativity that uses a trio of 2-forms as basic gravitational variables.

This formalism can be understood as a close relative of the first order tetrad formalism for general relativity. A trio of 2-forms is used as metric variable, replacing the tetrad. Moreover, rather than using the full spin connection, one uses only its self-dual part.

The 2-forms formalism has the advantage of being entirely 'chiral'. The meaning of chiral is in the sense that the local Lorentz representations involve only $SL(2, C)$, and not its conjugate $\overline{SL(2, C)}$. Therefore, it involves only 'one half' of the Lorentz group, reducing all redundant information. It has the disadvantage of being valid only in four dimensions, since the 2-forms are self-dual, and this is possible only in four dimensions. Self-duality also implies that, for a space-time of lorentzian signature we are using complex variables.

In this introduction, I begin with the familiar first order tetrad, or moving frame, formalism. To establish my notation, and in order to start from well known material, I review briefly the Cartan structure equations. Next, I discuss the first order tetradic variational principle for vacuum general relativity.

Exploiting the isomorphism between the Lorentz group and the direct product $SL(2, C) \times \overline{SL(2, C)}$, I reformulate first order tetrad gravity in terms of two-component spinors. I also describe a first order variational principle that involves only the self-dual part of the spin connection^{1,2}. I call this formalism 'semi-chiral', since the tetrad is kept as metric variable. It is worth noting that it was introduced as a covariant version of Ashtekar's canonical formulation of general relativity³.

From this semi-chiral variational principle I show how one can arrive to an entirely chiral formulation. The self-dual part of the spin connection remains as a field variable, but in the role of primary metric variable the tetrad is replaced by a trio of self-dual 2-forms.

The usefulness of 2-forms as variables for general relativity was recognized already in the early sixties, perhaps as a by-product of the Petrov algebraic classification of the Weyl tensor. The Einstein equations were written in terms of 2-forms by Debever⁴. Since then, this language has been used extensively, see *e.g.* Refs. [5]. In the seventies, Jerzy Plebański added an important new contribution, a variational principle for (complex) general relativity⁶. This was rediscovered in Ref. [7] (and in Refs. [8]). The formalism was further developed in Ref. [9], where, among other things, it is shown how to couple matter fields (including $N = 1$ super-gravity!), and how its canonical analysis produces the Ashtekar formulation in a very straightforward way. Furthermore, it was used in the derivation of a formulation of gravity purely in terms of a connection^{10,11} (see also Ref. [12]).

The 2-form formalism displays the Einstein-Hilbert action in a new form. Simple modifications lead to a class of 4-dimensional topological field theories¹³, that have come to be known as BF-theories. A different modification leads to a dynamical theory proposed by Husain and Kuchař in the context of the loop quantization of gravity¹⁴.

These observations indicate what may be the most interesting application of this formalism: the possibility of considering theories intermediate between topological field theories and general relativity, of which the Husain-Kuchař model is a prime example. One would be considering theories with local degrees of freedom, but, hopefully, simpler than general relativity. This would be the logical next step in an avenue towards quantum gravity that takes topological field theories as its starting point.

II. BACKGROUND

In this section, I review briefly Riemannian geometry in the tetrad formalism, and the first order variational principle for vacuum general relativity. Next, I present a translation into the two-component spinor formalism, and the Samuel-Jacobson-Smolín semi-chiral variational principle. No attempt is made at completeness in the discussion of these subjects, only the strictly necessary 'nuts and bolts' are introduced. All considerations are local.

Let M be an oriented 4-dimensional Riemannian manifold. An ortho-normal basis for the space of 1-forms at one point of M , θ_μ^a can be taken as the primary gravitational variable¹

$$g_{\mu\nu} = \theta_\mu^a \theta_\nu^b \eta_{ab}, \quad \eta^{ab} = \theta_\mu^a \theta_\nu^b g^{\mu\nu}. \quad (1)$$

¹Greek indices run from 1 to 4 and denote space-time indices. The $SO(3,1)$ indices $a, b, \dots$ take the value 0, 1, 2, 3. They are lowered and raised using the Minkowski metric $\eta_{ab} = \text{diag}(-1, +1, +1, +1)$ and its inverse η^{ab} . The $SO(3,1)$ valued Levi-Civita tensor density is denoted by ϵ^{abcd} , with $\epsilon^{0123} = +1$.

The first Cartan structure equation introduces the spin connection, ω^a_b ,

$$T^a = d\theta^a + \omega^a_b \wedge \theta^b =: D\theta^a, \quad (2)$$

where T^a is the torsion 2-form, and D the $SO(3,1)$ covariant derivative. In the following, I specialize to the case of Riemannian geometry, *i.e.* the torsion is assumed to vanish, $T^a = 0$, and the spin connection to be compatible with the 'group metric' η_{ab} , *i.e.* $d\eta_{ab} + \omega_a^c \eta_{cb} + \omega_b^c \eta_{ac} = \omega_{ab} + \omega_{ba} = 0$. Under these assumptions, equation (2) can be solved uniquely for the spin connection in terms of the tetrad, and its first derivative.

The curvature 2-form, Ω^a_b , of the connection ω^a_b is defined by the second Cartan structure equation

$$\Omega^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b. \quad (3)$$

In addition, the following consistency conditions hold (Bianchi identities)

$$DT^a := dT^a + \omega^a_b \wedge T^b = \Omega^a_b \wedge \theta^b, \quad (4)$$

$$D\Omega^a_b := d\Omega^a_b + \omega^a_c \wedge \Omega^c_b - \omega^c_b \wedge \Omega^a_c = 0. \quad (5)$$

The equations (2) through (5) provide an elegant reformulation of the basic equations of Riemannian geometry. They can be easily translated to space-time tensor language using the tetrad and its inverse, *e.g.* $\Omega^a_b = (1/2)\Omega^a_{bcd}\theta^c \wedge \theta^d$, and $R^\mu_{\nu\rho\sigma} = \theta^\mu_a \theta_\nu^b \theta_\rho^c \theta_\sigma^d \Omega^a_{bcd}$, where $R^\mu_{\nu\rho\sigma}$ is the Riemann tensor of M .

Consider now the first order action for vacuum general relativity

$$S[\theta^a, \omega^a_b] = \int \star(\theta^a \wedge \theta^b) \wedge \Omega_{ab}(\omega), \quad (6)$$

where the duality operator $\star$ acts on $SO(3,1)$ bivectors as $\star V^{ab} = (1/2)\epsilon^{ab}_{cd}V^{cd}$ and satisfies $\star^2 = -1$. The action (6) agrees with the second order vacuum Einstein-Hilbert action $\int d^4x \sqrt{-g} R$ for the metric $g_{\mu\nu} := \theta_\mu^a \theta_\nu^b \eta_{ab}$. The equation of motion for the connection is $D\star(\theta^a \wedge \theta^b) = 0$. This equation can be shown to be equivalent to $D\theta^a = 0$. When it is solved for the spin connection with respect to the tetrad θ^a and its derivatives, and the solution is substituted back into the action, one recovers the vacuum Einstein-Hilbert action.

I turn now to the two-component spinor formalism. This formalism exploits the 1-2 isomorphism between the Lorentz group $SO(3,1)$ and the direct product $SL(2, \mathbb{C}) \otimes \overline{SL(2, \mathbb{C})}$ ¹⁵.

A limited description consists in the replacement of the ortho-normal basis θ_μ^a , with an equivalent basis of 1-forms at a point x of the manifold M , $\theta_\mu^{AA'}$ ². In

²Upper case latin letters represent two component spinor indices. They are raised and lowered with the antisymmetric spinor ϵ_{AB} , with $\epsilon_{01} = +1$, together with its inverse and their conjugates according to the conventions $\lambda^A := \epsilon^{AB}\lambda_B$, $\mu_B := \mu^A\epsilon_{AB}$, etc.

this fixed basis $\theta_\mu^{AA'}$ is related to θ_μ^a by the constant matrices $\sigma_a^{AA'}$, i.e. $\theta_\mu^{AA'} = \sigma_a^{AA'} \theta_\mu^a$. These may be given explicitly by

$$\sigma_a^{AA'} := \frac{i}{\sqrt{2}} \tau_a^{AA'}, \quad (7)$$

where $\tau_0^{AA'}$ is the identity matrix and $\tau_i^{AA'}$, with $i = 1, 2, 3$ denote the usual Pauli matrices.

The matrices $\sigma_a^{AA'}$ satisfy the identities

$$\sigma_a^{AA'} \sigma_{bAA'} = \eta_{ab}, \quad (8)$$

$$\eta^{ab} \sigma_a^{AA'} \sigma_b^{BB'} = \epsilon^{AB} \epsilon^{A'B'}, \quad (9)$$

$$\epsilon^{abcd} \sigma_a^{AA'} \sigma_b^{BB'} \sigma_c^{CC'} \sigma_d^{DD'} = i(\epsilon^{AB} \epsilon^{CD} \epsilon^{A'C'} \epsilon^{B'D'} - \epsilon^{AC} \epsilon^{BD} \epsilon^{A'B'} \epsilon^{C'D'}). \quad (10)$$

The usefulness of the spinorial formalism, for our purposes, is given by the simplifications it introduces in the decomposition of an arbitrary $SO(3,1)$ bivector $V^{ab} = V^{[ab]}$ in its self-dual and anti self-dual parts. V^{ab} can be uniquely decomposed in its self-dual and anti self-dual part as

$$V^{ab} = \frac{1}{2}(V^{ab} - i \star V^{ab}) + \frac{1}{2}(V^{ab} + i \star V^{ab}) =: (+)V^{ab} + (-)V^{ab}, \quad (11)$$

where

$$\star (+)V^{ab} := i (+)V^{ab} \quad (12)$$

$$\star (-)V^{ab} = -i (-)V^{ab}. \quad (13)$$

In turn, the spinorial components of V^{ab} , $V^{AA'BB'} = \sigma_a^{AA'} \sigma_b^{BB'} V^{ab}$, can be uniquely decomposed as

$$V^{AA'BB'} = V^{AB} \epsilon^{A'B'} + \bar{V}^{A'B'} \epsilon^{AB}, \quad (14)$$

where $V^{AB} = V^{(AB)} = (1/2)V^{A_{A'}B_{A'}}^{A_{A'}B_{A'}}$ and $\bar{V}^{A'B'} = \bar{V}^{(A'B')} = (1/2)V_A^{A'AB'}$. In the general complex case V^{AB} and $V^{A'B'}$ are independent and not related by complex conjugation. This is the spinorial version of the decomposition (11) above, i.e. $V^{AB} \epsilon^{A'B'}$ represents the anti self-dual part of V^{ab} , and $V^{A'B'} \epsilon^{AB}$ its self-dual part, as one can verify directly using the identities (10).

These considerations can now be applied to the decomposition in self-dual and anti self-dual part of the space-time quantities of interest: the basis of 2-forms $\theta^a \wedge \theta^b$, the spin connection 1-form ω^{ab} , and its curvature Ω^a_b , all considered as $SO(3,1)$ bivectors.

The anti self-dual and the self-dual parts of $\theta^a \wedge \theta^b$ are represented spinorially by $\Sigma^{AB} \epsilon^{A'B'}$ and $\Sigma^{A'B'} \epsilon^{AB}$ respectively, where

$$\Sigma^{AB} := \frac{1}{2} \theta_{A'}^A \wedge \theta_{B'}^{B'}, \quad (15)$$

$$\Sigma^{A'B'} := \frac{1}{2} \theta_A^{A'} \wedge \theta_{B'}^{B'}. \quad (16)$$

Σ^{AB} is also anti-self-dual as a spacetime 2-form with respect to the duality operator $*$ constructed from the space-time metric $g_{\mu\nu} = \theta_\mu^{AA'} \theta_{\nu AA'}$,

$$* \Sigma_{\mu\nu}^{AB} := \frac{1}{2} \tau_{\mu\nu}^{\rho\sigma} \Sigma_{\rho\sigma}^{AB} = -i \Sigma_{\mu\nu}^{AB}. \quad (17)$$

The spin connection ω^{ab} can be decomposed in an anti self-dual part $\omega^{AB} \epsilon^{A'B'}$ and a self-dual part $\omega^{A'B'} \epsilon^{AB}$. Similarly, the curvature 2-form Ω^{ab} decomposes in $R^{AB} \epsilon^{A'B'}$ and $R^{A'B'} \epsilon^{AB}$. An important fact is that there are no cross terms, *i.e.* one can verify that the second Cartan's structure equation (3) gives

$$R^{AB} = d\omega^{AB} + \omega^{AC} \wedge \omega_C^B, \quad (18)$$

$$R^{A'B'} = d\omega^{A'B'} + \omega^{A'C'} \wedge \omega_{C'}^{B'}. \quad (19)$$

The Bianchi identities (4), (5), become

$$R_A^C \wedge \theta_{CA'} + R_{A'}^{C'} \wedge \theta_{AC'} = 0, \quad (20)$$

$$DR_{AB} := d\omega_{AB} + 2\omega_{(A}^C \wedge \omega_{B)C} = 0, \quad (21)$$

$$D'R_{A'B'} := d\omega_{A'B'} + 2\omega_{(A'}^{C'} \wedge \omega_{B')C'} = 0. \quad (22)$$

Using this language, the decomposition of the Riemann curvature in its irreducible parts is also simplified. Expanding the curvatures R^{AB} and $R^{A'B'}$ with respect to the basis for the space of 2-forms $\{\Sigma^{AB}, \Sigma^{A'B'}\}$, one finds

$$R_{AB} = \Psi_{ABCD} \Sigma^{CD} + \frac{1}{3} X \Sigma_{AB} + \Phi_{ABA'B'} \Sigma^{A'B'}, \quad (23)$$

$$R_{A'B'} = \Psi_{A'B'C'D'} \Sigma^{C'D'} + \frac{1}{3} X \Sigma_{A'B'} + \Phi_{ABA'B'} \Sigma^{AB}. \quad (24)$$

The totally symmetric spinor $\Psi_{ABCD} = \Psi_{(ABCD)}$ corresponds to the self-dual part of the Weyl tensor, and $\Psi_{A'B'C'D'} = \Psi_{(A'B'C'D')}$ to its anti self-dual part. The spinor $\Phi_{ABA'B'} = \Phi_{(AB)A'B'} = \Phi_{AB(A'B')}$ corresponds to the trace free part of the Ricci tensor, and X to its trace part. Note that R_{AB} and $R_{A'B'}$ have the same Ricci part because of the Bianchi identities (20).

The condition of Ricci flatness, *i.e.* $X = 0$, $\Phi_{ABA'B'} = 0$, can be written as an expression involving only the anti self-dual part of the curvature in one of the two equivalent ways

$$R_{AB} \wedge \theta^{AA'} = 0, \quad (25)$$

$$R_{AB} = \Psi_{ABCD} \Sigma^{CD}, \quad (26)$$

as can be verified from (23).

Consider now the following "semi-chiral" first order action for complex vacuum general relativity^{1,2}

$$S[\theta^{AA'}, \omega_{AB}] = -2i \int \theta^A_{A'} \wedge \theta^{B A'} \wedge R_{AB}(\omega) \quad (27)$$

In order to see that this action reproduces the Einstein field equations, consider the $\omega_A{}^B$ equation of motion

$$D(\theta^A{}_{A'} \wedge \theta^{B A'}) := d(\theta^A{}_{A'} \wedge \theta^{B A'}) + 2\omega^{C(A} \wedge \theta^{B)}{}_{A'} \wedge \theta_C{}^{A'} = 0. \quad (28)$$

This equation can be solved uniquely for $\omega_A{}^B$ with respect to the tetrad, and determines $\omega_{AB} \epsilon_{A'B'}$ as the self-dual part of the spin connection compatible with $\theta^{AA'}$. Thus, $R_{AB} \epsilon_{A'B'}$ is identified with the self-dual part of the curvature 2-form $\Omega^a{}_b$ of the spin connection compatible with the tetrad. The $\theta^{AA'}$ field equation, $\theta^{B A'} \wedge R_{AB} = 0$, then states that the spacetime metric $g_{\mu\nu} = \theta_\mu{}^{AA'} \theta_{\nu AA'}$ is Ricci flat.

The action (27) is complex. When (28) is satisfied and the tetrad is real, *i.e.* $\theta^{AA'} = \bar{\theta}^{AA'}$, one can check that the imaginary part of (27) vanishes identically. This follows from the Bianchi identities.

III. 2-FORMALISM FOR GRAVITY

In order to obtain an entirely chiral formulation, one would like to replace the tetrad with an $\text{SL}(2, \mathbb{C})$ valued field variable.

The key observation is that the tetrad appears in (27) only in the combination

$$\frac{1}{2} \theta^A{}_{A'} \wedge \theta^{B A'} =: \Sigma^{AB}, \quad (29)$$

where the 2-form Σ^{AB} is symmetric, *i.e.* $\Sigma^{AB} = \Sigma^{(AB)}$. One can therefore think of adopting this symmetric $\text{SL}(2, \mathbb{C})$ valued 2-form as the primary metric variable.

An obvious candidate action is

$$S[\Sigma, \omega] = -4i \int \Sigma^{AB} \wedge R_{AB}. \quad (30)$$

However, the Σ^{AB} equation of motion would then imply that the whole R_{AB} vanishes, not only its Ricci part. (This would be a sort of 4-dimensional analog of 3-dimensional gravity, see Sect. VI below.) What is needed is a constraint which enforces the fact that Σ^{AB} must ‘have come from some tetrad’, *i.e.* take the form (29) for some tetrad $\theta^{AA'}$. Since a chiral formulation is sought, the constraint should be invariant under $\text{SL}(2, \mathbb{C})$ transformations. A constraint which satisfies these requirements is given by

$$\Sigma^{(AB} \wedge \Sigma^{CD)} = 0. \quad (31)$$

This constraint is a necessary condition. The expression $\theta^{(A}{}_{A'} \wedge \theta^{B}{}_{B'} \wedge \theta^{C)}{}_{C'} = 0$ is totally antisymmetric in $[A'B'C']$, and hence vanishes since spin space is 2-dimensional. A dimension counting argument shows that it is also sufficient. (See *e.g.* Ref [9] for an explicit proof.)

Hence, in order to have an action equivalent to (27), in which the primary metric variable is a trio of 2-forms Σ^{AB} , one can append the constraint (31) to the action (30) via an arbitrary Lagrange multiplier. As shown by Plebański⁶, a chiral action for (complex) vacuum general relativity is then given by

$$S[\Sigma^{AB}, \omega_{AB}, \Psi_{ABCD}] = -4i \int [\Sigma^{AB} \wedge R_{AB} - \frac{1}{2} \Psi_{ABCD} \Sigma^{AB} \wedge \Sigma^{CD}], \quad (32)$$

where Ψ_{ABCD} is a totally symmetric spinor field, $\Psi_{ABCD} = \Psi_{(ABCD)}$.

The variation of this action gives the following equations of motion.

$$\Sigma^{(AB} \wedge \Sigma^{CD)} = 0, \quad (33)$$

$$D\Sigma^{AB} := d\Sigma^{AB} + 2\omega^{C(A} \wedge \Sigma^{B)C} = 0, \quad (34)$$

$$R_{AB} = \Psi_{ABCD} \Sigma^{CD}. \quad (35)$$

The first equation (33) is just the constraint (31) above. The preceding discussion shows that it implies that Σ^{AB} has the form (29) for some $\theta^{AA'}$. If the solution of the constraint is substituted back in the action, one recovers the action (27), which was already shown to be equivalent to (complex) vacuum general relativity.

From (26), it follows that the equations of motion identify the Lagrange multiplier Ψ_{ABCD} with the Weyl spinor, so (35) states that the curvature is pure Weyl.

In the presence of a cosmological constant Λ , the action (32) is modified to

$$S[\Sigma^{AB}, \omega_{AB}, \Psi_{ABCD}] = 4i \int [\Sigma^{AB} \wedge R_{AB} - \frac{1}{2} \Psi_{ABCD} \Sigma^{AB} \wedge \Sigma^{CD} - \frac{1}{6} \Lambda \Sigma^{AB} \wedge \Sigma_{AB}]. \quad (36)$$

The only modification in the field equations is in the Σ^{AB} field equation, which becomes

$$R_{AB} = \Psi_{ABCD} \Sigma^{CD} + \frac{1}{3} \Lambda \Sigma_{AB}. \quad (37)$$

When the constraint (31) is satisfied, Σ^{AB} is determined by the tetrad only up to $\overline{SL}(2, \mathbb{C})$ transformations on the primed indices. However, the metric is uniquely determined^{9,16}, (see also Ref. [17]). The metric density can be expressed directly in terms of Σ^{AB} with

$$\sqrt{g} g_{\mu\nu} = \frac{1}{3} \epsilon^{\alpha\beta\gamma\delta} \Sigma_{\mu\alpha}^{AB} \Sigma_{\beta\gamma B}^C \Sigma_{\delta\nu CA}, \quad (38)$$

where the 2-forms Σ^{AB} appear as a 'cubic root' of the metric density.

IV. REALITY CONDITIONS

The signature of the metric g can be fixed by imposing certain conditions on Σ^{AB} . Signature $(++--)$ corresponds simply to *real* Σ^{AB} . Euclidean signature is selected by imposing the condition that Σ^{AB} be hermitian with respect to some hermitian

inner product on the 2-dimensional spin space. This can be achieved by expanding Σ^{AB} as $\Sigma^{AB} = \Sigma^i \tau_i^{AB}$ with real coefficients Σ^i in a fixed basis of hermitian symmetric spinors τ_i^{AB} . Note that these conditions do not fix the overall sign of the metric, since under $\Sigma^{AB} \rightarrow -\Sigma^{AB}$ one has $g_{\mu\nu} \rightarrow -g_{\mu\nu}$. For these signatures, the factor of i should be dropped from the action (32), if it is to be real.

Conditions for lorentzian signature are not linear reality conditions on the 2-form Σ^{AB} . One option is to impose the condition, cubic in the Σ 's, that the expression (38) be i times a lorentzian signature metric. Alternatively, one can impose the following quadratic conditions

$$\Sigma^{AB} \wedge \bar{\Sigma}^{A'B'} = 0, \quad (39)$$

$$(\Sigma^{MN} \wedge \Sigma_{MN}) + (\bar{\Sigma}^{M'N'} \wedge \bar{\Sigma}_{M'N'}) = 0. \quad (40)$$

For all signatures, these conditions in turn imply reality conditions on the connection ω_{AB} , via the compatibility equation $D\Sigma^{AB} = 0$.

Now, is it possible to avoid the use of self-dual variables, and thus the need for reality conditions, while retaining the essential features of 2-form formalism? The answer is yes. A real formalism which mimics the chiral formalism presented here which uses a collection of six 2-forms as primary metric variables is introduced in Ref. [9].

V. SELF-DUAL GRAVITY

In a Symposium in honor of Jerzy Plebański, one should not miss the chance to talk about self-dual gravity.

In the special case of self-dual solutions, the field equations (33), (34), (35) for vacuum general relativity simplify considerably. For vanishing cosmological constant, the equations reduce to

$$\Sigma^{(AB} \wedge \Sigma^{CD)} = 0, \quad (41)$$

$$d\Sigma^{AB} = 0. \quad (42)$$

This formulation of the self-dual vacuum equations was employed by Plebański to reduce the self-dual case to the 'heavenly equations'¹⁸. Gindikin used this formulation to generate self-dual solutions¹⁹. It is also directly related to the canonical version of Ashtekar, Jacobson and Smolin²⁰,

In the case of non-vanishing cosmological constant, when the Weyl spinor Ψ_{ABCD} vanishes, (37) reduces to $R_{AB} = \frac{1}{3}\Lambda\Sigma_{AB}$. Given this, (34) is satisfied by virtue of the Bianchi identity $DR_{AB} = 0$ ²¹, so all that remains of the field equations is the quadratic constraint (41). Thus self-dual solutions with cosmological constant can be characterized by the condition that the spin connection satisfies the metric-independent equation^{22,23},

$$R_{(AB} \wedge R_{CD)} = 0. \quad (43)$$

The 2-form defined by $\Sigma^{AB} := \frac{3}{\Lambda} R^{AB}$ determines via (29) a metric which is a self-dual solution with cosmological constant, and all such solutions arise in this manner, at least locally.

VI. MODIFICATIONS OF THE CHIRAL ACTION

From the vantage point of the chiral formalism for gravity, formal similarities between general relativity and other field theories appear. Here I consider some field theories which result from natural modifications of the actions (32) and (36).

First, some 'truncated' versions of (32) and (36) correspond to some of the 4-dimensional topological field theories considered by Horowitz¹³.

A topological field theory is obtained by simply dropping the Ψ term in (32). The action takes a form analogous to the one for 3-dimensional gravity

$$S[\Sigma, \omega] = \int \Sigma^{AB} \wedge R_{AB}. \quad (44)$$

The equation of motion for Σ^{AB} implies that the connection is (locally) flat. The analogy with 3-dimensional gravity goes a step further. Because of the Bianchi identities, the action possesses an extra symmetry $\delta \Sigma^{AB} = D\tau^{AB}$, $\delta \omega_{AB} = 0$, for some 1-form τ^{AB} . If there are no topological obstructions, this symmetry can be used to set the 2-form Σ^{AB} equal to anything, in particular it can be set equal to zero.

A second type of topological field theory is obtained by dropping the Ψ term in (32) but adding a cosmological constant term, i.e.

$$S[\Sigma, \omega] = \int \Sigma^{AB} \wedge R_{AB} - \frac{1}{2} \Lambda \Sigma^{AB} \wedge \Sigma_{AB}. \quad (45)$$

The equations of motion are then $D\Sigma^{AB} = 0$ and $\Sigma^{AB} = \Lambda^{-1} R^{AB}$, which are satisfied by *any* connection. The action has now an 'extra' symmetry under $\delta \Sigma^{AB} = D\pi^{AB}$, $\delta \omega^{AB} = \Lambda \pi^{AB}$. If this is treated as a gauge symmetry, all connections can be gauged to zero, so although all connections are solutions, they are all gauge-equivalent.

An interesting type of field theory, introduced by Husain and Kuchař¹⁴ in the context of the loop representation for quantum gravity, is derived from a rather innocent looking modification of the action (32): If the cosmological constant is promoted to a Lagrange multiplier, then we have the action (32) *without* the condition that Ψ_{ABCD} be totally symmetric (tracefree). Its variational equation thus imposes a stronger algebraic constraint, $\Sigma^{AB} \wedge \Sigma^{CD} = 0$. This constraint implies that there exists a trio of 1-forms $\theta^{AB} = \theta^{(AB)}$ such that $\Sigma^{AB} = \theta^A_C \wedge \theta^{BC}$ (or equivalently that $\Sigma^{AB} = \alpha \wedge \theta^{AB}$, for some 1-form α). When this is substituted back in the action one obtains the action considered in Ref. [14],

$$S[\theta^{AB}, \omega_{AB}] = \int \theta^A_C \wedge \theta^{BC} \wedge R_{AB}. \quad (46)$$

This theory is itself a 'truncated' version of vacuum general relativity. As shown by Husain and Kuchař, in its canonical form it can be obtained from vacuum general relativity in the form proposed by Ashtekar by dropping the scalar constraint.

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