University of Cambridge

Selwyn College

Satake compactifications and the Schottky problem

Essay for the Smith-Knight and Rayleigh-Knight contests

Giulio Codogni
G.Codogni@dpmms.cam.ac.uk

January 2013

Preface

This is an essay for the Smith-Knight and Rayleigh-Knight contests. Section 1 is a reworking of the third chapter of [Fay73], it contains an extract of the paper [CSB11]. Section 2 is background material. Section 3 contains a result obtained in the joint work [CSB11] and some of its consequences, in particular section 3.1 is the third section of [CSB11]. Sections 4 and 5 are my own research. Pictures are taken from [Ber].

This work is done under the supervision of Prof. N. Shepherd-Barron.

Abstract

The Schottky problem is that of identifying the moduli space $\mathcal{M}_g$ of genus g curves inside the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties. We focus on modular forms and degenerations, this involves considering the Satake compactifications $\mathcal{M}_g^S$ and $\mathcal{A}_g^S$. Our aim is to give an insight into the relation between solutions of the Schottky problem in different genera.

The space $\mathcal{A}_g$ lies in the boundary of $\mathcal{A}_{g+m}^S$ for every m. As sets, the intersection of $\mathcal{A}_g$ and $\mathcal{M}_{g+m}^S$ is $\mathcal{M}_g$. Following [CSB11], we prove that the intersection between $\mathcal{M}_{g+m}^S$ and $\mathcal{A}_g$ contains the m-th infinitesimal neighbourhood of $\mathcal{M}_g$ in $\mathcal{A}_g$, this implies that stable equations for $\mathcal{M}_g$ do not exist.

With the same spirit, we look at the hyperelliptic locus Hyp_g. We consider its Satake compactification Hyp_g^S. As sets the intersection of $\mathcal{A}_g$ and Hyp_{g+1}^S is Hyp_g. In section 4 we show that $Hyp_{g+1}^S \cap \mathcal{A}_g$ is, scheme theoretically, equal to Hyp_g: this enables us to write down many millions of stable equations for Hyp_g.

We have similar results for Prym varieties of étale double covers and double covers branched at two points.

Most of our work relies upon variational formulæ. By variational formula we mean an explicit expression of (the first order part of) the periods of a one-parameter family of curves.

Contents

Introduction

We consider algebraic curves over the field of complex numbers. To a smooth genus g curve C one can associate its Jacobian variety $Jac(C)$, this is an abelian g-fold principally polarised by the Theta divisor. Torelli's theorem guarantees that is possible to reconstruct the curve from its Jacobian. Classical references for this are [Spr57], [Mum07], [Fay73] and [ACGH85].

Given a family of curves, it is possible to construct the relative Jacobian (see [ACG11] section XI.8), so we have the Torelli morphism

$$\begin{array}{cccc} T: & \mathcal{M}_g & \to & \mathcal{A}_g \\ & C & \mapsto & Jac(C) \end{array}$$

where $\mathcal{M}_g$ denotes the moduli space of smooth genus g curves and $\mathcal{A}_g$ the moduli space of g dimensional principally polarised abelian varieties. Dealing with global questions, there is an important difference between the moduli stack and the associated coarse space. In this essay, most of the time we will consider only coarse spaces: this means that we will consider just general curves.

The dimension of $\mathcal{M}_g$ is $3g-3$ and the dimension of $\mathcal{A}_g$ is $\frac{1}{2}g(g+1)$, so Torelli's map is dominant only for $g \leq 3$. The tangent space at a general point is the first order deformation space. In the case of a curve, it is isomorphic to $H^0(C, 2K_C)^\vee$, where K_C denotes the canonical bundle of C (cf. [ACG11] section XI.3). The first order deformation space of $Jac(C)$ can be interpreted either as the tangent space of a Grassmannian ([ACG11] section XI.8), or the deformation space of a polarised variety ([CvdG00] section 3). Using the second interpretation, the heat equation means that the first order deformation space of a principally polarised abelian variety X is $Sym^2 T_0 X^\vee$, where $T_0 X$ is the tangent space at the origin of X. Pick an automorphisms-free curve C, the co-differential of the Torelli map

$$dT(C)^\vee : (T_{Jac(C)}\mathcal{A}_g)^\vee = Sym^2 H^0(C, K_C) \to (T_C\mathcal{M}_g)^\vee = H^0(C, 2K_C)$$

can be identified with the natural multiplication map ([ACG11] XI.8), which is surjective if C is not hyperelliptic by Max Noether's theorem.

One advantage of working with $\mathcal{A}_g$ is its concrete presentation: it is the quotient of the Siegel upper half space $\mathfrak{H}_g$ (i.e. g-by-g complex symmetric matrices with positive definite imaginary part) by the action of the symplectic group $Sp(2g, \mathbb{Z})$. On this space we can define weight k modular forms, these are holomorphic functions on $\mathfrak{H}_g$ which transform in an appropriate way under the action of $Sp(2g, \mathbb{Z})$. Weight 1 modular forms are sections of an ample line bundle L on (the stack) $\mathcal{A}_g$, weight k modular forms are section of L^k. A standard reference is [Mum07].

The Schottky problem is to understand when an abelian variety is the Jacobian of a curve. This problem has many aspects, the one we are interested in is to find modular forms vanishing on $\mathcal{M}_g$, in other terms we are looking for the equations of $\mathcal{M}_g$ in $\mathcal{A}_g$. The Schottky-Jung relations give an ideal S_g of equations for $\mathcal{M}_g$. In [vG84] is proven that, for $g > 4$, the ideal S_g cuts out a reducible subscheme and $\mathcal{M}_g$ is one of its irreducible components. More in general, one can address the same question about other interesting sublocus of $\mathcal{A}_g$, such as the locus Hyp_g of Jacobians of hyperelliptic curves, or the locus $\mathcal{P}_g$ of Prym varieties of étale double covers.

In this essay, we will study degenerations of curves. This involves considering the Satake compactifications, which is the natural object to look at if one wants to understands how modular forms behave on degenerations.

The Satake compactification of $\mathcal{A}_g$ is constructed using modular forms, we denote it by $\mathcal{A}_g^S$. It is normal and every modular form on $\mathcal{A}_g$ extends to $\mathcal{A}_g^S$. The boundary of $\mathcal{A}_g^S$ is isomorphic to $\mathcal{A}_{g-1}^S$. A stable modular form F is the datum of weight n modular form F_g for every g, compatible with the restriction from $\mathcal{A}_g^S$ to $\mathcal{A}_{g-1}^S$. The Satake compactification of $\mathcal{M}_g$ (respectively of Hyp_g and $\mathcal{P}_g$) is denoted by $\mathcal{M}_g^S$ (Hyp_g^S, $\mathcal{P}_g^S$), it is the closure of $\mathcal{M}_g$ (Hyp_g, $\mathcal{P}_g$) inside $\mathcal{A}_g^S$. The variety $\mathcal{M}_g^S$ is equal, as set, to the union of all products $\mathcal{M}_{g_1} \times \cdots \times \mathcal{M}_{g_s}$ with $\sum g_i \leq g$. In particular, if we intersect $\mathcal{M}_{g+m}^S$ with $\mathcal{A}_g$ we obtain, as set, $\mathcal{M}_g$. The same is true for Hyp_g^S and $\mathcal{P}_g^S$. We will review Satake compactifications and stable modular forms in section 2, see also [Fre83].

Our first result is obtained in the joint work [CSB11].

Theorem 0.1 ([CSB11] Theorem 1.1). *The intersection of $\mathcal{M}_{g+m}^S$ and $\mathcal{A}_g$ is not transverse, it contains the m-th infinitesimal neighbourhood of $\mathcal{M}_g$ in $\mathcal{A}_g$.*

We will prove it in section 3.1, see theorem 3.7. The proof relies upon a variational formula proven in [Fay73], see theorem 1.10. By variational formula we mean an explicit expression of (the first order part of) the periods of a one-parameter family of curves.

A stable equation for $\mathcal{M}_g$ is a non-trivial stable modular form vanishing on $\mathcal{M}_g$ for every g, these are also called stable Schottky relations. The above theorem means that if a modular form F_{g+1} vanishes with multiplicity k on $\mathcal{M}_{g+1}$, then its restriction F_g on $\mathcal{A}_g$ vanishes with multiplicity at least $k+1$ on $\mathcal{M}_g$, see theorem 3.1. This immediately implies:

Corollary 0.2 ([CSB11]). *Stable equations for $\mathcal{M}_g$ do not exist.*

This corollary is a no-go result, but, as we will discuss in Section 3.2, it allows us to define many divisors on $\mathcal{M}_g$. In particular, one can associated to every even unimodular positive definite lattices Λ a Theta series Θ_Λ, this is a weight $\frac{1}{2}rk(\Lambda)$ stable modular form, so $\Theta_{\Lambda,g}$ is a modular form on $\mathcal{A}_g$ (see section 2.1).

Theorem 0.3 (= Theorem 3.13). *Let Λ and Γ be two even unimodular positive definite lattices of the same rank, there exists an integer g such that the modular form*

$$\Theta_{\Lambda,g} - \Theta_{\Gamma,g}$$

cuts out a divisor of finite slope on $\mathcal{M}_g$.

We have not the faintest idea about how to compute both g and the slope.

In Section 4, we carry out the same analysis for the hyperelliptic locus Hyp_g. In this case, the situation is completely different.

Theorem 0.4 (= Theorem 4.3). *The intersection of $\mathcal{A}_g^S$ and Hyp_{g+1}^S is transverse.*

Combining this theorem with a well-known slope argument, we can write down many stable equations for the hyperelliptic locus. Call μ_Γ the the minimal norm of non-trivial vectors of a lattice Γ.

Theorem 0.5 (= Theorem 4.9). *Let Λ and Γ be two even positive definite unimodular lattices of rank N and $\mu_\Lambda = \mu_\Gamma =: \mu$, if*

$$\frac{N}{\mu} \leq 8\,,$$

then

$$F := \Theta_\Lambda - \Theta_\Gamma$$

is a stable equation for the hyperelliptic locus. In other words, F_g vanishes on Hyp_g for every g.

The hypotheses on the lattices are quite restrictive, we will discuss them after the proof of the theorem. The couple (N, μ) can only take values (16,2), (32,4), (48,6). A lower bound for the number of lattices of rank 32 without roots (i.e. $\mu = 4$) is computed in [Kin03], so we have the following result.

Theorem 0.6 (= Theorem 4.10)**.** *There are more than 10000000 of linearly independent differences of Theta series vanishing on the hyperelliptic locus for every g.*

In section 5.2 we study the locus $\mathcal{P}_g$ of Prym varieties arising from étale double covers of curves. By proving some variational formulæ, we obtain the following results.

Theorem 0.7 (= Theorem 5.1)**.** *The intersection of $\mathcal{P}^S_{g+m}$ and $\mathcal{A}_g$ contains the m-th infinitesimal neighbourhood of $\mathcal{P}_g$ in $\mathcal{A}_g$.*

Theorem 0.8 (= Theorem 5.5)**.** *$\mathcal{P}^S_g$ contains the first infinitesimal neighbourhood of $\mathcal{M}_g$ in $\mathcal{A}_g$.*

Let us point out that both $\mathcal{M}^S_{g+1}$ and $\mathcal{P}^S_g$ contain the first infinitesimal neighbourhood of $\mathcal{M}_g$ in $\mathcal{A}_g$, this could be related to the Schottky-Jung relations.

In section 5.3 we carry out the same analysis for Prym varieties of double covers branched at two points.

Acknowledgement I would like to thank my advisor Nick Shepherd-Barron, without whom this work would not be possible. I would like to thank Enrico Arbarello: he first introduced me to the theory of algebraic curves, and I have benefited of several conversations with him during these years. I wish to thank Gabriele Benedetti, Gavril Farkas, Sam Grushenvsky, Marco Matone, John Christian Ottem, Riccardo Salvati Manni and Roberto Volpato for useful conversations. I thank the DPMMS, the EPSRC and Selwyn College for financial support.

1 Reading J. Fay's "Theta functions"

Our work has been heavily influenced by the reading of the third chapter of Fay's book [Fay73]. In this section, we rework some of his ideas. These constructions are also discussed in the informal lecture notes [Ber], other references are the second section of [CSB11] and appendix A of [MV10].

1.1 Local expansion of a differential

Let Δ_t be a small disc with parameter t around the origin in the complex plane. We consider a family

$$f : \mathcal{C} \to \Delta_t$$

of stable genus g curves, with singular central fibre. By definition of family, the total space $\mathcal{C}$ is smooth and the map f is proper. Call $\mathcal{C}_t$ the fibre over t.

First, let us study the topology of the family. We look for a symplectic basis $A_i(t), B_i(t)$ of the homology of each fibre $\mathcal{C}_t$. This basis should be a differentiable function of the base parameter t. It is possible to write down an explicit retraction of the family on its central fibre, see [ACG11] page 157. This means that the homology classes A_i, B_i of the central fibre can be extended to classes $A_i(t), B_i(t)$ on $\mathcal{C}_t$ for all t. It could happen that the central fibre has less than $2g$ cycles, this because the degeneration "squeeze" some non-trivial cycle. These "missing" classes we need to add are called vanishing cycles. As function of the base parameter, they are multivalued, and the monodromy is described by the Picard-Lefschetz formula, see [ACG11] page

143 or [BHPVdV04] section III.15 . It is also possible to study the topology of this family using Morse theory, see [Voi02] part V.

Now, we want to study the holomorphic differentials and their periods as holomorphic functions of t. The following proposition holds.

Proposition 1.1 ([Fay73] page 39). *Let $A_i(t), B_i(t)$ be a symplectic basis for $H_1(\mathcal{C}_t, \mathbb{Z})$. Up to shrink Δ_t, there exist holomorphic 2-forms $\Omega_i(t)$ on $\mathcal{C}$ such that their residues $\omega_i(t)$ on each fibre $\mathcal{C}_t$ form a normalised basis for the abelian differentials, i.e. for every t we have $\int_{A_j(t)} \omega_i(t) = \delta_{ij}$.*

The singularities of the central fibre are nodes and, locally around each node, $\mathcal{C}$ is isomorphic to the family of affine curve S defined by the equation $XY = t$ (cf [ACG11] page 84). Equivalently, S can be defined by $x^2 - y^2 = t$. This second co-ordinates are called pinching co-ordinates. We focus our attention on a single node n. Let us take a relative differential $\omega(t)$ on $\mathcal{C}$ and restrict it to S. A basis for the relative dualising sheaf of S at the origin, as $\mathcal{O}_{(S,0)}$ module, is $\frac{dx}{y}$ (see [ACG11] page 97), so locally we have a Taylor expansion

$$\omega(x, y, t) = \sum_{k=(k_1,k_2)} a_k(t) x^{k_1} y^{k_2} \frac{dx}{y} .$$

Replacing y with $\sqrt{x^2 - t}$ and relabelling the indexes we get

$$\omega(x, t) = \sum_{k\geq 0} a_k(t) x^k dx + \sum_{k\geq 0} b_k(t) \frac{x^k}{\sqrt{x^2 - t}} dx ,$$

where a_k and b_k are holomorphic functions of t.

Let

$$\nu : C \to \mathcal{C}_0$$

be the normalisation of the central fibre, call a and b the preimages of n, we have

$$\nu^* \omega(x, 0) = \sum_{k\geq 0} a_k(0) x^k dx \pm \sum_{k\geq 0} b_k(0) x^{k-1} dx ,$$

the sign depends if we are looking at the branch $x = y$ or $x = -y$.

The form $\nu^*\omega(x, 0)$ has poles of order one at a and b, with residues $\pm 2\pi i b_0(0)$. If $b_0(0) = 0$, which means $\nu^*\omega(x, 0)$ is regular at a and b, we can evaluate the form in term of dx. We get $\nu^*\omega(a, 0) = a_0(0) + b_1(0)$ and $\nu^*\omega(b, 0) = a_0(0) - b_1(0)$, so

$$\frac{1}{2} b_1(0) = \frac{\nu^*\omega(a, 0)}{dx} - \frac{\nu^*\omega(b, 0)}{dx} .$$

We apply the Gauss-Manin connection ∇ to $\omega(t)$ and we evaluate it at $t = 0$ (equivalently, we are considering the Lie derivative with respect to a lift of the tangent vector $\frac{d}{dt} |_{t=0}$ from Δ_t), the result is a well defined relative co-homology class $[\nabla\omega(0)]$ on $\mathcal{C}_0$. We want to understand if the meromorphic form $\nu^*\nabla\omega(0)$ defines a co-homology class on C and compute its periods.

As an aside, let us note that, on a smooth curve, a meromorphic form ω defines a co-homology class if and only if it has zero residue at every pole. Indeed, on can give sense to the integral

$$\int_\gamma \omega$$

choosing a representative of γ with a support disjoint from the poles. To check that the integral does not depends on the representative of γ we need to prove that, for every 2 simplex M in C, the integral of ω on the boundary of M is zero. This is equivalent to say that ω has zero residue at every pole, indeed the residue theorem says that

$$\int_{\partial M} \omega = 2\pi i \sum_{p \in M} Res_p(\omega)\,.$$

Let us go back to our original problem. Around a and b we have an expansion

$$\nu^* \nabla \omega(x,0) = \sum_{k \geq 0} a'(0) x^k dk \pm \sum_{k \geq 0} (\frac{1}{2} b_{k+1}(0) + x b_k'(0)) x^{k-2} dx\,,$$

where the dash means the derivative of the function with respect to t and (by abuse of notation) x is the pull-back of x via ν. The form $\nu^* \nabla \omega(0)$ has poles of order two and opposite residues at a and b. Let us focus on a. To compute the residue, let γ be a small loop on C around a. We can think at it as a loop $\gamma(0)$ in $S_0 \subset \mathcal{C}_0$, so we can extend it to a family of loop $\gamma(t)$ in $S_t \subset \mathcal{C}_t$. We have

$$\frac{1}{2\pi i} Res_a \nu^* \nabla \omega(0) = \int_\gamma \nu^* \nabla \omega(0) = \frac{d}{dt} |_{t=0} \int_{\gamma(t)} \omega(t)\,.$$

Let us summarise the results. We have a family $\mathcal{C}$ of stable curves over a disc Δ_t with singular central fibre, let

$$\nu : C \to \mathcal{C}_0$$

be the normalisation. We focus our attention on a node n of $\mathcal{C}_0$, call a and b its preimage. Choose pinching co-ordinates x and y such that, locally around n, the total space $\mathcal{C}$ is isomorphic to $xy = t$. Let $\gamma(t)$ be a family of loops such that the preimage of $\gamma(0)$ under ν is a loop around a. Call ∇ the Gauss-Manin connection.

Lemma 1.2. Let $\omega(t)$ be a relative differential on $\mathcal{C}$ and keep notations as above. Suppose that the integral $\int_{\gamma(t)} \omega(t)$ does not depend on t. Then, $\nu^* \nabla \omega(0)$ has a pole of order two and zero residue at a and b. Moreover, its leading coefficient with respect to the pinching co-ordinates is

$$\pm(\omega(a,0) - \omega(b,0))\,,$$

the sign depends on to the branch we are looking at, and ω is evaluated in term of the pinching co-ordinates.

There are at least two cases in which $\int_{\gamma(t)} \omega(t)$ does not depend on t. The first is when $\gamma(t)$ is topologically trivial on $\mathcal{C}_t$, so the integral vanishes identically. The second is when $\gamma(t)$ is one of the $A_i(t)$ cycles of a symplectic basis of the homology of $\mathcal{C}_t$, and ω is one of the normalised differentials $\omega_i(t)$, i.e. $\int_{A_j(t)} \omega_i(t) = \delta_{ij}$ (see proposition 1.1).

These differentials will turn out to be easy to handle. Let us recall a key and classical definition, see e.g. [Spr57] page 256-260 for a complete discussion and proofs. Let C be a smooth curve and A_i, B_i a symplectic basis for its homology.

Definition 1.3 (Normalized differentials of the second kind η_p). *A normalised differentials of the second kind η_p is a meromorphic differential on C with a unique pole at p such that*

- *the pole has order two and no residue,*
- *the integral $\int_{A_i} \eta_p$ is zero for every i.*

These differentials exist and they are unique up to a scalar. When a local co-ordinate around p is fixed, we denote by η_p the unique normalized differential of the second kind with leading coefficient one with respect to this co-ordinate. Their periods can be computed with the following classical formula.

Theorem 1.4 (Riemann's bilinear relations for differentials of the second kind)**.** *Keep notations as above, we have*

$$\int_{B_i} \eta_p = 2\pi i \frac{\omega_i}{dz_p}(p)\,,$$

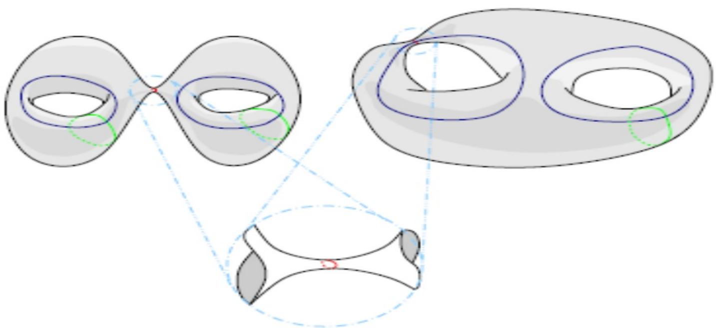
where z_p is a local coordinate around p such that the leading coefficient of η_p is 1, and $\int_{A_j} \omega_i = \delta_{ij}$.

The proof relies upon the residues theorem, see [Spr57] corollary 10.6 page 260.

Question 1.5. It would be interesting to have the same construction for family of higher dimensional varieties, e.g. K3 surfaces. Probably, one needs some analogues of Riemann's bilinear relations.

1.2 Variational formulæ for families of curves

In this section we construct some families of curves, and, using the argument of the previous section, we compute the first order part of their period matrix. We will mainly focus on two kind of degeneration, the picture to have in mind is the following.



On the left hand side, we are pinching a trivial homological cycle, on the right hand side a non-trivial one. In both cases, the local model is the bottom picture.

Pinching a non-trivial homological cycle Let us start with a genus g smooth curve C, two distinct points a and b and local co-ordinates z_a and z_b. Using these data, we can construct a family $\mathcal{C}$ of stable genus g curves degenerating to $C/a \sim b$. Roughly speaking, we perform a local surgery on the co-ordinate patch around a and b: we remove two small discs and we glue the remaining annuli according to the relation $z_a(x)z_b(y) = t$.

We reproduce the description of this family given in [CSB11], analogue descriptions can be found in [ACG11] page 184 or [Fay73] page 50. Let us stress that this is a construction "with parameters", this means that everything depends holomorphically on a, b, z_a and z_b.

> Start with a curve C of genus g. Let $\mathcal{V}$ be the infinite-dimensional variety whose points are quadruples (a, b, z_a, z_b), where a, b are distinct points on C and z_a, z_b are local holomorphic co-ordinates on C at a, b respectively.

The 2-torus $\mathbb{G}_m^2$ acts on $\mathcal{V}$ by

$$(\lambda, \mu)(a, b, z_a, z_b) = (a, b, \lambda^{-1} z_a, \mu^{-1} z_b).$$

Fix a non-empty finite-dimensional (in order to avoid irrelevant difficulties) and smooth subvariety V of $\mathcal{V}$ that is preserved under this torus action and that maps onto the complement U of the diagonal in $C \times C$.

We want to construct a family of morphisms $\{f_v : \mathcal{C}_v \to \Delta\}_{v \in V}$ that is parametrized by V, where Δ is a complex disc centred at 0, each $\mathcal{C}_v$ is a smooth complex surface, each f_v is proper and each fibre over 0 is a nodal curve $C/(a \sim b)$, every other fibre is a smooth curve of genus $g+1$ and the parametrization is holomorphic in V.

It is clearer to run through the construction without referring to the parameter space V. So fix the data C, a, b, z_a, z_b and choose $\delta > 0$ such that there are disjoint neighbourhoods U^a of a and U^b of b such that $z_a : U^a \to \mathbb{C}$ and $z_b : U^b \to \mathbb{C}$ are each an isomorphism to some open set that contains a disc of radius δ centred at $z_a(a) = 0$ and $z_b(b) = 0$, respectively.

Let Δ_δ, D_{δ^2} denote complex discs of radius δ, δ^2, respectively.

Take $W = W_\delta$ to be the open subset of $C \times D_{\delta^2}$ obtained by deleting the two closed subsets

$$\{(p,t) \big| p \in U^a,\ 0 \le \delta |z_a(p)| \le |t| \le \delta^2\},$$

$$\{(q,t) \big| q \in U^b,\ 0 \le \delta |z_b(q)| \le |t| \le \delta^2\}.$$

Lemma 1.6. If $\epsilon < \delta$ then $W_\epsilon \subset W_\delta$.

In W_δ, define open subsets

$$W^a = W^a_\delta = \{(p,t) \big| p \in U^a,\ 0 < |z_a(p)| < \delta \text{ and } |t| < \delta |z_a(p)|\},$$

$$W^b = W^b_\delta = \{(q,t) \big| q \in U^b,\ 0 < |z_b(q)| < \delta \text{ and } |t| < \delta |z_b(q)|\}.$$

Consider the complex surface $S = S_\delta \subset (\Delta_\delta)^2 \times D_{\delta^2}$ defined by the equation $XY = t$, where X, Y are co-ordinates on the two copies of Δ_δ and t is a co-ordinate on D_{δ^2}. Then there are isomorphisms

$$W^a_\delta \to S - (X = 0) : (p,t) \mapsto (z_a(p), t/z_a(p), t),$$

$$W^b_\delta \to S - (Y = 0) : (q,t) \mapsto (t/z_b(q), z_b(q), t).$$

Together these define an étale morphism $j : W^a_\delta \cup W^b_\delta \to S$, where the union is the disjoint union, taken inside $C \times D_{\delta^2}$. Let $i : W^a_\delta \cup W^b_\delta \to W_\delta$ be the inclusion.

If Z is a subspace of a space X, then $\overline{Z}$ denotes the closure of Z in X.

Lemma 1.7. $(i,j) : W^a_\delta \cup W^b_\delta \to W \times S$ is a closed embedding.

Proof. It is enough to show that the image of W^a_δ in $\overline{W^a_\delta} \times S$ is closed. Now points in $\overline{W^a_\delta} \times S$ are of the form (p, t_1, X, Y, t_2) with

$$\begin{aligned} \delta \ge |z_a(p)| \ge \quad & t_1/\delta,\ t_2 = t_1,\ X = z_a(p),\ Y = t_2/z_a(p), \\ & |X|, |Y| < \delta,\ |t_2| \le \delta^2,\ XY = t_2. \end{aligned}$$

But these conditions force $\delta > |z_a(p)| = |t_2|/|Y| > |t_2|/\delta$, and we are done. □

Now define $\mathcal{C} = \mathcal{C}_\delta$ by glueing W_δ to S_δ by the inclusion i and the étale map j. By the lemma, $\mathcal{C}$ is Hausdorff, [1] and by construction there is a morphism $f : \mathcal{C} \to D_{\delta^2}$ whose fibre over 0 is the nodal curve $C/(a \sim b)$.

Lemma 1.8. f is proper.

Proof. It is enough to show that, for any $r \in (0, \delta)$, the inverse image $Z_r = f^{-1}(\overline{D_{r^2}})$ is compact. By construction, Z_r is the union of the two compact spaces $\overline{W_\delta^1}$ and $\overline{S_r}$, where the subset $\overline{S_r}$ of S_δ is defined by $|X|, |Y| \leq r$. □

Lemma 1.9. The restriction of $f : \mathcal{C}_\delta \to D_{\delta^2}$ to the germ of the pair $(D_{\delta^2}, 0)$ is independent of δ.

Proof. This follows from the facts that, by Lemma 1.6 above, $\mathcal{C}_\epsilon$ is open in $\mathcal{C}_\delta$, and that $C/(a \sim b)$ is proper. □

Note that, by construction, W is open in $C \times D_{\delta^2}$, the image of the projection $pr_1 : W \to C$ is exactly $C - \{a, b\}$ and there is an étale morphism $\pi : W \to \mathcal{C}$.

Given cycles A_i, B_j on C that represent a symplectic basis of $H_1(C, \mathbb{Z})$ and are disjoint from $\{a, b\}$, we can then regard the A_i, B_j as cycles on $\mathcal{C}_t$ that represent part of a symplectic basis of $H_1(\mathcal{C}_t, \mathbb{Z})$ for $t \neq 0$ by taking $pr_1^{-1}(A_i) \cap pr_2^{-1}(t) = A_i \times \{t\}$ and the same thing for B_j. Define the cycle A_{g+1} on $\mathcal{C}_t$ by $A_{g+1} = \partial U^b \times \{t\}$; then $(A_1, ..., A_{g+1}, B_1, ..., B_g$ can be extended to a symplectic basis of $H_1(\mathcal{C}_t, \mathbb{Z})$ where B_{g+1} projects to a cycle on the nodal curve $\mathcal{C}_0 = C/(a \sim b)$ that passes through the node.

We want to extend this construction of a single degenerating pencil $f : \mathcal{C} \to D$ of curves to the construction of a family of such pencils, where the parameter space is V and the pencil depends holomorphically on V. This is merely a matter of enhancing the notation that we have just used, and the details are omitted. The end result of the construction is a parameter space D that is an open neighbourhood of $V \times \{0\}$ in $V \times \mathbb{C}$ and a proper flat morphism $\mathcal{C} \to D$ from an $(n+1)$-dimensional complex manifold to a complex n-manifold that is smooth outside $V \times \{0\}$ and whose restriction to $V \times \{0\}$ is trivial, with fibre $C/(a \sim b)$.

Summarising, we have a family $\mathcal{C} \to \Delta_t$ of curves, call

$$\nu : C \to \mathcal{C}_0 = C/a \sim b$$

the normalisation map. We have constructed a symplectic basis $A_i(t), B_i(t)$ for the homology of $\mathcal{C}_t$. The pull-back of $A_i(0), B_i(0)$, for $i < g+1$, is a symplectic basis for the homology of C, call τ the period matrix of C with respect to this basis. Combining lemma 1.2 and theorem 1.4, we can compute the first order part of the period matrix.

Theorem 1.10 ([Fay73] Corollary 3.8)**.** *The period matrix $T(t)$ of $\mathcal{C}_t$ with respect to the previous basis of the homology is*

$$T(t) = \begin{pmatrix} \tau & AJ(a-b) \\ {}^t AJ(a-b) & \frac{1}{2\pi i} ln(t) + c_0 \end{pmatrix} + t \begin{pmatrix} \sigma(a, b, z_a, z_b) & \cdots \\ \vdots & c_1 \end{pmatrix} + O(t^2)$$

where AJ is the Abel-Jacobi map, and σ is a holomorphic function on V given by

$$\sigma_{ij} = 2\pi i((\omega_i(a) - \omega_i(b))(\omega_j(a) - \omega_j(b))\,.$$

[1] Bourbaki, Top. Gén. TG I.55, Prop. 8. Thanks to MO.

The differentials are evaluated in term of dz_a and dz_b

Proof. Take the differentials $\omega_i(t)$ as in proposition 1.1. First, we compute the zero order term $T(t)$. The biggest block, i.e. $T_{ij}(0)$ for $i, j \leq g$, is the period matrix of C. This because, for $i, j \leq g$, the pull-back of $A_i(0)$ and $B_i(0)$ via ν form a basis for the homology of C, and $\nu^*\omega_i$ are a normalised basis of the holomorphic differentials. The bull back of $B_{g+1}(0)$ on C is a path from a to b, so we have the Abel-Jacobi map. To compute $T_{g+1,g+1}(0)$, we observe that, if we turn around the origin in Δ_t, the cycle $B_{g+1}(t)$ is increased by $A_{g+1}(t)$. In other terms the monodromy of $B_{g+1}(t)$ is $A_{g+1}(t)$, as prescribed by the Picard-Lefschetz formula (cf [ACG11] page 143). The integral of $\omega_{g+1}(t)$ along $A_{g+1}(t)$ is 1, so $T_{g+1,g+1}(t)$ has the same monodromy of $\frac{1}{2\pi i} ln(t)$. In other words, we can write $T_{g+1,g+1}(0)$ as $\frac{1}{2\pi i} ln(t)$ plus some unknown holomorphic functions.

To compute σ we need to apply lemma 1.2 and theorem 1.4. We are considering $i, j \leq g$. Since $\int_{A_{g+1}(t)} \omega_i(t) = 0$, the residue of the form $\nu^*\nabla(\omega_i)(0)$ at a and b is zero, we can thus apply lemma 1.2 (the role of $\gamma(t)$ is played by $A_{g+1}(t)$).

Moreover,

$$\int_{A_j(0)} \nu^*\nabla(\omega_i)(0) = \frac{d}{dt} |_{t=0} \int_{A_j(t)} \omega_i(t) = \frac{d}{dt} |_{t=0} \delta_{ij} = 0\,,$$

so the integral of $\nu^*\nabla\omega_i(0)$ along A_j is zero. Looking at the construction of $\mathcal{C}$, one sees that $\nu^*\nabla\omega_i(0)$ has not any pole other than a and b. Because of the uniqueness of the differentials defined in 1.3, we obtain that

$$\nu^*\nabla\omega_i(0) = (\omega_i(a) - \omega_i(b))(\eta_a - \eta_b)$$

The theorem follows from Riemann's bilinear relations 1.4. □

Recall that, because of the heat equation, the tangent space of $\mathcal{A}_{g-1}$ at $Jac(C)$ is isomorphic to $Sym^2 H^0(C, K_C)^\vee$. In a co-ordinates free fashion, we have

$$\sigma(\omega) = 2\pi i (\frac{\omega(a)}{dz_a} - \frac{\omega(b)}{dz_b})^2\,.$$

Let us consider the map

$$C \to \mathbb{P}H^0(C, K_C)^\vee \xrightarrow{Ver_2} \mathbb{P}Sym^2 H^0(C, K_C)^\vee\,,$$

where Ver_2 is the second Veronese embedding. The projectivization of σ is the image of a point on the line passing trough a and b.

Pinching a trivial homological cycle We can perform the same construction starting with two curves C and D of genus g and h, taking two points c and d and two local co-ordinates z_c and z_d. With the same procedure described above, we obtain a family $\mathcal{C}$ of genus $g + h$ curves degenerating to $C \sqcup D/c \sim d$. In this case there is not a vanishing cycle. Fix a basis for A_i, B_i the homology of C and D with the support disjoint from p and d. Call ω_i and ψ_i a basis for the normalised differentials of C and D, let τ and υ the period matrices with respect to these basis. Since the support of the basis is disjoint from the regions where we are performing the surgery, we can extend it to a basis $A_i(t), B_i(t)$ for $\mathcal{C}_t$ for every t

Theorem 1.11 ([Fay73] Corollary 3.2). *The period matrix $T(t)$ of $\mathcal{C}_t$ with respect to the previous basis of the homology is given by*

$$T(t) = \begin{pmatrix} \tau & 0 \\ 0 & \upsilon \end{pmatrix} + 2\pi i t R \otimes R + O(t^2)$$

where R is a holomorphic function of the parameters a, b, z_a and z_b given by

$$R = (\omega_1(0,c), \dots, \omega_g(0,c), -\psi_1(0,d), \dots, -\psi_h(0,d)) .$$

The differentials are evaluated in term of dz_c and dz_d

Proof. Let

$$\nu : C \sqcup D \to \mathcal{C}_0 = C \sqcup D / c \sim d$$

be the normalisation map.

$T(0)$ comes from pulling everything back on C and D.

Call $\omega_i(t)$ and $\psi_i(t)$ a basis for the normalised differentials of $\mathcal{C}_t$ such that $\nu^*\omega_i(0) = \omega_i$ and $\nu^*\psi_i(0) = \psi_i$. A small loop γ around c is homologically trivial on $\mathcal{C}_t$, so $\int_{\gamma(t)} \omega_i(t) = 0$, and thus $\int_\gamma \nu^*\nabla(\omega_i)(0) = 0$. We conclude that $\nu^*\nabla(\omega_i)(0)$ on C has zero residue at c. The same is true for $\nu^*\nabla(\psi_i)(0)$ at d. Using lemma 1.2 and arguing as in the proof of 1.10, we obtain that

$$\nu^*\nabla(\omega_i)(0) = \omega_i(c)(\eta_c - \eta_d)$$

$$\nu^*\nabla(\psi_i)(0) = -\psi_i(d)(\eta_c - \eta_d)$$

The theorem follows from Riemann's bilinear relations 1.4. □

An interesting case is when the genus of D is zero. In this case, the central fibre is not stable, the stable reduction contract D, so we get a family of smooth curves with central fibre C. The first order of the period matrix, as we are going to see, is the same of the Schiffer variation.

Schiffer's variations A Schiffer's variation at (p, z_p) of C is a particular family of smooth curves $\mathcal{C}$ over a disc Δ_t with central fibre C. To construct $\mathcal{C}_t$ consider a co-ordinate patch defined by z_p, remove a small disc and glue the remaining annulus with another disc via the formula

$$z^*(t) = z_p + \frac{t}{z_p} ,$$

where z^* is the co-ordinate on the new disc. For more details see [ACG11] page 175. We can fix a symplectic basis A_i, B_i for the homology of C with support disjoint from p, call τ the period matrix of C with respect to this basis. We extend the basis to a basis of $A_i(t), B_i(t)$ of the homology of $\mathcal{C}_t$ for every t.

Proposition 1.12 ([Pat63]). *The period matrix $T(t)$ of $\mathcal{C}_t$ is*

$$T(t) = \tau + t\sigma(p, z_p) + O(t^2) ,$$

where σ is a holomorphic function of p and z_p defined as

$$\sigma(p, z_p)_{ij} = 2\pi i \frac{\omega_i(p)\omega_j(p)}{dz_p^2} .$$

Proof. By definition

$$dz^*(t) = dz - \frac{t}{z^2} dz$$

Let us a fix a basis $\omega_i(t)$ of the relative abelian differential dual to $A_i(t)$. Locally in the annulus around p

$$\omega_i(t) = f_i(z,t)(dz - \frac{t}{z^2} dz) ,$$

where f_i is a holomorphic function on a disc around the origin. We apply the Gauss-Manin connection and specialise at $t = 0$ (i.e. we apply $\frac{d}{dt} |_{t=0}$), we get

$$\nabla\omega_i(0) = \frac{\partial f_i}{\partial t}(z,0)dz - f_i(z,0)\frac{1}{z^2}dz\,.$$

To compute the residue at p, take a small homologically trivial cycle γ around p, extend it to a family of cycles $\gamma(t)$ on $\mathcal{C}_t$, we have

$$\frac{1}{2\pi i}Res_p\nabla\omega_i(0) = \int_\gamma \nabla\omega_i(0) = \frac{d}{dt}|_{t=0}\int_{\gamma(t)}\omega_i(t) = \frac{d}{dt}|_{t=0}\, 0 = 0$$

The integral of $\nabla(\omega_i)$ along $A_j(0)$ is zero because of the same argument. Using lemma 1.2 and definition 1.3, we conclude that

$$\nabla(\omega_i)(0) = \frac{\omega_i(p)}{dz_p}\eta_p\,.$$

The proposition follows from Riemann's bilinear relations 1.4. □

The matrix $\sigma(p,z_p)$ belongs to $T_C\mathcal{A}_g = Sym^2H^0(C,K_C)^\vee$, in a co-ordinates free fashion, it is given by

$$\sigma(p,z_p)(\omega) = 2\pi i\frac{\omega}{dz_p}(p)^2.$$

The projectivization of $\sigma(p,z_p)$ does not depend on z_p, it is the image of p under the map

$$C \to \mathbb{P}H^0(C,K_C)^\vee \xrightarrow{Ver_2} \mathbb{P}Sym^2H^0(C,K_C)^\vee\,.$$

Schiffer's variations span all the tangent space to $\mathcal{M}_g$, see [ACG11] page 175, so we obtain this nice corollary.

Corollary 1.13. *The affine cone over the image of C spans all $T_C\mathcal{M}_g$.*

As explained in [ACG11] page 216-224, the co-differential of the Torelli's map can be identified with the multiplication map

$$\pi : Sym^2H^0(C,K_C) \to H^0(C,2K_C)\,,$$

so corollary 1.13 is equivalent to the classical Noether's theorem

Theorem 1.14 (Max Noether, [ACGH85] page 117)**.** *If C is not hyperelliptic, the map π is surjective.*

At the hyperelliptic locus the situation is more complicated.

Hyperelliptic families We can make the previous constructions work in the hyperelliptic case. Let us consider the first construction. We start with a hyperelliptic curve C, pick two points and local co-ordinates conjugated under the hyperelliptic involution (so, we are not taking Weierstrass points). The involution of the central fibre extends to an involution of the whole family. The quotient is a deformation of $\mathbb{P}^1$, which is trivial, so the family is a family of hyperelliptic curves. In this case, the tangent vector we get is $\sigma(p,\iota(p),z_p,\iota^*z_p) = 2\sigma(p,z_p)$, where $\sigma(p,z_p)$ is defined in 1.12.

The second construction works if and only if the points c and d are Weierstrass points.

For what concern Schiffer variation, if we perform simultaneously two variations, one at (p,z_p) and the other at $(\iota(p),\iota^*z_p)$, we obtain a family of hyperelliptic curve. The tangent vector is $\sigma(p,z_p)+\sigma(\iota(p),\iota^*z_p) = 2\sigma(p,z_p)$. We thus have the following result.

Lemma 1.15. The vector $\sigma(p, z_p)$ belongs to $T_C Hyp_g$, for every p in C.

Combining this with 1.13, we get a classical theorem.

Theorem 1.16 (Local Torelli's theorem for hyperelliptic curves - [ACG11] page 223-224)**.** *The differential of the Torelli's map at the hyperelliptic locus is not injective, its image is the tangent space to the hyperelliptic locus.*

When p is a Weierstrass point and $\iota^* z_p = -z_p$, the tangent vector $\sigma(p, z_p)$ can be interpreted as follow.

Theorem 1.17 (Rauch's variational formula - [Fay73] page 47 or [May69])**.** *Let C be the curve defined by the equation*

$$y^2 = (x - p) \prod_{i=1}^{2g+1} (x - p_i) \, ,$$

the vector in $T_C Hyp_g$ defined by the family

$$y^2(t) = (x - p - t) \prod_{i=1}^{2g+1} (x - p_i)$$

is $\sigma(p, z_p)$, where z_p is the local co-ordinate given by x.

Question 1.18. We would like to obtain similar constructions and formulæ for families of 3-gonal (or n-gonal) curves. Unfortunately, the degenerations we described above do not give rise to families of 3-gonal curves, unless we have points of total ramification.

2 Review of Satake compactification and stable modular forms

2.1 The construction

The moduli space $\mathcal{A}_g$ of principally polarised abelian g-folds is endowed with the ample line bundle L of weight one modular forms. Sections of L^n are weight n modular forms. The Satake compactification $\mathcal{A}_g^S$ is discussed in full details in [Fre83], see also the introduction of [CSB11]. We can define it as the *Proj* of the ring

$$\bigoplus_{n \geq 0} H^0(\mathcal{A}_g, L^n)$$

This compactification is normal and it is the one "seen" by modular forms. The boundary has codimension $g + 1$, it is singular and it has a bad moduli interpretation.

The Siegel operator is a map of graded rings

$$\Phi : \bigoplus_{n \geq 0} H^0(\mathcal{A}_g, L^n) \to \bigoplus_{n \geq 0} H^0(\mathcal{A}_{g-1}, L^n)$$

defined as

$$\Phi(F)(\tau) := \lim_{t \to +\infty} F(\tau \oplus it) \, .$$

It is surjective for n even and larger than $2g$, so it defines a stratification

$$\mathcal{A}_g^S = \mathcal{A}_g \sqcup \mathcal{A}_{g-1}^S = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \cdots \mathcal{A}_1 \sqcup \mathcal{A}_0 \, ,$$

in other words, the boundary $\partial\mathcal{A}_g^S$ is isomorphic to $\mathcal{A}_{g-1}^S$. A modular form is called cuspidal if it vanishes on the boundary of the Satake compactification, i.e. it is in the kernel of Φ.

Let $\overline{\mathcal{M}}_g$ be the Deligne-Mumford compactification of $\mathcal{M}_g$. The Torelli map can be extended to an application

$$T : \overline{\mathcal{M}}_g \to \mathcal{A}_g^S\,,$$

mapping a curve C to the Jacobian of its normalisation (see [BHPVdV04] section III.16). The image of $\overline{\mathcal{M}}_g$ is the Satake compactification $\mathcal{M}_g^S$. This compactification can be equivalently defined as the closure of $\mathcal{M}_g$ in $\mathcal{A}_g^S$, or the image of $\overline{\mathcal{M}}_g$ via the map defined by the Hodge bundle (see [ACG11] page 435). The compactification $\mathcal{M}_g^S$ as set is the union of all products $\mathcal{M}_{g_1}\times\cdots\times\mathcal{M}_{g_s}$ with $\sum g_i \leq g$. The component $\mathcal{M}_g$ is an open dense subset, the other products are boundary components. In particular, $\mathcal{M}_{g-1}$ is the image of the divisor Δ_0 of curves with one non-separating node. The component $\mathcal{M}_i\times\mathcal{M}_{g-i}$ is the image of the divisor Δ_i of curves whose normalisation is the disjoint union of a curve of genus i and a curve of genus $g-i$. Let us stress that by "the boundary component $\mathcal{M}_i\times\mathcal{M}_{g-i}$" we mean the intersection of $\mathcal{M}_g^S$ with $\mathcal{A}_i\times\mathcal{A}_{g-i}$, which, a priori, is equal to $\mathcal{M}_i\times\mathcal{M}_{g-i}$ just as a set: it might be singular. In particular, theorem 0.1 shows that the boundary component $\mathcal{M}_g^S\cap\mathcal{A}_{g-1}$ is non-reduced.

Similarly, the Satake compactification Hyp_g^S of the hyperelliptic locus is the closure of Hyp_g in $\mathcal{A}_g^S$, and the Satake compactification $\mathcal{P}_g^S$ of the Prym locus $\mathcal{P}_g$ is its closure in $\mathcal{A}_g^S$.

Using the stratification of $\mathcal{A}_g^S$, we can define stable modular forms.

Definition 2.1 (Stable modular form)**.** *A stable modular form F is the datum of a modular form F_g for every g such that $\Phi(F_{g+1}) = F_g$.*

There exists a surprising way to define stable modular forms from even positive definite unimodular lattices. We will remind some definitions and results, standard references are [CS99] and [Fre77], see also [CSB11].

A lattice is a couple (Γ, Q) (for short, we will write just Γ), where Γ is a free abelian group and Q is a bilinear $\mathbb{Z}$-valued form on Γ. The elements of Γ are called vectors. Γ is called of **Type II** or **even unimodular** if $Q(v,v)$ is even for all v in Γ and the determinant of Q is plus or minus one (i.e. Q is an isomorphism between Γ and its dual). A lattice is positive definite if Q is. One can prove that the rank of an even unimodular positive definite lattice is divisible by 8. Usually, lattices are presented as lattices inside an Euclidean space, and the quadratic form is the restriction of the standard one. An example of positive definite type II lattice is E_8.

Given a positive definite type II lattice (Γ, Q), we can define the associated Theta series Θ_Γ as follow

$$\Theta_{\Gamma,g}(\tau) := \sum_{x_1,\dots,x_g\in\Gamma} \exp\,(\pi i \sum_{i,j} Q(x_i,x_j)\tau_{ij})\,,$$

where τ belongs to the Siegel upper half space $\mathfrak{H}_g$. Theta series are stable modular form of weight $\frac{1}{2}rk(\Gamma)$, they verify the factorization properties

$$\Theta_{\Gamma\oplus\Lambda} = \Theta_\Gamma\Theta_\Lambda$$

and

$$\Theta_\Gamma(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}) = \Theta_\Gamma(A)\Theta_\Gamma(B)\,.$$

These identities are no longer true for linear combinations of Theta series. It is a theorem ([Fre77] Theorem 2.5) that the ring of stable modular forms is the polynomial ring in the irreducible positive definite even unimodular lattices.

Example 2.2 (The Schottky form)**.** There are exactly two even positive definite unimodular lattices of rank 16, D_{16}^+ and $E_8 \oplus E_8$. The difference of the associated Theta series is the Schottky form, this is the unique understood stable modular form. It behaves as follows:

- it is zero for g =0,1,2 and 3,
- it is not zero for $g = 4$, and it vanishes on $\mathcal{M}_4$ with multiplicity 1,
- it is not zero on $\mathcal{M}_5$: it cuts out a divisor of slope 8 (which, since the slope conjecture is true for $g = 5$, must be the divisor of trigonal curves) ([GSM11]),
- it vanishes on the hyperelliptic locus for every g ([Poo96]).

Example 2.3 (Witt's lattices)**.** Using the same definition of E_8, for every integer k one can define the Witt's lattices W_{8k}. The lattice W_{8k} has rank $8k$, it is equal to E_8 for $k = 1$ and to D_{16}^+ for $k = 2$. We have the following expansion

$$\Theta_{W_{8k},g}(\tau) = \sum_{\epsilon \text{ even}} \theta[\epsilon]^{8k}(\tau)\,,$$

where the sum runs over all the even Theta characteristics.

2.2 The singularities

In this section we study the singularities of $\mathcal{A}_{g+1}^S$. Let X be a generic point of $\mathcal{A}_g \subset \mathcal{A}_{g+1}^S$. The local ring $(\mathcal{A}_{g+1}^S, X)$ has been described in [Igu67a]. For the Fourier-Jacobi expansion see also [BvdGHZ08] sections III.8 and III.11. Let us recall the results.

The local ring of $(\mathcal{A}_{g+1}^S, X)$ is normal and it is isomorphic to the ring of convergent power series

$$\sum_{n=0}^{\infty} s_n(\tau, z) q^n\,,$$

where now X is identified with $\mathbb{C}^g/\mathbb{Z}^g \oplus \tau\mathbb{Z}^g$, z belongs to $\mathbb{C}^g$, s_n is a section of $H^0(X, 2n\Theta)$ and q belongs to a small disc around zero in the complex plane. The image of a modular form F_{g+1} of degree $g + 1$ in this local ring is the Fourier-Jacobi expansion of F_{g+1}. Let us be more explicit. Call $\mathfrak{H}_g$ be the Siegel upper half-space, for any element $T \in \mathfrak{H}_{g+1}$ write

$$T = \begin{pmatrix} \tau & z \\ {}^t z & t \end{pmatrix}\,,$$

with t in $\mathfrak{H}_1$ and τ in $\mathfrak{H}_g$. The Fourier-Jacobi expansion of F_{g+1} is

$$F_{g+1}(T) = f_0(\tau) + \sum_{n \geq 1} f_n(\tau, z) q^n\,,$$

where $q = \exp(2\pi i t)$ and $f_0 = \Phi(F_{g+1})$. The function f_n is the n-th Fourier-Jacobi coefficient of F_{g+1}, it belongs to $H^0(X, 2n\Theta)$. Being $(\mathcal{A}_{g+1}^S, X)$ normal, the derivation mapping a modular form to its first Fourier-Jacobi coefficient is surjective. The Fourier-Jacobi expansion of a Theta series can be written explicitly, the n-th coefficient is related to the vectors of norm $2n$. In particular, let us define

$$\mu_\Lambda := \min\{Q(v, v) \mid v \in \Lambda\,,\, v \neq 0\}\,.$$

A direct computation proves the following lemma.

Lemma 2.4. Let Λ be a positive definite even unimodular lattice, suppose $\Theta_{\Lambda,g}(\tau) = 0$, then the first non-trivial Fourier-Jacobi coefficient of $\Theta_{\Lambda,g+1}$ at τ is the $\frac{1}{2}\mu_\Lambda$-th coefficient.

The number μ_Λ is related to the so called "packing radius" of the lattice, the following bound holds

$$\mu_\Lambda \leq 2\lfloor\frac{rk(\Lambda)}{24}\rfloor + 2\,,$$

where "$\lfloor\,\rfloor$" is the round down, see [CS99] section 7.7 corollary 21.

The tangent space at X decomposes as

$$0 \to T_X\mathcal{A}_g \to T_X\mathcal{A}^S_{g+1} \to H^0(X, 2\Theta)^\vee \to 0$$

This means that the fibre of the normal bundle to $\mathcal{A}_g$ at X is $H^0(X, 2\Theta)^\vee$. The tangent cone to A^S_{g+1} in the normal bundle is the affine cone over the image of

$$\mid 2\Theta \mid : \quad X \quad \to \quad \mathbb{P}H^0(X, 2\Theta)^\vee$$

In other words, the singularity of $(\mathcal{A}^S_{g+1}, X)$ is isomorphic to the affine cone over the Kummer variety of X.

In the local ring $(\mathcal{A}^S_{g+1}, X)$, let $I_{\mathcal{A}_g}$ be the ideal of functions vanishing on $\mathcal{A}_g$. We have an exact sequence

$$0 \to I^{m+1}_{\mathcal{A}_g} \to (\mathcal{A}^S_{g+1}, X) \to \bigoplus_{n=0}^{m} H^0(X, 2n\Theta) \to 0$$

so, to compute the order of a modular form at the boundary, one should understand what is the first non-trivial Fourier-Jacobi coefficient.

3 The non-existence of stable Schottky relations

The following section is the third of [CSB11]. It contains the proof of theorem 0.1.

3.1 The failure of transversality - Third section of [CSB11]

Theorem 3.1. *If F_{g+1} has multiplicity at least m along M_{g+1} then F_g has multiplicity at least $m+1$ along M_g.*

Proof. Suppose that $N_{g+1}(\{x_{ij}\})$ is a homogeneous polynomial of degree d in the entries x_{ij} of a symmetric $(g+1) \times (g+1)$ matrix X. Our hypothesis is that for all $d \leq m-1$ and for all such N_{g+1}, the partial derivative

$$N_{g+1}(F_{g+1}) := N_{g+1}\left(\left\{\frac{\partial}{\partial T_{pq}}\right\}\right)(F_{g+1})$$

vanishes along M_{g+1} (rather, its inverse image in $\mathfrak{H}_{g+1}$) for $T = (T_{pq}) \in \mathfrak{H}_{g+1}$[2].

Given such N_{g+1}, we let N_g denote the polynomial obtained from it by setting the bottom row and last column of X equal to zero. Our goal is to show that for

[2]Siegel upper half space, i.e. $g+1$ - by - $g+1$ symmetric complex matrices with positive definite imaginary part

every such N_g of degree m, the partial derivative $N_g(F_g)$ vanishes at every point τ in $\mathfrak{H}_g$ that comes from a curve of genus g.

For any positive integer n, let S_n denote the set of $n \times n$ integer matrices that are symmetric, positive semi-definite and whose diagonal entries are even. Then recall that every Siegel modular form $F = F_{g+1}(T)$ of degree $g+1$ over a ring R has a Fourier expansion

$$F(T) = \sum_{X \in S_{g+1}} a(X) \exp \pi i \mathrm{Tr}\,(XT) = \sum_{X \in S_{g+1}} a(X) \exp \pi i \sum_{p,q=1}^{g+1} x_{pq} T_{pq}.$$

We write $X = (x_{pq})$ for $X \in S_{g+1}$. The Fourier coefficients $a(X) = a_F(X)$ lie in R. For us, $R = \mathbb{C}$.

Take T as above[3] and take N to have degree $m-1$; then

$$\frac{1}{(\pi i)^{m-1}} N_{g+1}(F_{g+1})(T) = \sum_{X \in S_{g+1}} a(X) N_{g+1}(\{x_{pq}\}) \exp \pi i \sum_{p,q=1}^{g+1} x_{pq} T_{pq}.$$

Our aim is to examine the coefficient of t in the expansion of this expression in powers of t, so calculate modulo t^2. Since $\exp 2\pi i T_{g+1,g+1} = \gamma_1 \gamma_2^t t$ modulo t^2, where $\gamma_j = \exp c_j$, it follows that

$$\frac{1}{(\pi i)^{m-1}} N_{g+1}(F_{g+1})(T) = \sum_{x_{g+1,g+1}=0} + \sum_{x_{g+1,g+1}=2},$$

since all terms with $x_{g+1,g+1} \geq 4$ vanish modulo t^2.

Lemma 3.2. If $X \in S_{g+1}$ and $x_{g+1,g+1} = 0$, then the right hand column and bottom row of X are both zero.

Proof. Immediate consequence of semi-positivity. □

Therefore

$$\sum_{x_{g+1,g+1}=0} = \sum_{X \in S_g} a(X) N_g(\{x_{pq}\}) \exp \pi i \sum_{p,q=1}^{g} x_{pq}(\tau_{pq} + t\sigma_{pq})$$

and

$$\begin{aligned}\sum_{x_{g+1,g+1}=2} = \quad & t\gamma_1\gamma_2^t \textstyle\sum_{X \in S_{g+1}, x_{g+1,g+1}=2} a(X) N_{g+1}(\{x_{pq}\}) \\ & . \left(\exp 2\pi i \textstyle\sum_{p=1}^{g} x_{p,g+1} \int_a^b \omega_p\right) \left(\exp \pi i \textstyle\sum_{p,q=1}^{g} x_{pq}\tau_{pq}\right)\end{aligned}$$

since we are calculating modulo t^2. So the coefficient of t that we seek is $A + \gamma_1 B$, where

$$A = \sum_{x_{g+1,g+1}=0} a(X) N_g(\{x_{pq}\}) \left(\pi i \sum_{p,q=1}^{g} x_{pq}\sigma_{pq}\right) \left(\exp \pi i \sum_{p,q=1}^{g} x_{pq}\tau_{pq}\right)$$

[3]This is the period matrix $T = T(t)$ studied in theorem 1.10. All the notations are the same.

and

$$B = \sum_{x_{g+1,g+1}=2} a(X) N_{g+1}(\{x_{pq}\}) \left(\exp 2\pi i \sum_{p=1}^{g} x_{p,g+1} \int_a^b \omega_p\right) \left(\exp \pi i \sum_{p,q=1}^{g} x_{pq}\tau_{pq}\right).$$

The quantities A, B, γ_1 are holomorphic functions on V and, by assumption, $A+\gamma_1 B$ vanishes identically.

Now rescale the local co-ordinates z_a, z_b; that is, given any non-zero scalars λ, μ, replace z_a by $\lambda^{-1}z_a$ and z_b by $\mu^{-1}z_b$. Such a rescaling will produce a different family $\mathcal{C} \to \Delta$, but the quantity $A + \gamma_1 B$ will still vanish for the rescaled family. Moreover, B is invariant under this rescaling, as is revealed by a cursory inspection. Also c_1 is a holomorphic function of λ, μ because the entries of a period matrix are holomorphic functions of the parameters.

On the other hand inspection also reveals that, because of the description above of σ_{pq}, A can be written as

$$A = C\lambda^2 + D\lambda\mu + E\mu^2$$

with C, D, E independent of λ, μ. So we have an identity

$$C\lambda^2 + D\lambda\mu + E\mu^2 = -B\exp\left(c_1(\lambda, \mu)\right)$$

of holomorphic functions on the 2-dimensional torus $\mathbb{G}_m^2 = \operatorname{Spec}\mathbb{C}[\lambda^{\pm}, \mu^{\pm}]$, where we regard B, C, D, E as constants.

Lemma 3.3. Suppose that f is a rational function on a complex algebraic variety X and that there is a holomorphic function h on some Zariski open subset U of X such that $f = \exp h$ on U. Then f is constant.

Proof. It is enough to show that f is constant on a general curve in X. So we can assume that $\dim X = 1$, and then that X is a compact Riemann surface. If f is not constant, then it has a zero, say at P, and in some neighbourhood U of P with a co-ordinate z we have $f = z^n f_1$ with f_1 holomorphic and invertible on U, and $n > 0$. Then $f_1 = \exp h_1$ with h_1 holomorphic on U, and h is holomorphic on $U - \{P\}$. Then z^n has a single-valued holomorphic logarithm on $U - \{P\}$, which is absurd. □

Corollary 3.4. *A and B vanish identically.*

In fact, we do not exploit the vanishing of B, although it is a key step in the argument of [GSM11] involving the linear system Γ_{00} of second order theta functions that vanish to order 4 at the origin and the heat equation.

Now A can also be written as

$$\begin{aligned} A = & \left.\frac{\partial}{\partial t}\right|_{t=0} \left(\textstyle\sum_{X\in S_g} a(X) N_g(\{x_{pq}\}) \exp \pi i \sum_{pq,=1}^{g} x_{pq}(\tau_{pq} + t\sigma_{pq})\right) \\ & = \left.\frac{\partial}{\partial t}\right|_{t=0} N_g(F_g(\tau + t\sigma)). \end{aligned}$$

That is, σ lies in the Zariski tangent space H at the point τ to the divisor in $\mathfrak{H}_g$ defined by the function $N_g(F_g) = N_g(\{\frac{\partial}{\partial \tau_{ij}}\})(F_g)$. It is important to note that, from this description, H depends upon C but is independent of the points a, b, the local co-ordinates z_a, z_b and the scalars λ, μ.

We let M_g^0 denote the open subvariety of M_g corresponding to curves with no automorphisms and A_g^0 the open subvariety of A_g corresponding to principally polarized abelian varieties with no automorphisms except ± 1. Then M_g^0 lies in A_g^0 and both are smooth varieties, and if C lies in M_g^0 there are natural identifications of tangent spaces given by

$$T_{[C]}M_g = H^0({\Omega_C^1}^{\otimes 2})^\vee,$$
$$T_{[C]}A_g = Symm^2 H^0(\Omega_C^1)^\vee.$$

The inclusion $T_{[C]}M_g \hookrightarrow T_{[C]}A_g$ is dual to the natural multiplication (which is surjective, by Max Noether's theorem) $Symm^2 H^0(\Omega_C^1) \to H^0({\Omega_C^1}^{\otimes 2})$.

We are aiming to prove that H, when regarded as a Zariski tangent space, is the whole of the tangent space $T_\tau \mathfrak{H}_g = Symm^2 H^0(C,\Omega^1)^\vee$. So assume otherwise; then H is a hyperplane. Projectivize: then $\sigma \in \mathbb{P}(H)$ and $\mathbb{P}(H)$ is a hyperplane in $\mathbb{P}(Symm^2 H^0(C,\Omega^1)^\vee)$.

Now comes the point at which information about abelian integrals is transformed into projective geometry and thence moduli.

The matrix σ is of rank 1 and is proportional to the tensor square of a vector:

$$\sigma = 2\pi i \left(\frac{\omega}{dz_a}(a) - \frac{\omega}{dz_b}(b)\right)^{\otimes 2},$$

where ω is the vector $(\omega_1, ..., \omega_g)$. Identify the curve C with its image in $\mathbb{P}^{g-1} = \mathbb{P}(H^0(C,\Omega^1)^\vee)$ under its canonical embedding $P \mapsto (\omega_1(P), ..., \omega_g(P))$, so that we have inclusions

$$C \hookrightarrow Sec(C) \hookrightarrow \mathbb{P}(H^0(C,\Omega^1)^\vee),$$

where $Sec(C)$ is the secant variety of C; recall that the secant variety $Sec(X)$ of a variety X in $\mathbb{P}^n$ is the closure of the union of all the secant lines $\langle x, y\rangle$ for pairs of distinct points $x, y \in X$. Then we see that the point σ in $\mathbb{P}(Symm^2 H^0(C,\Omega^1)^\vee)$ lies in the image under the second Veronese embedding

$$Ver_2 : \mathbb{P}(H^0(C,\Omega^1)^\vee) \hookrightarrow \mathbb{P}(Symm^2 H^0(C,\Omega^1)^\vee)$$

of the line $\langle a, b\rangle$ in $Sec(C)$. Moreover, fix local co-ordinates z_a^0 and z_b^0 and then rescale them by λ, μ; that is, write $z_a = \lambda^{-1} z_a^0$ and $z_b = \mu^{-1} z_b^0$. Then as λ, μ vary, the point σ in $\mathbb{P}(Symm^2 H^0(C,\Omega^1)^\vee)$ sweeps out an open piece of the line $\langle a, b\rangle$.

Since H is independent of the points $a, b \in C$ and the scalars λ, μ, the putative hyperplane $\mathbb{P}(H)$ contains $Ver_2(Sec(C))$. Then, by the nature of Ver_2, there is a quadric Q in $\mathbb{P}^{g-1}$ that contains $Sec(C)$. However, by Lemma 3.5 below, $Sec(C)$ has embedding dimension $g-1$ at every point of C, so that Q is singular along C. This is impossible, since the singular locus of any quadric is linear, and Theorem 3.1 is proved.

Lemma 3.5. Suppose that X is a non-degenerate subvariety of $\mathbb{P}^n$. Then $Sec(X)$ has embedding dimension n at every point of X.

Proof. Suppose that P is a point of X and that L is a hyperplane in $\mathbb{P}^n$ disjoint from P. The image of X under projection from P is then a non-degenerate subvariety X_P of L. The projective cone C_P over X_P with vertex P is then the union of the lines in $\mathbb{P}^n$ through P that are secant to X; each such line lies in $Sec(X)$ and so $Sec(X)$ contains C_P. Since X_P is non-degenerate, the embedding dimension of C_P at P is n, and then the same thing holds for $Sec(X)$. □

□

Theorem 0.1 is an immediate corollary of this and the following lemma in commutative algebra.

Lemma 3.6. Suppose that X is a closed subvariety of the variety Y defined by the ideal $I = I_{X/Y}$. Suppose that W is a smooth open subvariety of Y such that $W \cap X$ is smooth and non-empty and that J is an ideal of $\mathcal{O}_Y$ such that $J|_W = I^n|_W$. Then J is contained in $I^{[n]}$, the nth symbolic power of I.

Proof. First, recall that if X and Y are smooth over a field of characteristic zero, then $I^n = I^{[n]}$ and consists of the functions f on Y all of whose derivatives, with respect to local co-ordinates on Y, of order up to and including the $(n-1)$st, vanish along X.

We can assume that Y is affine, say $Y = Spec\, A$, so that A is an integral domain and I is prime. For any ideal $\mathfrak{a}$ of A, write $V(\mathfrak{a}) = Spec\,(A/\mathfrak{a})$.

We can increase J, provided that $J|_W$ is unchanged, so that in particular we can replace J by $J + I^{[n]}$. Then, without loss of generality, we can suppose that J contains $I^{[n]}$ and must prove that $J = I^{[n]}$. We have $V(J)_{red} \subset V(I^{[n]})_{red} = X$ and $V(J)_{red} \cap W = X \cap W$, so that $V(J)_{red} = X$, and therefore $\sqrt{J} = I$.

Recall that for any ideal $\mathfrak{a}$ with $\sqrt{\mathfrak{a}} = I$, there is a unique smallest I-primary ideal $\widetilde{\mathfrak{a}}$ containing $\mathfrak{a}$, given by the formula $\widetilde{\mathfrak{a}} = A \cap \mathfrak{a}.A_I$, where A_I is the localization of A at the prime ideal I. As before, we can increase J, and so assume that $J = \widetilde{J}$, that is, that J is I-primary. The symbolic power $I^{[n]}$ is $I^{[n]} = \widetilde{I^n}$.

By assumption, the generic point ξ of X lies in W and $A_I = \mathcal{O}_{Y,\xi}$, so that $J.A_I = I^n.A_I$. Intersecting both sides of this equation with A gives $J = \widetilde{J} = I^{[n]}$. □

Now regard the Satake compactifications A_g^S and M_{g+m}^S as closed subvarieties of A_{g+m}^S.

Theorem 3.7. *(= Theorem 0.1) The intersection $A_g^S \cap M_{g+m}^S$ contains the mth order infinitesimal neighbourhood of M_g^S in A_g^S.*

Proof. The ideal defining M_{g+m}^S inside A_{g+m}^S is generated by those Siegel modular forms F_{g+m} that vanish along M_{g+m}^S. From Theorem 3.1 and induction on m it follows that F_g and all its partial derivatives with respect to the co-ordinates τ_{pq} on $\mathfrak{H}_g$ of orders at most m vanish along M_g, which is just the statement of the corollary. □

Remark 3.8. For $m = 1$ this says that at a general point $[C]$ of M_g, the Zariski tangent space at $[C]$ to the $3g$-dimensional variety M_{g+1}^S contains the $g(g+1)/2$-dimensional tangent space $Symm^2 H^0(C, \Omega_C^1)^\vee$ at $[C]$ to A_g, where these tangent spaces both lie in $T_{[C]} A_{g+1}^S$.

3.2 Stable divisors

In this section we consider the weight n stable modular form

$$F_g := \Theta_{\Lambda,g} - \Theta_{\Gamma,g}\,,$$

where Λ and Γ are two even positive definite unimodular latices of rank $N = 2n$. Because of corollary 0.2, it cuts out a divisor on $\mathcal{M}_g$ for $g >> 0$. The only case in which we have a good understanding of this divisor is the Schottky form, see example 2.2.

This discussion makes sense also for more general linear combinations of Theta series, and one can consider $\mathcal{P}_g$ instead than $\mathcal{M}_g$. Being an ongoing project, for the sake of simplicity we will work just on differences of Theta series on $\mathcal{M}_g$.

We start from $g = 0$. On $\mathcal{A}_0$, which is just a point, we have $F_0 = 0$, because any Theta series values 1 on $\mathcal{A}_0$. There exists an integer g_1 such that $F_{g_1-1} = 0$ on $\mathcal{A}_{g_1-1}$ but F_{g_1} does not vanish. Because of remark 2.4 of [Fre77], the Siegel operator is an isomorphism for $2g > n$, so

$$g_1 \leq \frac{1}{4}N\,.$$

Here, the questions are the following.

Questions 3.9. It is possible to compute g_1? Can we found the order of F_{g_1} at the boundary of $\mathcal{A}_{g_1}$? Does F_{g_1} vanish on $\mathcal{M}_{g_1}$? If this is the case, with what multiplicity?

Lemma 2.4 implies this bound

$$\operatorname{mult}(F_{g_1}, \partial A^S_{g_1}) \geq \frac{1}{2}\min\{\mu_\Lambda, \mu_\Gamma\}\,.$$

Now, we restrict F_g to $\mathcal{M}_g$. Because of corollary 0.2, we know that there exists an integer g_2 such that $F_{g_2-1} = 0$ on $\mathcal{M}_{g_2-1}$ but F_{g_2} does not vanish on $\mathcal{M}_{g_2}$. A first important question is:

Question 3.10. Can we compute g_2?

For g between g_1 and g_2, we could expect that, in general, every time we restrict F_g to F_{g-1} the multiplicity of F_g on $\mathcal{M}_g$ increase by one. If this is the case, g_2 should be equal to g_1 plus the multiplicity of F_{g_1} on $\mathcal{M}_{g_1}$.

Let us give the following definition.

Definition 3.11 (Stable divisor). *A stable divisor D is the datum of a divisor D_g on $\mathcal{M}^S_g$ for every g, such that $D_g \cap \mathcal{M}^S_{g-1} = D_{g-1}$.*

The stable modular form F defines a non-trivial stable divisor, call it D. The divisor D_g is not trivial for $g \geq g_2$; since it is defined by a weight n modular forms, its cohomology class is n times the class of the determinant of the Hodge bundle. We can prove that, for some lattices, D_g contains the hyperelliptic locus for every g, see theorem 4.9.

We recall the definition of slope in the case of modular forms, standard references are [Far09] and [CFM12]. Let ϕ be a weight k modular form on $\mathcal{A}^S_g$ and suppose it is not zero on $\mathcal{M}^S_g$. Call a_i the multiplicity of ϕ on the boundary component $\mathcal{M}_i \times \mathcal{M}_{g-i}$ and a_0 its multiplicity on $\mathcal{M}_{g-1}$. (Recall that the boundary components are image of divisors Δ_i, they might have a non-reduced structure, see section 2.1) The slope of the divisor E defined by ϕ is

$$s(E) := \max_i \frac{k}{a_i}$$

If ϕ does not vanish on at least one boundary component, then the slope is ∞.

We want to understand the slope of the divisor D_g defined by $F_g = \Theta_{\Lambda,g} - \Theta_{\Gamma,g}$. On the boundary component $\mathcal{M}_{g-1}$, the value of F_g is F_{g-1}, so the slope of D_g is ∞ for $g > g_2$.

For $g = g_2$, we have the following result.

Lemma 3.12. The slope of D_{g_2} is finite.

Proof. We have to show that F_{g_2} vanishes on all the boundary components of $\mathcal{M}_{g_2}^S$. We know that F_{g_2} is zero on the boundary component $\mathcal{M}_{g_2-1}$, let us check the others. Any Theta series factor as

$$\Theta_{\Lambda,g}(\begin{pmatrix} \tau_i & 0 \\ 0 & \tau_{g-i} \end{pmatrix}) = \Theta_{\Lambda,i}(\tau_i)\Theta_{\Lambda,g-i}(\tau_{g-i}),$$

with τ_i in $\mathfrak{H}_i$. This property is no longer true for a linear combination of Theta series. Nevertheless, when

$$\tau := \begin{pmatrix} \tau_i & 0 \\ 0 & \tau_{g_2-i} \end{pmatrix} \in \mathcal{M}_i \times \mathcal{M}_{g_2-i}$$

we have $F_i(\tau_i) = 0$ (we are using $i < g_2$), so $\Theta_{\Lambda,i}(\tau_i) = \Theta_{\Gamma,i}(\tau_i)$, we conclude that

$$F_{g_2}(\tau) = \Theta_{\Lambda,i}(\tau_i)F_{g_2-i}(\tau_{g_2-i}) = 0.$$

□

The same argument proves that F_{2g_2+1} does not vanish on any boundary component of $\mathcal{M}_{2g_2+1}^S$ (equivalently, on any boundary divisor of $\overline{\mathcal{M}}_{2g_2+1}$).

Let us summarise the result in this way.

Theorem 3.13. *Let Λ and Γ be two even unimodular positive definite lattices, there exist an integer g such that*

$$\Theta_{\Lambda,g} - \Theta_{\Gamma,g}$$

cuts out a divisor of finite slope on $\mathcal{M}_g$.

Both g and the slope depend on Λ and Γ. The open questions are:

Questions 3.14. Can we compute the slope? Does this divisor have a geometrical interpretation?

In [GSM11], these questions are fully answered in the specific case of the Schottky form, see example 2.2.

These problems are also tackled in [GV09]. The ideas behind this paper come from string theory. Indeed, corollary 0.2 should mean that, looking at string of high enough genus, the partition function can distinguish two conformal field theories. The authors think that the values of g_2 could be related to the conformal field theories associated to Γ and Λ, and this could be useful as well to compute the slope.

3.3 Quadrics via degenerations

Let F_{g+1} be a degree $g+1$ modular form, as explained in section 2.2 we can consider its Fourier-Jacobi expansion

$$F_{g+1}(T) = f_0(\tau) + f_1(\tau, z)q + O(q^2)$$

The function f_1 is the first Fourier-Jacobi coefficient of F_{g+1} and it is a section of 2Θ.

Lemma 3.15. Suppose that F_{g+1} vanishes on $\mathcal{M}_{g+1}$, then for every curve C and points a and b we have

$$f_1(\tau, AJ(a-b)) = 0,$$

where τ is the period matrix of C and AJ is the Abel-Jacobi map.

Proof. Let $T(t)$ be the period matrix of the degeneration studied in 1.10. We have $F_{g+1}(T(t)) \equiv 0$, in particular $\frac{d}{dt}F_{g+1}(T(t)) \equiv 0$. Following computations and notations of section 3.1, we write

$$\frac{d}{dt}\,|_{t=0}\, F_{g+1}(T(t)) = A + \gamma_1 B\,.$$

Take N_{g+1} to have degree 0, we have

$$B = f_1(\tau, AJ(a-b))\,.$$

The lemma is now equivalent to corollary 3.4. □

If we let a tend to b, by continuity we get $f_1(\tau, 0) = 0$. We can now use the formula

$$f_1(\tau, a-b) = Q(a,b)E(a,b)$$

where Q is the second order part of f_1 and E is the prime form. (Actually, we should write $Q(dAJ(a), dAJ(b))$, where dAJ is the differential of the Abel-Jacobi map. This differential can be identified with the canonical map.) This formula is classical, for instance it can be found in the proof or proposition 2.1 of [vGBvdGG86] or at the bottom of page 44 of [MV10].

We know that $E(a,b) \neq 0$ for every a different from b; we conclude by continuity that $Q(x,x) = 0$ for every x in C, i.e. Q is a quadric containing the canonical model of C. Let us summarise the result.

Theorem 3.16. *Let F_{g+1} be a degree $g+1$ modular form vanishing on $\mathcal{M}_{g+1}$; for every period matrix τ of a genus g curve C, let $Q(\tau)$ be the second order part of the first Fourier-Jacobi coefficient of F_{g+1} at τ, then $Q(\tau)$ is a quadric containing the canonical model of C.*

4 The hyperelliptic locus

4.1 Projective invariants of hyperelliptic curves

In this section we introduce the projective invariants of a hyperelliptic curve and we prove the following well-known criterion.

Criterion 4.1. *Let F_g be a weight n and degree g modular form. Suppose it vanishes on $\mathcal{A}_{g-1} \cap Hyp_g^S$ with multiplicity at least k. If*

$$\frac{n}{k} < 8 + \frac{4}{g}$$

then F_g vanishes on Hyp_g.

This criterion can be proved using the slope of the hyperelliptic locus, see [CH88] theorem 4.12 or [CFM12] section 3.6. We will rather use projective invariants, references are [Igu67b], [Poo96], [AL02] and [Pas11] Chapter 2.

Let C be a smooth genus g hyperelliptic curve. A point p is called a Weierstrass point if $H^0(C, 2p)$ is not trivial. Equivalently, consider a two to one map π from C to $\mathbb{P}^1$, a point p is a Weierstrass point if and only if π ramifies at p. The map π is unique up to a projective automorphism of $\mathbb{P}^1$, it ramifies at $2g+2$ points. We can thus give the following definition.

Definition 4.2 (Projective invariants)**.** *The projective invariants of C are the image of the Weierstrass points under π, considered up to permutations and projective automorphisms of $\mathbb{P}^1$*

Starting from $2g+2$ distinct points on $\mathbb{P}^1$, one can construct a smooth genus g hyperelliptic curve with the prescribed projective invariants.

Call $\mathcal{B}_g$ the moduli space of $2g+2$ points on $\mathbb{P}^1$, up to permutation and projectivity. This space is a GIT quotient, the semi-stable locus (i.e. the $2g+2$-tuples $\mathcal{B}_g$ parametrises) consist of all the $2g+2$-tuples such that no more than $g+1$ coincide. $\mathcal{B}_g$ can be defined as the $Proj$ of the ring $S(2,2g+2)$. This ring is the ring of symmetric functions in $2g+2$ variables, which are semi-invariant under the natural action of $SL(2,\mathbb{C})$. See the references for more details. The discriminant Δ is an element of $S(2,2g+2)$ of degree $4g+2$, it cuts the divisor D parametrising the $2g+2$-tuples of points where at least two entries coincide.

Because of the previous discussion, we have an isomorphism

$$f_g : Hyp_g \to \mathcal{B}_g \setminus D$$

mapping a curve to its projective invariants. Following [AL02], this map extend to a map

$$f_g : \overline{Hyp}_g \to \mathcal{B}_g$$

where $\overline{Hyp}_g$ is the Deligne-Mumford compactification of Hyp_g. This map is a birational isomorphism between the boundary divisor Ξ_0 and D, it contracts all the other boundary divisors of $\overline{Hyp}_g$ to subvariety of co-dimension greater than 1. The divisor Ξ_0 parametrises curves of compact type, i.e. curves obtained starting with a genus $g-1$ curve C' and gluing two points conjugated under the hyperelliptic involution. Its image is the set of $2g+2$ points of the form $\{p_1,\dots,p_{2g},p,p\}$, the projective invariants of C' are $\{p_1,\dots,p_{2g}\}$, the glued points are the preimage of p.

We can consider the rational inverse of f_g, call $\bar{\rho}$ the composition

$$\bar{\rho} : \mathcal{B}_g \overset{f_g^{-1}}{\dashrightarrow} \overline{Hyp}_g \to Hyp_g^S$$

this map is the geometric version of the Igusa's morphism of projective invariants ρ defined in [Igu67b]. This morphism is a map of graded rings

$$\rho : \bigoplus_{n=0}^{\infty} H^0(\mathcal{A}_g, L^n) \to S(2,2g+2)$$

whose kernel is exactly the ideal of modular forms vanishing on the hyperelliptic locus. The degree of ρ is $\frac{1}{2}g$.

The image of the divisor Ξ_0 in Hyp_g^S is $Hyp_g^S \cap \mathcal{A}_{g-1}^S$, so the image of D under $\bar{\rho}$ is $Hyp_g^S \cap \mathcal{A}_{g-1}^S$. We can now prove the criterion.

Proof. (of criterion 4.1) Suppose F_g vanishes with multiplicity at least k on $Hyp_g^S \cap \mathcal{A}_{g-1}^S$. This means that $\bar{\rho}^* F_g$ vanishes with multiplicity at least k on D. In other words, Δ^k divides $\rho(F_g)$. The degree of the discriminant in $S(2,2g+2)$ is $4g+2$, the degree of $\rho(F_g)$ is $\frac{1}{2}gn$. Since, by hypothesis,

$$k(4g+2) > \frac{1}{2}gn$$

we obtain that $\rho(F_g)$ is equal to zero, so the claim. □

4.2 Transversality

The following result was suggested by variational formulæ 4.11 and 1.15. However, we will not use them in the proof.

Theorem 4.3. *The intersection of $\mathcal{A}_g^S$ and Hyp_{g+1}^S is transverse. In other words, scheme theoretically it is equal to Hyp_g^S.*

Let $I_{Hyp_{g+1}}$ be the ideal of modular forms on $\mathcal{A}_g$ vanishing on Hyp_g. The inclusion

$$\mathcal{A}_g^S \hookrightarrow \mathcal{A}_{g+1}^S$$

is induced by the Siegel operator Φ. We have to prove that the map

$$\Phi : I_{Hyp_{g+1}} \to I_{Hyp_g}$$

is surjective. For technical reasons, we will first prove the theorem on the finite cover defined by the level structure (4,8).

Let us recall few facts about level structures, cf. e.g. [Fre83] II.6. The group $\Gamma(4,8)$ is a normal co-finite subgroup of $\Gamma := Sp(2g,\mathbb{Z})$. Call G the finite quotient. The moduli space $\mathcal{A}_g(4,8)$ is the quotient of the Siegel upper half space by $\Gamma(4,8)$. A point of $\mathcal{A}_g(4,8)$ represent a principally polarised abelian variety with extra structures. Among these extra data, we have an isomorphism ϕ between the subgroup of two torsion elements and $(\mathbb{Z}/2\mathbb{Z})^{2g}$. On $\mathcal{A}_g(4,8)$ there is the ample line bundle L of weight one modular forms, whose sections are holomorphic functions on $\mathfrak{h}_g$ which transform appropriately under the action of $\Gamma(4,8)$. Using this line bundle, we can construct the Satake compactification $\mathcal{A}_g^S(4,8)$ of $\mathcal{A}_g(4,8)$. The boundary is composed by many irreducible components X_i, permuted by G.

For each components X_i, we have a Siegel operator Φ_i.

$$\Phi_i : H^0(\mathcal{A}_g(4,8), L^k) \to H^0(\mathcal{A}_{g-1}(4,8), L^k)$$

which realize an isomorphism between X_i and $A_{g-1}^S(4,8)$. There is a component, say X_0, called the "standard component", where the Siegel operator is given by the usual formula

$$\Phi_0(F)(\tau) := \lim_{t\to\infty} F(\tau \oplus it)$$

The others Siegel operators are obtained letting G act.

The hyperelliptic locus with level structure (4,8) is discussed for instance in [Igu67b] and [SM03]. We recall few facts. The space $Hyp_g(4,8)$ inside $\mathcal{A}_g(4,8)$ is the preimage of Hyp_g under the quotient map. This space splits in many irreducible components Y_j permuted by G. Call $Hyp_g(4,8)^S$ the closure of $Hyp_g(4,8)$ in $\mathcal{A}_g^S(4,8)$. The intersection of $Hyp_g(4,8)^S$ with any of the X_i is, set-theoretically, equal to $Hyp_{g-1}(4,8)^S$. We shall show that the equality is true as scheme.

A way to specify an irreducible component Y_j is to fix a special fundamental system of Theta characteristics $\mathfrak{m} = \{m_0, \ldots, m_{2g+1}\}$. This is a subset of $(\mathbb{Z}/2\mathbb{Z})^{2g}$ with some additional properties, see [SM03] for the definition. The relation between special fundamental system and irreducible components is the following. Call W the set of Weierstrass points of a hyperelliptic curve C. For any w in W, call AJ_w the Abel-Jacobi map with base point w. The set $AJ_w(W)$ is a subset of the 2-torsion subgroup of $Jac(C)$. The choice of a special fundamental system of Theta characteristic $\mathfrak{m}$, determines the component $Y_i = Y_{\mathfrak{m}}$ of abelian varieties $(Jac(C), \Theta, \phi)$ such that there exists a w in W for which $\phi(AJ_w(W)) = \mathfrak{m}$. Call $Y_{\mathfrak{m}}^S$ its closure in $\mathcal{A}_g^S(4,8)$.

Different choice of $\mathfrak{m}$ may determine the same component Y_i, this because of the freedom in the choice of the base point of the Abel-Jacobi map.

Fix a system of Theta characteristic $\mathfrak{m}$, so we have an irreducible component $Y_{\mathfrak{m}}$ of $Hyp_g(4,8)$. Let b be the sum of odd m_i in $\mathfrak{m}$. For every Theta characteristic m, the classical Thetanullerwerte θ_m is a well defined modular form on $\mathcal{A}_g(4,8)$ (but not on $\mathcal{A}_g$, this is the reason why we are using the level structure), see e.g. [Igu67b] or [SM03]. Our proof relies upon the following result.

Theorem 4.4 ([SM03] Theorem 1)**.** *The scheme $Y^S_{\mathfrak{m}}$ is ideal theoretically defined by the vanishing of Thetanullerwerte θ_{m+b} with $m = m_{i_1} + \cdots + m_{i_k}$, where $k \leq g$.*

We still need some more notations. As usual, we write a Theta characteristic as two vectors of size g. Call $\underline{0}$ the g dimensional zero vector. Define two $g+1$ dimensional Theta characteristics

$$p := \begin{bmatrix} \underline{0} & 0 \\ \underline{0} & 1 \end{bmatrix} \quad , \quad q := \begin{bmatrix} \underline{0} & 1 \\ \underline{0} & 1 \end{bmatrix}$$

For every g dimensional Theta characteristic $m = [\epsilon, \epsilon']$, let us define the $g+1$ dimensional Theta characteristic

$$\overline{m} := \begin{bmatrix} \epsilon & 0 \\ \epsilon' & 0 \end{bmatrix},$$

Moreover, for every special fundamental system of g dimensional Theta characteristic $\mathfrak{m}$, we pose

$$\overline{\mathfrak{m}} := p \cup q \cup \bigcup_{m \in \mathfrak{m}} \overline{m}.$$

This is a special fundamental system of $g+1$ dimensional Theta characteristics.

We can think $\mathcal{A}_g(4,8)$ as the standard cusp X_0 of $\mathcal{A}_{g+1}(4,8)^S$.

Lemma 4.5. Scheme theoretically, the intersection of $Y_{\overline{\mathfrak{m}}}$ and X_0 is isomorphic to $Y_{\mathfrak{m}}$.

Proof. By direct computation one sees that

$$\Phi_0(\theta \begin{bmatrix} \epsilon & 0 \\ \epsilon' & \delta \end{bmatrix}) = \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$$

for δ equal either to 0 or 1. Suppose that a g dimensional Theta characteristic m is of the form prescribed by theorem 4.4, for some special fundamental system $\mathfrak{m}$. The modular form θ_{m+b} vanishes on the irreducible component $Y_{\mathfrak{m}}$ of $Hyp_g(4,8)$, and the modular form $\theta_{\overline{m}+\bar{b}+p}$ vanishes on the irreducible component $Y_{\overline{m}}$ of $Hyp_{g+1}(4,8)$. We have

$$\Phi_0(\theta_{\overline{m}+\bar{b}+p}) = \theta_{m+b},$$

so, because of theorem 4.4,

$$Y_{\overline{\mathfrak{m}}} \cap X_0 \subset Y_{\mathfrak{m}}$$

Since the right hand side is irreducible and both terms have the same dimension, we get the equality. □

Proposition 4.6. *The intersection of $Hyp_{g+1}(4,8)^S$ and X_i is transverse for every i.*

Proof. For $i = 0$, the proposition follows from the lemma and the inclusion

$$Hyp_g(4,8) \subset Hyp^S_{g+1}(4,8) \cap X_0.$$

For a general i, it is enough to notice that G acts transitively on the boundary components and preserves the hyperelliptic locus. □

Let $I_{Hyp_g(4,8)}$ be the ideal of $Hyp_g(4,8)$ in $\mathcal{A}_g(4,8)$, by $I_{(Hyp_g(4,8),X_i)}$ we denote the ideal of $Hyp_g(4,8)$ in the boundary component X_i of $\mathcal{A}_{g+1}(4,8)^S$. We have a G equivariant map

$$\bigoplus_i \Phi_i : I_{Hyp_{g+1}(4,8)} \to \bigoplus_i I_{(Hyp_g(4,8),X_i)}$$

This map is surjective, because of the previous proposition. If we take G invariants, we get a map

$$\bigoplus_i \Phi : I_{Hyp_{g+1}} \to \bigoplus_i I_{Hyp_g}$$

which is surjective as well because G is finite and the base field has characteristic zero. We obtain the theorem projecting onto one of the factor.

4.3 Stable equations for the hyperelliptic locus

In this section, we combine the following two results to find stable equations for the hyperelliptic locus.

Theorem 4.7 (= Theorem 4.3)**.** *The intersection of $\mathcal{A}_g^S$ and Hyp_{g+1}^S is transverse.*

Criterion 4.8 (= Criterion 4.1)**.** *Let F_{g+1} be a weight n and degree $g+1$ modular form. Suppose it vanishes on $\mathcal{A}_g \cap Hyp_{g+1}^S$ with multiplicity at least k. If*

$$\frac{n}{k} \leq 8 + \frac{4}{g+1}$$

then F_{g+1} vanishes on Hyp_{g+1}.

We write stable equations as differences of Theta series introduced in section 2. Let Λ and Γ be two even positive definite unimodular lattice, call

$$\mu_\Lambda := \min\{Q(v,v) \mid v \in \Lambda; v \neq 0\}\,.$$

First, we look for a necessary condition. Suppose that the stable modular form

$$\Theta_{\Lambda,g} - \Theta_{\Gamma,g}$$

vanishes on Hyp_g for every g. This, in particular, means that it vanishes on $Hyp_1 = \mathcal{A}_1$, so

$$\Theta_{\Lambda,1} = \Theta_{\Gamma,1},.$$

Looking at the Fourier-Jacobi expansion for $g = 1$, the previous equality means that the two lattices have the same number of vectors of any given norm. In particular we have

$$\mu_\Lambda = \mu_\Gamma\,.$$

Our result is the following.

Theorem 4.9. *Let Λ and Γ be two even positive definite unimodular lattices of rank N and $\mu_\Lambda = \mu_\Gamma =: \mu$, if*

$$\frac{N}{\mu} \leq 8\,,$$

then

$$F := \Theta_\Lambda - \Theta_\Gamma$$

is a stable equation for the hyperelliptic locus. In other words, F_g vanishes on Hyp_g for every g.

Proof. The proof is by induction on g. The difference of two Theta series vanishes on $\mathcal{A}_0$. Suppose the statement true for g, we want to apply criterion 4.8 to F_{g+1}. Call $k := \frac{1}{2}\mu$, we need to prove that F_{g+1} vanishes at the boundary component $\mathcal{A}_g \cap Hyp^S_{g+1}$ with multiplicity at least k.

To stress why theorem 4.7 is important, let us first give the proof when $k = 2$. The argument is local, take a generic point τ of $\mathcal{A}_g \cap Hyp^S_{g+1}$. By induction we know that $F_{g+1}(\tau) = 0$, we want to prove that for every derivative D in $T_\tau Hyp^S_{g+1}$ we have $DF_{g+1}(\tau) = 0$. Since $k = 2$, the Fourier-Jacobi expansion of F_{g+1} looks like

$$F_{g+1} = F_g(\tau) + O(q^2)$$

so $DF_{g+1}(\tau) = DF_g(\tau)$. We can thus assume that D is tangent to $\mathcal{A}_g$. Now, we need to use theorem 4.7 to assume that D is tangent to Hyp_g. By inductive hypothesis F_g is zero on Hyp_g, so $DF_g(\tau) = 0$. (Remark that we can not run the same argument for $\mathcal{M}_g$: because of theorem 0.1, there are plenty of tangent vector to $\mathcal{M}^S_{g+1} \cap \mathcal{A}_g$ which are not tangent to $\mathcal{M}_g$)

For a general k the argument is pretty much the same. Suppose F_{g+1} vanishes on $Hyp^S_{g+1} \cap \mathcal{A}_g$ with order at least s smaller than k, we want to prove it vanishes with order at least $s+1$. In the local ring (Hyp^S_{g+1}, τ), consider the ideal I of elements vanishing on $Hyp^S_{g+1} \cap \mathcal{A}_g$. We know F_{g+1} belongs to I^s, we want to show that its class in I^s/I^{s+1} is trivial. The elements of I^s/I^{s+1} are symmetric s-linear forms on $T_\tau Hyp_{g+1}$, restricting them to $T_\tau(\mathcal{A}_g \cap Hyp^S_{g+1}) = T_\tau Hyp_g$ we get an exact sequence

$$H^0(\tau, 2s\Theta) \to I^s/I^{s+1} \xrightarrow{\Phi} Sym^s(T_\tau Hyp_g^\vee)\,.$$

The class of F_{g+1} is in the kernel of Φ, because by inductive hypothesis $\Phi(F_{g+1})$ vanishes on Hyp_g. Moreover, it is zero in $H^0(\tau, s\Theta)$, because $s < k$, so the conclusion. (We have used theorem 4.7 to replace $T_\tau(\mathcal{A}_g \cap Hyp^S_{g+1})$ with $T_\tau Hyp_g$.) □

The hypothesis

$$\frac{rk(\Lambda)}{\mu_\Lambda} \leq 8 \tag{1}$$

is quite restrictive. Indeed, given any even unimodular lattice Λ, there is a bound

$$\mu_\Lambda \leq 2\lfloor\frac{rk(\Lambda)}{24}\rfloor + 2$$

where "$\lfloor\,\rfloor$" is the round down (see [CS99] section 7.7 corollary 21); moreover μ is even and the rank, if Λ is positive definite, is divisible by 8. We conclude that if an even unimodular positive definite lattice Λ satisfies hypothesis 1, then the only possibilities for the couple $(rk(\Lambda), \mu_\Lambda)$ are (8,2),(32,4) and (48,6). All these lattices are **extremal**, which means

$$\mu_\Lambda = 2\lfloor\frac{rk(\Lambda)}{24}\rfloor + 2\,.$$

On the other hand, given two extremal lattices Λ and Γ, it is not true that $\Theta_{\Lambda,g} - \Theta_{\Gamma,g}$ vanishes on the hyperelliptic locus for every g. See [Oze88] for an example of 3 extremal lattices of rank 40 whose Theta series are different for $g = 2$. In the proof of theorem 4.9 we have not used the hypotesis $\mu_\Lambda = \mu_\Gamma$. However, as we have seen, hypoteses 1 and $rk(\Lambda) = rk(\Gamma)$ imply this fact.

There exist only two lattices of rank 8 and $\mu = 2$ (see example 2.2). A vector of norm 2 is sometime called a root, so a lattice of rank 32 and $\mu = 4$ is called a lattice without roots. In [Kin03] corollary 5, using a generalization of the mass formula, is shown that there are at least ten millions of lattices of type (32,4) (in this paper every lattice is tacitly assumed to be positive definite). The situation for lattices of type (48,6) is not clear, believably there exist many of them, see [Kin03] page 15. In any case, we can claim the following theorem.

Theorem 4.10. *There are more than 10000000 of linearly independent differences of Theta series vanishing on the hyperelliptic locus for every g.*

To find stable equations of higher weight, one probably needs a better understanding of the formal neighbourhood of Hyp_g in Hyp^S_{g+1}. Another possibility is to look for Theta series with the same first Fourier-Jacobi coefficients.

We would like to draw a comparison between $\mathcal{M}_g$ and Hyp_g. We know that stable equations for $\mathcal{M}_g$ do not exist, so the strategy we have used to find stable equations for Hyp_g can not work for $\mathcal{M}_g$. The main reason should be that theorem 4.7 does not hold for $\mathcal{M}_g$, see theorem 0.1. On the other hand, it is not known if an analogue of criterion 4.8 holds for $\mathcal{M}_g$. As we have seen, its proof relies upon a deep understanding of the projective invariants of the hyperelliptic locus, and we do not have any analogue neither of Igusa's morphism nor of Thomæ's formula for $\mathcal{M}_g$. This criterion is also equivalent to the fact that we have a positive lower bound on the slope of Hyp_g which does not depend on g, and this slope can be computed looking exclusively at the divisor Ξ_0.

4.4 Variational formula for a family of hyperelliptic curves

In this section, we compute explicitly a part of the period matrix $T(t)$ of the following family of genus $g+1$ hyperelliptic curves

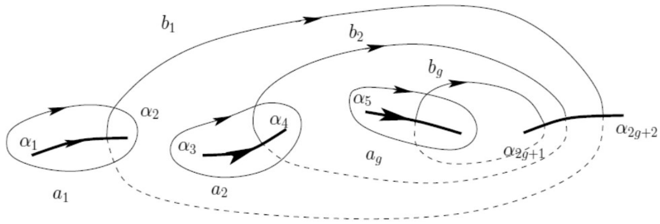
$$\mathcal{C}_t := \{y^2 = (x^2 - t) \prod_{i=0}^{2g+2} (x - p_i)\},$$

where p_i's are distinct complex numbers different from zero, and t belongs to a small disc Δ_t around the origin. The hyperelliptic involution ι maps (x, y) to $(x, -y)$. Call C the normalisation of $\mathcal{C}_0$, its equation is

$$C := \{y^2 = \prod_{i=0}^{2g+2} (x - p_i)\}.$$

Let p and $-p$ be two preimage of the singular point of $\mathcal{C}_0$, in co-ordinates $p = (0, \tilde{y})$ and $-p = (0, -\tilde{y})$, for a suitable $\tilde{y}$.

First, we must fix a symplectic basis for the homology of $H_1(\mathcal{C}_t, \mathbb{Z})$. A choice of an ordering of the Weierstrass points gives a standard choice $A_i(t), B_i(t)$ for the basis, as the following picture shows



In the picture, α_i denotes the image of the Weierstrass point on $\mathbb{P}^1$, bold lines are the classical "cuts" one performs to re-construct the curve (see [Ber] page 79). We choose as first two Weierstrass points the moving ones.

The hyperelliptic involution acts as -1 on the homology. The cycles $A_2(t), \ldots, A_{g+1}(t)$ and $B_2(t), \ldots B_{g+1}(t)$ give a symplectic basis for the homology of C. Call v_i the normalised basis for the holomorphic differentials on C and τ he period matrix.

The function x gives a local co-ordinate around p, call it z_p

Theorem 4.11. *Whit respect to the previous symplectic basis for the homology, we have*

$$T(t) = \begin{pmatrix} \frac{1}{2\pi i} ln(t) + c_0 & AJ(p - \iota(p)) \\ AJ(p - \iota(p)) & \tau \end{pmatrix} + t \begin{pmatrix} c_1 & \cdots \\ \vdots & \sigma(p, z_p) \end{pmatrix} + O(t^2),$$

where AJ is the Abel-Jacobi map and $\sigma(p, z_p)$ a holomorphic function of p and z_p given by

$$\sigma(p, z_p)_{ij} = 2\pi i \frac{v_i(p) v_j(p)}{dz_p^2}.$$

Remark 4.12. This is the same period matrix of theorem 1.10, when $a = p$, $b = \iota(p)$ and z_a and z_b are given by x.

A basis of the relative holomorphic differential on $\mathcal{C}_t$ (not dual to $\{A_i(t), B_i(t)\}$) is

$$\omega_i(x, t) := \frac{x^{i-1}}{y(t)} dx \qquad i = 1, \ldots, g + 1.$$

When $t = 0$, we consider the pull-back of ω_i on C: for $i = 1$ we get a meromorphic differential with simple poles on p and $-p$, for $i > 1$ we get a basis for the holomorphic differentials over C.

Let ∇ be the Gauss-Manin connection (which is nothing but $\frac{d}{dt}$). The key computation we will need is the following one.

Lemma 4.13. Keep notation as above,

$$\nabla \omega_j(x, 0) = \frac{1}{2x^2} \omega_j(x, 0)$$

Proof. Differentiating the equation that defines the family $\mathcal{C}_t$ we get

$$2y(0) \frac{dy}{dt}(0) = - \prod_{i=0}^{2g+2} (x - p_i),$$

moreover

$$\nabla \omega_j(x, 0) = - \frac{1}{y(0)^2} \frac{dy}{dt}(0) x^{i-1} dx.$$

The lemma follows using the equation $y(0)^2 = x^2 \prod (x - p_i)$. □

Because of proposition 1.1, there exist holomorphic forms $u_i(x, t)$ belonging to $\Omega^1_{\mathcal{C}/\Delta_t}$ such that, for $t \neq 0$, they form a normalised basis for the holomorphic differential of $\mathcal{C}_t$ with respect to $\{A_i, B_i\}$. For $t = 0$, the pull-back of $u_2, \ldots, u_{g+1}$ is the normalised basis $\{v_i\}$, and the pull back of u_1 is a meromorphic differential with only simple poles at p and $-p$, with residues respectively 1 and -1. For suitable holomorphic functions c_{ij} defined over Δ_t, we have

$$u_i(x, t) = \sum_j c_{ij}(t) \omega_j(x, t).$$

As usual, we will deal with poles of order two and zero residues. Let η_p be a normalised differential of the second kind with a pole of order 2 and zero residue at p, see 1.3. Its behaviour with respect to ι is the following.

$$\eta_{\iota(p)} = \iota^* \eta_p$$

$$\int_{B_i} \iota^* \eta_p = \int_{\iota(B_i)} \eta_p = -\int_{B_i} \eta_p \,.$$

Call $\nu : C \to \mathcal{C}_0$ the normalisation map.

Lemma 4.14. For $i > 1$, we have

$$\nu^* \nabla u_i(x, 0) = \frac{1}{2} \frac{u_i(p, 0)}{dx} (\eta_p(x) - \eta_{\iota(p)}(x)) \,,$$

where η_p has leading coefficient 1 with respect to the local co-ordinate x.

Proof. The expansion of u_i around $t = 0$ is

$$u_i(x, t) = u_i(x, 0) + t[\sum_j \frac{dc_{ij}}{dt}(0)\omega_j(0) + \frac{1}{2x^2} u_i(x, 0)] + O(t^2) \,.$$

Arguing as in the proof of 1.10, we prove that the residue of $\nu^* \nabla u_i(x, 0)$ at p and $-p$ is zero, so we have the expansion

$$\nu^* \nabla u_i(x, 0) = u_i(p, 0) \frac{1}{2x^2} dx + O(1) \quad \text{around}\, p,$$

and

$$\nu^* \nabla u_i(x, 0) = u_i(-p, 0) \frac{1}{2x^2} dx + O(1) \quad \text{around} \, -p,$$

where the differential u_i is evaluated using x as local co-ordinate. The involution acts as -1 on the abelian differentials, so

$$\nu^* u_i(-p, 0) = \nu^* u_i(\iota(p), 0) = -\nu^* u_i(p, 0) \,.$$

Being $u_i(x, t)$ normalised, we have

$$\int_{A_j(t)} u_i(x, t) = \delta_{ij} \,,$$

the right hand side does not depend on t, so

$$\int_{A_j} \nu^* \nabla u_i(x, 0) = 0 \,.$$

Because of the uniqueness of η_p we obtain the statement. □

We obtain σ using Riemann's bilinear relations 1.4.

We compute the monodromy of $T_{11}(t)$. If we rotate around the origin in Δ_t, the two Weierstrass points associated to $\sqrt{t}$ and $-\sqrt{t}$ are swapped: so the class of A_1 does not changed but the class of B_1 is increased by $\pm A_1$. We conclude that $T_{11}(t) = \frac{1}{2\pi i} ln(t) +$holomorphic function, see also [ACG11] page 143.

We consider the first row minus the element T_{11}. The pull-back of $B_1(0)$ over C is a path from p to $\iota(p)$, so we obtain the image of $p - \iota(p)$ under the Abel-Jacobi map. This is also a consequence of the corollary 10.4 page 258 of [Spr57]. This concludes the proof of theorem 4.11.

Variational formulæ 4.11 and 1.15 suggest a description of the normal bundle of Hyp_g in Hyp^S_{g+1}. Call X the Jacobian of C, we define a map Ψ as the composition:

$$\begin{array}{ccccccc} \Psi: & C & \xrightarrow{f} & C \times C & \xrightarrow{\delta} & X & \xrightarrow{|2\Theta|} & \mathbb{P}H^0(X, 2\Theta)^\vee \\ & p & \mapsto & (p, \iota(p)) & & & & \\ & & & (a,b) & \mapsto & AJ(a-b) & & \end{array}$$

This definition does not depend on the choice of the base point for the Abel-Jacobi map, the image of C is a rational curve, it contains 0_X with multiplicity $2g+2$.

We think that, at a general point, the tangent cone at of Hyp^S_{g+1} at C is the affine cone over $\Psi(C)$. We can prove that this cone spans a $2g$ dimensional vector space.

5 Prym varieties

In this section, we study degenerations of Prym varieties analogue to the degenerations studied in section 1.2.

5.1 Definitions and notations

We refer mainly to chapter 12 of [BL92], [ACGH85] appendix C or [Far12]. Prym varieties arise from double covers $\pi : \tilde{C} \to C$, unramified or branched at two points. Let g be the genus of C, the genus of $\tilde{C}$ is $2g+1$ if the cover is étale, $2g$ otherwise. The Prym variety associated to π can be defined as follows:

$$Pr(\tilde{C}/C) := H^1(\tilde{C}, K_{\tilde{C}})^{\vee -}/H^1(\tilde{C}, \mathbb{Z})^-$$

where "minus" indicates the minus one eigenspace of the involution ι of the cover. The differentials $H^1(\tilde{C}, K_{\tilde{C}})^-$ are called Prym differentials.

We describe a well-known way to compute periods of Prym (see e.g. Example 2.1 of [Far12]). Fix a symplectic basis $\tilde{A}_0, \tilde{B}_0, A_1^+, A_1^-, \ldots, A_g^+, A_g^-, B_1^+, B_1^- \ldots, B_g^+, B_g^-$ for the homology of $\tilde{C}$ with the following properties. The involution swaps A_i^+ with A_i^- and B_i^+ with B_i^-. The cycles $\tilde{A}_0$ and $\tilde{B}_0$ are present just in the unramified case and are fixed by the involution. Call A_i and B_i the cycles $\iota_*(A_i^+)$ and $\iota_*(B_i^+)$, and A_0 and B_0 half of $\iota_*(A_0)$ and $\iota_*(B_0)$. These cycles form a symplectic basis for the homology of C. Let $\omega_i^\pm$ (respectively ω_i) be the corresponding normalised basis for the holomorphic differentials on $\tilde{C}$ (C).

A symplectic basis for $H_1(\tilde{C}, \mathbb{Z})^-$ is given by $A_i^+ - A_i^-, B_i^+ - B_i^-$, for i from 1 to g. The corresponding basis for $H^1(\tilde{C}, K_{\tilde{C}})^-$ is given by

$$w_i := \frac{1}{2}(\omega_i^+ - \omega_i^-)\,,$$

so the periods of Prym differentials are

$$\frac{1}{2}\int_{B_i^+ - B_i^-} \omega_j^+ - \omega_j^- \qquad i,j = 1, \ldots, g\,.$$

We use the same notations for families. Given a one parameter family of double covers $\pi : \tilde{\mathcal{C}}_t \to \mathcal{C}_t$, we call $P(t)$ the corresponding family of Prym, and the periods are

$$P(t) = \frac{1}{2}\int_{B_i^+(t) - B_i^-(t)} \omega_j(t)^+ - \omega_j^-(t) \qquad i,j = 1, \ldots g\,.$$

By abuse of notations, we denote by $P(t)$ at the same time the Prym varieties and the matrices of periods of the Prym differentials. The local expansion of the period around $t = 0$ is

$$P(t) = \frac{1}{2}\int_{B_i^+(0)-B_i^-(0)} (\omega_j(0)^+ - \omega_j^-(0)) + \frac{1}{2}t\int_{B_i^+(0)-B_i^-(0)} \nabla(\omega_j(t)^+ - \omega_j^-(t)) \mid_{t=0} + O(t^2)\,,$$

where ∇ is the Gauss-Manin connection.

5.2 The étale case

We denote by $\mathcal{P}_g$ the coarse moduli space of Prym arising from étale double covers. The closure of $\mathcal{P}_g$ in $\mathcal{A}_g$ is its Satake compactification $\mathcal{P}_g^S$.

Pinching a non-trivial homological cycle We are going to prove the analogue result of 0.1. The intersection of $\mathcal{P}_{g+m}^S$ and $\mathcal{A}_g$ contains $\mathcal{P}_g$, we can give the following description.

Theorem 5.1. *The intersection of $\mathcal{P}_{g+m}^S$ and $\mathcal{A}_g$ contains the m-th infinitesimal neighbourhood of $\mathcal{P}_g$ in $\mathcal{A}_g$.*

The strategy of the proof is the same. We start with a 2 to 1 unramified cover $\tilde{C} \to C$, where C is automorphisms-free, call g the genus of C. We pick points a and b and local co-ordinates on C and we pull them back on $\tilde{C}$: call the preimages a^+, a^-, b^+ and b^-. We perform the construction of section 1.2 simultaneously on C and $\tilde{C}$, so we get a family of covers $\tilde{\mathcal{C}}_t \to \mathcal{C}_t$ over a disc Δ_t degenerating to $\tilde{C}/(a^+ \sim b^+, a^- \sim b^-) \to C/a \sim b$. Call $P(t)$ the corresponding family of Prym varieties.

Fix basis for the homology as in section 5.1. There are two vanishing cycles on $\tilde{\mathcal{C}}_t$: $A_{g+1}^+(t)$ and $A_{g+1}^-(t)$. They are swapped by the involution.

The pull back of $w_1(0), \ldots, w_{g-1}(0)$ is a basis for the Prym differentials of $\tilde{C}$.

Proposition 5.2 ([FS86] Section 2)**.** *Keep notation as above,*

$$P(0) = \begin{pmatrix} Pr(\tilde{C}/C) & AP(a-b) \\ {}^tAP(a-b) & \frac{1}{2\pi i}ln(t) + c_0 \end{pmatrix}$$

where $AP(a-b)$ is the Abel-Prym map.

Proof. The logarithm is due to the monodromy of the integral of $w_g(t)$ over $B_g^+(t) - B_g^-(t)$: turning around the origin of the disc Δ_t, $B_g^\pm$ is increased by $A_g^\pm$: the integral of $w_g(0)$ on $A_g^+ - A_g^-$ is 1, so we can write the entry $P(t)_{g,g}$ as $\frac{1}{2\pi i}ln(t)$ plus some holomorphic function.

Let us compute the rest of the first row. We have to integrate w_i on $B_1^+ - B_1^-$, for $i = 1, \ldots, g-1$. To do this, we pull back w_i on $\tilde{C}$, and take the difference of the integral from a^+ to b^+ and from a^- to b^-. This is nothing but the Abel-Prym map.

The biggest block of the matrix comes from pulling everything back to $\tilde{C}$. □

Proposition 5.3. *The period matrix of $P(t)$ is*

$$P(t) = \begin{pmatrix} Pr(\tilde{C}/C) & AP(a-b) \\ {}^tAP(a-b) & \frac{1}{2\pi i}ln(t) + c_0 \end{pmatrix} + t\begin{pmatrix} \sigma & \cdots \\ \vdots & c_1 \end{pmatrix} + O(t^2)$$

where σ is a holomorphic function of the parameters given by

$$\sigma_{ij} = 2\pi i(w_i(a^+) - w_i(b^+))(w_j(a^+) - w_j(b^+))\,.$$

The differentials are evaluated in term of the pull backs of dz_a and dz_b

Proof. The zero order term comes from the previous proposition. Call

$$\nu : \tilde{C} \to \tilde{\mathcal{C}}_0$$

the normalisation map. To compute σ we apply lemma 1.2 to $\nu^* \nabla w_i(0)$, for $i = 1, \dots, g-1$. We obtain

$$\nu^* \nabla w_i(0) = ((w_i(a^+) - w_i(b^+))(\eta_{a^+} - \eta_{b^+}) + (w_i(a^-) - w_i(b^-))(\eta_{a^-} - \eta_{b^-}) .$$

Recall that

$$w_i(a^-) = -w_i(a^+) \qquad w_i(b^-) = -w_i(b^+) .$$

The proposition follows from Riemann's bilinear relations 1.4.

□

The tangent space to $P(0)$ at the origin is $H^0(C, K_C + \eta)^\vee$, where η is the theta characteristic of the cover $\pi : \tilde{C} \to C$. Because of the heat equation, the tangent space $T_{Pr(\tilde{C}/C)}\mathcal{A}_{g-2}$ is isomorphic to $Sym^2 H^0(C, K_C + \eta)^\vee$, In a co-ordinates free way, σ is given by

$$\sigma(w) = 2\pi i \left(\frac{w}{\pi^* dz_a}(a^+) - \frac{w}{\pi^* dz_b}(b^+)\right)^2 .$$

Recall that Prym differentials can be interpreted at the same time as differentials on $\tilde{C}$ or as sections of a line bundle over C. We can interpret the projectivization of σ as follows. Consider the map

$$C \xrightarrow{K_C+\eta} \mathbb{P}H^0(K_C + \eta)^\vee \xrightarrow{Ver_2} \mathbb{P}Sym^2 H^0(K_C + \eta)^\vee$$

Then σ is the image of a point on the secant variety of C under Ver_2. The image of C via $K_C + \eta$ is, by definition, the Prym canonical model of the curve. If $Cliff(C) \geq 3$ this line bundle is very ample ([SV02] page 10), so generically it is.

Proposition 5.4. *Varying the choice of a, b, z_a, and z_b, σ spans all $T_{Pr(\tilde{C}/C)}\mathcal{A}_{g-2}$.*

Proof. In order to apply the argument of the last part of section 3.1, we just need to know that the image of C is a one dimensional variety not contained in any hyperplane of $\mathbb{P}H^0(K_C + \eta)^\vee$. □

Now, we can use exactly the same argument of section 3.1. We suppose that a modular form F_{g-1} vanishes along $\mathcal{P}_{g-1}$ with multiplicity at least k, we take its Fourier expansion, we restrict its derivatives to P_t, and we can apply the previous proposition in order to prove theorem 5.1.

Pinching a homologically trivial cycle As we will see, the moduli space $\mathcal{M}_g$ is contained in $\mathcal{P}_g^S$ (cf. [BL92] page 376), we want to prove the following theorem.

Theorem 5.5. *The Stake compactification $\mathcal{P}_g^S$ contains the first infinitesimal neighbourhood of $\mathcal{M}_g$ in $\mathcal{A}_g$. In other words, let C be a point of $\mathcal{M}_g$, we have $T_C\mathcal{P}_g^S = T_C\mathcal{A}_g$.*

This result could be related to the Schottky-Jung relations. Indeed, these relations, morally, relate the vanishing on $\mathcal{M}_{g+1}$ to the vanishing on $\mathcal{P}_g$, and we know that both $\mathcal{M}_{g+1}^S$ and $\mathcal{P}_g^S$ contain the first infinitesimal neighbourhood of $\mathcal{M}_g$.

To prove the theorem, we compute a variational formula for the family described in [BL92] page 376. First, we describe the central fibre of the family.

We start with a smooth curve C of genus g and automorphisms-free. We pick two points a and b, call $\mathcal{C}_0$ the curve $C/a \sim b$. We construct an unramified 2 to 1 cover $\tilde{\mathcal{C}}_0 \to \mathcal{C}_0$ taking two copies of C, call them C^+ and C^-, and gluing a^+ with b^-, obtaining a node n_1, and a^- with b^+, obtaining a node n_2. Let $P(0)$ be the associated Prym.

Lemma 5.6. The Prym variety $P(0)$ is isomorphic to the Jacobian of C.

Proof. Keep notations as in section 5.1. On $\tilde{\mathcal{C}_0}$ we have $\int_{B_j^+} \omega_i^- = \int_{B_j^-} \omega_i^+ = 0$, so

$$\frac{1}{2}\int_{B_j^+ - B_j^-} \omega_i^+ - \omega_i^- = \int_{B_j} \omega_i$$

□

We construct a family of Pryms degenerating to $P(0)$. Pick local co-ordinates z_a and z_b on C. Using the procedure described in section 1.2, we can obtain a family $\tilde{\mathcal{C}_t} \to \mathcal{C}_t$ of covers degenerating to $\tilde{\mathcal{C}_0} \to \mathcal{C}_0$. Let $P(t)$ be the corresponding family of Prym. This degeneration together with lemma 5.6 show that $\mathcal{M}_g$ is contained in $\mathcal{P}_g^S$.

Fix a basis for the homologies of $\tilde{\mathcal{C}_t}$ and $\mathcal{C}_t$ as in section 5.1. Remark that both $A_0(t)$ and $\tilde{A}_0(t)$ are vanishing cycles. Since $P(0)$ is isomorphic to the Jacobian of C, we denote by τ the period matrix of C and the period matrix of $P(0)$.

Proposition 5.7. *The period matrix of* $P(t)$ *is*

$$P(t) = \tau + t\sigma + O(t^2)\,,$$

where σ *is a holomorphic function of* a*,* b*,* z_a *and* z_b *given by*

$$\sigma_{ij} = 2\pi i((\omega_i(a) - \omega_i(b)(\omega_j(a) - \omega_j(b))\,.$$

The differentials ω_i *are a normalised basis for the abelian differentials of* C*, they are evaluated in term of* dz_a *and* dz_b*.*

Remark 5.8. This is the same σ we get in 1.10.

Proof. The zero order term has already been computed.

Call

$$\nu : C^+ \sqcup C^- \to \tilde{\mathcal{C}_0}$$

the normalisation map. Keep notations as in section 5.1. We have to identify the differentials $\nu^*\nabla\omega_i^{\pm}$. We use lemma 1.2. The differentials ω_i^+ live on C^+, and we have

$$\nu^*\omega_i^+(a^+) = \omega_i(a)\ ,\ \nu^*\omega_i^+(a^-) = 0\,,$$
$$\nu^*\omega_i^+(b^+) = \omega_i(b)\ ,\ \nu^*\omega_i^+(b^-) = 0\,.$$

Loops around $a^{\pm}$ and $b^{\pm}$ correspond to homologically trivial cycles on $\tilde{\mathcal{C}_0}$, so the residues of $\nu^*\nabla(\omega_i^{\pm})(0)$ at $a^{\pm}$ and $b^{\pm}$ are zero. Moreover, we do not have any other pole. The integrals along the cycles $A_i^{\pm}$ are zero, because of the usual argument, see the proof of 1.10. We conclude that on C^+

$$\nu^*\nabla\omega_i^+ = \frac{1}{4}(\omega_i(a)\eta_a + \omega_i(b)\eta_b)\,,$$

and on C^-

$$\nu^*\nabla\omega_i^+ = -\frac{1}{4}(\omega_i(b)\eta_a + \omega_i(a)\eta_b)\,.$$

Arguing as above, we obtain on C^+

$$\nu^*\nabla\omega_i^- = -\frac{1}{4}(\omega_i(b)\eta_a + \omega_i(a)\eta_b)\,,$$

and on C^-

$$\nu^*\nabla\omega_i^- = \frac{1}{4}(\omega_i(a)\eta_a + \omega_i(b)\eta_b)\,.$$

The result follows from Riemann's bilinear relations 1.4. □

Theorem 5.5 follows from the following result and lemma 3.6.

Proposition 5.9. *Varying the choice of* a*,* b*,* z_a*, and* z_b*,* σ *spans all* $T_C\mathcal{A}_{g-1}$*.*

The proof of this proposition is in section 3.1.

Schiffer's variations We conclude with a variational formula for Schiffer's variations. Consider a 2 to 1 unramified cover $\pi : \tilde{C} \to C$, call $P(0)$ the period matrix of the associated Prym. Pick a point p on C and let $\{p^+, p^-\}$ be $\pi^{-1}(p)$. Perform simultaneously a Schiffer variation at (p, z_p), $(p^+, \pi^* z_p)$ and $(p^-, \pi^* z_p)$. We get a family of covers $\pi : \tilde{\mathcal{C}}_t \to \mathcal{C}_t$ over a disc Δ_t.

Theorem 5.10. *The Prym* $P(t)$ *of the family constructed above is*

$$P(t) = P(0) + t\sigma(p, z_p) + O(t^2)\,,$$

where

$$\sigma_{ij} = 2\pi i \frac{w_i(p) w_j(p)}{dz_p^2}$$

Proof. The proof is as in theorem 1.12. The only difference is that $\nabla\omega_i(0)$ will be a meromorphic differential with two poles of order two without residues at p^+ and p^-, with appropriate leading coefficients. □

The tangent space $T_{P(0)}\mathcal{A}_g$ is $Sym^2 H^0(K_C + \eta)^\vee$, so σ is given by

$$\sigma(p, z_p)(\omega) = 2\pi i (\frac{\omega}{dz_p}(p))^2\,.$$

The projectivization of $\sigma(p, z_p)$ does not depend on z_p, it is the image of p under the composition

$$C \xrightarrow{K_C+\eta} \mathbb{P}H^0(K_C + \eta)^\vee \xrightarrow{Ver_2} \mathbb{P}Sym^2 H^0(K_C + \eta)^\vee\,.$$

The moduli space of étale double covers is a finite cover of $\mathcal{M}_g$, so its tangent space is generated by Schiffer's variations. We have the following corollary

Corollary 5.11. *The tangent space of* $\mathcal{P}_g$ *at the Prym variety of the cover* $\tilde{C} \to C$ *is generated by the affine cone over the image of the map*

$$C \to \mathbb{P}H^0(K_C + \eta)^\vee \xrightarrow{Ver_2} \mathbb{P}Sym^2 H^0(K_C + \eta)^\vee\,.$$

The line bundle $K_C + \eta$ is very ample if the Clifford index of C is greater than or equal to 3. An analysis of the Prym canonical model is carried out in [Deb89].

Question 5.12. We do not know a refined version of local Torelli problem for Prym. In the case of hyperelliptic curve, we are thinking about the paper [OS80].

A survey about Torelli's problems for Prym is [SV02].

5.3 The ramified case

Let ${}^r\mathcal{P}_g$ be the coarse moduli space of Prym arising from double covers branched at two points. As usual, ${}^r\mathcal{P}_g^S$ will denote its closure in $\mathcal{A}_g^S$.

Pinching a non-trivial homological cycle We start with a two to one cover $\tilde{C} \to C$ ramified at two points. We picks two points (distinct from the branch locus) and two local-co-ordinates on C, we pull them back on $\tilde{C}$, and we perform the usual surgery. The construction and the computation go exactly as in 5.3, and we get the following theorem.

Theorem 5.13. *The intersection of* ${}^r\mathcal{P}^S_{g+m}$ *and* $\mathcal{A}_g$ *contains the* m*-th infinitesimal neighbourhood of* ${}^r\mathcal{P}_g$ *in* $\mathcal{A}_g$.

Pinching a trivial homological cycle As for the étale case, the moduli space $\mathcal{M}_g$ is a boundary component of ${}^r\mathcal{P}^S_g$. We want to study the singularity. Fix a genus g automorphisms-free curve C and a point p. Take two copies of (C,p), call them (C^+,p^+) and (C^-,p^-). Let $\tilde{\mathcal{C}}_0$ the curve obtained gluing p^+ and p^-, call n the node. The curve $\tilde{\mathcal{C}}_0$ is a double cover of C. Fix a local co-ordinate z_p at p. Using this co-ordinate we construct a family $\tilde{\mathcal{C}}_t$ degenerating at $\tilde{\mathcal{C}}_0$. The involution on $\tilde{\mathcal{C}}_0$ extends to an involution of the entire family, so $\tilde{\mathcal{C}}$ is a double cover of a family $\mathcal{C}_t$ with central fibre isomorphic to C. (We have not an explicit description of $\mathcal{C}_t$.) Call $P(t)$ the corresponding family of Prym.

Lemma 5.14. The Prym variety $P(0)$ is isomorphic to the Jacobian of C.

Proof. Fix basis for the homology as in section 5.1, on the central fibre we have $\int_{B_i^-}\omega_j^+ = \int_{B_i^+}\omega_j^- = 0$, so

$$\frac{1}{2}\int_{B_i^+-B_i^-}\omega_j^+ - \omega_j^- = \int_{B_i}\omega_j$$

□

Fix a basis for the homologies of $\tilde{\mathcal{C}}_t$ and $\mathcal{C}_t$ as in section 5.1. We do not have any vanishing cycle. Since $P(0)$ is isomorphic to the Jacobian of C, we denote by τ the period matrix of C and the period matrix of $P(0)$.

Proposition 5.15. *The periods of the Prym differentials of the family* $\tilde{\mathcal{C}}_t \to \mathcal{C}$ *are*

$$P(t) = \tau + t\sigma(p,z_p) + O(t^2)\,,$$

where σ *is given by*

$$\sigma(p,z_p)_{ij} = 2\pi i\frac{\omega_i(p)\omega_j(p)}{dz_p^2}\,.$$

The ω_i *are normalized abelian differential on* C.

Proof. Call

$$\nu : C^+ \sqcup C^- \to \tilde{\mathcal{C}}_0$$

the normalisation map. As usual, we want to apply lemma 1.2 to the differentials $\nu^*\nabla w_i$. All the differentials are evaluated with respect to dz_p. We have

$$\nu^*\omega_i^+(p^+) = \omega_i(p)\ ,\ \nu^*\omega_i^+(p^-) = 0\,,$$

so on C^+

$$\nu^*\nabla\omega_i^+ = \frac{1}{2}\omega_i(p)\eta_p$$

and on C^-

$$\nu^*\nabla\omega_i^+ = -\frac{1}{2}\omega_i(p)\eta_p\,.$$

Moreover

$$\nu^*\nabla\omega_i^- = -\nu^*\nabla\omega_i^+\,.$$

The statement follows from Riemann's bilinear relations 1.4. □

The matrix $\sigma(p, z_p)$ is the tangent vector to a Schiffer variation of C at (p, z_p), see proposition 1.12. To obtain a more general degeneration to $P(0)$, the only thing we can do is to perform some Schiffer's variations on C away from p, and pull them back to $\widetilde{C}$. The tangent vector we get in this way is a linear combination of Schiffer's variations, so we have the following result.

Theorem 5.16. *Let C be a generic point of $\mathcal{M}_g$, then*

$$(T^r_C \mathcal{P}^S_g)^{arc} = T_C \mathcal{M}_g \,,$$

where $(T^r_C \mathcal{P}^S_g)^{arc}$ is the subspace of $T^r_C \mathcal{P}^S_g$ spanned by vectors coming from arcs.

This situation reminds the local Torelli's theorem for hyperelliptic curves, Torelli problem in the ramified case is discussed in [MP12].

References

[ACG11] E. Arbarello, M. Cornalba, and P. A. Griffiths. *Geometry of algebraic curves. Volume II*, volume 268 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011. With a contribution by J. D. Harris.

[ACGH85] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. *Geometry of algebraic curves. Vol. I*, volume 267 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.

[AL02] D. Avritzer and H. Lange. The moduli spaces of hyperelliptic curves and binary forms. *Mathematische Zeitschrift*, 242(4):615–632, 2002.

[Ber] M. Bertola. Riemann surfaces and theta function. http://www.mathstat.concordia.ca/faculty/bertola/.

[BHPVdV04] W. P. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven. *Compact complex surfaces*. Springer-Verlag, Berlin, second edition, 2004.

[BL92] C. Birkenhake and H. Lange. *Complex abelian varieties*, volume 302 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[BvdGHZ08] J. H. Bruinier, G. van der Geer, G. Harder, and D. Zagier. *The 1-2-3 of modular forms*. Universitext. Springer-Verlag, Berlin, 2008.

[CFM12] D. Chen, G. Farkas, and I. Morrison. *Effective divisors on moduli spaces of curves*. 2012. arXiv:1205.6138v2.

[CH88] M. Cornalba and J. Harris. Divisor classes associated to families of stable varieties, with applications to the moduli space of curves. *Ann. Sci. École Norm. Sup. (4)*, 21(3):455–475, 1988.

[CS99] J. H. Conway and N. J. A. Sloane. *Sphere packings, lattices and groups*, volume 290 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1999.

[CSB11] G. Codogni and N. Shepherd-Barron. The abscence of stable schottky forms. *arXiv:1112.6137v2*, 2011.

[CvdG00] C. Ciliberto and G. van der Geer. The moduli space of abelian varieties and the singularities of the theta divisor. In *Surveys in differential geometry*, Surv. Differ. Geom., VII, pages 61–81. Int. Press, Somerville, MA, 2000.

[Deb89] O. Debarre. Sur le probleme de Torelli pour les varieties de Prym. *Amer. J. Math.*, 111(1):111–134, 1989.

[Far09] G. Farkas. *Birational aspects of the geometry of* $\overline{\mathcal{M}}_g$, volume 14 of *Surv. Differ. Geom.*, pages 57–110. Int. Press, Somerville, MA, 2009.

[Far12] G. Farkas. Prym varieties and their moduli. In *Contributions to algebraic geometry*, EMS Ser. Congr. Rep., pages 215–255. Eur. Math. Soc., Zürich, 2012.

[Fay73] J. Fay. *Theta Functions on Riemann Surfaces.* Lecture Notes in Mathematics, Vol. 352. Springer-Verlag, Berlin, 1973.

[Fre77] E. Freitag. Stabile Modulformen. *Math. Ann.*, 230(3):197–211, 1977.

[Fre83] E. Freitag. *Siegelsche Modulfunktionen*, volume 254 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 1983.

[FS86] R. Friedman and R. Smith. Degenerations of Prym varieties and intersections of three quadrics. *Invent. Math.*, 85(3):615–635, 1986.

[GSM11] S. Grushevsky and R. Salvati Manni. The superstring cosmological constant and the Schottky form in genus 5. *Amer. J. Math.*, 133(4):1007–1027, 2011. And arXiv:0809.1391v5.

[GV09] M. R. Gaberdiel and R. Volpato. Higher genus partition functions of meromorphic conformal field theories. *J. High Energy Phys.*, (6):048, 43, 2009.

[Igu67a] J. Igusa. A desingularization problem in the theory of. siegel modular functions. *Math. Annalen*, 168:228–260, 1967.

[Igu67b] J. Igusa. Modular forms and projective invariants. *Amer. J. Math.*, 89:817–855, 1967.

[Kin03] O. D. King. A mass formula for unimodular lattices with no roots. *Mathematics of Computation*, 72(242):839–863 (electronic), 2003.

[May69] A. Mayer. Rauch's variational formula and the heat equation. *Math. Ann.*, 181:53–59, 1969.

[MP12] V. O. Marcucci and G. P. Pirola. Generic Torelli theorem for Prym varieties of ramified coverings. *Compos. Math.*, 148(4):1147–1170, 2012.

[Mum07] D. Mumford. *Tata lectures on theta. I.* Modern Birkhäuser Classics. Birkhäuser Boston Inc., Boston, MA, 2007. With the collaboration of C. Musili, M. Nori, E. Previato and M. Stillman, Reprint of the 1983 edition.

[MV10] M. Matone and R. Volpato. Getting superstring amplitudes by degenerating Riemann surfaces. *Nuclear Phys. B*, 839(1-2):21–51, 2010.

[OS80] F. Oort and J. Steenbrink. The local Torelli problem for algebraic curves. In *Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, pages 157–204. Sijthoff & Noordhoff, Alphen aan den Rijn, 1980.

[Oze88] M. Ozeki. Examples of even unimodular extremal lattices of rank 40 and their Siegel theta-series of degree 2. *J. Number Theory*, 28(2):119–131, 1988.

[Pas11] S. Pascolutti. *Boundaries of Moduli Space of Hyperelliptic Curves with a Level 2 Structure*. PhD thesis, La Sapienza, Universita' di Roma, 2011.

[Pat63] C. Patt. Variations of Teichmueller and Torelli surfaces. *J. Analyse Math.*, 11:221–247, 1963.

[Poo96] C. Poor. Schottky's form and the hyperelliptic locus. *Proc. Amer. Math. Soc.*, 124(7):1987–1991, 1996.

[SM03] R. Salvati Manni. Modular forms vanishing on the hyperelliptic locus. *Japan. J. Math. (N.S.)*, 29:135–142, 2003.

[Spr57] G. Springer. *Introduction to Riemann surfaces*. Addison-Wesley Publishing Company, Inc., Reading, Mass., 1957.

[SV02] R. Smith and R. Varley. The Prym Torelli problem: an update and a reformulation as a question in birational geometry. In *Symposium in Honor of C. H. Clemens (Salt Lake City, UT, 2000)*, volume 312 of *Contemp. Math.*, pages 235–264. Amer. Math. Soc., Providence, RI, 2002.

[vG84] B. van Geemen. Siegel modular forms vanishing on the moduli space of curves. *Inventiones Mathematicae*, 78(2):329–349, 1984.

[vGBvdGG86] van Geemen B. and van der Geer G. Kummer varieties and the moduli spaces of abelian varieties. *Amer. J. Math.*, 108(3):615–641, 1986.

[Voi02] C. Voisin. *Théorie de Hodge et géométrie algébrique complexe*, volume 10 of *Cours Spécialisés [Specialized Courses]*. Société Mathématique de France, Paris, 2002.