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Relativistic soliton collisions of axion type dark matter

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ABSTRACT

Axion-like scalar fields and the Lane–Emden (LE) truncation of their periodic potential are analyzed as a toy model of dark matter halos. Then, collisions of the well-known kinks in $(1+1)$ spacetime dimensions can be mapped to those of localized lumps of the LE equation. Here, we generalize this mapping to $(2+1)$D or even $(3+1)$D and discuss a challenging intrinsic inelastic effect during relativistic soliton collisions.

1. Introduction

Head-on collisions of massive galaxy clusters, like those occurring in the so-called *bullet cluster* [8] are a challenge for the cold dark matter (CDM) paradigm, inasmuch as either the individual galaxies or the hot X-ray emitting interstellar gas are only partially anchored to the DM lumps.

Here, we are going to probe axion-like particles [35–37] as dark matter candidates. In particular, we continue our investigations [4,5] whether or not two dark matter halos pulling towards one another can be modeled via soliton-type collisions, but now in a more realistic setting of $(2+1)$ or even 4 spacetime dimensions.

Since solitons or lumps appear to be rather stable entities, they behave effectively like colliding particles, i.e. after leaving the interaction region where they may deform due to a temporally inelastic mechanism studied here in a relativistic setting, they ultimately return to their original shapes and velocities.

Recently, the previous proposal of Kolb and Tkachev [16,17] that DM may be composed of a gas of ‘axion mini clusters’ or mini axion stars has been adopted and developed further [21,22,2]. Moreover, for such Bose–Einstein (BE) type condensates one needs some self-interaction and, in view of the self-similarity of solitons, one would end up with much larger configurations resembling lump-type halos to be considered here.

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2. Nonlinear Klein–Gordon equation

As one of the most representative relativistic models, the Klein–Gordon equation has found applications in many areas: Besides the nonlinear version that results in the known ϕ^4 model and higher-order field theories, it is used to describe successive phase transitions on realistic systems. Such processes are known to occur in ferroelastic and ferroelectric crystals and in meson physics. Some of these models present different kind of solitons; topological and nontopological, kinks, lump and breather type.

In the Klein–Gordon case, we depart from the wave operator

$$\Box := \nabla \bullet \nabla - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{1}$$

of Jean-Baptiste le Rond d'Alembert, still being linear in its derivatives. Thus it is invariant under the general Lorentz transformations (boosts)

$$ct \to \gamma\left(ct - \vec{\mathbf{v}}\cdot\vec{\mathbf{x}}/c\right) \tag{2}$$

$$\vec{\mathbf{x}} \to \vec{\mathbf{x}} + \frac{\gamma-1}{v^2}(\vec{\mathbf{v}}\cdot\vec{\mathbf{x}})\vec{\mathbf{v}} - \gamma\vec{\mathbf{v}}t, \tag{3}$$

where $\vec{\mathbf{x}} := (x, y, z)$ is the radius vector of an event and

$$\gamma = 1/\sqrt{1-\vec{\mathbf{v}}\cdot\vec{\mathbf{v}}/c^2} \tag{4}$$

the Lorentz factor, cf. [29].

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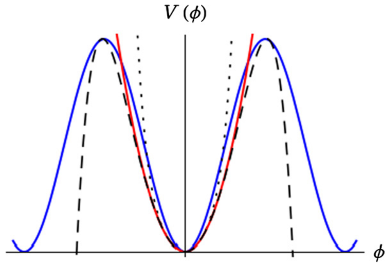

Fig. 1. (Color online.) Nonlinear potentials: sine-Gordon (blue), ϕ^2 (red), ϕ^4 (dotted) and ϕ^6 (dashed).

The challenge comes in the election of the potential $V = V(\phi)$ in the nonlinear Klein–Gordon (NLKG) equation

$$\Box\phi = \frac{\partial V(\phi)}{\partial\phi}. \tag{5}$$

For a start, let us consider another well known model, the sine-Gordon equation: The exact and complete potential has received many applications, and different truncations of (7) may approximate rather well other models: The first term of the Taylor series corresponds to the mass in the original Klein–Gordon equation; the next one would give rise to the famous ϕ^4-theory. Combining the first and third term can be reckoned as the modified ϕ^6-theory of Lane–Emden, and so on [14]. By plotting some of the potentials we obtained an approximated view of the field-theoretical 'vacuum' at the center, see Fig. 1.

In particular, we are interested in the LE form of ϕ^6-theory, a model which had been use before in astrophysics and has the mathematical virtue that some analytic expressions are available for 2 till 4 spacetime dimensions.

So far, the study of such solitons has had many advances in $1+1$ dimensional systems, but higher dimensions remain in many cases unsolvable. One of the hardest problems arise due to the lack of integrability, which, at times, is fully or partially lost.

Since the DM distributions in the bullet cluster appear not to be affected during merging, we postulate an axion-like scalar component of galaxies and clusters and analyze its solitary wave behavior, or as in our previous 2D toy model [4], its Lane–Emden (LE) truncation [23]

$$V_{\rm LE}(\phi) = \frac{m^2}{2}\phi^2\left(1 - \chi\phi^4\right). \tag{6}$$

Although globally unbounded due to $\chi = -\lambda^2/(360m^4)$, the LE truncation (6) has the advantage that, in 4D, it admits exact spherically symmetric static solutions [23] which model quite well [25] DM halos of individual galaxies. Moreover, more than a century ago[1] it was considered as approximate models for the density of the sun as well as for the distribution of stars in globular clusters.

3. Soliton collisions in $(1+1)$D

The *axions* of Quantum Chromodynamics (QCD) with inertial mass m are self-interacting via the *effective* [15,30] periodic potential

$$V_a(\phi) = \frac{m^4}{\lambda}\left[1 - \cos\left(\frac{\sqrt{\lambda}}{m}\phi\right)\right]$$

$$\simeq V_{\rm LE}(\phi) - \frac{\lambda}{4!}\phi^4 - \cdots \tag{7}$$

which deviates from the LE potential at higher order of $|\phi|$.

When several solitons collide, one expects that they recuperate their initial shapes and velocities after some time has passed. Remnants from crossing the scattering region may be the concomitant phase shifts or displacements of the centers of the individual solitary waves due the nonlinear, partially inelastic interaction.

For constructing multi-solitons, the well-established *Bäcklund transformation* (BT), cf. Refs. [38,32,13] is employed which may also bridge between different types of nonlinear equations.

Let us depart from the sine-Gordon (sG) equation [31] which in dimensionless light-cone coordinates $\xi := \frac{1}{2}(\tilde{x} + c\tilde{t})$ and $\eta := \frac{1}{2}(\tilde{x} - c\tilde{t})$ acquires the form

$$\theta_{\xi\eta} = \sin\theta \tag{8}$$

and is CPT invariant. In a moving frame, it has the exact kink solution

$$\theta = 4C\arctan\left[\exp\gamma\left(\tilde{x} - u\tilde{t}\right)\right], \tag{9}$$

for $C = 1$ and anti-kink for $C = -1$. Since its spatial derivative

$$\theta_{\tilde{x}} = 2\gamma C\,\mathrm{sech}\left[\gamma\left(\tilde{x} - u\tilde{t}\right)\right] \tag{10}$$

becomes localized and square-integrable, its absolute value will facilitate a subsequent comparison with the scattering behavior of solitons or lumps regarded as Bose–Einstein condensates [24,28] of DM.

Due to Bianchi's permutability theorem of BTs, there results the 'nonlinear superposition' principle

$$\tan\left[(\theta_3 - \theta_0)/4\right] = \mathbb{B}\tan\left[(\theta_1 - \theta_2)/4\right], \tag{11}$$

where $\mathbb{B}$ is a common or 'average' relativistic velocity of the superposed solitons. This allows us to algebraically construct multi-kink solutions of the sG equation a la Perring and Skyrme [26,41].

The CPT invariance of our relativistic KG equation, will allow us to distinguish solitons from anti-solitons:

Here we focus on the collision of two kinks (instead of a collision of a kink and its CP odd anti-kink, as in Ref. [4]) and obtain from the trivial seed solution $\theta_0 = 0$ the exact solution

$$\theta_{\rm kk} = 4\arctan\left[K(\zeta_1, \zeta_2)\right], \tag{12}$$

where the kinetic factor

$$\begin{aligned} K(\zeta_1, \zeta_2) :&= \mathbb{B}\frac{\exp(\zeta_1) + C\exp(\zeta_2)}{\exp(\zeta_1 + \zeta_2) - C} \\ &= \frac{\exp(\zeta_1 + \gamma_1\delta_1) + C\exp(\zeta_2 + \gamma_2\delta_2)}{\exp(\zeta_1 + \zeta_2) - C} \\ &\simeq \exp(\zeta_1 + \gamma_1\delta_1) + C\exp(\zeta_2 + \gamma_2\delta_2) \end{aligned} \tag{13}$$

depends on the initial velocities and the inverse Lorentz transformations $\zeta_i := \gamma_i(\tilde{x} + u_i\tilde{t})$. The former is also known as Hirota's formula [11,12].

At large separations from the interaction region, cf. Fig. 2, the solution (12) clearly decouples asymptotically into a (non-interacting) kink–kink or kink–antikink pair [7] distinguished by the sign $C = \pm 1$ of the topological charge.

3.1. *Collisions of Lane–Emden lumps*

For the Lane–Emden equation of interest in astrophysics, an exact auto-Bäcklund transformation has yet not been found. Instead a generalized transformation or *mapping* will serve us as a guide in constructing multi-lump solution. Consider the Lane–Emden potential (6) as a truncation of (7), then the corresponding non-linear KG equation in light-cone coordinates simplifies to

[1] Jonathan Homer Lane: "On the theoretical temperature of the Sun under the hypothesis of a gaseous mass maintaining its volume by its internal heat and depending on the Laws of Gases known to Terrestrial experiment". The American Journal of Science and Arts. 2 50 (1870) 57–74.

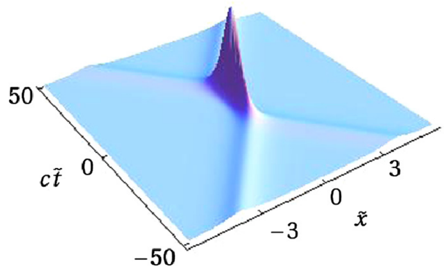

Fig. 2. (Color online.) Kink–kink collision monitored via the absolute value of its spatial derivative.

$$\phi_{\xi\eta} = \frac{\partial V}{\partial \phi} = \phi\left(1 - 3\chi\phi^4\right). \tag{14}$$

Integration leads us to the explicit exact solution

$$\phi = \frac{1}{\sqrt[4]{\chi}}\sqrt{\operatorname{sech}(2\zeta)}, \tag{15}$$

where an integration constant has been suppressed.

Due to the identity $\sin(2\arctan y) = \operatorname{sech}(\ln y)$ for (hyperbolic) trigonometric functions, the Lane–Emden solution (15) is related to the kink of the sG equation via the nonlinear *mapping*

$$\phi = \phi(\theta) = \frac{1}{\sqrt[4]{\chi}}\left[\sin(\theta/2)\right]^{1/2} = \frac{1}{\sqrt[4]{\chi}}\sqrt{\operatorname{sech}(2\zeta)}. \tag{16}$$

This suggest to generate a two soliton solution

$$\begin{aligned}\phi = (\theta_{\rm kk})\sqrt[4]{\chi} &= \left(\sin\left\{2\arctan\left[K(\zeta_1,\zeta_2)\right]\right\}\right)^{1/2} \\ &= \left(\operatorname{sech}\left\{\ln\left[K(\zeta_1,\zeta_2)\right]\right\}\right)^{1/2}\end{aligned} \tag{17}$$

of the LE equation via the same mapping, using again identities for trigonometric (hyperbolic) functions. So far, a more precise approximation has not yet been found.

In the resulting spacetime diagram, the scattering of two solitons behaves as expected: After crossing the collision region, the individual solitons regain their original velocities, their trajectories are asymptotically the same as the initial ones, merely the centers of the lumps suffer the phase shift or displacement

$$\delta_i := \ln|\mathbb{B}|/\gamma_i. \tag{18}$$

For solitons, some temporary 'bouncing' of the center of a lump in the collision region occurs, whereas an anti-soliton may 'tunnel' through a soliton during merging, cf. Ref. [4] for details.

Thus, interacting solitons behave more like 'extended particles' where, due to the temporarily inelastic effects discussed above, a continuous interchange of inertia (invariant mass) and interaction energy occurs. In Ref. [7], this non-Newtonian behavior is referred to as a *local version of Mach's principle*, inasmuch as the total inertia of a lump depends also on the relative motion of other solitons in its vicinity.

Although Eq. (17) is not an exact solution to the nonlinear KG equation (14), the same relativistic factor occurs in the LE truncation.

4. Solitons in 2 + 1 dimensions

Generalizing solutions of nonlinear equations from (1 + 1)D to more spatial dimensions is rather complicated due to many reasons: One of them is that the use of nonlinear tools is not widely developed or is lacking. An extrapolation of our previous mapping to 3 or 4 spacetime dimensions, is not direct in the most cases. Commonly, integrability is fully or partially lost, cf. Ref. [33].

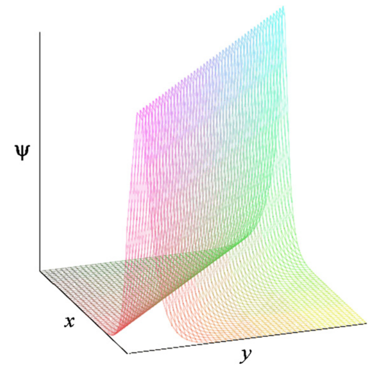

Fig. 3. (Color online.) Crest like 3D soliton in a rest frame.

Introducing the LE potential and a scalar field of the form $\phi(t,x,y)$ in (5) and scaling coordinates to $(\tilde{t},\tilde{x},\tilde{y}) = m(t,x,y)$, the NLKG as a dimensionless wave equation can be written explicitly as

$$\left(\frac{\partial^2}{\partial\tilde{x}^2} + \frac{\partial^2}{\partial\tilde{y}^2} - \frac{1}{c^2}\frac{\partial^2}{\partial\tilde{t}^2}\right)\phi = \phi\left(1 - 3\chi\phi^4\right) \tag{19}$$

However, the previous mapping is not restricted to the degree of the spatial dimensions. Adopting the Ansatz

$$\tan(\phi/4) = \exp\left[y - f(x;t)\right] \tag{20}$$

of Refs. [10,27], we can, as a result, depart from a solution of the form

$$\phi\left(\tilde{t},\tilde{x},\tilde{y}\right) = \pm\frac{1}{\chi^{1/4}}\sqrt{\operatorname{sech}\left[2\left(\tilde{y} - f\left(\tilde{t},\tilde{x}\right)\right)\right]}, \tag{21}$$

using dimensionless coordinates.

Then the terms entering in Eq. (19) can explicitly be written as

$$\begin{aligned}\frac{\partial^2}{\partial\tilde{x}^2}\phi\left(\tilde{t},\tilde{x},\tilde{y}\right) &= \frac{1}{\chi^{1/4}}\operatorname{sech}^{5/2}\left(\frac{\mathfrak{B}}{2}\right) \\ &\quad\times\left[\left(\frac{\partial f}{\partial\tilde{x}}\right)^2\mathfrak{A} \pm \frac{\sinh(\mathfrak{B})}{2}\left(\frac{\partial^2 f}{\partial\tilde{x}^2}\right)\right], \\ \frac{1}{c^2}\frac{\partial^2}{\partial\tilde{t}^2}\phi\left(\tilde{t},\tilde{x},\tilde{y}\right) &= \frac{1}{\chi^{1/4}}\operatorname{sech}^{5/2}\left(\frac{\mathfrak{B}}{2}\right) \\ &\quad\times\left[\left(\frac{1}{c}\frac{\partial f}{\partial\tilde{t}}\right)^2\mathfrak{A} \pm \frac{\sinh(\mathfrak{B})}{2}\left(\frac{1}{c^2}\frac{\partial^2 f}{\partial\tilde{t}^2}\right)\right],\end{aligned} \tag{22}$$

$$\frac{\partial^2}{\partial\tilde{y}^2}\phi\left(\tilde{t},\tilde{x},\tilde{y}\right) = \frac{\mathfrak{A}}{\chi^{1/4}}\operatorname{sech}^{5/2}\left(\frac{\mathfrak{B}}{2}\right), \tag{23}$$

$$\phi\left(1 - 3\chi\phi^4\right) = \frac{\mathfrak{A}}{\chi^{1/4}}\operatorname{sech}^{5/2}\left(\frac{\mathfrak{B}}{2}\right), \tag{24}$$

where $\mathfrak{B} = 4\tilde{y} - 4f\left(\tilde{t},\tilde{x}\right)$ and $\mathfrak{A} = \pm\left[\cos(\mathfrak{B}) - 5\right]/2$.

It turns out that ϕ fulfills the nonlinear Eq. (19) for functions $f(t,x)$ which obey the two conditions

$$\frac{1}{c^2}\frac{\partial^2 f}{\partial\tilde{t}^2} = \frac{\partial^2 f}{\partial\tilde{x}^2}, \tag{25}$$

$$\left(\frac{1}{c}\frac{\partial f}{\partial\tilde{t}}\right)^2 = \left(\frac{\partial f}{\partial\tilde{x}}\right)^2. \tag{26}$$

In Fig. 3 the simple choice $f(\tilde{t},\tilde{x}) = -\left(\tilde{x} - c\tilde{t}\right)$ is considered which, however, does not provide a localization in the perpendicular direction. It can be regarded as a *wave crest* after a rotation in the xy-plane. The scalar field is normalized to $\Psi = \chi^{1/4}\phi$.

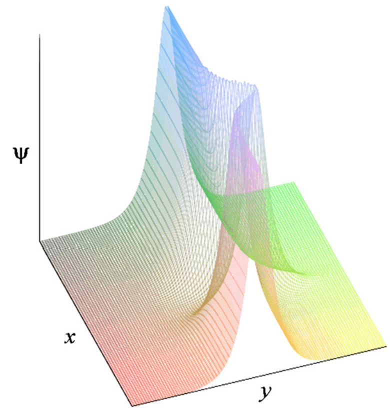

Fig. 4. (Color online.) Partially localized 3D lump-like soliton in a rest frame.

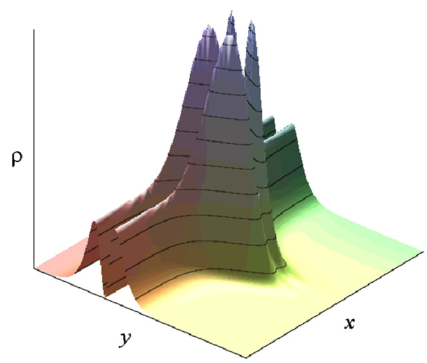

Fig. 5. (Color online.) Energy density of the partially localized 3D lump.

In Fig. 4 we considered the choice $f\left(\tilde{t},\tilde{x}\right) = k\text{sech}\left(c\tilde{t}-\tilde{x}\right) = k\text{sech}\left(\tilde{\xi}\right)$ which provides a lump localized only in y direction. Again, the scalar field is normalized to $\Psi = \chi^{1/4}\phi$.

The energy density of such lumps has the standard form:

$$\rho\left(\tilde{t},\tilde{x},\tilde{y}\right) = \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial\tilde{x}}\right)^2 + \left(\frac{\partial\phi}{\partial\tilde{y}}\right)^2 + \left(\frac{1}{c}\frac{\partial\phi}{\partial\tilde{t}}\right)^2\right] + V\left(\phi\right), \quad (27)$$

and yields

$$\rho\left(\tilde{t},\tilde{x},\tilde{y}\right) = \frac{m^2}{\sqrt{\chi}}\left(\text{sech}^3\left[F\right] - \text{sech}\left[F\right]\right) \times \left\{k^2\gamma^2\left(\text{sech}^4\zeta - \text{sech}^2\zeta\right) - 1\right\} \quad (28)$$

where $F := (2\tilde{y}\cosh\zeta - k)/\cosh\zeta$. The density ρ in Fig. 5 is normalized by $\sqrt{\chi}/m^2$.

Colliding LE lumps of initially equal velocities and masses can, eventually, be generated via the nonlinear mapping

$$\tan\left(\frac{\arcsin\left(\chi^{1/2}\phi^2\right)}{2}\right) = \frac{c\sinh\left[\gamma m v t\right]}{v\cosh\left[\gamma m\left(x+y\right)\right]}. \quad (29)$$

Mathematically, however, wave equations in odd or even dimensions can behave differently with respect to the Huygens principle, e.g. for $(2+1)$ spacetime dimensions there may occur "lacunas", cf. Ref. [9] for details.

5. Radial 4D lump in the Lane–Emden truncation

Let us recall that the LE potential

$$\tilde{V}\left(\phi\right) = \phi^2\left(1-\chi\phi^4\right), \quad (30)$$

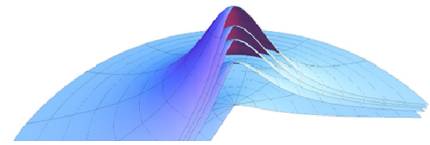

Fig. 6. (Color online.) Radial profile of exact lump type 4D solutions of the Lane–Emden equation in a rest frame.

is highly nonlinear in $|\phi|$ and its behavior dependents on the coupling constant χ. (So if $\chi \le 0$ it has only one minimum and for $\chi > 0$ it adds two maxima; rather similar to the sine case at the origin.)

Using the spherical symmetry Ansatz

$$\phi = P\left(r\right)Y\left(\theta,\varphi\right)\exp(-i\omega t) \quad (31)$$

in (5), and applying an 'averaging' procedure [20] over the angular dependence in the spherical harmonics $Y\left(\theta,\varphi\right)$, the classical LE equation in the radial coordinate r is recovered. The result

$$P'' + \frac{2}{r}P' + 3m^2\chi P^5 = 0, \quad (32)$$

is familiar from the in theory of polytropic gases. Since this radial Ansatz reduces the 4D equation to *effective* $(1+1)$ spacetime dimensions, in the rest frame it can be solved exactly[2] and analytically, after rescaling the radial coordinate via $x = mr$, by

$$P\left(r\right) = \pm\chi^{-1/4}\sqrt{\frac{A}{1+A^2x^2}}. \quad (33)$$

Here $A = \sqrt{\chi}P_0^2$ is associated to the central peak and P_0 is the value at $r = 0$.

More specifically, the LE equation provides us in $(3+1)$D with the exact [19] static radial solution (33) which is completely nonsingular and localized in all spatial directions. This is referred to as a meta-stable lump, cf. Fig. 6 or 12 from Ref. [4], which cannot easily be confounded with domain walls.

Thus it appears prospective to generalize the scattering of two Lane–Emden solitons in 2D to more realistic 3D (head-on) collisions of self-gravitating lumps [23,3]. In 4D one would like to depart from those depicted in Fig. 6, albeit formidable requirements on the computational resources.

6. Discussion

As is well-known [20], the LE truncation provides only metastable lumps, whose decay time $\tau \simeq \lambda_{\text{Compt}}/c$ is proportional to the Compton wave length. Since this is shorter than the collision time as well as the age of the Universe, such lumps need to be stabilized via their self-generated gravity [18,34], similarly as in the case of colliding *boson stars* [6] or *axion miniclusters* [22,39,40]. Nevertheless, we could report here some progress in the semi-analytical study of solitons collisions modeling, to a certain extent, DM halos.

Acknowledgements

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[2] Numerical approximations [1] for some LE equations can also be obtain in terms of Legendre polynomials.

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