THE MAPPING CLASS GROUP: HOMOLOGY AND LINEARITY

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The mapping class group, namely the group of components of the group $M_g=\pi_0 \mathrm{Diff}_+(\Sigma_g)$ of isotopy classes of orientation preserving differentiable self-homeomorphisms of a Riemann surface Σ_g of genus g, has played a more and more important role in the statistical mechanics of the 3-dimensional Ising model[1] and in string theory[2].

In the former case $M_g=\mathrm{Homeo}(\Sigma_g)/\mathrm{Isot}(\Sigma_g)$ enters the picture when the model is defined on a lattice L homogeneous under some finitely presented finite group G.

Indeed, when the solution is formulated in terms of the so called Pfaffian (or dimer) method, the three relevant steps to be performed are the following:

i) the positional degrees of freedom of the decorated lattice L_d, obtained from L by the Fisher procedure[3], are to be relabelled in terms of a set of anticommuting Grassmann variables, in one-to-one correspondence with the elements of G.

ii) The group G is then to be extended to a group $\underline{G}$ in such a way that all the orientations of the bonds of L_d compatible with the combinatorial requirements expressed by the global generalization of the Kasteleyn's theorem to a non-planar situation, and only those, might be obtained as the invariant set of configurations of the oriented graph matching L_d.

iii) The partition function of the Ising model on L is then reduced to the evaluation of the Pfaffian associated with the incidence matrix $\underline{A}$ of L_d, extended with respect to $\underline{G}$:

$$Z(L) = \mathrm{Pf}\ \underline{A} \qquad (1)$$

iv) Since the regular representation R of a finite group is the direct sum of its irreducible representations (labelled by the index J), each contained as many times as its dimension j, (1) can be naturally reduced -when G is finite- to the evaluation of a finite number of finite determinants:

$$Z(L) = \prod_{J_F} (\det R[\underline{A}^{(J_F)}])^{\frac{1}{2} j_F}, \qquad (2)$$

where the sub-index F refers to the fermionic representations, and $\underline{A}^{(J)}$ is a matrix of rank j.

There are several constraints in the choice of L and $\underline{G}$, imposed by both topological and combinatorial limitations. The former essentially consists in the requirement that L should be embeddable in a two-dimensional orientable compact surface Σ_g of genus g (such that the coordination of the 3-dimensional lattice sites is locally preserved on Σ_g). It was shown in [1] that the most general extension $\underline{G}$ of G satisfying all the requisites is of the form

$$\underline{G} = C \circledR S_{2g} \qquad (3)$$

where (wr) denotes the wreath product[4], S_n is the permutation group of n objects and $C=M_g/H$, M_g being the mapping class group of Σ_g and H the stabilizer subgroup of M_g, namely the group of diffeomorphisms of Σ_g which preserves the isotopy class of a maximal unordered non separating system of g disjoint smoothly embedded cycles c_i, i=1,...,g (non contractible and non isotopic).

It should be noted that maps and spaces are to be thought of in the piecewise-linear cathegory.
The homeomorphism ext: $G \to \underline{G}$ acts locally by attaching a Kasteleyn phase to the circuits on Σ_g homotopic to zero, and globally by an extension by the fundamental group -i.e. mapping $\pi_1(\Sigma_g)$ to $\mathbf{Z}_2$-.
Since H has a finite presentation[5], also $\underline{G}$ is finitely presented. Minimal presentations for M_g and for C have been discussed by Thurston[6], Wajnryb[7] and in [1] respectively. The interesting property emerging from these is that both groups are generated by H and the elements representing the homology exchange between any pair of cycles c_i, c_j; $i \neq j$; $i,j=1,\dots,g$; and all the relations derive from a set of subrelations supported in subsurfaces of Σ_g of genus at most 2.
In string theory, closed string amplitudes at critical dimensions are obtained by computing the correlation functions between vertex operators on a Riemann surface and then summing first over inequivalent Riemann surfaces of fixed topology, (i.e. of a given genus g), then over all topologies[8].
The property mentioned above that the generators of $M_g(\Sigma_g)$ for higher genus surfaces involve Dehn-Lickorish twists around either a single handle or two handles, but never around more than two handles implies a fact of tremendous physical importance: in the analysis of modular invariant amplitudes in string theory it is enough to take into account one- and two-loop diagrams[9]. Moreover string amplitudes can be analyzed in terms of the complex analytic structure of moduli space Mod(g) of inequivalent Riemann surfaces of genus g; in particular, to obtain the partition function for the bosonic string theory, one has to integrate over Mod(g) with respect to a measure constructed in terms of the determinants of the various fields living on the string world sheet.

Thus in the two applications described of the mapping class group the main questions are:

i) to derive from some finite presentation a faithful representation of M_g -which is residually finite- as a group of matrices with entries in a suitable field;

ii) to produce a complete description of the moduli space Mod(g) of smooth curves of genus g. The latter is the quotient of the action of M_g on the Teichmuller space $\mathbf{T}^{(3g-3)}$. Such an action is proper, discontinuous, with finite isotropy groups[10].

Both these questions run into great difficulties which hinge on some of the most crucial questions of the modern mathematical theory of Riemann surfaces.

As for question i) one has an affirmative answer for g=1, in which case $M_1 \sim SL(2,\mathbb{Z})$, the modular group. That this is the case can be readily checked by recalling that a lattice Γ in $\mathbb{C}$ can be thought of as a free abelian group of rank 2, generated by two complex numbers z_1, z_2, linearly independent over $\mathbb{R}$ (i.e. such that z_1/z_2 is non real). Then Γ is a (discrete) subgroup of $\mathbb{C}$ thought of as a topological group, and the quotient $\mathbb{C}/\Gamma$, together with the complex structure thus inherited, is a compact Riemann surface of genus 1 (a torus). Conversely, any torus T can always be represented in this way, by selecting a basis -say {a,b}- for the first homology group $H_1(T,\mathbb{Z})$ (which is itself a free abelian group of rank 2) and a holomorphic one-form w on T: then the pair of numbers $\int_a w$, $\int_b w$ generate a lattice Γ' such that $T \sim \mathbb{C}/\Gamma'$.

Denoting by $|z\rangle = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ an element of the space of generators, two elements $|z\rangle$, $|z'\rangle$ generate the same lattice Γ -up possibly to an irrelevant change of basis- if $|z'\rangle = g|z\rangle$ with g some matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $GL(2,\mathbb{Z})$. Moreover if $|z'\rangle = c|z\rangle$, c a nonzero complex constant, the corresponding Riemann surfaces are obviously conformally equivalent: if one decides to identify them, the equivalence class is obtained simply passing from

$|z\rangle$ to $\theta=z_1/z_2\in\{\mathbf{C}-\mathbf{R}\}$. On the other hand points θ and $-\theta$ represent the same surface with opposite orientations, and as parameter space for the classes of generating sets one only needs to consider the upper half complex plane $\mathbf{H}=\{\theta\,|\,\mathrm{Im}\theta>0\}$.

In this g=1 case, the moduli space Mod(1) is then a space whose points correspond to conformal isomorphism classes of tori.

In order to construct it, one has to consider the result of the action of $GL(2,\mathbf{Z})$ on $\mathbf{H}$: $|z'\rangle=g|z\rangle$ with θ and θ' both in $\mathbf{H}$ if and only if $\theta'=(a\theta+b)/(c\theta+d)$ with $ad-bc=1$; in other words the action is that of the (inhomogeneous) group of Möbius transformations $PSL(2,\mathbf{Z})=SL(2,\mathbf{Z})/\{\mathbf{I},-\mathbf{I}\}$.

An answer to i) holding in general for any g is not equally easy to settle. The difficulty lies deep in the topology. Let $S(\Sigma_g)$ be the set of isotopy classes of non-oriented, closed curves embedded (as one manifolds) in Σ_g, and let L_g be a foliation of Σ_g whose leaves are geodesics for a hyperbolic metric on Σ_g (which has negative Euler characteristics), with a transverse measure. The latter is a real positive function p assigning to each arc α in Σ_g, transverse to the leaves of L_g and with endpoints in $\{\Sigma_g-L_g\}$, an invariant weight such that:

- $p(\alpha)=p(\beta)$ if α is homotopic to β through arcs transverse to L_g and with endpoints in $\{\Sigma_g-L_g\}$;
- if $\alpha=\cup_i\alpha_i$, with $\alpha_i\cap\alpha_j\subset\partial\alpha_i\cap\partial\alpha_j$, $p(\alpha)=\Sigma_i p(\alpha_i)$;
- $p(\alpha)\neq 0$ if $\alpha\cap L_g\neq\emptyset$.

The collection of all these measured geodesic foliations constitute a space $\mathbf{L}_g$ on which M_g acts in the natural way. If $\Phi\in M_g$, one says that Φ is periodic if it is of finite order in M_g, reducible if there is a point of $S(\Sigma_g)$ which is invariant under Φ, pseudo-Anosov if there exist mutually transverse measured geodesic foliations $L_g^{(s)},L_g^{(u)}\in\mathbf{L}_g$ (s stands for stable, u for unstable) such that $\Phi(L_g^{(s)})=1/\delta\, L_g^{(s)}$ and

$\Phi(L_g^{(u)})=\delta L_g^{(u)}$ for some real $\delta>1$.

A general answer to question i) requires as a prerequisite proving that no normal subgroup NM_g of M_g can have all its non-identity elements which are pseudo-Anosov. Indeed in such a case -since the only possible overlap in the classification given above of the mapping class group elements is between periodic and reducible mapping classes- one might know when a given $\Phi\in NM_g$ fixes some $\sigma\in S(\Sigma_g)$, and there would be no obstructions to constructing an induced faithful representation of M_g as a group of matrices.

Even though no exhaustive theorem was proven so far, providing a global answer to the question, we could at least show that indeed some normal subgroup of M_g can be constructed with the desired "non-Anosov" property. We sketch hereafter the essential steps of the proof and state the interesting results.

One should recall first[11] that every piecewise-linear orientation preserving homeomorphism of a closed oriented surface of genus g is isotopic to a product of maps D_{σ_i} (Dehn's twists) of the following form. Let N_i be a neighbourhood of the simple closed curve c_i defined previously, C_o an oriented closed cylindrical surface parametrized by coordinates (r,θ), $-1\leqslant r\leqslant 1$, $0\leqslant\theta<2\pi$, e_i: $C_o\rightarrow\Sigma_g$ the orientation preserving embedding such that $e_i(C_o)=N_i$, $e_i(\{0,\theta\})=c_i$ and μ: $C_o\rightarrow C_o$ the map defined by $\mu(r,\theta)=(r,\theta+\pi(r+1))$; then $D_{\sigma_i}=e_i\mu\, e_i^{-1}$ (notice that D_{σ_i} is the identity on the two boundary curves of N_i).If γ is a path which crosses the curve c_i at a finite number of points $\{a_1^{(i)},\ldots,a_r^{(i)}\}$, the effect of D_i on γ is to break it at each point $a_k^{(i)}$ and insert there a copy of c_i in such a way that it coalesces -including orientation- with γ.

One further recalls that on any surface Σ_g one may find a pair of essential simple closed curves c, c' which fill the surface, with the further property that there is an essential closed curve ç', disjoint from c', such that cUç' does not fill Σ_g. If ç is an essential simple curve with no intersections with cUç', it is known[12] that $D_c D_{c'}^{-1}$ is isotopic to a pseudo-Anosov map. Then $D_c D_{c'}^{-1} \circ ç'$ is a curve, say c", disjoint from ç. Thus, there exists a map $D_{ç'}^{-1} D_c D_{c'}^{-1} D_{ç'} . D_{c'} . D_c^{-1} = D_{ç'}^{-1} D_{c''}$ fixing ç and hence <u>not</u> pseudo-Anosov.

Then the proof proceeds by the following steps:

- Looking at the action of M_g on the projective space $\mathbb{L}_g$ of measured geodesic foliations, Dehn's twists can be recognized and treated as maps with parabolic action: indeed they are locally conjugate to the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in PSL(2,\mathbb{Z})$.
- Resorting to the presentation of M_g as given in refs.[1],[6] and [7], and recalling that the elements of the fundamental group which act parabolically on the hyperbolic projective space are those which may be freely homotoped into the cusps and that just these elements are non-Anosov, one can find a class of Riemann surfaces such that: a) M_g has a geometrically finite subgroup SM_g of finite index on which M_g acts by conjugation, and b) in the action of M_g on the projective space of measured geodesic foliations, Dehn's twists can be recognized and treated as maps with parabolic action[13].

Then, if the normal closure in $\pi_1(\Sigma_g)$ of the elements of the action of M_g on SM_g does not exclude all the cusp generators (because of the Jørgensen's inequality the elements of a discrete group can never get too close to the identity[14]) not all of its (non-identity) elements are pseudo-Anosov. This conclusion does not hold for M_1, whose linearity was proven above, but it does for $g \geqslant 2$ when M_g has a non empty set of elementary homeomorphisms equivalent to global braids. The corresponding matrix representation is that induced from the monodromy representation associated with the Lefschetz fibration[15] of Σ_g.

It is interesting to mention the connection of the approach above with the theory of representations of braid groups.

We turn now briefly to the question ii) which is much more complicated, and for which the results presently avaliable amount to a sound working scheme, essentially geometric in its conceptual structure, and a few conjectures.

It has been shown by Mumford[16], that working in the framework of the piece-wise-linear category one can define universal cohomology classes as the (n+1)-fold (simplicial) cup product of the first Chern classes of the tangent bundle to the universal Σ_g bundle with fiber Σ_g, integrated along the fibres. Such classes map the elements of the (2n+1)-th cohomology group of Σ_g into those of $H^{2n}(M_g,\mathbf{Z})$, and allow therefore to study the symplectic homomorphisms with integer coefficients induced by diffeomorphisms of Σ_g.

It is then quite natural[17] to try to detect nonzero classes in the universal bundle of $\mathrm{Diff}_+(\Sigma_g)$ by pulling back classes from that of the group $Sp(2g,\mathbf{Z})$. Now, the real symplectic group $Sp(2g,\mathbf{R})$ has the unitary group $U(g)$ as maximal compact subgroup, thus the inclusion $U(g)\rightarrow Sp(2g,\mathbf{R})$ induces the homotopy equivalence of the corresponding universal bundles. There follows that a map of such bundles is induced by the following diagram, summarizing the inclusion and group homomorphisms described:

$$\begin{array}{ccc} \mathrm{Diff}_+(\Sigma_g) & \rightarrow & Sp(2g,\mathbf{Z}) \\ & & \downarrow \\ U(g) & \rightarrow & Sp(2g,\mathbf{R}) \end{array}$$

On the other hand the homology of the universal bundle of $U(g)$ is a polynomial algebra under the Whitney sum, thus μ can be used to detect the possible polynomial generators of the homology groups. It has been conjectured that the number of even-dimensional generators might indeed increase exponentially with g. Much less can be said, so far, on the odd-dimensional homology of M_g.

References

[1] M.Rasetti, Ising Model on Finitely Presented Groups, in "Group Theoretical Methods in Physics", M.Serdaroglu and E.Inönü, eds.; Springer-Verlag Lecture Notes in Physics 180; Berlin, 1983, p.513; and references therein.

[2] L.Alvarez-Gaumé, G.Moore and C.Vafa, Commun.Math.Phys. 106, 1(1986)

[3] M.E.Fisher, J.Math.Phys. 7, 1776(1966)

[4] A.Kerber, "Representations of Permutation Groups", Springer-Verlag Lecture Notes in Mathematics 240; Berlin, 1971.

[5] F.Laudenbach, Astérisque 66-67, 267(1979)

[6] A.Hatcher and W.Thurston, Topology 19, 221(1980)

[7] B.Wajnrib, Israel J.Math 45, 157(1983)

[8] A.M.Polyakov, Phys.Lett. 103 B, 207-211(1981)

[9] E.Witten, Global Anomalies in String Theory, in "Anomalies, Geometry, Topology", W.A.Bardeen and A.R.White, eds.; World Scientific Publ.Co.; Singapore, 1985, p.61.

[10] J.Harer, Ann. of Math. 121, 215(1985)
C.J.Earle and J.Eells, J.Diff.Geometry 3, 19(1969)

[11] J.S.Birman, The Algebraic Structure of Surface Mapping Class Groups, in "Discrete Groups and Automorphic Functions", W.J.Harvey, ed.; Academic Press; London, 1977, p.163.

[12] A.Fathi, F.Laudenbach and V.Poenaru, Astérisque 66-67, 33(1979)

[13] A.Casson, "Automorphisms of Surfaces after Nielsen and Thurston", University of Texas at Austin, 1983; Lecture Notes by S.Bleiler.

[14] T.Jørgensen, Amer.J.Math. 98, 839(1976)

[15] R.Mandelbaum and J.R.Harper, Can.Math.Soc.Conf.Proc. 2, 35(1982)

[16] D.Mumford, Towards an Enumerative Geometry of the Moduli Space of Curves, in "Arithmetic and Geometry", vol.II, Birkhauser; Boston, 1983, p.271.

[17] M.F.Atiyah, The Signature of Fibre Bundles, in "Global Analysis" (Papers in honor of K.Kodaira); Tokyo University Press; Tokyo, 1969, pp.73-84.