Abstract

An elementary review of string theory aimed at physicists in general rather than theorists in particular is given. It is explained how string theory can provide a consistent theory of gravity and quantum mechanics and may also unify all the forces of nature. The relationship between critical phenomena, conformal field theory and string theory is explained. More recent developments involving W-algebras and integrable models are summarized.
1 Introduction

Although string theory originally arose as an attempt to describe hadronic physics, it is now regarded as a theory that could describe all the fundamental interactions. There are two main reasons for this belief: certain string theories not only solve the long standing problem of combining gravity and quantum mechanics, but they also naturally contain massless particles of spins from two to zero. As such string theory provides, for the first time, a consistent theory that could unify all of physics in that it can, potentially, describe the four forces and the matter that are observed in nature. We begin by explaining these remarks in more detail.

1.1 The Problem of Gravity and Quantum Mechanics

Soon after the discovery of quantum mechanics, it was realised that the calculation of any generic process in a relativistic quantum field theory leads to an infinite result. In a relativistic quantum theory, we must sum over all possible processes, and these include all processes where particles are created and destroyed. It is inevitable that in these sums there will occur graphs which have closed loops. Although momentum is conserved at each vertex of the graph there is still an undetermined momentum circulating around any loop. We must also sum over all possible loop momenta and it is this sum which is frequently divergent in quantum field theory. To be concrete we now give an example of such a process within the context of Quantum Electrodynamics which has the action

\[ A = \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{i}{2} \bar{\psi} \left( \partial_\mu - ieA_\mu \right) \gamma^\mu \gamma^5 - m \bar{\psi} \right\}, \]

although the details of the calculation will not be required to what follows. A typical scattering process is given in the figure 1 which is the lowest order correction to the electron propagator. In this process an electron creates a photon and re-absorbs it at a latter time. Using factors proportional to \( \frac{1}{p^2} \) and \( \frac{1}{k^2} \) for the photon and fermion propagators with momentum \( \mu \) respectively and a factor of \( ie\gamma^\mu \) for the vertices and summing over the loop momentum the graph of figure 1 evaluates to the expression

\[ \int d^4k \frac{f^2 (\gamma_\mu \gamma_5)}{k^2 (\gamma_\mu \gamma_5)} \frac{\eta_{\mu\nu}}{(p + k)^2} \times \text{constant.} \]

This integral is clearly divergent. Putting a cut-off \( \Lambda \) on the high momentum, the above integral is proportional to \((c\mu + d) \ln \Lambda\) where \( c \) and \( d \) are constants. This process is a correction to \( \frac{1}{p + m} \) which is the electron propagator. We recognize the second term as a correction. albeit infinite, to the mass and the first term as an infinite correction to the wave function normalisation. A similar process gives a correction to the charge of the electron.

The principles of quantum mechanics ensure that processes in which particles are con-
tinually created and destroyed, even in the vacuum, always take place. As a result one can never, in any experiment, measure the parameters that appear in an action. For Q.E.D. these parameters are the mass, wave function normalization and the charge. This is most intuitively obvious for the charge of the electron, since it is surrounded by the creation of pairs of electrons and positrons that shield the charge in a way similar to that in a dielectric. Consequently, in order to relate the parameters of the theory which are relevant to experiment we must modify the parameters in the action by taking into account processes involving the creation and destruction of particles. It is obvious in the case of the process of figure 1 that these infinite corrections are absorbed into the mass and wave function normalization when one carries out this redefinition, or renormalization. In fact, for quantum electrodynamics it can be shown that when one renormalizes the theory, all such infinities that occur in the theory are absorbed into the parameters of the theory. The proof of this statement is rather complicated and indeed this solution was realised only in 1959, some thirty years after the discovery of the problem of infinities. Although many physicists, notably Dirac, found this procedure rather unappealing the most accurately known agreement between theory and experiment in any area of physics are based on such calculations.

We now examine the case of gravity. Gravity when viewed from the perspective of a particle physicist arises due to the exchange of a spin two particle which is described by the two index field $h_{\mu\nu}$. By writing the metric $g_{\mu\nu}$ as $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, where $\kappa$ is proportional to the gravitational coupling constant, that is Newton constant. Einstein’s action is of the generic form

$$\int d^4x \left\{ \frac{1}{2} \partial^2 h + \kappa h^2 \partial^2 h + \ldots \right\}.$$ 

In this equation we have not displayed the indices on $h_{\mu\nu}$ or $\partial_\mu$ and $\ldots$ stands for terms containing higher powers of $h$. The existence of the three vertex in the above action gives rise to the quantum process, displayed in figure 2, where one graviton creates and then absorbs another graviton.

This correction to the gravitational propagator has the generic form

$$\kappa^2 \int d^4k \frac{k^2}{k^2 (p + k)^2},$$

since the propagator and the vertices contribute factors proportional to $\frac{1}{k^2}$ and $k^2$ respectively. Introducing a cut-off $\Lambda$ for the high momentum one readily finds that this graph diverges like $\Lambda^4$. This is much more divergent than the corresponding graph in Q.E.D. Indeed, it is straightforward to see that the divergence of a graph with $V$ propagators and $L$ loops is $\Lambda^r$ where $r = 2V + 4L - 2F = 2L + 2$. To derive this last equation, we have used the topological relation $L = P - V + 1$ which is valid for any graph.
Consequently, unlike in Q.E.D. where the degree of divergence does not increase with the number of loops, for gravity one finds that one encounters worse and worse divergences as the number of loops increases. In fact, the degree of divergence is less than that given above since the final result must be general coordinate invariant. However, taking this into account only reduces the degree of divergence of the two graviton process by two powers of $\Lambda$. A similar, but smaller reduction occurs for the many graviton process. In fact, one encounters new types of divergences at every higher loops order. As one might expect, these divergences cannot be absorbed into the parameters of the theory in the renormalization procedure. This is the celebrated problem of quantum gravity.

It could have happened that the coefficients of the above divergences, which arise in a complicated way from the Feynman rules of the theory, were in fact zero. However, at two loops the coefficient of a divergence which cannot be absorbed has been shown to be non-zero. The problem for gravity can be traced to the fact that the coupling constant $\kappa$ has the dimension of mass$^{-1}$ in units where $\hbar = c = 1$. As a result, higher order corrections have more powers of $\kappa$ and so can have more powers of momentum, and in particular the cut-off, in their numerator. In Q.E.D. the coupling is dimensionless and so the powers of the cut-off that can occur is limited.

It was hoped for some time that supergravity would solve the problem of quantum gravity. Supersymmetric theories are composed of fermions and bosons that are related by a symmetry. In the case of the simplest supergravity theory, supersymmetry relates the graviton to the gravitino which is a spin $3/2$ particle. Long ago, Pauli noticed that fermion loops have associated with them a $-1$ factor and as a result there may be cancellations of divergences between graphs involving fermions and bosons. This is what happens in supersymmetric theories and the degree of the divergences in these theories is less than that one might expect from a naive count. Indeed there even exist certain relativistic four dimensional supersymmetric quantum field theories that have no infinities at all. These theories involve gauge particles of spin 1, fermions of spin $1/2$ and spin 0 particles. Unfortunately, although supergravity theories do have fewer divergences than Einstein gravity it is thought very likely that they have an infinite number of divergences that cannot be absorbed into the parameters of the theory.

1.2 Unification of the forces

The history of modern physics has been characterized by increasing unification which has been achieved by encoding into our theories more and more symmetry. These developments fit well with our natural affinity for symmetric structures, however, it is important
to remember that these unifications have generally come about as a result of efforts to reconcile inconsistencies between different theories and to account for new experimental results. As we explain below, the discovery of the standard model is perhaps the prototype example of this phenomenon.

Although it is not strictly necessary to combine the four forces there are a number reasons, which we now give, to suggest that they should be unified. The standard model by itself is most probably inconsistent. What I have in mind is the problem of triviality. The theory of one scalar field \( \phi \) in four dimensions with interaction term \( \int d^4x \phi^4 \) is known to be renormalizable in the sense discussed above, namely at any order of perturbation theory if one redefines the parameters of the theory in terms of physical quantities then all the infinities of the theory are absorbed. However, a more careful consideration of the full theory has shown that although this theory appears to be well defined in perturbation theory it is in fact trivial. To be more precise, because of the presence of infinities one must regulate the theory, however, having renormalized the theory, if one then removes the regulator then the renormalized coupling constant of the theory vanishes. For example, if one regulates the theory by placing the theory on a lattice, one finds that the physical coupling constant goes to zero as one takes the lattice spacing to zero. Put another way the only consistent theory of this type is free.

Triviality is thought to be related to a very old problem that was first found to occur in Q.E.D by Landau. If one sums to all orders a certain subset of the graphs of perturbation theory then a pole corresponding to a particle with negative mass squared occurs. This disease of the theory called a Landau pole. goes away if the coupling of the theory vanishes. Despite, the fact that this problem was not shown to be present in the full theory it was one of the factors that encouraged some to abandon quantum field theory and pursue the S-matrix approach. With the return of quantum field theory as a result of the successes of the standard model and Q.C.D. the Landau problem was temporally forgotten, but we now realize, partly as a result of its relation to the triviality problem, that it should be taken seriously. A Landau ghost occurs in all theories which do not possess an ultra-violet fixed point at the origin of coupling constant space, or, in other words, theories that are not asymptotically free. Thus it occurs not only in Q.E.D., but also in \( \lambda \phi^4 \) and in theories which contain a \( U(1) \) gauge part. It does not occur, however in non-abelian gauge theories which do not have too many fermions. It is thought that Q.E.D, but also \( \lambda \phi^4 \) and \( U(1) \) gauge theories also suffer from the problem of triviality. Consequently, the standard model itself is thought to suffer from these problems since it contains a \( U(1) \) factor. To obtain a consistent theory of the weak and electromagnetic forces on must embed the standard model in some larger structure, for example a grand unified theory based on a non-abelian group.
Although gauge invariance has become universally accepted as the determining symmetry principle of the forces of nature, it is instructive to remember that our theories, apart from general relativity, were not discovered by adopting this viewpoint. Indeed, the standard model was the result of the synthesis of a great deal of experimental evidence, encoded in four Fermi-theory, and subsequently, the quest to find a consistent theory which coincided with this theory at low energies. The major problem afflicting four Fermi-theory was that it contained the type of infinities discussed above which could not be absorbed by the renormalization process. The resolution of these problems was that the forces must be transmitted by spin one bosons which carried a gauge symmetry and that their masses should result from the introduction of the spin 0 Higgs particle. However, the gauge principle is so far a mathematical principle for which there is no transparent physical requirement. This suggests that the process of unification is not complete in the sense that we are missing certain principles, which would explain why gauge invariance is so essential.

Another drawback of the standard model is that it contains about 20 free parameters, most of these are associated with the way the Higgs particle interacts with the rest of the model in particular the quarks and leptons. Unlike the gauge bosons, whose interactions are determined by the gauge principle there is no known corresponding principle which determines the couplings of the Higgs. One may hope that in unifying the forces one will find such a principle.

At first sight it might seem that supersymmetry, being a Fermi-Bose symmetry, could play a role in relating the spin 0 Higgs to the spin 1/2 quarks and leptons. Unfortunately this hope turns out not to be realized. Supersymmetry does, however, appear to be required for two other reasons. The first reason assumes that we should unify the nuclear strong and weak and electromagnetic forces. Recently, the strength of the strong nuclear coupling constant has now been more accurately measured and it would appear that the two coupling constants of the standard model and that of the strong nuclear force do not meet when evolved to high energy as was once thought to be the case. However, the coupling constants do meet if one starts from theories that are supersymmetric just above the weak nuclear scale. The second reason for believing in supersymmetry is associated with the quantum corrections to the mass of the Higgs. Unlike other particles the Higgs’s mass is not protected by any symmetry in the sense that if it is initially small it need not remain small after quantum corrections. This can be traced to the fact that these corrections are quadratically divergent before they are renormalized. As such, the mass of the Higgs gets swept up to the highest mass scale in the theory as a result of quantum corrections and we must fine tune its value at the classical level in order to negate the effect of the quantum corrections. It turns out that in a supersymmetric extension of the standard model the mass of the Higgs is protected and so no artificial fine tuning is necessary. Note that this
problem is likely to occur in any non-supersymmetric extension of the standard model and the fine tuning will be required as soon as one considers a scale appreciably above that of the weak scale.

In fact it is natural to include gravity when one unifies the strong nuclear force with the nuclear weak and electromagnetic forces. One reason for this is that the unification scale of the forces without gravity is just below the Planck scale. One interpretation of this coincidence resulting from the coupling constants taking the values they do is that when unifying the forces one should also include the force of gravity. An interesting aspect of supersymmetric models of the strong nuclear and electromagnetic forces is that the only natural way to provide the necessary symmetry breaking structures is to include gravity.

Any theory that proposes to unify the forces must include within it the spin 1/2 quarks and leptons, the spin 1 gauge bosons of the strong and weak and electromagnetic forces and the spin 2 gauge boson of gravity. It should also probably contain the spin 0 Higgs and the spin 3/2 gravitino which is the superpartner of the graviton. It is truly remarkable that certain string theories do contain massless particles of precisely all these spins and no more. Strings can be of various types they can be open or closed or they can be supersymmetric or bosonic. In the table below we give the massless particles of these strings, taking in the third column a particular class of closed superstrings.

<table>
<thead>
<tr>
<th>string / spin</th>
<th>open bosonic</th>
<th>closed bosonic</th>
<th>superstring</th>
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<tr>
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<td>3/2</td>
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Massless States of Various Strings

The open bosonic string contains a massless spin one and so must contain the gauge invariance of Yang-Mills theories. The closed bosonic string contains a massless spin two and a massless spin zero. The former particle implies that this string theory contains, in its low energy limit, Einstein's principle of general coordinate invariance. Certain superstrings contain not only the above particles and their corresponding symmetries, but also the required massless fermions; a spin 3/2, or gravitino, which implies that the theory possess supersymmetry and spin 1/2's. Of course, it is not sufficient to find the correct number of spin 1/2's and spin 1's as the known quarks and leptons and gauge bosons respectively;
they must belong to the correct representations of the gauge group $SU(3) \otimes SU(2) \otimes U(1)$ of the standard model and the theory must also give the known fermions their masses, presumably by spontaneously breaking the gauge symmetry. This involves accounting for masses which vary over $10^5$ orders of magnitude. In a large class of string theories one does find gauge groups, such as $E_8$, which do contain the gauge group of the standard model and one can find realistic models which contain the standard model in some detail. However, as we will explain shortly, there are many many string theories in four dimensions all of which are believed to be consistent and the discovery that the standard model is contained amongst some of them is perhaps a less convincing test of string theory than was once hoped.

It is not entirely clear why string theory should contain all the symmetries thought to exist in nature, but it perhaps hints that string theory is part of some deeper principle that embodies these symmetries. It is interesting to note that the quest to find a consistent theory of quantum gravity has resulted in a theory, namely string theory, which involves a number of additional particles to those already observed. These include the gravitino which brings with it the supersymmetry. This can be viewed as somewhat analogous to the search to find a consistent model of weak interactions that resulted in the prediction of the gauge bosons of $SU(3) \otimes SU(2) \otimes U(1)$ and the spin 0 Higgs.

2 The Spectrum of the Bosonic String

We now wish to explain why quantum strings possess the massless spectra discussed above by showing that the open quantum bosonic string possess in its spectrum a massless spin one particle. Many of the features of the calculation of the spectrum are present in the corresponding, but much simpler calculation for the point particle which we shall therefore discuss first. As the point particle moves through space-time it sweeps out a line $x^\mu$ in space-time which we parameterise by $t$. Its motion is so as to extremise the length $A$ of the world-line which is given by

$$A = -m \int ds = -m \int dt \sqrt{-\dot{x}^\mu \dot{x}^\nu \eta_{\mu \nu}}$$

where $\dot{x}^\mu = \frac{dx^\mu}{dt}$. The action or length of the world-line is invariant under reparameterisations $t \rightarrow f(t)$. This invariance is to be expected since our choice of parameterisation has no physical significance. The momentum corresponding to $x^\mu$ is

$$p^\mu = \frac{\delta A}{\delta \dot{x}^\mu} = m \frac{\dot{x}^\mu}{\sqrt{-\dot{x}^\mu \dot{x}^\nu \eta_{\mu \nu}}}.$$

It is straightforward to verify that the momentum satisfies the constraint

$$p^\nu p_\nu + m^2 = 0.$$
The presence of this constraint can be thought of as a consequence of the reparameterisation invariance of the world line. Reparameterisation invariance tells us that the theory is invariant when the dynamical variables are pushed forward in time. It is usually, however, the role of the dynamics is to determine the time evolution. This coincidence of roles for the dynamics and the symmetry for the point particle manifests itself in that the above constraint is, in effect, the only remaining feature of the dynamics.

To quantize the point particle, we replace the classical Poisson Brackets \( \{x^\mu, p^\mu\} = \eta^\mu\nu \) by \( [x^\mu, p^\mu] = \eta^\mu\nu \) in accord with the Dirac rule. We represent this relation by setting \( x^\mu \rightarrow x^\mu + \partial^\mu t \rightarrow -i\hbar \frac{\partial}{\partial x^\mu} \). Upon substituting these expressions into the constraint, it becomes a differential operator which we impose on the wavefunction \( \varphi(x^\mu) \).

\[ (-\partial^2 + m^2)\varphi = 0. \]

This we recognize as the Klein-Gordon equation which describes a spin zero particle. Thus the quantum point particle describes a spin 0 particle.

Carrying out a similar calculation for a super point particle which moves through the superspace with coordinates \( x^\mu, \psi^\mu \) one finds that, upon quantization, it describes the Dirac equation for a spin 1/2 particle.

We now repeat the above calculation for the bosonic string. The bosonic string can be either open or closed as shown in the figure 3. As the string moves, it sweeps out in space-time a two dimensional surface \( z^\mu \), referred to as the world-sheet, which we can parameterise by \( \tau \) and \( \sigma \). This motion is so as to extremise the area \( A \) swept out. The area, which we may take as the action, turns out to be given by

\[ A = -\frac{1}{2\pi\alpha'} \int \sqrt{-\det[\partial_\mu z^\nu \partial_\nu z^\mu]} . \]

In the above expression \( \partial_\mu z^\nu = (\frac{\partial z^\mu}{\partial \tau}, \frac{\partial z^\mu}{\partial \sigma}) = (\dot{z}^\mu, \ddot{z}^\mu) \) and \( \alpha' \) is a constant which has the dimensions of mass^{-2}.

The action is invariant under the transformations of the Poincare group, namely \( z^\mu \rightarrow \Lambda^\mu_\nu z^\nu + a^\mu \) and under arbitrary reparameterisations of the world-sheet. The momentum conjugate to \( z^\mu \) is

\[ p^\mu = \frac{\delta A}{\delta \partial_\mu z^\nu} = \frac{1}{2\pi\alpha'} \frac{\dot{z}^\mu (z^\nu \cdot \dot{z}^\nu) - \ddot{z}^\mu \dot{z}^\nu}{\sqrt{-\det[\partial_\mu z^\nu \partial_\nu z^\mu]}} . \]

Just as for the point particle, the reparameterisation invariance leads to constraints, which are given by

\[ p \cdot p + \frac{1}{(2\pi\alpha')^2} z^\nu \cdot \dot{z}^\nu = 0. \]
and 

\[ x' \cdot \rho = 0. \]

These constraints are easily seen to hold by straightforward substitution of the above expression for the momentum.

Before proceeding with the formal quantisation of the string, we give a heuristic argument which explains some of the most important features of the spectrum of the string. Let us consider a string of length \( L \) that is rigid and rotates about its mid-point with constant angular velocity. It can be shown that this classical configuration solves the above constraints and the equations of motion of the string. A short calculation finds that the energy \( E \) of such a string is \( E = \frac{J^2}{2L^2} \), while the angular momentum \( J \) is \( J = \frac{E^2}{L^2} \). These results, up to constants of proportionality, also follow from dimensional analysis and the fact that since both the energy and angular momentum have only one power of momentum and they can contain only one inverse power of \( \alpha' \). As such, the energy and angular momentum obey the relation \( J = \alpha' E^2 \). When we quantize the string, the angular momentum is quantized in units of \( \hbar \), namely \( J = n\hbar \) and as a result \( \alpha' E^2 = n\hbar \). We may, however interpret the energy bound up in the string, whose center of mass is at rest, as its mass \( m \) and the angular momentum as its spin \( J \). These variables then obey the relation 

\[ \alpha' m^2 = n\hbar \]

We would therefore conclude that the quantum string contains an infinite number of particles whose mass\(^2\) rises linearly with their spin. The above argument also suggests that the string contains other particles which correspond to the possibility that the string can vibrate as it rotates. Despite the very heuristic nature of the above argument, the results are generically correct, although the argument is not sufficiently accurate to tell us which particles are massless since the relation between mass and spin above is missing an additive constant which we will see results from a careful consideration of normal ordering.

Let us now extend the quantization procedure of the point particle to the open bosonic string. The Poisson brackets are replaced by the commutators as well as vanishing commutators for \( x^\mu(\sigma) \) with \( x^\nu(\sigma') \) and \( p^\mu(\sigma) \) with \( p^\nu(\sigma') \). We have chosen \( \hbar = 1 \). These relations can be represented by the replacements

\[ x^\mu(\sigma) \to x^\mu(\sigma), \quad p^\mu(\sigma) \to -i\frac{\delta}{\delta x^\mu(\sigma)} \]
The constraints now become differential operators that we impose on the wave function $\psi$ which is now a functional of $x^\mu(\sigma)$. To analyse these equations, in detail we restrict ourselves to the open bosonic string, for which we take $\sigma$ to range from 0 to $\pi$. As is often the case with a one dimensional extended object, we introduce the normal modes of the string $x^\mu(\sigma) = x^\mu_0 + \sum_{n \neq 0} x^\mu_n \cos n\sigma$. The wave function $\psi$ can be thought of as a function of the $x^\mu_n$ and can be written in terms of the complete set of polynomials of $x^\mu_n$, $n \geq 1$ with coefficients which are functions of $x^\mu_0$, namely

$$\psi(x^\mu_n) = [\psi(x^\mu_0) + x^\mu_1 A_\mu(x^\mu_0) + \ldots] x^\mu_0.$$ 

In this equation $\ldots$ indicates the presence of terms that contain $x^\mu_n, n \geq 2$ and higher polynomials in $x^\mu_n$ and $\psi$ is a fixed function of $x^\mu_0, n \geq 1$ which we will discuss shortly. We recognize the infinite number of fields $\varphi, A_\mu, \ldots$ as the usual type of fields one encounters in conventional field theories since we can identify $x^\mu_0$ with the coordinates $x^\mu$ of space-time. Imposing the constraints, one finds that the fields $\varphi$ and $A_\mu$ obey the equations

$$(\partial^2 + \frac{1}{\alpha})\varphi = 0, \quad \partial^2 A_\mu = 0, \quad \partial^\alpha A_\mu = 0,$$

where $\partial^\alpha = \partial^\mu \partial_\mu$ and $\partial_\mu = \frac{\partial}{\partial x^\mu}$. One also finds an infinite set of equations which are the equations of motion for the infinite number of fields that occur higher up in the expansion of the wave function. We therefore conclude that the quantum open string contains a tachyon $\varphi$, which is a particle whose mass is negative, a massless spin 1, $A_\mu$ and an infinite number of massive modes.

In order to derive the above equations of motion we must introduce the Fourier transform of the above constraints.

$$L_n = \pi \alpha' \int_0^\pi d\sigma [p \cdot p + \frac{1}{(2\pi \alpha')^2} x' \cdot x'] \cos(n\sigma) + i \frac{1}{2\pi \alpha'} x' \cdot p \sin(n\sigma)].$$

As in electrodynamics, one cannot impose all the constraints on the wave function since it turns out that this would imply that the wave function itself vanished. The strongest set of constraints one can impose is

$$L_n \psi = 0, \quad n \geq 1, \quad (L_0 - 1) \psi = 0.$$ 

When defining the quantum string one must make a choice of ordering for the operators $x^\mu(\sigma)$ and $p^\mu(\sigma)$ that appear in the constraints. This normal ordering plays an important role in string theory and is associated with the specific choice of 1 in the latter equation. Making the change of variables from $x^\mu(\sigma)$ to $x^\mu_n$ one finds that

$$p^\mu_n = \frac{-i}{\pi} \frac{\partial}{\partial x^\mu_n} \left[ \frac{\partial^\alpha}{\partial x^\alpha_n} + 2 \sum_{n=1}^{\infty} \cos(n\sigma) \frac{\partial}{\partial x^\alpha_n} \right].$$
Substituting this expression into the constraints, and adopting a suitable normal ordering
constant in \( L_0 \), gives the expressions

\[
L_0 = -\alpha' \partial^\mu \partial_\mu - 2\alpha' \frac{\partial}{\partial x_1^1} \frac{1}{8\alpha'} + \frac{1}{2} + \cdots
\]

and

\[
L_1 = -2\alpha' \partial^\mu \frac{\partial}{\partial x_1^1} \frac{1}{2} x_1^1 \partial_\mu + \cdots
\]

To obtain simple equations of motion we should choose an expansion for the wave function
such that the result of \( L_0 \) acting on it does not reshuffle the terms of the expansion. To
ensure this we choose the fixed function \( \psi_0 = \exp\left(-\frac{x_0^2}{8\alpha'}\right) \). Applying \( L_0 \) and \( L_1 \) to the
wave function and substituting the result into the quantum constraints then leads directly
to the equations of motion above.

In fact, the equations of motion for the higher modes are much easier to find if one
begins by using Hermite polynomials, and their corresponding harmonic oscillators, as a
complete basis for the wave function rather than the ordinary polynomials used above.
This procedure has the advantage that it takes care of the diagonalisation of \( L_0 \) and the
normal ordering problems in a systematic way.

One of the remarkable features of string theory, is that the physical states of the string,
as defined by the above constraints on the wave function, all possess positive norm if the
dimension of space-time was 26 or less. Since the norm can be interpreted as a probability,
a positive norm is clearly a physical requirement. However, with higher spin particles this
is far from easy to achieve. In fact, the scattering amplitudes of the bosonic string develop
other problems with unitarity if the dimension of space-time is not equal to 26.

Although we have only considered the open bosonic string, the calculation for the closed
bosonic string is very similar once one has adopted the appropriate mode expansion. The
additional massless states that occur for the superstrings arise from the presence of the
anti-commuting degrees of freedom that live on these strings.

3 Conformal Symmetry

In the early days of string theory it was thought that string theories were almost unique;
indeed only the open and closed bosonic strings, discussed above, and the open or closed
superstrings had been constructed. The closed superstring could be of two types depending
whether it possess either one or two supersymmetries, but all of these superstrings could
exist only in 10 dimensions. It is now apparent, however, that there exist many many
possible string theories in four dimensions. To understand how this is possible we must first understand the role of conformal symmetry in string theory.

A conformal transformation in $D$ dimensions is a general coordinate transformation $\zeta^\mu \rightarrow \zeta^\mu (\xi^\nu)$ that preserves the Minkowski line element up to a scale factor which may be space-time dependent:

$$ds^2 = d\zeta^\mu d\zeta^\nu \eta_{\mu\nu} = \Omega(\zeta^\nu) ds^2 = \Omega(\xi^\nu) d\zeta^\mu d\zeta^\nu \eta_{\mu\nu}$$

Conformal transformations include those of the Poincare group, which by definition have $\Omega = 1$. Also included are the dilations $\zeta^\mu \rightarrow \lambda \zeta^\mu$ and it is straightforward to verify that the infinitesimal transformations $\zeta^\nu \rightarrow \zeta^\nu + \xi^\nu \xi - a^\mu$ are also conformal transformations. When the dimension of space-time is greater than two the above transformations are all the possible conformal transformations.

Theories that are conformally invariant do not possess a scale since dilation invariance implies that all scales are treated equally. Consider, for example, a scalar field theory in four dimensions. The kinetic term $\int d^4\zeta(-\frac{1}{2} \partial \phi \cdot \partial \phi)$ is invariant provided that $\phi \rightarrow \lambda^{-1} \phi$. This transformation leaves invariant the interaction term $g \int d^4\zeta \phi^4$, but it does not preserve the mass term $m^2 \int d^4\zeta \phi^2$.

In two dimensions, the conformal group is infinite dimensional. This important difference with higher dimensions is most easily seen by writing the line element in terms of the light-cone coordinates $z = \zeta^0 + \zeta^1$, $\bar{z} = \zeta^0 - \zeta^1$:

$$ds^2 = -(d\zeta^0)^2 + (d\zeta^1)^2 = -dzd\bar{z}.$$ 

The conformal transformations are clearly given by

$$z \rightarrow f(z), \quad \bar{z} \rightarrow g(\bar{z}).$$

Infinitesimal conformal transformations are of the form $z \rightarrow z + az^{n+1}$ where $a$ is a constant parameter, as well as a similar transformation for $\bar{z}$. Since the two types of transformations for $z$ and $\bar{z}$ behave in identical ways we will from now on restrict our attention to the former type. These transformations are generated by $L_n = -z^{n+1} \frac{\partial}{\partial z}$ which obey the infinite dimensional Lie algebra

$$[L_n, L_m] = (n - m) L_{n+m}.$$ 

We now examine how conformal symmetry manifests itself in a Poincare invariant field theory with corresponding energy-momentum tensor $T_{\mu\nu}$ which is conserved, $\partial_{\nu} T_{\mu\nu} = 0$.
and can be taken to be symmetric. Let us now also suppose that the energy-momentum tensor is traceless, $T^\mu_\mu = 0$. In any dimension, one can then show that the theory is conformally invariant. This fact is most easily shown in two dimensions by using the light-cone coordinates introduced above. Originally $T^\mu_\nu$ had three components, but with vanishing trace $T^\mu_\mu = T_{\mu\mu} = 0$ the only two remaining components are $T \equiv T_{\mu\nu}$ and $\bar{T} \equiv T_{\nu\mu}$. The conservation of the energy-momentum tensor then becomes

$$\frac{\partial}{\partial z} T = 0, \quad \frac{\partial}{\partial \bar{z}} \bar{T} = 0$$

The infinite number of currents, which generate the above conformal transformations, are given by

$$J^{\mu+1} = T^\mu_{\mu+1}$$

They are obviously conserved, since

$$\frac{\partial}{\partial z} (x^{\mu+1} T) = 0.$$  

The corresponding charges are given by

$$L_n = \int dz x^{\mu+1} T$$

and they obey the Lie algebra for the classical conformal transformations given above.

In a Poincare invariant quantum field theory, we can also consider the case when the energy-momentum tensor is traceless. All the above discussion of the classical case also applies to the quantum case with the exception that the charges $L_n$ obey the modified algebra

$$[L_n, L_m] = (n - m) L_{n+m} + \frac{c}{12} n(n^2 - 1) \delta_{n+m,0}.$$  

The additional term is the result of the normal ordering that is required in the quantum theory. The constant $c$ is called the central charge and it measures the breaking of conformal symmetry by quantum corrections and as such it is an anomaly. However, provided one works in Minkowski space, that is not in curved space-time, one does not see this anomaly since the currents required to break conformal symmetry are not present in the Minkowski space theory.

The simplest example of this phenomenon is provided by a free scalar field theory. The action

$$\int d^2 \zeta (-\frac{1}{2} \partial^2 \phi \partial \phi)$$
is classically conformally invariant and in light cone-coordinates the non-vanishing components of the energy-momentum tensor are given by \( T = -\frac{1}{2} \partial_\alpha \phi \partial_\beta \phi \) and \( T = -\frac{1}{2} \partial_\phi \phi \partial_\phi \phi \).

The conserved charges \( L_n \) then satisfy the above algebra with central charge \( c = 1 \).

### 4 Conformal Symmetry and String Theory

We now explain the role of conformal symmetry in string theory. Let us consider \( D \) scalar fields \( \phi^i \), \( i = 1, 2, \ldots, D \) coupled to gravity in \( d \) dimensions. The corresponding action is

\[
A = \int d^d \zeta \sqrt{-\det g_{\alpha \beta}} [g^{\alpha \beta} \partial_\alpha \phi^i \partial_\beta \phi^i + m^2]
\]

The last term in this expression is a cosmological constant. In one dimension we only have the time, \( \zeta^0 = t \) and we set the one component of the metric \( g_{00} = -V^2 \). In terms of these variables \( \sqrt{-\det g_{\alpha \beta}} = V \), \( g^{00} = -V^{-2} \) and the above action becomes

\[
A = \int dt [-V^{-1} \dot{\phi}^i \dot{\phi}^i + m^2 V].
\]

The equation of motion for \( V \) is \( V^2 \dot{\phi}^i \dot{\phi}^i + m^2 = 0 \). Since this equation is algebraic it allows us to eliminate \( V \) from the above action to find the alternative action

\[
A = -m \int dt \sqrt{-\dot{\phi}^i \dot{\phi}^i}
\]

If we replace \( \phi^i \) by \( x^\mu \) we recognise this as the action for the point particle of mass \( m \).

Provide that \( m \neq 0 \) the above two actions are equivalent. However, while we cannot take \( m = 0 \) in this latter action we can in the former which we take to be the action for a massless point particle. Thus a massless point particle can be thought of as \( D \) scalar fields coupled to one dimensional gravity without a cosmological term.

The action of \( D \) scalar fields \( x^\mu \) coupled to two dimensional gravity, with an appropriate constant of proportionality, is

\[
A = -\frac{1}{4\pi\alpha'} \int d^2 \zeta \sqrt{-\det g_{\alpha \beta}} [g^{\alpha \beta} \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu \nu}].
\]

Eliminating \( g_{\alpha \beta} \), by using its equation of motion, we recover the action for the bosonic string, namely the area swept out. Thus we can consider the bosonic string as \( D \) scalar fields coupled to two dimensional gravity. In this latter formulation, the equation of motion for \( g_{\alpha \beta} \) leads directly to the constraints since they set the energy-momentum tensor \( T_{\alpha \beta} \) for \( D \) free scalar fields to zero. The latter action possess another symmetry, namely Weyl symmetry, under which

\[
g_{\alpha \beta} \rightarrow \Omega(\zeta)g_{\alpha \beta}, \quad x^\mu \rightarrow x^\mu.
\]
When matter is coupled to gravity in such a way that it is invariant under general coordinate and Weyl transformations it is referred to as being coupled to conformal gravity.

In the previous section, we found that classically conformally invariant theories developed anomalies due to quantum corrections. Anomalies do not present a problem for the consistency of a theory unless the corresponding symmetry is a local symmetry, since in this case, the currents which give rise to the anomaly occur in the theory itself. As a result, the anomaly occurs in the scattering amplitudes of the theory and the local classical symmetry is no longer a symmetry of the quantum theory. This situation arises in a theory of electro-weak forces which contains quarks one of which is a singlet rather than in an $SU(2)$ doublet; for then the $SU(2) \times U(1)$ is anomalous. Indeed, when only three quarks had been found, the existence of a fourth quark was predicted using this argument.

As such, in string theory we should insist that there is no conformal anomaly; or that the central charge should vanish. When examining if this is the case we must take into account that we must add ghosts corresponding to the local symmetries such as the reparameterization invariance of the theory. These ghosts contribute to the conformal anomaly, however the form of the ghosts in the action is independent of the matter fields used to construct the theory. If the only local symmetry is reparameterization invariance then the ghosts contribute $-26$ to the value of the central charge. Hence, if we construct the string from $D$ free scalar fields, as is the case for the usual bosonic string, then the central charge will vanish if $D = 26$ since each scalar field contributes one to the central charge.

We now use the lessons learnt for the bosonic string to formulate the general procedure for constructing a consistent string theory. We can construct a consistent string theory by

a) taking any two dimensional conformally invariant classical theory and couple it to conformal gravity, and by

b) demanding that it has no conformal anomaly after taking into account the contribution of the ghosts.

Since one can add two conformally invariant theories to obtain a third which has a central charge that is the sum of the two, we can add such theories until we have cancelled the contribution of the ghosts to the conformal anomaly. In general, the above two criterion do not by themselves guarantee a consistent string theory since such a theory may have anomalies in its space-time symmetries such as the gauge invariance associated with its massless spin one particles. Such anomalies can be eliminated by demanding a third
condition, namely
c) the string is modular invariant.

We will not discuss this last condition further. It is commonly believed that any string theory which satisfies these three requirements will be consistent. More recently, string theories which do have a conformal anomaly have also been considered. It has been found, however that these so called non-critical string theories, upon quantization, develop an additional degree of freedom such that it together with the original theory possess no conformal anomaly.

If we wish to build a superstring theory, then we should start with a theory that is invariant under both conformal symmetry and world-sheet supersymmetry or superconformally invariant. couple this system to superconformal gravity and demand that it have no superconformal anomalies. Such a theory can be constructed from $D$ scalars $x^\mu$ and $D$ fermions $\chi^\mu$ which possess the superconformally invariant action

$$
\int d^2\xi \left( -\frac{1}{2} \left( \partial_\nu x^\mu \partial^\nu x^\mu + \bar{\chi}^\mu \partial_\nu \chi^\mu \right) \right) \eta_{\mu\nu}.
$$

We then couple this system to superconformal gravity and add the ghosts corresponding to the reparameterization and local supersymmetry transformations. These ghosts contribute $-26$ and $+11$ respectively to the central charge. Thus we may cancel the anomaly by taking $D = 10$, since a single free real fermion contributes $\frac{1}{2}$ to the central charge.

Unlike in higher dimensions, the two dimensional world-sheet has only one space dimension and consequently motion to the left or the right has a Lorentz invariant meaning. This fact was exploited in the construction of the heterotic string which contained the fields $x_L^\mu$, $x_R^\mu$ and $\chi_L^\mu$, $\chi_R^\mu$ where the subscripts L and R denote fields that are left and right moving respectively. While the left-handed fields carry a representation of superconformal group, the right-handed fields carry only a representation of the conformal group. As such there are no conformal anomalies if the indices take the ranges $\mu = 0, 1, ... 9$ and $i = 1, 2, ... 32$. The heterotic string possess a spectrum that includes massless particles of spins 2, 3/2, 1, 1/2 and 0. Those of spin 1 belong to the gauge group $E_8 \times E_8$.

The above superstrings possess ten dimensional Poincare invariance, however since we only see four dimensions we require only four dimensional Poincare symmetry. We may also wish to have space-time supersymmetry and it turns out that this will be the case if either the left or right handed coordinates carry a representation of world-sheet supersymmetry. Thus we may construct string theories in four dimensions which are constructed from the
fields \( x_{\mu}^i, \chi_{\mu}^i, \chi_{\mu}^j \), \( \mu = 0, 1, 2, 3 \), and any other fields which comprise a conformal system and cancel the conformal anomaly. There are many many four dimensional string theories that can be constructed in this way. Although among these string theories there are some which are realistic in that they appear to contain the standard model, the uniqueness that string theory appeared to provide has been lost.

5 Critical Phenomena and String Theory

The graph given in figure 4 shows the phase diagram for a substance such as \( H_2O \). Examining the gas-liquid transition one observes that the line terminates at a certain point referred to as the critical point. As one moves up the line towards this point the latent heat required to convert liquid to gas becomes less and less until at the critical point there is no latent heat required at all. Such a point is called a second order phase transition and it occurs in many different substances and for many different types of transition. Perhaps the most celebrated being the occurrence of spontaneous magnetization. Over the years a large amount of data has been collected on the behavior of such systems in the neighborhood of their critical points. It has emerged that the deviation of quantities from their values at their critical point exhibit a power law behavior. For example for the liquid-gas transition, the deviations of the density and temperature from their critical values \( \rho_c \) and \( T_c \) respectively obey the relationship

\[
(\rho_L - \rho_c) \propto (T - T_c)^\beta
\]

Two remarkable features emerged from such systematic observations concerning the powers, such as \( \beta \), which were referred to as critical indices. These critical indices were not integers and for two systems that possessed the same symmetry and lived in the same dimension they were the same. The latter fact is referred to as universality as it occurred for systems that seemed to have no connection apart from the fact that they both possessed a critical point.

The explanation of these observations relied on an application of the renormalization group. It was realized that a system at a critical point is dominated by large scale fluctuations corresponding to the existence of massless excitations. This suggested that no scale was preferred and that the system at the critical point corresponds to a system at an infra-red fixed point of the renormalization group. Such a system is known to be dilation invariant, but it is also thought to be invariant under the full conformal group. Universality is explained by the fact that if one considers the space of field theories of the same symmetry and in a given dimension, then there are very few infra-red fixed points. Consequently, for large regions of this space it does not matter which theory, or equivalently,
where in the space one starts one is always driven to the same infra-red fixed point as one approaches the critical point. The non-integer nature of the critical exponents reflects a non-analyticity of the dependence of the partition function on the external parameters of the system. This can only originate from a sum over an infinite number of configurations, which in turn must arise from a system with an infinite number of degrees of freedom. This observation is also consistent with the presence of large scale fluctuations.

The partition function of the system, near the critical point, can be viewed as a conformal Euclidean quantum field theory:

$$Z = \sum_{\text{configurations}} e^{-\beta H} = \int \mathcal{D}\phi e^{-\beta H}.$$  

This is true even for lattice models such as the Ising model, since near the critical point large scale fluctuations allow us to replace the discrete lattice by a continuum. Thus systems at a critical point correspond to a conformal field theory.

One of the most important developments in the eighties was the discovery, by Belavin, Polyakov and Zamolodchikov, that they could solve a large class of conformally invariant two dimensional quantum conformal field theories. This was possible as a result of the infinite dimensional nature of the two dimensional conformal group discussed above. Solve meant that one could write down the explicit expressions for the Green's functions of the theory. In their procedure an important role was played by the central charge which had to belong to specific series in order that the theory was solvable. The simplest such series is for unitary theories and it is given by

$$c = 1 - \frac{6}{m(m+1)}, \quad m = 3, 4, \ldots$$

This series contained many of the most interesting models, for example the Ising model which has $c = \frac{1}{2}$.

In the previous section, we saw that we could build string theories out of conformally invariant theories. Clearly, we can use the conformal theories in the above series. In fact, one finds many string theories in this way. One very interesting conjecture is that certain of these string theories appear to be the same as starting from the heterotic string and taking 6 of the $x^i$ to belong to a Calabi-Yau manifold. This appears to make a fascinating connection between the properties of two dimensional field theory and the geometry of these ill-understood manifolds. Also one can hope to make use of the knowledge acquired in the study of critical phenomenon, such as the ability to flow from one conformal field theory to another.
W-Algebras

We have seen above the crucial role that the conformal algebra has played in string theory. It has been found that there exist new algebras, called W-algebras, that are extensions of the conformal algebra.

Consider a two dimensional quantum field theory which possess spin two and spin three conserved currents denoted $T_{\mu\nu}$ and $W_{\mu\nu}$, respectively. We will assume that these currents are symmetric in their indices. Let us assume in addition that $T_{\mu\nu} = 0$ and $W_{\mu\nu} = 0$. In this case, we have only two components of $T_{\mu\nu}$ and $4 - 2 = 2$ components of $W_{\mu\nu}$. Using the light-cone coordinates $z, \bar{z}$, as before, these components are given by $T = T_{++}, T = T_{+\bar{z}}$, and $W = W_{++}, W = W_{+\bar{z}}$. The conservation laws now read

$$\frac{\partial}{\partial z} T = 0, \quad \frac{\partial}{\partial \bar{z}} W = 0,$$

as well as similar equations for $\bar{T}$ and $\bar{W}$. It obviously follows that the doubly infinite set of quantities $z^n T$ and $\bar{z}^n W$ are also conserved leading to charges $L_n$ and $W_n$. It will prove useful to note that any polynomial in $T$ and $W$ is also conserved.

Given the explicit expressions for the charges $L_n$ and $W_n$ in any field theory with the above properties we can calculate their algebra. The commutators are of the form

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12} n(n^2 - 1)\delta_{n+m,0}$$

$$[L_n, W_m] = (2n - m)W_{n+m}$$

as well as a more complicated one which has

$$[W_n, W_m] = \frac{16}{22 + 5c}(n - m)A_{n+m}$$

$$+ (n - m) \left[ \frac{1}{15} (n + m + 2)(n + m + 3) - \frac{1}{6} (n+2)(m+2) \right] L_{n+m} + \frac{c}{360} n(n^2 - 1)(n^2 - 4)\delta_{n+m,0}.$$ 

The first commutator is the same as we found previously, while the second expresses the simple fact that $W$ has spin 3. The final result is rather complicated, but it has one very important property, namely it is not linear in the generators. Consequently, this W-algebra is not a Lie algebra. Nonetheless it is a consistent algebra in that it obeys the Jacobi identities.

One can repeat the above exercise by supposing the theory also has a conserved, but symmetric traceless rank $n$ tensor $W_{\mu_1...\mu_n}$. This tensor has only two components since
it initially has \( \frac{(n+1)!}{n!} = n + 1 \) components, but we must subtract \( \frac{-1}{n-2} \) components due to the trace. In light-cone coordinates these components are \( W_{ij} \) and \( W_{ij} \) and one again finds an infinite set of conserved quantities. In general, one can have a number of generators with spins higher than two, but for every such consistent theory one finds that the corresponding algebra is not a lie algebra. The one exception to this is when one has an infinite number of higher spin generators. As the above suggests, there exist many such \( W \)-algebras and it is not clear at present how to classify them.

The \( W \)-algebras first made their appearance in the context of the minimal models discussed in the previous section. It turned out that the minimal model with central charge \( \frac{1}{2} \) has, in addition to its energy momentum tensor, a conserved spin three field which obeys the algebra above. Other \( W \)-algebras were found to be embedded in other conformal field theories. However, they were also found to appear in integrable models non-perturbative string theory, the fractional quantum Hall effect and a number of other areas.

After the discovery of \( W \)-algebras, new string theories based upon them were constructed. This was achieved by following the strategy laid out in the section 4, but the role played by the Virasoro or conformal algebra was played by the \( W \) algebra. It has been found that these theories have physical states and scattering amplitudes which contain Ising model states and correlation functions respectively. The significance of this result is still unclear.

7 Integrable Models

A classical system with \( n \) degrees of freedom is called integrable if it possess \( n \) conserved quantities. In this case, one can find a canonical transformation to a new set of variables such that the \( n \) momenta are the \( n \) conserved quantities. In these new coordinates, the dynamics is trivial.

For many years a number of systems with an infinite number of degrees of freedom that possess an infinite number of conserved quantities have been known. Two of the most well known are the KdV equation

\[
\frac{\partial}{\partial t} u = \frac{1}{8} \frac{\partial^3 u}{\partial x^3} - \frac{3}{8} \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2},
\]

and the non-linear Schrodinger equation,

\[
\frac{\partial}{\partial t} \psi + \frac{\partial^2}{\partial x^2} \psi + 2 \alpha \psi^* \psi = 0.
\]
Although there are a number of methods of finding the infinite number of conserved quantities for the above and other such classical theories, these methods do not generalize when these systems are quantized. Nonetheless, considerable evidence suggests that these quantum systems do possess an infinite number of conserved quantities.

What has also emerged is that such systems are connected to the $W$-algebras discussed in the previous section. The KdV equation is related to the Virasoro or conformal algebra, in that it possess a Hamiltonian formulation such that the Poisson bracket algebra for the Fourier components $u_n$ of the $u$ obey the same algebra as the $L_n$ given above. In fact, the KdV equation is the first of an infinite hierarchy of equations. The $p$th member of this hierarchy involves $p$ variables $u_1, \ldots, u_p$ which have a Poisson bracket algebra that is a $W$-algebra that involves generators with spins $2, 3, \ldots, p+1$. More recently, it has emerged that a number of other integrable systems have been found to be associated with $W$-algebras. For example, the non-linear Schrödinger equation is associated with a $W$ algebra that contains an infinite number of a generators one for every spin.

It would now seem likely that all integrable systems are associated with a $W$-algebra and that integrability may be seen as a consequence of this connection. Indeed it is possible that $W$ algebras may be used to classify integrable systems. The connection with $W$-algebras should also help in finding the conserved quantities in the quantum case.

The relation between integrability and $W$-algebras seems less mysterious as a result of some work of Zamolodchikov. He showed that one can start from certain conformal field theories and perturb them by special terms which are such that the resulting theory still possess an infinite number of conserved quantities even though the additional terms are not themselves conformally invariant. These infinite number of conserved quantities, which are polynomials in $T$ and the $W$'s, are inherited from amongst the infinite number of conserved quantities of the conformal theory since they are the ones that commute with the perturbation. For example, the KdV equation is of this type and its conserved quantities can be viewed as the set of commuting polynomials in the energy momentum tensor which survived the introduction of the perturbation. The above suggests that all integrable models can be obtained in this way.

Let us consider the space of all field theories. Some of the points in this space will be conformal field theories and from these special points we may flow along integrable perturbations to find the integrable field theories. This picture is also of interest to string theory since some of the conformal points may form part of different string theories. Thus
certain flows in the space of all field theories may relate different string theories. Indeed, it has been found that the flows between certain string theories constructed from minimal models are governed by the KdV and the higher such equations. Given the vast number of different string theories it is of interest to find ways of relating them in the hope that a mechanism similar to the Higgs mechanism, will provide a way of distinguishing between them and that the favoured theories will be those that contain the standard model and even perhaps fix some of the 20 or so parameters of the theory.

While it is possible that the above strategy will be successful, there is another camp of opinion which believes that string theory is part of a much deeper process that will lead to a theory built from concepts some of which are radically different from those that we currently use. The most obvious casualty in this process is space-time which there are many reasons to believe is not a fundamental concept. Indeed there are some indications that string theory would like to abandon space-time. The problem is that if we do not use space-time what should we use. One interesting suggestion is to replace our space-time by some operator algebra, but the way this is to be achieved has yet to be spelt out. It is conceivable that some of the ideas discussed above will be useful in this process.
Fig. 1  Correction to the Electron Propagator

Fig. 2  Correction to the Graviton Propagator

Fig. 3  Types of Bosonic String

Fig. 4  Phase Diagram of H₂O