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CLASSICAL AND SEMI-CLASSICAL SOLUTIONS OF THE YANG-MILLS THEORY

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CONTENTS

This Review summarizes what is known at present about classical solutions to Yang-Mills theory both in Euclidean and Minkowski space. The quantal meaning of these solutions is also discussed. Solutions in Euclidean space expose multiple vacua and tunnelling of the quantum theory. Those in Minkowski space-time provide a semi-classical spectrum for a conformal generator.

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I INTRODUCTION

The strategy recently evolved for extracting new information from a quantum field theory consists first of ignoring the quantal character of the operator fields, and solving field equations as non-linear partial differential equations for classical functions. These solutions are then studied by various semi-classical approximations methods to yield information about the quantized theory.[1]

Classical solutions can be conveniently characterized by their space-time dependence. Most familiar are time- and space-independent solutions; these are constants which satisfy the field equations. The quantum significance of such fields has been known for some time: when they describe stable configurations of finite [zero] energy, these constants are approximations to the vacuum expectation value of the quantum field and frequently signal spontaneous symmetry breaking.

Attention has been drawn also to time-independent but space-dependent solutions, such as the kink in one spatial dimension and the monopole in three. These — the solitons — are stable finite-energy field configurations and their quantal significance is known: classical soliton solutions signal the existence, in the quantum theory, of coherent bound states which describe heavy particles, the quantum solitons. The classical soliton energy is a weak coupling approximation to the quantum soliton's mass, while the Fourier transform of the classical soliton field approximates the matrix element of the quantum field between one-soliton momentum eigenstates.

Time- and space-dependent solutions are much harder to come by, simply because the non-linear partial differential equations are sufficiently complicated to prevent a complete analysis. A prominent exception is the sine-Gordon theory where time dependent periodic and scattering multi-soliton solutions have been obtained. A semi-classical quantization provides information about the bound states and the S matrix for quantum solitons. Moreover, the model turns out to be sufficiently transparent so that a complete quantum-mechanical solution is possible. The exact results provide an important check on the approximate ones: the two agree where expected — for weak coupling. [In fact the WKB bound state spectrum turns out to be exact!] No such success has been achieved for realistic models in three dimensions. However most recently some very interesting space and time-dependent solutions of the Yang-Mills equations have been found. Their quantum meaning is now being explored, and some of our [tentative] ideas regarding them will be presented in Section IV.

The study of the physics of solitons has exposed several fascinating effects,which should be briefly recalled. For weak coupling strength g, there are three scales of interaction strength. The interactions of the ordinary particles of the theory are weak, $O(g)$; the solitons however interact strongly $O(g^{-1})$; the interaction between solitons and ordinary particles is of intermediate strength, $O(g^0)$. New types of conserved

quantum numbers have been discovered. These insure the stability of the quantum soliton, but do not arise from local conservation laws of the Noether variety; rather they reflect topological properties of the field configurations. Furthermore, a startling phenomenon has been found: conversion of bosons to fermions, and correspondingly conversion of internal symmetry degrees of freedom into spin degrees of freedom. Finally,the coupling of Dirac fermions to the solitons has produced peculiar zero-energy bound states with profound effects on the theory.

While we have clearly learned that a quantum field theory gives rise to a much richer variety of phenomena than previously seen in perturbative Feynman-Dyson expansions, the fact remains that in theories which are presently intensely studied as possible candidates for a fundamental theory of natural processes — the Yang-Mills gauge models of strong interactions or of unified weak and electromagnetic interactions — no soliton solutions have been found. Indeed there are non-existence theorems which indicate that something different must be done, if one wants to apply semi-classical ideas to these models.[2]

There is yet a further type of classical solution that can be considered; a solution not of the original equations, but rather of modified equations which are obtained by replacing time by imaginary time $t=x^0 \to -ix_4$ [and similarly changing the time components of all tensors]. What then is the quantal significance of these Euclidean fields? For an immediate answer recall that practical calculations in quantum field theory are most frequently performed in Euclidean space which is reached by a Wick rotation. Thus one may expect classical Euclidean fields to contain some information about the quantum theory. More precisely, one may formulate the quantum theory in Euclidean space by a functional integral or by an operator method, compute various semi-classical amplitudes, and continue back to Minkowski space. Moreover, there is a more physical reason for finding Euclidean [imaginary-time] solutions. It is well-known that such classical solutions signal, in the corresponding quantum theory, the occurrence of tunnelling — i.e. there is motion which, though classically forbidden, is allowed quantum mechanically.

Imaginary-time solutions to pure Yang-Mills theory have indeed been found; in these Lectures we review both the original example of Belavin, Polyakov, Schwartz and Tuypkin,[3] and the later generalizations.The physical import of the original solution has been now established; it gives evidence of a rich non-perturbative structure to the quantum theory. The following description emerges.

For the classical SU(2) Yang-Mills theory in the gauge $A_a^o=0$, a=1,2,3, the classical vacua — that is, the classical zero-energy configurations — are gauge potentials, which themselves are pure gauges. The gauge functions $g(\vec{r})$ are time-independent 2×2 unitary matrices; they need be characterized further to obtain a complete description of the classical vacuum. An important

gauge function is the constant one, leading to a vanishing gauge potential. One need not consider those $g(\vec{r})$ which are singular functions of $\vec{r}$, nor those that do not tend to a constant for large $\vec{r}$, since the corresponding gauge potentials are separated by an infinite energy barrier from the vanishing one and are presumably irrelevant to the physical sector which includes the vanishing potential. The remaining gauge functions — those that do tend to a constant at spatial infinity — are mappings from 3-dimensional space [augmented by the point at infinity] to the SU(2) group, and as such can be arranged into homotopically inequivalent classes characterized by the integers $n=0,\pm1,\ldots$. Gauge functions belonging to distinct homotopy classes cannot be continuously deformed into each other.

Pure gauge potentials constructed from the gauge functions can thus be classified by the integers, n. Gauge potentials within a class are gauge-deformable one into another by gauge transformation built from gauge functions of the trivial, class, n=0; these gauge transformations are called "small" gauge transformations. On the other hand two gauge potentials which belong to different classes can be gauge-deformed into each other only by gauge transformations built from gauge functions of some non-trivial class, $n\neq0$; these are called the "large" gauge transformations.

In the quantum theory, there are distinct states $|n\rangle$ describing the gauge potential in each characteristic class n. No physical significance is given to the degrees of freedom associated with small gauge transformations. Indeed conventional gauge fixing procedures in the $A_a^o=0$ gauge are tantamount to imposing Gauss's law on the physical states and as a consequence the states $|n\rangle$ are invariant under small gauge transformations. Under the large gauge transformation $\mathcal{G}$, $|n\rangle$ transforms to $|n+1\rangle$, and a linear superposition must be taken to obtain a gauge invariant description for physical processes. However there is no requirement that the physical states be invariant under a large gauge transformation; gauge invariance is still achieved if the action of $\mathcal{G}$ on a physical state produces a phase. Therefore the linear superposition must be of the form $|\theta\rangle=\sum_n e^{in\theta}|n\rangle$ and $\mathcal{G}|\theta\rangle=e^{-i\theta}|\theta\rangle$.

To complete the description of the quantal low-lying states, we must ascertain whether the states $|\theta\rangle$ are degenerate in energy, or whether tunnelling splits them. It is here that the pseudoparticle solution becomes relevant. One notes that this solution interpolates, as its imaginary time parameter passes from $-\infty$ to ∞, between a vanishing gauge potential — one that evidently belongs to the $n=0$ class — and a gauge potential belonging to the $n=1$ class. We conclude that in the quantum theory there is tunnelling; the energy levels acquire a θ dependence and exhibit a band spectrum.

This Bloch wave picture is dramatically altered when massless fermions are included in the theory. The anomaly of the axial vector current renders the conserved chiral charge $\tilde{Q}_5$ gauge non-invariant under large gauge transformations. Specifically one finds $\mathcal{G}\tilde{Q}_5\mathcal{G}^{-1}=\tilde{Q}_5+2$. It is impossible to diagonalize simultaneous H, $\mathcal{G}$ and $\tilde{Q}_5$. Physical considerations require that energy

eigenstates diagonalize $\mathcal{G}$; hence they are chirally non-invariant. Indeed chiral transformations shift θ; since they also commute with H, the θ dependence of energy eigenvalue disappears and tunnelling is suppressed.

Detailed calculations based on the above physical picture have been performed and were described by 't Hooft.[4] The original pseudoparticle solution is used in an approximate evaluation of the Euclidean functional integral. The physical significance of more general Euclidean solutions has not as yet been established; they appear to give insignificant corrections to the amplitudes described by 't Hooft. Nevertheless, we feel that it is important to undertake a detailed study of all solutions to Yang-Mills theory, for several reasons. Firstly, it is self-evident that any information about the theory will be helpful in establishing its physical content. Let us recall, especially, that computations of the dominant effects are not completely satisfactory since they suffer from uncontrollable infra-red divergences which reflect the infra-red instability of the theory. Secondly, our analysis of this system has put us in contact with parallel developments in pure mathematics. That there should be a conjunction of interests between modern mathematics and physics is truly a gratifying circumstance, and we are happy to be participating in it. The collaborative physical-mathematical efforts yield a new understanding of the axial-vector-current anomaly through the Atiyah-Singer index theorem. Also they give some hints that the Yang-Mills equations are considerably simpler than appear at first, and may even in some sense be linear. Further understanding of these possibilities shall certainly be substantial progress towards a solution of the theory.

II PSEUDOPARTICLE CONFIGURATIONS

We begin these Lectures by a study of solutions to the Yang-Mills field equations in Euclidean four-space. We shall consider an SU(2) gauge group and represent the potentials and field strengths as anti-hermitian matrices in the space of infinitesimal generators, with the gauge coupling constant e scaled out.

$$\frac{1}{e} A^{\mu} = A^{\mu}_{a} \frac{\sigma^{a}}{2i} \tag{2.1a}$$

$$\frac{i}{e} F^{\mu\nu} = F^{\mu\nu}_{a} \frac{\sigma^{a}}{2i} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} + [A^{\mu}, A^{\nu}] \tag{2.1b}$$

σ^a (a=1,2,3) are the Pauli matrices, and summation over repeated indices is implied.

The Yang-Mills field equations

$$D^{A}_{\mu} F^{\mu\nu} = \partial_{\mu} F^{\mu\nu} + [A_{\mu}, F^{\mu\nu}] = 0 \tag{2.2}$$

follow from the requirement that the action S be stationary.

$$S = -\frac{1}{2} \int d^4x \; F^{\mu\nu} F_{\mu\nu} \tag{2.3}$$

A. Topological Considerations

As in Ref. 3, we shall proceed by establishing a lower bound on S. The bound is saturated if the fields satisfy a set of first order non-linear differential equations, which of course imply the second order Eqs. (2.2). Throughout this section we shall consider only solutions to the Yang-Mills equations which minimize the action by saturating its lower bound. No other solutions with finite action have been found in the Euclidean domain, and one may conjecture that the class we consider is exhaustive of all finite action solutions.

We define the dual $*F^{\mu\nu}$ of the field strength by the totally antisymmetric tensor $\varepsilon_{\mu\nu\rho\sigma}(\varepsilon_{1234}=1)$.

$$*F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \quad (2.4)$$

The inequality

$$-\frac{1}{4}\int d^4x \, tr\left(F^{\mu\nu} \pm *F^{\mu\nu}\right)\left(F_{\mu\nu} \pm *F_{\mu\nu}\right) \geq 0 \quad (2.5)$$

together with the algebraic identity $F^{\mu\nu}F_{\mu\nu}=*F^{\mu\nu}*F_{\mu\nu}$, establishes a lower bound on S.

$$S \geq \left|\frac{1}{2}\int d^4x \, tr \, *F^{\mu\nu}F_{\mu\nu}\right| \quad (2.6)$$

We shall soon show that if S is finite the right hand side of this inequality does not depend on the detailed features of the field configuration, but only on general topological properties of the boundary values of the potentials A^{μ}. More precisely, it will emerge that the requirement of finite action separates all possible field configurations into equivalence classes of potentials which can be continuously distorted into each other. Within each class the quantity

$$q = -\frac{1}{16\pi^2}\int d^4x \, tr \, *F^{\mu\nu}F_{\mu\nu} \quad (2.7)$$

takes a definite integer value, called the "Pontryagin index" of the field configuration. Postponing the proof of these statements for a moment, we see that the bound on the action

$$S \geq 8\pi^2|q| \quad (2.8)$$

is saturated if

$$F^{\mu\nu} = \pm *F^{\mu\nu}, \quad (2.9)$$

i.e., if the field is self-dual or anti-self-dual. Eq. (2.9) implies the Yang-Mills equations (2.2). This follows also from $*F^{\mu\nu}=\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ and the Bianchi identity satisfied by $*F^{\mu\nu}$.

$$D^A_\mu *F^{\mu\nu} = \partial_\mu *F^{\mu\nu} + [A_\mu, *F^{\mu\nu}] = 0 \quad (2.10)$$

What we learn here is that the self-duality or anti-self-duality condition is the equation for the absolute minimum of the action within a definite Pontryagin class. In the literature the self-dual field configurations with Pontryagin index q=1 are often referred to as pseudoparticles [or instantons]; those with $|q|>1$ are called multi-pseudoparticle configurations.

The proof that q is a topological invariant, to which we now return, proceeds as follows. Notice first that $tr*F^{\mu\nu}F_{\mu\nu}$ is a divergence

$$tr \, *F^{\mu\nu}F_{\mu\nu} = \partial_\mu X^\mu \quad (2.11)$$

with

$$X^\mu = 4\varepsilon^{\mu\alpha\beta\gamma} tr\left(\frac{1}{2}A_\alpha\partial_\beta A_\gamma + \frac{1}{3}A_\alpha A_\beta A_\gamma\right) \quad (2.12)$$

Assuming for the moment that A^μ has no singularities, the integral in Eq. (2.7) can be replaced by a boundary surface integral,

$$-\frac{1}{16\pi^2}\int d^4x\, \mathrm{tr}\, {}^*F^{\mu\nu} F_{\mu\nu} = \lim_{R\to\infty} -\frac{1}{16\pi^2}\int_\sigma d^3\sigma\, n_\mu X^\mu \tag{2.13}$$

where σ is a surface enclosing a sphere of radius R and n^μ is its outward normal. The requirement that the action be finite demands that $F^{\mu\nu}$ approach zero faster than $|x|^{-2}$ as $|x|\to\infty$. The field configuration then becomes integrable and one can find a unitary matrix $g(x)$ such that, for $|x|\to\infty$,

$$A^\mu = g^{-1}\partial^\mu g + O(|x|^{-2}) \tag{2.14}$$

$g(x)$ defines a mapping from the points of any boundary surface σ into the manifold G of the SU(2) group. This mapping determines q; substituting Eq. (2.14) into Eqs. (2.12) and (2.13), we find

$$q = \lim_{R\to\infty}\frac{1}{24\pi^2}\int_\sigma d^3\sigma\, \varepsilon^{\mu\alpha\beta\gamma} n_\mu\, \mathrm{tr}\left\{(g^{-1}\partial_\alpha g)(g^{-1}\partial_\beta g)(g^{-1}\partial_\gamma g)\right\} =$$

$$\lim_{R\to\infty}\frac{1}{24\pi^2}\int_\sigma d^3\sigma\,\frac{\partial(g)}{\partial(\sigma)} = \lim_{R\to\infty}\frac{1}{24\pi^2}\int_G d^3g \tag{2.15}$$

where d^3g is the invariant volume element of the SU(2) group. Thus

we see that q counts how many times the volume of the group manifold G is covered by $g(x)$ as we let the argument x^μ span the whole surface at infinity. The fact that A^μ must approach a pure gauge at infinity follows from the requirement of finite action, and then it is obvious that no continuous deformation of A^μ preserving the finiteness of the action can modify the value of q.

The equivalence classes of potentials A^μ that can be continuously distorted into each other are in one-to-one correspondence with the equivalence classes of mappings $\sigma\to G$ which can be continuously related. Because σ is topologically equivalent to a three-dimensional sphere S^3, these classes are also in one-to-one correspondence with the elements of the third homotopy group of G.

A realization of the boundary conditions leading to $q=1$ that will be very convenient for our analysis is obtained by demanding that as $|x|\to\infty$ $A^\mu\to g^{-1}\partial^\mu g$, with

$$g(x) = \frac{\alpha\cdot x}{\sqrt{x^2}} \qquad g^{-1}(x) = \frac{\bar\alpha\cdot x}{\sqrt{x^2}} \tag{2.16}$$

$$\alpha^\mu = (-i\vec\sigma, I) \qquad \bar\alpha^\mu = (\alpha^\mu)^\dagger = (i\vec\sigma, I) \tag{2.17}$$

Let us define

$$A_0^\mu = g^{-1}\partial^\mu g = -2i\,\sigma^{\mu\nu}x_\nu/x^2 \tag{2.18}$$

$$\sigma^{\mu\nu} = \frac{1}{4i}[\bar\alpha^\mu, \alpha^\nu] = {}^*\sigma^{\mu\nu}, \quad \sigma^{i4} = \frac{\sigma^i}{2}, \quad \sigma^{ij} = \varepsilon^{ijk}\frac{\sigma^k}{2} \tag{2.19}$$

Inserting A^μ into Eq. (2.12) we find

$$X^{\mu} = -8x^{\mu}/x^{4} \tag{2.20}$$

and

$$\lim_{R\to\infty} -\frac{1}{16\pi^{2}} \int_{\Omega} R^{3}\, d\Omega\, \frac{X^{\mu}x_{\mu}}{R} = 1 \tag{2.21}$$

which shows that a regular field A^{μ} approaching A_{0}^{μ} at infinity has q=1.

This computation also shows that we must recognize an important point. The field strength associated with the pure gauge potential A_{0}^{μ} of course vanishes, so that q, for this case, must be zero. Written as a surface integral, q receives a contribution +1 from the boundary term at infinity. This must be cancelled by some other surface contribution at finite x^{μ}. Indeed, A_{0}^{μ} is singular also at the origin, as is apparent from Eq. (2.18), and the contribution to q coming from a small surface enclosing the origin is -1. We retain from this example the fact that the Pontryagin index, expressed as a surface integral over the group volume [Eq. (2.15)], may receive contributions from all the singularities of the gauge potential A^{μ}. This will be very relevant in the following.

B. One-Pseudoparticle Solution

From the discussion presented above we also see that a field configuration with q=1 is obtained if A_{0}^{μ} is multiplied by a function $f(x^{2})$ such that $f(0)=0$, $f(\infty)=1$. Let us consider then

$$A^{\mu} = -2i\, f(x^{2})\, \frac{\sigma^{\mu\nu}x_{\nu}}{x^{2}} \tag{2.22}$$

Symmetry considerations, which will be expanded later, suggest that the functional form of this Ansatz is compatible with the self-duality equation, leading to an equation for $f(x^{2})$. From Eq. (2.22) we evaluate

$$F^{\mu\nu} = 4i\left(f - f^{2}\right)\frac{\sigma^{\mu\nu}}{x^{2}} + 4i\left(f' - \frac{f}{x^{2}} + \frac{f^{2}}{x^{2}}\right)\left(\sigma^{\mu\rho}x_{\rho}x^{\nu} - \sigma^{\nu\rho}x_{\rho}x^{\mu}\right) \tag{2.23}$$

The matrix valued tensor $\sigma^{\mu\nu}$ is self-dual and the condition $F^{\mu\nu}={}^{*}F^{\mu\nu}$ is satisfied if and only if

$$x^{2}f' - f + f^{2} = 0 \tag{2.24a}$$

This equation is solved by

$$f(x^{2}) = \frac{x^{2}}{\lambda^{2} + x^{2}} \tag{2.24b}$$

with λ^{2} being an arbitrary scale. The resulting field strength is

$$F^{\mu\nu} = \frac{4i\lambda^{2}}{(\lambda^{2}+x^{2})^{2}}\,\sigma^{\mu\nu} \tag{2.25}$$

If the function $f(x^{2})$ is subject instead to the boundary conditions $f(0)=1$, $f(\infty)=0$, the ensuing field configuration will have q=-1. The anti-self-duality condition $F^{\mu\nu}=-{}^{*}F^{\mu\nu}$ can then be imposed and it is solved if

$$f(x^2) = \frac{\lambda^2}{\lambda^2 + x^2} \tag{2.26}$$

leading to

$$F^{\mu\nu} = \frac{4i\lambda^2}{(\lambda^2+x^2)^2 x^2}\left[\sigma^{\mu\nu}x^2 - 2\sigma^{\mu\rho}x^\nu x_\rho + 2\sigma^{\nu\rho}x^\mu x_\rho\right]. \tag{2.27}$$

Notice that the right hand sides of (2.24b) and (2.26) add to 1 and that the sum of the corresponding gauge potentials is the pure gauge A_0^μ.

Field configurations with q=-1, but without a singularity at the origin, or with q=+1, and with a singularity at the origin, can be obtained by replacing the matrices $\sigma^{\mu\nu}$ in Eqs. (2.18) to (2.27) by matrices $\bar{\sigma}^{\mu\nu}$ defined as

$$\bar{\sigma}^{\mu\nu} = -*\bar{\sigma}^{\mu\nu} = \frac{1}{4i}\left[\alpha^\mu, \bar{\alpha}^\nu\right],$$

$$\bar{\sigma}^{i4} = -\sigma^{i4}, \quad \bar{\sigma}^{ij} = \sigma^{ij}. \tag{2.28}$$

The matrices $\bar{\sigma}^{\mu\nu}$ are obtained from $\sigma^{\mu\nu}$ by a parity inversion of the 4th axis and are anti-self-dual.

C. Symmetries of the One-Pseudoparticle Solution

The action density of the pseudoparticle

$$-\frac{1}{2}\,\mathrm{tr}\,F^{\mu\nu}F_{\mu\nu} = \frac{48\lambda^4}{(\lambda^2+x^2)^4} \tag{2.29}$$

is spherically symmetric. This suggests that the field configuration itself may be symmetric under O(4) rotations. The expressions for A^μ and $F^{\mu\nu}$ are not explicitly symmetric; for instance, the right hand side of Eq. (2.25) remains invariant under an O(4) rotation, whereas $F^{\mu\nu}$ must transform as a second rank tensor. However, the apparent non-symmetry may be compensated by an appropriate gauge transformation. Let us consider a combined rotation in the Euclidean and SU(2) spaces, generated by

$$\Theta = \frac{1}{2}\left(M^{\mu\nu} + \sigma^{\mu\nu}\right)\theta_{\mu\nu}, \tag{2.30a}$$

where $M^{\mu\nu}$ denotes the operators that effect space rotations; i.e., $M^{\mu\nu}$ contains both an orbital component, which acts upon the position dependence of a field, and a spin component, which acts upon its tensor indices. $\sigma^{\mu\nu}$ are the matrices of Eq. (2.19); one shows that they [as well as the $\bar{\sigma}^{\mu\nu}$] obey commutation relations identical to those of $M^{\mu\nu}$. It is easy to verify that

$$\left[\Theta, A^\mu\right] = \left[\Theta, F^{\mu\nu}\right] = 0 \tag{2.30b}$$

for the fields of the pseudoparticle, which proves the rotational symmetry of the configuration. [The commutator notation is symbolic; it does not represent quantum mechanical commutation, but rather the infinitesimal action of the transformation.]

Notice that the group of combined O(4) and SU(2) transformations is a group of covariance of the self-duality equations; Eq. (2.22) gives the most general <u>Ansatz</u> invariant under these rotations. This explains why the <u>Ansatz</u> is compatible with the self-duality constraint.

Because the theory we are considering is covariant under the full O(5,1) group of conformal transformations, any conformal transformation of the pseudoparticle will still solve the self-duality equations. Of course, as we have just verified for the O(4) subgroup of rotations, some conformal transformations may not give a different solution, but just a gauge transform of the original one. We investigate now whether the group of symmetries of the pseudoparticle is actually larger than O(4).

The conformal group in Euclidean four-space has, as infinitesimal generators, the operators $M^{\mu\nu}$, P^{μ}, $K^{\mu}=\mathcal{J} P^{\mu} \mathcal{J}$ and D, which generate, respectively, rotations, translations, special conformal transformations and dilatations, with $\mathcal{J}$ the inversion operator. Of these, a dilatation changes the scale λ of the pseudoparticle and is thus not a candidate for an invariance. If we denote by $\mathcal{A} = \frac{48\lambda^4}{(x^2+\lambda^2)^2}$ the action density of Eq. (2.29), we readily determine the effect on $\mathcal{A}$ of the remaining generators to be

$$[P^{\mu}, \mathcal{A}] = -\frac{8x^{\mu}}{\lambda^2+x^2}\mathcal{A} \tag{2.31a}$$

and

$$[K^{\mu}, \mathcal{A}] = -\frac{8\lambda^2 x^{\mu}}{\lambda^2+x^2}\mathcal{A}. \tag{2.31b}$$

While it is clear that translations and special conformal transformations are not symmetries of the solution, it is apparent that $\mathcal{A}$ is invariant under the transformations generated by

$$R^{\mu} = \frac{1}{2}\left(\frac{1}{\lambda}K^{\mu} + \lambda P^{\mu}\right) \tag{2.32}$$

The commutators of R^{μ} and $M^{\mu\nu}$ close into the algebra of an O(5) subgroup of the conformal group.

The proof that the field configuration itself is also O(5) symmetric is slightly less trivial than for the O(4) subgroup; one must perform a space-dependent gauge transformation together with an O(5) conformal transformation to achieve form invariance of A^{μ} and $F^{\mu\nu}$. One verifies[5] that A^{μ} and $F^{\mu\nu}$ are symmetric under the combined conformal and gauge transformation generated by

$$\mathcal{R}^{\mu} = R^{\mu} + \frac{\sigma^{\mu\nu}x_{\nu}}{\lambda} \tag{2.33}$$

D. <u>O(5) Formalism</u>

The symmetry of the pseudoparticle under the O(5) subgroup of conformal transformations can be made manifest by an extension of the formalism,[5] which is very convenient for computational purposes and which we shall now illustrate.

The O(5) subgroup of conformal transformations can be realized as an ordinary group of rotations, if we introduce coordinates $r_a, a=1,\ldots,5$, $r_a r^a=1$, related to the Euclidean coordinates x^{μ} by a projective transformation.

$$\tau^{\mu} = \frac{2\lambda x^{\mu}}{\lambda^2 + x^2}$$

$$\tau^{5} = \frac{\lambda^2 - x^2}{\lambda^2 + x^2} \tag{2.34}$$

Eqs. (2.34) effect a projection of Euclidean four-space onto the surface of an hypersphere, S^4, imbedded in a five-dimensional Euclidean space. The rotations of this hypersphere induce, via Eqs. (2.34), conformal transformations belonging to the O(5) group on the coordinates x^{μ}. Gauge fields A^{μ} [which can be considered as the components of a matrix valued form $A = A_{\mu} dx^{\mu}$] transform in a mapping of manifolds like the derivatives of a function [which are the components of the differential $df = \frac{df}{dx^{\mu}} dx^{\mu}$]. Differentials over the surface of S^4 are conveniently expressed by tangential derivatives

$$df = \hat{\partial}_a f \, d\tau^a$$

$$\hat{\partial}_a = \frac{\partial}{\partial \tau^a} - \tau_a \left(\tau^b \frac{\partial}{\partial \tau^b} \right) \tag{2.35}$$

Correspondingly, we introduce a five component gauge field[6] $\hat{A}_a$, which is related to the usual four component gauge field A_{μ} by

$$A_{\mu} = \hat{A}_a \frac{\partial \tau^a}{\partial x^{\mu}} \tag{2.36a}$$

Eq. (2.36a) gives four relations; to specify the five-component object $\hat{A}_a$ completely, we need one more. This is clearly

$$\tau^a \hat{A}_a = 0 \tag{2.36b}$$

which insures that the gauge potentials are, like the derivatives, transverse. [The metric on the hypersphere is Euclidean; hence, there is no distinction between upper and lower indices.]

With $\hat{A}_a$ we construct covariant derivatives over the hypersphere

$$\hat{D}_a^{\hat{A}} = \hat{\partial}_a + \hat{A}_a \tag{2.37}$$

and the field strength is related as usual to the non-commutativity of these. We must, however, take into account the non-commutativity of the tangential derivatives themselves,

$$[\hat{\partial}_a, \hat{\partial}_b] - \tau_a \hat{\partial}_b + \tau_b \hat{\partial}_a = 0 \tag{2.38}$$

We therefore define $\hat{F}_{ab}$ as the value the left hand side of Eq. (2.38) takes when $\hat{\partial}_a$ is replaced by the covariant derivative $\hat{D}_a^{\hat{A}}$.

$$\hat{F}_{ab} = \hat{\partial}_a \hat{A}_b - \hat{\partial}_b \hat{A}_a - \tau_a \hat{A}_b + \tau_b \hat{A}_a + [\hat{A}_a, \hat{A}_b] \tag{2.39}$$

Notice that $\hat{F}_{ab}$ is anti-symmetric and tangential; it has six independent components.

The configuration of Eq. (2.22) is invariant under combined space and SU(2) rotations because the algebra of the $\sigma^{\mu\nu}$ matrices is isomorphic to the algebra of the rotation generators $M^{\mu\nu}$. The addition of $\sigma^{\mu\nu}$ to $M^{\mu\nu}$ in Eq. (2.30) produces, upon commutation with $\sigma^{\mu\nu}x_\nu$, terms which compensate the variation of the expression due to space rotations. This suggests that the O(5) symmetry might be made manifest if the gauge and tensorial degrees of freedom of the fields could be combined into an expression of the form $\Sigma_{ab}r^b$, where now the Σ_{ab} matrices obey the algebra of the generators of O(5) rotations. This algebra cannot be realized by 2×2 matrices, but we can still achieve a manifestly O(5) invariant expression for the pseudoparticle by the device of putting together a pseudoparticle and an anti-pseudoparticle in an extended gauge system.

Let us define 4×4 matrices

$$\Sigma^{\mu\nu} = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix} \tag{2.40}$$

and extend the SU(2) gauge theory to a theory with $SU(2)\times SU(2)\cong O(4)$ gauge group. The potentials and field strengths are represented by 4×4 matrices.

$$A^\mu = \frac{1}{i} A^\mu_{\alpha\beta}\, \Sigma^{\alpha\beta} \tag{2.41a}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + [A^\mu, A^\nu] \tag{2.41b}$$

The expression

$$A^\mu = \frac{-2i\,\Sigma^{\mu\nu} x_\nu}{\lambda^2 + x^2} \tag{2.42}$$

is block diagonal and contains, in the upper diagonal block, the

SU(2) gauge field of the pseudoparticle, and in the lower diagonal block the field of the anti-pseudoparticle [c.f. Eqs. (2.22), (2.24) and the remarks preceding Eq. (2.28)]. The right-hand-side of Eq. (2.42) is neither self-dual nor anti-self-dual [although the individual blocks are] and has q=0. It does, however, satisfy the Yang-Mills equations of motion. The advantage of having combined the pseudoparticle and anti-pseudoparticle into a single expression is that, within the space of 4×4 matrices, we can find a representation of the algebra of O(5) generators and, as we now proceed to show, the field configuration of Eq. (2.42) can be cast in a manifestly O(5) symmetric form.

We obtain a matrix representation of the O(5) algebra by enlarging the set of matrices $\Sigma^{\mu\nu}$ to a set Σ^{ab}, with the four new matrices $\Sigma^{\mu 5}$ given by

$$\Sigma^{\mu 5} = \frac{1}{2}\begin{pmatrix} 0 & -i\bar{\alpha}^\mu \\ i\alpha^\mu & 0 \end{pmatrix} \tag{2.43}$$

The ten independent matrices Σ^{ab} have the commutation relations of the O(5) infinitesimal generators. In the hyperspherical formalism it is now possible to write down a field configuration, with O(5) gauge group, which is invariant under combined space and global gauge rotations; it is given by the <u>Ansatz</u>

$$\hat{A}_a = i\alpha\, \Sigma_{ab}\, r^b \tag{2.44}$$

where the only freedom is in the constant α. The corresponding field strength is

$$\hat{F}_{ab} = -i(\alpha^2 + 2\alpha)\left(\Sigma_{ab} - r_a r_c \Sigma_{cb} - \Sigma_{ac} r_c r_b\right) \tag{2.45}$$

and the Yang-Mills equations of motion $\hat{D}_a^{\hat{A}}\hat{F}^{ab}=0$ are satisfied if $(\alpha+1)(\alpha^2+2\alpha)=0$. The two solutions $\alpha=0$ and $\alpha=-2$ correspond to pure gauge fields; $\alpha=-1$ gives a non trivial field configuration.

$$\hat{A}_a = -i\,\Sigma_{ab}\,r^b \tag{2.46a}$$

$$\hat{F}_{ab} = i\left(\Sigma_{ab} + r_a\,\Sigma_{bc}\,r^c - r_b\,\Sigma_{ac}\,r^c\right) \tag{2.46b}$$

This expression has manifest O(5) symmetry. It remains for us to verify whether it describes the field configuration of Eq. (2.42). We notice first that the potentials A^μ of Eq. (2.42) belong to the subalgebra spanned by the O(4) generators $\Sigma^{\mu\nu}$, whereas $\hat{A}_a$ in Eq. (2.46a) involves all the Σ^{ab} matrices. If they describe the same system, it must be possible to gauge transform $\hat{A}_a$ to the O(4) gauge space, that is to gauge $\hat{A}^a_{\alpha s}$ to zero. This can be done by the gauge transformation

$$\hat{A}_a \to \hat{A}_a' = U^{-1}\hat{A}_a U + U^{-1}\hat{\partial}_a U \tag{2.47}$$

where

$$U = \exp\left[i\,\frac{\cos^{-1} r_s}{\sqrt{1-r_s^2}}\;\Sigma_{\alpha s}\,r^\alpha\right] \tag{2.48}$$

If we now evaluate the explicit form of $\hat{A}_a'$ and insert it into Eq. (2.36a), we obtain precisely the A^μ field of Eq. (2.42). This proves that $\hat{A}_a=-i\Sigma_{ab}r^b$ does, indeed, describe the fields of a pseudoparticle and an anti-pseudoparticle.

As we have mentioned before, the hyperspherical formalism is very convenient for computations involving the field of the pseudoparticle. We illustrate how to evaluate the eigenvalues and eigenfunctions of the operator that describes the propagation of a massless scalar, iso-vector field coupled to the pseudoparticle.[7] For further applications of the formalism to various systems of spinor and vector fields see Refs. 5, 7, 8 and 9.

We represent a scalar iso-vector field by anti-hermitian matrices in the space of group generators.

$$\varphi = \varphi_a \frac{\sigma^a}{2i} \tag{2.49}$$

and define a corresponding field $\hat{\phi}$ over the surface of the hypersphere by

$$\hat{\varphi}(r) = \frac{\lambda^2+x^2}{2\lambda}\,\varphi(x) \tag{2.50}$$

The weight factor on the right-hand-side of this equation guarantees that the conformal transformations of the O(5) subgroup (in which ϕ transforms as a field of dimension 1) are represented by scalar rotations of $\hat{\phi}$.

We have seen that to exploit the O(5) formalism it is necessary to imbed the SU(2) gauge group in a larger gauge group O(5). We therefore define a more general field

$$\hat{\Phi} = \frac{1}{i}\,\Phi_{ab}\,\Sigma^{ab} \tag{2.51}$$

Σ^{ab} being the matrices of Eqs. (2.40) and (2.43). We shall study the propagation of $\hat{\Phi}$ over the hypersphere, in the background provided by the pseudoparticle field $\hat{A}_a=-i\Sigma_{ab}r^b$. Notice that the gauge transformation U of Eq. (2.48) reduces $\hat{A}_a$ to a block diagonal form, indicating that the system consists of independent pseudoparticle and anti-pseudoparticle. We shall require that the field $\hat{\Phi}$ also reduce to block diagonal form after the gauge transformation

$$\hat{\Phi}' = U^{-1}\hat{\Phi}U \qquad \hat{\Phi}'_{\alpha\delta} = 0 \tag{2.52}$$

This condition insures that the upper and lower diagonal blocks of $\hat{\Phi}'$ describe scalar, iso-vector fields coupled to the pseudoparticle and anti-pseudoparticle, respectively.

The requirement that $\hat{\Phi}'$ be block diagonal can be expressed by the equation

$$[\hat{\Phi}', \Gamma^5] = 0 . \tag{2.53}$$

where

$$\Gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}. \tag{2.54}$$

The matrices $\Gamma^\mu = 2i\Sigma^{\mu 5}\Gamma^5$ and Γ^5 transform as the five components of a vector Γ^a in rotations generated by the matrices Σ^{ab}. This implies that

$$U\Gamma^5 U^{-1} = \Gamma^a \tau_a , \tag{2.55}$$

and Eq. (2.53) is therefore equivalent to the condition

$$[\hat{\Phi}, \Gamma^a\tau_a] = 0 . \tag{2.56}$$

With these premises in mind, we now determine eigenfunctions and eigenvalues of the operator

$$\{(\hat{D}^{\hat{A}})^2 - 2\}\hat{\Phi} = \hat{\partial}_a\hat{\partial}^a\hat{\Phi} + \hat{\partial}_a[\hat{A}^a, \hat{\Phi}] + [\hat{A}_a, \hat{\partial}^a\hat{\Phi}] + [\hat{A}_a, [\hat{A}^a, \hat{\Phi}]] - 2\hat{\Phi} \tag{2.57}$$

with $\hat{A}_a=-i\Sigma_{ab}r^b$. After gauge transforming by U and projecting back to Euclidean space, the equation

$$\{(\hat{D}^{\hat{A}})^2 - 2 \quad + \mu\}\hat{\Phi} = 0 \tag{2.58a}$$

takes the form

$$\left\{ (D^A)^2 + \frac{4\lambda^2\mu}{(\lambda^2+x^2)^2} \right\} \Phi = 0 \tag{2.58b}$$

As discussed in Refs. 8 and 10, Eq. (2.58b), although it contains space dependent coefficients, is more advantageous than the standard eigenvalue equation $\{(D^A)^2+\mu\}\Phi=0$ for the computations relevant to the pseudoparticle system. The equation (2.58a) has the useful property that, if $\hat{\Phi}$ is a solution with a definite eigenvalue μ, so are

$$\hat{\Phi}_1 = \Gamma_a \tau^a \hat{\Phi}, \tag{2.59a}$$

$$\hat{\Phi}_2 = \hat{\Phi}\Gamma_a\tau^a . \tag{2.59b}$$

This can be verified directly or, more simply, by noticing that after the gauge transformation U all matrix valued fields appearing in $(\hat{D}^{\hat{A}})^2$ take block diagonal form, and Eqs. (2.59) reduce to

$$\hat{\Phi}'_1 = \Gamma^5 \hat{\Phi}' = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \hat{\Phi}' \qquad (2.60a)$$

$$\hat{\Phi}'_2 = \hat{\Phi}' \Gamma^5 = \hat{\Phi}' \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \qquad (2.60b)$$

This shows in particular that it is consistent to constrain $\hat{\Phi}$ by (2.56). If $\hat{\Phi}$ satisfies this equation, the two projections

$$\hat{\Phi}_\pm = \frac{1 \mp \Gamma_a r^a}{2} \hat{\Phi} \qquad (2.61)$$

describe the fields coupled to the pseudoparticle and anti-pseudoparticle, respectively.

Eq. (2.57) has a very simple group theoretical meaning, as we now proceed to show. Substituting into it the explicit form of $\hat{A}_a$, we obtain

$$\hat{\partial}_a \hat{\partial}^a \hat{\Phi} - 2i \left[\Sigma_{ab}, r^b \hat{\partial}^a \hat{\Phi}\right] - \left[\Sigma_{ab} r^b, \left[\Sigma^{ac} r_c, \hat{\Phi}\right]\right] + (\mu - 2)\hat{\Phi} = 0. \qquad (2.62)$$

If we define

$$L_{ab} = -i r_a \hat{\partial}_b + i r_b \hat{\partial}_a, \qquad (2.63)$$

then

$$\hat{\partial}_a \hat{\partial}^a = -\frac{1}{2} L_{ab} L^{ab} \qquad (2.64)$$

Moreover L_{ab}, Σ_{ab} and the operator J_{ab} where

$$J_{ab} \hat{\Phi} = L_{ab} \hat{\Phi} + \left[\Sigma_{ab}, \hat{\Phi}\right] \qquad (2.65)$$

all satisfy the algebra of the infinitesimal generators of the O(5) group. Using the fact that $\hat{\Phi}$ is a linear combination of Σ_{ab} matrices it is straightforward to show

$$\left[\Sigma_{ab}, \left[\Sigma^{ab}, \hat{\Phi}\right]\right] = 12 \hat{\Phi} \qquad (2.66)$$

so that, from Eqs. (2.64), (2.65) and (2.66), we obtain

$$-\frac{1}{2} J_{ab} J^{ab} \hat{\Phi} = \hat{\partial}_a \hat{\partial}^a \hat{\Phi} - 2i \left[\Sigma_{ab}, r^b \hat{\partial}^a \hat{\Phi}\right] - 6 \hat{\Phi} \qquad (2.67a)$$

If $\hat{\Phi}$ satisfies Eq. (2.56), it must be of the form $\hat{\Phi} = \frac{1}{2}\phi_{ab}\Sigma^{ab}$ with $r_a \hat{\phi}^{ab} = \hat{\phi}_{ab} r^b = 0$, which implies

$$\left[\Sigma_{ab} r^b, \left[\Sigma_{ac} r^c, \hat{\Phi}\right]\right] = 2 \hat{\Phi} \qquad (2.67b)$$

Putting all these results together, we see that Eq. (2.58) is equivalent to

$$\frac{1}{2} J_{ab} J^{ab} \hat{\Phi} = (\mu + 2) \hat{\Phi} \qquad (2.68)$$

i.e., $\hat{\Phi}$ must be an eigenfunction of the operator

$$C^{(1)} = \frac{1}{2} J_{ab} J^{ab} \qquad (2.69)$$

which is the first Casimir operator of the O(5) group generated by J_{ab}. If λ is the eigenvalue of $C^{(1)}$, then $\mu=\lambda-2$.

The eigenfunctions of $C^{(1)}$ are easily found as follows. Let us consider any tensor harmonic $Y^{(\lambda,\lambda',m)}_{a_1\ldots a_N}(\omega)$ of the O(5) rotation group [λ and λ' are the eigenvalues of the two Casimir operators of O(5), m stands for the "magnetic" quantum numbers, ω denotes the hyperspherical angles and $a_1\ldots a_N$ are tensor indices] and let us saturate all the indices $a_1\ldots a_N$ with Γ^a matrices to form the covariant

$$\mathcal{Y}^{(\lambda,\lambda',m)} = Y^{(\lambda,\lambda',m)}_{a_1\ldots a_N}\,\Gamma^{a_1}\ldots\Gamma^{a_N} \tag{2.70}$$

Then it is straightforward to show that

$$J_{ab}\,\mathcal{Y}^{(\lambda,\lambda,m)} = \left(L_{ab}+S_{ab}\right)\mathcal{Y}^{(\lambda,\lambda',m)}\ , \tag{2.71}$$

where the spin operators S_{ab} act on the tensor indices of $Y^{(\lambda\lambda'm)}_{a_1\ldots a_N}$. It follows that all the covariants $\mathcal{Y}^{(\lambda,\lambda',m)}$ will be eigenfunctions of the operator $\frac{1}{2}J_{ab}J^{ab}$ with eigenvalue λ. A complete set of eigenfunctions for the expansion of an arbitrary field $\hat{\Phi}$ is given by the covariants $\mathcal{Y}^{n,m}_i$, which are linear combinations of the $\mathcal{Y}^{(\lambda,\lambda',m)}$.[7]

$$\begin{aligned}
\mathcal{Y}_1^{n,m} &= i\,\Sigma^{ab}L_{ab}\,Y^{n,m}\\
\mathcal{Y}_2^{n,m} &= \Sigma^{ab}\,r_a\,\mathcal{Y}_b^{n,m}\\
\mathcal{Y}_3^{n,m} &= \Sigma^{ab}\,\hat{\partial}_a\,\mathcal{Y}_b^{n,m}\\
\mathcal{Y}_4^{n,m} &= i\,\varepsilon^{abcde}\,\Sigma_{ab}\,L_{cd}\,\mathcal{Y}_e^{n,m}\ .
\end{aligned} \tag{2.72}$$

where $Y^{n,m}$ is a scalar hyperspherical harmonic, with $\lambda=n(n+3)$, and $\mathcal{Y}^{n,m}_a$, $n>1$, are vector hyperspherical harmonics, as given by Adler,[6] with $\lambda=(n+1)(n+2)$. It is easy to check that the covariants $\mathcal{Y}^{n,m}_1$ do not satisfy Eq. (2.56), that the covariants $\mathcal{Y}^{n,m}_4$ do satisfy the constraint, and that only a definite linear combination of the covariants $\mathcal{Y}^{n,m}_2$ and $\mathcal{Y}^{n,m}_3$ does. This linear combination is most easily found by multiplying $\mathcal{Y}^{n,m}_4$ by $\Gamma_a r^a$. From

$$\Gamma_f\,\Sigma_{ab} = \frac{1}{2i}\left(\delta_{af}\,\Gamma_b-\delta_{bf}\,\Gamma_a\right)-\frac{1}{2}\,\varepsilon_{abfcd}\,\Sigma^{cd} \tag{2.73}$$

we obtain $\Gamma_a r^a\mathcal{Y}^{n,m}_4=2(\mathcal{Y}^{n,m}_2-\mathcal{Y}^{n,m}_3)$.

We conclude that the matrix valued fields $\hat{\Phi}=\mathcal{Y}^{n,m}_2-\mathcal{Y}^{n,m}_3$ and $\hat{\Phi}=\mathcal{Y}^{n,m}_4$ are the sought eigenfunctions of the operator $\{(\hat{D}^{\hat{A}})^2-2\}$ with eigenvalues $\mu=n^2+3n$.

E. Multi-Pseudoparticle Solutions

A system of n pseudoparticles is, by definition, a field configuration which has Pontryagin index n and satisfies the self-duality constraint. Whereas it is straightforward to write down field configurations with arbitrary values of the Pontryagin index q [$A^\mu=f(x^2)g^{-n}\partial^\mu g^n$, with g and f as given in Eqs. (2.16) and (2.24), for instance, has q=n], it is not obvious that there exist self-dual configurations with q>1. In this sub-Section we exhibit explicit self-dual field configurations with arbitrary [integer] values of the Pontryagin index, and study their properties.

The search for multi-pseudoparticle solutions starts with an analysis of the single pseudoparticle potential

$$A^\mu = -2i \; \frac{\sigma^{\mu\nu} x_\nu}{\lambda^2 + x^2} \tag{2.74}$$

or

$$A^\mu = -2i \; \frac{\bar{\sigma}^{\mu\nu} x_\nu \lambda^2}{(x^2+\lambda^2)\, x^2} \tag{2.75}$$

These expressions are of the form

$$A^\mu = i\,\sigma^{\mu\nu} a_\nu \tag{2.76}$$

and

$$A^\mu = i\,\bar{\sigma}^{\mu\nu} a_\nu \tag{2.77}$$

with an appropriate four-vector field a^μ. Notice that A^μ is a matrix-valued vector field, with twelve independent components. Eq. (2.76) couples the space index of the gauge potential with its isospin indices [implicit in the matrix structure] so as to re-express the twelve components of A^μ in terms of the four components of the vector a^μ. We are dealing with representations of the $O(4) \widetilde{\sim} SU(2)\times SU(2)$ group of rotations together with the SU(2) gauge group; thus Eqs. (2.76) and (2.77) represent definite couplings of tensorial components. This point will be elaborated in Section III.

We try to generalize the pseudoparticle solutions by assuming an Ansatz [11] as in Eqs. (2.76,2.77), and checking whether the self-duality constraint is solved with an appropriate choice of a^ν.

From Eqs. (2.76) and (2.77) we find

$$F^{\mu\nu} = i\left[(\partial^\mu a_\rho - a^\mu a_\rho)\sigma^{\nu\rho} - (\partial^\nu a_\rho - a^\nu a_\rho)\sigma^{\mu\rho} - a_\rho a^\rho \sigma^{\mu\nu}\right] \tag{2.78}$$

with $\sigma^{\mu\nu}$ replaced by $\bar{\sigma}^{\mu\nu}$ if we start from $A^\mu = i\bar{\sigma}^{\mu\nu} a_\nu$. The expression for $*F^{\mu\nu}$ can be simplified using the self-duality (anti-self-duality) property of $\sigma^{\mu\nu}$ ($\bar{\sigma}^{\mu\nu}$). Using the identities

$$\begin{aligned} \varepsilon^{\mu\nu\alpha\beta}\sigma_{\nu\rho} &= \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\nu\rho\gamma\delta}\sigma^{\gamma\delta} \\ &= -g^\mu_\rho \sigma^{\alpha\beta} - g^\alpha_\rho \sigma^{\beta\mu} - g^\beta_\rho \sigma^{\mu\alpha} \end{aligned} \tag{2.79a}$$

$$\begin{aligned} \varepsilon^{\mu\nu\alpha\beta}\bar{\sigma}_{\nu\rho} &= -\tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\nu\rho\gamma\delta}\bar{\sigma}^{\gamma\delta} \\ &= g^\mu_\rho \bar{\sigma}^{\alpha\beta} + g^\alpha_\rho \bar{\sigma}^{\beta\mu} + g^\beta_\rho \bar{\sigma}^{\mu\alpha} \end{aligned} \tag{2.79b}$$

we obtain from (2.76)

$$\begin{aligned} {}^*F^{\mu\nu} = -i\big[(\partial_\rho a^\mu - a_\rho a^\mu)\sigma^{\nu\rho} - (\partial_\rho a^\nu - a_\rho a^\nu)\sigma^{\mu\rho} \\ + \partial_\rho a^\rho \sigma^{\mu\nu}\big] \end{aligned} \tag{2.80}$$

or from (2.77)

$$\begin{aligned} {}^*F^{\mu\nu} = i\big[(\partial_\rho a^\mu - a_\rho a^\mu)\bar{\sigma}^{\nu\rho} - (\partial_\rho a^\nu - a_\rho a^\nu)\bar{\sigma}^{\mu\rho} \\ + \partial_\rho a^\rho \bar{\sigma}^{\mu\nu}\big] \end{aligned} \tag{2.81}$$

The self-duality constraint $F^{\mu\nu} = {}^*F^{\mu\nu}$ can now be converted into equations for the vector field a^μ. In deriving these we must pay attention to the fact that the matrices $\sigma^{\mu\nu}$ ($\bar{\sigma}^{\mu\nu}$) are not independent. The proper procedure is to multiply the equation $F^{\mu\nu} - {}^*F^{\mu\nu} = 0$ by $\sigma^{\alpha\beta}$ ($\bar{\sigma}^{\alpha\beta}$) and use the identities

$$\operatorname{tr}\sigma^{\alpha\beta}\sigma^{\mu\nu} = \frac{1}{2}\left(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\nu}g^{\beta\mu} + \varepsilon^{\alpha\beta\mu\nu}\right) \tag{2.82a}$$

$$\operatorname{tr}\bar{\sigma}^{\alpha\beta}\bar{\sigma}^{\mu\nu} = \frac{1}{2}\left(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\nu}g^{\beta\mu} - \varepsilon^{\alpha\beta\mu\nu}\right) \tag{2.82b}$$

Thus starting from the Ansatz $A^\mu = i\sigma^{\mu\nu}a_\nu$ we obtain

$$\partial^\mu a^\nu + \partial^\nu a^\mu - 2a^\mu a^\nu = \frac{1}{2}g^{\mu\nu}\left(\partial_\alpha a^\alpha - a_\alpha a^\alpha\right) \tag{2.83}$$

as the self-duality condition. If $A^\mu = i\bar{\sigma}^{\mu\nu}a_\nu$, the self-duality condition leads to two equations.

$$\partial^\mu a^\nu - \partial^\nu a^\mu = \varepsilon^{\mu\nu\rho\sigma}\partial_\rho a_\sigma \tag{2.84a}$$

and

$$\partial_\mu a^\mu + a_\mu a^\mu = 0 \tag{2.84b}$$

From now on, we shall consider the two cases separately. Eq. (2.83) constitutes a set of nine non-linear first order differential equations for the four components of a^μ. The integrability conditions give rise to a set of constraints involving the six independent components of $f^{\mu\nu} \equiv \partial^\mu a^\nu - \partial^\nu a^\mu$, the algebraic complexity of which makes it difficult to analyze them completely; however, since the number of independent equations is large, it is plausible that the only solution is $f^{\mu\nu}=0$. We assume this to be true. If $f^{\mu\nu}=0$, then a^μ may be written in the form $a^\mu = -\partial^\mu a$. Inserting this into Eq. (2.83), we get

$$\partial_\mu\partial_\nu a = \frac{g_{\mu\nu}}{4}\Box a \tag{2.85a}$$

or

$$\partial_\mu \Box a = 0 \tag{2.85b}$$

Eq. (2.85b) requires $\Box a$ to be constant, and (2.85a) shows that a is at most a quadratic polynomial in x, $a = \alpha[\lambda^2 + (x-y)^2]$, so that

$$a^\mu = \frac{-2(x-y)^\mu}{\lambda^2 + (x-y)^2} \tag{2.86}$$

Thus for $A^\mu = i\sigma^{\mu\nu}a_\nu$ we recover only the single pseudoparticle solution.

We turn now to the set of coupled differential equations (2.84). First we decompose a^μ into transverse and longitudinal parts,

$$a^\mu = \partial^\mu \ln\rho + b^\mu, \quad \partial_\mu b^\mu = 0 \tag{2.87}$$

Eq. (2.84a) states that the Abelian "field strength" $f^{\mu\nu} = \partial^\mu b^\nu - \partial^\nu b^\mu$ derived from the potential b^μ is self-dual,

$$f^{\mu\nu} = {}^*f^{\mu\nu} \tag{2.88a}$$

while Eq. (2.84b) becomes

$$\frac{1}{\rho}\left(\partial_\mu + b_\mu\right)\left(\partial^\mu + b^\mu\right)\rho = 0 \tag{2.88b}$$

A general solution of (2.88a) is obtained as follows. Owing to its transversality, b^μ may always be written as a divergence of an anti-symmetric tensor $h^{\mu\nu}$. But $h^{\mu\nu}$ has six independent components, while b^μ has only three, so three conditions may be imposed: it is convenient to demand that $h^{\mu\nu}$ be anti-self-dual. Consequently b^μ is represented as follows.

$$b^\mu = \partial_\nu h^{\mu\nu} \tag{2.89a}$$

$$^*h^{\mu\nu} = -h^{\mu\nu} \tag{2.89b}$$

It is then trivial to verify that Eq. (2.88a) reduces to the requirement that $h^{\mu\nu}$ be a harmonic function. Thus the non-linear self-duality equation has been linearized with the help of the Ansatz $A^\mu = i\bar{\sigma}^{\mu\nu}a_\nu$: we are to choose any harmonic, anti-self-dual tensor $h^{\mu\nu}$, form b^μ and solve the linear equation (2.88b). However a non-trivial *global* problem still remains. The functions $h^{\mu\nu}$ and ρ necessarily have singularities which may induce singularities in the potential A^μ. These singularities must be arranged so that the gauge invariant quantity tr $F^{\mu\nu}F_{\mu\nu}$ is non-singular. Later, in considering small deformations of a given potential, we shall encounter singularities of exactly the same type, which appear as pure gauge artifacts. However, we do not know how to arrange for this to happen *in* the general case; indeed it is not clear that this is possible for non-vanishing $h^{\mu\nu}$.

So we proceed with the assumption that a^μ is a gradient. Setting b^μ to zero Eq. (2.88a) is of course satisfied, and Eq. (2.88b) reduces to

$$\frac{1}{\rho}\Box\rho = 0 \tag{2.90}$$

It is still true that the harmonic function ρ will possess singularities, but now it is easy to find a form for them so that gauge invariant quantities are non-singular. We take

$$\rho = \sum_{i=1}^{m} \frac{\lambda_i^2}{(x-y_i)^2} \tag{2.91}$$

Note that with this super-position of poles Eq. (2.90) is satisfied everywhere, even at the poles, due to the prefactor ρ^{-1}.

Summarizing, we have found that the formula

$$A^\mu = i\bar{\sigma}^{\mu\nu}\partial_\nu \ln\rho \tag{2.92}$$

with ρ as in Eq. (2.91) gives origin to a self-dual field configuration.[12,13,14]

We must still verify that the singularities introduced in A^μ by the poles of ρ are pure gauges and evaluate the Pontryagin index of the field configuration. Near a singularity, which we take for convenience at the origin, A^μ behaves as

$$A^\mu \approx i\bar{\sigma}^{\mu\nu}\partial_\nu \ln\frac{1}{x^2} = -2i\bar{\sigma}^{\mu\nu}x_\nu/x^2 \tag{2.93}$$

Comparing with Eq. (2.18) [with $\sigma^{\mu\nu}$ replaced by $\bar{\sigma}^{\mu\nu}$] we see that the singularity at the origin is indeed of a pure gauge form. Also, if we evaluate the Pontryagin index q as a surface integral, this singularity will contribute one unit to it [cf. also the discussion after Eqs. (2.21), (2.27)]. The behavior of the field at infinity, where $\rho \sim \frac{1}{x^2}\sum_{i=1}^{m}\lambda_i^2$, is still of the form given by Eq. (2.93). We *conclude* that the Pontryagin index is m-1, because the m singularities at $x=y_i$, $i=1,\ldots m$, contribute +m to q, whereas a contribution −1 comes from the surface at infinity.

The value of q may also be found using the elegant formula

$$\mathrm{tr}\ {}^{*}F^{\mu\nu}F_{\mu\nu} = \Box\Box \ln\rho \tag{2.94}$$

which can be derived when A^{μ} is given by Eq. (2.92). Eq. (2.94) shows in particular that $\mathrm{tr}{}^{*}F^{\mu\nu}F_{\mu\nu}$ is not changed when ρ is multiplied by a factor $(x-y)^2$. Thus q may be evaluated as

$$q = -\frac{1}{16\pi^2}\int d^4x\ \Box\Box \ln\tilde{\rho} \tag{2.95a}$$

where $\tilde{\rho}$ is a polynomial of degree 2m−2, and this immediately gives

$$q = -\lim_{R\to\infty}\frac{1}{16\pi^2}\int d\Omega\ R^2 R_{\mu}\partial^{\mu}\Box\ln\left(R^{2m-2}+\dots\right) = m-1 \tag{2.95b}$$

The number of parameters appearing in the expression of the self-dual field A^{μ} with Pontryagin index n is 5n+4. This is apparent from Eqs. (2.91) and (2.92): the parameters are the 5m=5n+5 scales λ_i and positions y_i^{μ}, minus one overall scale, which can be modified by an additive change of $\ln\rho$. The number 5n+4 is surprising if one thinks that the field configuration is obtained putting together n pseudoparticles, each characterized by a position and a scale. One may indeed consider a limit where the (n+1)th scale λ_{n+1} and coordinate y^{μ}_{n+1} go to infinity simultaneously with $\lim \lambda^2_{n+1}/y^2_{n+1}=1$, in which case ρ takes the form[13]

$$\rho = 1 + \sum_{i=1}^{n}\frac{\lambda_i^2}{(x-y_i)^2} \tag{2.96}$$

For small λ_i, i=1,...,n, one can then identify the y_i^{μ}'s as approximate positions of peaks in the action density, with width λ_i^2. A conformal transformation of the field configuration (which of course preserves the self-duality of $F^{\mu\nu}$) would re-introduce the more general form of Eq. (2.91).

It may be verified indeed that the class of field configurations represented by Eqs. (2.91) and (2.92) is closed under conformal transformations in the following sense.[12] In a finite special conformal transformation where

$$x^{\mu} \to \tilde{x}^{\mu} = \frac{x^{\mu} - c^{\mu}x^2}{1-2c\cdot x + c^2x^2} \tag{2.97a}$$

we let ρ(x) transform as a scalar density of dimension +1, i.e.

$$\rho(x) \to \tilde{\rho}(x) = \frac{1}{1-2c\cdot x + c^2x^2}\,\rho(\tilde{x}) \tag{2.97b}$$

For infinitesimal $c^{\mu}=\varepsilon^{\mu}$,

$$\delta\rho = \left(2\varepsilon\cdot x\, x^{\alpha} - x^2\varepsilon^{\alpha}\right)\partial_{\alpha}\rho + 2\varepsilon\cdot x\,\rho \tag{2.97c}$$

and, with simple algebra, one verifies that the induced transformation of a^{μ} takes the form

$$\delta a^{\mu} = \delta\,\partial^{\mu}\ln\rho = \delta_c a^{\mu} + 2\varepsilon^{\mu} \tag{2.97d}$$

where $\delta_c a^{\mu}$ is the conformal variation of a vector field of dimension +1. If we replace a^{ν} with $a^{\nu}+\delta_c a^{\nu}+2\varepsilon^{\nu}$ in the Ansatz $A^{\mu}=i\bar{\sigma}^{\mu\nu}a_{\nu}$ and perform then an infinitesimal gauge transformation

(2.98a)

with parameter

$$\Theta = 2i\, \varepsilon_\alpha x_\beta \bar{\sigma}^{\alpha\beta} \tag{2.98b}$$

again it is a matter of straightforward algebra to check that the total variation of A^μ is precisely the conformal change $\delta_c A^\mu$ of a vector field of dimension 1.

Summarizing, a conformal transformation of the gauge potential A^μ of Eq. (2.92) can be obtained changing ρ first according to Eq. (2.97) and then performing a suitable gauge transformation. But, starting from Eqs. (2.91) and (2.97) with an explicit computation we find

$$\tilde{\rho}(x) = \sum_{i=1}^{m} \frac{\tilde{\lambda}_i^2}{(x-\tilde{y}_i)^2} \tag{2.99a}$$

$$\tilde{\lambda}_i^2 = \frac{\lambda_i^2}{1+2c\cdot y_i + c^2 y_i^2} \tag{2.99b}$$

$$\tilde{y}_i^\mu = \frac{y_i^\mu + c^\mu y_i^2}{1+2c\cdot y_i + c^2 y_i^2} \tag{2.99c}$$

We see therefore that a conformal transformation changes any field configuration of the class defined by Eqs. (2.91) and (2.92) into another field configuration of the same class, modulo a gauge transformation. The superpotential $\tilde{\rho}(x)$ of the new field configuration is obtained from the old one by a conformal transformation of the scales λ_i^2 and the positions y_i^μ. In particular, the function ρ of Eq. (2.96) can be considered as the limiting form of the more general ρ of Eq. (2.91) obtained when one of the singularities

has moved to infinity; it is transformed into a ρ of the more general class by a conformal transformation.

F. Small Deformations of Self-Duality Condition

The realization that all the 5n+4 parameters present in the expression A^μ are necessary to have an explicit representation of the conformal group still leaves open the possibility that some of the parameters are unphysical, i.e., that the values of the λ_i^2 and y_i^μ may be modified by a gauge transformation. We know after all that the single pseudoparticle configuration depends on five physical parameters, whereas the present analysis gives a number of parameters equal to nine for n=1. We shall see later that there are indeed situations where some of the constants y_i^μ and λ_i^2 may be modified by performing a gauge transformation on the fields, but it is convenient to postpone the study of this residual gauge freedom. Instead, we consider now the problem of finding the most general infinitesimal deformation of the fields which preserves the self-duality of $F^{\mu\nu}$.[15] This analysis will also provide an answer to the question of the residual gauge freedom; as will become apparent, in general all the 5n+4 parameters are physical.

A small variation of the potential A^μ

$$A^\mu \to A^\mu + \delta A^\mu \tag{2.100a}$$

generates a variation of $F^{\mu\nu}$

$$F^{\mu\nu} \to F^{\mu\nu} + \delta F^{\mu\nu} \tag{2.100b}$$

$$\delta F_{\mu\nu} = D^A{}_\mu \delta A_\nu - D^A{}_\nu \delta A_\mu$$

$$D^A_\mu \delta A_\nu = \partial_\mu \delta A_\nu + [A_\mu, \delta A_\nu] \tag{2.100c}$$

We take A^μ to be given by Eqs. (2.91) and (2.92) and require that $\delta F^{\mu\nu}$ be self dual. The most general δA^μ can be represented by

$$\delta A^\mu = i\bar{\sigma}^{\alpha\beta} X^\mu{}_{\alpha\beta} \tag{2.101}$$

where $X^\mu_{\alpha\beta}$ is anti-symmetric and anti-self-dual in the indices $\alpha\beta$. But it is not convenient to consider an expression as general as (2.101), because all infinitesimal gauge transformations of A^μ

$$\delta_{gauge} A_\mu = D^A{}_\mu \, i\bar{\sigma}^{\alpha\beta}\omega_{\alpha\beta} \tag{2.102}$$

would appear as uninteresting solutions of

$$\delta F^{\mu\nu} = {}^*\delta F^{\mu\nu} \tag{2.103}$$

We fix the gauge by requiring that δA^μ be of the form

$$\delta A^\mu = i\bar{\sigma}^{\mu\nu}\partial_\nu \frac{\delta\rho}{\rho} + i\bar{\sigma}_{\alpha\beta}\,\rho\,\partial_\nu Y^{\mu\nu\,\alpha\beta} \tag{2.104a}$$

where $Y^{\mu\nu\,\alpha\beta}$ is a tensor field, anti-symmetric and anti-self-dual in both pairs of indices $\mu\nu$, $\alpha\beta$ and constrained by

$$Y_{\mu\nu}{}^{\mu\nu} = 0 \tag{2.104b}$$

This apparently arbitrary choice of a gauge is motivated by the fact that it simplifies the algebra. The first term in the right-hand-side of Eq. (2.104a) is included because we want

to find among the infinitesimal deformations those induced by a variation of the parameters of ρ. The condition $Y_{\mu\nu}{}^{\mu\nu}=0$ removes one of the nine independent components of $Y^{\mu\nu\,\alpha\beta}$ so as to leave the correct number of variable functions — nine — in the Ansatz of Eq. (2.104).

The tensor $Y^{\mu\nu\,\alpha\beta}$ can be decomposed into a symmetric, traceless and an anti-symmetric part.

$$Y^{\mu\nu\,\alpha\beta} = S^{\mu\nu\,\alpha\beta} + A^{\mu\nu\,\alpha\beta} \tag{2.105a}$$

$$S^{\mu\nu\,\alpha\beta} = S^{\alpha\beta\,\mu\nu} \tag{2.105b}$$

$$A^{\mu\nu\,\alpha\beta} = -A^{\alpha\beta\,\mu\nu} = \frac{1}{4}\left(g^{\mu\alpha}V^{\nu\beta} - g^{\nu\alpha}V^{\mu\beta} + g^{\nu\beta}V^{\mu\alpha} - g^{\mu\beta}V^{\nu\alpha}\right) \tag{2.105c}$$

$V^{\alpha\beta}$ is anti-symmetric and anti-self-dual. After non-trivial algebraic manipulations that make use of many identities satisfied by anti-self-dual quantities, one finds that Eqs. (2.100), (2.103) and (2.104) imply

$$\Box S^{\mu\nu\,\alpha\beta} = 0 \tag{2.106}$$

All non-trivial solutions of this equation introduce in δA^μ singularities which cannot be removed by a gauge transformation, and therefore we set $S^{\mu\nu\,\alpha\beta}=0$. When $S^{\mu\nu\,\alpha\beta}$ vanishes, one finds that the anti-self-duality of $F^{\mu\nu}$ implies for $\delta\rho$ and $V^{\mu\nu}$ the equations

$$\Box \rho V^{\mu\nu} = 0 \tag{2.107a}$$

$$\Box\,\delta\rho + 2\rho\,\partial_\mu V^{\mu\nu}\partial_\nu \rho = 0 \tag{2.107b}$$

These are solved by

$$V^{\mu\nu} = \frac{1}{\rho}\sum_{i=1}^{n+1} \frac{k_i^{\mu\nu}}{(x-y_i)^2} \tag{2.108a}$$

$$\delta\rho = \sum_{i=1}^{n+1}\sum_{\substack{j=1\\ j\neq i}}^{n+1} \frac{2}{(y_i-y_j)^2}\,\frac{k_i^{\mu\nu}\lambda_j^2 (x-y_i)_\mu (x-y_j)_\nu}{(x-y_i)^2 (x-y_j)^2} + \delta\tilde{\rho} \tag{2.108b}$$

where the $k_i^{\mu\nu}$ are constant anti-self-dual tensors and $\delta\tilde{\rho}$ is the variation induced by a change of the parameters of ρ.

Inserting $V^{\mu\nu}$ and $\delta\rho$ as given by Eqs.(2.108) into Eqs. (2.104) and (2.105), one finds an expression for δA^μ which is singular near the poles y_i^μ of ρ. It is possible to show, however, that the singularity can be removed by a suitable gauge transformation,[15] so that δA^μ represents an acceptable infinitesimal deformation of the self-dual field configuration.

It is very interesting to observe that upon performing an infinitesimal gauge transformation [according to Eq. (2.102)] with parameter

$$\omega^{\alpha\beta} = \frac{1}{4}\rho\, V^{\alpha\beta} \tag{2.109a}$$

the infinitesimal variation of the potential becomes

$$\begin{aligned}\delta A'^\mu &= \delta A^\mu + \delta_{gauge} A^\mu \\ &= i\,\bar{\sigma}^{\mu\nu}\left[\partial_\nu\left(\frac{\delta\rho}{\rho}\right) + \partial^\lambda(\rho V_{\lambda\nu})\right]\end{aligned} \tag{2.109b}$$

which is of the form

$$\delta A'^\mu = i\,\bar{\sigma}^{\mu\nu}\,\delta a_\nu \tag{2.109c}$$

In this gauge, the infinitesimal deformation appears as a first order variation of the original Ansatz, where δa^ν consists of both a gradient term and a divergenceless term. The gauge transformation leading to this form of the potential is singular and $\delta A'^\mu$ behaves as $|x-y_i|^{-3}$ near the poles of ρ. Because of these singularities, the representation of the infinitesimal deformations provided by $\delta A'^\mu$ is not very useful to study the finite physical deformations, but it is extremely convenient for an analysis of the residual gauge freedom.

To expose possible gauge artifacts among the infinitesimal deformations $\delta A'^\mu$, we perform an additional gauge transformation with parameter $\omega'^{\alpha\beta}$ and inquire whether

$$\delta A''^\mu = \delta A'^\mu + \delta'_{gauge} A^\mu \tag{2.110}$$

can still be of the form

$$\delta A''^\mu = i\,\bar{\sigma}^{\mu\nu}\left(\delta a_\nu + \delta'_{gauge} a_\nu\right) \tag{2.111}$$

With some algebra, one finds that the form of the Ansatz is preserved only if

$$\omega'^{\alpha\beta} = \rho\, \tilde{\omega}^{\alpha\beta} \tag{2.112a}$$

with
$$\tilde{\omega}^{\alpha\beta} = 2x^\alpha A^{\beta\gamma} x_\gamma - 2x^\beta A^{\alpha\gamma} x_\gamma + x^2 A^{\alpha\beta} + B^\alpha x^\beta - B^\beta x^\alpha + \varepsilon^{\alpha\beta\gamma\delta} B_\gamma x_\delta + C^{\alpha\beta} \tag{2.112b}$$

where B^α is a constant vector, $A^{\alpha\beta}$ and $C^{\alpha\beta}$ are constant self-dual and anti-self-dual tensors, respectively. The variation of $\delta'_{gauge} a^\nu$ is then given by

$$\delta'_{gauge}\, a^\nu = -4\, \tilde{\omega}^{\nu\alpha} \partial_\alpha \rho - \frac{4}{3} \rho\, \partial_\alpha \tilde{\omega}^{\nu\alpha} \tag{2.113a}$$

which in terms of $V^{\alpha\beta}$ and ρ, reads

$$\delta'_{gauge}\, \rho V^{\alpha\beta} = 4 \sum_{i=1}^{n+1} \frac{\lambda_i^2}{(x-y_i)^2}\, \tilde{\omega}^{\alpha\beta}(y_i) \tag{2.113b}$$

$$\frac{1}{\rho}\, \delta'_{gauge}\, \rho = \frac{4}{3} \sum_{i=1}^{n+1} \frac{\lambda_i^2}{(x-y_i)^2} (x^\alpha - y_i^\alpha)\, \partial^\beta\, \tilde{\omega}_{\alpha\beta}(y_i) \tag{2.113c}$$

Since $\tilde{\omega}^{\alpha\beta}$ contains ten independent constants, we conclude that ten of the independent components of the tensors $k_i^{\alpha\beta}$ in Eq. (2.108a) can be modified by a gauge transformation. Therefore the dimensionality of the space of physical small deformations of a given solution is 8(n+1)-10-1 (-1 because of

the arbitrariness of an overall rescaling of the λ_i's], =8n-3, which, by continuity, must also be the dimensionality of [a connected component of] the full manifold of solutions. The number 8n-3 has a *nice* interpretation: the n-pseudoparticle solution appears parametrized by the positions, scales and relative group orientations of the pseudoparticles.

Notice that if we start from any of the 5n+4 solutions described by the <u>Ansatz</u> $A^\mu = i\bar{\sigma}^{\mu\nu}\partial_\nu \ln\rho$ and perform an infinitesimal gauge transformation, Eq. (2.113b) tells us that we shall not preserve the functional form of the <u>Ansatz</u> unless the positions of the poles y_i^μ satisfy

$$\tilde{\omega}^{\alpha\beta}(y_i) = 0 \tag{2.114}$$

But this is the equation of a definite circle [or of a straight line as a limiting case] in 4-space, and therefore if the poles y_i^μ are more than three in number, and in general positions, then all the 5n+4 parameters represent physical degrees of freedom. On the other hand, when the poles y_i^μ lie on a circle, one can perform a gauge transformation which moves the singularities around the circle [see Eq. (2.113c)]. Through three points one can always draw a circle, so that if n=2, one of the 5n+4=14 parameters is always a gauge artifact, and the 2-pseudoparticle solution depends on 13 gauge invariant parameters. Through two points one can draw a three-dimensional variety of circles, and therefore four of the 5+4=9 parameters describing the single pseudoparticle within this Ansatz are gauge artifacts, in agreement with the fact that a single pseudoparticle is characterized by only five parameters, position and scale. These considerations indicate that for $n>3$ there certainly exist solutions to the self-duality equations beyond the ones given by Eqs. (2.91) and (2.92); as yet they have not been found.

III FURTHER MATHEMATICAL DEVELOPMENTS

A. Spinorial Formalism

As mentioned earlier, the solutions to the various equations that have been discussed take the form they do as a consequence of the coupling of internal degrees of freedom, [SU(2)], to kinematical degrees of freedom, [O(4)]. For Yang-Mills theory, this coupling can be made explicit in the context of a spinorial formalism, which we describe in this Section. The formalism is also important since it exposes features of the self-duality condition which are used as a point of departure for an analysis by methods of algebraic geometry. Furthermore, with the help of this formalism, we shall be able to simplify considerably the Dirac equation for zero-eigenvalue modes of a fermion, with arbitrary iso-spin, and to solve it completely for iso-spin ½ and 1.[16]

The spinorial formulation begins with the observation that the O(4) invariants of interest in Euclidean four-space may be designated by SU(2)×SU(2) representation labels. Also, the internal SU(2) gauge group gives rise to such labels. Hence all objects with which we are concerned are SU(2) multi-spinors, and equations are simplified when the various SU(2) groups are cunningly coupled to each other.

In this formalism all objects carry spinor labels A, B, C ... , which take on two values and describe the spin and iso-spin degrees of freedom. An anti-symmetric metric tensor with two upper indices is defined by

$$\varepsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma^2 \qquad (3.1a)$$

The negative inverse of this matrix is a metric tensor with lower indices.

$$\varepsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \varepsilon^{AB} \qquad (3.1b)$$

A spinor may have lower or upper indices, which can be raised or lowered with the metric tensors according to the following rules [repeated indices are summed].

$$\xi^{A} = \varepsilon^{AB}\,\xi_{B} \qquad (3.2a)$$

$$\xi_{B} = \xi^{A}\,\varepsilon_{AB} \qquad (3.2b)$$

Covariant summations always involve one upper and one lower index. Note that $\xi^{A}{}_{A} = -\xi_{A}{}^{A}$. For every pair of indices we may define a symmetric and anti-symmetric part

$$\xi_{AB} = \frac{1}{2}\,\varepsilon_{AB}\,\xi_{C}{}^{C} + \frac{1}{2}\,\xi_{\underset{\smile}{AB}} \qquad (3.3)$$

where the symbol $\underset{\smile}{AB}$ denotes the symmetric sum $\xi_{AB}+\xi_{BA}$. More generally for a multi-indexed object, $\xi_{A_1A_2\ldots A_n}$, symmetric in $A_2\ldots A_n$,

$$\xi_{\underbrace{A_1 A_2 \ldots A_n}} \equiv \xi_{A_1A_2\ldots A_n} + \xi_{A_2A_1\ldots A_n} + \ldots + \xi_{A_nA_2\ldots A_1} \quad \text{(n terms)}. \qquad (3.4)$$

An O(4) two-component spinor is described by a spinor with one index. To every O(4) tensor with indices $\mu, \nu, \ldots$, there corresponds a spinor with index pairs $AA', BB', \ldots$. The rule

of association is given through the α matrices defined in (2.17).

$$\xi_{\mu}\,(\alpha^{\mu})_{AA'} = \xi_{AA'} \qquad (3.5a)$$

$$\xi_{\mu}\,(\bar{\alpha}^{\mu})_{A'A} = \xi^{AA'} \qquad (3.5b)$$

[That the two definitions are consistent with (3.2) is easily established from the properties of the Pauli matrices]. The O(4) covariants may be regained from the spinors by projecting with the appropriate α matrix. The above holds also for derivatives $\partial_{\mu} \leftrightarrow \partial_{AA'}$.

Iso-spinor indices are represented as follows. Iso-spin $\frac{1}{2}$ objects are described by one-index spinors. For iso-spin 1, a two-index spinor, symmetric in the indices, is used. In general an iso-spin T object is described by a totally symmetric spinor with $2T$ indices, so that there are $2T+1$ independent components. The correspondence between the conventional description and the spinorial one is immediate for iso-spin $\frac{1}{2}$ — the two coincide. For unit iso-spin the correspondence is

$$\xi^{a}\left(\frac{\sigma_a}{2i}\right)_{UV} = -\,\xi^{V}{}_{U} \qquad (3.6)$$

A consequence is that ξ_{VU} and ξ^{VU} are symmetric in $U \leftrightarrow V$ and that $\varepsilon_{abc}\xi^{b}\bar{\xi}^{c}$ corresponds $\xi_{\underset{\smile}{UW}}\bar{\xi}^{W}{}_{V}$. The relations for higher iso-spin are more complicated, and will not be given here.

B. Gauge Field Equations in Spinorial Formalism

The gauge potential A^{μ}_{a} is described by $A_{AA';UV}$; the gauge field $F_a^{\mu\nu}$, by $F_{AA',BB';UV}$ which is anti-symmetric in the interchange $A\leftrightarrow B$, $A'\leftrightarrow B'$. Both expressions are symmetric in $U\leftrightarrow V$, and the formula relating the two is

$$F_{AA',BB';UV} = \partial_{AA'} A_{BB';UV} - \partial_{BB'} A_{AA';UV} + A_{AA';UW} A_{BB';}{}^{W}{}_{V} \tag{3.7}$$

Due to its anti-symmetry properties, F may be split into two parts

$$F_{AA',BB';UV} = \tfrac{1}{2}\varepsilon_{AB} F^{+}_{A'B';UV} + \tfrac{1}{2}\varepsilon_{A'B'} F^{-}_{AB;UV} \tag{3.8}$$

where $F^{+}_{A'B';UV}$ is symmetric in $A'\leftrightarrow B'$, and $F^{-}_{AB;UV}$ is symmetric in $A\leftrightarrow B$; these are just the self-dual and anti-self-dual parts of the gauge field, as is seen by noting that the definition $*F^{\mu\nu}=\tfrac{1}{2}\varepsilon^{\mu\nu\;\alpha\beta}F_{\alpha\beta}$ becomes in the spinorial formalism

$$*F_{AA',BB';UV} = F_{AB',BA';UV} \tag{3.9a}$$

or

$$*F^{+}_{A'B';UV} = F^{+}_{A'B';UV} \tag{3.9b}$$

$$*F^{-}_{AB;UV} = -F^{-}_{AB;UV} \tag{3.9c}$$

It follows from (3.7) and (3.8) that

$$F^{+}_{A'B';UV} = \partial_{AA'} A^{A}{}_{B';UV} + A_{AA';UW} A^{A}{}_{B';}{}^{W}{}_{V} \tag{3.10a}$$

$$F^{-}_{AB;UV} = \partial_{AA'} A_{B}{}^{A'}{}_{;UV} + A_{AA';UW} A_{B}{}^{A'}{}_{;}{}^{W}{}_{V} \tag{3.10b}$$

The self-duality condition demands that F^{-} vanishes. Hence a self-dual gauge potential satisfies

$$\partial_{AA'} A_{B}{}^{A'}{}_{;UV} + A_{AA';UW} A_{B}{}^{A'}{}_{;}{}^{W}{}_{V} = 0 \tag{3.11}$$

The conformal solution to this equation is

$$A_{AA';UV} = \tfrac{1}{2}\varepsilon_{AU}\partial_{VA'}\ln\rho \qquad \frac{1}{\rho}\Box\rho = 0 \tag{3.12}$$

Thus far we have merely transcribed into new formalism results which already exist in the conventional approach. We wish now to make some further observations about self-dual gauge fields. These form the starting point for an analysis of self-dual gauge fields by methods of algebraic geometry.

Consider a special set of complex bi-spinors [4-vectors] $x_{AA'}$, which can be written as $\ell_A z_{A'}$, where ℓ_A is fixed and $z_{A'}$ varies. It is clear that all points described by such coordinates are light-like with respect to each other: $x_{AA'}\tilde{x}^{AA'}=\ell_A z_{A'}\ell^{A}\tilde{z}^{A'}=0$. This set of points for fixed ℓ defines a light-like plane. Next let us project the gauge field onto such a light-like plane.

$$F^{\ell}_{UV} = x^{AA'}\tilde{x}^{BB'} F_{AA',BB';UV} = \ell^{A} z^{A'} \ell^{B} \tilde{z}^{B'} F_{AA',BB';UV} \tag{3.13}$$

But a self-dual field takes the form

$$F_{AA',BB',UV} = \frac{1}{2}\,\varepsilon_{AB}\,F^{+}_{A'B';UV} \tag{3.14}$$

so for self-dual configurations the projection (3.13) vanishes. We thus come to the important conclusion that on arbitrary light-planes the gauge potential, for self-dual fields, is integrable.

$$\ell^A A_{AA'} = g_\ell^{-1}\,\ell^A \partial_{AA'}\,g_\ell \tag{3.15}$$

The program of reconstructing $A_{AA'}$ from the above by methods of algebraic topology is being pursued actively, but we shall not discuss this topic further.[17]

C. Dirac Equations in Spinorial Formalism

An important feature of pseudoparticle configurations is that they produce zero-eigenvalue modes in the Dirac equation for a [Euclidean] fermion in the pseudoparticle field; that is one can solve

$$i\,\gamma^\mu(\partial_\mu + A_\mu)\,\Psi = 0 \tag{3.16}$$

with several normalizable functions. Here, ψ has 2T+1 components which transform according to some definite, irreducible representation of SU(2)

$$\delta\Psi = i\,T^a\,\Psi\,\Theta_a$$

$$[T^a, T^b] = i\,\varepsilon_{abc}\,T^c \tag{3.17}$$

and A^μ is the Yang-Mills potential in an anti-hermitian matrix

representation: $iA^\mu = A^\mu_a T^a$. In general A^μ need not solve the Yang-Mills equations, but is always taken to be sufficiently well-behaved that the Yang-Mills action is finite; consequently the gauge configuration is characterized by an integer valued Pontryagin index.

When the gauge potential is the conformal, self-dual configuration (3.12), the spinorial formalism may be used to simplify Eq. (3.16) considerably. We now present this analysis and solve Eq. (3.17) completely for iso-spin $\frac{1}{2}$ and 1.

The Dirac matrices in (3.16), satisfy Euclidean anti-commutation relations $\{\gamma^\mu,\gamma^\nu\} = 2\delta^{\mu\nu}$ which can be realized in a fashion such that γ_5 is diagonal.

$$\gamma^\mu = \begin{bmatrix} 0 & \alpha^\mu \\ \bar{\alpha}^\mu & 0 \end{bmatrix}$$

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \tag{3.18}$$

In this representation Eq. (3.16) decouples into two separate equations for two-component spinors of definite chirality.

$$\psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}$$

$$i\,\alpha^\mu(\partial_\mu + A_\mu)\,\psi^- = 0 \tag{3.19a}$$

$$i\,\bar{\alpha}^\mu(\partial_\mu + A_\mu)\,\psi^+ = 0 \tag{3.19b}$$

When we discuss the Atiyah-Singer index theorem in the next sub-

Section, we shall show that (3.19b) has no normalizable solutions, and, therefore, only (3.19a) need be considered.

In the spinorial formalism,(3.19a) transcribes into

$$\partial_{AA'}\psi^{A'}{}_{;U_1\ldots U_{2T}} + A_{AA';U_1V}\,\psi^{A'}{}_{;}{}^{V}{}_{U_2\ldots U_{2T}} \qquad (3.20a)$$

Here the spinor carries the index A', describing two spatial components; it is entirely symmetric in the 2T indices U_i which refer to the 2T+1 components of iso-spin. Substitution of the conformal solution into the above gives the equation we analyze.

$$\frac{1}{2T}\partial_{AA'}\psi^{A'}{}_{;U_1\ldots U_{2T}} + \frac{1}{2}\varepsilon_{AU_1}(\partial_{BB'}\ln\rho)\psi^{B'}{}_{;}{}^{B}{}_{U_2\ldots U_{2T}} - \frac{1}{2}(\partial_{U_1A'}\ln\rho)\psi^{A'}{}_{;A\ldots U_{2T}} + \text{permutations} = 0 \qquad (3.20b)$$

A straightforward but lengthy sequence of manipulations of indices, which among other things involves separating the above into symmetric and anti-symmetric parts in (A,U_1), yields the result that a normalizable solution necessarily has the form[16]

$$\psi_{A';U_1\ldots U_{2T}} = \rho^{T}\partial_{U_1A'}\rho^{-2T}\chi_{U_2\ldots U_{2T}} + \text{permutations} \qquad (3.21a)$$

where χ satisfies

$$\partial_{U_2B'}\rho^{-2T+1}\,\partial^{CB'}\chi_{CU_3\ldots U_{2T}} + \text{permutations} = 0 \qquad (3.21b)$$

We now specialize to iso-spin ½ and 1. For the former

$$\psi_{A';U} = \rho^{1/2}\partial_{UA'}\rho^{-1}\chi$$

$$\partial_{UB'}\partial^{CB'}\chi = 0$$

or

$$\Box\chi = 0 \qquad (3.22)$$

Of course only singular functions solve the harmonic equation; however we can tolerate singularities, provided they are absent from the gauge-invariant norm density $\psi^{A';U}\psi_{A';U}$, so that the spinor is normalizable. Therefore we can allow in χ only poles which are already present in ρ. In this way we get n+1 solutions for χ .

$$\chi^{(i)} = \frac{\lambda_i^2}{(x-y_i)^2}, \quad i = 1,\ldots,n+1 \qquad (3.23)$$

Of these only n $\psi^{(i)}$'s are independent since $\sum_i\chi^{(i)} = \rho$, and $\sum_i\psi^{(i)} = 0$.[18]

For iso-spin 1

$$\psi_{A';U_1U_2} = \rho\,\partial_{U_1A'}\rho^{-2}\chi_{U_2}$$

$$\partial_{U_2B}\rho^{-1}\,\partial^{CB'}\chi_C = 0 \qquad (3.24)$$

Solutions are conveniently exhibited by setting $\chi_i = M_{CC'}u^{C'}$, where $u^{C'}$ is a constant spinor, with two arbitrary components which provide two solutions χ for each matrix M. Eq. (3.24) is solved by the following $n+1$ expressions for M.

$$M^{(1)(i)}_{cc'} = \frac{\lambda_i^2}{(x-y_i)^4}\,(x-y_i)_{cc'}$$

$$i = 1,\dots,n+1 \tag{3.25a}$$

An additional $n+1$ forms are

$$M^{(2)(i)}_{cc'} = -\rho\frac{\lambda_i^2}{(x-y_i)^2}\varepsilon_{cc'} - \sum_{\substack{j=1\\ j\neq i}}^{n+1} \frac{\lambda_i^2\lambda_j^2}{(y_i-y_j)^2}\left[\frac{(x-y_i)_{cc''}+(x-y_j)_{cc''}}{(x-y_i)^2(x-y_j)^2}\right](y_i-y_j)_{c'}{}^{c''}$$

$$i = 1,\dots,n+1 \tag{3.25b}$$

However of these 2n+2 matrices, only 2n are linearly independent, since the following relationships are readily established.

$$\sum_{i=1}^{n+1} M^{(2)(i)}_{cc'} = -\rho^2\varepsilon_{cc'} \tag{3.26a}$$

$$\sum_{i=1}^{n+1}\lambda_i^2 M^{(1)(i)}_{cc'} + \sum_i^{n+1} M^{(2)(i)}_{cc''}(y_i)^{c''}{}_{c'} = \rho^2 x_{cc'} \tag{3.26b}$$

One finds all the solutions to be normalizable; thus there are 4n zero-eigenvalue modes for iso-spin 1 Fermi fields.

We conclude this discussion of the solutions to the Dirac equation by noting that, since the eigenfunctions have definite

chirality, bilinears $\psi^\dagger\Gamma\psi$ vanish for a vectorial Dirac matrix Γ. In particular $\psi^\dagger T^a\gamma^\mu\psi$ is zero; hence the functions A^μ and ψ are also solutions to the coupled Yang-Mills fermion equations, when A^μ has definite duality and ψ is a chiral eigenstate solution of the Dirac equation. [We show below that such solutions exist not only for iso-spin ½ and 1, but also for arbitrary iso-spin T.]

D. Atiyah-Singer Index Theorem

In the two examples discussed earlier — iso-spin ½ and 1 Fermi fields moving in a self-dual Yang-Mills potential — we found a number of zero-eigenvalue modes of definite chirality. The existence of these modes has far-reaching physical consequences; moreover it is related to the anomaly of the axial vector current[1a] and to topological properties of the gauge fields. This unexpected connection between physics and mathematics is best understood with the help of the "Atiyah-Singer index theorem", which we now explain.

Consider a linear differential operator L and its adjoint $L^\dagger$; further suppose that the number of normalizable zero-eigenvalue modes of L is n_- and that of $L^\dagger$ is n_+. The "index" is the quantity $n_- - n_+$, and the index theorem evaluates this object in terms of the properties of L. In order to make these considerations relevant to our Dirac equation (3.16), let us write it in block form, using the γ matrices in the representation

(3.18), which diagonalizes chirality:

$$\begin{bmatrix} 0 & L \\ L^\dagger & 0 \end{bmatrix}\begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} = 0 \tag{3.27a}$$

$$L\psi^- = 0 \tag{3.27b}$$

$$L^\dagger \psi^+ = 0 \tag{3.27c}$$

$$L = i\alpha^\mu(\partial_\mu + A_\mu) \tag{3.27d}$$

$$L^\dagger = i\bar{\alpha}^\mu(\partial_\mu + A_\mu) \tag{3.27e}$$

Thus we see that n_+ (n_-) is the number of positive (negative) chirality zero-eigenvalue solutions of the Dirac equation. The index theorem, which we derive below, when applied to (3.27) states

$$n_- - n_+ = -\frac{1}{16\pi^2}\int d^4x \; tr \; {}^*F^{\mu\nu} F_{\mu\nu}$$
$$= \frac{1}{16\pi^2} \; tr \; T_a T_b \int d^4x \; {}^*F_a^{\mu\nu} F_{b\,\mu\nu} \tag{3.28a}$$

For fermions with total iso-spin T and gauge fields with Pontryagin index n, the above is evaluated with the help of

$$tr \; T_a T_b = \frac{1}{3}\, T(T+1)(2T+1)\,\delta_{ab} \tag{3.28b}$$

and we find that the index is

$$n_- - n_+ = \frac{2}{3}\, T(T+1)(2T+1)\, n \tag{3.28c}$$

We have assumed that the gauge potential leads to finite action and that it carries Pontryagin index n; in all other respects, it is arbitrary. However when A^μ is self-dual or anti-self-dual, the index theorem may be strengthened by showing that only n_- or n_+ is non-zero: apply $L^\dagger$ to (3.27b) and L to (3.27c) to get

$$\left[(\partial_\mu + A_\mu)^2 + 2i\,\bar{\sigma}_{\mu\nu}\, F^{\mu\nu}\right]\psi^+ = 0 \tag{3.29a}$$

$$\left[(\partial_\mu + A_\mu)^2 + 2i\,\sigma_{\mu\nu}\, F^{\mu\nu}\right]\psi^- = 0 \tag{3.29b}$$

The duality properties of $\bar{\sigma}^{\mu\nu}(\sigma^{\mu\nu})$ [see (2.19) and (2.28)] assure that $\bar{\sigma}_{\mu\nu}F^{\mu\nu}$ ($\sigma_{\mu\nu}F^{\mu\nu}$) vanishes for self-dual (anti-self-dual) gauge fields. Since $(\partial_\mu + A_\mu)^2$ is a positive definite operator, the differential equation without the gauge-field term does not have normalizable solutions. All known Yang-Mills solutions with finite action are self-dual or anti-self-dual; hence for these potentials there are precisely $\frac{2}{3}\,T(T+1)(2T+1)n$ zero-eigenvalue modes with chirality determined by the gauge field's duality properties. Of course this general result reproduces, for $T=\frac{1}{2}$ and 1, the numbers found before: n and $4n$.

When n_+ or n_- vanishes, a "vanishing theorem" is said to hold. We have seen that such a theorem can be always established when the gauge field is self-dual or anti-self-dual; however, it is not yet known whether the vanishing theorem is valid for more general field configurations.[20]

We now derive the index theorem, by a method which makes reference to the anomaly of the axial-vector current.[19] First a local version of (3.28a) is obtained; upon integration over all space, (3.28a) is regained. The derivation begins with a consideration of the full eigenvalue problem for the Dirac operator.

$$i\gamma^\mu(\partial_\mu + A_\mu)\psi_E = E\psi_E \tag{3.30}$$

It is clear that γ_5, which anti-commutes with the left-hand-side of (3.30), takes eigenfunctions ψ_E into ψ_{-E}, while the zero-eigenvalue modes can be chosen to be eigenstates of γ_5.

$$\int d^4x\, \psi_E^\dagger(x)\gamma_5\psi_E(x) = 0 \qquad E \neq 0 \tag{3.31a}$$

$$\int d^4x\, \psi_0^\dagger(x)\gamma_5\psi_0(x) = (\pm)1 \quad \begin{pmatrix}\text{positive}\\ \text{negative}\end{pmatrix}\text{chirality} \tag{3.31b}$$

To proceed, we construct the resolvent of the differential operator in (3.30).

$$R(x,y,\mu) = \sum_E \frac{\psi_E(x)\psi_E^\dagger(y)}{E+i\mu} \tag{3.32a}$$

$$\left[i\gamma^\mu(\partial_\mu + A_\mu) + i\mu\right]R(x,y;\mu) = \delta^4(x-y) \tag{3.32b}$$

We shall want to take x and y coincident, which may produce infinities and ambiguities that must be regulated. A convenient, gauge invariant regularization is the Pauli-Villars scheme; from (3.32a) the same expression is subtracted with μ replaced by M, and and at the end of the calculation M is passed to infinity. [It happens that one regulator mass is sufficient for the problem at hand.]

$$R_{Reg}(x,y;\mu) = \lim_{M\to\infty}\left[R(x,y,\mu) - R(x,y,M)\right] \tag{3.32c}$$

Next we form an axial-vector projection of the resolvent — which we call the "axial vector current" — and also its divergence.

$$J_5^\mu(x) = \operatorname{tr} i\gamma^\mu\gamma_5\, R_{Reg}(x,x;\mu) \tag{3.33}$$

A simple calculation, based on (3.30) gives

$$\partial_\mu J_5^\mu(x) = 2i\mu\sum_E \frac{\psi_E^\dagger(x)\gamma_5\psi_E(x)}{E+i\mu} - \lim_{M\to\infty}\left(2iM\sum_E \frac{\psi_E^\dagger(x)\gamma_5\psi_E(x)}{E+iM}\right) \tag{3.34}$$

To complete the calculation we need to evaluate the limit. [Formally it is given by the ambiguous expression $2i\sum_E\psi_E^\dagger(x)\gamma_5\psi_E(x) = 2i\operatorname{tr}\gamma_5\sum_E\psi_E(x)\psi_E^\dagger(x) = 2i(0)\delta^4(0)$.] It is here that we can use the results about the anomaly of the axial-vector current operator constructed from quantum Fermi fields which interact with an external classical vector field. Of course in the above we are not dealing with a quantum field theory; rather, we are studying differential equations in Euclidean space. Nevertheless, the objects we have constructed are recognized to be precisely the [Wick rotated] quantal amplitudes. Thus the resolvent R is exactly the [Wick rotated] propagator for a massive Fermi field in an external gauge potential, and the axial-vector current J_5^μ is the [Wick rotated] vacuum expectation value of the axial-vector current operator for that theory. Hence we arrive at[19]

$$\partial_\mu J_5^\mu(x) = 2i\mu\, J_5(x) - \frac{1}{8\pi^2}\, tr\, {}^*F^{\mu\nu} F_{\mu\nu}$$

$$J_5(x) = \sum_E \frac{\psi_E^\dagger(x)\,\gamma_5\,\psi_E(x)}{E + i\mu} \tag{3.35}$$

This anomalous divergence of the axial-vector current is also the local form of the index theorem.

To derive the global relation, Eq. (3.35) is integrated over all x.[21]

$$\int d^4x\, \partial_\mu J_5^\mu(x) = 2i\mu \int d^4x \sum_E \frac{\psi_E^\dagger(x)\,\gamma_5\,\psi_E(x)}{E + i\mu} - \frac{1}{8\pi^2}\int d^4x\; tr\, {}^*F^{\mu\nu} F_{\mu\nu} \tag{3.36}$$

When it is assumed that the integral on the left-hand-side produces no surface terms, and that the integral on the right-hand-side can be evaluated term-by-term with the help of (3.31), Eq. (3.28a) is regained.

The above derivation also exposes circumstances which may modify the simple, integrated expression (3.28a). The surface term for the integral of $\partial_\mu J^\mu_5$ need not vanish; the term-by-term integration may be illegitimate. In that case an additional contribution is present in (3.28a); it is called the "signature defect". We expect that such pathologies occur when long-range potentials are present in the Dirac equation. In our example the gauge potential can be long-range, but gauge-invariant quantities see only the short-range gauge field, and the simple result (3.28a) is expected to hold, as is indeed the case in the explicit computations for iso-spin ½ and 1. A more mathematical formulation states that the index theorem should be applied only to compact manifolds without boundaries. In that case there obviously is no surface term on the left-hand-side of (3.36); the summation over eigenvalues on the right-hand-side is truly a sum over discrete eigenvalues, and the term-by-term integration may be justified. In our example, we are on the non-compact manifold of Euclidean 4-space. However, as explained in Section II, the conformal invariance of the theory and the assumption that the gauge fields decrease rapidly at infinity allow our problem to be mapped onto the surface of a 4-dimensional hypersphere, and a signature defect is not expected. [This consideration introduces the following subtlety: The normalizability condition for the Dirac equation in an O(5) covariant formulation requires only that $\int \frac{d^4x}{1+x^2}\psi^\dagger(x)\psi(x)$ converge, while $\int d^4x\psi^\dagger(x)\psi(x)$ may diverge. However, we have not encountered a situation in 4 dimensions where this distinction makes a difference.]

Even though the signature defect is absent in the present application of index theory, it will play a role in other physical situations. We have encountered Dirac equations in an odd number of Euclidean dimensions, where no anomaly exists, yet there are zero-eigenvalue modes.[22] These examples involve soliton-monopole potentials which include a long-range Higgs field and provide physically interesting applications of the signature defect.[23]

To illustrate the utility of the index theorem, we derive once more the result that the self-duality equation $F^{\mu\nu}={}^*F^{\mu\nu}$ has 8n-3 gauge invariant deformations. An infinitesimal variation of this equation, caused by an arbitrary variation δA^{μ} about a self-dual gauge potential A^{μ}, is [see (2.100) and (2.103)],

$$\delta F^{\mu\nu} - \delta\, {}^*F^{\mu\nu} = 0$$

$$\delta F_{\mu\nu} = D_{\mu}^{A}\,\delta A_{\nu} - D_{\nu}^{A}\,\delta A_{\mu} \tag{3.37a}$$

Since the matrix $\bar{\sigma}^{\mu\nu}$ is anti-symmetric and anti-self-dual, the above is entirely equivalent to

$$\bar{\sigma}^{\mu\nu}\, D_{\mu}^{A}\,\delta A_{\nu} = 0 \tag{3.37b}$$

Next we write $\bar{\sigma}^{\mu\nu}$ as $\frac{1}{2i}(\alpha^{\mu}\bar{\alpha}^{\nu}-\delta^{\mu\nu})$ and impose the background gauge condition on the small variations.

$$D_{\mu}^{A}\,\delta A^{\mu} = 0 \tag{3.37c}$$

Hence the equation one is left to solve is

$$\alpha^{\mu} D_{\mu}^{A}\left(\bar{\alpha}^{\nu}\,\delta A_{\nu}\right) = 0 \tag{3.37d}$$

But we recognize (3.37d) to be two decoupled Dirac equations for two Dirac two-component spinors in the adjoint (T=1) representation — these two spinors make up the two columns of the matrix $\bar{\alpha}^{\nu}\delta A_{\nu}$ and move in the external potential A^{μ}. In other

words, the above demonstrates that if ψ_a, a=1,2,3, solves

$$i\alpha^{\mu}(\partial_{\mu} + A_{\mu})\psi = 0 \tag{3.38}$$

with self-dual A_a^{μ}, then $A_a^{\mu}+u^{\dagger}\alpha^{\mu}\psi_a$ is self-dual to first order, with u being a constant arbitrary two-component spinor. The index theorem states that there are 4n solutions to (3.38); and, by the construction, 8n small deformations are found. One shows that they can be arranged into exactly 8n linearly independent, real combinations, and one further finds that 3 of them are infinitesimal gauge transformations $D_{\mu}^{A}\theta$. Hence the number of infinitesimal deformations is 8n-3, in agreement with our previous computation.[24] The explicit solution of the Dirac equation for iso-vector fermions, presented earlier, provides therefore explicit formulas for the small deformations of a self-dual gauge potential, now in the familiar background gauge, rather than in the somewhat obscure gauge employed previously.

The startling relationship between solutions of the Dirac equation for iso-vector fermions and small deformations of the self-dual gauge potential goes even further. The following fact is easily establihsed by reducing products of gamma matrices. If $F_a^{\mu\nu}$ solves the Yang-Mills equation, then $\psi_a=F_a^{\mu\nu}\gamma_{\mu}\gamma_{\nu}u$ and $\psi_a=F_a^{\mu\nu}\gamma_{\mu}\gamma_{\nu}\gamma\cdot xu$ solve the iso-spin 1 Dirac equation where u is an arbitrary constant 4-component spinor. [Notice that when $F_a^{\mu\nu}$ is self-dual (anti-self-dual) only the negative (positive) chiral components of these spinors are non-vanishing.] These curious connections between gauge fields and Fermi fields are related to the super-symmetry properties of iso-vector fermions.[25]

IV MINKOWSKI SPACE SOLUTIONS

In this last Section we shall discuss some solutions to the Yang-Mills equations in Minkowski space which have recently been found. At the present time it is not clear what information about the quantum theory is contained in these classical field configurations; towards the end of the presentation we shall describe some tentative ideas that we have about this question.

A. O(4)×O(2) Formalism

Rather than recording the solutions straight-away, we first develop a kinematical framework in which their elegance and significance is manifest. The Yang-Mills theory possesses the O(4,2) conformal group of invariances. Under conformal transformations the coordinates x^μ transform non-linearly. But as is well known, one may introduce a light-like six-vector ξ^A, A=1...6, $\xi^2=\xi_1^2+\xi_2^2+\xi_3^2-\xi_4^2+\xi_5^2-\xi_6^2=0$, which has the property that (pseudo) rotations of ξ^A correspond to conformal transformations of $\xi^\mu/(\xi_5+\xi_6)$, μ=1,2,3,4. The action of the conformal group is thus linearized on this null-cone, and it becomes convenient to use the ξ^A's as coordinates, rather than the conventional x^μ. The relationship between the two coordinate systems contains of course a large amount of ambiguity. For example, one possible mapping is

$$\begin{aligned}
\xi^i &= \frac{2x^i\lambda}{\lambda^2-x^2}\,f \qquad i=1,2,3\\
\xi^4 &= \frac{2t\lambda}{\lambda^2-x^2}\,f\\
\xi^5 &= \frac{\lambda^2+x^2}{\lambda^2-x^2}\,f\\
\xi^6 &= f
\end{aligned} \tag{4.1}$$

where $x^2=t^2-\vec{x}^2$, λ is an arbitrary scale and f is an arbitrary function of x which parametrizes the ambiguity. Conventionally the ambiguity is removed by setting homogeneity conditions on all interesting objects of the theory. An alternate way to remove the ambiguity is to fix the value of $\vec{\xi}^2+\xi_5^2=\xi_4^2+\xi_6^2$. This we do here; we set that quantity to unity, which forces ξ^A to lie on a six-dimensional hypertorus. Thus the mapping introduces two Euclidean vectors; a 4-component $\hat{R}^\mu$ and a 2-component $\hat{r}^a$ of unit magnitude. Explicitly one has

$$\begin{aligned}
\hat{R}^i &= \frac{2x^i\lambda}{\omega} \qquad i=1,2,3\\
\hat{R}_4 &= \frac{\lambda^2+x^2}{\omega}\\
\hat{r}_1 &= \frac{2t\lambda}{\omega}\\
\hat{r}_2 &= \frac{\lambda^2-x^2}{\omega}\\
\omega^2 &= (\lambda^2-x^2)^2+4t^2\lambda^2
\end{aligned} \tag{4.2}$$

and all of Minkowski space is mapped, two-to-one, onto the hypertorus $\hat{r}^2=\hat{R}^2=1$. The action of the O(4)×O(2) subgroup of the conformal group is then represented by independent rotations of $\hat{R}^\mu$ and $\hat{r}^a$, while the remaining conformal transformations mix the $\hat{R}^\mu$'s with the $\hat{r}^a$'s. [The metric of the $\hat{R}^\mu$ coordinates, as well as that of the $\hat{r}^a$ coordinates is Euclidean.]

The ordinary derivatives $\frac{\partial}{\partial x^\mu}$ and gauge potentials A^μ are mapped into derivatives and gauge potentials tangential to the surface of the torus. Thus we have

$$\hat{\partial}_\mu = \frac{\partial}{\partial \hat{R}^\mu} - \hat{R}_\mu \left(\hat{R}^\nu \frac{\partial}{\partial \hat{R}^\nu} \right)$$
$$\hat{\Delta}_a = \frac{\partial}{\partial \hat{r}^a} - \hat{r}_a \left(\hat{r}^b \frac{\partial}{\partial \hat{r}^b} \right)$$
$$\frac{\partial}{\partial x^\mu} = \frac{\partial \hat{R}_\nu}{\partial x^\mu} \frac{\partial}{\partial \hat{R}^\nu} + \frac{\partial \hat{r}_a}{\partial x^\mu} \frac{\partial}{\partial \hat{r}^a} \tag{4.3}$$

$$A^\mu dx_\mu = \hat{A}^\mu d\hat{R}_\mu + \hat{a}^a d\hat{r}_a$$
$$\hat{R}_\mu \hat{A}^\mu = \hat{r}_a \hat{a}^a = 0 \tag{4.4}$$

[Compare with the similar mapping discussed in Section IID for the O(5) formalism in Euclidean space.]

From $\hat{A}^\mu$ and $\hat{a}^a$ one constructs an "electric" field

$$\hat{E}_{a\mu} = \hat{\Delta}_a \hat{A}_\mu - \hat{\partial}_\mu \hat{a}_a + [\hat{a}_a, \hat{A}_\mu] \tag{4.5a}$$

and a "magnetic" field.

$$\hat{H}_{\mu\nu} = \hat{\partial}_\mu \hat{A}_\nu - \hat{\partial}_\nu \hat{A}_\mu - \hat{R}_\mu \hat{A}_\nu + \hat{R}_\nu \hat{A}_\mu + [\hat{A}_\mu, \hat{A}_\nu] \tag{4.5b}$$

Both are tangential, i.e., $\hat{R}^\nu \hat{H}_{\mu\nu}=0$, $\hat{R}^\nu \hat{E}_{a\nu}=0$, $\hat{r}^a \hat{E}_{a\mu}=0$. The nomenclature is derived from the fact that near the origin $\hat{E}_{2i}$ and $\hat{H}_{ij}$ are proportional to the electric F_{oi} and magnetic F_{ij} components of $F_{\mu\nu}$.

It is convenient to parametrize the two-dimensional vector by an angle.

$$\hat{r}_1 = \cos\tau \qquad \hat{r}_2 = \sin\tau \tag{4.6}$$

Then $\hat{\Delta}_a = -\varepsilon_{ab}\hat{r}^b \frac{\partial}{\partial\tau}$, and we may similarly set

$$\hat{a}_a = -\varepsilon_{ab} \hat{r}^b \hat{a}$$
$$\hat{E}_{a\mu} = -\varepsilon_{ab} \hat{r}^b \hat{E}_\mu$$
$$\hat{E}_\mu = \dot{\hat{A}}_\mu - \hat{\partial}_\mu \hat{a} + [\hat{a}, \hat{A}_\mu] \tag{4.7}$$

where the dot refers to differentiation with respect to τ. The gauge potentials can be modified by a gauge transformation.

$$\hat{A}_\mu \to g^{-1} \hat{A}_\mu g + g^{-1} \hat{\partial}_\mu g$$
$$\hat{a} \to g^{-1} a g + g^{-1} \dot{g} \tag{4.8}$$

In particular, one may gauge transform $\hat{a}$ to zero.

The Minkowski-space Yang-Mills action

$$I = \frac{1}{2} \int d^4x \; \mathrm{tr}\, F^{\mu\nu} F_{\mu\nu} \tag{4.9a}$$

becomes in terms of new variables

$$I = -\frac{1}{2} \int d\tau \, d\Omega \; \mathrm{tr} \left[\hat{E}^\mu \hat{E}_\mu - \frac{1}{2} \hat{H}^{\mu\nu} \hat{H}_{\mu\nu} \right] \tag{4.9b}$$

The range of the τ integration is from 0 to 2π; the remaining integration is over the surface of the sphere $\hat{R}^2=1$. Since the mapping $x^\mu \to \{\tau,\Omega\}$ is two-to-one there appears an additional factor of ½ in (4.9b). For the same reason, there is no periodicity requirement in τ. Note that the range of integration is compact, hence finiteness of the action is guaranteed when the fields are non-singular on the τ circle and on the Ω sphere. The Yang-Mills field equations of motion, which follow from varying the fields in (4.9b), are

$$\hat\partial^\mu \hat H_{\mu\nu} - \dot{\hat E}_\nu + [\hat A^\mu, \hat H_{\mu\nu}] - [\hat a, \hat E_\nu] = 0$$

$$\hat\partial^\mu \hat E_\mu + [\hat A^\mu, \hat E_\mu] = 0 \tag{4.10}$$

B. Invariant Solutions

Having developed this hypertorodial formalism, which makes explicit the O(4)×O(2) group of symmetries of the problem, we may look for solutions which are themselves invariant under interesting subgroups of O(4)×O(2). Specifically O(4) invariant field configurations are obtained by setting

$$\begin{aligned} \hat A^\mu &= i\sigma^{\mu\nu} \hat R_\nu f(\tau) \\ a &= 0 \end{aligned} \tag{4.11}$$

The Yang-Mills equations then reduce to

$$\ddot f + 2f(f+1)(f+2) = 0 \tag{4.12}$$

which is identical to the equation of motion of a point particle in a two-minimum potential, symmetric about f=-1.

The solutions are obvious. A first integral is immediate.

$$\frac{1}{2}\dot f^2 + \frac{1}{2}\left((f+1)^2 - 1\right)^2 = \varepsilon \tag{4.13}$$

There are τ independent solutions: f=0,-1,-2; these lead to O(4)×O(2) invariant Yang-Mills potentials. f=0 gives the trivial,

vanishing potential; f=-2 is a pure gauge; f=-1 is the solution found by deAlfaro, Fubini and Furlan.[27] In the mechanical analog problem, f=0 and -2 correspond to the particle sitting at the minima of the potential; f=-1 is in unstable equilibrium at the maximum. The τ dependent solutions have been found by Lüscher and Schechter.[26] These are periodic functions of $\tau-\tau_0$, where τ_0 is another integration constant. Two different types of solution are seen: for $\varepsilon<\frac{1}{2}$ there are separate oscillations about each of the two minima, for $\varepsilon>\frac{1}{2}$ the oscillations range widely across the central hump. The analytic expression, which we do not record here, involves Jacobi elliptic functions. For $\varepsilon=\frac{1}{2}$ a simple formula holds

$$f = -1 \pm \frac{\sqrt{2}}{\cosh\sqrt{2}(\tau-\tau_0)} \tag{4.14}$$

It is clear that the action of these solutions is finite.

By using the formulas (4.2) and (4.4) the Yang-Mills fields may be given in conventional variables. We do not carry out the projection, since the resulting configurations do not exhibit any noteworthy features; the fields have finite energy, but dissipate in time in accordance to general theorems which tell us that no soliton solutions exist in the pure Yang-Mills theory.[2] Rather we prefer to remain with the hypertorodial coordinates where the solutions are constant or periodic, not in time, to be sure, but in the new variable τ.

C. Alternate Quantization of Yang-Mills Theory

Let us recall that the evolution of a dynamical system need not necessarily be described by time evolution. Other combinations of t and $\vec{x}$ are possible evolution variables, provided all space-time is covered. In a Lorentz invariant theory one may use any time-like vector for describing evolution of initial data specified on a space-like surface. In a conformally invariant theory there are further possibilities, and in particular one can use τ to describe evolution of data specified on the Ω surface. The generator of τ translations is easily determined; it is $R = \frac{1}{2}(\frac{1}{\lambda} K^0 + \lambda P^0)$.

When these considerations are brought to bear on a quantum theory they lead to the well-known conclusion that in a Lorentz invariant quantum theory there are alternate methods of quantization which do not rely on a Hamiltonian evolving the quantal system in time. Indeed some years ago, light-cone quantization was profitably employed to analyze deep-inelastic scattering processes.[28] Similarly, alternate quantization methods have been suggested for conformally invariant quantum theories in Euclidean space.[29] In the present context it appears very interesting to take τ as the quantization variable. Correspondingly the Hamiltonian is R, which we now shall call H_τ, and the states which diagonalize H_τ become the "static" basis for the Hilbert space [rather than the energy eigenstates of conventional quantum theory]. Precisely this alternative for conformally invariant theories has been advocated by Fubini.[30] It is emphasized that a new quantum theory is not being developed, rather the conventional theory is discussed in terms of a new set of basis states. Indeed it was explicitly demonstrated by deAlfaro, Fubini and Furlan,[31] in the simple example of the conformally invariant quantum mechanics of a point particle in a $1/r^2$ potential, that the new approach is entirely equivalent to the conventional one.

Here we consider quantizing Yang-Mills theory with τ as the evolution variable and H_τ as the Hamiltonian.

$$H_\tau = -\frac{1}{2}\int d\Omega\; \mathrm{tr}\left[\hat{E}^\mu \hat{E}_\mu + \frac{1}{2}\hat{H}^{\mu\nu} H_{\mu\nu}\right] \tag{4.15}$$

The canonical quantization procedure is entirely straightforward. The gauge $\hat{a}=0$ is very convenient; the canonical coordinates are then $\hat{A}^\mu$ with conjugate momenta $\hat{E}^\mu = \dot{\hat{A}}^\mu$. Then Eq. (4.10), $\hat{D}_\mu \hat{E}^\mu = 0$, becomes Gauss' law which has to be imposed as a condition on the physical states. Equal-τ commutators involve delta functions on the Ω surface and in the non-interacting case H_τ can be explicitly diagonalized. One important result emerges: the spectrum is discrete, the eigenvalues are dimensionless integers. This of course is a consequence of the fact that the space is compact; infra-red divergence has been tamed.

D. Semi-classical Quantization of Solutions

We shall not go into the obvious details of the canonical approach. Rather we want to inquire to what extent the solutions which we have previously discussed can be used to perform a semi-classical analysis of the quantum theory. Clearly the constant solutions f=0 and -2, which are pure gauges, correspond to the vacuum; the latter being a non-vanishing, pure gauge potential.

$$\hat{A}^\mu = -2i\,\sigma^{\mu\nu}\,\hat{R}_\nu$$
$$= g^{-1}\,\hat{\partial}^\mu g$$
$$g = \alpha\cdot R \qquad (4.16a)$$

This gauge has unit winding number, and obviously describes one of the many classically degenerate vacua of Yang-Mills theory. The other vacua are O(3) invariant configurations.

$$\hat{A}^\mu_n = g^{-n}\,\hat{\partial}^\mu g^n \qquad (4.16b)$$

Of course there is tunnelling between the vacua; the pseudoparticle solution which exists for imaginary τ insures this. [The pseudoparticle is just the kink solution of (4.12) with $\ddot{f}$ replaced by $-\ddot{f}$.]

The f=-1 solution is seen to correspond to an unstable vacuum, hence no quantum state is associated with it. Nevertheless we can use this solution to compute the height of the barrier which separates the two minima; it is $\frac{3\pi^2}{2}$. [Another curious feature of this solution can be noted. If the pseudoparticle is continued from imaginary τ to real τ, we obtain a complex self-dual gauge potential. Since the equations are nonlinear one does not expect, a priori, that the real and imaginary

parts of this complex configuration satisfy the Yang-Mills equation. Nevertheless, the real part of the self-dual complex potential is just the f=-1 solution.[32]]

Of course it is the periodic solutions that offer the most interesting probe into the quantum theory since it is possible to quantize them by the Bohr-Sommerfeld method, thus obtaining the semi-classical spectrum of H_τ. Before proceeding, let us review the Bohr-Sommerfeld method as applied to field theory.

For quantum mechanics of a point particle in one-dimensional motion, the WKB quantization condition reads

$$(n+\tfrac{1}{2})\pi = \int_{q_1}^{q_2} dq\; p(q) \qquad (4.17a)$$

where p(q) is the local momentum $\sqrt{2E-2V(q)}$ and the q_i are the turning points of the bound motion. The quantity $n\pi$ on the left-hand-side arises from the correspondence principle; the quantity $\frac{1}{2}\pi$ is derived from the details of one-dimensional motion, and is specific to that problem. The approximation is presumed accurate for large n. When one drops $\frac{1}{2}$ compared with n, one is left with the Bohr-Sommerfeld quantization, which after a change of variable from q to t may also be written as

$$n\pi = \int dt\; \dot{q}(t)\, p(t) \qquad (4.17b)$$

where the integration is now over a semi-period.

Although the full WKB condition may also be derived for many degrees of freedom,[33] as well as for a field theory with infinite degrees of freedom,[34] we remain with the simpler Bohr-Sommerfeld condition. For several degrees of freedom (4.17b) generalizes to

$$n\pi = \int dt \sum_m p_m(t)\, \dot{q}_m(t) \tag{4.18a}$$

while for field theory we take

$$n\pi = \int dt \int dV\, \Pi(t,V)\, \dot{\Phi}(t,V) \tag{4.18b}$$

Eq. (4.18b) instructs us to find a periodic solution Φ, multiply its time derivative by the canonical momentum Π and integrate over all variables V, save the evolution parameter t. The resulting quantity, $\int dV \Pi\dot{\Phi}$, depending on t as well as various constants of integration, is then integrated over a semi-period of t and set equal to $n\pi$, thus achieving one quantization condition on the constants of motion.

For the Yang-Mills theory governed by the action (4.9b), the Bohr-Sommerfeld condition reads

$$n\pi = -2\int d\tau \int d\Omega\; \mathrm{tr}\left\{ \frac{\delta \mathcal{L}}{\delta \dot{\hat{A}}^{\mu}}\, \dot{\hat{A}}^{\mu} + \frac{\delta \mathcal{L}}{\delta \dot{\hat{a}}}\, \dot{\hat{a}} \right\} \tag{4.19a}$$

Since

$$\frac{\delta \mathcal{L}}{\delta \dot{\hat{a}}} = 0\,, \quad \frac{\delta \mathcal{L}}{\delta \dot{\hat{A}}^{\mu}} = \frac{1}{2}\, \hat{E}_{\mu} \tag{4.19b}$$

(4.19a) becomes

$$\begin{aligned} n\pi &= -\int d\tau \int d\Omega\; \mathrm{tr}\; \hat{E}^{\mu} \dot{\hat{A}}_{\mu} \\ &= -\int d\tau \int d\Omega\; \mathrm{tr}\left\{ \hat{E}^{\mu}\hat{E}_{\mu} + \hat{E}^{\mu}\left(\hat{\partial}_{\mu}\hat{a} - [\hat{a}, \hat{A}^{\mu}] \right) \right\} \end{aligned} \tag{4.19c}$$

An integration by parts and use of the equation for $\hat{E}^{\mu}$ shows that the second term in the curly brackets can be set to zero, and we are left with the gauge-invariant quantization condition.

$$n\pi = -\int d\tau \int d\Omega\; \mathrm{tr}\; \hat{E}^{\mu}\hat{E}_{\mu} \tag{4.19d}$$

We insert into this formula the known periodic solutions, for which

$$\begin{aligned} \hat{A}^{\mu} &= i\,\sigma^{\mu\nu}\hat{R}_{\nu}\, f(\tau) \\ \hat{E}^{\mu} &= i\,\sigma^{\mu\nu}\hat{R}_{\nu}\, \dot{f}(\tau) \end{aligned} \tag{4.20a}$$

with f satisfying

$$\dot{f}^2 = 2\varepsilon - f^2(f+2)^2 \tag{4.20b}$$

and get

$$n\pi = 6\pi \int d\tau\, \dot{f}^2(\tau) = 6\pi \int df \sqrt{2\varepsilon - f^2(f+2)^2} \tag{4.21}$$

where the f integration ranges between the two turning points of the classical motion. The meaning of ε is clear. If we evaluate the τ-Hamiltonian, Eq. (4.15), for the solution (4.20), we find $3\pi^2\varepsilon$. Evidently our semi-classical procedure provides the semi-classical eigenvalues of H_τ, which we call $E_\tau = 3\pi^2\varepsilon$.

To determine the dependence of E_τ on n, the f integration in (4.21) must be performed. The formulas involve the complete elliptic integrals of the first and second kind. Two distinct expressions emerge depending whether ε is less than or greater than $\frac{1}{4}$. We record here only the asymptotic forms. For small n, doubly degenerate levels are found with

$$E_\tau = n \tag{4.22}$$

For large n, there is no degeneracy and

$$E_\tau \propto n^{4/3} \tag{4.23}$$

(When small oscillations about any of the vacuum configurations are canonically quantized one also obtains a linear spectrum.) The quantal significance of the periodic solution may be given: when it is expanded in a Fourier series, the Fourier coefficients provide a semi-classical approximation to the matrix elements of the quantum field $\hat{A}^\mu$ between successive bound states.[35]

What corrections to these semi-classical results are envisioned? It is clear that tunnelling removes the degeneracy; this is clear and causes no conceptual problems. Much more problematical are the questions which arise if one confronts this entire program with the realities of Yang-Mills perturba-

tion theory. The problem is of course that the well-known anomalies prevent the theory from being conformally invariant — the renormalization procedure introduces conformal symmetry breaking.[36] In other words it is not obvious how to relate results of the alternate quantization method to the physically relevant, Poincaré covariant theory.

One of two approaches is possible. The theory is regulated in a conventional way; H_τ acquires a τ dependence; bound states disappear but perhaps some kind of adiabatic perturbation theory can be used to study further properties of the spectrum.[37] Alternatively, a non-conventional regularization scheme may be adopted such that H_τ remains a constant of motion. The theory loses translation covariance, since translation generators acquire a τ dependence. Let us suppose however that even in the renormalized theory it is true that $P^\mu = \lim_{\lambda\to\infty} \frac{1}{\lambda} R^\mu$. Then information about the translationally covariant theory could be regained in the limit.

E. Reduced Yang-Mills Theories

To conclude these considerations, we put aside the serious obstacles which still exist in assessing the physical relevance of our alternate proposals, and proceed to another suggestion for obtaining results about Yang-Mills theory. It is apparent that at the present time the model appears too complicated for a complete solution, either classically or quantum mechanically. Nevertheless it has been possible to obtain complete classical solutions which respect a symmetry. A suggestion for the analysis of the quantum theory is to reduce the full degrees of freedom to those that are invariant under a subgroup of the conformal group. In this way we obtain quantum systems which are considerably simpler than the full Yang-Mills theory, yet may retain some of the physical properties of the complete theory.

A very simple model is obtained if we freeze out all but the O(4) invariant degrees of freedom. The Ansatz (4.11) leads to the action

$$I = 6\pi^2 \int d\tau \left[\tfrac{1}{2} \dot{f}^2 - \tfrac{1}{2} f^2 (f+2)^2 \right] \tag{4.24}$$

The quantum mechanics, though trivial, already exhibits some of the features of the complete theory: classical degeneracy, vacuum tunnelling, an anharmonic oscillator bound state spectrum.

Richer is the model where only O(3) symmetry is imposed. The most general O(3) Ansatz is

$$\begin{aligned} \hat{A}^\mu &= i\sigma^{\mu\nu} \hat{R}_\nu f_1 + i P^{\mu\alpha} \sigma_{\alpha\beta} \hat{C}^\beta f_2 \\ &\quad + i P^{\mu\nu} \hat{C}_\nu \hat{C}_\alpha \sigma^{\alpha\beta} \hat{R}_\beta f_3 \\ \hat{a} &= i \hat{R}_\alpha \sigma^{\alpha\beta} \hat{C}_\beta f_4 \\ P^{\mu\nu} &= g^{\mu\nu} - \hat{R}^\mu \hat{R}^\nu \end{aligned} \tag{4.25}$$

Here $\hat{C}$ is a unit 4-vector which picks out the direction of O(4) breakdown to O(3). We set $R\cdot C = \cos\theta$, $0 \le \theta \le \pi$, and the f_i's

depend on τ and θ. With the redefinitions

$$\begin{aligned} \varphi_1 &= 1 + \sin^2\theta\, f_1 \\ \varphi_2 &= \sin\theta\, (\cos\theta\, f_1 + f_2) \\ A^\theta &= f_1 + \cos\theta\, f_2 + \sin^2\theta\, f_3 \\ A^\tau &= \sin\theta\, f_4 \end{aligned} \tag{4.26}$$

the action becomes

$$\begin{aligned} I = 2\pi \int d\tau\, d\theta \Big\{ & |(\partial_\mu + i A_\mu) \Phi|^2 - \tfrac{1}{4} \sin^2\theta\, F^{\mu\nu} F_{\mu\nu} \\ & - \frac{1}{2\sin^2\theta} \left(1 - |\Phi|^2\right)^2 \Big\} \end{aligned} \tag{4.27}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, $\Phi = \phi_1 + i\phi_2$, and the metric is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with the first coordinate being τ, the second θ. The above is an Abelian Higgs model in a 2-dimensional space of constant curvature, with a 3-parameter O(2,1) conformal invariance group of coordinate transformations.

$$\begin{aligned} \delta\xi^\alpha :\ \delta\tau &= a + b \sin\tau \cos\theta + c \cos\tau \cos\theta \\ \delta\theta &= \phantom{a +{}} b \cos\tau \sin\theta - c \sin\tau \sin\theta \\ \delta\Phi &= (\delta\xi^\alpha)\partial_\alpha \Phi \\ \delta A^\mu &= (\delta\xi^\alpha)\partial_\alpha A^\mu + (\partial^\mu \delta\xi^\alpha) A_\alpha \end{aligned} \tag{4.28}$$

The gauge theory possesses an infinite set of topologically inequivalent classical vacua, just as does the complete theory.

$$\Phi = e^{2in\theta}$$
$$A^r = 0$$
$$A^\theta = 2n \qquad (4.29)$$

The imaginary time $[\tau \to i\tau]$ version of the theory has n-pseudo-particle solutions. [The Euclidean model is a conformal transformation of Witten's Lagrangian.[14]] Thus the simplified theory gives rise to the multiple vacua and tunnelling of the complete theory.

At present nothing more is known about the reduced classical or quantum mechanical theory (4.27), beyond of course its O(4) invariant "sub-theory". There is a fine tradition in theoretical physics of studying 2-dimensional models for clues to realistic 4-dimensional ones. We hope that an understanding of our 2-dimensional model (4.27) will help unravel the complexities of the full Yang-Mills theory from which, after all, it was obtained.

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