

Article

The Role of Riemann’s Zeta Function in Mathematics and Physics ^{†,‡}

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[†] This article is dedicated to Martin Reuter on the occasion of his 60th birthday, as well as to a mutual hero of ours: Bernhard Riemann.

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Abstract: In particular, Riemann’s impact on mathematics and physics alike is demonstrated using methods originating from the theory of numbers and from quantum electrodynamics, i.e., from the behavior of an electron in a prescribed external electromagnetic field. More specifically, we employ Riemann’s zeta function to regularize the otherwise infinite results of the so-called Heisenberg–Euler Lagrangian. As a spin-off, we also calculate some integrals that are useful in mathematics and physics.

Keywords: Riemann; zeta function; functional equation; Heisenberg-Euler Lagrangian

1. Usefulness of Riemann’s Functional Equation for the Zeta Function in Physics

Riemann introduced in his talk in Berlin in 1859 [1] (p. 147)

$$\Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \int_1^\infty dx \psi(x) \left(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}\right) - \frac{1}{s(1-s)}. \quad (1)$$

where $\psi(x)$ is related to one of Jacobi’s θ functions. Notice that there is no change of the right-hand side under $s \rightarrow (1-s)$. $\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$ has simple poles at $s = 0$ (from Γ) and $s = 1$ (from ζ). To remove these poles, we multiply by $\frac{1}{2}s(s-1)$. This is the reason why Riemann defines

$$\tilde{\zeta}(s) = \frac{1}{2}s(s-1) \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s), \quad (2)$$

which is an entire function (ζ is a meromorphic function). Obviously, we have

$$\tilde{\zeta}(s) = \tilde{\zeta}(1-s)$$

together with the symmetrical form of the functional equation, which was proved by Riemann for all complex s :

$$\Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \Gamma\left(\frac{1-s}{2}\right) \pi^{-\frac{1}{2}(1-s)} \zeta(1-s). \quad (3)$$

Notice that the right-hand side is obtained from the left-hand side by replacing s by $1-s$.

Before we continue with Equation (3), we make use of two important formulae due to Euler and Legendre.

Legendre’s duplication formula: Γ functions of argument $2s$ can be expressed in terms of Γ functions of smaller arguments.

$$\begin{aligned} \Gamma(2s) &= (2\pi)^{-\frac{1}{2}} 2^{2s-\frac{1}{2}} \Gamma(s) \Gamma\left(s + \frac{1}{2}\right) \\ &= \frac{1}{\sqrt{\pi}} 2^{2s-1} \Gamma(s) \Gamma\left(s + \frac{1}{2}\right). \end{aligned} \tag{4}$$

When we replace $s \rightarrow \frac{s}{2}$, we obtain:

$$\begin{aligned} \Gamma(s) &= \frac{1}{\sqrt{\pi}} 2^{s-1} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) \\ \text{or } \frac{\sqrt{\pi}}{2^{s-1}} \Gamma(s) &= \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right). \end{aligned} \tag{5}$$

Euler’s reflection formula for the Γ function:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}. \tag{6}$$

Replace $s \rightarrow \frac{s+1}{2} = \frac{s}{2} + \frac{1}{2}$:

$$\begin{aligned} \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(1 - \frac{s+1}{2}\right) &= \frac{\pi}{\sin\left(\frac{\pi s}{2} + \frac{\pi}{2}\right)} \\ \text{or } \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{1-s}{2}\right) &= \frac{\pi}{\cos \frac{\pi s}{2}}. \end{aligned} \tag{7}$$

Now multiply Equation (3) by $\Gamma\left(\frac{s+1}{2}\right)$ to obtain

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s). \tag{8}$$

Then, we get from Equations (5) and (7) the following equations:

$$\begin{aligned} \pi^{-\frac{s}{2}} \frac{\sqrt{\pi}}{2^{s-1}} \Gamma(s) \zeta(s) &= \pi^{-\frac{1-s}{2}} \frac{\pi}{\cos \frac{\pi s}{2}} \zeta(1-s), \\ \zeta(1-s) &= \frac{2}{(2\pi)^s} \cos \frac{\pi s}{2} \Gamma(s) \zeta(s). \end{aligned} \tag{9}$$

Here, we replace $s \rightarrow 1-s$ thereby obtaining

$$\zeta(s) = \frac{2}{(2\pi)^{1-s}} \cos \frac{\pi(1-s)}{2} \Gamma(1-s) \zeta(1-s) \tag{10}$$

$$\text{or } \zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s). \tag{11}$$

The latter equation is of great importance in the following sections.

2. Correction of the Classical Electromagnetic Lagrangian by Vacuum Electrons

In 1936, Werner Heisenberg and Hans Euler [2] wrote down the first effective Lagrangian in quantum field theory, which incorporates a quantum correction to the classical Lagrangian of a constant electromagnetic field; this correction is due to the polarization of the quantum vacuum (Dirac’s idea), i.e., the effect of an external constant electromagnetic field on the motion of the vacuum electrons.

To simplify matters, we only consider a constant magnetic field in z direction. For this special case, the modified Lagrangian takes the form—in Schwinger’s representation:

$$\begin{aligned} \mathcal{L}(B) &= \mathcal{L}^{(0)} + \mathcal{L}^{(1)}, \quad \mathcal{L}^{(0)} = -\frac{1}{2}B^2, \\ \mathcal{L}^{(1)}(B) &= \frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-im^2s} \left[(eBs) \cot(eBs) + \frac{1}{3}(eBs)^2 - 1 \right]. \end{aligned} \tag{12}$$

The integral was explicitly calculated for the first time in [3] by dimensional regularization and thereafter in [4] by the so-called zeta-function regularization. The findings of these two different methods agree exactly, whereby the result obtained by the zeta-function regularization is finite without the usual subtraction of divergent counterterms. The result turns out to be

$$\begin{aligned} \mathcal{L}^{(1)}(B) &= -\frac{1}{32\pi^2} \left\{ -3m^4 + 4(eB)^2 \left(\frac{1}{3} - 4\zeta'(-1) \right) + 4m^2(eB)(\ln 2\pi - 1) \right. \\ &\quad - 2m^4 \ln \frac{2eB}{m^2} - 4m^2(eB) \ln \frac{2eB}{m^2} - \frac{4}{3}(eB)^2 \ln \frac{2eB}{m^2} \\ &\quad \left. - 16(eB)^2 \int_1^{1+\frac{m^2}{2eB}} dx \ln \Gamma(x) \right\}. \end{aligned} \tag{13}$$

For those values of the field strength, i.e., for strong fields $\frac{eB}{m^2} \gg 1$, the integral over the logarithm of the gamma function only yields a constant.

With $b = \frac{eB}{m^2}$, we obtain

$$\begin{aligned} b^2 \int_1^{1+\frac{1}{2b}} dx \ln \Gamma(x) &\approx b^2 \int_1^{1+\frac{1}{2b}} dx \left[\ln \Gamma(1) + \frac{d}{dx} \ln \Gamma(x) \Big|_{x=1} (x-1) \right] \\ &= b^2 \Psi(1) \int_1^{1+\frac{1}{2b}} dx (x-1) = \frac{1}{8} \Psi(1) = -\frac{1}{8} C. \end{aligned} \tag{14}$$

Here, C is Euler’s number, $C = \ln \gamma = 0.57721566490$, and the digamma function

$$\Psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)} \quad \text{at } x = 1 \quad \text{is given by } \Psi(1) = -\ln \gamma.$$

Therefore, by only considering the dominant forms for large magnetic field strength, we obtain for the asymptotic form of the one-loop effective Lagrangian in spinor quantum electrodynamics (QED)

$$\begin{aligned} \lim_{\frac{eB}{m^2} \rightarrow \infty} \mathcal{L}^{(1)}(B) &= -\frac{1}{32\pi^2} \left\{ 4(eB)^2 \left(\frac{1}{3} - 4\zeta'(-1) \right) - \frac{4}{3}(eB)^2 \ln \frac{2eB}{m^2} \right\} \\ &= \frac{\alpha B^2}{6\pi} \left\{ \ln \frac{eB}{m^2} + 12\zeta'(-1) - 1 + \ln 2 \right\}, \quad \alpha = \frac{e^2}{4\pi}. \end{aligned} \tag{15}$$

On the other hand, we find in Ritus’ paper [5] under Formula (60) the expression

$$\lim_{\frac{eB}{m^2} \rightarrow \infty} \mathcal{L}^{(1)}(B) = \frac{\alpha B^2}{6\pi} \left\{ \ln \frac{eB}{\gamma \pi m^2} + \frac{6}{\pi^2} \zeta'(2) \right\}. \tag{16}$$

Since Equations (15) and (16) are just two different representations of the same strong-field Lagrangian $\mathcal{L}^{(1)}(B)$, we have the equality

$$\ln \frac{2eB}{m^2} - 1 + 12\zeta'(-1) = \ln \frac{2eB}{m^2 2\pi\gamma} + \frac{6}{\pi^2}\zeta'(2)$$

or

$$-1 + 12\zeta'(-1) = -\ln(2\pi\gamma) + \frac{6}{\pi^2}\zeta'(2). \tag{17}$$

This important equation is in fact a direct consequence of a variant of the famous functional equation of Riemann’s zeta function [1]

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s). \tag{18}$$

Note that the result in Equation (17), which arises from long, complicated field theoretic calculations, follows from solving a physics problem—not from analytical theory of numbers, namely by studying the behavior of vacuum electrons in presence of a constant external strong classical magnetic field. (Later, we meet $\zeta'(-1)$ and $\zeta'(2)$ again in connection with the Glaisher–Kinkelin constant.) Thus far, we consider nonlinear spinor QED where spin $\frac{1}{2}$ particles with mass m are coupled to an external constant magnetic field. The corresponding effective Lagrangian is given by Equation (12).

Now, we study charged spinless particles with mass m , associated with a complex scalar field, which interact with a constant magnetic field. Here, the starting point is given by the Heisenberg–Euler effective Lagrangian, which in Schwinger’s proper time representation reads:

$$\mathcal{L}_{scalar}^{(1)}(B) = -\frac{1}{16\pi^2} \int_0^\infty \frac{1}{s^3} e^{-im^2s} \left[\frac{eBs}{\sin(eBs)} - \frac{1}{6}(eBs)^2 - 1 \right] ds. \tag{19}$$

Without going into detail, we just repeat the former results for spinor particles obtained for strong magnetic fields, when performing the calculation in both the ζ function and proper time regularization. Here are the results for scalar QED:

$$\lim_{\frac{eB}{m^2} \rightarrow \infty} \mathcal{L}_{scalar}^{(1)}(B) = \frac{\alpha B^2}{24\pi} \left[\ln \frac{2eB}{m^2} + (12\zeta'(-1) - 1 + \ln 2) \right] \tag{20}$$

$$= \frac{\alpha B^2}{24\pi} \left[\ln \frac{2eB}{m^2} + \left(-\ln \gamma\pi + \frac{6}{\pi^2}\zeta'(2) \right) \right] \tag{21}$$

This brings us back to Equation (17), which is equivalent to Riemann’s functional equation for the zeta function. This functional equation is evidently independent of the masses involved, be they fermionic or scalar. Were it not for the factors $\frac{1}{24\pi}$ instead of $\frac{1}{6\pi}$ and $\ln \frac{2eB}{m^2}$ instead of $\ln \frac{eB}{m^2}$ in the spinor case, we could have guessed Equations (20) and (21). However, arriving from the proper-time integrals of Equation (12) or Equation (19) at Equations (15), (16), (20), and (21) is a highly challenging undertaking.

Finally, let us mention that the results of Equations (15) and (20) can be used in the Callan–Szymanski renormalization equation to calculate the $\beta_\zeta(\alpha)$ function for spinor and scalar QED to result in

$$\beta_\zeta(\alpha) = \frac{2}{3} \frac{\alpha}{2\pi} \text{ (spinor)}, \quad \beta_\zeta(\alpha) = \frac{1}{6} \frac{\alpha}{\pi} \text{ (scalar)}. \tag{22}$$

This confirms the correctness of the results in Equation (22) calculated otherwise.

3. Proof of Equation (17) from the Functional Equation of Riemann’s Zeta Function

Since Equation (17) contains derivatives of the $\zeta(s)$ function at $s = -1$ and $s = 2$, we need the derivative of Equation (18). Writing $2^s \pi^{s-1} = e^{s \ln 2} e^{(s-1) \ln \pi}$, we obtain

$$\begin{aligned} \zeta'(s) &= (\ln 2 + \ln \pi) 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \\ &+ \frac{\pi}{2} 2^s \pi^{s-1} \cos\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \\ &- 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) (\Gamma'(1-s) \zeta(1-s) + \Gamma(1-s) \zeta'(1-s)). \end{aligned}$$

For $s = -2$, we find

$$\begin{aligned} \zeta'(-2) &= \frac{\pi}{2} 2^{-2} \pi^{-3} \cos(-\pi) \Gamma(3) \zeta(3), \quad \Gamma(3) = 2, \\ \zeta'(-2) &= \frac{-\zeta(3)}{4\pi^2} \end{aligned} \tag{23}$$

or

$$\zeta(3) = -4\pi^2 \zeta'(-2),$$

a result we use below.

For $s = -1$, we employ $\zeta(2) = \frac{\pi^2}{6}$ (Euler’s Basel problem) and $\Gamma'(2) = 1 - C = 1 - \ln \gamma$, $\Gamma(2) = 1$, so that

$$\zeta'(-1) = -\ln 2 \times \frac{1}{12} - \ln \pi \times \frac{1}{12} + (1 - \ln \gamma) \times \frac{1}{12} + \frac{1}{2} \times \frac{1}{\pi^2} \zeta'(2)$$

or

$$\zeta'(-1) = \frac{1}{12} \left[1 - \ln(2\pi\gamma) + \frac{6}{\pi^2} \zeta'(2) \right], \tag{24}$$

that can also be written as

$$-1 + 12\zeta'(-1) = -\ln(2\pi\gamma) + \frac{6}{\pi^2} \zeta'(2), \tag{25}$$

which is exactly the equation that followed from the two regularization methods, which produced the physically motivated results Equations (15) and (16).

4. Asymptotic Expansions of $\ln \Gamma(x + 1)$ and $\ln \Gamma_1(x + 1)$

On the way to calculating the constant L_1 (for the definition cf. Equation (33)), we start with the asymptotic expansion of $\ln \Gamma(x + 1)$ [6]:

$$\begin{aligned} \ln(x!) &= \sum_{x=1}^x \ln x = L_0 + \left(x + \frac{1}{2}\right) \ln x - x + \frac{B_2}{1 \times 2x} + \frac{B_4}{3 \times 4x^3} + \frac{B_6}{5 \times 6x^5} + \dots \\ &= \ln \Gamma(x + 1). \end{aligned} \tag{26}$$

This is called the Moivre–Stirling formula. Actually, it was discovered by Moivre (1667–1754) with the aid of Euler’s (1707–1783) summation formula. Stirling (1692–1770) “only” showed, using Wallis’ (1616–1703) product formula, that the constant L_0 is given by

$$L_0 = \ln \sqrt{2\pi} = 0.918938533 \dots \tag{27}$$

The Bernoulli numbers in Equation (26) are

$$\begin{aligned}
 B_1 &= -\frac{1}{2}, & B_{2n+1} &= 0, & n &= 1, 2, 3, \dots, \\
 B_2 &= \frac{1}{6}, & B_4 &= -\frac{1}{30}, & B_6 &= \frac{1}{42}, & B_8 &= -\frac{1}{30} \dots
 \end{aligned}
 \tag{28}$$

Useful integrals of $\ln \Gamma(x + 1)$ are due to Raabe (1801–1859):

$$\int_x^{x+1} dt \ln \Gamma(t) = \int_{x-1}^x dt \ln \Gamma(t + 1) = x \ln x - x + \ln \sqrt{2\pi}.
 \tag{29}$$

In particular,

$$\begin{aligned}
 \int_0^1 dx \ln \Gamma(x + 1) &= \ln \sqrt{2\pi} - 1, \\
 \int_0^1 dx \ln \Gamma(x) &= \ln \sqrt{2\pi}.
 \end{aligned}$$

The generalized Γ function $\Gamma_1(x)$ shows up in the value of the integral

$$\int_0^x dt \ln \Gamma(t) = \ln \Gamma_1(x) - \frac{x}{2}(x - 1) + x \ln \sqrt{2\pi}.
 \tag{30}$$

One could use Equation (30) as defining equation for $\ln \Gamma_1(x)$. Putting $x = \frac{1}{2}$, we obtain from Equation (30)

$$\begin{aligned}
 \int_0^{\frac{1}{2}} dx \ln \Gamma(x) &= \ln \Gamma_1\left(\frac{1}{2}\right) + \frac{1}{8} + \frac{1}{2} \ln \sqrt{2\pi} \\
 &= \frac{3}{2}L_1 + \frac{5}{24} \ln 2 + \frac{1}{4} \ln \pi,
 \end{aligned}
 \tag{31}$$

where we use

$$\ln \Gamma_1\left(\frac{1}{2}\right) = \frac{3}{2}L_1 - \frac{1}{8} - \frac{1}{24} \ln 2,
 \tag{32}$$

which can be looked up in Ref. [6]. The constant L_1 remains to be determined, which is the main goal of this paper.

Next, we introduce the asymptotic expansion of $\ln \Gamma_1(x + 1)$ [6]:

$$\begin{aligned}
 \ln \left(1^1 \times 2^2 \times 3^3 \dots x^x\right) &= \sum_{k=1}^x k \ln k =: \ln \Gamma_1(x + 1) \\
 &= L_1 + \left[\frac{x}{2}(x + 1) + \frac{1}{12}\right] \ln x - \frac{1}{4}x^2 - \left(\frac{B_4}{2 \times 3 \times 4x^2} + \frac{B_6}{4 \times 5 \times 6x^4} + \frac{B_8}{6 \times 7 \times 8x^6} + \dots\right).
 \end{aligned}
 \tag{33}$$

As pointed out in the title of this paper, the constant L_1 is determined by comparing different regularization methods arising from physics arguments.

The generalized Γ function of the first kind satisfies the relations

$$\ln \Gamma_1(x + 1) = \int_0^x dt \ln \Gamma(t + 1) + \frac{x}{2}(x + 1) - x \ln \sqrt{2\pi}, \tag{34}$$

$$\ln \Gamma_1(x) = \int_0^x dt \ln \Gamma(t) + \frac{x}{2}(x - 1) - x \ln \sqrt{2\pi}. \tag{35}$$

$\Gamma_1(x)$ satisfies the functional equation

$$\Gamma_1(x + 1) = x^x \Gamma_1(x) \tag{36}$$

with the constraints $\Gamma_1(0) = \Gamma_1(1) = \Gamma_1(2) = 1$.

The resemblance to the usual Γ function (of the zeroth kind $\Gamma_0 \equiv \Gamma$) becomes obvious in observing that Γ_1 for integer values of the argument reduces to

$$\Gamma_1(n + 1) = 1^1 \times 2^2 \times 3^3 \dots n^n, \quad n > 0.$$

The generalized Γ function of the k th kind satisfies

$$\begin{aligned} \Gamma_{k+1}(x + 1) &= x^{x^{k+1}} \Gamma_{k+1}(x), \\ \ln \Gamma_{k+1}(x + 1) &= x^{k+1} \ln x + \ln \Gamma_{k+1}(x), \\ k = 0: \quad \ln \Gamma_1(x + 1) &= x \ln x + \ln \Gamma_1(x). \end{aligned} \tag{37}$$

The analog of Raabe’s formula is given by

$$\int_{x-1}^x dt \ln \Gamma_1(t + 1) = \frac{x^2}{2} \ln x - \frac{x^2}{4} + L_1 - \frac{1}{12}. \tag{38}$$

In particular,

$$\int_0^1 dx \ln \Gamma_1(x + 1) = L_1 - \frac{1}{3}, \tag{39}$$

$$\int_0^1 dx \ln \Gamma_1(x) = L_1 - \frac{1}{12}. \tag{40}$$

Using $x = \frac{1}{2}$ in Equation (34) and Equation (32) in

$$\ln \Gamma_1\left(\frac{3}{2}\right) = \frac{1}{2} \ln \frac{1}{2} + \ln \Gamma_1\left(\frac{1}{2}\right) \tag{41}$$

we obtain

$$\ln \Gamma_1\left(\frac{3}{2}\right) = -\frac{13}{24} \ln 2 + \left(\frac{3}{2}\right) L_1 - \frac{1}{8},$$

so that

$$\int_0^{\frac{1}{2}} dx \ln \Gamma(x + 1) = \ln \Gamma_1\left(\frac{3}{2}\right) - \frac{3}{8} + \frac{1}{4} \ln 2 + \frac{1}{4} \ln \pi$$

$$= -\frac{7}{24} \ln 2 + \frac{3}{2} L_1 - \frac{1}{2} + \frac{1}{4} \ln \pi. \tag{42}$$

5. Numerical Value of L_1 from Γ_1 Function Regularization

Here, we make substantial use of the results contained in the rather elaborate paper of the authors of Ref. [7], from which we can extract the Γ_1 function regularization for $\mathcal{L}^{(1)}(B)$ limiting ourselves to strong fields, $eB/m^2 \gg 1$.

Here is the result:

$$\lim_{\frac{eB}{m^2} \rightarrow \infty} \mathcal{L}^{(1)}(B) = \frac{\alpha B^2}{6\pi} \left\{ \frac{eB}{m^2} + \ln 2 - 12L_1 \right\}. \tag{43}$$

Comparing the right-hand side with the results from the ζ function regularization in Equation (15) or from Ritus' formula in Equation (16), the equality of the three different regularization methods teaches us

$$L_1 = \frac{1}{12} - \zeta'(-1) \tag{44}$$

or

$$L_1 = \frac{C}{12} + \frac{1}{12} \ln 2\pi - \frac{1}{2\pi^2} \zeta'(2). \tag{45}$$

Upon using the constants

$$\zeta'(-1) = -0.165421, \quad \zeta'(2) = -0.93754$$

and Euler's constant

$$\ln \gamma = C = -\Psi(1) = 0.57721566490,$$

we obtain

$$L_1 = 0.248754477. \tag{46}$$

This number can be inserted into the text wherever we meet the constant L_1 .

Given the explicit expression in Equation (46), it is worthwhile looking at the so-called Glaisher–Kinkelin constant A . Here is one of the many representations:

$$A = e^{\frac{1}{12} - \zeta'(-1)} \quad \text{from} \quad \ln A = \frac{1}{12} - \zeta'(-1) = L_1. \tag{47}$$

When we take the value of L_1 from Equation (45), we obtain:

$$A = (2\pi)^{\frac{1}{12}} \left[e^{\frac{\pi^2}{6} \ln \gamma - \zeta'(2)} \right]^{\frac{1}{2\pi^2}} = 1.2824271291 \tag{48}$$

Here are two more representations:

$$\int_0^{\frac{1}{2}} \ln \Gamma(x) dx = \frac{3}{2} \ln A + \frac{5}{24} \ln 2 + \frac{1}{4} \ln \pi \tag{49}$$

yields

$$A = 2^{-\frac{5}{36}} \pi^{-\frac{1}{6}} \exp \left[\frac{2}{3} \int_0^{\frac{1}{2}} \ln \Gamma(x) dx \right]. \tag{50}$$

$$\int_0^{\frac{1}{2}} \ln \Gamma(x+1) dx = \frac{3}{2} \ln A - \frac{7}{24} \ln 2 + \frac{1}{4} \ln \pi - \frac{1}{2} \quad (51)$$

results in

$$A = 2^{\frac{7}{36}} \pi^{-\frac{1}{6}} \exp \left[\frac{1}{3} + \frac{2}{3} \int_0^{\frac{1}{2}} \ln \Gamma(x+1) dx \right]. \quad (52)$$

A final remark should be made concerning the importance of $\zeta(3)$, i.e., the infinite sum of all the inverse cubes,

$$\zeta(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \dots \quad (53)$$

No complex numbers are needed because $\zeta(3)$ lies in the original Euler domain of convergence, i.e., $x > 1$. No wonder that Euler became interested in the fundamentals of its value, e.g. is it rational, transcendental, etc.? Remarkably, it took until the late 20th century before a major breakthrough was achieved in 1979 by the 61-year-old French mathematician Roger Apéry, who was able to prove the irrationality of $\zeta(3)$. Today, due to his proof, the constant $\zeta(3)$ is known as Apéry constant. Its numerical value is given by

$$\zeta(3) \approx 1.202056903159\dots \quad (54)$$

Hence, everywhere we meet $\zeta(3)$ in the foregoing text, one can use this value. However, $\zeta(3)$ is not only of great interest to researchers working in number theory, but is also of immense value for people doing calculations in QED, e.g. the second- and third-order (in α) radiative corrections of the electron (and muon) anomalous magnetic moment, which is one of the best-measured and -calculated numbers in all of physics. Furthermore, $\zeta(3)$ is needed in the second-order radiative corrections to the so-called triangle anomaly, which is essential for the understanding of the π^0 decay into two photons.

Finally, while knowledge of the numerical value of Apéry's constant $\zeta(3) = -4\pi^2 \zeta'(-2) \approx 1.202056903159$ is absolutely necessary for computing the just-mentioned elementary particle processes, the constants $\zeta'(-1) \approx -0.165421$ and $\zeta'(2) \approx -0.93754$ —tied together by the relation in Equation (25)—are likewise of utmost importance for evaluating effects arising from the effective Lagrangian in QED, i.e., from the Heisenberg–Euler non-linear contribution. An example of this is the impact of the QED birefringent quantum vacuum structure on the propagation of light in the universe in the neighborhood of neutron stars, which constitute the laboratory for studying physical processes in superstrong magnetic fields.

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