Charged black holes coupled to non-linear electrodynamics in scalar-tensor theories of gravity with massive scalar field *

Ivan Zh. Stefanov†, **Stoytcho S. Yazadjiev**‡ and

Dept. of Theoretical Physics, Faculty of Physics,
University of Sofia "St. Kliment Ohridski" ,
5, James Bourchier Blvd., 1164 Sofia, BULGGARIA

Michail D. Todorov§

Faculty of Applied Mathematics and Computer Science,
Technical University of Sofia,
8, Kliment Ohridski Blvd., 1000 Sofia, BULGARIA

Abstract

The no-scalar-hair theorems do not apply in the case when non-linear electrodynamics is included in the theory. In the current work, some preliminary numerical results describing charged black holes coupled to Born-Infeld type non-linear electrodynamics in scalar-tensor theories of gravity with massive scalar field are presented.

1. Introduction

Scalar-tensor theories (STT) of gravity are the most natural generalization of General Relativity (GR) and arise naturally from different alternative theories of gravity [1]. Different modifications of STT are attracting much interest also in cosmology and astrophysics. A most natural question is whether STT would predict the existence of new objects or phenomena which are absent in the GR. If we constrain ourselves to black holes, which are among the most interesting and characteristic objects for GR we would probably like to know whether they would have different properties in frame of STT and if so, how we can distinguish between the black holes in these theories. According to the no-hair conjecture in GR in the exterior of a black hole the only information available regarding the black hole may

* This work was partially supported by the Bulgarian National Science Fund, Grants MUF 04/05 (MU 408) and VU-F-205/06, and the Sofia University Research Fund N60.

† e-mail address: zhivkov@phys.uni-sofia.bg

‡ e-mail address: yazad@phys.uni-sofia.bg

§ e-mail address: mtod@tu-sofia.bg

be that of its mass, charge, and angular momentum. Since in STT an additional field is present, namely the scalar field, we would expect that new preserved quantities might exist which would be characteristic of the STT black holes. It appears, however, that the no-hair conjecture holds also for a large class of STT and the scalar field does not induce new preserved charges. The hypothesis that 'Black holes have no hair' was introduced by Wheeler for the first time in the early 1970's. Since then different versions of no-scalar-hair theorems have been proved (See[3] and references therein). A simple and relatively recent proof of a no-scalar-hair theorem for static, spherically symmetric, asymptotically flat, neutral black holes for a large class of STT was proposed by Saa [2]. He applied an explicit, covariant method to generate the exterior solutions for these theories through conformal transformations from the minimally coupled case. The scalar field in these theories becomes constant if one demands that the essential singularity at the center of symmetry is hidden by an event horizon.

A similar theorem treating also the case of charged scalar field with self-interaction was proved by Bekenstein [3]. Saa's theorem was generalized for the case of charged black holes in linear electrodynamics by Banerjee and Sen [4].

Several solutions that seem to circumvent no-scalar hair conjecture have been found. Most of them, however, are either unstable or asymptotically non-flat. One possible chance to find solutions describing hairy black holes is to consider theories in which the lagrangian of the electromagnetic field is non-linear. The nonlinear electrodynamics was first introduced by Born and Infeld in 1934 to obtain finite energy density model for the electron [5]. They proposed the following Lagrangian

$$L_{BI} = 2b\left[1 - \sqrt{1 + \frac{1}{4b}F_{\mu\nu}F^{\mu\nu} - \frac{1}{64b^2}(F_{\mu\nu} \star F^{\mu\nu})^2}\right], \tag{1}$$

where the star "$\star$" stands for the Hodge operator. In recent years nonlinear electrodynamics models are attracting much interest, too. The reason is that the nonlinear electrodynamics arises naturally in open strings and D-branes [6]. Nonlinear electrodynamics models coupled to gravity have been discussed in different aspects (see, for example, [7]–[15] and references therein).

In the non-linear electrodynamics the energy-momentum tensor of the electromagnetic field has a non-vanishing trace a sequence of which is that the electromagnetic field is non-trivially coupled to the scalar field. In other words, in these theories the electromagnetic field could act as a source of the scalar field. In the present work we prove that our assumption is correct and find numerical solutions describing black holes with a non-trivial massive scalar field in the non-linear electrodynamics. In this paper we consider a particular example of non-linear electrodynamics, namely the Born-Infeld non-linear electrodynamics. For further reading on black holes with massive dilaton we refer the reader to [12] and [13].

2. Formulation of the problem

The general form of the extended gravitational action in scalar-tensor theories is

$$S = \frac{1}{16\pi G_*}\int d^4x\sqrt{-\tilde{g}}\left(F(\Phi)\tilde{\mathcal{R}} - Z(\Phi)\tilde{g}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - 2U(\Phi)\right) + S_m\left[\Psi_m;\tilde{g}_{\mu\nu}\right]. \tag{2}$$

For the sake of mathematical convenience, it is usual for scalar-tensor theories to be studied and analyzed with respect to the conformally related Einstein frame given by the metric:

$$g_{\mu\nu} = F(\Phi)\tilde{g}_{\mu\nu}. \tag{3}$$

Further, let us introduce the scalar field φ (the so-called dilaton) via the equation

$$\left(\frac{d\varphi}{d\Phi}\right)^2 = \frac{3}{4}\left\{\frac{d\ln[F(\Phi)]}{d\Phi}\right\}^2 + \frac{Z(\Phi)}{2F(\Phi)}, \tag{4}$$

and define

$$\mathcal{A}(\varphi) = F^{-1/2}(\Phi)\ \ ,2V(\varphi) = U(\Phi)F^{-2}(\Phi). \tag{5}$$

These are not symmetry transformations but rather a transition to a different set of fields. In the Einstein frame action (2) takes the form

$$S = \frac{1}{16\pi G_*}\int d^4x\sqrt{-g}\left[\mathcal{R} - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - 4V(\varphi)\right] + S_m[\Psi_m;\mathcal{A}^2(\varphi)g_{\mu\nu}], \tag{6}$$

where R is the Ricci scalar curvature with respect to the Einstein metric $g_{\mu\nu}$.

With the use of (4) and (5) the nonlinear electrodynamics action takes the following form in the Einstein frame:

$$S_m = \frac{1}{4\pi G_*}\int d^4x\sqrt{-g}\mathcal{A}^4(\varphi)L(X,Y), \tag{7}$$

where

$$X = \frac{\mathcal{A}^{-4}(\varphi)}{4}F_{\mu\nu}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}, \qquad Y = \frac{\mathcal{A}^{-4}(\varphi)}{4}F_{\mu\nu}\left(\star F\right)^{\mu\nu} \tag{8}$$

and "$\star$" stands for the Hodge dual with respect to the Einstein frame metric $g_{\mu\nu}$.

The action (6) with (7) yields the following field equations

$$
\begin{aligned}
&\mathcal{R}_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi + 2V(\varphi)g_{\mu\nu} - 2\partial_X L(X,Y)\left(F_{\mu\beta}F_\nu^\beta - \frac{1}{2}\,g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)\\
&\qquad\quad - 2\mathcal{A}^4(\varphi)\left[L(X,Y) - Y\partial_Y L(X,Y)\right]g_{\mu\nu},\\
&\nabla_\mu\left[\partial_X L(X,Y)F^{\mu\nu} + \partial_Y L(X,Y)(\star F)^{\mu\nu}\right] = 0, \qquad\qquad (9)\\
&\nabla_\mu\nabla^\mu\varphi = \frac{dV(\varphi)}{d\varphi}\\
&\qquad\quad - 4\alpha(\varphi)\mathcal{A}^4(\varphi)\left[L(X,Y) - X\partial_X L(X,Y) - Y\partial_Y L(X,Y)\right],
\end{aligned}
$$

where $\alpha(\varphi) = \frac{d\mathcal{A}(\varphi)}{d\varphi}$.

In what follows we consider the truncated[1] Born-Infeld electrodynamics described by the Lagrangian

$$
L_{BI}(X) = 2b\left(1 - \sqrt{1 + \frac{X}{b}}\right). \tag{10}
$$

Here we take potential of the scalar field in the form

$$
V(\varphi) = \frac{1}{2}\frac{\varphi^2}{\Lambda^2}, \tag{11}
$$

where Λ is the Compton wavelength of the scalar field.

3. Basic equations

The metric of a static, spherically symmetric spacetime can be written in the form

$$
ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -f(r)e^{-2\delta(r)}dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{12}
$$

where

$$
f = 1 - \frac{2m(r)}{r}. \tag{13}
$$

In the considered Born-Infeld type non-linear electrodynamics an electric-magnetic duality exists which means that the solutions in the magnetically charged case and the electrically charged case coincide. We will study the magnetically charged black holes for which the electromagnetic field is given by

$$
F = P\sin\theta d\theta\wedge d\phi \tag{14}
$$

[1] Here we consider the pure magnetic case for which $Y = 0$.

and the magnetic charge is denoted by P.

The field equations reduce to the following coupled system of ordinary differential equations:

$$\frac{d\delta}{dr} = -r\left(\frac{d\varphi}{dr}\right)^2, \tag{15}$$

$$\frac{dm}{dr} = r^2\left[\frac{1}{2}f\left(\frac{d\varphi}{dr}\right)^2 + V(\varphi) - \mathcal{A}(\varphi)^4 L(X)\right], \tag{16}$$

$$\frac{d}{dr}\left(r^2 f\frac{d\varphi}{dr}\right) = r^2\left\{\frac{dV(\varphi)}{dr} - 4\alpha(\varphi)\mathcal{A}^4(\varphi)\left[L(X) - X\partial_X L(X)\right] - rf\left(\frac{d\varphi}{dr}\right)^3\right\}, \tag{17}$$

where X reduces to:

$$X = \frac{\mathcal{A}^{-4}(\varphi)}{2}\frac{P^2}{r^4}. \tag{18}$$

We will be searching for solutions for which the dilaton field φ is regular on the event horizon. In the present work we will consider only theories for which $\alpha(\varphi) = \text{const}$. Constant coupling parameter corresponds to the Brans-Dicke theory.

The present solution has a much richer causal structure in comparison to the case with massless scalar field, since it admits the presence of inner horizons and extremal solutions. The conditions for existence of extremal solutions are given by the following algebraic equations:

$$\frac{1}{2} = r_e^2\left[V(\varphi_e) - \mathcal{A}(\varphi_e)^4 L(X_e)\right], \tag{19}$$

$$0 = r_e^2\left\{\frac{dV(\varphi_e)}{dr} - 4\alpha(\varphi_e)\mathcal{A}^4(\varphi_e)\left[L(X_e) - X_e\partial_X L(X_e)\right]\right\}, \tag{20}$$

These relations are obtained from equations (15)-(17) by taking into account that on the extremal horizon

$$f(r_e) = 0, \qquad \frac{dm(r_e)}{dr} = \frac{1}{2}. \tag{21}$$

We solve equations (19)-(20) with respect to r_e^2 and obtain:

$$r_e^2 = \frac{-F_1 \pm \sqrt{F_1^2 - 4\alpha F_2}}{2F_2}, \tag{22}$$

where

$$F_1 = \left(\frac{dV}{d\varphi} - 4\alpha V\right), \qquad F_2 = \left(4b\,\frac{dV}{d\varphi}e^{4\alpha\varphi_e} - 2V\frac{dV}{d\varphi} + 4\alpha V^2\right). \tag{23}$$

The root for which $0 < r_e^2 < \infty$ for all values of φ_e should be chosen. Then we substitute it in equation (19) and obtain the following sophisticated non-linear equation for φ_e

$$1 = \frac{-F_1 + \sqrt{F_1^2 - 4\alpha F_2}}{2F_2} \times \left\{ 2V - 4b\, e^{4\alpha\varphi_e} \left[1 - \sqrt{1 + \frac{P^2}{2b}\, e^{-4\alpha\varphi_e} \left(\frac{2F_2}{-F_1 + \sqrt{F_1^2 - 4\alpha F_2}} \right)^2} \right] \right\}, \tag{24}$$

which we treat numerically. The solution for φ_e is substituted back in (22) which, on its turn, gives us the radius if the degenerate horizons.

4. Numerical results

The nonlinear system (15)-(17) is inextricably coupled and the event horizon r_H is *a priori* unknown boundary. In order to be solved, it is recast as a equivalent first order system of ordinary differential equations. Following the physical assumptions of the matter under consideration the asymptotic boundary conditions are set, i.e.,

$$\lim_{r\to\infty} m(r) = M \quad (M \text{ is the mass of the black hole in the Einstein frame}),$$

$$\lim_{r\to\infty} \delta(r) = \lim_{r\to\infty} \varphi(r) = 0.$$

At the horizon both the relationship

$$f(r_H) = 0$$

and the regularization condition

$$\left(\frac{df}{dr} \cdot \frac{d\varphi}{dr} \right)\Big|_{r=r_H} = \Big\{ 4\alpha(\varphi)\mathcal{A}^4(\varphi)[X\partial_X L(X) - L(X)] \Big\}\Big|_{r=r_H}$$

concerning the spectral quantity r_H must be held. For the treating the above posed boundary-value problem (BVP) the Continuous Analog of Newton Method (see, for example [16],[17],[11]) is used. After an appropriate linearization the original BVP is rendered to solving a vector two-point BVP. On a discrete level sparse (almost diagonal) linear algebraic systems with regard to increments of sought functions $\delta(r)$, $m(r)$, and $\varphi(r)$ have to be inverted.

We studied the parametric space for fixed value of the coupling parameter $\alpha = 0.01$ (this value is close to the one established on the bases of experimental data) and for several values of the magnetic charge.

In Fig.(1) radius and the temperature of the horizon are shown. In the case of massless scalar field the the radius of the event horizon turns to zero for

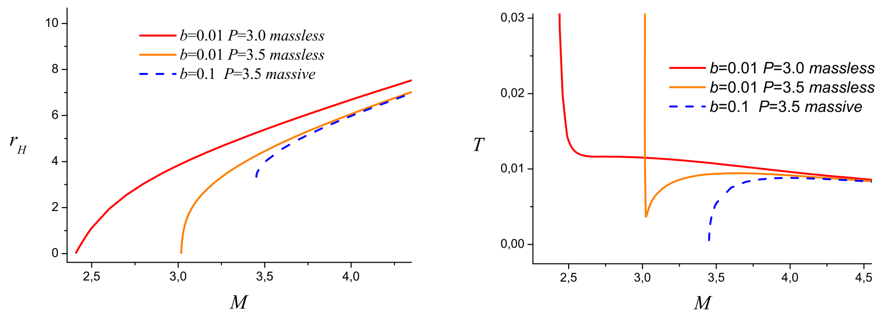

Figure 1: The radius r_H and the temperature T of the black hole as a function of its mass M.

a finite value of the black hole's mass M. For the massive scalar field, however, extremal solutions are present for some values of the parameters α, b, P and M. We have shown one particular example of an extremal solution on the same figure–the graphic with the dashed line. Extremal black holes have zero temperature. This can also be seen in the same figure.

Equation (24) for φ_e has two roots. The two corresponding solutions r_{e_1} and r_{e_2} represent an external, degenerate horizon (an extremal event horizon) and an inner, degenerate horizon, respectively. It should be noted, however, that the analysis for existence of degenerate horizons does not take in consideration the boundary conditions. In other words, it is possible that for some values of the parameters the extremal black-hole solutions are not asymptotically flat. The dependence of the degenerate horizons on the magnetic charge P and on the inverse Compton length of the scalar field Λ (i.e. on the mass of the scalar field) are presented in Fig. (2).

5. Conclusion

In the present work numerical solutions describing charged black holes coupled to non-linear electrodynamics in the scalar-tensor theories with massless scalar field were found. Since an electric-magnetic duality is present in the used electrodynamics, only purely magnetically case was studied here. For the Lagrangian of the non-linear electrodynamics the truncated Born-Infeld Lagrangian was chosen and scalar-tensor theories with massive scalar field and positive coupling parameter were considered. As a result of the numerical investigation, some general properties of the solutions were found. The theory we considered admits the existence of extremal black-hole solutions unlike the case with massless scalar field.

References

[1] T. Damour and A. Polyakov, Nucl. Phys. **B423**, 532 (1994)

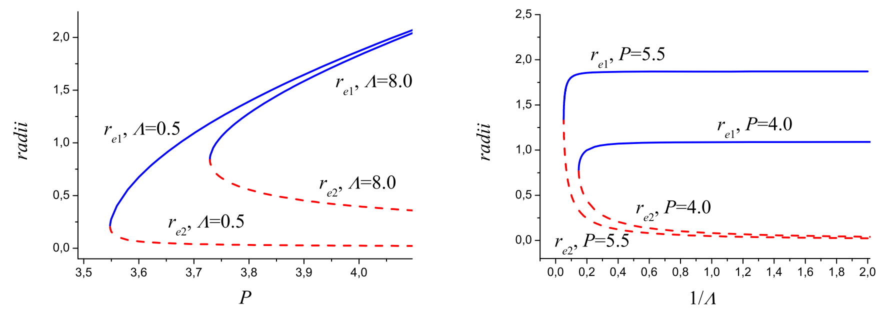

Figure 2: The radius of the horizon for the extremal black holes as a function of the magnetic charge P of the black hole and inverse Compton length $1/\Lambda$ of the scalar field. r_{e1} (solid line) designates an exterior degenerate horizon while r_{e2} (dashed line) is an inner degenerate horizon.

[2] A. Saa, J. Math. Phys. **37**, 2346 (1996).

[3] A. Mayo and J. Bekenstein, Phys. Rev. **D54**, 5059 (1996).

[4] N. Banerjee and S. Sen, Phys. Rev. **D47**, 104024 (1998).

[5] M. Born and L. Infeld, Proc. R. Soc. London **A143**, 410 (1934).

[6] R. Leigh, Mod. Phys. Lett. **A4**, 2767 (1989).

[7] M. Demianski, *Found.Phys.* **16**, 187 (1986)

[8] G. Gibbons and D. Rasheed, Nucl. Phys. **B454**, 185 (1995).

[9] T. Tamaki, T. Torii, Phys. Rev. **D64**, 024027 (2001)

[10] H. Yajima and T. Tamaki, Phys. Rev. **D63**, 064007 (2001).

[11] S. Yazadjiev, P. Fiziev, T. Boyadjiev and M. Todorov, Mod. Phys. Lett. **A16**, 2143 (2001).

[12] J. H. Horne and G. T. Horowitz, Nucl. Phys. **B399**, 169-196 (1992).

[13] R. Gregory and J. Harvey, Phys. Rev. **D47**, 2411 (1993).

[14] S. Yazadjiev, Phys. Rev. **D72**, 044006 (2005)

[15] A. Sheykhi, N. Riazi and M. H. Mahzoon, Phys. Rev. **D74**, 044025 (2006)

[16] M. K. Gavurin, Izvestia VUZ, Matematika **14**(6), 18–31 (1958) (in Russian) (see also Math.Rev. **25**(2), #1380 (1963)).

[17] E. P. Jidkov, G. I. Makarenko and I. V. Puzynin, in *Physics of Elementary Particles and Atomic Nuclei* (JINR, Dubna, 1973) vol. 4, part I, 127–166 (in Russian), English translation: *American Institute of Physics*, p.53.