Linear higher derivative field theories, entanglement entropy and quantum phase transition

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Abstract. Quantum systems described by linear, higher order spatial derivatives of the field exhibit a new kind of phase transition as the coupling strength of the higher order terms varies. This essentially happens because a fundamental change in the ratio of ground state energy eigenvalues, between linear and non-linear regime, takes place which leads to excitation of higher modes. The new phase has inverse relation with area.

Introduction: Entanglement entropy plays crucial role in understanding quantum behavior of macroscopic (black-holes) and microscopic systems [1, 2]. The so-called area law [3, 4, 5, 2] raises the possibility to interpret the Bekenstein-Hawking entropy as the entanglement entropy. Entanglement is also an important factor in understanding quantum phase transitions [6, 7]. It is an useful measure to quantify various aspects of the transitions.

In general, theory of phase-transitions assumes that free energy contains non-linear self-interactions of the order parameter: $F(\phi) = a_2\phi^2 + a_4\phi^4 + \cdots$ [8]. One can generalize the above model by including higher spatial derivatives of the order-parameter [9]. Setting the self-interactions terms to zero ($a_2 = a_4 = 0$), the Hamiltonian for a real scalar field (ϕ) is given by [10, 11, 12, 13]:

$$H = \frac{1}{2}\int d^3x \left[\pi^2(x) + \sum_{m=1}^{\infty} b_m \left(\mathcal{D}^m\phi\right)\left(\mathcal{D}_m\phi\right)\right], \qquad (1)$$

the entanglement entropy shows a sudden discontinuity due to the presence of the term corresponding to $m = 3$. The operator $\mathcal{D}^m$ in the above Hamiltonian is

$$\mathcal{D}_m \equiv \nabla_1 \nabla_2 \ldots \nabla_m, \qquad (2)$$

where $b_m \equiv \pm\kappa^{-2(m-1)}$ are the appropriate coupling coefficients and κ is the frequency across which the dispersion relation changes from linear to non-linear or vice-versa. To extract the physics, we will consider the following dispersion relation:

$$\omega^2 = k^2 + \frac{\epsilon}{\kappa^2}k^4 + \frac{\tau}{\kappa^4}k^6, \qquad (3)$$

keeping the terms up to $m = 3$ in Hamiltonian (1), where ϵ, τ are (dimensionless) constants. The results presented below have already been submitted for publication [15].

`Simulation technique and results:` The Discretised Hamiltonian (1) along the radial direction of a spherical lattice with lattice spacing a, such that $r \to r_i = ia$; $r_{i+1} - r_i = a$ is given by:

$$H = \sum_j H_j = \frac{1}{2a} \sum_{i,j}^{N} \delta_{ij} \pi_j^2 + \varphi_j \, K_{ij} \, \varphi_i \qquad (4)$$

where the interaction between them are contained in the off-diagonal elements of the matrix K_{ij}.

Total system size is denoted by a large but finite N. An intermediate point n is chosen, which represents the boundary surface (the *horizon*) with radius $\mathcal{R}$ ($= an$), that separates the lattice points between two parts. The lattice spacing a appears as an overall factor in the Hamiltonian (4) and in the coupling factor $P \equiv \frac{1}{a^2 \kappa^2}$. This implies that the results for a particular value of N (say, $N = 300$) can be mapped to larger value of N (say, 3000) for smaller value of a. Note that a should be greater than the cutoff scale ($1/\kappa$) implies $0 \le P \le 1$.

Entanglement is computed as the von Neumann entropy for the reduced density matrix $\rho(r, r')$ [1, 2]: $S^{(P)} = \text{Tr} \, (\rho \ln \rho)$. We compute the entanglement entropy for the discretized Hamiltonian (4) for different P. The computations are done using Matlab for the lattice size $N = 300, 50 \le n \le 295$ and the relative error in the computation is 10^{-5}. Figures below show the results for the following two different scenarios:
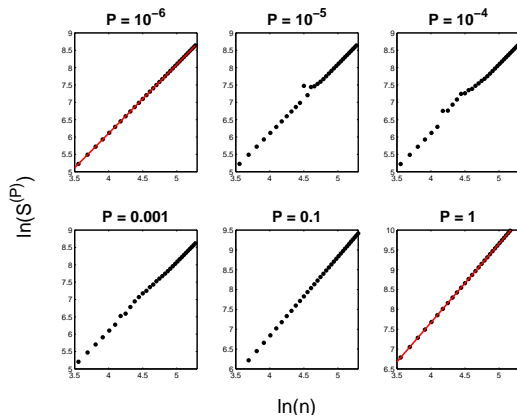

Figure 1. Log-log plot of von Neumann entropy, S, versus the scaled radius of the sphere $\mathcal{R}/a = n$, for $\tau = 0$, $\epsilon = 1$ and different P. The dots represent the numerical output and solid lines denote lines of best fit.
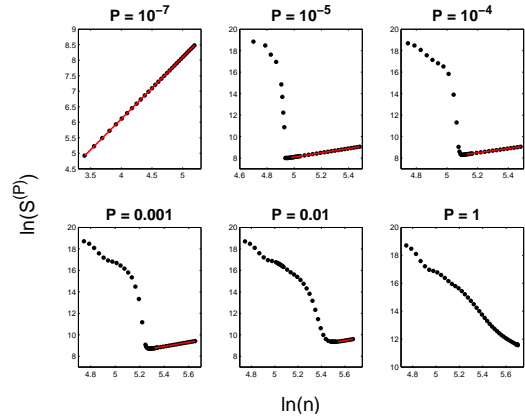
Figure 2. Log-log plot of von Neumann entropy, S, versus the scaled radius of the sphere $\mathcal{R}/a = n$, for $\tau = 1$, $\epsilon = 0$ and different P. n_c for $P = 10^{-5}, 10^{-4}, 10^{-3}$, respectively, are 139, 167 and 197.

(1) $\tau = 0$, $\epsilon = 1$: The best fit (solid) curves in the two asymptotic regimes, $P \to 0$ and $P \to 1$, show that the entropy scales approximately as area $S \sim (\mathcal{R}/a)^2$. With increasing P, however, the prefactor increases. For $P = 10^{-5}$ and 10^{-4}, around the transition region between linear and non-linear, entropy increases by an order.

(2) $\tau = 1$, $\epsilon = 0$: (i) For small P, entropy scales as area. (ii) As we increase P, above a *critical* value of P, say P_c (here $P_c \sim 10^{-6}$) a new phase appears in the $\log-\log$ plots of entropy vs n. (iii) For $P > P_c$, scaling of $S^{(P)}$ changes across some *critical* value of n, say n_c (e.g. $n_c \sim 137$ for $P = 10^{-5}$). For $n > n_c$, $S^{(P)}$ increases with n and scales approximately to area. However, for $n < n_c$, $S^{(P)}$ increases with decreasing n [1] (iv) With increasing P, n_c increases.

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[1] Note that as $n \to 0$, the entropy becomes zero as the number of degrees of freedom gradually vanishes.

The above results indicate that higher derivative term for $m = 3$ modifies the vacuum such that the entropy jumps by few orders of magnitude close to the critical point.

Fig. (3) shows the entropy distribution per partial wave for two values of n — $n = 200$ (which falls in the *area-law* region in Fig. 2) and $n = 137$ (from the transition region in Fig. 2) for $\epsilon = 0, \tau = 1$ and $P = 10^{-5}$. For $n = 200$, the contribution of large ℓ falls off rapidly implying that only low ℓ-modes contribute to the entropy. However, for $n = 137$, higher partial waves contribute significantly to the entropy compared to low ℓ.
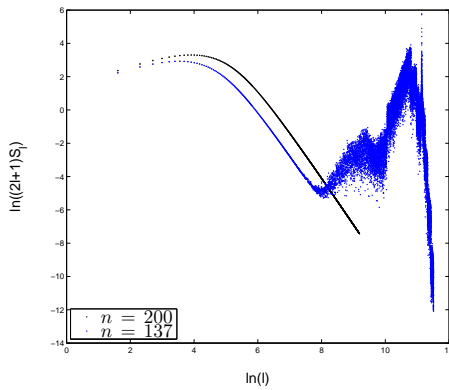

Figure 3. Log-Log plot of the entropy distribution per partial wave $(2\ell + 1)S_\ell$ vs ℓ for $n = 200$ and 137 for $P = 10^{-5}$. For a specific n and P, the asymptotic analysis shows that modes higher than a critical value $\ell_c \sim P^{-1/2}n$ contribute more than the ones lower than ℓ_c. Which also implies that for a fixed mode ℓ_c, this cross-over appear at higher n with increasing P as found in Fig. (2).

Undestanding the results: The higher derivative terms dominate with decreasing length and hence, for a fixed P the inverse scaling phase appears below certain n_c. To understand the cause of excitation of high ℓ-modes (for $P > P_c$), let us consider a simple quantum mechanical model of a particle in a box. The Schrödinger equation for a non-relativistic particle in 1-dimensional infinite square well with a general dispersion relation, $F(k) \propto k^m$, is given by

$$\frac{1}{\kappa^{2(m-1)}}\frac{d^{2m}\psi}{dx^{2m}} + E\psi = 0, \tag{5}$$

Eq. (5) can be solved for all m with appropriate boundary conditions at $x = 0$ and $x = L$. Let us consider that this system obeys the linear dispersion relation ($F(k) = k$) at low-energies and non-linear dispersion relation ($F(k) = k^m$) at high energies. Then Eq. (5) implies that the ratio of ground state energy eigenvalues in these two regimes is:

$$\frac{\text{linear dispersion ground state energy}}{\text{non-linear dispersion ground state energy}} = \left[\frac{L\kappa}{\pi}\right]^{2(m-1)} \tag{6}$$

Eq. (6) implies that, for $\kappa < \pi/L$, the ground state energy eigenvalue of the system satisfying linear dispersion relation is lower compared to that of non-linear dispersion relations. As in the field theory model, where there is a cross-over from the linear to non-linear regime, with increasing P (the condition $\kappa < \pi/L$ implies that the cross-over occurs for $P > P_c$), the system needs to readjust in such a way that the ground state energy of the system increases. In other words, the cross-over of the dispersion relation catalyses *larger population* of higher energy quantum states compared to low-energy states.

We also argue that the change in the ground state is related to a *quantum phase transition* based on the following fact. The two point correlation (Wightman) Function $G^+(x^\mu, y^\mu)$ for $m = 1$ and $m = 2$, respectively, are $\frac{1}{4\pi^2 r^2}$ and $\frac{\kappa}{8\pi r}$ where $r = |\vec{x} - \vec{y}|$. Note that in linear dispersion regime the interaction is *localised* which explains the area-law behavior of the entropy. When

the second order correction dominates, correlation *decays* slowly (w.r.t. the linear dispersion scenario) with increasing distance which is approximately same as having nearest neighbour interaction. This implies that including the 3rd order derivative term will further increase the correlation length.

`Discussions:` We have shown that linear, higher spatial derivative theories give rise to a new kind of quantum phase transition. The new phase, with inverse scaling, appears due to excitation of higher modes which can be understood by considering a toy model of particle in a box where a cross-over beween linear and non-linear dispersion happens.

As the size of a black-hole decreases, due to Hawking radiation, the curvature of the event-horizon increases and hence one needs to include higher curvature terms [14, 16, 17, 18]. Our analysis shows that as the size of the black-hole decreases, below a critical radius, the scaling of entropy changes from area to an inverse-scaling wrt size of the horizon.

Currently we are investigating how n_c/N varies with $N \to \infty$ (the thermodynamic limit). We hope to report on more detailed understanding of this phenomenon elsewhere [15].

Acknowledgments

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