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COMPLETE SOLUTION OF THE CHIRAL SCHWINGER MODEL HILBERT SPACE, CONSTRAINTS AND WESS-ZUMINO TERMS*

DANIEL BOYANOVSKY

Physics Department
University of California, Davis, CA 95616

and

Stanford Linear Accelerator Center
Stanford University, Stanford, California, 94305

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Abstract

The chiral Schwinger Model is completely solved by bosonization. The Hilbert space is constructed and it is found to be of indefinite metric. The quantum constraints that define the physical subspace are determined and the physical operators (those that commute with the constraints) are found. We compute their correlation functions and find that there is non-trivial fermion wave function renormalization constant (Z_2) and vertex renormalization constant (Z_1^{-1}) and that $Z_2 = Z_1$ although the theory has lost its gauge invariance because of the chiral anomaly.

The addition of a Wess-Zumino (W-Z) term is studied and the modifications of the constraints introduced by this term is analyzed. The physical gauge invariant correlation functions in the theory with the W-Z term are found to be the same as the physical correlation functions of the theory without the W-Z terms.

Section 1:

Introduction and the Questions:

Chiral anomalies play a fundamental role in the physics of gauge theories, and their cancellation severely restricts the fermionic content of such theories. The fermions have to be in anomaly-free representations of the gauge group and this determines the family structure.[1]

Mechanisms for anomaly cancellation that do not involve a prescribed assignment of fermions are not readily available and this leaves few roads of action.

In a remarkable series of papers, D'Hoker and Farhi[2] found that if in an anomaly free theory there is a fermionic sector that is very heavy, the attempt of decoupling this heavy sector leaves behind a Wess-Zumino (W-Z) term. This W-Z term[3] restores gauge invariance to the theory, since the light fermions are no longer in an anomaly free representation.

In a parallel development Faddeev and Satashvili[4] proposed that in a theory where gauge invariance is lost by the anomalies, it can be restored by adding ad hoc a W-Z term to the action.

More recently Jackiw and Rajaraman[5] (J.R.) studied the Chiral Schwinger model in $1 + 1$ dimensions, in this model only right <u>or</u> left handed fermions couple to the gauge fields. These authors found that despite the fact that gauge invariance is lost because of the anomaly, the theory seems to be unitary. This result sparked interest and some authors studied the problem further.[6–8] In ref. (7) the model of right and left handed fermions coupled to the gauge fields with different couplings was quantized. It was found in this reference that quantizing in non-covariant gauges, the spectrum is not relativistic, signaling the lack of gauge invariance. However these authors noticed that adding a W-Z term to the action, gauge invariance is restored and the spectrum is relativistic. These results agree with those of ref. (8), where the current commutators were evaluated.

More recently a number of authors realized that the W-Z term does not need to be introduced by hand but that it is already contained in the path integral hidden in the integration over the gauge orbits.[9] These authors pointed out that the W-Z term arises after using the Faddeev-Popov procedure for integrating over the gauge orbits and by the non-invariance of the fermionic measure under gauge transformations in a chiral theory. Therefore the results of these authors seem to indicate that the theory is gauge invariant and unitary when integration over the gauge orbits is taken into account. The collection of the results mentioned above raises a series of questions that we want to address in this paper.

First of all on the theory without the W-Z fields as was studied by (J.R.). Since the theory has lost gauge invariance one cannot fix a gauge to quantize it, since in so doing degrees of freedom may be left out. Proceeding covariantly, it is expected that the Hilbert space will be of indefinite metric as in the covariant quantization of Q.E.D. If this is the case, one must understand the nature of the constraints at the quantum level that determine the physical subspace and the physical operators.

In a recent paper Girotti et. al.[10] looked at the structure of the fermionic correlation functions as well as the constraints of the theory in Dirac's formalism. These authors found that there is nontrivial fermion wave function renormalization, and this raises questions about the renormalizability aspects of the theory, for example vertex renormalization and Ward identities. This aspect is non-trivial since now there is an explicit mass term for the photon,[5] and the theory is no longer superrenormalizable (see section 4).

With regard to the Wess-Zumino term: Does the W-Z term change the nature of the constraints of the theory? What are the physical correlation functions? The W-Z field seems to be a new dynamical field, however in the approach of references (9), it appears after using a procedure that cannot introduce new physics. Therefore is the W-Z field physical? The paper is organized as follows: in section 2 we bosonize the fermions and derive the bosonized Lagrangian. In section 3 the Hilbert space is constructed and the quantum constraints that define the physical subspace are defined.

In section 4 we study the fermionic correlation functions and in particular wave function and vertex renormalizations. The equality $Z_2 = Z_1$ is established exactly. In section 5 we analyze the physical operators of the theory i.e. those that commute with the constraints.

Section 6 is devoted to studying the theory with the W-Z term, in particular the constraints and the physical gauge invariant operators. Their correlation functions are computed.

Finally we summarize the results and conclusions.

Section 2: Bosonization

We will use the technique of bosonization in order to solve completely the chiral Schwinger model. This approach has been used several times and in particular (JR) found the expression for the bosonized Lagrangian. However the explicit expression for the fermion fields in terms of the corresponding bosonic fields has been only partially given in ref (10). One of the aims of the present paper is to compute fermion correlation functions and for these we need to construct the fermion fields in terms of the bosonic fields appearing in the bosonized Lagrangian. For gauge theories the bosonization is slightly more subtle because the usual bosonization rules involve the canonical momentum conjugate to the boson field, but the interaction is given by a derivative coupling and therefore the canonical momentum does not coincide with the time derivative of the field.

Therefore we will carry out the bosonization procedure in the interaction picture[11] in which all the operators are written in terms of free fields. Once the interaction piece of the Hamiltonian is known in the interaction picture the unitary transformation to the Heisenberg fields is performed by Dyson's time evolution operator. Then the next task is to bosonize free fermions. The prescription for this procedure has been given in the literature and we refer the reader to references (11-14) for a detailed presentation. Here we review the technicalities that are relevant for our purpose.

Free massless fermions obey the Dirac equation

$$i\partial\!\!\!/\psi = 0 \tag{2.1}$$

We use a representation in which

$$\gamma^0\gamma^1 = \gamma^5 = \sigma_3 \tag{2.2}$$

In this (chiral) representation the spinors solutions are of the form

$$\psi(x,t) = \begin{pmatrix} \psi_R(x-t) \\ \psi_L(x+t) \end{pmatrix} \tag{2.3}$$

In $1+1$ dimensions a massless free bosonic field (solution to the massless Klein-Gordon equation) is written as

$$\phi(x,t) = \phi_R(x-t) + \phi_L(x+t) \tag{2.4}$$

From ϕ_R and ϕ_L we construct the dual field

$$\tilde{\phi}(x,t) = \phi_R(x-t) - \phi_L(x+t) \tag{2.5}$$

$\tilde{\phi}$ satisfies

$$\frac{\partial\tilde{\phi}}{\partial x} = -\frac{\partial\phi}{\partial t} \tag{2.6}$$

For free fields the relation (2.6) can be written as

$$\frac{\partial\tilde{\phi}}{\partial x} = -\pi \tag{2.7}$$

with $\pi(x,t)$ being the canonical momentum conjugate to $\phi(x,t)$. In terms of the fields

$$\begin{aligned} \phi_R(x-t) &= \frac{1}{2}(\phi+\tilde{\phi}) \\ \phi_L(x+t) &= \frac{1}{2}(\phi-\tilde{\phi}) \end{aligned} \tag{2.8}$$

the fermion fields are written as[12,14]

$$\begin{aligned} \psi_R(x,t) &= \frac{1}{\sqrt{L}} : \; e^{i\sqrt{4\pi}\phi_R(x,t)} : \\ \psi_L(x,t) &= \frac{1}{\sqrt{L}} : \; e^{-i\sqrt{4\pi}\phi_L(x,t)} : \end{aligned} \tag{2.9}$$

The quantization of the field theory is carried out in a one-dimensional "box" of length L, the factor $1/\sqrt{L}$ in (2.9) restores the proper dimensions to ψ. The dots in equation (2.9) represent normal ordering with respect to the quanta of the bose fields. We recall the relationship[13]

$$e^{i\sqrt{4\pi}\phi(x,t)} =: e^{i\sqrt{4\pi}\phi(x,t)} : \; e^{+2\pi[\phi^{(-)}(x,t),\phi^{(+)}(x,t)]} \tag{2.10}$$

where $\phi^{(+)}(\phi^{(-)})$ is the positive (negative) frequency part of ϕ.

Strictly speaking since the right and left going waves are independent $\phi_R(x,t)$ and $\phi_L(x,t)$ commute, therefore ψ_R and ψ_L in (2.9) commute rather than anti-commute. The anticommutation of ψ_R and ψ_L is arranged for by introducing a Klein factor,[12,15] for example by multiplying ψ_R by $e^{i\pi Q}$ where Q is the fermion number operator.[14] This is irrelevant for our purposes in the next few sections. The Klein factor will be omitted for the moment and it will be recovered in Section (4), see the discussion after equation (4.5). The free fermion number current is defined as

$$J^{\mu}(x) =: \overline{\psi}(x)\gamma^{\mu}\psi(x) : \tag{2.11}$$

where in equation (2.11) the normal ordering is understood with respect to the bose quanta in (2.9). As usual the current $J^{\mu}(x)$ is computed as

$$J^{\mu}(x) = \lim_{\epsilon\to 0} \overline{\psi}(x+\epsilon)\gamma^{\mu}\psi(x) - < 0|\overline{\psi}(x+\epsilon)\gamma^{\mu}\psi(x)|0 > \tag{2.12}$$

In terms of the boson fields the definition (2.11) amounts to carrying out the operator product expansion (OPE) of the operators given in eq. (2.9) and subtracting the singular c-number piece proportional to $1/\epsilon$.

The result for the free current is

$$J^{\mu}(x) = \frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_{\nu}\phi(x) \tag{2.13}$$

When the fermions are coupled to gauge fields there is an ambiguity in defining the current. For a gauge invariant result the path ordered exponential of the gauge field needs to be inserted in (2.12) since this expression involves fields at different points. The exponential of the line integral of the gauge fields modifies the result (2.13) because of the singularities $1/\epsilon$ in the OPE.

Since we expect that gauge invariance is lost in the Chiral Schwinger model, the definition of the current is ambiguous since there is no principle of gauge invariance that selects a regularization prescription for the expressions (2.12) and (2.13). By Lorentz covariance this ambiguity can only be of a form of a mass term for the gauge fields.

The Lagrangian density for the Chiral Schwinger Model is

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + i\overline{\psi}\partial\!\!\!/\psi + g\overline{\psi}\gamma^{\mu}(1-\gamma_5)\psi A_{\mu} \tag{2.14}$$

Using the fact that $\gamma^{\mu}\gamma^{5} = -\epsilon^{\mu\nu}\gamma_{\nu}$ ($\epsilon^{01} = +1$) in $1+1$ dimensions, the bosonized Heisenberg Lagrangian is

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2}(\partial_{\mu}\phi)^2 + \frac{g}{\sqrt{\pi}}A^{\mu}(\partial_{\mu}\phi + \epsilon_{\mu\nu}\partial^{\nu}\phi) + \frac{1}{2}\frac{g^2}{\pi}aA_{\mu}A^{\mu} \tag{2.15}$$

This is the bosonized Lagrangian proposed by Jackiw and Rajaraman. The last term (explicit mass term for the gauge field) represents the ambiguity in the regularization of the currents in the bosonization procedure and it is parametrized by the parameter a.

Alternatively we can think of this term as arising from the ambiguity (of the form $g^{\mu\nu}$) in the current-current correlation function (vacuum polarization tensor).[5,8,16]

Now that we know the interaction Hamiltonian we can pass to the Heisenberg picture. The Heisenberg picture fermion field operators are written as in equation (2.9) in terms of the Heisenberg bose fields.

In particular for example, in the Heisenberg picture

$$\pi_{\phi} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi} + \frac{g}{\sqrt{\pi}}(A_o - A_x) \tag{2.16}$$

Although the reader may consider all the above steps confusing, they are necessary to obtain the fermion fields and eventually to construct their correlation functions.

Section 3: The Hilbert Space and the Constraints.

The equations of motion obtained from the Lagrangian in equation (2.15) are

$$\Box\phi + \frac{g}{\sqrt{\pi}}(\partial_\mu A^\mu - \epsilon_{\nu\mu}\partial^\nu A^\mu) = 0 \qquad (3.1-a)$$

$$\partial_\mu F^{\mu\nu} = J^\nu = -\frac{g}{\sqrt{\pi}}(g^{\nu\mu} + \epsilon^{\nu\mu})\partial_\mu\phi - a\frac{g^2}{\pi}A^\nu \qquad (3.1-b)$$

Since we expect gauge invariance to be lost, the "longitudinal" part of the gauge field may acquire dynamics. To understand this we exploit the fact that in $1+1$ dimensions we can write

$$\begin{aligned} A^\mu(x) &= \partial^\mu\lambda(x) + \epsilon^{\mu\nu}\partial_\nu\chi(x) \\ \partial_\mu A^\mu &= \Box\lambda; \quad F^{\mu\nu} = -\epsilon^{\mu\nu}\Box\chi \end{aligned} \qquad (3.2)$$

Under a gauge transformation

$$\begin{aligned} A_\mu(x) &\to A_\mu(x) + \partial_\mu\alpha(x) \\ \lambda(x) &\to \lambda(x) + \alpha(x) \\ \chi(x) &\to \chi(x) \end{aligned} \qquad (3.3)$$

In terms of λ and χ the Lagrangian (2.15) reads

$$\begin{aligned} \mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}\Box\chi\Box\chi - \frac{g}{\sqrt{\pi}}\partial_\mu\phi\partial^\mu(\chi-\lambda) + \frac{a}{2}\frac{g^2}{\pi}(\partial_\mu\lambda)^2 \\ - \frac{a}{2}\frac{g^2}{\pi}(\partial_\mu\chi)^2 \end{aligned} \qquad (3.4)$$

To obtain the above Lagrangian several surface terms have been dropped. The Lagrangian density in equation (3.4) can be cast in terms of free fields by diagonalizing the mixing terms by the following canonical transformations

$$\phi = \phi' + \frac{g}{\sqrt{\pi}}(\chi - \lambda) \qquad (3.5)$$

$$\lambda = \lambda' - \frac{1}{a-1}\chi \tag{3.6}$$

$$\chi_1 = \frac{\sqrt{\pi(a-1)}}{ag}\Box\chi \tag{3.7}$$

$$\chi_2 = \frac{\sqrt{\pi(a-1)}}{ag}\left(\Box + \frac{g^2a^2}{\pi(a-1)}\right)\chi \tag{3.8}$$

The Lagrangian density written in terms of φ', λ', χ_1 and χ_2 reads (after dropping surface terms)

$$\mathcal{L} = -\frac{1}{2}\phi'\Box\phi' - \frac{1}{2}\chi_1(\Box + m^2)\chi_1 + \frac{1}{2}\chi_2\Box\chi_2 - \frac{1}{2}\frac{g^2}{\pi}(a-1)\lambda'\Box\lambda' \tag{3.9}$$

with

$$m^2 = \frac{g^2a^2}{\pi(a-1)} \tag{3.10}$$

The transformations (3.7) and (3.8) are typical of a higher derivative theory.[11,17]

The Lagrangian in equation (3.9) has many interesting features. First notice that λ' is the field that changes under gauge transformations, hence only for $a = 1$ the theory seems to be gauge invariant, but at this particular value of a the mass given by (3.10) diverges. For $a < 1$ the theory has tachyonic excitations and λ' has to be quantized with negative metric. Even for $a > 1$ notice that the field χ_2 has to be quantized with negative metric and therefore the Hilbert space will be of indefinite metric. The field χ_2 is also present in the covariant operator solution of the Schwinger model as given by Lowenstein and Swieca[18] and also by Halpern[11] and Kogut and Susskind.[19] In these references it is clearly shown that the Hilbert space of the (vector-like) Schwinger model quantized covariantly is of indefinite metric.

The last term in equation (3.9) is further simplified by the transformation

$$\frac{g}{\sqrt{\pi}}\sqrt{a-1}\,\lambda' = \eta \tag{3.11}$$

where now η is a free massless canonical field quantized with positive metric.

The content of the original theory is now completely determined by three massless free bose fields: ϕ' which is a gauge singlet quantized with positive metric, χ_2, a gauge singlet quantized with negative metric and η that transforms under gauge transformations and is quantized with positive metric, and finally χ_1 a free massive bose field quantized with positive metric.

The Hilbert space of the theory is of indefinite metric and is Fock, it is given by a tensor product of the Fock spaces for ϕ', χ_1, χ_2 and η

$$\mathcal{H} = \mathcal{H}_{\phi'} \otimes \mathcal{H}_{\chi_1} \otimes \mathcal{H}_{\chi_2} \otimes \mathcal{H}_{\eta} \tag{3.12}$$

However as is the case in the covariant solution of the Schwinger model we expect the Hilbert space $\mathcal{H}$ given by (3.12) to be too large.[18] The equation of motion (3.1-b) written in terms of the free fields reads

$$\partial_\mu F^{\mu\nu} = J^\nu \tag{3.1 - b}$$

$$\partial_\mu F^{\mu\nu} = -\frac{\sqrt{\pi(a-1)}}{ga}\epsilon^{\mu\nu}\partial_\mu\Box(\chi_2 - \chi_1) \tag{3.13}$$

$$\begin{aligned} J^\nu =& -\frac{g}{\sqrt{\pi}}(\partial^\nu\phi' + \epsilon^{\nu\mu}\partial_\mu\phi') + \frac{g}{\sqrt{\pi}}\frac{1}{\sqrt{a-1}}\left((1-a)\partial^\nu\eta + \epsilon^{\nu\mu}\partial_\mu\eta\right) \\ &- \frac{ga}{\sqrt{\pi}\sqrt{a-1}}\epsilon^{\nu\mu}\partial_\mu(\chi_2 - \chi_1) \end{aligned} \tag{3.14}$$

Notice that the current J^ν is conserved by the equations of motion (ϕ' and η are massless fields).

However combining (3.1-b) with (3.13) and (3.14) we find

$$\frac{1}{m}\partial_\mu\epsilon^{\mu\nu}(\Box\chi_1 + m^2\chi_1) = J_F^\nu \tag{3.15}$$

$$\begin{aligned} J_F^\nu =& -\frac{g}{\sqrt{\pi}}(\partial^\nu\phi' + \epsilon^{\nu\mu}\partial_\mu\phi') + \frac{g}{\sqrt{\pi}}\frac{1}{\sqrt{a-1}}\left((1-a)\partial^\nu\eta + \epsilon^{\nu\mu}\partial_\mu\eta\right) \\ &- \frac{g}{\sqrt{\pi}}\frac{a}{\sqrt{a-1}}\epsilon^{\nu\mu}\partial_\mu\chi_2 \end{aligned} \tag{3.16}$$

Therefore the equations of motion that define the physical theory are not satisfied, much in the same way as in the Schwinger model, as shown by Lowenstein and Swieca.[18]

We would be tempted to set $J_F^\nu = 0$ to define the physical subspace, however this would rule out the existence of a vacuum state of the theory, and furthermore it does not commute with other operators of the theory (ϕ', η, χ_2). Therefore since J_F^ν is a linear combination of derivatives of free fields, the corresponding constraint is of the Gupta-Bleuler type

$$J_F^{\nu(+)}|P>\equiv 0 \tag{3.17}$$

Where $|P>$ is any physical state and $J_F^{\nu(+)}$ is the positive frequency (annihilation) part of J_F^ν. The quantum condition (3.17) defines the physical subspace $\mathcal{H}_{\text{phys}}$ of the total Hilbert space given by equation (3.12).

The equation of motion (3.1-a) is satisfied by the canonical transformation (3.5).

The physical operators are those that commute with $J_F^{\nu(+)}$ in (3.17), they create physical excitations out of the vacuum. These operators will be studied in section (5).

The current J_F^ν given by equation (3.16) creates zero norm states out of the vacuum since

$$<0 \mid J_F^\nu(x) J_F^\mu(y) \mid 0>=0 \qquad \forall x, y \tag{3.18}$$

Again this fact should be compared with the covariant solution of the Schwinger model.[18]

Section 4: The Fermion Fields: Wave Function and Vertex Renormalization.

In order to compute correlation functions of Fermi fields, these have to be written in terms of the bosonic free fields ϕ', χ_1, χ_2 and η in the Heisenberg picture. The canonical momentum conjugate to ϕ is given by equation (2.16) (see eq. 2.15)

$$\pi_\phi = \dot{\phi} + \frac{g}{\sqrt{\pi}}(A_o - A_x) \tag{2.16}$$

In terms of the "longitudinal" and "transverse" components of A_μ given by equation (3.2) and the transformation given by (3.5) we find

$$\pi_\phi = \dot{\phi}' + \frac{g}{\sqrt{\pi}}\partial_x(\chi - \lambda) \tag{4.1}$$

All the fields are in the Heisenberg picture. Since ϕ' is a free massless field it can be written as

$$\phi' = \phi'_R(x - t) + \phi'_L(x - t)$$

therefore as it was discussed in section 2 for free fields, we can define the dual field $\tilde{\phi}'$ by

$$\frac{\partial \tilde{\phi}'}{\partial x} = -\dot{\phi}'$$

The canonical momenta in equation (4.1) is thus written as

$$\pi_\phi = -\frac{\partial \tilde{\phi}'}{\partial x} + \frac{g}{\sqrt{\pi}}\frac{\partial}{\partial x}(\chi - \lambda) \tag{4.2}$$

But $-\pi_\phi$ is the space derivative of the dual field $\tilde{\phi}$ in the Heisenberg picture, (see eq. 2.7) hence

$$-\frac{\partial \tilde{\phi}}{\partial x} = -\frac{\partial \tilde{\phi}'}{\partial x} + \frac{g}{\sqrt{\pi}}\frac{\partial}{\partial x}(\chi - \lambda) \tag{4.3}$$

Therefore in the Heisenberg picture, and using (3.5)

$$\phi_R = \frac{1}{2}(\phi + \tilde{\phi}) = \phi'_R(x - t) \tag{4.4}$$

$$\phi_L = \frac{1}{2}(\phi - \tilde{\phi}) = \phi'_L(x + t) + \frac{g}{\sqrt{\pi}}(\chi - \lambda) \tag{4.5}$$

Therefore equations (2.9) and (4.4) imply that the right handed component of the Fermi field is free, this is of course a consequence of the fact that only the lft handed component is coupled to the gauge field in the Chiral Schwinger model.

Therefore we write the Heisenberg fermion fields as

$$\psi_R = \frac{C}{\sqrt{L}}\, e^{i\sqrt{4\pi}\phi'_R}$$
$$\psi_L = \frac{C}{\sqrt{L}}\, e^{-i\sqrt{4\pi}\phi'_L}\, e^{-i2\beta}$$
$$\beta = g(\chi - \lambda)$$

The constant C is related to normal ordering, see equation (2.10). The above expressions can be written in a compact form as

$$\psi = e^{-i\beta P_-}\psi_o$$

with $P_- = (1 - \gamma_5)$ and ψ_o a <u>free</u> massless spinor by equation (2.9), and (2.10).

At this stage we can recover the Klein factor. It is the same that ensures the anticommutation of ψ_R, ψ_L in equation (2.9), since ψ_o is a <u>free</u> massless spinor, and anticommutation of ψ_o will ensure anticommutation of ψ since $(\chi - \lambda)$ does not involve time derivatives.

In terms of the canonical free fields ϕ'_L, η, χ_1 and χ_2, the component ϕ_L of the Heisenberg field ϕ reads

$$\phi_L = \phi'_L - \frac{1}{\sqrt{a-1}}\eta + \frac{1}{\sqrt{a-1}}(\chi_2 - \chi_1) \tag{4.6}$$

However the normal ordering definition in equation (2.9) is with respect to the quanta of the field ϕ in the <u>interaction picture</u>, i.e., ϕ is a free <u>massless</u> field in equation (2.9).

Using the relation given by equation (2.10) we can write in the Heisenberg picture

$$\psi_L = \frac{C}{\sqrt{L}} e^{-i\sqrt{4\pi}\phi_L(x,t)} \tag{4.7}$$

with ϕ_L given by equation (4.6) and

$$C = e^{-2\pi[\alpha_L^{(-)}(x,t),\alpha_L^{(+)}(x,t)]} \tag{4.8}$$

In equation (4.8) α_L is the left handed component of a free massless field (see equation 2.10). Therefore

$$\psi_L = \frac{C}{\sqrt{L}} e^{-i\sqrt{4\pi}\phi'_L} \; e^{-i\sqrt{\frac{4\pi}{a-1}}(\chi_2-\eta)} e^{i\sqrt{\frac{4\pi}{a-1}}\chi_1} \tag{4.9}$$

and since ϕ'_L is the left component of a free massless field and χ_2 and η are free massless fields quantized with opposite metric, we see by using (2.10), (4.7) and (4.8) that

$$\psi_L = \frac{1}{\sqrt{L}} : e^{-i\sqrt{4\pi}\phi'_L} \; e^{-i\sqrt{\frac{4\pi}{a-1}}(\chi_2-\eta)} : e^{i\sqrt{\frac{4\pi}{a-1}}\chi_1} \tag{4.10}$$

Writing

$$e^{i\frac{\sqrt{4\pi}}{a-1}\chi_1} =: e^{i\frac{\sqrt{4\pi}}{a-1}\chi_1} : e^{\frac{2\pi}{a-1}[\chi_1^{(-)},\chi_1^{(+)}]} \tag{4.11}$$

with

$$[\chi_1^{(-)}(x,t),\chi_1^{(+)}(x,t)] = -\int \frac{dk}{2\pi 2w_k} = -i\Delta_F(o,m) \tag{4.12}$$

We find[20]

$$\psi_L = \frac{1}{\sqrt{L}} : e^{-i\sqrt{4\pi}\phi_L} : e^{-\frac{1}{2}(\frac{4\pi i}{a-1})\Delta_F(o,m)} \tag{4.13}$$

with ϕ_L given by (4.6). The normal ordering in (4.13) is now with respect to the quanta of the free Heisenberg fields ϕ', η, χ_1 and χ_2.

Equation (4.13) suggests that we introduce a wave function renormalization constant (see footnote page 19 and reference 20)

$$Z_2 = e^{\frac{4\pi i}{a-1}\Delta_F(o,m)} = \left(\frac{\Lambda^2}{m^2}\right)^{\frac{1}{a-1}} ; \tag{4.14}$$

such that $Z_2^{\frac{1}{2}}\psi_L$ has finite matrix elements, in equation (4.14) Λ is an ultraviolet cut-off. In particular

$$Z_2 < 0 \mid T\psi_L^+(x)\psi_L(y) \mid 0 >= \frac{1}{L} < 0 \mid T : e^{i\sqrt{4\pi}\phi_L(x)} :: e^{-i\sqrt{4\pi}\phi_L(y)} :\mid 0 > \tag{4.15}$$

The right hand side of equation (4.15) can be computed easily by using Wick's theorem and by recalling the χ_2 and η are quantized with opposite metric and that the term

$$\frac{1}{\sqrt{L}} : e^{-i\sqrt{4\pi}\phi'_L} :$$

is a bosonized free left handed fermion. Therefore we find

$$Z_2 < 0 \mid T\psi_L^+(x)\psi_L(y) \mid 0 >= iS_{F,L}(x-y)e^{\frac{4\pi i}{a-1}\Delta_F(x-y,m)} \tag{4.16}$$

with $S_{F,L}$ the free propagator for left handed fermions and Δ_F the free propagator for massive bosons. This result agrees with that of reference 10.

The Vertex Renormalization Constant:

Following the procedure used for the wave function renormalization constant we can compute the vertex renormalization correction. For this purpose we notice that the current coupled to the gauge field only involves the operator

$$\rho_L(x) = \psi_L^+(x)\psi_L(x) \tag{4.17}$$

This operator has two sources of singularities. The first being the fact that the unrenormalized operator ψ_L has singular matrix elements by the second term on the right hand side of (4.17) (wave function renormalization). The second source of singularities is the fact that (4.17) is a product of operators at the same point. The first problem is handled by using the renormalized fields (or multiplying $\rho_L(x)$ by Z_2)

$$Z_2\rho_L(x) = \frac{1}{L} : e^{i\sqrt{4\pi}\phi_L(x)} :: e^{-i\sqrt{4\pi}\phi_L(x)} : \tag{4.18}$$

The second problem is handled by using Mandelstam's[13] approach and defining

$$\tilde{\rho}_L(x) = \lim_{x\to y} Z_2 \left\{ | \, cm(x-y) \, |^\sigma \, \psi_L^+(x)\psi_L(y) + F(x-y) \right\} \tag{4.19}$$

The function $F(x-y)$ and the constant σ are chosen such that the limit $x \to y$ is finite.

Using the formula

$$: e^A :: e^B := e^{[A^{(+)}, B^{(-)}]} : e^{A+B} :$$

and the fact that χ_2 and η have opposite metric we find[13,20]

$$Z_2\psi_L^+(x)\psi_L(y) \underset{x\to y}{=} : e^{i\sqrt{4\pi}(\phi_L(x)-\phi_L(y))} : \frac{(c^2m^2|x-y|^2)^{\frac{-1}{a-1}}}{2\pi i(x-y)} \tag{4.20}$$

with c a numerical constant (related to Euler's constant).

Therefore we see that if we choose $\sigma = 2/(a-1)$ and $F(x-y) = 1/2\pi i(x-y)$ then

$$\tilde{\rho}_L(x) = \frac{1}{\sqrt{\pi}} \frac{\partial \phi_L}{\partial x} \tag{4.21}$$

The term $|cm(x-y)|^\sigma$ in equation (4.19) plays the role of the vertex renormalization constant Z_1^{-1}. Introducing an ultraviolet cut-off in the limit $x \to y$ as

$|x - y| = (c\Lambda)^{-1}$, we find that*

$$Z_1^{-1} = \left(\frac{m^2}{\Lambda^2}\right)^{\frac{1}{a-1}} \tag{4.22}$$

or that

$$Z_1 = Z_2 \tag{4.23}$$

this is one of the main results of this paper. In vector-like electrodynamics the relation (4.23) is a consequence of the Ward identities, a result of gauge invariance. Despite the fact that there is fermion wave function and vertex renormalization, there is no coupling constant renormalization in this theory. This is understood from the fact that there is no wave function renormalization for the "photon", and from equation (4.23) (there is only "mass" renormalization parametrized by a). This can be seen to be the same reason by which the coupling constant in the Thirring model is not renormalized.

An obvious question that arises is why is there a wave function and vertex renormalization constant?

In fact this question can be answered in perturbation theory. Let us first consider the fermion self energy. The full photon propagator can be found by integrating out the field ϕ in the Lagrangian (2.15), it is found to be given by[5]

$$G_{\mu\nu}(k) = \frac{i}{k^2 - m^2}\left\{-g_{\mu\nu} + \frac{1}{a-1}\left[k_\mu k_\nu\left(\frac{\pi}{g^2} - \frac{2}{k^2}\right)\right] - \epsilon_{\mu\alpha}\frac{k^\alpha k_\nu}{k^2} - \epsilon_{\nu\alpha}\frac{k^\alpha k_\mu}{k^2}\right\} \tag{4.24}$$

with m given by equation (3.10).

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* Instead of writing $|x - y| = (c\Lambda)^{-1}$ we can use Coleman's regularization (see equation 2.9 in reference 20) $\Delta(x, m) - \Delta(x, \Lambda) = \Delta(x, m, \Lambda)$ such that as $|x - y| \to 0$, $\Delta(o, m, \Lambda) = -\frac{1}{4\pi}\ln(\frac{m^2}{\Lambda^2})$. This yields the same result as given in equation (4.22).

The high momentum behavior of the above propagator is similar to the one in a Proca theory (massive vector field)

$$\lim_{|k|\to\infty} G_{\mu\nu}(k) \sim \frac{i\pi k_\mu k_\nu}{k^2 g^2(a-1)} \tag{4.25}$$

Now consider the second order fermion self energy correction with the full photon propagator (4.24)

$$\begin{gathered}\Sigma(p) \sim g^2 \int \frac{d^2k}{(2\pi)^4}\gamma_\mu P_- \frac{G_{\mu\nu}(k)}{\not k + \not p}\gamma_\nu P_- \\ P_- = (1-\gamma_5)\end{gathered} \tag{4.26}$$

Because of (4.25), $\Sigma(p)$ seems to be linearly divergent, however by chiral symmetry it has to be of the form

$$\Sigma(p) = \not p \epsilon(p) \tag{4.27}$$

and in fact $\epsilon(p)$ is logarithmically divergent. The divergence arises from the term in equation (4.25), notice that the g^2 in the denominator of equation 4.25, and in the numerator of equation 4.26 cancel. Therefore we find

$$Z_2 = 1 + \frac{A}{a-1}\ln\frac{\Lambda^2}{m^2} + \cdots \sim \left(\frac{\Lambda^2}{m^2}\right)^{\frac{A}{a-1}} \tag{4.28}$$

where A is a numerical constant arising from the Feynman integral. These arguments explain clearly the power $\left(\frac{1}{a-1}\right)$ and the absence of coupling constants in Z_2. A similar analysis can be carried out for the vertex function with $G_{\mu\nu}$ given above, and in fact it is very easy to see to this order that $Z_1 = Z_2$ by the usual analysis since $Z_2 = (\partial\Sigma(p)/\partial\not p)_{\not p=0}$ corresponds to the insertion of a zero momentum photon, therefore the logarithmic singularities are the same for the vertex and self-energy corrections.

Before closing this section we want to remark that the correlation function given by (4.16), Z_1 and Z_2 and the relation (4.23) are exact. In a gauge invariant theory the fermion correlation function can only be computed by specifying the choice of gauge, if no gauge is chosen the fermion correlation function must vanish by gauge invariance. Therefore equation (4.16) reflects once again the lack of gauge invariance in the theory.

Section 5: The Physical Operators:

We have seen in section 3 that the full Hilbert space of the theory is of indefinite metric, but there exists a quantum constraint that restricts the full Hilbert space to a physical subspace in which the equations of motion (3.1a,b) are satisfied.

The constraint is given by equations (3.17) with J_F^ν given by (3.16).

A physical operator $\hat{P}(y)$ is defined such that

$$\left[J_F^{\nu(+)}(x),\ \hat{P}(y)\right] = 0 \quad \forall\ x, y \quad \nu = 0, 1 \tag{5.1}$$

Condition (5.1) ensures that the states obtained by applying $\hat{P}$ onto the vacuum are annihilated by $J_F^{\nu(+)}$, therefore satisfying the constraint (3.17).

For a massless field $\phi(x,t) = \phi_R(x-t) + \phi_L(x+t)$

$$\partial^\nu \phi + \epsilon^{\nu\mu}\partial_\mu \phi = 2\epsilon^{\nu\mu}\partial_\mu \phi_L(x+t) \tag{5.2}$$

Hence J_ν^F can be written as

$$\frac{\sqrt{\pi}}{g} J_F^\nu = -2\epsilon^{\nu\mu}\partial_\mu \phi'_L + 2\frac{(1-a)}{\sqrt{1-a}}\epsilon^{\nu\mu}\partial_\mu \eta_L + \frac{a}{\sqrt{a-1}}\epsilon^{\nu\mu}\partial_\mu(\eta - \chi_2) \tag{5.3}$$

Since J_F^ν does not depend on ϕ'_R and χ_1 and

$$[\phi'_R(x),\ \phi'_L(y)] = 0 \quad \forall\ x, y \tag{5.4}$$

then

$$[J_F^{\nu(+)}(x),\ \phi'_R(y)] = [J_F^{\nu(+)}(x),\ \chi_1(y)] = 0 \quad \forall\ x, y \tag{5.5}$$

Therefore ϕ'_R and χ_1 are physical operators. Consequently ψ_R and $F_{\mu\nu}$ (see equations (3.2) and (3.7)) are physical operators.

It is a matter of straightforward algebra to show that because η_L and η_R commute and because χ_2 and η are quantized with opposite metrics then (ϕ_L is defined in equation 4.6)

$$[J_F^{\nu(+)}(x),\ \phi_L(y)] = 0 \quad \forall\ x, y \tag{5.4}$$

Consequently ψ_L is a physical operator. This should again be contrasted to the case of the gauge-invariant Schwinger model quantized in a covariant gauge where the fermion field is not a physical operator.[18]

Physical states are obtained by applying polynomials of ϕ'_R, $F_{\mu\nu}$, χ_1 and ϕ_L onto the vacuum of the full Hilbert space $\mathcal{H}$ given by (3.12).

However neither ϕ'_L, η or χ_2 create physical excitations. We see that since η and χ_2 appear in the combination $\eta - \chi_2$ in ϕ_L, the physical Hilbert space is of positive metric. Therefore in S-matrix elements of physical operators η and χ_2 only enter as intermediate states in the combination $\eta - \chi_2$, cancelling each other because of the opposite metrics.[19]

Section 6: The Physics of the Wess-Zumino Term

As was mentioned in the introduction a different mechanism that is proposed to cancel the anomaly and to render the theory gauge invariant is the addition of a Wess-Zumino (W-Z) term to the action. In references (7-9) it was shown that incorporating this Wess Zumino (W-Z) term in fact restores the gauge symmetry in the Chiral Schwinger model. More recently several authors[9] suggested the possibility that in fact the W-Z term does not have to be put in the theory by hand, but that it naturally emerges from the path-integration over all gauge orbits. In particular the W-Z (pseudo) scalar field arises after using the Fadeev-Popov trick to take into account the integration over the gauge orbits.

To the present author this result has to be taken with care for at least two reasons. The first is that since the theory has quantum constraints, it is not clear whether integration over the full gauge orbits respects the constraints. The second reason is perhaps more disturbing. The Fadeev-Popov trick is a way of writing 1 in an adequate way, but it should not introduce new physics in the theory. However there is a new field, the W-Z field, that seems to have acquired the status of a physical field, therefore it is legitimate to ask is the W-Z field physical? Furthermore, we have learned that the full Hilbert space of the theory is of indefinite metric, how does the W-Z field modify the constraints? And last but not least which are the physical correlation functions in the theory with the W-Z terms?

To answer these questions we adopt the attitude of just adding the W-Z form to the action (2.15) ad hoc to study its consequences. The W-Z term for the Chiral Schwinger model was given several times in the literature,[8,9] it is given by

$$\mathcal{L}_{W-Z} = \frac{1}{2}(a-1)(\partial_\mu \theta)^2 - \frac{g}{\sqrt{\pi}}\theta[(a-1)\partial_\mu A^\mu + \epsilon^{\mu\nu}\partial_\mu A_\nu] \tag{6.1}$$

and the total Lagrangian density is

$$\mathcal{L} = \mathcal{L}_o + \mathcal{L}_{W-Z}$$

with $\mathcal{L}_o$ given by equation (2.15). Notice that $\mathcal{L}$ is explicitly invariant under the transformation

$$\begin{aligned} A_\mu &\to A_\mu + \partial_\mu \alpha(x) \\ \theta &\to \theta - \frac{g}{\sqrt{\pi}}\alpha(x) \end{aligned} \tag{6.2}$$

therefore $\mathcal{L}_{W-Z}$ has restored the gauge invariance of the theory. The equation of motion for θ is

$$(a-1)\Box\theta = -\frac{g}{\sqrt{\pi}}[(a-1)\partial_\mu A^\mu + \epsilon^{\mu\nu}\partial_\mu A_\nu] \tag{6.3}$$

Using equations (3.2), and (3.6) we see that there is a canonical transformation that diagonalizes the mixing term in (6.1). Defining

$$\theta = \frac{\varsigma}{\sqrt{a-1}} - \frac{g}{\sqrt{\pi}}\lambda' \tag{6.4}$$

with ς a free massless field. In terms of ς, ϕ', χ_1, and χ_2

$$\mathcal{L} = -\frac{1}{2}\phi'\Box\phi' - \frac{1}{2}\chi_1(\Box + m^2)\chi_1 + \frac{1}{2}\chi_2\Box\chi_2 - \frac{1}{2}\varsigma\Box\varsigma \tag{6.5}$$

Comparing (6.5) with (3.9) we see that the field ς replaces the field η (or λ' by (3.11)). However by the transformation laws (6.2) and by (3.3) and (3.6) we see that the field ς is a <u>singlet</u> under gauge transformations as are ϕ', χ_1 and χ_2. So that the field λ' (or η) which is the <u>only</u> field that transforms under gauge transformation does not appear in $\mathcal{L}$ in (6.5). Of course this is a consequence of the gauge invariance of $\mathcal{L}$. The contribution of the W-Z field to the current in (3.1b) is given by

$$J^\nu_\theta = -\frac{g}{\sqrt{\pi}}[(a-1)\partial^\nu\theta - \epsilon^{\nu\mu}\partial_\mu\theta] \tag{6.6}$$

And since (6.6) does not depend on χ_1, this term contributes to J^ν_F in equation (3.16) by just adding J^ν_θ to J^ν_F, therefore the right hand side of equation (3.15) is now given by

$$\underset{\sim}{J}^\nu_F = J^\nu_F + J^\nu_\theta \tag{6.7}$$

with J^ν_F given by equation (3.16) and J^ν_θ by (6.6). In terms of the fields in (6.5) we find

$$J_F^\nu = -\frac{g}{\sqrt{\pi}}(\partial^\nu \phi' + \epsilon^{\nu\mu}\partial_\mu \phi') + \frac{g}{\sqrt{\pi}}\frac{1}{\sqrt{a-1}}[(1-a)\partial^\nu \varsigma + \epsilon^{\nu\mu}\partial_\mu \varsigma]$$
$$-\frac{g}{\sqrt{\pi}}\frac{a}{\sqrt{a-1}}\epsilon^{\nu\mu}\partial_\mu \chi_2 \tag{6.8}$$

Notice again that the only change that the W-Z term introduces is the replacement of the field η that transforms under gauge transformations by the field ς which is a gauge singlet. Now J_F^ν is gauge invariant.

Now for the theory with the Wess-Zumino field given by the Lagrangian density $\mathcal{L}_o + \mathcal{L}_{W-Z}$ the quantum constraint that defines the physical subspace of the full Hilbert space is analogous to (3.17) with ${J_F}^\nu$ replacing J_F^ν

$$J_F^{\nu(+)}|P>= 0 \tag{6.9}$$

The physical operators are χ_1, ϕ'_R, $F_{\mu\nu}$ and

$$\phi_L = \phi'_L - \frac{\varsigma}{\sqrt{a-1}} + \frac{1}{\sqrt{a-1}}(\chi_2 - \chi_1) \tag{6.10}$$

Since the field ς is a gauge singlet, now the physical operators are gauge invariant. However notice that ς is not a physical operator, in the same way as η (λ') was not a physical operator in the theory without the Wess-Zumino term.

Therefore we conclude that the Wess-Zumino field θ in $\mathcal{L}_{W-Z}$ (eq. 6.1) is not associated to a physical operator in the quantum theory. And as in the theory without $\mathcal{L}_{W-Z}$ the physical Hilbert space contains only positive metric states, since ς and χ_2 are quantized with opposite metrics.

Since the theory is now gauge invariant, the only meaningful correlation functions are those of gauge invariant objects. Let us define the following gauge invariant quantity

$$\psi_L = \frac{1}{Z_2\sqrt{L}} : e^{i\sqrt{4\pi}\theta} e^{-i\sqrt{4\pi}\phi_L} : \tag{6.11}$$

where θ is the W-Z field and ϕ_L is given by equation (4.6). Normal ordering is understood with respect to the respective quanta. By using (6.4) and (3.11) the expression (6.11) can be written as

$$\psi_L = \frac{1}{Z_2\sqrt{L}} : e^{-i\sqrt{4\pi}\phi_L} : \tag{6.12}$$

with ϕ_L given by (6.10). It is now evident that ψ_L is gauge invariant, and that since ς and χ_2 have opposite metric the wave function renormalization Z_2 in (6.11-6.12) is given by (4.14). It is now clear that the correlation function

$$Z_2 < 0|T\psi_L^{+}(x)\psi_L(y)|0 > \tag{6.13}$$

is exactly given by the right hand side of equation (4.16). This is obvious since the only change that the Wess-Zumino term has introduced is the replacement of the free massless field η by the gauge singlet free massless field ς everywhere, and η and ς are quantized with positive metric.

Therefore we find that in the theory with the W-Z term, the physical operators are gauge invariant and that the correlation functions of these gauge invariant physical operators are exactly the same as the correlation functions of the physical operators in the theory without the W-Z term. Hence in this sense we can think of the theory without the W-Z term, as the theory with this term but quantized in the "unitary" gauge $\theta = 0$, as proposed in ref. (7).

Therefore we see that the W-Z term restores gauge invariance in the theory, but does not simplify the constraints that define the physical Hilbert space. And in particular the W-Z field θ that is a gauge singlet is not a physical quantum field.

Perhaps the only advantage of introducing the W-Z term is to restore gauge invariance, ensuring the Ward identity $Z_2 = Z_1$ in equation (4.23) (although it is not clear to the present author how the gauge invariance brought about by the W-Z field is responsible for this Ward identity).

However, although gauge invariance is restored, this does not relieve the theory from the severe constraints that determine the physical Hilbert space and the physical operators. The constraints are given by equations (6.8) and (6.9) (in the theory with the W-Z field), the physical Hilbert space is of positive metric, hence unitarity is obeyed in this subspace.

Summary of the Results, Conclusions and a Glimpse at Four Dimensions.

In this paper, the Chiral Schwinger model (chiral QED) is completely solved in $1+1$ dimensions. The theory is not gauge invariant because of the anomaly and there is a renormalized mass for the vector field parametrized by a constant $\underline{a}$ (and the coupling constant). This mass is a parameter of the theory that has to be specified. However only for $a > 1$ are the massive excitations physical.

The full Hilbert space is constructed and it is a tensor product of Fock spaces for free bosons. This Hilbert space is of indefinite metric, however, there is a quantum constraint that defines the physical states and physical operators. This constraint not only involves the gauge fields but also the fermion fields.

The fermionic fields are written in terms of physical operators and their correlation functions calculated exactly. We find that (for $a > 1$) there is an ultraviolet divergent wave function renormalization constant for the (left handed) fermions Z_2 and a vertex renormalization constant Z_1^{-1} and that the identity $Z_2 = Z_1$ is fulfilled exactly despite the lack of gauge invariance. The right handed component of the fermions is a free field.

These non-trivial renormalizations can be seen to arise in perturbation theory from the poor high energy behavior of the full photon propagator, which is the same as in a Proca theory. For the gauge fields, there is only mass renormalization, absorbed in the parameter $\underline{a}$ and no wave function renormalization, hence no coupling constant renormalization.

A Wess-Zumino term was added ad hoc into the theory to study its consequences. These W-Z terms restore gauge invariance to the theory by replacing the gauge non-singlet part of the gauge field by a massless gauge-singlet field. However the constraints of the theory remain slightly modified and they define the physical Hilbert space of the theory with the W-Z field. In particular we find that the W-Z field itself is not-physical (does not commute with the constraints).

The gauge invariant fermionic correlation functions (see eqs. (6.11), (6.12) and (6.13)) computed in the theory with the W-Z term, are exactly equal to the fermionic correlation functions of the original theory without the W-Z term (not gauge invariant).

In fact we conclude that correlation functions of physical operators (determined by the constraint) are the same in both theories, with and without the Wess-Zumino terms. However the physical operators are not determined by the gauge invariance of the theory with the W-Z term, but by a set of constraints that are equivalent in both theories.

We conjecture that many of these features may survive in chiral theories in four dimensions. In particular a mass for the "photon" will be a physical parameter of the theory.

Since the gauge invariance would be lost, the physical degrees of freedom cannot be exposed by fixing a noncovariant gauge since this would break Lorentz covariance. Hence the theory would have to be quantized covariantly, and we expect the Hilbert space to be of indefinite metric, with an ensuing constraint that would define the physical subspace of the full Hilbert space. In this physical subspace unitarity is expected to hold.

Recently Rajaraman[21] and Rajeev[22] have partially studied the question of unitarity and Lorentz invariance in four dimensional chiral theories. These authors have modified the theory by introducing W-Z-type terms. Although the nature of the Hilbert space and physical operators have not been completely elucidated, their results seem to confirm (partially) our expectations.

Regarding renormalizability, the $1+1$ dimensional theory strongly suggests that the gauge-invariant Ward identities may be maintained. And although the divergence structure of the theory is expected to be different (worse) from that of a gauge theory, if the Ward identities are still satisfied, the theory may still be renormalizable.

Of course this has to be studied further, the main worry would be overlapping divergences. The next question would be to understand the nature of the constraints, and this may constitute a considerable task.

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