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Symmetries of Systems of Nonlinear Reaction-Diffusion Equations

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We present the complete analysis of classical Lie symmetries of systems of two nonlinear diffusion equations with $1+m$ independent variables $t, x_1, \ldots, x_m$, whose nonlinearities do not depend on t and x.

1 Introduction

Coupled systems of nonlinear diffusion equations have many important applications in mathematical physics, chemistry and biology. These systems are very complex in nature and admit fundamental particular solutions (for example, traveling waves and spiral waves) which have a clear group-theoretical interpretation and which can be obtained using the classical Lie approach. The existence of such solutions predetermines an important role for the group theoretical approach in the analysis of systems of reaction diffusion equations. However, to the best of our knowledge, a comprehensive group analysis has not been undertaken previously although analyses of some special cases do exist.

In the present paper we investigate Lie symmetries of equations in the general form

$$\frac{\partial u}{\partial t} - \Lambda \sum_i \frac{\partial^2 u}{\partial x_i^2} - f(u) = 0, \tag{1.1}$$

where the dependent variable $u = \text{column}(u_1, u_2, \ldots, u_n)$ is a n-component vector-function, dependent on $m+1$ variables $t, x_1, x_2, \ldots, x_m$. Also, $f = \text{column}(f_1, f_2, \ldots, f_n)$ is an arbitrary vector-function of u and Λ is a $n \times n$ constant matrix which is non-singular.

Classical Lie symmetries of equation (1.1) with, $n = m = 1$, were investigated by Ovsiannikov [1] whose results were completed by Dorodnitsyn [2]. The related conditional (nonclassical) symmetries were described by Fushchych and Serov [3] and Clarkson and Mansfield [4]. Symmetries of equation (1.1) with $m > 1$ and (or) $n > 0$ were partly investigated in papers [5–7]. We notice that it was equation (1.1) for $m = n = 1$, $f \equiv 0$, was the subject of a group analysis by Sophus Lie [8].

An investigation of the symmetries of the general equation (1.1) can be undertaken within the framework of the classical Lie algorithm (see, for example, [9, 10]) which reduces the problem of determining symmetry to the solution the systems of linear over-determined equations for the coefficients of the symmetry operators. We will show that when applied to systems (1.1), this algorithm admits a rather simple formulation which may also be applied to an extended class of partial differential equations.

2 Determining equations for symmetries of the system (1.1)

We require form-invariance of the system of reaction diffusion equations (1.1) with respect to the one-parameter group of transformations:

$$t \to t'(t,x,\varepsilon), \qquad x \to x'(t,x,\varepsilon), \qquad u \to u'(t',x',\varepsilon), \tag{2.1}$$

where ε is a group parameter. In other words, we require that $u'(t',x',\varepsilon)$ satisfies the same equation, as $u(t,x)$:

$$L'u' = f(u'), \qquad L' = \frac{\partial}{\partial t'} - \frac{\partial^2}{\partial x'^2}. \tag{2.2}$$

From the infinitesimal transformations:

$$\begin{aligned} &t \to t' = t + \Delta t = t + \varepsilon\eta, \qquad x_a \to x'_a = x_a + \Delta x_a = x_a + \varepsilon\xi^a, \\ &u_a \to u'_a = u_a + \Delta u_a = u_a + \varepsilon\pi_a \end{aligned} \tag{2.3}$$

we obtain the following representation for the operator L'.

$$L' = \left[1 + \varepsilon\left(\eta\frac{\partial}{\partial t} + \xi^a\frac{\partial}{\partial x_a}\right)\right] L \left[1 - \varepsilon\left(\eta\frac{\partial}{\partial t} + \xi^a\frac{\partial}{\partial x_a}\right)\right] + O\left(\varepsilon^2\right). \tag{2.4}$$

Using the classical Lie algorithm, it is possible to find the determining equations for the functions η, ξ_a and π_a, which specify the generator X of the symmetry group:

$$X = \eta\frac{\partial}{\partial t} + \xi^a\frac{\partial}{\partial x_a} - \pi^a\frac{\partial}{\partial u_a}, \tag{2.5}$$

where a summation from 1 to m is assumed over repeated indices. This system will not be reproduced here but we note that three of the equations are:

$$\frac{\partial\eta}{\partial u_a} = 0, \qquad \frac{\partial\xi^a}{\partial u_b} = 0, \qquad \frac{\partial^2\pi^a}{\partial u_c\partial u_b} = 0. \tag{2.6}$$

So from (2.6), η and ξ^a are functions of t and x_a and π^a is linear in u_a. Thus:

$$\pi^a = -\pi^{ab}u_b - \omega^a, \tag{2.7}$$

where π^{ab} and ω^a are functions of t and $x = (x_1, x_2, \ldots x_m)$.

From (2.6) it is possible to deduce all the remaining determining equations. Indeed, substituting (2.3), (2.7) into (2.4), using (1.1) and neglecting the terms of order ε^2, we find that

$$[Q,L]\,u + L\omega = \pi f + \frac{\partial f}{\partial u_a}\left(-\pi^{ab}u_b - \omega^a\right), \qquad Q = \eta\frac{\partial}{\partial t} + \xi^a\frac{\partial}{\partial x_a} + \pi, \tag{2.8}$$

and π is a matrix whose elements π^{ab} are defined by the relation (2.7).

To guarantee that equation (2.8) is compatible with (1.1) and does not impose new nontrivial conditions for u in addition to (1.1), it is necessary to suppose that the commutator $[Q,L]$ admits the representation:

$$[Q,L] = \Lambda L + \varphi(t,x), \tag{2.9}$$

where Λ and φ are $n \times n$ matrices dependent on (t, x_a).

Substituting (2.9) into (2.8) the following determining equations for f are obtained:

$$\left(\Lambda^{kb} - \pi^{kb}\right) f^b + \varphi^{kb}u^b + L\omega^k = -\left(\omega^a + \pi^{ab}u_b\right)\frac{\partial f^k}{\partial u^a}. \tag{2.10}$$

Thus, to find all nonlinearities f^k generating Lie symmetries for equation (1.1) it is necessary to solve the operator equation (2.9) for L, Q given in (2.2), (2.8) and determine the corresponding matrices Λ, π, φ and functions η and ξ. In the second step the nonlinearities f^a may be found by solving the system of first order equations (2.10) with their known coefficients.

Equation (2.9) is a straight forward generalization of the invariance condition for the *linear* system of diffusion equations (1.1) with $f(u) = 0$, so that:

$$[Q, L] = \Lambda L$$

which may readily be solved. By means of this "linearization" the problem of investigating symmetries of systems of nonlinear diffusion equations is reduced to the, rather simple, application of elements of matrix calculus in order to classify non-equivalent solutions of the determining equations.

We note also that calculations of the *conditional* (nonclassical) symmetries for the system (1.1) may be reduced to the solution of the determining equations (2.10) where now Λ, π, φ, η and ξ are defined as solutions of the following relationship:

$$[Q, L] = \Lambda L + \varphi(t, x) + \mu(t, x)Q, \tag{2.11}$$

where $\mu(t, x)$ is an unknown function of the independent variables.

3 General form of symmetry operators

We now determine the general solutions for matrices Λ, φ, π and also the functions ξ, η, π which satisfy (2.10), (2.9). Evaluating the commutator in (2.9) and equating the coefficients for linearly independent differential operators, we obtain the determining equations:

$$2A\xi^a_b = -\delta_{ab}(\Lambda A + [A, \pi]), \tag{3.1}$$

$$\dot{\eta}_a = 0, \qquad \dot{\eta} = \Lambda, \tag{3.2}$$

$$\dot{\xi}^a - 2A\pi_a - A\xi^a_{nn} = 0, \tag{3.3}$$

$$\varphi = A\pi_{nn} - \dot{\pi}. \tag{3.4}$$

Here the dots denotes derivatives with respect to t and subscripts denote derivatives with respect to the spatial variables, so for example, $\eta_a = \frac{\partial \eta}{\partial x_a}$.

From (3.2) Λ is proportional to the unit matrix, $\Lambda = \lambda I$. Moreover, it follows from (3.1) that $[A, \pi] \equiv 0$. Indeed, choosing in (3.1) $a = b$ we obtain

$$\pi - A^{-1}\pi A = (2\xi^a_a - \lambda)I. \tag{3.5}$$

The trace of the left hand side of (3.5) is equal to zero, and so $2\xi^a_a - \lambda \equiv 0$ and $A\pi - \pi A = 0$.

Equations (3.1)–(3.4) contain matrices which commute, and so they may easily be integrated using, for example, the method of characteristics. The general solution of (3.1)–(3.4) is:

$$\begin{gathered} \xi^a = C^{[ab]}x_b + \dot{d}x^a + g^a, \qquad \eta = -2d, \\ \pi = \frac{1}{2}A^{-1}\left(\frac{\ddot{d}}{2}x^2 + \dot{g}^a x^a\right) + C, \qquad \Lambda = -2\dot{d}I, \\ \varphi = \frac{m}{2}\ddot{d} - \dot{C} - \frac{1}{2}A^{-1}\left(\frac{\dddot{d}}{2}x^2 + \ddot{g}^a x^a\right), \end{gathered} \tag{3.6}$$

where d, g^a are arbitrary functions of t and C is a t-dependent matrix commuting with A.

By considering the x-dependence of functions (3.6) it is convenient to represent a still unknown function ω, occuring in (2.10), as:

$$\omega_a = \omega_2^a x^2 + \omega_1^{ab} x_b + \omega_0^a + \mu^a, \tag{3.7}$$

where ω_2^a, ω_1^{ab}, ω_0^a are functions of t, and μ^a is a function of t and x, $x^2 = x_1^2 + x_2^2 + \cdots + x_m^2$. Without loss of generality we suppose that all terms in the right hand side of (3.7) are linearly independent. Then comparing with (2.10), (3.6) the functions μ^k have to satisfy:

$$L\mu^k = \lambda^{kb}\mu^b + \xi_0^k + \xi_1^{kb} x_b + \xi_2^k x^2, \tag{3.8}$$

where λ^{kb} are constants and ξ_0^k, ξ_1^{kb}, ξ_2^k are functions of t.

The final step is to substitute (3.5), (3.7) into (2.10) and equate coefficients for all different powers of x_a. As a result we obtain the system of equations:

$$\ddot{d}\left(A^{-1}\right)^{kb} f^b + \dddot{d}(A^{-1})^{kb} u^b - \ddot{d}\left(A^{-1}\right)^{ab} u^b \frac{\partial f^k}{\partial u^a} = 4\left(\dot{\omega}_2^k + \xi_2^k - \omega_2^b \frac{\partial f^k}{\partial u_b}\right) = 0, \tag{3.9}$$

$$\dot{g}^a\left(A^{-1}\right)^{kb} f^b + \ddot{g}^a\left(A^{-1}\right)^{kb} u^b - \dot{g}^a\left(A^{-1}\right)^{kb} u^b \frac{\partial f^k}{\partial u^a} = 2\left(\dot{\omega}_1^{ka} + \xi_1^k a - \omega_1^{ba}\frac{\partial f^k}{\partial u_a}\right) = 0, \tag{3.10}$$

$$\left(2\dot{d}\delta^{kb} + C^{kb}\right) f^b + \left(\dot{C}^{kb} - \frac{m}{2}\ddot{d}\delta^{kb}\right) u^b - \left(\omega_0^a + C^{ab}u^b\right)\frac{\partial f^k}{\partial x_a} = \omega_0^k - 2mA^{kb}\omega_2^b - \omega_0^b \frac{\partial f^k}{\partial u_b}, \tag{3.11}$$

$$\frac{\partial f^k}{\partial u_b}\mu^b = \lambda^{kb}\mu^b. \tag{3.12}$$

Thus, the general form of symmetry group generators for equation (1.1) is given by relations (2.5), (3.6), (3.7), where d, g^a, C^{ab}, ω_0^k, ω_1^{kb}, ω_2^a, μ^a are functions of t to be specified using equations (3.9)–(3.12).

4 Nonlinearities and symmetries

We will not give the detailed calculations but present the general solution of relations (3.9)–(3.12) in the form of the following Tables 1–3.

In Table 1 the Greek letters denote arbitrary coefficients while D_μ, G_a^i and $\bar{G}_a^i$, X_A, Y_a, $\hat{F}$, $\hat{B}$ are various types of dilatation, Galilei and special transformation generators as follows:

$$\begin{gathered}
D_0 = 2t\frac{\partial}{\partial t} + x_a\frac{\partial}{\partial x_a}, \qquad D_1 = D_0 - \frac{2}{k}\hat{F}, \qquad D_2 = D_0 - \frac{2s}{\sqrt{k^2+s^2}}\left(u\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}\right),\\
D_3 = D_0 - \frac{2}{k}\left(\frac{\partial}{\partial u_1} - 2nu_1\frac{\partial}{\partial u_1}\right), \qquad D_4 = D_0 - \frac{2}{k}\omega_a\frac{\partial}{\partial u_a},\\
G_a = t\frac{\partial}{\partial x_a} - \frac{1}{2}x_a\left(A_1^{-1}\right)^{nb} u_b\frac{\partial}{\partial u_n}, \qquad \hat{G}_a = \exp(nt)\left(\frac{\partial}{\partial x_a} - \frac{1}{2}nx_a\left(A_1^{-1}\right)^{nb} u_b\frac{\partial}{\partial u_n}\right),\\
X_0 = \alpha\frac{\partial}{\partial t} + \beta_a\frac{\partial}{\partial x_a} + \nu^{[a,b]}x_a\frac{\partial}{\partial x_b}, \qquad \nu^{[a,b]} = -\nu^{[b,a]}, \qquad Y_1 = nt\hat{F} - \hat{B},\\
Y_2 = \exp(st)\left(u_1\frac{\partial}{\partial u_1} + n\frac{\partial}{\partial u_2}\right), \qquad Y_2 = u_1\frac{\partial}{\partial u_2} - n\frac{\partial}{\partial u_1}, \qquad Y_3 = u_1\frac{\partial}{\partial u_2} - 2n\frac{\partial}{\partial u_2},\\
Y_4 = \exp(kt)\left(u_1\frac{\partial}{\partial u_2} + \frac{nx^2}{2m}\frac{\partial}{\partial u_2}\right),
\end{gathered} \tag{4.1}$$

where k, n, s are parameters used in the definitions of the nonlinear terms,

$$\hat{F} = F^{ab}u_b\frac{\partial}{\partial u_a}, \qquad \hat{B} = B^{ab}u_b\frac{\partial}{\partial u_a}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \tag{4.2}$$

and B is one of the matrices:

$$\begin{aligned} &I. \quad B = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}; \qquad && II. \quad B = \begin{pmatrix} d & -1 \\ 1 & d \end{pmatrix}; \\ &IIIa. \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \qquad && IIIb. \quad B = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix}. \end{aligned} \tag{4.3}$$

In Tables 2 and 3 we form a two-dimensional Lie algebra based upon the matrices, F and B, classified the categories:

$$\begin{aligned} &I. \quad F = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, \qquad && B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; \\ &IIa. \quad F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad && B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \\ &IIb. \quad F = \begin{pmatrix} d & -1 \\ 1 & d \end{pmatrix}, \qquad && B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \\ &IIIa. \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad && B = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}; \\ &IIIb. \quad F = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix}, \qquad && B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{4.4}$$

Here, $\kappa = \text{const}$, $\Delta = k_0n_1 - n_0k_1$, $\delta = \frac{1}{4}(k_0 - n_1)^2 + k_1n_0$ and:

$$\left(\frac{\partial}{\partial t} - \sum_{i=1}^{m}\frac{\partial^2}{\partial x_i^2}\right)\psi_n = n\psi_n, \qquad n = 0, b. \tag{4.5}$$

The generators D_μ, $\hat{A}_\alpha$, G_a, $\hat{G}_a$, X_ν, Y_s when not specified in (4.1) are given by:

$$\hat{A}_0 = t^2\frac{\partial}{\partial t} + tx_a\frac{\partial}{\partial x_a} - \frac{1}{4}x^2\left(A^{-1}\right)^{ab}u_b\frac{\partial}{\partial u_a} - \frac{m}{2} + \left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\right),$$

$$\hat{A}_1 = \hat{A}_0 + nt^2\hat{F} - \frac{m}{2}\hat{B}, \qquad D_5 = D_0 - \frac{m}{2}\left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\right),$$

$$D_6 = D_0 - 2tn\hat{F} - \frac{2}{k}\hat{B}, \qquad D_7 = D_0 + \frac{2}{r}\left(\frac{kst}{r} - 1\right)u_1\frac{\partial}{\partial u_1} - \frac{2st}{r}\frac{\partial}{\partial u_2},$$

$$D_8 = D_0 - 2st(u_2 - 1)\frac{\partial}{\partial u_1} - 2\frac{\partial}{\partial u_2}, \qquad D_9 = D_0 - \left(\frac{p}{m}x^2 + 2qt\right)\frac{\partial}{\partial u_1} - 2\frac{\partial}{\partial u_2},$$

$$D_{10} = D_0 - \frac{1}{n}\left(u_1\frac{\partial}{\partial u_1} + 2u_2\frac{\partial}{\partial u_2}\right) + \frac{t}{2sn}\frac{\partial}{\partial u_1} - \frac{t}{s}u_1\frac{\partial}{\partial u_2},$$

$$D_{11} = D_0 - \frac{1}{k}\left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} - s(2k+1)x^2\frac{\partial}{\partial u_1}\right),$$

$$X_1 = \mu \exp(\lambda^+ t)\hat{\mathcal{F}}_1 + \nu \exp(\lambda^- t)\hat{\mathcal{F}}_2, \qquad \mathcal{F}_1 = \left[\frac{1}{2}(k_o - n_1) + \sqrt{\delta}\right] F + n_0 B, \qquad n = \lambda^+,$$

$$\mathcal{F}_2 = k_1 F - \left[\frac{1}{2}(k_o - n_1) + \sqrt{\delta}\right] B, \qquad n = \lambda^-, \qquad \lambda^\pm = \frac{1}{2}(k_0 + n_1) \pm \delta,$$

$$X_2 = \mu \exp(nt)\hat{\mathcal{F}}_3 + \nu\hat{\mathcal{F}}_4, \qquad n = k_0 + n_1, \qquad \mathcal{F}_3 = k_0 F + n_0 B, \qquad \mathcal{F}_4 = k_0 B - k_1 F;$$

$$X_3 = \mu \exp(nt)\hat{\mathcal{F}}_5 + \nu\hat{\mathcal{F}}_6, \qquad n = k_0 + n_1, \qquad \mathcal{F}_5 = k_1 F + n_0 B, \qquad \mathcal{F}_6 = n_1 F - n_0 B,$$

$$X_4 = \exp(nt)\hat{\mathcal{F}}_7, \qquad n = \frac{1}{2}(k_0 + n_1), \qquad \mathcal{F}_7 = \mu F + \nu B,$$

$$X_5 = \exp(nt)\left[\mu(k_1 t\hat{F} + \hat{B}) + \nu\hat{F}\right],$$

$$X_6 = \exp(nt)\left[\mu\left(\hat{F} + n_0 t\hat{B}\right) + \nu\hat{B}\right], \qquad n = \frac{1}{2}(k_0 + n_1),$$

$$X_7 = \exp(nt)\mu\left[\left(\sqrt{-n_0 k_1}t + 1\right)\hat{F} + n_0 t\hat{B}\right] + \nu\left[\left(k_1\hat{F} - \sqrt{-n_0 k_1}\hat{B}\right)t + \hat{B}\right],$$

$$n = \frac{1}{2}(k_0 + n_1),$$

$$X_8 = \nu\left[k_1 t\hat{F} + (1 - k_0 t)\hat{B} + \mu\left(k_1\hat{F} - k_0\hat{B}\right)\right],$$

$$X_9 = \exp(nt)\left[\frac{1}{2}(n_1 - k_0)\cos(\omega t) + \omega\sin(\omega t)\right]\hat{F} - n_0\cos(\omega t)\hat{B},$$

$$X_{10} = \exp(nt)\left[\omega\sin(\omega t) + \frac{1}{2}(k_0 - n_1)\cos\omega t\right]\hat{B} - k_1\cos(\omega t)\hat{F}, \qquad n = \frac{1}{2}(k_0 - n_1),$$

$$Y_5 = \exp(nt)\left(u_1\frac{\partial}{\partial u_2} - \frac{q}{2p}\left(\frac{sx^2}{2m}\frac{\partial}{\partial u_2} - \frac{\partial}{\partial u_1}\right)\right), \qquad Y_6 = \exp(nt)\left(u_1\frac{\partial}{\partial u_2} - \frac{\partial}{\partial u_1}\right),$$

$$Y_7 = u_1\frac{\partial}{\partial u_1} + nt\frac{\partial}{\partial u_2}, \qquad Y_8 = \exp(kt)\left(u_1\frac{\partial}{\partial u_1} + nt\frac{\partial}{\partial u_2}\right),$$

$$Y_9 = \exp(kt)\left(u_1\frac{\partial}{\partial u_1} + \frac{n}{k-b}\frac{\partial}{\partial u_2}\right), \qquad Y_{10} = ku_1\frac{\partial}{\partial u_1} - r\frac{\partial}{\partial u_2},$$

$$Y_{11} = \exp(nt)\left(\sin(pt)u_1\frac{\partial}{\partial u_1} + \cos(pt)\frac{\partial}{\partial u_2}\right),$$

$$Y_{12} = \exp(nt)\left(\cos(pt)u_1\frac{\partial}{\partial u_1} - \sin(pt)\frac{\partial}{\partial u_2}\right),$$

$$Y_{13} = \exp(nt)\left(ptu_1\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}\right), \qquad Y_{14} = \exp(nt)u_1\frac{\partial}{\partial u_1}.$$

Table 1. Nonlinearities with arbitrary functions

No	Nonlinear terms	Type of matrix B (4.3)	Arguments of φ_1, φ_2	Conditions for parametrs	Symmetries $A^{-1} \neq \kappa B$	Symmetries $A^{-1} = \kappa B$
1.	$f^1 = u_1^{k+1}\varphi_1$ $f^2 = u_1^{k+d}\varphi_2$	I	$\frac{u_2}{u_1^d}$	$k \neq 0$	$X_0 + \nu D_1$	$X_0 + \nu D_1$
2.	$f^1 = \exp(k\theta)[\varphi_1 u_2 + \varphi_2 u_1]$ $f^2 = \exp(k\theta)[\varphi_1 u_1 - \varphi_2 u_2]$	II	$R\, e^{-d\theta}$	$k \neq 0$	$X_0 + \nu D_1$	$X_0 + \nu D_1$
3.	$f^1 = \varphi_1 u_1^{k+1}$ $f^2 = (\varphi_1 \ln u_1 + \varphi_2)\, u^{k+1}$	$IIIb$, $d = 1$	$u_1 e^{-\frac{u_2}{u_1}}$	$k \neq 0$	$X_0 + \nu D_1$	$X_0 + \nu D_1$
4.	$f^1 = \exp\left(k\frac{u_2}{u_1}\right)\varphi_1 u_1$ $f^2 = \exp\left(k\frac{u_2}{u_1}\right)(\varphi_1 u_2 + \varphi_2)$	$IIIa$	u_1	$k \neq 0$	$X_0 + \nu D_1$	
5.	$f^1 = u_1(n \ln u_1 + \varphi_1)$ $f^2 = u_2(n \ln u_2 + \varphi_2)$	I	$\frac{u_2}{u_1^d}$	$n \neq 0$	$X_0 + \mu e^{nt}\hat{B}$	$X_0 + \nu_a \tilde{G}_a + \mu e^{nt}\hat{B}$
				$n = 0$	$X_0 + \mu\hat{B}$	$X_0 + \mu\hat{B} + \nu_a G_a$
6.	$f^1 = \varphi_1 u_2 + \varphi_2 u_1$ $\quad + \frac{n}{2}\left(\frac{1}{d}\ln R + \theta\right)(du_1 - u_2)$ $f^2 = \varphi_1 u_1 - \varphi_2 u_2$ $\quad + \frac{n}{2}\left(\frac{1}{d}\ln R + \theta\right)(du_2 + u_1)$ $R^2 = u_1^2 + u_2^2$, $\theta = \arctan\left(\frac{u_2}{u_1}\right)$	II, $d \neq 0$	$R\, e^{-d\theta}$	$n \neq 0$	$X_0 + \mu e^{nt}\hat{B}$	$X_0 + \nu_a \hat{G}_a + \mu e^{nt}\hat{B}$
				$n = 0$	$X_0 + \mu\hat{B}$	$X_0 + \mu\hat{B} + \lambda_a G_a$
7.	$f^1 = (\varphi_1 - n\theta)u_2 + \varphi_2 u_1$ $f^2 = \varphi_1 u_1 - \varphi_2 u_2$	II, $d = 0$	R	$n \neq 0$	$X_0 + \mu e^{nt}\hat{B}$	$X_0 + \nu_a \hat{G}_a + \mu e^{nt}\hat{B}$
				$n = 0$	$X_0 + \mu\hat{B}$	$X_0 + \mu\hat{B} + \lambda_a G_a$
8.	$f^1 = \varphi_1 u_1 + nu_2$ $f^2 = (\varphi_1 u_2 + u_1\varphi_2) + nu_2\left(1 + \frac{u_2}{u_1}\right)$	$IIIb$, $d = 1$	$\frac{u_2}{u_1} - \ln u_1$	$n \neq 0$	$X_0 + \mu e^{nt}\hat{B}$	$X_0 + \nu_a \hat{G}_a + \mu e^{nt}\hat{B}$
				$n = 0$	$X_0 + \mu\hat{B}$	$X_0 + \mu\hat{B} + \lambda_a G_a$

Table 1 (continued). Nonlinearities with arbitrary functions

No	Nonlinear terms	Type of matrix B (4.3)	Arguments of φ_1, φ_2	Conditions for parametrs	Symmetries $A^{-1} \neq \kappa B$	Symmetries $A^{-1} = \kappa B$
9.	$f^1 = \varphi u_1$ $f^2 = \varphi u_2 - n u_1$	$IIIa$, $d=0$	u_1		$X_0 + \mu \hat{B} + \nu Y_1$	
10.	$f^1 = \varphi u_1$ $f^2 = \varphi_1 u_2 + \varphi_2 u_1 + n u_2$	$IIIa$	u_1		$X_0 + \mu e^{nt} \hat{B}$	
11.	$f^1 = u_1^{\left(k+\frac{k^2}{\sqrt{k^2+s^2}}\right)}$ $\times \exp\left[\left(s + \frac{sk}{\sqrt{k^2+s^2}} u_2\right)\right] \varphi_1$ $f^2 = u_1^k \exp(s u_2) \varphi_2$	I, $d=0$	$s \ln u_1 + k u_2$		$X_0 + \nu D_2$	
12.	$f^1 = \varphi_1 \exp\left(\frac{n u_2}{n^2+1}\right) u_1^{\frac{1}{n^2+1}}$ $+\frac{s}{n^2+1} u_1 (n u_2 + \ln u_1)$ $f^2 = \varphi_2 + \frac{ns}{n^2+1}(n u_2 + \ln u_1)$	I, $d=0$	$u_2 - n \ln u_1$		$X_0 + \nu Y_2$	
13.	$f^1 = \exp(k u_1) \varphi_1$ $f^2 = \exp(k u_1)(\varphi_2 - 2n \varphi_1 \ln u_1)$	$IIIa$	$n u_1^2 + u_2$	$k \neq 0$	$X_0 + \lambda D_3$	
14.	$f^1 = \varphi_1 + s u_1$ $f^2 = \varphi_2 - 2n \varphi_1 u_1 + 2s u_2$	$IIIa$	$n u_1^2 + u_2$		$X_0 + \lambda Y_3$	
15.	$f^1 = n$ $f^2 = k u_2 + \varphi_2$	$IIIa$	u_1		$X_0 + \lambda Y_4 + \psi_k \frac{\partial}{\partial u_2}$	
16.	$f^\alpha = \exp\left[\frac{k}{\omega}(\omega_1 u_1 + \omega_2 u_2)\right] \varphi^\alpha$ $\alpha = 1, 2,\ \omega^2 = \omega_1^2 + \omega_2^2$	*any*	$\omega_1 u_2 - \omega_2 u_1$	$k \neq 0$	$X_0 + \nu D_4$	$X_0 + \nu D_4$
17.	$f^1 = \varphi_1$ $f^2 = \varphi_2$	*any*	(u_1, u_2)		X_0	X_0

Table 2. Non-linearities with arbitrary parameters which generate symmetry with respect to dilatation

No	Nonlinear terms	Conditions for parameters	Symmetries, $A^{-1} \neq \kappa F$	Symmetries, $A^{-1} = \kappa F$	Matrix class and symmetry generator parameters
1	$f^1 = (gu_1^q u_2^r - s)\, u_1$ $f^2 = \left(pu_1^q u_2^r - \frac{rs}{q}\right) u_2$	$s = 0,\ q \neq 0$ $q + r = \frac{4}{m}$	$X_0 + \mu\hat{F} + \nu D_5$	$X_0 + \mu\hat{F} + \nu D_5 + \sigma_a G_a + \lambda A_0$	$I,\ d = -\frac{q}{r}$
		$s = 0,\ q \neq 0,\ r \neq 0$ $0 \neq q + r \neq \frac{4}{m}$	$X_0 + \nu\hat{F} + \mu D_5$	$X_0 + \nu\hat{F} + \mu D_5 + \sigma_a G_a$	$I,\ k = r + q$ $d = -\frac{q}{r}$
		$s \neq 0,\ q \neq 0$	$X_0 + \nu\hat{F} + \mu D_6$	$X_0 + \nu\hat{F} + \mu D_6 + \sigma_a G_a$	$I, k = r$ $d = -\frac{q}{s}$
2	$f^1 = e^{q\theta} R^r\,(gu_1 - pu_2)$ $+su_2 - lu_1$ $f^2 = e^{q\theta} R^r\,(gu_2 + pu_1)$ $-su_1 - lu_2$ $R^2 = u_1^2 + u_2^2$ $\theta = \tan^{-1}\left(\frac{u_2}{u_1}\right)$	$s = l = 0$ $r = \frac{4}{m}$	$X_0 + \nu\hat{F} + \mu D_5$	$X_0 + \nu\hat{F} + \mu D_5 + \sigma_a G_a + \lambda A_0$	$IIb,\ k = r$ $d = -\frac{q}{r}$
		$r \neq \frac{4}{m},\ r \neq 0$ $s = l = 0$	$X_0 + \nu\hat{F} + \mu D_5$	$X_0 + \nu\hat{F} + \mu D_5 + \sigma_a G_a$	$IIb,\ k = \frac{4}{m}$ $d = -\frac{q}{r}$
		$l = sq\left(1 + \frac{1}{r}\right)$ $s \neq 0,\ r \neq 0$	$X_0 + \nu\hat{F} + \mu D_6$	$X_0 + \nu\hat{F} + \mu D_6 + \sigma_a G_a$	$IIb,\ k = r$ $n = sq$ $d = q\left(1 + \frac{1}{r}\right)$
		$s = 0,\ l \neq 0$ $q \neq 0,\ r = 0$	$X_0 + \nu\hat{F} + \mu D_6$	$X_0 + \nu\hat{F} + \mu D_6 + \sigma_a G_a$	$IIa,\ k = q$ $n = lq$
3	$f^1 = \left(pu_1^r e^{q\frac{u_2}{u_1}} - s\right) u^1$ $f^2 = e^{q\frac{u_2}{u_1}}\,(pu_2 + gu_1)\, u_1^r$ $-s\left(u_2 - \frac{r}{q}u_1\right)$	$r = -q = \frac{4}{m}$ $l = s = 0$	$X_0 + \nu\hat{F} + \mu D_5$	$X_0 + \nu\hat{F} + \mu D_5 + \alpha_a G_a + \lambda A_1$	$IIIb,\ d = 1$ $k = \frac{4}{m}$
		$-q = r \neq \frac{4}{m}$ $l = s = 0,\ r \neq 0$	$X_0 + \nu\hat{F} + \mu D_5$	$X_0 + \nu\hat{F} + \mu D_5 + \alpha_a G_a$	$IIIb,\ d = 1$ $k = r$
		$q \neq 0,\ s \neq 0$	$X_0 + \nu\hat{F} + \mu D_6$	$X_0 + \nu\hat{F} + \mu D_6 + \sigma_a G_a$	$IIIb,\ k = q$ $n = sq,\ d = -\frac{n}{q}$

Table 2 (continued). Non-linearities with arbitrary parameters which generate symmetry with respect to dilatation

No	Nonlinear terms	Conditions for parameters	Symmetries, $A^{-1} \neq \kappa F$	Symmetries, $A^{-1} = \kappa F$	Matrix class and symmetry generator parameters
4	$f^1 = pu_1^{k+1}$ $f^2 = u_1^k\left(pu_2 + qu_1^d\right) - \frac{n}{d+k-1}u_1$	$d+k \neq 1$ $k \neq 0,\ q \neq 0$	$X_0 + \nu\hat{F} + \mu D_6$		$IIIa$
		$k \neq 0,\ n = 0$ $q = 0$	$X_0 + \nu\hat{F} + \mu D_5 + \lambda\hat{B}$		$IIIa,\ d = 0$
		$k \neq 0,\ n = 0$ $q = 0$	$X_0 + \nu\hat{F} + \mu D_5 + \lambda Y_1$		$IIIa,\ d = 0$
5	$f^1 = pu_1^{k+1}$ $f^2 = pu_1^k u_2 + qu_1 + nu_1 \ln u_1$	$k \neq 0,\ n \neq 0$	$X_0 + \nu\hat{F} + \mu D_6$		$IIIa,\ d = 1-k$
6	$f^1 = qu_1^{r+1}e^{ku_2} + \frac{ks}{r^2}u_1$ $f^2 = pu_1^r e^{ku_2} + \frac{r}{s}$	$r \neq 0, -1\ p \neq 0$	$X_0 + \nu D_7 + \mu Y_7$		I
7	$f^1 = e^{u_2} + su_2 + q$ $f^2 = n$	$q = 0$	$X_0 + \nu D_7 + \mu Y_7 + \psi_n \frac{\partial}{\partial u_1}$		$I,\ k = -r = 1$
		$s = 0, q \neq 0$ $n \neq 0$	$X_0 + \nu D_1$ $+\mu\left(Y_7 - qt\frac{\partial}{\partial u_1}\right) + \psi_0\frac{\partial}{\partial u_1}$		$I, d = 0, k = 1$
		$q = n = 0$ $s \neq 0$	$X_0 + \nu D_8 + \mu\hat{F} + \psi_0\frac{\partial}{\partial u_1}$		$IIIa$
8	$f^1 = k_1 e^{u_2} - pA^{21}$ $f^2 = k_2 e^{u_2} - pA^{11} + q$		$X_0 + \lambda D_9 + \psi_0\frac{\partial}{\partial u_1}$	$X_0 + \nu D_9 + \psi_0\frac{\partial}{\partial u_1}$	any
9	$f^1 = p\left(u_2 + nu_1^2\right)^{s+\frac{1}{2}} + \frac{1}{2n(2s+1)}$ $f^2 = q\left(u_2 + nu_1^2\right)^{s+1} - \frac{1}{2s+1}u_1$ $-2npu_1\left(u_2 + nu_1^2\right)^{s+\frac{1}{2}}$	$s \neq 0,\ s \neq \frac{1}{2}$ $p \neq 0,\ n \neq 0$	$X_0 + \nu D_{10}$		I
10	$f^1 = pu_1^{2k+1} - 2msA^{11}$ $f^2 = qu_1^{2k+1} - 2msA^{21}$	$k \neq 0$	$X_0 + \nu D_{11} + \psi_0\frac{\partial}{\partial u_2}$		any

Table 3. Further non-linearities with arbitrary parameters

No	Nonlinear terms	Matrix Class (4.4)	Conditions for parameters	Symmetries, $A^{-1} \neq \kappa F$	Symmetries, $A^{-1} = \kappa F$
1	$f^1 = (k_0 \ln u_1 + k_1 \ln u_2 + q)\, u_1$ $f^2 = (n_0 \ln u_1 + n_1 \ln u_2 + p)\, u_2$	$I,\ d = 0$	$\delta > 0,\ \Delta \neq 0$ $\mathcal{F} = \mathcal{F}_1, \mathcal{F}_2$	$X_0 + \nu X_1$	$X_0 + \lambda X_1 + \sigma_a \hat{G}_a$
			$\delta > 0,\ \Delta = 0$ $n_1 = 0,\ k_1 \neq 0,\ \mathcal{F} = \mathcal{F}_3$	$X_0 + \lambda X_2$	$X_0 + \lambda X_2 + \sigma_a \hat{G}_a$
2	$f^1 = (k_0 u_1 - n_0 u_2) \ln R$ $+\theta\, (k_1 u_1 - n_1 u_2) + p u_1 - q u_2$ $f^2 = (k_0 u_2 + n_0 u_1) \ln R$ $+\theta\, (n_1 u_1 + k_1 u_2) + q u_1 + p u_2$	IIa	$\delta > 0,\ \Delta = 0$ $n_1 = 0,\ k_1 \neq 0,\ \mathcal{F} = \mathcal{F}_4$	$X_0 + \lambda X_2$	$X_0 + \lambda X_2 + \sigma_a G_a$
			$\delta > 0,\ \Delta = 0$ $n_1 \neq 0,\ \mathcal{F} = \mathcal{F}_5$	$X_0 + \lambda X_3$	$X_0 + \lambda X_3 + \sigma_a \hat{G}_a$
			$\delta > 0,\ \Delta = 0$ $n_1 \neq 0,\ \mathcal{F} = \mathcal{F}_6$	$X_0 + \lambda X_3$	$X_0 + \lambda X_3 + \sigma_a G_a$
3	$f^1 = (k_0 \ln u_1 + q)\, u_1 + k_1 u_2$ $f^2 = (n_0 u_1 + k_0 u_2) \ln u_1$ $+k_1 \frac{u_2^2}{u_1} + p u_1 + (n_1 + q)\, u_2$	$IIIa$	$\delta = 0,\ \Delta \neq 0$ $k_1 = n_0 = 0,\ \mathcal{F} = \mathcal{F}_7$	$X_0 + \lambda X_4$	$X_0 + \lambda X_4 + \sigma_a \hat{G}_a$
			$\delta = 0,\ \Delta \neq 0$ $n_0 = 0,\ k_1 \neq 0,\ \mathcal{F} = F$	$X_0 + \lambda X_5$	$X_0 + \lambda X_5 + \sigma_a \hat{G}_a$
			$\delta = 0,\ \Delta \neq 0$ $k_1 = 0,\ n_0 \neq 0,\ \mathcal{F} = B$	$X_0 + \lambda X_6$	$X_0 + \lambda X_6 + \sigma_a \hat{G}_a$
			$\delta = 0,\ \Delta \neq 0$ $n_0 k_1 < 0$	$X_0 + \lambda X_7$	
			$\delta = 0 = \Delta$ $k_1 = 0,\ n_0 = 0$ $\mathcal{F} = \alpha F + \mu B$	$X_0 + \nu \hat{F} + \mu \hat{B}$	$X_0 + \nu \hat{F} + \mu \hat{B} + \sigma_a G_a$
			$\delta = 0 = \Delta$ $k_1 = 0,\ n_0 \neq 0,\ \mathcal{F} = B$	$X_0 + \nu \hat{B}$ $+\mu \left(\hat{F} + n_0 t \hat{B}\right)$	$X_0 + \nu \hat{B}$ $+\mu \left(\hat{F} + n_0 t \hat{B}\right) + \sigma_a G_a$
			$\delta = 0, \Delta = 0$ $k_1 \neq 0,\ n_0 = 0,\ \mathcal{F} = F$	$X_0 + \nu \hat{F}$ $+\mu \left(\hat{B} + k_1 t \hat{F}\right)$	$X_0 + \nu \hat{F}$ $+\mu \left(\hat{B} + k_1 t \hat{F}\right) + \sigma_a G_a$
			$\delta = 0, \Delta = 0,\ n_0 k_1 \neq 0$ $\mathcal{F} = k_1 F - k_0 B$	$X_0 + \nu X_8$	$X_0 + \nu X_8 + \sigma_a G_a$
			$\delta = -\omega^2 < 0$	$X_0 + \nu X_9 + \mu X_{10}$	

Table 3 (continued). Further non-linearities with arbitrary parameters

No	Nonlinear terms	Matrix Class (4.4)	Conditions for parameters	Symmetries, $A^{-1}\neq\kappa F$	Symmetries, $A^{-1}=\kappa F$
4	$f^1=qu_1$ $f^2=nu_2+pu_1^d su_1+k$	$IIIa$	$d\neq 1,2,k=0$ $s=0,\ n\neq q,\ \mathcal{F}=B$	$X_0+\nu\hat{B}+\psi_n\frac{\partial}{\partial u_2}$	$X_0+\nu\hat{B}+\psi_n\frac{\partial}{\partial u_2}+\sigma_aG_a$
			$d\neq 0,2,n=0$ $s=0,\mathcal{F}=B$	$X_0+\psi_0\frac{\partial}{\partial u_2}+\nu\left(\hat{B}-dkt\frac{\partial}{\partial u_2}\right)$	$X_0+\psi_0\frac{\partial}{\partial u_2}$ $+\nu\left(\hat{B}-dkt\frac{\partial}{\partial u_2}\right)+\sigma_aG_a$
			$d\neq 0,1,2,\ n=q$ $k=0,\ s=\frac{1}{1-d}$ $\mathcal{F}=B$	$X_0+\nu\hat{F}+\mu\hat{B}+\psi_n\frac{\partial}{\partial u_2}$	$X_0+\nu\hat{F}+\mu\hat{B}$ $+\psi_n\frac{\partial}{\partial u_2}+\sigma_aG_a$
			$d=2,\ n=q$ $k=0,\ s\neq 0$	$X_0+\mu Y_5+\psi_n\frac{\partial}{\partial u_2}$	
			$d=2,\ n=q$ $k=s=0,\ \mathcal{F}=B$	$X_0+\nu\hat{B}+\psi_0\frac{\partial}{\partial u_2}$	$X_0+\nu\hat{B}+e^{nt}\hat{F}$ $+\psi_0\frac{\partial}{\partial u_2}+\sigma_aG_a$
			$n=2\,(q+p)\,,\ d=2$ $k=s=0,\ \mathcal{F}=B$	$X_0+\nu Y_6+\mu\hat{B}+\psi_0\frac{\partial}{\partial u_2}$	$X_0+\nu Y_6+\mu\hat{B}$ $+\psi_0\frac{\partial}{\partial u_2}+\sigma_aG_a$
			$d=2,\ p=-q$ $n=s=0$	$X_0+\nu Y_6+\psi_0\frac{\partial}{\partial u_2}$ $+\mu\left(\hat{B}-dkt\frac{\partial}{\partial u_2}\right)$	
5	$f^1=ku_1\ln u_1+pu_1$ $f^2=bu_2+n\ln u_1+q$	I	$b=k=q=0$ $p\neq 0$	$X_0+\nu Y_7+\psi_0\frac{\partial}{\partial u_2}$	
			$q=p=0$ $b=k$	$X_0+\nu Y_8+\psi_b\frac{\partial}{\partial u_2}$	
			$q=p=0$ $b\neq k,\ b\neq 0$	$X_0+\nu Y_9+\psi_b\frac{\partial}{\partial u_2}$	
			$k\neq 0,\ b=p=0$	$X_0+\nu Y_9+\psi_0\frac{\partial}{\partial u_2}$	
6	$f^1=pu_1u_2+nu_1\ln u_1$ $f^2=nu_2-q\ln u_1$	I	$q=p$	$X_0+\nu Y_{11}+\mu Y_{12}$	
			$q=0$	$X_0+\nu Y_{13}+\mu Y_{14}$	

5 Discussion

Thus we have found all possible versions of systems of diffusion equations which admit a non-trivial Lie symmetry. These results can be used to construct mathematical models with required symmetry properties in, for example, physics, biology, chemistry. On the other hand, our results give ad hoc solution of problems of group analysis of all models using systems of diffusion equations. As an example consider the nonlinear Schrödinger equation in m-dimensional space

$$\left(i\frac{\partial}{\partial t}-\sum_{i=1}^{m}\frac{\partial^2}{\partial x_i^2}\right)\psi=F\left(\psi,\psi^*\right)\psi \tag{5.6}$$

which is also is a particular case of (1.1). If we denote

$$u_1=\frac{1}{2}(\psi+\psi^*),\qquad u_2=\frac{1}{2i}\left(\psi-\psi^*\right) \tag{5.7}$$

then (5.6) reduces to the form (1.1) with $A=i\sigma_2$, and

$$f^1=\frac{1}{2}\left(F^*+F\right)u_2+\frac{1}{2i}\left(F-F^*\right)u_1,$$

$$f^2=\frac{1}{2i}\left(F-F^*\right)u_2-\frac{1}{2}\left(F+F^*\right)u_1.$$

In other words, any solution given in Tables 2, 3 with matrices belonging to Classes I, II give rise to the nonlinearity

$$F=\frac{1}{R^2}\left[\left(u_2f^1-u_1f^2\right)+i\left(u_2f^2+f^1u^1\right)\right]$$

for the nonlinear Schrödinger equation (5.6) which admits a nontrivial Lie symmetry. We see that the number of nonlinearities which guarantee a non-trivial symmetry for the non-linear Schödinger equation is very extended and exceeds one hundred.

We notice that the nonlinear Schrödinger equations with *ad hoc* required symmetry with respect to the (extended) Galilei group where described in [11, 12].

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