

# Taking off the square root of Nambu–Goto action and obtaining Filippov–Lie algebra gauge theory action

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Received: 23 June 2009 / Published online: 12 September 2009

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**Abstract** We propose a novel prescription to take off the square root of the Nambu–Goto action for a  $p$ -brane, which generalizes the Brink–Di Vecchia–Howe–Tucker, also known as the Polyakov method. With an arbitrary decomposition,  $d + n = p + 1$ , our resulting action is a modified  $d$ -dimensional Polyakov action, which is gauged and possesses a Nambu  $n$ -bracket squared potential. We first spell out how the  $(p + 1)$ -dimensional diffeomorphism is realized in the lower dimensional action. Then we discuss a possible gauge fixing of it to a direct product of  $d$ -dimensional diffeomorphism and  $n$ -dimensional volume preserving diffeomorphism. We show that the latter naturally leads to a novel Filippov–Lie  $n$ -algebra based gauge theory action in  $d$  dimensions.

## 1 Introduction

A  $p$ -brane is a spatially extended object propagating in a target spacetime. The number  $p$  counts the spatial dimensions of the brane such that  $p = 0, 1, 2, \dots$  correspond to point-like particle, string, membrane, etc. The geodesic motion of a point particle, i.e. a  $p = 0$  brane, minimizes the relativistic length of the trajectory in the target spacetime. The Nambu–Goto action for a  $p$ -brane then generalizes this geometric significance: the induced worldvolume of the brane is to be minimized.

With an embedding of  $(p + 1)$ -dimensional worldvolume coordinates into a  $D$ -dimensional target spacetime,

$$X(\xi) : \xi^m \rightarrow X^M, \quad (1)$$

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where  $m = 0, 1, \dots, p$  and  $M = 0, 1, \dots, D - 1$ , the Nambu–Goto action for a  $p$ -brane is [1, 2]

$$S_{\text{N.G.}} = - \int d^{p+1}\xi \sqrt{-\det \mathcal{G}_{mn}}. \quad (2)$$

Here  $\mathcal{G}_{mn}$  is the induced metric on the worldvolume, such that the action measures the relativistic worldvolume of the  $p$ -brane in the target spacetime,

$$\mathcal{G}_{mn} := \partial_m X^M \partial_n X^N G_{MN}(X). \quad (3)$$

For simplicity we set the brane tension to unity.

Despite its elegant geometric significance, the Nambu–Goto action is hard to quantize due to the presence of a highly nonlinear structure, the square root. An equivalent but far more convenient action is available, thanks to Deser–Zumino [3], Brink–Di Vecchia–Howe [4] and Howe–Tucker [5], by introducing an auxiliary worldvolume metric  $h_{mn}$ :

$$S_{\text{Poly.}} = - \frac{1}{2} \int d^{p+1}\xi \sqrt{-h} \left[ h^{mn} \partial_m X^M \partial_n X_M + 1 - p \right]. \quad (4)$$

This action is often dubbed the Polyakov action. Integrating out the auxiliary worldvolume metric using its equation of motion,  $h_{mn} \equiv \partial_m X^M \partial_n X_M$  for  $p \neq 1$  or  $h_{mn} \propto \partial_m X^M \partial_n X_M$  for  $p = 1$ , the Polyakov action reduces to the Nambu–Goto action  $S_{\text{Poly.}} \equiv S_{\text{N.G.}}$ . Here and henceforth we denote the on-shell equality as well as gauge fixings by ‘ $\equiv$ ’ and the defining equality by ‘ $:=$ ’. Both the Nambu–Goto and the Polyakov actions (2), (4) is manifestly invariant under the  $(p + 1)$ -dimensional worldvolume diffeomorphisms.

In the present paper we generalize the Brink–Di Vecchia–Howe–Tucker–Polyakov method and construct an action whose characteristic features are, compared to the Polyakov action, the appearance of *gauge covariant derivatives* and a *Nambu-bracket squared potential*. After some gauge fixing

we show that our action can be identified as a lower dimensional gauge theory action based on the *Filippov–Lie algebra*.

Previous works on related topics include the light-cone gauge fixed action of a  $p$ -brane [6–9].<sup>1</sup> Taking the light-cone gauge means fixing the light-cone variable to a classical on-shell value. Hence the light-cone gauge action describes only a sector of classically fixed light-cone momentum and breaks the full background isometry. In contrast, our resulting action is covariant and the full background isometry survives. Furthermore, the covariant derivative in our action takes a different form and is based on the Filippov–Lie algebra.

In particular, applying our result to the Nambu–Goto action for a five-brane we obtain a Filippov three-algebra based gauge theory action in three dimensions. As we will see, the precise form of the gauge covariant derivative and the presence of the three-algebra squared potential are identical to the Bagger–Lambert–Gustavsson description of multiple M2-branes [10–13].

## 2 General analysis

Our prescription to generalize the Brink–Di Vecchia–Howe–Tucker–Polyakov method starts with dividing formally the  $p$ -brane worldvolume dimension into two parts,

$$1 + p = d + n, \tag{5}$$

which corresponds to the decomposition of the worldvolume coordinates into two sets:

$$\{ \xi^m \} = \{ \sigma^\mu, \varsigma^i \}, \tag{6}$$

where  $\mu = 0, 1, \dots, d - 1$  and  $i = 1, \dots, n$ . The decomposition here is *a priori* arbitrary as for any positive integers  $d, n$ . One natural application of the splitting will be the case where the  $p$ -brane is extended over both compact and non-compact directions: in this case we reserve  $\varsigma^i$  for compact directions and  $\sigma^\mu$  for non-compact directions, including time.

According to the splitting, the induced metric (3) decomposes into the following  $d \times d$ ,  $d \times n$  and  $n \times n$  blocks  $K, B, V$ , defined by

$$K_{\mu\nu} := \mathcal{G}_{\mu\nu}, \quad B_{\mu i} := \mathcal{G}_{\mu i}, \quad V_{ij} := \mathcal{G}_{ij}. \tag{7}$$

<sup>1</sup>The appearance of a gauge connection from diffeomorphism invariance is also well known in Kaluza–Klein theory; see [14] and references therein. However, the gauge field in Kaluza–Klein theory originates from the spacetime metric which is dynamical in gravity, while in our action the gauge field is introduced as a non-dynamical auxiliary variable.

The first crucial step in our formalism is to express the determinant of the  $(p + 1) \times (p + 1)$  induced metric as a product of two determinants of the smaller  $d \times d$  and  $n \times n$  matrices:

$$\det \mathcal{G}_{mn} = \det \tilde{K} \det V, \quad \tilde{K} := K - B V^{-1} B^T. \tag{8}$$

This follows from the following simple observation:

$$\begin{aligned} \det \begin{pmatrix} K & B \\ B^T & V \end{pmatrix} &= \det \left[ \begin{pmatrix} 1 & -B V^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} K & B \\ B^T & V \end{pmatrix} \right] \\ &= \det \begin{pmatrix} \tilde{K} & 0 \\ B^T & V \end{pmatrix}. \end{aligned} \tag{9}$$

The resulting Nambu–Goto action (2),

$$S_{\text{N.G.}} = - \int d^{p+1} \xi \sqrt{-\det \tilde{K} \det V}, \tag{10}$$

can be now reformulated in a square root free form, if we introduce an auxiliary variable  $\rho$ :

$$\int d^{p+1} \xi \left( \rho \det \tilde{K} - \frac{1}{4} \rho^{-1} \det V \right). \tag{11}$$

To proceed, we introduce a  $d \times d$  auxiliary matrix  $\varphi_{\mu\nu}$  and apply the Brink–Di Vecchia–Howe–Tucker–Polyakov method to the determinant of  $\tilde{K}_{\mu\nu}$  in order to have our semi-final action:

$$\int d^{p+1} \xi \left[ \rho \det \varphi \left( \varphi^{\mu\nu} \tilde{K}_{\mu\nu} + 1 - d \right) - \frac{1}{4} \rho^{-1} \det V \right]. \tag{12}$$

At this point it is convenient to reparameterize the auxiliary variables  $\rho, \varphi_{\mu\nu}$  by a new auxiliary scalar  $\omega$  and a  $d$ -dimensional ‘worldvolume metric’  $h_{\mu\nu}$ :

$$\omega^{-1} := \left( -\rho^2 \det \varphi \right)^{\frac{1}{d-2}}, \tag{13}$$

$$h_{\mu\nu} := \left( -\rho^2 \det \varphi \right)^{\frac{1}{d-2}} \varphi_{\mu\nu}.$$

Now we are ready to spell our novel action, which we propose in order to reformulate the Nambu–Goto action for a  $p$ -brane:

$$\begin{aligned} S_{\text{New}} &= \int d^d \sigma \text{Tr} \left( \sqrt{-h} \mathcal{L}_{\text{New}} \right), \quad \text{Tr} := \int d^n \varsigma, \\ \mathcal{L}_{\text{New}} &= -h^{\mu\nu} D_\mu X^M D_\nu X_M - \frac{1}{4} \omega^{d-1} \det V + (d - 1) \omega. \end{aligned} \tag{14}$$

In addition to  $\omega$  and  $h_{\mu\nu}$ , here we introduced one more auxiliary field  $A_\mu^i$  which defines the ‘covariant derivative’:

$$D_\mu X^M := \partial_\mu X^M - A_\mu^i \partial_i X^M. \tag{15}$$

The corresponding field strength reads

$$F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i - A_\mu^j \partial_j A_\nu^i + A_\nu^j \partial_j A_\mu^i. \tag{16}$$

In terms of the Nambu  $n$ -bracket which is defined by [15]

$$\{Y_1, Y_2, \dots, Y_n\}_{\text{N.B.}} := \epsilon^{i_1 i_2 \dots i_n} \partial_{i_1} Y_1 \partial_{i_2} Y_2 \dots \partial_{i_n} Y_n, \quad (17)$$

the ‘potential’  $\det V$  takes the form<sup>2</sup>

$$\det V = \frac{1}{n!} \{X^{M_1}, X^{M_2}, \dots, X^{M_n}\}_{\text{N.B.}} \times \{X_{M_1}, X_{M_2}, \dots, X_{M_n}\}_{\text{N.B.}}. \quad (18)$$

In the above  $\epsilon^{i_1 i_2 \dots i_n}$  is the totally anti-symmetric  $n$ -dimensional tensor of the normalization  $\epsilon^{12 \dots n} = 1$ .

The auxiliary variables assume the on-shell values

$$A_\mu^i \equiv (BV^{-1})_\mu^i \quad \omega^{2-d} \equiv \frac{1}{4} \det V, \quad (19)$$

$$h_{\mu\nu} \equiv \omega^{-1} \tilde{K}_{\mu\nu}.$$

Plugging these into the action (14), we recover the Nambu–Goto action (10),  $S_{\text{New}} \equiv S_{\text{N.G.}}$ . In particular, we have the following on-shell relations:

$$\partial_i X^M D_\mu X_M \equiv 0, \quad D_\mu X^M D_\nu X_M \equiv \tilde{K}_{\mu\nu}. \quad (20)$$

The former is nothing but the Euler–Lagrange equation for  $A_\mu^i$ , which is solved by  $A \equiv BV^{-1}$  and prescribes that  $D_\mu X^M$  should be orthogonal to  $\partial_i X^M$  on-shell. The latter holds since  $P_N^M := \delta_N^M - \partial_i X^M (V^{-1})^{ij} \partial_j X_N$  is a projector satisfying  $P^2 = P$ . Note also that  $P_N^M \partial_i X^N = 0$  and  $P_N^M \partial_\mu X^N \equiv D_\mu X^M$ .

Although not manifest, our novel action (14) enjoys the full  $(p + 1)$ -dimensional diffeomorphism symmetry like the Nambu–Goto action (2), irrespective of the arbitrary splitting of the worldvolume coordinates. Under an arbitrary infinitesimal coordinate transformation,  $\delta \xi^m = -v^m$  or  $\delta \partial_m = \partial_m v^n \partial_n$ , all the fields transform as

$$\begin{aligned} \delta X^M &= 0, \\ \delta A_\mu^i &= D_\mu v^\nu A_\nu^i + D_\mu v^i + \frac{1}{4} \omega^{d-1} \det V h_{\mu\nu} \partial_j v^\nu V^{-1ji}, \\ \delta \omega &= -\frac{2}{d-2} \omega (\partial_i v^\lambda A_\lambda^i + \partial_i v^i), \\ \delta h_{\mu\nu} &= D_\mu v^\lambda h_{\lambda\nu} + D_\nu v^\lambda h_{\mu\lambda} \\ &\quad + \frac{2}{d-2} (\partial_i v^\lambda A_\lambda^i + \partial_i v^i) h_{\mu\nu}. \end{aligned} \quad (21)$$

Note that this transformation rule is consistent with the on-shell relations (19), and further that we assume the ‘active’ form of the diffeomorphism. The dual ‘passive’ diffeomorphism, which is directly relevant to the Noether symmetry,

<sup>2</sup>For the curved target spacetime manifold having the metric  $G_{MN}(X)$ , one should bear in mind that  $D_\mu X_N = D_\mu X^M G_{MN}$  and  $\{X_{N_1}, X_{N_2}, \dots, X_{N_n}\}_{\text{N.B.}} = \{X^{M_1}, X^{M_2}, \dots, X^{M_n}\}_{\text{N.B.}} G_{M_1 N_1} G_{M_2 N_2} \dots G_{M_n N_n}$ .

is given by  $\delta_{\text{pssv.}} \partial_m = 0$  and  $\delta_{\text{pssv.}} \Phi = \delta_{\text{activ.}} \Phi + v^m \partial_m \Phi$  for each field  $\Phi$ .

Apparently from (13), the above formalism is singular if  $d = 2$ , essentially due to the Weyl invariance in two dimensions. In this case, we return to (12); let  $h_{\mu\nu} := \varphi_{\mu\nu}$  and introduce a dilaton  $e^{-\phi} := \rho \sqrt{-h}$ . The proposed action for  $d = 2$  case becomes, rather than (14):

$$\begin{aligned} S_{\text{New}}^{d=2} &= \int d^2 \sigma \text{Tr} \left( \sqrt{-h} \mathcal{L}_{\text{New}}^{d=2} \right), \quad \text{Tr} := \int d^n \zeta, \\ \mathcal{L}_{\text{New}}^{d=2} &= -e^{-\phi} h^{\mu\nu} D_\mu X^M D_\nu X_M - \frac{1}{4} e^\phi \det V + e^{-\phi}. \end{aligned} \quad (22)$$

Like (19) the auxiliary variables assume the following on-shell values:

$$A_\mu^i \equiv (BV^{-1})_\mu^i, \quad e^{-2\phi} \equiv \frac{1}{4} \det V, \quad (23)$$

$$h_{\mu\nu} \equiv \tilde{K}_{\mu\nu}.$$

Plugging these into the action (22) we recover the Nambu–Goto action (2) again. The full  $(p + 1)$ -dimensional diffeomorphism symmetry has the following two-dimensional realization:

$$\begin{aligned} \delta X^M &= 0, \\ \delta A_\mu^i &= D_\mu v^\nu A_\nu^i + D_\mu v^i + \frac{1}{4} e^{2\phi} \det V h_{\mu\nu} \partial_j v^\nu V^{-1ji}, \\ \delta \phi &= -\partial_i v^\lambda A_\lambda^i - \partial_i v^i, \\ \delta h_{\mu\nu} &= D_\mu v^\lambda h_{\lambda\nu} + D_\nu v^\lambda h_{\mu\lambda}. \end{aligned} \quad (24)$$

Although the action (14) is still valid except  $d = 2$ , the case of  $d = 1$  is special: the auxiliary scalar  $\omega$  drops from the action as well as from the diffeomorphism transformations. In other words, when  $d = 1$ , we need only two types of auxiliary fields to take off the square root of the Nambu–Goto action of a  $p$ -brane: an einbein  $e$  and a gauge field  $A_\tau^i$ ,  $i = 1, 2, \dots, p$ . With a worldline parameter  $\tau$ , the action (14) reduces to

$$S_{\text{New}}^{d=1} = \int d\tau \text{Tr} \left( e^{-1} D_\tau X^M D_\tau X_M - \frac{1}{4} e \det V \right). \quad (25)$$

The on-shell values of the auxiliary fields are then

$$A_\tau^i \equiv (BV^{-1})_\tau^i, \quad e \equiv 2\sqrt{-D_\tau X^M D_\tau X_M / \det V}. \quad (26)$$

In the case of  $d = 1$  the full  $(p + 1)$ -dimensional diffeomorphism takes the following form:

$$\begin{aligned} \delta X^M &= 0, \\ \delta A_\tau^i &= D_\tau v^\tau A_\tau^i + D_\tau v^i - \frac{1}{4} e^2 \det V \partial_j v^\tau V^{-1ji}, \\ \delta e &= e (D_\tau v^\tau - A_\tau^i \partial_i v^\tau - \partial_i v^i). \end{aligned} \quad (27)$$

### 3 Gauge fixing to Filippov–Lie $n$ -algebra

Although our resulting actions for a  $p$ -brane, (14) for  $d \geq 3$ , (22) for  $d = 2$  and (25) for  $d = 1$ , are written in the form of a  $d$ -dimensional gauge theory with  $d$  being less than  $p + 1$ , they are invariant under the full  $(p + 1)$ -dimensional diffeomorphism. They are still identified as  $(p + 1)$ -dimensional models. In order to be identified as genuine lower dimensional gauge theories, it is necessary to break the full  $(p + 1)$ -dimensional diffeomorphism to a direct product of the  $d$ -dimensional diffeomorphism and the  $n$ -dimensional volume preserving diffeomorphism. The latter then corresponds to a local gauge symmetry of the  $d$ -dimensional action. In fact, for each value of  $d$  we can impose a pair of gauge fixing conditions:<sup>3</sup>

- For  $d \geq 3$ ,

$$\partial_i A_\mu^i \equiv 0, \quad \omega \equiv 1. \tag{28}$$

The unbroken local symmetry is then the direct product of the  $d$ -dimensional diffeomorphism and the  $n$ -dimensional volume preserving gauge symmetry, generated by the infinitesimal transformations satisfying  $\partial_i v^\mu = 0$  and  $\partial_j v^j = 0$ .

- For  $d = 2$ ,

$$\partial_i A_\mu^i \equiv 0, \quad \phi \equiv 0. \tag{29}$$

The unbroken local symmetry is the direct product of the two-dimensional diffeomorphism and the  $(p - 1)$ -dimensional volume preserving gauge symmetry.

- For  $d = 1$ ,

$$\partial_i A_\tau^i \equiv 0, \quad e \equiv 2. \tag{30}$$

As we fix the einbein, the unbroken local gauge symmetry is given by the  $p$ -dimensional volume preserving diffeomorphism only.

In each case, from (21), (24) and (27), the former  $d$ -number of conditions can be essentially achieved by a diffeomorphism with the  $d$ -number of  $v^\mu$  generators satisfying  $\partial_i v^\mu \neq 0$ , while the latter single condition can be met by  $\partial_i v^i \neq 0$ .

The divergence free condition  $\partial_i A_\mu^i \equiv 0$  must be imposed once we demand the covariant derivative  $D_\mu = \partial_\mu - A_\mu^i \partial_i$  to be an anti-Hermitian differential operator, allowing for the usual *integration by parts*. Furthermore, the volume preserving diffeomorphism generators also satisfy the divergence

<sup>3</sup>When  $d = 1$ , besides (30), it is also possible to set  $e \equiv 2$  and  $A_\tau^i \equiv 0$  for all  $i = 1, 2, \dots, p$ , utilizing the full  $(p + 1)$ -dimensional world-volume diffeomorphism. Then the case of  $p = 1$  coincides with the well known conformally gauge fixed Polyakov string action. We thank Kanghoon Lee for pointing this out [16].

free condition  $\partial_i v^i = 0$ . That is to say, as usual, the gauge connection assumes the same ‘‘Lie algebra’’ value as the volume preserving gauge symmetry generators.

Now it is crucial to note that the volume preserving gauge symmetry generator as well as the covariant derivative can be represented by the Nambu  $n$ -bracket:<sup>4</sup> with the functional basis  $T^a(\zeta)$ ,  $a = 1, 2, 3, \dots$ , for the  $n$ -dimensional manifold we have

$$\begin{aligned} v^i \partial_i &= v_{a_1 a_2 \dots a_{n-1}} \{T^{a_1}, T^{a_2}, \dots, T^{a_{n-1}}\}_{\text{N.B.}}, \\ D_\mu &= \partial_\mu - A_{\mu a_1 a_2 \dots a_{n-1}} \{T^{a_1}, T^{a_2}, \dots, T^{a_{n-1}}\}_{\text{N.B.}}. \end{aligned} \tag{31}$$

Note that here  $v_{a_1 a_2 \dots a_{n-1}}$  and  $A_{\mu a_1 a_2 \dots a_{n-1}}$  are  $d$ -dimensional fields, being independent of the  $\zeta^i$  coordinates. Furthermore, the  $n$ -dimensional manifold is assumed to be compact.

As is well known (see e.g. [17, 18]), the Nambu  $n$ -bracket provides an explicit realization of an infinite dimensional Filippov–Lie  $n$ -algebra [19] defined by the  $n$ -bracket satisfying the totally anti-symmetric property:

$$\begin{aligned} [X_1, \dots, X_i, \dots, X_j, \dots, X_n] \\ = -[X_1, \dots, X_j, \dots, X_i, \dots, X_n], \end{aligned} \tag{32}$$

and the Leibniz rule, also known as a fundamental identity:

$$\begin{aligned} [X_1, \dots, X_{n-1}, [Y_1, \dots, Y_n]] \\ = \sum_{j=1}^n [Y_1, \dots, [X_1, \dots, X_{n-1}, Y_j], \dots, Y_n]. \end{aligned} \tag{33}$$

In the Nambu-bracket representation of a Filippov–Lie algebra, we may employ the structure constant through

$$\{T^{a_1}, T^{a_2}, \dots, T^{a_n}\}_{\text{N.B.}} = f^{a_1 a_2 \dots a_n}_b T^b. \tag{34}$$

The structure constant is then totally anti-symmetric for the upper indices and satisfies, from the Leibniz rule (33),

$$\begin{aligned} f^{a_1 a_2 \dots a_n}_c f^{b_1 b_2 \dots b_n}_{a_n} \\ = \sum_{j=1}^n f^{a_1 a_2 \dots a_{n-1} b_j}_e f^{b_1 \dots b_{j-1} e b_{j+1} \dots b_n}_c. \end{aligned} \tag{35}$$

Now from (31) and (34), expanding the dynamical variables by the functional basis  $X^M(\sigma, \zeta) = X^M_a(\sigma) T^a(\zeta)$ , the covariant derivative can be rewritten as

$$\begin{aligned} D_\mu X^M &= (D_\mu X^M)_a T^a, \\ (D_\mu X^M)_a &= \partial_\mu X^M_a - X^M_b \tilde{A}^b_{\mu a}, \end{aligned} \tag{36}$$

<sup>4</sup>From the Poincaré lemma the divergence free volume preserving generator is given by  $v^i \partial_i = \epsilon^{i_1 i_2 \dots i_n} \partial_{i_1} \hat{v}_{i_2 \dots i_{n-1}} \partial_{i_n}$ , which can be further organized to take the form (31).

where we set

$$\tilde{A}_{\mu a}^b := A_{\mu c_1 c_2 \dots c_{n-1}} f^{c_1 c_2 \dots c_{n-1} b}{}_a. \tag{37}$$

In this way, after the gauge fixings, our final actions (14), (22), (25) reduce to genuine lower dimensional Filippov–Lie  $n$ -algebra based gauge theory actions, where the potential is given by the  $n$ -Lie bracket squared (18) and the covariant derivative is given by (36). Furthermore, at this point, we may generalize the actions to assume an arbitrary (finite or infinite dimensional) Filippov–Lie  $n$ -algebra as a gauge symmetry. With  $\tilde{v}^b{}_a := v_{c_1 c_2 \dots c_{n-1}} f^{c_1 c_2 \dots c_{n-1} b}{}_a$ , from the passive transformation of (21) and the expression (31), the Filippov–Lie  $n$ -algebra based gauge transformation is given by

$$\begin{aligned} \delta X_a^M &= X_b^M \tilde{v}^b{}_a, \\ \delta A_{\mu a_1 a_2 \dots a_{n-1}} &= \partial_\mu v_{a_1 a_2 \dots a_{n-1}} + (-1)^n (n-1) A_{\mu c [a_1 a_2 \dots a_{n-2} \tilde{v}^c}_{a_{n-1}]} \end{aligned} \tag{38}$$

of which the latter induces, from (35),

$$\delta \tilde{A}_{\mu a}^b = \partial_\mu \tilde{v}^b{}_a - \tilde{v}^b{}_c \tilde{A}_{\mu a}^c + \tilde{A}_{\mu c}^b \tilde{v}^c{}_a. \tag{39}$$

Especially, taking  $n = 3$ , equations (36) and (37) precisely coincide with the definition of the covariant derivative in the Bagger–Lambert–Gustavsson description of multiple M2-branes *via* the Filippov three-algebra gauge interaction [10, 13].

#### 4 Comments

A Filippov–Lie  $n$ -algebra is normally equipped with a bilinear inner product. This might be a potential problem whilst identifying our final actions (14), (22), (25) after the gauge fixing (28), (29), (30) as a  $d$ -dimensional gauge theory based on a genuine Filippov–Lie  $n$ -algebra, since the actions are not generically quadratic. For example, the kinetic term reads

$$\sqrt{-h} h^{\mu\nu} D_\mu X^M D_\nu X_M.$$

Again the Nambu-bracket provides a solution by simply generalizing the bi-linear inner product to multi-linear inner products or the “trace” [20]:

$$\text{Tr}(T^a T^b \dots T^c) = \int d^n \zeta T^a T^b \dots T^c, \tag{40}$$

which is invariant under the Filippov–Lie  $n$ -algebra gauge transformation. For arbitrary  $m = 1, 2, 3, \dots$ ,

$$\begin{aligned} \sum_{k=1}^m \text{Tr}(Y_1, Y_2, \dots, Y_{k-1}, [X_1, \dots, X_{n-1}, Y_k], \\ Y_{k+1}, \dots, Y_m) = 0, \end{aligned} \tag{41}$$

or equivalently

$$\begin{aligned} \sum_{k=1}^m \int f^{a_1 a_2 \dots a_{n-1} b_k}{}_c \\ \times \text{Tr}(T^{b_1} T^{b_2} \dots T^{b_{k-1}} T^c T^{b_{k+1}} \dots T^{b_m}) = 0. \end{aligned} \tag{42}$$

Our work manifests the general phenomenon, commonly known as Myers effect [21], that non-Abelian structure of lower dimensional gauge theories can capture the description of a higher dimensional brane:<sup>5</sup> a single  $p$ -brane can be described not only by a  $(p + 1)$ -dimensional Polyakov action but also by a gauged  $d$ -dimensional Polyakov action based on Filippov–Lie  $n$ -algebra with  $p + 1 = d + n$ . Since the functional basis of the  $n$ -dimensional manifold is infinite dimensional, the corresponding gauge group based on the Filippov–Lie  $n$ -algebra is *a priori* infinite dimensional. However, we emphasize that our final action admits a simple generalization taking *any* Filippov–Lie algebra as a gauge symmetry.<sup>6</sup>

If we turn off the Filippov–Lie  $n$ -algebra gauge interaction, our  $d$ -dimensional action corresponds simply to a Polyakov action for  $(d - 1)$ -brane. This suggests the following physical picture behind our formalism: the description of a single  $p$ -brane as a condensation of infinitely many lower dimensional branes through Filippov–Lie algebra gauge interactions.

In particular, the action (25) provides a description of a  $p$ -brane *via* infinitely many interacting relativistic point particles (see [26] for a related earlier work). Especially if we apply our formalism to an M2-brane in eleven dimensions we obtain, with the choice of  $d = 1$ ,

$$\begin{aligned} S_{M2} = \int d\tau \text{Tr} \left( e^{-1} D_\tau X^M D_\tau X_M \right. \\ \left. - \frac{1}{8} e [X^M, X^N] [X_M, X_N] \right). \end{aligned} \tag{43}$$

Since there are eleven scalars  $M = 0, 1, 2, \dots, 10$ , this action corresponds to a covariant version of the  $\mathcal{M}$ -theory matrix model [27] (see also [28]), without the light-cone gauge fixing.

Furthermore, in the case of  $p = 5$  and  $d = n = 3$ , our results have common features with the Bagger–Lambert–Gustavsson description of multiple M2-branes [10, 13]: the Filippov three-algebra naturally arises, the definition of the covariant derivative precisely coincides and the potential is given by the three-bracket squared. This supports the idea

<sup>5</sup>Since the trace (40) is invariant under any permutation of its arguments, the word ‘non-Abelian’ might be improper. A more relevant structure appears to be the Filippov–Lie  $n$ -algebra.

<sup>6</sup>For the discussion on the uniqueness of finite dimensional Filippov–Lie algebra, see [22–25].

that the Bagger–Lambert–Gustavsson action with infinite dimensional gauge group may describe a M5-brane as a condensation of infinitely many interacting M2-branes, as explored in [29–37].

**Acknowledgements** We wish to thank Xavier Bekaert, Kanghoon Lee, Dmitri Sorokin for useful comments and especially Choonkyu Lee for encouraging us to look for the full diffeomorphism. The work is in part supported by the Center for Quantum Spacetime of Sogang University with grant number R11-2005-021, and also by the Korea Science and Engineering Foundation grant funded by the Korea government (R01-2007-000-20062-0).

## References

1. Y. Nambu, Lectures at the Copenhagen symposium (1970)
2. T. Goto, Prog. Theor. Phys. **46**, 1560 (1971)
3. S. Deser, B. Zumino, Phys. Lett. B **65**, 369 (1976)
4. L. Brink, P. Di Vecchia, P.S. Howe, Phys. Lett. B **65**, 471 (1976)
5. P.S. Howe, R.W. Tucker, J. Phys. A **10**, L155 (1977)
6. J. Hoppe, PhD thesis. MIT (1982)
7. J. Hoppe, Elem. Part. Res. J. (Kyoto) **80**, 145 (1989)
8. J. Hoppe, Helv. Phys. Acta **70**, 302 (1997). [arXiv:hep-th/9602020](#)
9. E. Bergshoeff, E. Sezgin, Y. Tani, P.K. Townsend, Ann. Phys. **199**, 340 (1990)
10. J. Bagger, N. Lambert, Phys. Rev. D **75**, 045020 (2007). [arXiv:hep-th/0611108](#)
11. J. Bagger, N. Lambert, Phys. Rev. D **77**, 065008 (2008). [arXiv:0711.0955](#) [hep-th]
12. J. Bagger, N. Lambert, J. High Energy Phys. **0802**, 105 (2008). [arXiv:0712.3738](#) [hep-th]
13. A. Gustavsson, Nucl. Phys. B **811**, 66 (2009). [arXiv:0709.1260](#) [hep-th]
14. A. Salam, J.A. Strathdee, Ann. Phys. **141**, 316 (1982)
15. Y. Nambu, Phys. Rev. D **7**, 2405 (1973)
16. K. Lee, J.-H. Park, In preparation for the case of  $p > 1$
17. L. Takhtajan, Commun. Math. Phys. **160**, 295 (1994). [arXiv:hep-th/9301111](#)
18. H. Awata, M. Li, D. Minic, T. Yoneya, J. High Energy Phys. **0102**, 013 (2001). [arXiv:hep-th/9906248](#)
19. V.T. Filippov, n-Lie algebras. Sib. Mat. Zh. **26**(6), 126–140 (1985)
20. K. Lee, J.-H. Park, J. High Energy Phys. **0904**, 012 (2009). [arXiv:0902.2417](#) [hep-th]
21. R.C. Myers, J. High Energy Phys. **9912**, 022 (1999). [arXiv:hep-th/9910053](#)
22. J.M. Figueroa-O'Farrill, G. Papadopoulos, [arXiv:math/0211170](#)
23. G. Papadopoulos, J. High Energy Phys. **0805**, 054 (2008). [arXiv:0804.2662](#) [hep-th]
24. J.P. Gauntlett, J.B. Gutowski, [arXiv:0804.3078](#) [hep-th]
25. G. Papadopoulos, Class. Quantum Gravity **25**, 142002 (2008). [arXiv:0804.3567](#) [hep-th]
26. H.S. Yang, I. Kim, B.H. Lee, Phys. Rev. D **58**, 085018 (1998). [arXiv:hep-th/9806112](#)
27. T. Banks, W. Fischler, S.H. Shenker, L. Susskind, Phys. Rev. D **55**, 5112 (1997). [arXiv:hep-th/9610043](#)
28. C. Sochichiu, Lect. Not. Phys. **698**, 189 (2006). [arXiv:hep-th/0506186](#)
29. P.M. Ho, Y. Matsuo, J. High Energy Phys. **0806**, 105 (2008). [arXiv:0804.3629](#) [hep-th]
30. P.M. Ho, Y. Imamura, Y. Matsuo, S. Shiba, J. High Energy Phys. **0808**, 014 (2008). [arXiv:0805.2898](#) [hep-th]
31. C. Krishnan, C. Maccaferri, J. High Energy Phys. **0807**, 005 (2008). [arXiv:0805.3125](#) [hep-th]
32. J. Gomis, G. Milanesi, J.G. Russo, J. High Energy Phys. **0806**, 075 (2008). [arXiv:0805.1012](#) [hep-th]
33. S. Benvenuti, D. Rodriguez-Gomez, E. Tonni, H. Verlinde, J. High Energy Phys. **0901**, 078 (2009). [arXiv:0805.1087](#) [hep-th]
34. P.M. Ho, Y. Imamura, Y. Matsuo, J. High Energy Phys. **0807**, 003 (2008). [arXiv:0805.1202](#) [hep-th]
35. I. Jeon, J. Kim, N. Kim, S.W. Kim, J.-H. Park, J. High Energy Phys. **0807**, 056 (2008). [arXiv:0805.3236](#) [hep-th]
36. I.A. Bandos, P.K. Townsend, Class. Quantum Gravity **25**, 245003 (2008). [arXiv:0806.4777](#) [hep-th]
37. I.A. Bandos, Phys. Lett. B **669**, 193 (2008). [arXiv:0808.3568](#) [hep-th]