A Generalization of the Fourier Transform and Applications to Quantum Field Theory

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Abstract

Using a generalization of the usual Fourier transform on Minkowski space we demonstrate an $SO_0(1,4)$ ($SO_0(3,2)$) equivalence between a massless spin 0 or spin 1/2 particle on de Sitter space (anti-de Sitter space) and corresponding particles of mass $-\frac{1}{4R^2}(\frac{1}{4R^2})$ or $-\frac{2}{R^2}(\frac{2}{R^2})$, respectively. Using these results we consider an interpretation of Feynman's theory of relativistic cut-off as a theory of interaction of matter with massless virtual bosons in de Sitter or anti-de Sitter space. This interpretation leads to some interesting results about the electromagnetic mass differences of hadrons.

1 Introduction

Conformal invariance of the laws of physics places four space-time manifolds, Minkowski space, de Sitter space, anti-de Sitter space and the universal cosmos, on a certain equal footing, in that the actions of the conformal group on them are projective ones, and in this sense they share a certain uniqueness [1]. These manifolds and the corresponding actions of the conformal group occur naturally in various decompositions of the conformal group and in various associated parallelizations, respectively [2]. Although additional physical principles such as causality and positivity of energy eliminate some of these space-time models as plausible ones [3], we wish to consider only conformal invariance plus as few as possible additional postulates, which are physically necessary, in describing virtual particles and perhaps short-lived resonances. If we only demand conformal invariance there is nothing which prevents us from taking the view that such particles have their existences in any of the space-time models mentioned above. However, it is necessary to postulate that the usual definition of an elementary particle applies to these particles, namely, that an elementary particle is given by a ray in an irreducible projective ray representation of the Poincaré group. Now we have shown that a free particle in de Sitter space (V^4) or anti-de Sitter space (AdS) fulfills this definition of an elementary particle [4]. Thus virtual particles can be considered as particles in Minkowski space, (M^4), or V^4 or AdS. (We can also view them as particles in the universal cosmos, analogously; however the essential ingredient, as we shall see, is the introduction of a radius, and this information is already obtained by considering V^4 or AdS.)

In order to describe interactions of such virtual particles with real particles in the Minkowski space of our objective reality, we introduce a generalized Fourier transform, which transfers quantized fields on V^4 or AdS to corresponding fields on Minkowski space. Massless fields on V^4 are associated to tachyons in M^4, and those on AdS are associated to real (positive) mass particles in M^4. Our main results are summarized in the table. Using these results we consider the theory of the interaction of matter with the corresponding *massive* virtual particles (tachyons in the V^4 case and real mass particles in the AdS case) as describing the interaction of matter in M^4 with massless virtual particles in V^4 or AdS. We conclude with some applications to the electromagnetic structure of hadrons.

2 The Generalized Fourier Transform

We refer to reference [2] for the standard definitions of V^4 and AdS and also for the description of the representations of the conformal group and its $SO_0(3,1)$ and $SO_0(4,1)$ subgroups on V^4 and AdS. The space of solutions of the "mass" zero wave equation on V^4 (AdS),

$$\left(\tilde{\Box} + \frac{k}{6} + \mu^2\right) f(\xi) = 0 \quad \left(k = \frac{12}{R^2}\right), \tag{1}$$

($\xi \in V^4$ (AdS) and $f\colon V^4 \to \mathbf{C}$ ($f\colon \text{AdS} \to \mathbf{C}$), $\tilde{\Box}$ is the Laplace Beltrami operator on V^4 (AdS)) defines an invariant subspace of the representation space of the conformal group, whose complement is not invariant for $\mu = 0$ (indecomposable representation). Note that the restriction of the representation of the conformal group to $SO_0(1,4)$ ($SO_0(3,2)$) is just a linear representation of the subgroup. We may also construct a multiplier representation of $SO_0(1,4)$ ($SO_0(3,1)$) on T^3, the unit mass hyperboloid [4]; and the equivalence between this $SO_0(1,4)$ ($SO_0(3,1)$) representation on T^3 with the corresponding one on the space of solutions of the zero mass wave equation on V^4 (AdS) is established with the help of the generalized Fourier transform:

$$f(\xi) = (\pi^\nu \psi)(\xi) = c^\lambda(\nu) \int_{T^3} dT^3\, \psi\left(\frac{p}{m}\right) |\mu(g,p)|^{-\nu-\frac{3}{2}} \quad (\lambda = \frac{1}{R};\, \nu = \frac{m}{R}) \tag{2}$$

Here $\psi\colon T^3 \to \mathbf{C}$, $\frac{p}{m} \in T^3$ and $c^\lambda(\nu)$ is a constant which is the Plancherel measure on V^4 or AdS. We have the following key result [6]:

$$\begin{aligned}&\left(\tilde{\Box}_s \pi^\nu \phi\right)(\xi)\\ &= \left[\pi^\nu\left\{B_\mu B^\mu - \frac{\lambda^2}{2} L_{\mu\nu} L^{\mu\nu}\right\}\phi\right](\xi) = \left[\pi^\nu\left\{P^2 + \frac{9}{4}\lambda^2 - \lambda^2 s(s+1)\right\}\phi\right](\xi)\end{aligned} \tag{3}$$

($P^2 = P_\mu P^\mu = M^2$) where

$$B_\mu = P_\mu + \frac{\lambda}{2m}\{P^\rho, L_{\rho\mu}\}, \quad L_{\mu\nu} = M_{\mu\nu} + S_{\mu\nu}. \tag{4}$$

This defines an equivalence at least for $s = 0$ and $s = \frac{1}{2}$: for $s = 0$, $\tilde{\Box}_s = \tilde{\Box}$ and $\tilde{\Box}_{1/2} = \frac{1}{2R^2} L_{ab} L^{ab}$ with $L_{ab} = M_{ab} + S_{ab}$. Using this one readily establishes the results compiled in the table.

Table: Some Quantities for Massless Spin Zero and Spin $\frac{1}{2}$ Fields on V^4 and AdS.

		V^4 or AdS	associated Momentum space	Minkowski space
wave equation	$s=0$	$(\tilde{\Box}+\frac{2}{R^2})f(\xi)=0$	$(P^2+\frac{1}{4R^2})\phi(p)=0$	$(\Box-\frac{1}{4R^2})\phi(x)=0$
	$s=\frac{1}{2}$	$D^{1/2}f(\xi)=0$	$(P^2+\frac{2}{R^2})\phi(p)=0$	$(\Box-\frac{2}{R^2})\phi(x)=0$
mass	$s=0$	$\tilde{m}^2=0$	$m^2=\tilde{m}^2-\frac{1}{4R^2}$	$m^2=\tilde{m}^2-\frac{1}{4R^2}$
	$s=\frac{1}{2}$	$\tilde{m}^2=0$	$m^2=\tilde{m}^2-\frac{2}{R^2}$	$m^2=\tilde{m}^2-\frac{2}{R^2}$

($D^{1/2}=\frac{1}{2R^2}L_{ab}L^{ab}-\frac{1}{2R^2}$, $D^{1/2}f(\xi)=0$ is the conformal invariant wave equation.)

3 Applications to Quantum Field Theory

We may summarize our results as follows: a massless spin zero or spin 1/2 field in de Sitter (anti-de Sitter) space is associated with a massive spin zero or spin 1/2 field of mass^2 $-\frac{1}{4R^2}(+\frac{1}{4R^2})$ or $-\frac{2}{R^2}(+\frac{2}{2R^2})$, respectively. Denote a free scalar field of mass^2 $-\frac{1}{4R^2}$ by $\phi(x;R)$, where now R may also be imaginary, in order to include the AdS case. Add to the usual interaction Lagrangian, which describes the interaction of a charged spin 1/2 particle and a pseudo-scalar photon [5], the following additional term:

$$\mathcal{L}'(x)=ke\int\tilde{\psi}(x)\gamma_5\psi(x)\cdot\phi(x;R)\rho(R^2)dR^2\quad(k^2=-1)\tag{5}$$

We may interpret this additional term as describing the contribution from virtual photons, which have imaginary mass ($R^2>0$) or real mass ($R^2<0$), or equivalently, as describing the contribution from massless scalar de Sitter or andti-de Sitter fields. The choice of $\rho^2(R^2)=\delta(R^2+R_0^2)$ leads to Feynman's result for the self energy of the electron [5]:

$$\triangle m_{\mathrm{SE}}=\frac{\alpha}{4\pi}\frac{m}{8\pi}\left[\log\left(\frac{{\Lambda_0}^2}{m^2}\right)-\frac{1}{2}\right]\quad({\Lambda_0}^2=-\frac{1}{4{R_0}^2})\tag{6}$$

Classically, we find by a standard calculation [6],

$$\triangle m_{\mathrm{SE}}^{\mathrm{class}}=\frac{1}{4\pi}\int_0^\infty\left[T^{00}(x)+T^{00}(x;R^0)\right]d^3x=\frac{\alpha}{2}\left(\frac{\Lambda_0}{m}\right)m\tag{7}$$

($T^{\mu\nu}(x)$ is the usual energy-momentum tensor for the massless scalar field, and $T^{\mu\nu}(x;R^0)$ is the energy-momentum tensor for the massive scalar fiedl.) Thus we may view $\frac{1}{\Lambda_0}$ as an effective radius for a point charge.

Now we give some applications to the electromagnetic structure of hadrons. For the masses of the $\pi^\pm$ or $K^\pm$ we take the formula

$$m^\pm=m_0+\frac{(\triangle m)^2}{2m_0}\tag{8}$$

where m_0 is the mass of the corresponding natural particle i.e. π_0 or K_0. This formula is better suited for describing electromagnetic mass differences of bosons than the formula $m = m_0 + \triangle m$, since bosons obey the Klein-Gordon equation [6]. Next substitute (7) into (8). For Λ_0 real we obtain the correct sign difference for the $m^{\pm} - m_0$ of the pion system, and for Λ_0 pure imaginary we obtain the correct sign difference for the $m^{\pm} - m_0$ of the kaon system. We also obtain reasonable quantitative agreement from these simple classical arguments [6]. The quantum mechanical treatment is complicated by the fact that we must consider the quantum theory of interaction of matter with tachyons [7]. However we have been able to obtain the $\pi^{\pm} - \pi_0$ mass difference from a quantum mechanical treatment [6].

References

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