CONFORMAL GROUP, QUANTIZATION, AND THE KEPLER PROBLEM

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§1. INTRODUCTION. This is a report on some joint work with Shlomo Sternberg. We consider a variation on geometric quantization for the orthogonal groups $SO(2,n)$, realizing certain of their representations on the nonzero cotangent bundle of the $(n-1)$-sphere. Here the elliptic orbits of the Kepler problem (with collision orbits regularized) appear as $SO(2)$-orbits. Another viewpoint, related by a geometric Cayley transform, gives the hyperbolic orbits as $SO(1,1)$-orbits in the nonzero cotangent bundle of real hyperbolic $(n-1)$-space. This gives a correspondence between the classical bound states and the classical scattering states for the hydrogen atom.

Our group-theoretic considerations are valid with only minor changes for the unitary groups $U(2,n)$, the special unitary groups $SU(2,n)$, and the unitary symplectic groups $Sp(2,n)$. While there is a connection with the harmonic oscillator, the physical interpretations are not always so clear. In any case, here I just indicate the situation for $SO(2,n)$. Complete details will appear elsewhere.

§2. A NILPOTENT CO-ADJOINT ORBIT. Let $R^{2,n}$ denote the real vector space with standard basis $\{e_{-1}, e_0, e_1, \ldots, e_n\}$ and inner product $\langle u,v\rangle = u_{-1}v_{-1} + u_0v_0 - (u_1v_1 + \cdots + u_nv_n)$. $O(2,n)$ is the orthogonal group of $R^{2,n}$, $G = SO(2,n)$ denotes its identity component, and the alternating tensor square $\Lambda^2(R^{2,n})$ is identified with the Lie algebra $\mathfrak{g} = \mathfrak{o}(2,n)$ under

$$u \wedge v : x \mapsto \langle x,u\rangle v - \langle x,v\rangle u \ .$$

Here the adjoint representation is given by $Ad(g)(u \wedge v) = gu \wedge gv$.

If $\xi \in \mathfrak{g}$ let E_ξ denote its range. If E_ξ is 2-dimensional and totally isotropic, then $\xi^2 = 0$, and E_ξ projects onto $R^{2,0} = \text{span}\{e_{-1}, e_0\}$, so ξ has unique expression

$$(2.1)\qquad \xi = s(e_{-1} + p) \wedge (e_0 + q) \text{ where } \begin{cases} p,\ q \in R^{0,n} = \text{span}\{e_1, \ldots, e_n\} \\ \|p\|^2 = \|q\|^2 = -1,\ \langle p,q\rangle = 0 \end{cases}$$

All such ξ form a single $O(2,n)$-orbit. Here $s = \langle \xi, e_{-1} \wedge e_0 \rangle$, and that single orbit falls into two G-orbits as $s > 0$ or $s < 0$. We will use the orbit

$$(2.2) \qquad \mathcal{V} = \{ \xi \in \mathfrak{g} \text{ as in } (2.1) : s > 0 \}.$$

The semisimple Lie algebra $\mathfrak{g}$ is identified with its dual space $\mathfrak{g}^*$ under the Killing form, and we view $\mathcal{V}$ as a (co-adjoint) orbit of G on $\mathfrak{g}^*$. That gives $\mathcal{V}$ the structure of G-homogeneous symplectic manifold.

In the notation (2.1), think of q as a point on the unit sphere $S^{n-1} = \{ x \in R^{0,n} : \|x\|^2 = -1 \}$ and sp as an arbitrary nonzero cotangent vector to S^{n-1} at q. This identifies $\mathcal{V}$ with the bundle $T^+(S^{n-1})$ of nonzero cotangent vectors to S^{n-1}. In this identification, the subgroup

$$G_1 = SO(1,n) = \{ g \in G : ge_{-1} = e_{-1} \}$$

is visibly transitive on $T^+(S^{n-1})$, and thus on $\mathcal{V}$. Furthermore $\xi = s(e_{-1} + p) \wedge (e_0 + q) \longmapsto sp \wedge (e_0 + q)$ is a bijection of $\mathcal{V}$ onto the principal nilpotent coadjoint orbit of G_1, which is

$$(2.3) \qquad \mathcal{V}_1 = \{ \xi_1 \in \mathfrak{g}_1 : \dim E_{\xi_1} = 2 \text{ and } \dim(E_{\xi_1} \cap E_{\xi_1}^{\perp}) = 1 \}.$$

$\mathcal{V}$ now carries three symplectic structures: as co-adjoint orbit of G, from the natural symplectic structure on the cotangent bundle of S^{n-1}, and from the natural symplectic structure of $\mathcal{V}_1$. Here our result is

THEOREM. The three symplectic structures on $\mathcal{V}$ coincide. In particular, the natural symplectic structure on $T^+(S^{n-1})$ is invariant under the action of $G = SO(2,n)$.

§3. ORBITS FOR THE KEPLER PROBLEM. We have $R^{2,n} = R^{2,0} \oplus R^{0,n}$ as above, and the G-stabilizer of this splitting is the maximal compact subgroup $K = SO(2) \times SO(n)$. Here $SO(n)$ acts on $\mathcal{V}$ through its usual action on the tangent bundle $T(S^{n-1})$,

$$A : s(e_{-1} + p) \wedge (e_0 + q) \longmapsto s(e_{-1} + Ap) \wedge (e_0 + Aq),$$

and $SO(2)$ acts by rotations, the rotation r_φ through an angle φ sending $s(e_{-1} + p) \wedge (e_0 + q)$ to

$$\begin{aligned} &s(\cos\varphi\, e_{-1} + \sin\varphi\, e_0 + p) \wedge (-\sin\varphi\, e_{-1} + \cos\varphi\, e_0 + q) \\ &= s(e_{-1} + \cos\varphi\, p - \sin\varphi\, q) \wedge (e_0 + \sin\varphi\, p + \cos\varphi\, q). \end{aligned}$$

On (co)-tangent vectors of length s, this rotation r_φ is geodesic flow $f_{\varphi/s}$ at time φ/s. The infinitesmal generator of the geodesic flow $\{f_t\}$ is the vector field V_H corresponding (by exterior derivative and the symplectic form) to $H = -s^2/2$, so $\{r_\varphi\}$ has infinitesmal generator that is the Hamiltonian field for $(-2H)^{1/2} = s$. Since the SO(2)-orbits are the orbits of the geodesic flow, they are the elliptic orbits of the Kepler problem with collision orbits regularized.

Similarly $R^{2,n} = R^{1,1} \oplus R^{1,n-1}$ where $R^{1,1} = \text{span}\{e_{-1}, e_n\}$ and $R^{1,n-1} = \text{span}\{e_0, e_1, \ldots, e_{n-1}\}$. The G-stabilizer of this splitting is a two-component group with identity component $K' = SO(1,1) \times SO(1,n-1)$, and $\mathcal{U}$ is the union of three K'-invariant sets

$$\mathcal{U}^+ = \{t(e_{-1}+p)\wedge(e_n+q) : t>0,\ p,q \in R^{1,n-1}, \|p\|^2=-1, \|q\|^2=1, p\perp q, \langle e_0,q\rangle>0\},$$

$$\mathcal{U}^0 = \{\xi \in \mathcal{U} : E_\xi \cap R^{1,1} \neq 0\}, \text{ and}$$

$$\mathcal{U}^- = \{t(e_{-1}+p)\wedge(e_n+q) : t<0,\ p,q \in R^{1,n-1}, \|p\|^2=-1, \|q\|^2=1, p\perp q, \langle e_0,q\rangle<0\}.$$

Let H_+^{n-1} (resp. H_-^{n-1}) denote the real hyperbolic (n-1)-space that is is the sheet $\langle e_0,q\rangle > 0$ (resp. $\langle e_0,q\rangle < 0$) of the mass hyperboloid $\|q\|^2 = 1$ in $R^{1,n-1}$. Then $\mathcal{U}^+$ (resp. $\mathcal{U}^-$) is identified with its bundle $T^+(H_+^{n-1})$ (resp. $T^+(H_-^{n-1})$) of nonzero cotangent vectors, $\xi = t(e_{-1} + p) \wedge (e_n + q)$ corresponding to the vector tp of length $|t|$ at q. Here SO(1,n-1) acts through its usual action by isometries and SO(1,1) acts, as before, by hyperbolic rotations proportional to the geodesic flow. So the SO(1,1)-orbits on $\mathcal{U}^\pm$ are the hyperbolic orbits of the Kepler problem.

If one interprets the SO(2)-orbits on $\mathcal{U}$ as the classical bound states for the hydrogen atom, and the SO(1,1)-orbits on $\mathcal{U}^\pm$ as the scattering states, then the Cayley transform relating SO(2) to SO(1,1) gives a sort of correspondence between those states. The geometric picture for this Cayley transform comes from noting that the sign condition on $\langle e_0,q\rangle$ identifies H_+^{n-1} with the upper hemisphere of S^{n-1} and H_-^{n-1} with the lower hemisphere:

$$t(e_{-1}+p)\wedge(e_n+q) = t\langle e_0,q\rangle\{(e_{-1}+p)\wedge(e_0 + \langle e_0,q\rangle^{-1}(q-\langle e_0,q\rangle e_0+e_n))\}.$$

Another interesting picture comes from taking p for base point and tq for (co)-tangent vector.

§4. GEOMETRIC QUANTIZATION. We turn to the question of quantizing the action of $G = SO(2,n)$ on its co-adjoint orbit $\mathcal{V} = T^{+}(S^{n-1})$.

The standard Kostant-Souriau quantization procedure does not work here because there is no G-invariant polarization. In effect, a result of Ozeki and Wakimoto says that any such polarization would be a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}_{C}$, a result of mine would then say $\mathfrak{q} = \mathfrak{p}_{C}$ for some parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, and of course $\mathfrak{p}$ would necessarily have codimension n-1 in $\mathfrak{g}$. But the maximal parabolic subalgebras of $\mathfrak{g}$ are the stabilizers of null lines, which have codimension n, and the stabilizers of null planes, which have codimension 2n-1, and so $\mathfrak{p}$ does not exist.

There are several possibilities for circumventing this lack of polarizations:

(i) weaken the definition of polarization,

(ii) view $\mathcal{V}$ as a limit of polarized co-adjoint orbits,

(iii) use the Kostant-Sternberg-Blattner half-form method.

In the first approach, one takes the usual definition of invariant polarization as complex subalgebra $\mathfrak{q}$ of $\mathfrak{g}_{C}$, but no longer requires that $\mathfrak{q} + \bar{\mathfrak{q}}$ be an algebra; that is done implicitly in N. Woodhouse's report at this conference. In the second approach, one has a smooth family $\mathcal{V}_t$ of co-adjoint orbits with $\mathcal{V} = \mathcal{V}_0$, with representations π_t associated to $\mathcal{V}_t$ for $t \neq 0$, in such a way that one can make sense of $\pi_0 = \lim \pi_t$ and associate it to $\mathcal{V}$; in E. Onofri's report here, that is done for elliptic semisimple approximating orbits and holomorphic discrete series approximating representations, and I have a comment on this in §6 below. Sternberg and I use the third approach.

§5. HALF FORMS AND VARYING POLARIZATIONS. Let P denote the standard polarization on $T^{+}(S^{n-1})$; its maximal integral manifolds are the cotangent spaces with origin deleted. Then $G_1 = SO(1,n)$ is the stabilizer of P in $G = SO(2,n)$, and the G-translates of P are parameterized by the mass shell $H = \{x \in R^{2,n} : \|x\|^2 = 1\}$:

$$G/G_1 = SO(2,n)/SO(1,n) \cong SO(2,n)(e_{-1}) = H .$$

Given $x = ge_{-1} \in H$, let P_x denote the image $g(P)$. The half form method gives a family of Hilbert spaces $\mathcal{H}_x$, and nondegenerate pairings between them, stable under the action of G. Here G_1 has natural irreducible unitary representation ψ on $\mathcal{H} = \mathcal{H}_{e_{-1}}$ by standard geometric quantization using $P = P_{e_{-1}}$; in fact ψ is the principal series representation that corresponds to the trivial character on the minimal parabolic subgroup, and $\mathcal{H}$ is $L^2(S^{n-1})$. More generally, if $g \in G$ then g carries $\mathcal{H}$ to $\mathcal{H}_x$, $x = ge_{-1}$, and we pair this back to $\mathcal{H}$ using the half forms. Thus G acts on $L^2(S^{n-1})$, and this action π restricts to the representation ψ of G_1. **Sternberg and I still have to clarify some technical matters with the half form pairing here.**

§6. LIMIT METHOD. I'll close by exhibiting the representation π of G, corresponding to the co-adjoint orbit $\mathcal{U} = T^+(S^{n-1})$, as a limit of spherical principal series representations. This has the advantage of simplicity over Onofri's procedure with the holomorphic discrete series, but the disadvantage of obscuring the place of G_1 and $L^2(S^{n-1})$ as compared with the half form method.

Fix $\xi = (e_{-1}+e_{n-1})\wedge(e_0+e_n) \in \mathfrak{g}$. Its matrix is $\begin{pmatrix} J & 0 & -J \\ 0 & 0 & 0 \\ J & 0 & -J \end{pmatrix}$ where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then $\eta = -\frac{1}{4}(e_{-1}-e_{n-1})\wedge(e_0-e_n)$ is another nilpotent element of $\mathfrak{g}$. It has matrix $\frac{1}{4}\begin{pmatrix} -J & 0 & -J \\ 0 & 0 & 0 \\ J & 0 & J \end{pmatrix}$, and so $h = [\xi,\eta]$ has matrix $\begin{pmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ I & 0 & 0 \end{pmatrix}$ where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Now

$$[h,\xi] = 2\xi\,, \quad [h,\eta] = -2\eta \quad \text{and} \quad [\xi,\eta] = h\,.$$

So $\{h,\xi,\eta\}$ is a standard generating triple for a split three dimensional simple subalgebra (TDS) in $\mathfrak{g}$, that is

$$h \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \xi \longrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \eta \longrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

defines a Lie algebra isomorphism of $\mathrm{span}\{h,\xi,\eta\}$ onto $\mathfrak{sl}(2;\mathbb{R})$. From this we see that

$\xi_t = \xi + th$ is semisimple with real eigenvalues for $t \neq 0$. Let B be a minimal parabolic subgroup of G whose Lie algebra contains ξ and h, and denote

$\mathcal{U}_t = \mathrm{Ad}(G)\cdot\xi_t$ viewed as a co-adjoint orbit,

π_t : the corresponding principal series representation $(t \neq 0)$,

φ_t : the positive definite spherical function for π_t $(t \neq 0)$.

Then the π_t, $t \neq 0$, are irreducible unitary representations of G

on $L^2(G/B)$ given by formulas that depend smoothly on t , and one has

$\pi = \lim_{t \to 0} \pi_t$: unitary representation of G on $L^2(G/B)$.
Here π_t corresponds to the orbit $\mathcal{V}_t$ for $t \neq 0$, and so π corresponds to $\mathcal{V} = \mathcal{V}_0$.

One obtains the same limit with the spherical functions. For φ_t defines π_t in the standard manner when $t \neq 0$, and $\varphi = \lim_{t \to 0} \varphi_t$ is a positive definite spherical function and thus defines a limit representation π .

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