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New Tetrads in Riemannian Geometry and New Ensuing Results in Group Theory, Gauge Theory and Fundamental Physics in Particle Physics, General Relativity and Astrophysics

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A new tetrad is introduced within the framework of geometrodynamics for non-null electromagnetic fields. This tetrad diagonalizes the electromagnetic stress-energy tensor and allows for maximum simplification of the expression of the electromagnetic field. The Einstein-Maxwell equations will also be simplified. New group isomorphisms are proved. The local group of electromagnetic gauge transformations is isomorphic to the new group LB1. LB1 is the group of local tetrad transformations comprised by SO(1,1) plus two different kinds of discrete transformations. The local group of electromagnetic gauge transformations is also isomorphic to the local group of tetrad transformations LB2, which is SO(2), as well. Therefore, we proved that LB1 is isomorphic to LB2. These group results amount to proving that the no-go theorems of the sixties like the S. Coleman- J. Mandula, the S. Weinberg or L. ORaifeartagh versions are incorrect. Not because of their internal logic, but because of the assumptions made at the outset of all these versions. These new tetrads are useful in astrophysics spacetime evolution algorithms since they introduce maximum simplification in all relevant objects, specially in stress-energy tensors.

Keywords: Einstein-Maxwell Spacetimes, Non-Null Electromagnetic fields, New Tetrads, Covariant Diagonalization of Stress-Energy Tensor, New Groups, New Group Isomorphisms.

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1. Diagonalization of the Stress-Energy Tensor

Throughout the paper we use the conventions of Ref.[1] In particular we use a metric with sign conventions -+++. The only difference in notation with Ref.[1] will be that we will call our geometrized electromagnetic potential A_μ, where $f_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu}$

is the geometrized electromagnetic field $f_{\mu\nu} = (G^{1/2}/c^2)\, F_{\mu\nu}$. The stress-energy tensor according to Eq. (14a) in Ref.,[1] can be written as,

$$T_{\mu\nu} = f_{\mu\lambda}\;\, f_{\nu}{}^{\lambda} + *f_{\mu\lambda}\;\, *f_{\nu}{}^{\lambda}\ , \tag{1}$$

where

$$*f_{\mu\nu} = \frac{1}{2}\,\epsilon_{\mu\nu\sigma\tau}\, f^{\sigma\tau}\ ,$$

is the dual tensor of $f_{\mu\nu}$. The local duality rotation given by Eq. (59) in paper[1]

$$f_{\mu\nu} = \xi_{\mu\nu}\;\cos\alpha + *\xi_{\mu\nu}\;\sin\alpha\,,$$

allows us to express the stress-energy tensor in terms of the extremal field

$$T_{\mu\nu} = \xi_{\mu\lambda}\;\, \xi_{\nu}{}^{\lambda} + *\xi_{\mu\lambda}\;\, *\xi_{\nu}{}^{\lambda}\,.$$

We can express the extremal field as,

$$\xi_{\mu\nu} = e^{-*\alpha} f_{\mu\nu}\;\; = \cos\alpha\; f_{\mu\nu} - \sin\alpha\;\, *f_{\mu\nu}. \tag{2}$$

Extremal fields are local gauge invariants in the electromagnetic sense as it can be noticed from Eq. (2). Extremal fields satisfy the equation

$$\xi_{\mu\nu} * \xi^{\mu\nu} = 0\ . \tag{3}$$

This a condition imposed on extremal fields in order to find a local scalar named the complexion α. The explicit expression for the complexion, which is also a local electromagnetic gauge invariant, can be given when imposing condition (3), by

$$\tan(2\alpha) = -f_{\mu\nu}\;\, *f^{\mu\nu} / f_{\lambda\rho}\, f^{\lambda\rho}\,.$$

Through the use of the general identity,

$$A_{\mu\alpha}\, B^{\nu\alpha} - *B_{\mu\alpha}\;\, *A^{\nu\alpha} = \frac{1}{2}\,\delta_{\mu}{}^{\nu}\, A_{\alpha\beta}\, B^{\alpha\beta}\ \ , \tag{4}$$

which is valid for every pair of antisymmetric tensors in a four-dimensional Lorentzian spacetime,[1] when applied to the case $A_{\mu\alpha} = \xi_{\mu\alpha}$ and $B^{\nu\alpha} = *\xi^{\nu\alpha}$, it can be proved that condition (3) yields the equivalent condition,

$$\xi_{\alpha\mu}\;\, *\xi^{\mu\nu} = 0\ . \tag{5}$$

The extremal field $\xi_{\mu\nu}$ and the scalar complexion α have been previously defined through Eqs. (22-25) in Ref. [1] It is our purpose to find a tetrad in which the stress-energy tensor is diagonal. This tetrad would simplify the analysis of the geometrical properties of the electromagnetic field. There are four tetrad vectors that at every point in spacetime diagonalize the stress-energy tensor in geometrodynamics,

$$V^{\alpha}_{(1)} = \xi^{\alpha\lambda}\,\xi_{\rho\lambda}\, X^{\rho}\,; \tag{6}$$

$$V^{\alpha}_{(2)} = \sqrt{-Q/2}\,\xi^{\alpha\lambda}\, X_{\lambda}\,; \tag{7}$$

$$V^{\alpha}_{(3)} = \sqrt{-Q/2} \; * \xi^{\alpha\lambda} \, Y_{\lambda} \, ; \tag{8}$$

$$V^{\alpha}_{(4)} = *\xi^{\alpha\lambda} \; * \xi_{\rho\lambda} \, , \, Y^{\rho} \; , \tag{9}$$

where $Q = \xi_{\mu\nu} \, \xi^{\mu\nu} = -\sqrt{T_{\mu\nu} T^{\mu\nu}}$ according to Eq. (39) in manuscript Ref.[1] Q is assumed not to be zero, because we are dealing with non-null electromagnetic fields. We are free to choose the vector fields X^{α} and Y^{α}, as long as the four vector fields (6-9) are not trivial. Two equations in the extremal field are going to be used extensively in this work, in particular, to prove that tetrad (6-9) diagonalizes the stress-energy tensor. The first equation is given by (64) in Ref.,[1] also given in Eq. (5). When we replace $A_{\mu\alpha} = \xi_{\mu\alpha}$ and $B^{\nu\alpha} = \xi^{\nu\alpha}$ in (4), the second identity is found,

$$\xi_{\mu\alpha} \, \xi^{\nu\alpha} - *\xi_{\mu\alpha} \; * \xi^{\nu\alpha} = \frac{1}{2} \, \delta_{\mu}^{\ \ \nu} \; Q \; . \tag{10}$$

When we make iterative use of Eqs. (5) and (10) we find,

$$V^{\alpha}_{(1)} \, T_{\alpha}^{\ \ \beta} = \frac{Q}{2} \, V^{\beta}_{(1)} \, ; \tag{11}$$

$$V^{\alpha}_{(2)} \, T_{\alpha}^{\ \ \beta} = \frac{Q}{2} \, V^{\beta}_{(2)} \, ; \tag{12}$$

$$V^{\alpha}_{(3)} \, T_{\alpha}^{\ \ \beta} = -\frac{Q}{2} \, V^{\beta}_{(3)} \, ; \tag{13}$$

$$V^{\alpha}_{(4)} \, T_{\alpha}^{\ \ \beta} = -\frac{Q}{2} \, V^{\beta}_{(4)} \, . \tag{14}$$

In paper[1] the stress-energy tensor was diagonalized through the use of a Minkowskian frame in which the equation for this tensor was given in Eqs. (34) and (38). In this work, we give the explicit expression for the tetrad in which the stress-energy tensor is diagonal. The freedom we have to choose the vector fields X^{α} and Y^{α}, represents available freedom that we have to choose the tetrad. If we make use of Eqs. (5) and (10), it is straightforward to prove that (6-9) is a set of orthogonal vectors.

2. Electromagnetic Potentials in Geometrodynamics

Our goal is to simplify as much as we can the expression of the electromagnetic field through the use of an orthonormal tetrad, so its geometrical properties can be understood in an easier way. As it was mentioned above we would like to show this simplification through an explicit example by making a convenient and particular choice of the vector fields X^{α} and Y^{α}. In geometrodynamics, the Maxwell equations,

$$f^{\mu\nu}{}_{;\nu} = 0 \, ,$$

and

$$*f^{\mu\nu}{}_{;\nu} = 0 \, ,$$

are telling us that two potential vector fields exist,[3]

$$f_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu}\,,$$

and

$$*f_{\mu\nu} = *A_{\nu;\mu} - *A_{\mu;\nu}\,.$$

For instance, in the Reissner-Nordstrom geometry the only non-zero electromagnetic tensor component is

$$f_{tr} = A_{r;t} - A_{t;r}\,,$$

and its dual

$$*f_{\theta\phi} = *A_{\phi;\theta} - *A_{\theta;\phi}\,.$$

The symbol ";" stands for covariant derivative with respect to the metric tensor $g_{\mu\nu}$ and the star in $*A_\nu$ is just a name, not the dual operator, meaning that

$$*A_{\nu;\mu} = (*A_\nu)_{;\mu}\,.$$

The vector fields A^α and $*A^\alpha$ represent a possible choice in geometrodynamics for the vectors X^α and Y^α. It is not meant that the two vector fields have independence from each other, it is just a convenient choice for a particular example. A further justification for the choice $X^\alpha = A^\alpha$ and $Y^\alpha = *A^\alpha$ could be illustrated through the Reissner-Nordstrom geometry. In this particular geometry, $f_{tr} = \xi_{tr}$ and $*f_{\theta\phi} = *\xi_{\theta\phi}$, therefore, $A_\theta = 0$ and $A_\phi = 0$. Then, for the last two tetrad vectors (8-9), the choice $Y^\alpha = *A^\alpha$ becomes meaningful under the light of this particular extreme case, when basically there is no magnetic field.

3. Gauge Geometry. Gauge Transformations on Blades One and Two

Once we make the choice $X^\alpha = A^\alpha$ and $Y^\alpha = *A^\alpha$ the question about the geometrical implications of electromagnetic gauge transformations of the tetrad vectors (6-9) arises. We first notice that a local electromagnetic gauge transformation of the "gauge vectors"

$$X^\alpha = A^\alpha\,,$$

and

$$Y^\alpha = *A^\alpha\,,$$

can be just interpreted as a new choice for the gauge vectors

$$X_\alpha = A_\alpha + \Lambda_{,\alpha}$$

and

$$Y_\alpha = *A_\alpha + *\Lambda_{,\alpha}\,.$$

When we make the transformation,

$$A_\alpha \to A_\alpha + \Lambda_{,\alpha}\,,$$

$f_{\mu\nu}$ remains invariant, and the transformation,

$$*A_\alpha \to *A_\alpha + *\Lambda_{,\alpha}\,,$$

leaves $*f_{\mu\nu}$ invariant, as long as the local functions Λ and $*\Lambda$ are scalars. It is valid to ask how the tetrad vectors (6-7) are going to transform under

$$A_\alpha \to A_\alpha + \Lambda_{,\alpha}\,,$$

and (8-9) under

$$*A_\alpha \to *A_\alpha + *\Lambda_{,\alpha}\,.$$

Schouten defined what he called, a two-bladed structure in a spacetime.[4] These local blades or planes are the planes determined by the pairs $(V^\alpha_{(1)}, V^\alpha_{(2)})$ and $(V^\alpha_{(3)}, V^\alpha_{(4)})$.

Given the space constraint in these proceedings we will limit ourselves to show a few illustrative results as far as tetrad transformations for gauge vector choice given by electromagnetic gauge transformations. The whole analysis is given in manuscript Ref.[2] In order to simplify the notation we are going to write $\Lambda_{,\alpha} = \Lambda_\alpha$. First we study the change in (6-7) under

$$A_\alpha \to A_\alpha + \Lambda_{,\alpha}\,.$$

Using the following notation,

$$C = (-Q/2)\, V_{(1)\sigma}\, \Lambda^\sigma / (\, V_{(2)\beta}\, V^\beta_{(2)}\,)\,,$$

and

$$D = (-Q/2)\, V_{(2)\sigma}\, \Lambda^\sigma / (\, V_{(1)\beta}\, V^\beta_{(1)}\,)\,,$$

several cases arise on blade one. We would like to calculate the norm of the transformed vectors $\tilde{V}^\alpha_{(1)}$ and $\tilde{V}^\alpha_{(2)}$,

$$\tilde{V}^\alpha_{(1)}\, \tilde{V}_{(1)\alpha} = [(1+C)^2 - D^2]\, V^\alpha_{(1)}\, V_{(1)\alpha}\,; \tag{15}$$

$$\tilde{V}^\alpha_{(2)}\, \tilde{V}_{(2)\alpha} = [(1+C)^2 - D^2]\, V^\alpha_{(2)}\, V_{(2)\alpha}\,, \tag{16}$$

where the relation $V^\alpha_{(1)}\, V_{(1)\alpha} = -V^\alpha_{(2)}\, V_{(2)\alpha}$ has been used and $V^\alpha_{(1)}$ assumed timelike for simplicity. In order for these transformations to keep the timelike or spacelike character of $V^\alpha_{(1)}$ and $V^\alpha_{(2)}$ the condition

$$[(1+C)^2 - D^2] > 0\,,$$

must be satisfied. If this condition is fulfilled, then we can normalize the transformed vectors $\tilde{V}^\alpha_{(1)}$ and $\tilde{V}^\alpha_{(2)}$ as follows,

$$\frac{\tilde{V}^\alpha_{(1)}}{\sqrt{-\tilde{V}^\beta_{(1)}\, \tilde{V}_{(1)\beta}}} = \frac{(1+C)}{\sqrt{(1+C)^2 - D^2}}\, \frac{V^\alpha_{(1)}}{\sqrt{-V^\beta_{(1)}\, V_{(1)\beta}}} + \frac{D}{\sqrt{(1+C)^2 - D^2}}\, \frac{V^\alpha_{(2)}}{\sqrt{V^\beta_{(2)}\, V_{(2)\beta}}} \tag{17}$$

$$\frac{\tilde{V}^{\alpha}_{(2)}}{\sqrt{\tilde{V}^{\beta}_{(2)}\ \tilde{V}_{(2)\beta}}} = \frac{D}{\sqrt{(1+C)^2-D^2}}\ \frac{V^{\alpha}_{(1)}}{\sqrt{-V^{\beta}_{(1)}\ V_{(1)\beta}}} + \frac{(1+C)}{\sqrt{(1+C)^2-D^2}}\ \frac{V^{\alpha}_{(2)}}{\sqrt{V^{\beta}_{(2)}\ V_{(2)\beta}}}\,. \tag{18}$$

The condition $[(1+C)^2 - D^2] > 0$ allows for two possible situations, $1+C > 0$ or $1+C < 0$. For the particular case when $1+C > 0$, the transformations (17-18) are telling us that an electromagnetic gauge transformation on the vector field A^{α}, that leaves invariant the electromagnetic field $f_{\mu\nu}$, generates a boost transformation on the normalized tetrad vector fields

$$\left(\frac{V^{\alpha}_{(1)}}{\sqrt{-V^{\beta}_{(1)}\ V_{(1)\beta}}}, \frac{V^{\alpha}_{(2)}}{\sqrt{V^{\beta}_{(2)}\ V_{(2)\beta}}}\right),$$

The case $1+C < 0$, represents the composition of two transformations. An inversion of the normalized tetrad vector fields

$$\left(\frac{V^{\alpha}_{(1)}}{\sqrt{-V^{\beta}_{(1)}\ V_{(1)\beta}}}, \frac{V^{\alpha}_{(2)}}{\sqrt{V^{\beta}_{(2)}\ V_{(2)\beta}}}\right),$$

and a boost. If the case under study is that $[(1+C)^2 - D^2] < 0$, the vectors $V^{\alpha}_{(1)}$ and $V^{\alpha}_{(2)}$ are going to change their timelike or spacelike character,

$$\tilde{V}^{\alpha}_{(1)}\ \tilde{V}_{(1)\alpha} = [-(1+C)^2 + D^2]\,(-V^{\alpha}_{(1)}\ V_{(1)\alpha})\,; \tag{19}$$

$$(-\tilde{V}^{\alpha}_{(2)}\ \tilde{V}_{(2)\alpha}) = [-(1+C)^2 + D^2]\ V^{\alpha}_{(2)}\ V_{(2)\alpha}\ . \tag{20}$$

These are improper transformations on blade one. They have the property of being a composition of boosts and a discrete transformation given by $\Lambda^{o}{}_{o} = 0$, $\Lambda^{o}{}_{1} = 1$, $\Lambda^{1}{}_{o} = 1$, $\Lambda^{1}{}_{1} = 0$. We notice that this discrete transformation is not a Lorentz transformation. They might also be composed with an inversion, see Ref.[2] for the whole analysis. On blade or plane two, the choice $Y_{\alpha} = *A_{\alpha} + *\Lambda_{,\alpha}$ induces just local spatial tetrad vector transformations. We reiterate that local tetrad electromagnetic gauge transformations can be interpreted as new or different gauge choices $X_{\alpha} = A_{\alpha} + \Lambda_{,\alpha}$ and $Y_{\alpha} = *A_{\alpha} + *\Lambda_{,\alpha}$. In order to summarize this section we state that there is a new local group LB1 composed by the boosts $SO(1,1)$ and two discrete transformations. One of the discrete transformations is the full inversion or just minus the identity. The other one is the "'switch" given by the components $\Lambda^{o}{}_{o} = 0$, $\Lambda^{o}{}_{1} = 1$, $\Lambda^{1}{}_{o} = 1$, $\Lambda^{1}{}_{1} = 0$. The "switch" is not a Lorentz transformation. The local group LB2 is just $SO(2)$.

4. Group Isomorphisms

We will just limit ourselves to state these new theorems proved in detail in Ref.[2]

Theorem 1. *The mapping between the group of electromagnetic gauge transformations and the group LB1 defined above is isomorphic.*

Theorem 2. *The mapping between the group of electromagnetic gauge transformations and the group LB2 defined above is isomorphic.*

5. Conclusions

We can state the following conclusions as a summary.

- New orthonormal tetrad for non-null electromagnetic fields in four-dimensional curved Lorentzian spacetimes. This tetrad diagonalizes locally and covariantly the Einstein-Maxwell stress-energy tensor. Astrophysical applications in spacetime evolution[5,6].
- Isomorphisms between the local electromagnetic gauge group and the local groups of tetrad spacetime transformations LB1 and LB2. There is an isomorphism between kinematic states and gauge states of the gravitational fields, locally.
- Maximum simplification of relevant tensors and field equations.
- New tetrads encode gravitational and electromagnetic gauge information.
- We are introducing an explicit "link" between the "internal" and the "spacetime", so far detached from each other.
- Extension or generalization to non-Abelian theories with gauge group $SU(2) \times U(1)$, see Ref.[7]
- Hypotheses made at the outset of the no-go theorems[8−10] proved incorrect. Therefore, the no-go theorems are incorrect.

We have proven that the local group of electromagnetic gauge transformations is isomorphic to the new group LB1. The local group of electromagnetic gauge transformations is isomorphic to the local group of tetrad transformations LB2 as well. Therefore, we proved that LB1 is isomorphic to LB2. These group results amount to proving that the no-go theorems[8−10] of the sixties like the S. Coleman- J. Mandula, the S. Weinberg or L. ORaifeartagh versions are incorrect. Not because of their internal logic, but because of the assumptions made at the outset of all these versions. The explicit isomorphic link between the Abelian local "internal" electromagnetic gauge transformations and the local tetrad transformations on special orthogonal local planes is manifest evidence of these incorrect assumptions as it has been proved.[2] Simply because the Lorentz transformations on a local plane in a four-dimensional curved Lorentzian spacetime do not commute with Lorentz transformations on a different local plane in general, element of contradiction with the no-go theorems assumptions. LB1 isomorphic to LB2 which is SO(2) means that the boosts plus two discrete transformations can be put in a one to one relation to SO(2) which also contradicts the assumptions made at the outset of the no-go theorems.

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