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LIMITATIONS OF PROTON BEAM CURRENT IN A STRONG FOCUSING LINEAR ACCELERATOR ASSOCIATED WITH THE BEAM SPACE CHARGE

I. M. Kapchinskij and V. V. Vladimirskij

Institute for Theorical and Experimental Physics, Academy of Science, Moscow

(presented by V. V. Vladimirskij)

I. INTRODUCTION

The obtaining of the required average beam current in a pulse operated proton linear accelerator with high duty-ratio may necessitate the injection of pulsed high intensity proton currents. At a vacuum of the order of 10^{-6} mm and with a pulse duration not surpassing several tens of microseconds, the neutralisation of the proton charge by slow electrons is impossible for lack of time [1]. In this case the beam space charge materially influences the motion of particles in the accelerator. With a given external focusing field and a definite phase volume of particles, the maximum cross-section of the beam grows with the beam current.

It is well known that at injection energies up to 1 MeV, the focusing of the proton beam by longitudinal magnetic field requires much greater electric power than the focusing by a quadrupole lens field. The focusing of the proton beam by a longitudinal magnetic field proves to be practically inexpedient even when the space charge is absent [2], since the stability of the beam with due consideration for the defocusing action of the accelerating gaps may be ensured only by extremely strong fields. Nevertheless, by means of the longitudinal fields, there always exists, for an arbitrary beam current, at least a theoretical possibility to obtain the beam cross-section as small as wanted if the magnetic field intensity is sufficiently high. In case of strong focusing, the permissible field of quadrupole lenses is not only limited by the technical difficulties, but is restricted by the first region of stability. This, in principle, is a limitation for the maximum possible beam current at a given channel aperture. The aperture of the channel within the drift tubes is restricted by the requirements associated firstly with the necessity of ensuring a sufficiently high transit-time factor, and secondly, with the possibility of creating the quadrupole lens field gradient required to obtain a stable beam throughout the range of phase oscillations of the particles with the available magnetic materials (in the case of magnetic lenses), or with the permissible tension (in the case of electrostatic lenses). Thus, the demands imposed on the aperture of the channel predetermine the maximum beam cross-section and thereby limit the beam current in the strong focusing channel.

With a given beam current the space charge, generally speaking, does not affect the stability of particle motion but determines the beam cross-section in the stationary conditions for a definite phase volume of injected particles. This paper presents the estimates of the dependence of the beam sizes upon the proton current at different values of quadrupole lens focusing fields and phase volumes of the injected particles, taking into consideration the longitudinal particle scattering associated with the presence of synchrotron oscillations in the accelerator. A channel without acceleration is a special case. This paper discusses magnetic quadrupole lenses. but all the results may be easily applied to the case of electrostatic quadrupole lenses. Inasmuch as the co-solution of the equations of motion of many particles, taking into account the Coulomb interaction, in a general case is very difficult we have studied a number of special cases which are distinguished by the fact that, thanks to the special choice of the initial phase space distribution of particles, the equations of motion remain linear.

II. EQUATIONS OF TRANSVERSE OSCILLATIONS

The following simplifying assumptions have been made in deducing the equations:

1. The beam cross-section is appreciably smaller than the variation period of the focusing field;

2. The particles move only in the linear region of the external field close to the accelerator's axis.

3. Since the influence of the space charge is essential only at relatively small velocities, it is assumed that the particle velocity is much below the velocity of light;

4. The energy increment in a gap is sufficiently small, so that the period of synchrotron oscillations is much greater than the acceleration period.

In these assumptions the equations of transversal oscillations may be expressed in the following form

$$\frac{d^2x}{d\tau^2}+Q_x(\tau)\cdot x+\frac{e\lambda^2}{mc^2}\cdot\frac{\partial U_K}{\partial x}=0 \qquad (1)$$

$$\frac{d^2y}{d\tau^2}+Q_y(\tau)\cdot y+\frac{e\lambda^2}{mc^2}\cdot\frac{\partial U_K}{\partial y}=0 \qquad (2)$$

The axis z is directed along the axis of the accelerator. The following dimensionless value has been assumed as the independent variable

$$\tau=\frac{ct}{\lambda} \qquad (3)$$

In these equations

e, m are charge and the mass of a particle,

λ is the wavelength of the high frequency accelerating field in free space and

$U_k(x,y,z)$ is the potential of the field of the beam space charge.

Functions $Q_x(\tau)$ and $Q_y(\tau)$ are equal to

$$Q_x(\tau)=Q_1(\tau)+Q_2(\tau) \qquad (4)$$

$$Q_y(\tau)=Q_1(\tau)-Q_2(\tau) \qquad (5)$$

The function $Q_1(\tau)$ is determined by the defocusing action of the accelerating gaps:

$$Q_1(\tau)=\frac{e\lambda^2}{2mc^2}\cdot\frac{\partial E_z(r,z,\tau)}{\partial z} \qquad (6)$$

E_z is the longitudinal component of the accelerating field. The function $Q_2(\tau)$ is determined by the focusing field:

$$Q_2(\tau)=K^2\cdot g(\tau) \qquad (7)$$

$$K=\lambda\cdot\sqrt{\frac{e\beta}{mc^2}\cdot G} \qquad (8)$$

The function $g(\tau)$ has the period of the focusing field and changes within the limits of $|g(\tau)|\leq 1$; G is the magnetic field gradient on the axis in the middle portion of the lens.

The period of the beam cross-section change has the same value order as the period of the focusing field; since the beam cross-section is substantially smaller than the period of cross-section change, the field of the space charge $\vec{E_k}$ changes much slower along the axis z than along the axes x,y:

$$\left|\frac{\partial E_{kz}}{\partial z}\right|\ll\left|\frac{\partial E_{xk}}{\partial x}\right|,\quad\left|\frac{\partial E_{yk}}{\partial y}\right|.$$

Thus,

$$\operatorname{div}\vec{E_k} \approx \frac{\partial E_{xk}}{\partial x} + \frac{\partial E_{yk}}{\partial y},$$

from which

$$\frac{\partial^2 U_K}{\partial x^2} + \frac{\partial^2 U_K}{\partial y^2} = -4\pi \cdot \rho(x,y,z) \tag{9}$$

Let us assume that $n(x,y,z,\dot{x},\dot{y},\dot{z},)$ is the density of particle distribution in the phase space of co-ordinates and velocities. According to Liouville's theorem the function $n(x,\ldots\dot{z})$ remains constant along the trajectory of the particle motion in the phase space, so that n depends only on the integrals of motion. The space charge density in the beam $\rho(x,y,z)$ is determined by the integral

$$\rho(x,y,z) = e\iiint n(x,y,z,\dot{x},\dot{y},\dot{z})\, d\dot{x}d\dot{y}d\dot{z} \tag{10}$$

Thus due to the fourth assumption

$$\left|\frac{\dot{z}-v_s}{v_s}\right| = \frac{1}{2\pi}\left|\frac{d\phi}{d\tau}\right| \ll 1, \tag{11}$$

where v_s is the speed of the equilibrium particle and ϕ is the phase of the high frequency field at the passage of the particle in the electrical centre of the gap. The deviation of the velocity $\frac{dz}{dt}$ of the particle from that of the synchronous particle v_s may be neglected, by introducing into Eq. (10) $\frac{dz}{dt} = v_s$ for all the particles.

Then

$$\rho(x,y,z) = e\iint n(x,y,z,\dot{x},\dot{y})\, d\dot{x}d\dot{y} \tag{12}$$

The current in the beam along the axis z

$$I = v_s\iint \rho(x,y,z)\, dxdy = \text{const.} \tag{13}$$

The Equations (1), (2) should be solved jointly with the Equations (9), (10), which determine the potential of the space charge field $U_k(x,y,z)$, provided the phase density of particles n is a function of the integrals of motion of the Equations (1), (2). The coordinate of the particle z, owing to our assumptions is a function of the variable τ only.

It is expedient to simplify the expression (6) for the function $Q_1(\tau)$ by replacing the action of the defocusing force, which varies along the accelerating gap, by the action of the equivalent defocusing force constant in the gap which is described by the gap averaged value of the function $Q_1(\tau)$

$$\bar{Q}_1 = \frac{1}{b_\tau}\int_{b_\tau} Q_1(\tau)\, d\tau \tag{14}$$

b_τ is the length of the gap along the axis τ; inasmuch as throughout the gap $\beta \approx$ const, $b_\tau = b/\beta\lambda$, where b is the length of the gap in cm. The longitudinal component of the accelerating field may be expressed as

$$E_z = E\cdot h(r,z)\cdot \cos 2\pi\tau \tag{15}$$

Moreover

$$\bar{Q}_1 = \frac{eE\lambda^2}{2mc^2 b}\int_b \frac{\partial h}{\partial z}\cos\left(\frac{2\pi z}{\beta\lambda} + \phi - \phi_s\right) dz \tag{16}$$

We obtain

$$\bar{Q}_1 = -\frac{\gamma}{b_\tau} \tag{17}$$

where

$$\gamma = -\frac{\pi k W_\lambda}{\beta}\cdot\frac{\sin\phi}{\cos\phi_s} \tag{18}$$

Here

$$W_\lambda = \frac{eE_0 T\lambda\cos\phi_s}{mc^2} \tag{19}$$

is the parameter introduced by Alvarez [3], equal to the relation of the energy increment of the synchronous particle on the wavelength λ to the rest energy;

$$E_0 = \frac{E}{k\beta\lambda}\int_b h(z)dz$$

is the average amplitude of the accelerating field; T is the transit-time factor; k is the acceleration period multiplicity. The para- meter γ determines the defocusing action of the gaps at a given phase of passage of the centre of the gap by the particle. The minus sign in the righthand part of Equation (18) is due to the fact that the phases ϕ are counted from the maximum value of the fields o that the stable phase of the equilibrium particle is negaıve,i $\phi_s < 0$. The same parameter γ, obtained owing to somewhat different considerations has been used in papers by Smith and Gluckstern [2] and Teng [4]. Thus in the indicated approximation $Q_1 = -\gamma/b_\tau$ in the accelerating gap, and $Q_1 = 0$ outside the gap.

The integration of Equations (1), (2) will be materially simplified if we introduce a special assumption regarding the law of particle distribution in the phase space; namely, if we choose such a dependence of phase density upon the integrals of motion that the Equations (1), (2) become linear in the separated variables. As we shall show, a function similar to the micro-canonic distribution proves to be this dependence.

Let us consider Equation (1). If it is linear with regard to the function $x(\tau)$, then it is possible to choose a pair of complex-conjugated fundamental solutions

$$\left.\begin{aligned} \chi_x(\tau) &= \delta_x(\tau)\cdot e^{i\psi_x(\tau)} \\ \chi_x^*(\tau) &= \delta_x(\tau)\cdot e^{-i\psi_x(\tau)} \end{aligned}\right\} \tag{20}$$

which make it possible to express any real solution with arbitrary initial conditions:

$$x(\tau) = A_x\cdot\delta_x(\tau)\cdot\cos\left[\psi_x(\tau)+\theta_x\right] \tag{21}$$

The Wronskian of the functions χ_x, χ_x^* is a well known a constant value. Let us put:

$$\chi_x\cdot\frac{d\chi_x^*}{d\tau}-\chi_x^*\frac{d\chi_x}{d\tau} = -2i\,. \tag{22}$$

It follows from Eq. (22) that

$$\frac{d\psi_x}{d\tau} = \frac{1}{\delta_x^2}\,. \tag{23}$$

The solution of the equation (2) has the analogous form

$$\left.\begin{aligned} y(\tau) &= A_y\delta_y(\tau)\cos\left[\psi_y(\tau)+\theta_y\right] \\ \frac{d\psi_y}{d\tau} &= \frac{1}{\delta_y^2} \end{aligned}\right\} \tag{24}$$

The values $A_x, A_y, \theta_x, \theta_y$ depend only upon the initial conditions and consequently are the integrals of motion. The phases of the transversal oscillations of particles θ_x, θ_y are cyclic functions of the initial coordinates and velocities. It is natural to suppose that the function of density distribution n depends only upon the integrals of motion A_x, A_y which are characteristic of the transversal oscillation intensity. The oscillation phases θ_x, θ_y may be of arbitrary value. We shall obtain the expressions for the integrals A_x, A_y by eliminating from the functions $x(\tau)$, $\frac{dx}{d\tau}$ and $y(\tau)$, $\frac{dy}{d\tau}$ the phases θ_x, θ_y:

$$(\delta_x\dot{x}-\dot{\delta}_x x)^2+\left(\frac{x}{\delta_x}\right)^2 = A_x^2 \tag{25}$$

$$(\delta_y\dot{y}-\dot{\delta}_y y)^2+\left(\frac{y}{\delta_y}\right)^2 = A_y^2 \tag{26}$$

Let us now assume that the particle distribution density in phase space depends upon the integrals of motion in the following way:

$$n = n_0\cdot\delta(F-F_0)\,, \tag{27}$$

where in a general case

$$F = A_x^2+s\cdot A_y^2 \tag{28}$$

and δ is the usual δ-function.

Inasmuch as there are no grounds to expect in the majority of practical cases that the phase volumes of the particles in the planes x, $\dot{x}$ and y, $\dot{y}$ should differ, we shall assume further on that $s = 1$; so we get

$$F = A_x^2+A_y^2 = F_o \tag{29}$$

where F_o is a certain constant. Its numerical value will be determined further on. Thus, due to the assumption, all the particles in a four-dimensional phase space x, y, $\dot{x}$, $\dot{y}$ are situated on the surface of the ellipsoid

$$(\delta_x\dot{x}-\dot{\delta}_x x)^2+\left(\frac{x}{\delta_x}\right)^2+(\delta_y\dot{y}-\dot{\delta}_y y)^2+\left(\frac{y}{\delta_y}\right)^2 = F_0 \tag{30}$$

The projections of the representing points on the plane x, $\dot{x}$ and y, $\dot{y}$ fill up the corresponding ellipses

$$(\delta_x\dot{x}-\dot{\delta}_x x)^2+\left(\frac{x}{\delta_x}\right)^2 = A_{x\,\max}^2 \tag{31}$$

$$(\delta_y\dot{y}-\dot{\delta}_y y)^2+\left(\frac{y}{\delta_y}\right)^2 = A_{y\,\max}^2 \tag{32}$$

At the same time, according to Eq. (29)

$$A_{x\max}^2 = A_{y\max}^2 = F_0\,. \tag{33}$$

By introducing the distribution function (27) into the expression (12), which determines the space charge density in the beam, we shall obtain

$$\rho = en_0\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\delta(F-F_0)\,d\dot{x}\,d\dot{y} \tag{34}$$

or after replacing the variables of integration in the function F:

$$\delta_x \dot{x} - \dot{\delta}_x x = \alpha \cos \Psi \ ; \quad \delta_y \dot{y} - \dot{\delta}_y y = \alpha \sin \Psi \ ;$$

$$\rho = \frac{en_o}{\delta_x \delta_y} \int_0^{2\pi} \int_0^{\infty} \delta \left[\alpha^2 - \left(F_0 - \frac{x^2}{\delta_x^2} - \frac{y^2}{\delta_y^2} \right) \right] \alpha d\alpha d\Psi = \frac{\pi e n_o}{\delta_x \delta_y}.$$

Thus the function of particle distribution Eq. (27) corresponds to the uniform distribution of the charge in every section of the beam; the space charge density does not depend upon x, y and is a function of time only: $\rho = \rho(\tau)$. Any cross-section of the beam in the xOy plane, according to Eq. (30), is of an elliptical shape.

The potential of the field of the beam space charge is determined with Eq. (9) by the equation:

$$\frac{\partial^2 U_K(x,y,\tau)}{\partial x^2} + \frac{\partial^2 U_K(x,y,\tau)}{\partial y^2} = -4\pi \cdot \rho(\tau) \qquad (35)$$

Inasmuch as the beam has an elliptical cross-section, the function $U_k(x,y,\tau)$ may be approximately determined as the potential of the field of a uniformly charged elliptical cylinder. If the cross-section half axes are $r_x(\tau)$ and $r_y(\tau)$, respectively, it can be shown that

$$U_K(x,y,\tau) = -\pi \cdot \rho(\tau) \cdot \left[x^2 + y^2 - \frac{r_x - r_y}{r_x + r_y}(x^2 - y^2) \right]. \qquad (36)$$

According to Eq. (13)

$$\rho(\tau) = \frac{J}{\pi r_x r_y v_s}, \qquad (37)$$

from which we get

$$U_K(x,y,\tau) = -\frac{J}{r_x r_y v_s} \left[x^2 + y^2 - \frac{r_x - r_y}{r_x + r_y}(x^2 - y^2) \right]. \qquad (36a)$$

It is of interest that the sign of the quadrupole (proportional to $x^2 - y^2$) member in the potential coincides with the sign of the external quadrupole member Q_2, since in the x-focusing magnetic lenses $r_x > r_y$. It can be said, in this sense, that the Coulomb interaction, far from having a screening effect, even increases to a certain extent the external quadrupole field.

By substituting Eq. (36a) into Equations (1), (2) and by introducing the notation

$$r_a = \lambda \sqrt{\frac{2eJ}{mc^3 \beta}} \qquad (38)$$

we shall arrive at the following motion equations of each individual particle:

$$\frac{d^2 x}{d\tau^2} + \left[Q_x(\tau) - \frac{2r_a^2}{r_x(r_x + r_y)} \right] x = 0 \qquad (39)$$

$$\frac{d^2 y}{d\tau^2} + \left[Q_y(\tau) - \frac{2r_a^2}{r_y(r_y + r_x)} \right] y = 0 \qquad (40)$$

If the initial conditions for all the particles have been chosen in accordance with the distribution function Eq. (27) then the solutions of Equations (39) and (40) are self-consistent.

Let us assume that ng is the number of accelerating gaps in one focusing period. Then the length of the focusing period S in cm is $S = kng\ \beta\lambda$. The duration of the focusing period on the τ axis is equal to $S_\tau = S/\beta\lambda = kng$ and is constant along the accelerator axis. Thus, the functions Q_x, Q_y determined by the equalities (Eq. (4) to (7)) are periodical (if the phase ϕ is constant) with a period of kng. If the channel with strong focusing does not have accelerating gaps we may merely assume that $kng = 1$ and that $\lambda = S/\beta$.

If the Coulomb forces in the beam are negligibly small

$$\frac{r_a}{r_x}, \ \frac{r_a}{r_y} \ll K$$

then the Equations (39), (40) are reduced to the equations of Mathieu-Hill with the given periodical coefficients Q_x, Q_y. It is possible to find by regular methods the Floquet functions of these equations fully describing the transversal particle oscillations by arbitrary initial conditions. However, if the Coulomb members in the Equations (39), (40) are not small, these equations cannot be solved directly because they incorporate the functions $r_x(\tau)$, $r_y(\tau)$ which are thus far unknown and which represent the envelopes of the beam in the corresponding planes xOz and yOz. Essentially speaking, the principal problem is to calculate the envelopes $r_x(\tau)$, $r_y(\tau)$ with the given functions of Q_x, Q_y and the given "Coulomb radius" r_a or, on the contrary, to calculate the maximum permissible "Coulomb radius" r_a at a given aperture of the channel ($r_{x\,\max}$, $r_{y\,\max}$) and the chosen parameters K, γ, which ensure a sufficient stability reserve for transversal oscillations. When the envelopes $r_x(\tau)$, $r_y(\tau)$ are found, it is possible, by using the Equations (39), (40), to appreciate the main characteristics of the movement which are necessary for the calculation of design tolerances.

III. EQUATIONS FOR BEAM ENVELOPES

If we make a special choice of the fundamental solutions of Equations (39) (40), then the envelopes r_x, r_y prove to be connected in a simple way with the modules of these fundamental functions. This enables us to obtain a system of equations for r_x, r_y proceedind girectly from Equations (39), (40).

Let us take on a phase plane x, $\dot{x}$ an ellipse surrounding the projections of the representing points of all particles at a certain time moment τ_o. Let us choose the values $\delta_x(\tau_o)$, $\dot{\delta}_x(\tau_o)$ and A_x so that the ellipse (25) coincides with the border ellipse given on the plane x, $\dot{x}$. Then the integral of motion A_x for all the points on the border ellipse will have the same value equal to its maximum value $A_{x\,\max} = \sqrt{F_0}$. Since at any moment of time the oscillation phases θ_x of different particles may be of any value, then it follows from the expression (21) that the beam envelope in the xOz plane is determined by an equality

$$r_x(\tau) = A_{x\,\max}\cdot\delta_x(\tau) = \sqrt{F_o}\cdot\delta_x(\tau) \qquad (41)$$

In a similar way we shall obtain for the envelope in the yOz plane :

$$r_y(\tau) = A_{y\,\max}\cdot\delta_y(\tau) = \sqrt{F_o}\cdot\delta_y(\tau) \qquad (42)$$

Let us assume that the big and small half axes of the ellipse which restricts the phase volume on the x, $\dot{x}$ plane are M_x and N_x, respectively, while the inclination angle of the big half axes to the Ox axis is α_x (Fig. 1). Then, as can be easily shown by transforming the ellipses (31), (32) to the canonic axes,

$$F_o = M_x\cdot N_x = M_y\cdot N_y \qquad (43)$$

$$r_x = \sqrt{M_x^2\cos^2\alpha_x + N_x^2\sin^2\alpha_x} \qquad (44)$$

$$\frac{dr_x}{d\tau} = \frac{1}{2r_x}(M_x^2 - N_x^2)\sin 2\alpha_x \qquad (45)$$

The expressions for r_y, $\frac{dr_y}{d\tau}$ are obtained from Eq. (44), (45) by replacing the indices. The value F_o which is proportional to the area of the ellipses on

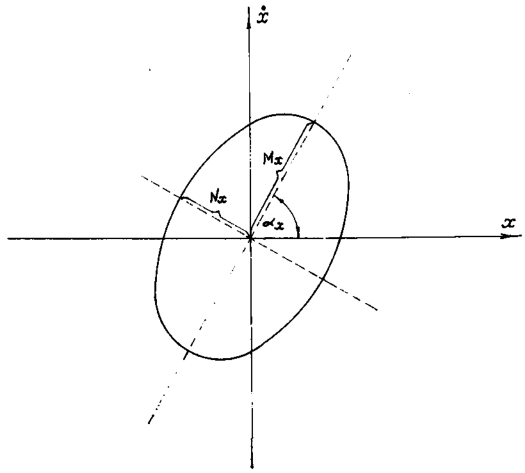

Fig. 1 Phase volume on x, $\dot{x}$ plane. Notations.

the planes x, $\dot{x}$ and y, $\dot{y}$ will be called later on, for convenience sake, the beam phase volume.

By substituting the fundamental functions $\sqrt{F_o}\cdot\chi_x(\tau) = r_x e^{i\psi_x}$, $\sqrt{F_o}\cdot\chi_y(\tau) = r_y e^{i\psi_y}$, into the Equations (39) and (40) and by making use of the condition (22) we obtain the following system of equations for the envelopes

$$\frac{d^2r_x}{d\tau^2} + Q_x(\tau)\cdot r_x - \frac{F_o^2}{r_x^3} - \frac{2r_a^2}{r_x+r_y} = 0 \qquad (46)$$

$$\frac{d^2r_y}{d\tau^2} + Q_y(\tau)\cdot r_y - \frac{F_o^2}{r_y^3} - \frac{2r_a^2}{r_x+r_y} = 0 \qquad (47)$$

The initial conditions for solving these equations can be determined if we proceed from the initial phase volumes on the x, $\dot{x}$ and y, $\dot{y}$ planes by means of Eq. (44), (45).

It is expedient to make several preliminary remarks with regard to the system of Equations (46), (47).

1. At a zero phase volume ($N_x = N_y = 0$) we have $F_0 = 0$ and the Equations (46), (47) are reduced to the following :

$$\left.\begin{aligned} \frac{d^2r_x}{d\tau^2} + Q_x r_x - \frac{2r_a^2}{r_x+r_y} = 0 \\ \frac{d^2r_y}{d\tau^2} + Q_y r_y - \frac{2r_a^2}{r_x+r_y} = 0 \end{aligned}\right\} . \qquad (48)$$

In the absence of the Coulomb forces ($r_a = 0$) the equations for the envelopes are the same as the equations for the trajectories of the individual particles. Thus, in case of a zero phase volume and in the absence of space charge the particle which was on the surface of the beam is always the external one.

The Equations (48), in the far-fetched assumption that the beam in the strong focusing channel always has a circular cross-section ($r_x \equiv r_y$), are reduced to an equation obtained in the paper by Mendel [5] for the trajectory of the "external" particle.

2. Let us suppose that when $r_a = 0$ the Equations (46), (47) have a stable stationary solution which is not on the edge of the stability region. If the channel aperture is not limited, then, generally speaking, a certain stable stationary solution will be present at any value of $r_a > 0$, since the total of $r_x + r_y$ is always reaching a sufficiently big value when the Coulomb members in the equations will not disrupt the stability conditions.

3. There are always some initial conditions for which the solutions of the Equations (46), (47) are periodical. These solutions correspond to the periodical envelope of the beam with the period equal to the focusing field period. In this case the equations of individual trajectories (39), (40) turn into the equations of Mathieu-Hill type at $r_a > 0$ also, while the functions

$$\delta_x = \frac{r_x}{\sqrt{F_0}}, \quad \delta_y = \frac{r_y}{\sqrt{F_0}}$$

become the modules of the corresponding Floquet functions. When $r_a = 0$ the Floquet functions completely describe the maximum beam cross-section. Indeed due to the absence of interaction between the particles, any other ellipses enclosing the representing points on the planes x, $\dot{x}$ and y, $\dot{y}$ "rotate" within the surrounding Floquet ellipses, When $r_a > 0$, though, the Equations (39), (40) in case of non-periodical envelopes are not Mathieu-Hill equations and for them the Floquet ellipses do not exist in general. Therefore, in contrast to the case when $r_a = 0$, at $r_a > 0$ we cannot restrict ourselves merely to the consideration of the periodical solutions of the Equations (46), (47).

4. The value r_a in the Equations (46), (47) depends on the longitudinal velocity of particles β. The value γ, which is incorporated in the functions Q_x, Q_y, depends upon β and the phase of the particles ϕ. This is why the values r_a and γ are generally speaking functions of τ, in the presence of an accelerating field in the strong focusing channel. We suppose, however, that β and ϕ alternate slowly along the accelerator axis. To appreciate the influence of Coulomb forces and the defocusing in the gaps, upon the beam sizes, the values r_a and γ may be left constant in the equations. One may attribute to them different values along the accelerator axis in the final solutions.

IV. THE CASE OF "SMOOTH" ENVELOPES

The system of Equations (46), (47) may be approximately integrated in the case of weak lenses ($K \ll 1$), with small idle intervals when the modulation of the envelopes with the period of the focusing field is small, and the period of the transversal oscillations of the particles is substantially bigger than the focusing field period.

Suppose

$$\left.\begin{aligned} r_x(\tau) &= \bar{r}_x(\tau) + \xi(\tau) \\ r_y(\tau) &= \bar{r}_y(\tau) + \eta(\tau) \end{aligned}\right\} \qquad (49)$$

where $\bar{r}_x$, $\bar{r}_y$ are the constant components of the functions r_x and r_y in each period of focusing; ζ, η are the terms which oscillate with the frequency of the focusing field $\Omega = 2\pi/S_\tau$. The functions $\bar{r}_x$ and $\bar{r}_y$ are the "slow" functions of τ. Their frequency ω is much smaller than Ω. Furthermore, ζ, $\eta \ll \bar{r}_x$, $\bar{r}_y$.

Following Landau and Lifshits [6], let us introduce the expressions (49) in the Equation (46)., If we put

$$Q_x \cdot \xi = \overline{Q_x \cdot \xi} + \kappa_x,$$

where κ_x is the oscillating part with the frequency Ω, we get in the first approximation (considering $\xi/\bar{r}_x$, $\eta/\bar{r}_y$ as small):

$$\frac{d^2 \bar{r}_x}{d\tau^2} + \frac{d^2\xi}{d\tau^2} + \bar{Q}_x \bar{r}_x + \overline{Q_x \xi} + \kappa_x - F_0^2 \frac{1 - 3\xi/\bar{r}_x}{\bar{r}_x^3} - \frac{2r_a^2(1 - \xi + \eta/\bar{r}_x + \bar{r}_y)}{\bar{r}_x + \bar{r}_y} = 0$$

Separating in the last equation the slow and the rapid oscillating members and neglecting the small oscillating members we arrive at the following equations:

$$\frac{d^2\bar{r}_x}{d\tau^2}+\overline{Q_x\xi}-\frac{F_o{}^2}{\bar{r}_x^3}-\frac{2r_a^2}{\bar{r}_x+\bar{r}_y}=0 \tag{50}$$

$$\frac{d^2\xi}{d\tau^2}+Q_x\bar{r}_x=0\,. \tag{51}$$

Let us introduce the periodical functions $q_x(\tau)$ and $q_y(\tau)$ (with the frequency Ω) in the following way:

$$\frac{d^2q_x}{d\tau^2}=-Q_x(\tau)\;;\quad \frac{d^2q_y}{d\tau^2}=-Q_y(\tau)\,. \tag{52}$$

Then the function $\xi(\tau)=q_x(\tau)\cdot\bar{r}_x(\tau)$ will be the approximate solution of the Equation (51). Indeed, by the order of value

$$\frac{dq_x}{d\tau}\sim-\frac{Q_x}{\Omega};\quad q_x\sim\frac{Q_x}{\Omega^2};\quad \frac{d\bar{r}_x}{d\tau}\sim\omega\bar{r}_x\,;\quad \frac{d^2\bar{r}_x}{d\tau^2}\sim-\omega^2\bar{r}_x$$

and

$$\frac{d^2\xi}{d\tau^2}=\frac{d^2q_x}{d\tau^2}\bar{r}_x+2\frac{dq_x}{d\tau}\cdot\frac{d\bar{r}_x}{d\tau}+q_x\frac{d^2\bar{r}_x}{d\tau^2}\approx$$

$$\approx -Q_x\bar{r}_x-\frac{\omega}{\Omega}\left(2+\frac{\omega}{\Omega}\right)Q_x\bar{r}_x$$

The function $\xi=q_x\bar{r}_x$ satisfies the Equation (51) approximately since $\omega\ll\Omega$. From which it follows that:

$$\overline{Q_x\xi}=\overline{q_xQ_x}\cdot\bar{r}_x\;;\qquad \overline{Q_y\eta}=\overline{q_yQ_y}\cdot\bar{r}_y$$

If the parameter γ is sufficiently small, it can be easily shown that the average values of the periodical functions $q_x\cdot Q_x$ and $q_y\cdot Q_y$ during the period are similar:

$$\overline{q_xQ_x}=\overline{q_yQ_y}=\omega_0^2 \tag{53}$$

In the accepted approximation we have

$$\omega_0=\frac{\mu}{kng} \tag{54}$$

where μ is the phase increment of the Floquet function in the focusing period for the given focusing field when $r_a=0$. The number of the focusing periods for one period of transversal oscillations at $r_a=0$ is $2\pi/\mu$. The value μ (expressed in the same way) has been used in the paper by Smith and Gluckstern [2)].

By introducing Eq. (53) into the Equation (50), we shall obtain finally the following equation for the slow component of the envelope:

$$\frac{d^2\bar{r}_x}{d\tau^2}+\omega_0^2\bar{r}_x-\frac{F_o{}^2}{\bar{r}_x^3}-\frac{2r_a^2}{\bar{r}_x+\bar{r}_y}=0 \tag{55}$$

From Equation (47) we shall obtain in a similar way

$$\frac{d^2\bar{r}_y}{d\tau^2}+\omega_0^2\bar{r}_y-\frac{F_o{}^2}{\bar{r}_y^3}-\frac{2r_a^2}{\bar{r}_x+\bar{r}_y}=0 \tag{56}$$

Finally, by introducing the above-found expressions for ξ, η into the equalities (49), we obtain

$$\left.\begin{aligned} r_x(\tau)&=[1+q_x(\tau)]\cdot\bar{r}_x(\tau)\\ r_y(\tau)&=[1+q_y(\tau)]\cdot\bar{r}_y(\tau)\end{aligned}\right\}. \tag{57}$$

The functions (57) are the first approximation to the solution of the Equations (46), (47), the functions q_x, q_y are directly known from the equalities (52), while the functions $\bar{r}_x$, $\bar{r}_y$ should be found from the system of equations (55), (56).

The Equations (55), (56) are more simple than the initial Equations (46), (47) since they are autonomous, and the first integral can be derived for them immediately. To simplify the subsequent calculation, let us assume that the beam of particles which enters the strong focusing channel is symmetrical so that the initial conditions for the averaged functions $\bar{r}_x$ and $\bar{r}_y$ are similar. Then, due to the complete symmetry of the Equations (55), (56), $\bar{r}_x(\tau)\equiv\bar{r}_y(\tau)$ later on too. By denoting $\bar{r}_x=\bar{r}_y=R(\tau)$ we shall obtain instead of two equations (55), (56), one, namely

$$\frac{d^2R}{d\tau^2}+\omega_0^2R-\frac{F_0^2}{R^3}-\frac{r_a^2}{R}=0 \tag{58}$$

while the solutions (57) will be expressed as

$$\begin{aligned} r_x(\tau)&=[1+q_x(\tau)]\cdot R(\tau)\\ r_y(\tau)&=[1+q_y(\tau)]\cdot R(\tau)\,.\end{aligned} \tag{59}$$

In this case the " slow " components of the envelopes have the same phase while the " rapid components " differ by their phase. Since $K\ll1$ and $\gamma\ll1$ we find that $|q_x|$, $|q_y|\ll1$.

Let us discuss the Equation (58). The first integral of this equation (the analogue of the energy integral) is

$$\left(\frac{dR}{d\tau}\right)^2+V(R)=C\,, \tag{60}$$

where the function

$$V(R)=\omega_0^2R^2+\frac{F_0^2}{R^2}-r_a^2\ln\left(\frac{\omega_0R}{r_a}\right)^2 \tag{61}$$

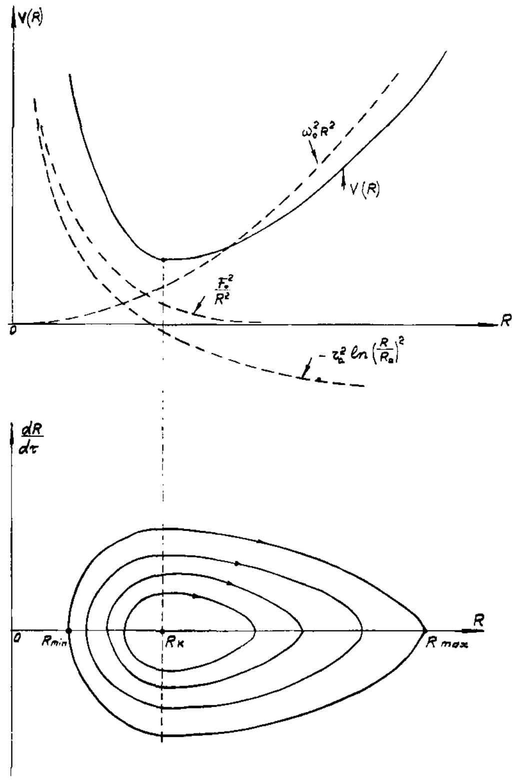

Fig. 2 Integral curves determined by the equation of the " slow " component of the envelopes.

is the analogue of the potential energy. Fig. 2 shows the behaviour of the function $V(R)$ and the family of the integral curves of the Equation (58) on the plane R, $dR/d\tau$. The equilibrium state $R = R_k$, $\dot{R} = 0$ is determined from the condition $dV/dR = 0$:

$$R_K = \sqrt{\frac{r_a^2 + \sqrt{r_a^4 + 4\omega_0^2 F_0^2}}{2\omega_0^2}} \qquad (62)$$

The state of equilibrium and the periodical motions in the plane R, $\dot{R}$ are stable (after Lyapunov) for arbitrary values of the parameters, provided $\omega_0^2 > 0$. However, the maximum swing of the envelope at the given initial conditions depends materially upon r_a. Suppose $\dot{R}(0) = 0$. Then, if $R(0) > R_k$, the maximum value of the slow envelope component in the subsequent motion of the beam in the system of quadrupole lenses does not exceed $R(0)$. However, when $R(0) < R_k$, then the maximum value of the envelope surpasses the initial value of $R(0)$ and exceeds it the more, the less R_0 is compared with the " critical radius " R_k. If $R(0) = R_k$, $\dot{R}(0) = 0$, then $R(\tau) \equiv R_k$ and the envelopes (59) are periodical with the frequency Ω of the focusing field; at any other initial conditions the maximum value of $R(\tau)$ surpasses R_k. With the given initial values of $R(0) = R_0$, $\dot{R}(0) = \dot{R}_0$, the extremal values of the function $R(\tau)$ are determined by the equation

$$V(R) = \dot{R}_0^2 + V(R_0) \qquad (63)$$

It follows from the expression (62) that in case we have a sufficiently great phase volume $\sqrt{2\omega_0 F_0} \gg r_a$ we can neglect the space charge of the beam; in this case

$$R_K = \sqrt{\frac{F_0}{\omega_0}} \qquad (64)$$

If the current of the beam is great $r_a \gg \sqrt{2\omega_o F_o}$ it is possible to assume that the beam has the zero phase volume; in this case

$$R_K = R_a = \frac{r_a}{\omega_0} \qquad (65)$$

In a general case

$$R_K = R_a \cdot \sqrt{\frac{1 + \sqrt{1 + 4/p^4}}{2}} \qquad (66)$$

where the parameter p is determined by the relation between the values of the phase volume, beam current and the lens power:

$$p = \frac{r_a}{\sqrt{\omega_0 F_0}} \qquad (67)$$

By making use of Eqs. (65), (67), we obtain from the expression (61):

$$V_1(R) = \frac{V(R)}{r_a^2} = \left(\frac{R}{R_a}\right)^2 + \frac{1}{p^4}\left(\frac{R_a}{R}\right)^2 - \ln\left(\frac{R}{R_a}\right)^2 \qquad (68)$$

The function V_1 of the variable R/R_a depends only upon one parameter p. According to Eq. (63), when $\dot{R}_0 = 0$ and $R_0 = R_{min} < R_k$ the maximum swing R_{max} of the envelope $R(\tau)$ is determined by the equation $V_1(R_{max}) = V_1(R_{min})$. Fig. 3 shows the diagrams which present the relation of the maximum beam size to its initial radius R_{max}/R_{min} versus the relative value of the initial radius R_{min}/R_a for the different values of p when $\dot{R}_0 = 0$ and $R_0 = R_{min} < R_k$.

For example, let us consider a channel without accelerating gaps when $\beta = 0.04$, the focusing period $S = 20$ cm and $\cos\mu = 0.99$ ($\omega_o = 0.144$). The parameters of the input beam are: $R_0 = 0.5$ cm,

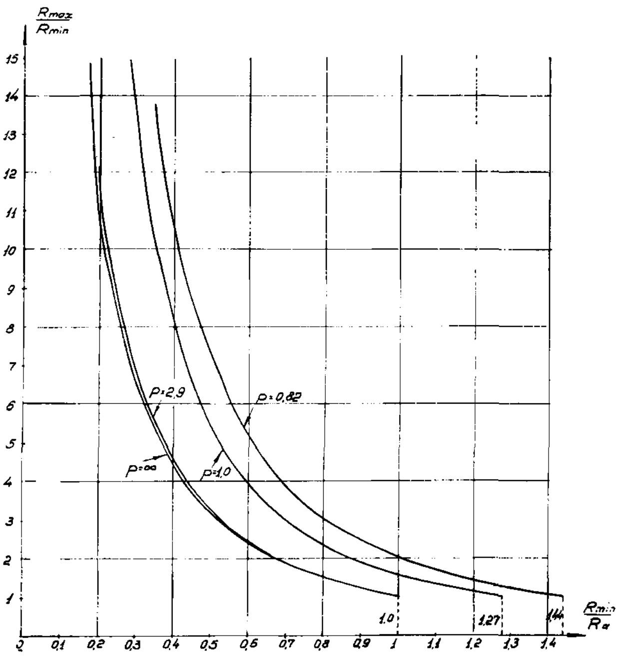

Fig. 3 The dependence of the relation of the maximum beam radius to the initial one upon the relative magnitude of the initial radius (for the case of the " smooth " envelope).

$\dot{R}_0 = 0$; the angular spread $\pm 10^{-3}$ radian. Then $M = 0.5$ cm; $N = 0.02$ cm; $F_0 = 0.01$ cm^2. When the beam current is $I = 100$ mA we have $p = 5.3$ and $R_k \approx R_a \approx 1.4$ cm; this beam grows wider in the channel, its radius reaching 2.6 cm. A beam with a current within 12 mA may be let through practically without widening.

The solution corresponding to the " smooth " envelope for the case $F_0 = 0$ and $R_0 \approx R_a$ has been obtained in the paper by Clogston and Heffner [7]. This solution corresponds to the results presented in this section.

V. RESULTS OF NUMERICAL INTEGRATION OF EQUATIONS FOR ENVELOPES

Those cases when the modulation of the envelopes with the field focusing period is not small are the most interesting for linear proton accelerators. In this general case, the Equations (46), (47) were integrated numerically by means of an electronic computer.

For numerical integration, it is expedient to diminish the number of independent parameters in the Equations (46), (47), passing to dimensionless envelopes.

$$\delta_x = \frac{r_x}{\sqrt{F_0}}; \quad \delta_y = \frac{r_y}{\sqrt{F_0}} \tag{69}$$

The current values of functions $\delta_x(\tau)$ and $\delta_y(\tau)$ do not depend upon the absolute value of the phase volume of the particles F_0. By introducing the expression (69) into (46), (47) we obtain :

$$\left.\begin{aligned} &\frac{d^2\delta_x}{d\tau^2} + Q_x(\tau)\cdot\delta_x - \frac{1}{\delta_x^3} - \frac{2\delta_a^2}{\delta_x+\delta_y} = 0 \\ &\frac{d^2\delta_y}{d\tau^2} + Q_y(\tau)\cdot\delta_y - \frac{1}{\delta_y^3} - \frac{2\delta_a^2}{\delta_x+\delta_y} = 0 \end{aligned}\right\} \tag{70}$$

Here the parameter

$$\delta_a = \frac{r_a}{\sqrt{F_0}} \tag{71}$$

represents a dimensionless " Coulomb radius. "

The initial conditions for the solutions of the system of Equations (70) follow from the relations (43) to (45) :

$$\left.\begin{aligned} \delta_x &= \sqrt{\eta_x \cos^2 \alpha_x + \frac{1}{\eta_x} \sin^2 \alpha_x} \\ \frac{d\delta_x}{d\tau} &= \frac{1}{2\delta_x}\left(\eta_x - \frac{1}{\eta_x}\right) \sin 2\alpha_x \\ \eta_x &= \frac{M_x}{N_x} \end{aligned}\right\} \tag{72}$$

The expressions for δ_y, $\dot{\delta}_y$ and η_y are obtained from Eq. (72) by changing the indices.

The numerical solution of the equation system (70) has been carried out for an idealised focusing period represented by its functions Q_x and Q_y in Fig. 4.

As has been shown by numerical calculations the solutions of the Equations (70) restricted at $\delta_a = 0$, remain limited at any value $\delta_a > 0$. When $\delta_a = 0$, the stability region of the solutions for the given channel are within the following limits : when $\gamma = 0$, $0 < K < 1.98$ and when $\gamma = 0.326$, $1.27 < K < 2.09$. The value $\cos \mu = 0$ corresponds to the following values of K : $K = 1.72$ (when $\gamma = 0$) and $K = 1.82$ (when $\gamma = 0.326$).

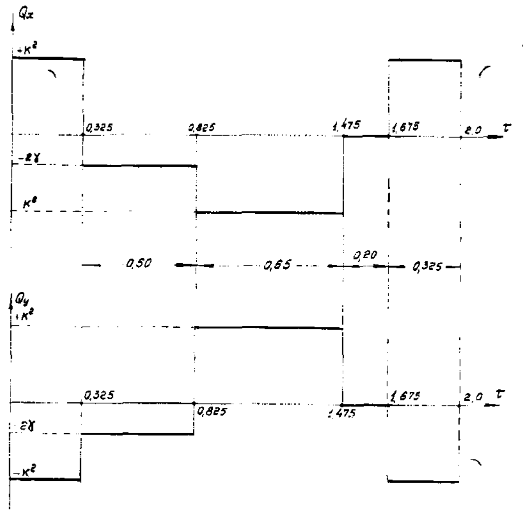

Fig. 4 Diagram of functions $Q_x(\tau)$ and $Q_y(\tau)$, used in the numerical integration of the equations for the envelopes.

For every combination of the values of the parameters δ_a and K, γ (which are within the region of stability) it is possible to find periodical solutions for the envelopes with the period of the focusing field. These solutions take place at definite initial conditions which depend on the values of the parameters K, γ and δ_a. The initial conditions according to Eq. (72) may be taken either as the values δ_x, δ_y, $\dot{\delta}_x$, $\dot{\delta}_y$, which describe directly the initial beam envelope at a given phase volume F_0, or by the relations of half axis of the ellipses η_x, η_y and the angles of inclination of the ellipses α_x, α_y on the planes x, $\dot{x}$ and y, $\dot{y}$ (Fig. 1). The dependence of the values η_x, η_y, α_x, α_y upon δ_a at different parameters of the channel is shown in Fig. 5 and 6. If the relations of half axes and the angles of inclination of the projections of the phase volume on the planes x, $\dot{x}$ and y, $\dot{y}$ correspond to the periodical solutions for the envelopes of the beam, then this phase volume is called further on " matched " with the channel and the beam is called a " matched " beam.

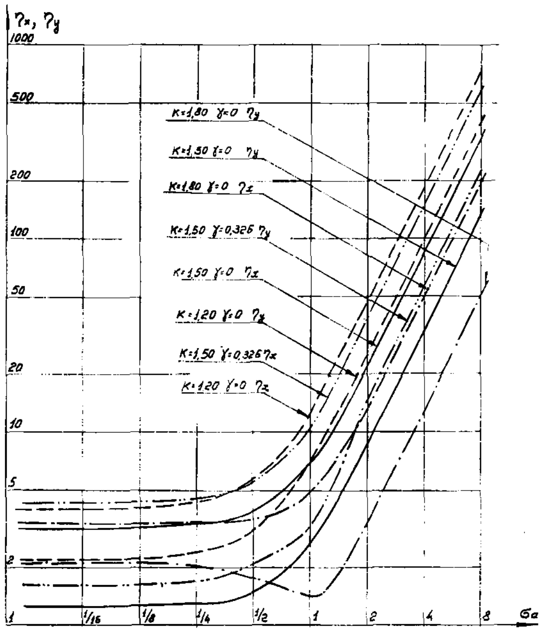

Fig. 5 The dependence of the matched initial conditions upon the dimensionless " Coulomb radius ". The relations of the half axes of phase ellipses.

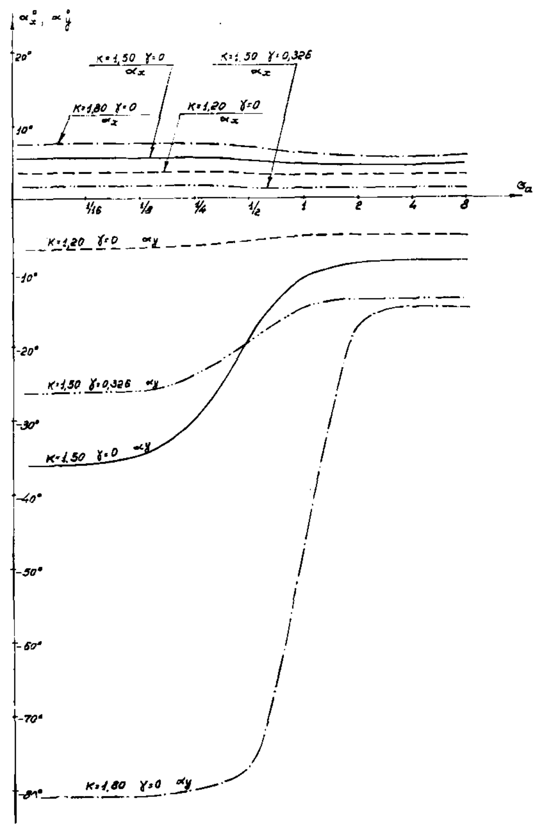

Fig 6 The dependence of the matched initial conditions upon the dimensionless " Coulomb radius ". The angles of inclination of the phase ellipses.

With a small beam current and a big phase volume ($r_a \ll \sqrt{F_0}$) the matched volume does not depend upon $r_a/\sqrt{F_0}$ and corresponds to the case when the space charge is absent. In another extreme case when the beam current is great and the phase volume value is small ($r_a \gg \sqrt{F_0}$), the angles of inclination of the ellipses α_x, α_y as formerly do not depend upon

$r_a/\sqrt{F_0}$ (but have different characteristic values); as to the half axes of the matched ellipses, they grow proportionally to the square of $r_a/\sqrt{F_0}$. Thus, the smaller the phase volume F_0 is, compared with the " Coulomb radius " square r_a^2, the more elongated the matched ellipses ε on the planes x, $\dot{x}$ and y, $\dot{y}$ are. In other words, the greater the beam current (with a given value of the phase volume) is, the smaller the scattering of the transversal velocities of the particles in the " matched " input beam should be.

The maximum values of r_x and r_y in the focusing period for the " matched " beam are the same. In Fig. 7 the relation of the maximum transversal size r_{max} of the envelopes of the " matched " beam to $\sqrt{F_0}\,\delta_{max} = r_{max}/\sqrt{F_0}$ is plotted on the axis of ordinates. The presented diagrams help to trace the dependence of r_{max} upon the " Coulomb radius " r_a at any fixed value of the phase volume F_0. Fig. 8 shows the dependence of $r_{max}/r_a = \delta_{max}/\delta_a$ for the " matched " beam upon the value $\sqrt{F_0}/r_a = 1/\delta_a$. It follows from the diagrams given in both figures that if $r_a \ll \sqrt{F_0}$ the transversal sizes of the " matched " beam do not depend upon the beam current but depend linearly upon $\sqrt{F_0}$, while when $r_a \gg \sqrt{F_0}$ the transversal sizes of the " matched " beam do not depend on the phase volume value and depend linearly upon the " Coulomb radius " r_a.

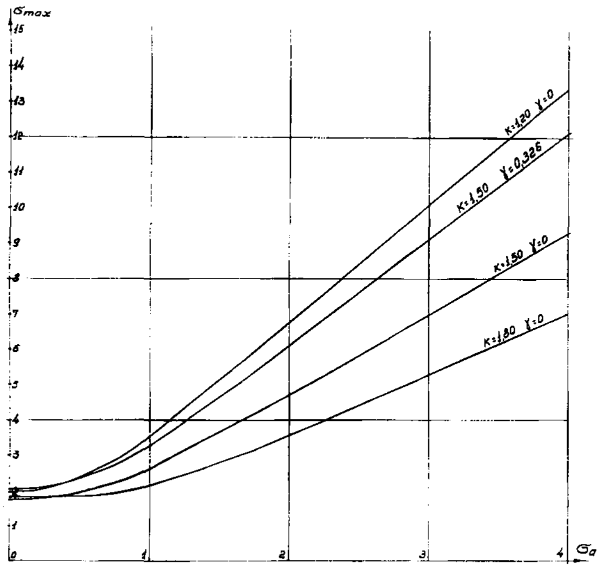

Fig. 7 The dependence of the maximum size of the matched beam upon " Coulomb radius " (with a fixed phase volume value).

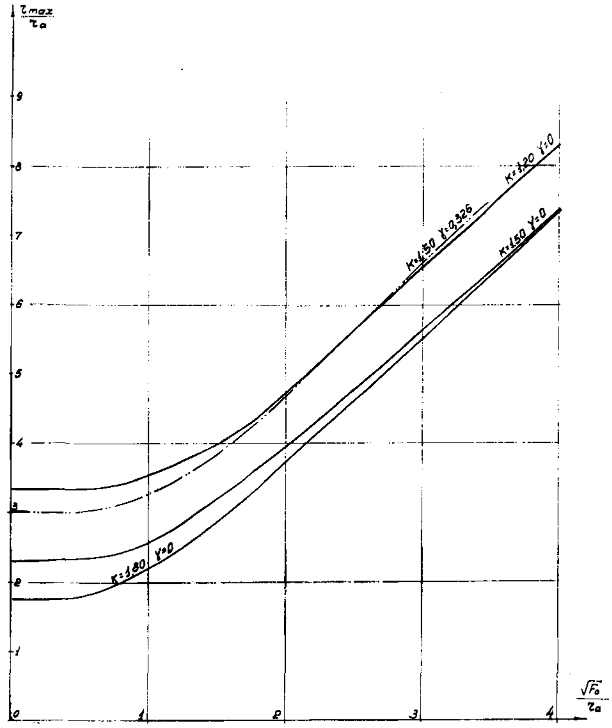

Fig. 8 The dependence of the maximum size of the matched beam upon phase volume (with a fixed value of " Coulomb radius ").

If the biggest beam size r_{max} is given, then the diagrams on Fig. 7 for the accepted channel characteristic help to determine $r_a/\sqrt{F_0}$ depending upon $r_{max}/\sqrt{F_0}$. Consequently, with a known value of the phase volume F_0, it becomes possible to determine the maximum permissible value of the " Coulomb radius " r_a. If the phase volume F_0 is small it is more convenient to use the diagrams of the type shown in Fig. 8 which make it possible to determine the maximum permissible value of r_a at a given r_{max} and at any F_0 (including $F_0 = 0$).

Let us discuss an example. Suppose $\beta = 0.04$, $\lambda = 200$ cm, $K = 1.50$, $r_{max} = 0.5$, $F_0 = 4\times10^{-3}$ cm^2. Then $\delta_{max} = 7.9$ which, at $\gamma = 0$, gives $\delta_a = 3.4$. This gives us for the maximum permissible " Coulomb radius " the value of $r_a = 0.21$ cm or for the beam current $I = 725$ mA. The " matched " beam is determined by the following initial conditions: $\eta_x = 58$, $\eta_y = 22$, $\alpha_x \approx 4.5°$, $\alpha_y \approx -9°$. In the absence of a space charge, the beam with the same phase volume matches when $\eta_x = 3.2$, $\eta_y = 1.2$, $\alpha_x = 5.5°$, $\alpha_y = -36°$ and has the maximum size $r_{max} = 0.1$ cm.

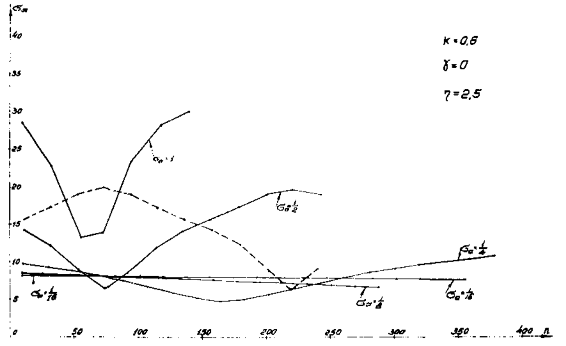

Fig. 9 The oscillations of the main maxima of the unmatched beam envelope.

To estimate the beam sizes when the phase volume is unmatched with the channel we have considered a case when a parallel beam of circular cross-section arrives at the inlet of the strong focusing channel (Fig. 4) : $\eta_x = \eta_y = \eta$, $\alpha_x = \alpha_y = 0$. In all the cases, the channel began from the middle of x-focusing lens as is shown in Fig. 4. The only exception is the estimate of the oscillations of the main maximums (Fig. 9); in this estimate, due to immaterial technical reasons, the channel began from the middle of the idle interval before the x-focusing lens.

As has been shown by numerical calculations, in the case of unmatched initial conditions, the local maxima of the envelopes which take place practically in every period of the focusing field (in some of the periods the maximum may be absent) vary with a certain period which is very slightly dependent upon the value δ_a. For briefness sake we shall call further on the greatest values of the local maxima in each period of oscillations of these maxima—the main maxima. In a long channel, the main maxima, in their turn, alternate with a big period. This complex pattern of oscillations is shown in Fig. 9 where the numbers of the focusing periods are plotted along the absciss axis and the main maxima of the functions $\delta_x(\tau)$ are plotted along the ordinate axis. The diagrams are given

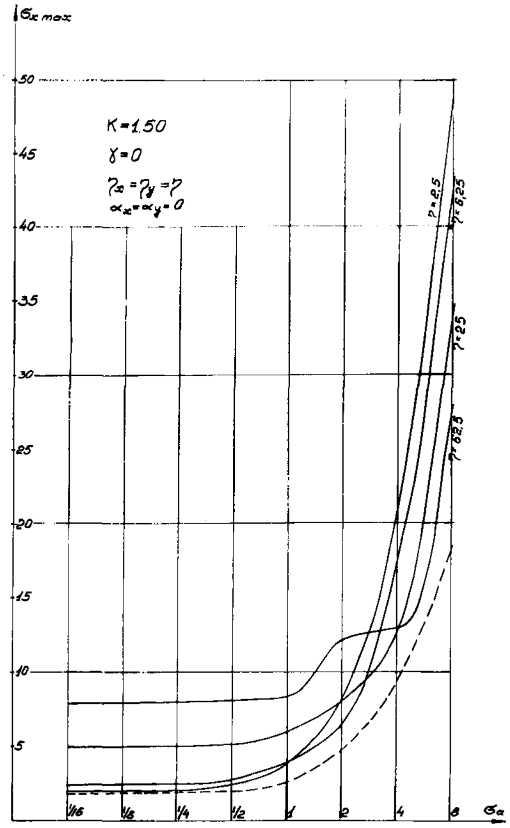

Fig. 10 The dependence of the first main maximum in the xOz plane upon the dimensionless " Coulomb radius ". The case when $K = 1.50$, $\gamma = 0$.

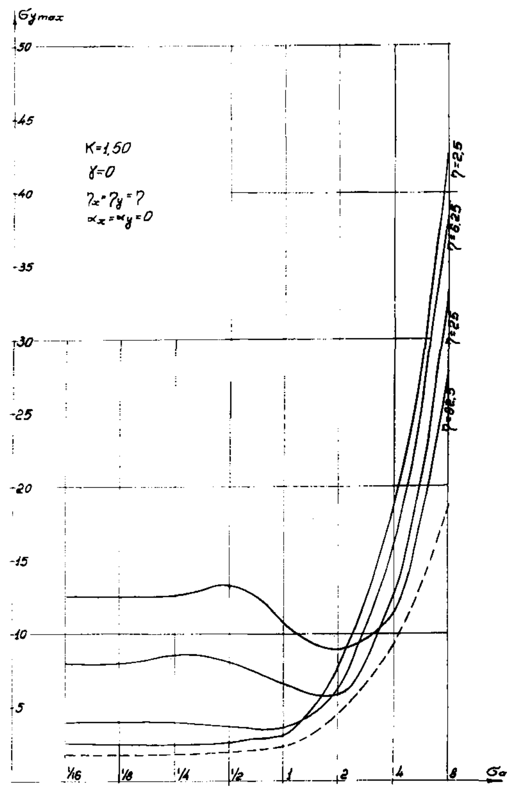

Fig. 11 The dependence of the first main maximum in the yOz plane upon the dimensionless " Coulomb radius ". The case when $K = 1.50$, $\gamma = 0$.

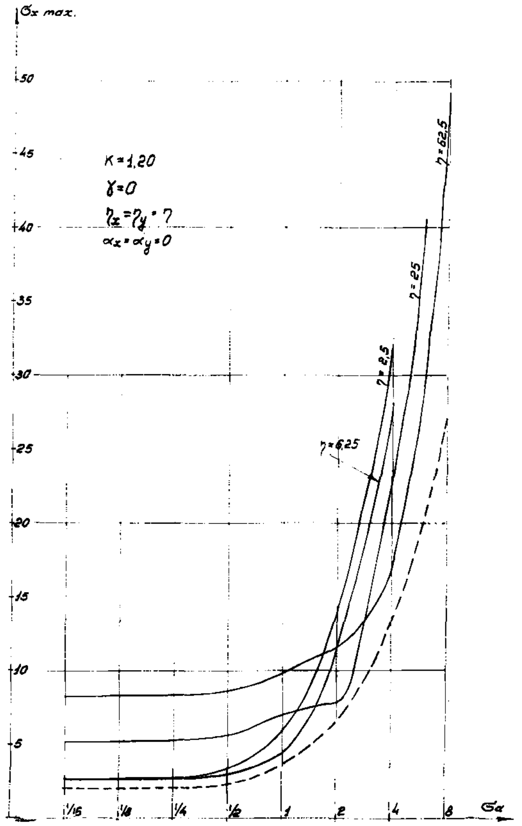

Fig. 12 The dependence of the first main maximum in the xOz plane upon the dimensionless " Coulomb radius ". The case when $K = 1.20$, $\gamma = 0$.

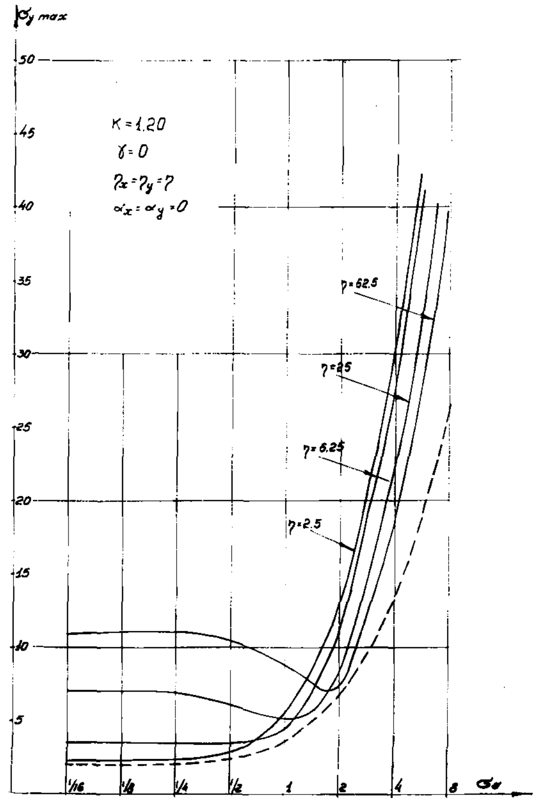

Fig. 13 The dependence of the first main maximum in the yOz plane upon the dimensionless " Coulomb radius ". The case when $K = 1.20$, $\gamma = 0$.

for the case when $K = 0.6$, $\gamma = 0$, $\eta = 2.5$ at different values of δ_a. The functions $\delta_y(\tau)$ are shifted by half a period; a dotted line in Fig. 9 shows the behaviour of the main maxima of the function $\delta_y(\tau)$ for $\delta_a = 1/2$.

In practical cases, the length of the strong focusing channel is usually much below the period of the main maxima. One should also bear in mind that in linear accelerators the length within which the values δ_a and γ remain approximately constant are, ordinarily, substantially less than the length of the accelerator. This is why the value of the first main maximum describes sufficiently well the maximum beam size in the channel at the given channel-parameters.

In Figs. 10 to 13 the diagrams show the dependence of the first main maximum of the functions $\delta_x(\tau)$ and $\delta_y(\tau)$ upon the parameter δ_a for $\gamma = 0$, with the two values of $K = 1.20$, $K = 1.50$ and four values of η. The dotted curves show the dependence of the maximum values of δ_x, δ_y upon the parameter δ_a for the phase volume matched in each case. It follows from these diagrams that for each combination of the values of parameters K, γ, δ_a the maximum size for the matched beam is smaller than for the " unmatched " beams while the phase volume F_0 remains the same. This conclusion fully coincides with similar conclusions known for the beam in the absence of the space charge and those obtained in the previous section for the beam with a " smooth " envelope. Further, when the parameter δ_a has small values, the beams with a small value of η are closer to the " matched " beams. As the parameter δ_a grows the beams with increased values of η are closer to the " matched " one. It should be noted that when $\delta_a \gg 1$, the beam with a given value of η may be close to the " matched " one only in a short interval of the parameter δ_a alternation; in other words, at big beam currents, the matching of the phase volume requires greater accuracy than in the case of weak currents.

The maximum size of the unmatched beam at the same values of the current and the phase volume may surpass substantially the size of the " matched " beam. So, with the conditions of the previous example ($\beta = 0.04$, $\lambda = 200$ cm, $K = 1.50$, $\gamma = 0$ and $F_0 = 4 \times 10^{-3}$ cm^2) when $\eta_x = \eta_y = 62.5$, $\alpha_x = \alpha_y = 0$ and when the beam current is $I = 725$ mA, we have $r_{max} = 0.8$ cm. For this beam we would have $r_{max} = 0.5$ cm with a current of $I = 62$ mA. This indicates how expedient it is to prepare the beam by a system of matching lenses which should becalculated taking into account the Coulomb repulsion.

In conclusion we shall note that since at $\delta_a \gg 1$ the beam cross-sections are practically independent of the phase volume, the quantitative appreciations for this case will be approximately the same for any assumption regarding the law of particle distribution in phase space.

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DISCUSSION

SYMON : I would like to ask Vladimirskij how close the calculated currents agree with those found in actual accelerators, when one takes into account the space-charge effects?

VLADIMIRSKIJ : Unfortunately, we do not yet have an actual accelerator.

SEIDL : I would like to report about a kind of space-charge instability observed in toroidal electron beams.(*)

BLEWETT, J. P. : Could Vladimirskij summarize the graphs by saying how much current one can get through a 1 cm aperture in a linear accelerator?

VLADIMIRSKIJ : The maximum current in our type of accelerator is about 300 mA.

(*) This contribution is fully reported on p. 327 of the Proceedings.