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On dynamical three-dimensional higher-spin gravity solutions

By

Manuel Vielma

Advisor: Prof. Julian Sonner - University of Geneva

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Abstract: We present a framework that allows to find dynamical higher-spin gravity solutions in three dimensions. The dynamics is provided by particles which are coupled to the gravitational background. This coupling is performed in the language of gauge theory, a possibility that arises in three dimensions due to the fact that gravity can be recast in this specific case as a particular gauge theory known as Chern-Simons. The key feature of this formulation is that it is conceivable to extend the gauge group, thus passing from pure (spin-2) gravity to higher-spin gravity. Building on a setup that mirrors the ones leading to certain models of gravitational collapse, we derive a set of equations of motion for the gauge connections whose integrals correspond to dynamical solutions, a characteristic absent in the higher-spin solutions known to date. Via the gauge/gravity duality, the kind of dynamical gravity solutions we discuss are used to describe progress towards thermalization of a field theory that has been injected some energy. By investigating higher-spin generalizations of these solutions we therefore hope to shed some light on further aspects of that very process.

Keywords: 3D gravity, Higher-spin black holes, Chern-Simons gravity, Vaidya solution.

A mis hermanos Greta y Jhon,
y a la memoria de Olexis Vielma.

Contents

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Chapter 1

Introduction

A duality relating gravity theories on asymptotically Anti-de-Sitter (AdS) spacetimes to conformal field theories (CFTs) was conjectured by the physicist Juan Maldacena near the end of the 1990's [1]. In this duality, dubbed AdS/CFT, a higher dimensional bulk – the gravity theory – is mapped into a lower dimensional boundary – the CFT. In this sense, the aforementioned duality is then a particular realization of the holographic principle, which in the context of semi-classical considerations for quantum gravity asserts that the information stored in a volume V_{d+1} is encoded on its boundary area A_d measured in units of the Planck area [2]. In its strongest version, AdS/CFT implies the complete physical equivalence between the gravity theory and the quantum field theory in consideration, meaning that the parameters of each theory are identified with those of the other for generic values of those parameters. However, since different regimes of the parameters are usually associated with perturbative or non-perturbative sectors of the theories, explicit computations are out of reach for arbitrary ranges, particularly in the non-perturbative part. A possible simplification consists in specializing to the weakly coupled regime of string theory, which corresponds to a classical gravity theory (and some possible corrections thereof). It turns out that this specialization can be done keeping the field theory in its strongly coupled regime. In this particular, more tractable limit AdS/CFT then becomes an example of a strong-weak coupling duality, providing a tool (the gravity theory) to gain new insights into the highly non-trivial non-perturbative sector of the quantum field theory. This remarkable feature has allowed the AdS/CFT correspondence to find domains of applicability ranging from hydrodynamics [3] to quark-gluon plasma [4], and to superfluidity and superconductivity [5], among others[1].

Particularly interesting setups into which the AdS/CFT correspondence can offer new insights are those involving dynamics. Examples of phenomena intrinsically related to the presence of dynamics in the field theory side which have been addressed holographically include thermalization, transport and chaos. For instance, the correspondence has been used as a tool to probe the scale dependence of thermalization in two-dimensional strongly coupled field theories which have been injected some finite energy [6]. Probes of thermalization can be achieved by comparing the instantaneous values of non-local quantities such as two-point functions or entanglement entropy with the values at thermal equilibrium[2]. While calculating these quantities from first principles in the quantum field theory is challenging, they can be computed relatively easily using holography. Through the AdS/CFT correspondence, the approach to thermal equilibrium in the boundary is related to the process of black hole formation in the bulk. This is so because states of finite temperature in the CFT correspond to black hole solutions in the bulk, or at least to solutions sharing some of their characteristic features [2]. A simple model of black hole formation by collapse in AdS which has offered insight into the dynamics of two-dimensional strongly field theories is the one described by the ingoing Vaidya metric in three dimensions:

$$ds^2 = -F(r,v)dv^2 + 2drdv + r^2 d\varphi^2, \tag{1.1}$$

where $F(r,v) = \left(1 + \frac{r^2}{\ell^2} - M(v)\right)$, ℓ is the AdS radius, v is the ingoing Eddington-Finkelstein coordinate, and r and φ are the standard spherical coordinates. Vaidya metrics are solutions to the Einstein equations with a null energy-momentum tensor of the form $T_{\mu\nu} = \frac{M'(v)}{2r}\delta^v_\mu \delta^v_\mu$. As we will see, these metrics are to be viewed as non-static generalizations of the Schwarzschild

[1]For a thorough and pedagogical introduction to Maldacena's correspondence the reader can refer to the recent book [2]. We shall not try to give a detailed exposition of the AdS/CFT correspondence as its role in our considerations is of a more indirect, inspirational nature. Suffice it to say, for the advanced reader, that in Maldacena's seminal paper the map $g^2_{YM} = 2\pi g_s$ and $2g^2_{YM}N = L^4/\alpha'^2$ is conjectured between the free parameters of $\mathcal{N} = 4$ Super Yang-Mills theory and of type IIB superstring theory. The more tractable limit to which we have referred involves taking $N \to \infty$ and $g_s \to 0$. The strongly coupled regime of the field theory corresponds to taking $g^2_{YM}N \to \infty$, which amounts to considering the limit $\alpha'/L \to 0$ in the string theory side.

[2]Notice that a local gauge invariant quantity such as the energy momentum tensor ($T^{\mu\nu}$) would not provide information about progress towards thermalization. For instance, for spatially homogeneous states of a weakly interacting massless scalar field, $T^{\mu\nu} = \int \frac{d\mathbf{k}}{k^0} k^\mu k^\nu n(\mathbf{k})$, where $n(\mathbf{k})$ denotes the occupation number of a momentum mode of the scalar field. $T^{\mu\nu}$ then does not tell us anything about the particle distribution, unlike, for example, the equal-time two-point function, given by $G(x) = \int \frac{d\mathbf{k}}{k^0}[n(\mathbf{k}) + 1]\exp(i\mathbf{k}\cdot x)$.

solution to Einstein's field equations, in which case M is constant (see [8] for a detailed treatment of Vaidya metrics). A particular example of an ingoing Vaidya metric which will play an important role in our explorations is that of a thin null shell falling along $v = v_0$, which corresponds to the mass profile $M(v) = m\,\Theta(v - v_0)$, with m constant.

Via the AdS/CFT dictionary, the computation of the equal-time two-point function can be reduced to some intrinsically geometrical bulk calculation. Indeed, following [9], equal-time Green functions in CFT_2 can be computed via a path integral as

$$\langle \mathcal{O}(t,\mathbf{x})\mathcal{O}(t,\mathbf{x}')\rangle = \int \mathcal{D}\mathcal{P} e^{i\Delta L(\mathcal{P})} \approx \sum_{\text{geodesics}} e^{-\Delta \mathcal{L}}, \tag{1.2}$$

where the paths, of proper length $L(\mathcal{P})$, begin and end at the boundary points $(t,\mathbf{x})$ and $(t,\mathbf{x}')$. The last expression is a saddlepoint approximation, effective for $\Delta \gg 1$, in which $\mathcal{L}$ is the length of *geodesics* connecting the boundary points. Here Δ is a quantity known as the conformal dimension of the operator $\mathcal{O}(t,\mathbf{x})$. It is given by $\Delta = \frac{1}{2}(d + \sqrt{d^2 + 4m^2})$, where d is the dimensionality of the CFT and m is the mass of the scalar field that is holographically related to the operator $\mathcal{O}(t,\mathbf{x})$ [6].

In addition to the correlation function, a quantity that constitutes a powerful probe of dynamics of a field theory is the entanglement entropy. Consider a quantum mechanical system in a state $|\Phi\rangle$ that consists of two disjoint parts A and B. The entanglement entropy of the system restricted to A is given by the von Neumann entropy

$$S(A) = -\mathrm{tr}_A(\rho_A \ln \rho_A), \tag{1.3}$$

where ρ_A is the reduced density matrix obtained by taking a partial trace over the subsystem B, i.e $\rho_A = \mathrm{tr}_B|\Psi\rangle\langle\Psi|$. The entanglement entropy provides us with a convenient measureof the entanglement of a given wavefunction $|\Psi\rangle$ is[3], and its time evolution has proved to be a useful tool in the understanding of dynamics in CFTs, as illustrated by Cardy and Calabrese in their remarkable work [10] (see [11] and references therein for computations using holography). A proposal to compute holographically the entanglement entropy of a

[3]For illustration, consider a bipartite system with composite Hamiltonian $H_A \otimes H_B$. A pure state in this system which can be written in the form $|\Psi\rangle = |\phi\rangle_A \otimes |\varphi\rangle_B$ is called separable and the reduced states are simply $\rho_A = |\phi\rangle_A\langle\phi|_A$, and similarly for ρ_B. If the state is not separable in the sense just mentioned, then it is said to be an entangled state, and ρ_A, ρ_B are necessarily mixed. It is this mixture that the von Neumann entropy measures. A characteristic feature of entanglement is that in this case the von Neumann entropy of the global system is smaller than the sum of the von Neumann entropies of the parts.

region A was developed by Ryu and Takayanagi [14, 15]. Just as for the correlation function, such computation rests on a geometrical quantity defined in the bulk. Explicitly, the Ryu-Takayanagi prescription allows to compute the entanglement entropy S_{EE} in a d-dimensional CFT via

$$S_{EE} = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+1)}} \tag{1.4}$$

where γ is the $d-1$-dimensional minimal surface in AdS_{d+1} whose boundary is given by ∂A, and $G_N^{(d+1)}$ is the $d+1-$dimensional Newton constant. The proposal (1.4) suggests a highly non-trivial relation between entanglement and geometry that has inspired a number of works aiming to understand whether the connectedness of the bulk spacetime emerges from the entanglement of the boundary degrees of freedom [16].

In contrast to what happens in higher dimensions, there is a lot of analytic control over two-dimensional conformal field theories. This makes of $\text{AdS}_3/\text{CFT}_2$ a particularly rich arena allowing to explore both sides of the correspondence. Furthermore, AdS_3 has the remarkable property of being topological, meaning that it has no local propagating degrees of freedom. A quick counting argument supports our claim: In d dimensions, the phase space of general relativity is parametrized by a spatial metric at constant time, which has $d(d-1)/2$ components, and its conjugate momentum, which has the same number of components. However, d of the Einstein equations are not dynamical equations but constraints, and d more degrees of freedom can be eliminated by coordinate choices [19]. This leaves us with $d(d-1) - 2d = d(d-3)$ degrees of freedom per point. If $d = 4$, we have the four phase space degrees of freedom corresponding to the two gravitational polarizations and their conjugate momenta. If $d = 3$, *there are no local degrees of freedom*.

As we shall see, many tasks are enormously simplified in three dimensions due to the absence of local degrees of freedom. It should be pointed out, however, that this property does not render the theory trivial, as was believed for many years. The major non-triviality comes from the fact that, as shown by Bañados, Teitelboim and Zanelli [17], three-dimensional gravity has a black hole solution. We mentioned before that black holes are dual to finite-temperature states in the CFT. This means that the existence of this solution, known as the BTZ black hole, is tantamount to the possibility of studying thermodynamic aspects of CFT_2 via AdS/CFT. Such a possibility calls also for possible generalizations of the BTZ black hole which would account for a corresponding generalization of the CFT scenario. In this thesis we will be mostly concerned with attempts at generalizing this solution to scenarios with higher-spin. The key point that allows to attempt such a generalization is the realization that, as a consequence of its topological nature, AdS_3 turns out to be closely related to a gauge theory known as Chern-Simons [18, 19]. The relation is established via the identification

$$\begin{aligned} S_{EH} =& \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{Tr}\left(\mathcal{A}\wedge d\mathcal{A}+\frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right) \\ =& \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{Tr}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right)-\frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{Tr}\left(\bar{A}\wedge d\bar{A}+\frac{2}{3}\bar{A}\wedge\bar{A}\wedge\bar{A}\right) \end{aligned} \tag{1.5}$$

where $\mathcal{A}\in so(2,2)$, and k, called the Chern-Simons level, takes values in a discrete subgroup of the real numbers[4]. In the last line we have made use of the fact that $SO(2,2)$ can be written as $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$, implying that $A,\bar{A}\in sl(2,\mathbb{R})$.

Among the various advantages of the gauge theory reformulation of 3D gravity of (1.5) there is the striking fact that one can study gravity coupled to higher-spin fields by simply promoting the gauge group from $SL(2,\mathbb{R})$ to $SL(N,\mathbb{R})$ [21]. A related interesting feature of these $SL(N)$ theories is that the number of higher-spin fields is controlled by the choice of the rank of the group. In this respect these theories differ fundamentally from different approaches to higher-spin solutions, where an infinite tower of spins is necessary to ensure causality [13][5]. When dealing with the $SL(N)$-type theories we have mentioned, something important to note is that, while the spin-2 metric is invariant under $sl(2,\mathbb{R})$ transformations (diffeomorphisms), it is does not remain unchanged under $sl(N,\mathbb{R})$ transformations. This is so because, under these higher-spin transformations, the metric gets mixed with the higher spin fields, which implies that intrinsically geometric notions such as black hole horizons or geodesics are no longer gauge invariant. This last feature was investigated in e.g. [24], where it was shown that a higher spin solution corresponding to a black hole metric in a given gauge leads to a wormhole metric in another gauge.

If geometric notions are not gauge invariant in the $SL(N)$ higher spin gravity theories, then the prescriptions to holographically probe field theory dynamics that we discussed above become useless. In particular, since different choices of gauge yield to different metrics, the Ryu-Takayanagi formula (1.4) cannot be used to compute the entanglement entropy. In view of this limitation, Ammon, Castro and Iqbal proposed in [25] a gauge invariant generalization

[4]See [20] for a review of general aspects of Chern-Simons theory.

[5]This could actually be a negative feature of the $SL(N)$-type theories, since from some string-theory motivated considerations one expects an infinite tower of higher-spins. See [12] for some recent comments on this issue.

of Ryu-Takayanagi according to which the entanglement entropy of a single interval X (a subdomain in the CFT) is

$$S_{EE} = -\log(W_{\mathcal{R}}(C)) = -\log\left(\mathrm{tr}_{\mathcal{R}}(\mathcal{P}\exp\int_C \mathcal{A})\right), \tag{1.6}$$

where $W_{\mathcal{R}}(C)$ is a bulk Wilson line in a (infinite-dimensional) representation $\mathcal{R}$ of the gauge group $SL(N,\mathbb{R})$ along a curve C that ends at the boundary at ∂X, and $\mathcal{P}$ denotes path ordering. If this proposal generalizes the Ryu-Takayanagi formula, then it should reproduce the latter in the $SL(2,\mathbb{R})$ case, which corresponds to pure gravity. This means that in this case (1.6) should encode information about, for example, the length of a geodesic, which is the kind of quantity involved in the Ryu-Takayanagi formula. However, after a quick look at the equation (1.6), it seems hopeless to find that kind of information encoded on the Wilson line, in particular because the only geometrical element present, the path C, is actually irrelevant (the result of the integral is the same regardless of the path C; only the endpoints will matter). To see how the information about geodesics could be encoded on Eq. (1.6), we must recall that these curves are nothing but the paths followed by probe particles in a given geometry. So, if we manage to read from (1.6) some information about paths followed by point particles, then we would have a way to understand how this equation reduces to the Ryu-Takayanagi proposal in the $SL(2,\mathbb{R})$ case. For this, of course, we first need to find out where in (1.6) such particles would be presumably hiding.

The answer to the puzzle just posed above is in the choice of the representation $\mathcal{R}$ in (1.6). Indeed, as is well known, infinite dimensional representations of certain groups can be built using as a basis the Hilbert space of a free particle[6]. This means that if one chooses an infinite-dimensional representation $\mathcal{R}$, the probe particles could naturally arise from the auxiliary fields U of the Hilbert space which would serve as a basis. This not only "explains" why (1.6) reduces sensibly to the Ryu-Takayanagi proposal, but also provides a way to compute the 'tr' in our infinite-dimensional representation[7] by means of a trace over the Hilbert space that defines the partition function in the Hamiltonian formalism[8]. For us, the important

[6] The canonical example is the representation of the Poincaré group on one-particle states. See footnote 6 for more details.

[7] The particular infinite-dimensional representations used by Ammon, Castro and Iqbal are known as the *highest-weight* representations of $SL(N,\mathbb{R})$. See [25] for details

[8] More precisely, we want to replace the quantum mechanical trace over $\mathcal{R}$ in (1.6) by a path integral over the auxiliary field U which is attached to the curve C and coupled to the connection $\mathcal{A}$ as a background field, so that schematically $W_{\mathcal{R}}(C) = \int \mathcal{D}U e^{[-S(U;\mathcal{A})_C]}$, with S the action of the system. The realization that the Wilson line can be computed in this way was actually the original motivation to introduce the auxiliary quantum mechanical system, and it constitutes one of the many

bit of this discussion is the presence of the probe particles and the possibility of coupling them to the background connections. As we shall see, in the $SL(2,\mathbb{R})$ (pure gravity) case, this coupling has a natural general relativity-like realization (one simply adds the action of the particle to the Einstein Hilbert action). This general relativity-inspired coupling can be generalized to scenarios with higher spins. In any case, the key point is that, as proposed in [41], the action of this particle can be written in the language of group manifolds in terms of our $SL(N,\mathbb{R})$-valued field U and its (algebra-valued) "conjugate momentum" P. As we shall see, this all follows from the fact that the metric of a manifold M of Lie group G can be written as

$$g_{\mu\nu} = \mathrm{Tr}(U^{-1}\partial_\mu U U^{-1}\partial_\nu U), \tag{1.7}$$

where $U \in G$.

With these particles at hand, one could imagine (minimally) coupling to our Chern-Simons gravity a shell of matter formed by an infinite number of them. If this shell follows null geodesics (i.e. collapses at the speed of light), then this would exactly mirror the kind of setup that leads to the Vaidya-like solutions (1.1), as we shall review in detail. Moreover, since, as we discussed, we can extend the gauge group to any $SL(N,\mathbb{R})$ we like, these particles could carry higher-spin charges as determined by the values of the Casimirs of the representation[9]. The question then arises as to the possibility of building in this way Vaidya-like solutions with non-zero higher spin charge. Since the Vaidya metric is non-static, in this way we would be building higher-spin solutions which, being Vaidya-inspired, are *dynamical*, a feature absent in all the higher-spin solutions known to date. To address this question is the purpose of this thesis.

The presentation of our explorations regarding the dynamics of higher-spin gravity theories is divided into four main parts. A first part, condensed in Chapter 2, is devoted to the aforementioned special features of gravity in three dimensions. We review the considerations leading to the recast of 3D gravity as a Chern-Simons theory and we present a number

gems of [22]. Our way of introducing the Hilbert space was mostly guided by the necessity of the probe particles, but both approaches are of course intrinsically related.

[9] A Casimir operator is an operator that commutes with all the generators of a given algebra. The eingenvalues of the Casimirs label the representation and are associated with the spin charges that arise in the system under study. For example, in the (total) angular momentum algebra, J^2 is a quadratic Casimir and its eigenvalues are related to the total angular momentum quantum numbers. In analogy to the square one, the cubic, quartic,... Casimirs control the value of the higher spin charges. See footnote 6 for some further comments.

of known three-dimensional gravity solutions, including the BTZ black hole as well as a higher-spin example. Chapter 3 contains the second part of this work, which deals with the possibility of adding dynamics by considering probes following geodesics in a Chern-Simons background. We do this building on the example of a shell sourcing the Vaidya metric. As we said, the translation of this example into the gauge theory language demands that we recast the action of the particle in a group-theoretical fashion; we review how this is done starting from (1.7). The equations of motion of the total action obtained by coupling the particle action thus obtained to the Einstein-Hilbert one are also derived in this chapter. Possible ways of solving these very equations is the subject we address in Chapter 4, which contains the main original contributions of this thesis. Specifically, we see how to recover the Vaidya metric from our gauge-theoretical equations, and we initiate the study of solutions carrying non-zero spin-3 current. Our presentation concludes with a final chapter devoted to a general discussion of our results.

Chapter 2

$2+1-$ dimensional gravity as a Chern-Simons theory

2.1 The vielbein formalism

A smooth manifold M comes naturally with its tangent spaces T_p, which can be identified with the space of directional derivative operators along curves through points p of the manifold. Intuitively, one says that the tangent spaces contain the possible "directions" at which one can tangentially pass through the points p. If one considers a coordinate chart with coordinates x^μ, then there is an obvious set of directional derivatives at a given point, namely the partial derivatives ∂_μ, which form a particular basis for the tangent space T_p. Nothing prevents us, however, from setting up different sets of bases. In particular, one can conceive an orthonormal basis $\hat{e}_{(a)}$ defined by the orthonormality condition

$$g(\hat{e}_{(a)}, \hat{e}_{(b)}) = \eta_{ab}, \tag{2.1}$$

or equivalently

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}, \tag{2.2}$$

where η_{ab} is the canonical form of the metric (e.g. Minkowski in a Lorentzian space) and $e_\mu{}^a$, known as the vielbeins, are the matrices that take us from the old to the new basis, i.e.

$$\hat{e}_{(\mu)} \equiv \partial_\mu = e_\mu{}^a \hat{e}_{(a)}. \tag{2.3}$$

A notion associated with the one of the tangent space is that of the cotangent space T_p^*, which is defined as the set of linear maps $\omega : T_p \to \mathbb{R}$. The elements of the cotangent space are known as one-forms. The canonical example of a one-form is the gradient of a function $\mathrm{d}f$. Indeed, given a coordinate chart, the most natural basis for the cotangent space is provided by the gradients dx^μ, since in this case one has $\mathrm{d}x^\mu(\partial_\nu) = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu$. This last expression is precisely what defines in general the *duality* condition between the basis of the tangent space and that of the cotangent space.

Because the gradients $\mathrm{d}x^\mu$ of the coordinate functions are a natural coordinate basis for the cotangent space T_p^*, the vielbeins can be thought of as the components of a $(1,1)$ tensor of the form:

$$e = e_\mu{}^a \mathrm{d}x^\mu \otimes \hat{e}_{(a)} \tag{2.4}$$

Note that, as long as the orthonormality condition (2.1) is satisfied, the orthonormal basis can be changed independently of the coordinates. The transformations that leave the flat metric η_{ab} invariant are, according to its signature, orthogonal or Lorentzian transformations of the form

$$\Lambda^a{}_{a'}\Lambda^b{}_{b'}\eta_{ab} = \eta_{a'b'} \tag{2.5}$$

A mixed tensor transforms then as follows:

$$T^{a'\mu'}{}_{b'\nu'} = \Lambda^{a'}{}_a \frac{\partial x^{\mu'}}{\partial x^\mu} \Lambda^b{}_{b'} \frac{\partial x^\nu}{\partial x^{\nu'}} T^{a\mu}{}_{b\nu} \tag{2.6}$$

As in a coordinate basis, the covariant derivative of a tensor written in the orthonormal basis is given by its partial derivative plus connection terms:

$$\nabla_\mu X^a{}_b = \partial_\mu X^a{}_b + \omega_\mu{}^a{}_c X^c{}_b - \omega_\mu{}^c{}_b X^a{}_c, \tag{2.7}$$

where the $\omega_\mu{}^a{}_b$, known for historical reasons as the spin connections, are the equivalent, in the non-coordinate basis, of the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$. A relation between the spin connection, the vielbeins, and the Christoffel symbols can then be found by writing ∇X in different bases. In the coordinate basis one has:

$$\nabla X = (\partial_\mu X^\nu + \Gamma^\nu_{\mu\lambda} X^\lambda)\mathrm{d}x^\mu \otimes \partial_\nu, \tag{2.8}$$

whereas in a mixed basis this becomes:

$$\begin{aligned}\nabla X &= (\nabla_\mu X^a)\mathrm{d}x^\mu \otimes \hat{e}_{(a)} \\ &= (\partial_\mu(e_\nu{}^a X^\nu) + \omega_\mu{}^a{}_b e_\lambda{}^b X^\lambda)\mathrm{d}x^\mu \otimes (e^\sigma{}_a \partial_\sigma) \\ &= (\partial_\mu X^\nu + e^\nu{}_a \partial_\mu e_\lambda{}^a X^\lambda + e^\nu{}_a e_\lambda{}^b \omega_\mu{}^a{}_b X^\lambda)\mathrm{d}x^\mu \otimes \partial_\nu .\end{aligned}$$

Equating both expressions leads to the relation

$$\Gamma^\nu_{\mu\lambda} = e^\nu{}_a \partial_\mu e_\lambda{}^a + e^\nu{}_a e_\lambda{}^b \omega_\mu{}^a{}_b, \tag{2.9}$$

or equivalently

$$\omega_\mu{}^a{}_b = e_\nu{}^a e^\lambda{}_b \Gamma^\nu_{\mu\lambda} - e^\lambda{}_b \partial_\mu e_\lambda{}^a. \tag{2.10}$$

The map between the coordinate and noncoordinate bases allows us to rewrite any quantity we like in terms of the vielbeins and spin connections. For example, for the torsion and curvature (Riemann tensor) one has[1]

$$T^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b \tag{2.11}$$

and

$$R^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \tag{2.12}$$

where the exterior derivative d allows us to differentiate p-form fields to obtain $p+1$-form fields. It is defined as the following normalized, antisymmetrized partial derivative:

$$(\mathrm{d}A)_{\mu_1\cdots\mu_{p+1}} = (p+1)\partial_{[\mu_1} A_{\mu_2\cdots\mu_{p+1}]} \tag{2.13}$$

where the pair of square brackets denotes antisymmetrization. For example:

$$T_{[abc]} = \frac{1}{3!}(T_{abc} - T_{acb} + T_{bca} - T_{bac} + T_{cab} - T_{cba}) \tag{2.14}$$

The wedge product $A \wedge B$ is defined as the antisymmetrized tensor product:

$$(A \wedge B)_{\mu_1\cdots\mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu_1\cdots\mu_p} B_{\mu_{p+1}\cdots\mu_{p+q}]} \tag{2.15}$$

[1]For illustration, let us check that one recovers the usual expression of the torsion in the coordinate basis: $T_{\mu\nu}{}^\lambda = e^\lambda{}_a T_{\mu\nu}{}^a = e^\lambda{}_a(\partial_\mu e_\nu{}^a - \partial_\nu e^a_\mu + \omega_\mu{}^a{}_b e_\nu{}^b - \omega_\nu{}^a{}_b e_\mu{}^b) = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}$, which is precisely the expression we were looking for.

The two equations (2.11) and (2.12) are known as the Cartan structure equations. In practice, the torsion-free condition

$$\mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0 \tag{2.16}$$

is the most convenient way to find the spin connection from the expressions of the vielbeins, as we will see later on.

2.2 Einstein-Hilbert as a Chern-Simons action

The action in Lagrangian form which leads to the Einstein field equations in vacuum is of course the Einstein-Hilbert action[2]:

$$S_{EH} = \frac{1}{16\pi G_N} \int d^3x \sqrt{g} \left(R + \frac{2}{\ell^2} \right) \tag{2.17}$$

where ℓ is related to the cosmological constant Λ via $\Lambda = -1/\ell^2$. Indeed, extremization of this action with respect to the spacetime metric $g_{\mu\nu}$ yields

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left(R + \frac{2}{\ell^2} \right) = 0, \tag{2.18}$$

which, in three dimensional space, determines the full Riemann tensor as

$$R_{\mu\nu\lambda\rho} = -\frac{1}{\ell^2} (g_{\mu\lambda} g_{\nu\rho} - g_{\nu\lambda} g_{\mu\rho}), \tag{2.19}$$

describing a symmetric space of *constant* negative curvature.

With the vielbein formalism at hand, it is natural to extend our map and rewrite the Einstein-Hilbert action in terms of objects in the noncoordinate basis. Such procedure goes as follows [39]:

[2] Note that we are omitting possible surface terms. For a discussion on boundary terms and asymptotic symmetries in AdS_3 see [40].

$$\begin{aligned}
S_{EH} &= \frac{1}{16\pi G_N}\int d^3x\sqrt{|g|}\left(R+\frac{2}{\ell^2}\right)\\
&= \frac{1}{16\pi G_N}\int d^3x\,|e|\left(\frac{1}{2}\epsilon_{\lambda\mu\nu}\epsilon^{\lambda\rho\sigma}R^{\mu\nu}{}_{\rho\sigma}\frac{2}{\ell^2}\right)\\
&= \frac{1}{16\pi G_N}\int d^3x\,\epsilon_{abc}\,e^a{}_\lambda e^b{}_\mu e^c{}_\nu\left(\frac{1}{2}\epsilon^{\lambda\rho\sigma}R^{\mu\nu}{}_{\rho\sigma}+\epsilon^{\lambda\mu\nu}\frac{2}{3!\ell^2}\right)\\
&= \frac{1}{16\pi G_N}\int d^3x\,\epsilon_{abc}\,e^a\wedge\left(R^{bc}+\frac{1}{3\ell^2}e^b\wedge e^c\right). \qquad (2.20)
\end{aligned}$$

Here $|e| = \frac{1}{3!}\epsilon^{\mu\nu\rho}\epsilon_{abc}e^a{}_\mu e^b{}_\nu e^c{}_\rho$ denotes the determinant of $e^a{}_\mu$, which is also equal to $\sqrt{|g|}$. Recall that $R^{ab} = \mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}$, and so the last line implies that, if we interpret e and ω as gauge fields, we should compare this to a gauge action $\int(AdA + A\wedge A\wedge A)$. Remarkably, forms of this kind do appear in the context of gauge theories, specifically in what is known as Chern-Simons theories [20]. This then suggests that we treat the vielbein and the spin connection as gauge fields, but a more precise translation will require some further considerations[3].

The connection between (2.20) and Chern-Simons theory is most easily seen after introducing the definition

$$\omega^a = \frac{1}{2}\epsilon^a{}_{bc}\omega^{bc}, \qquad (2.21)$$

in terms of which the action can be recast as

$$S_{EH} = \frac{1}{16\pi G_N}\int d^3x\left(2e^a\mathrm{d}\omega_a + \epsilon_{abc}e^a\wedge\left(\omega^b\wedge\omega^c+\frac{1}{3\ell^2}e^b\wedge e^c\right)\right) \qquad (2.22)$$

Define now

$$A^a = \omega^a + \frac{1}{\ell}e^a, \quad \bar{A}^a = \omega^a - \frac{1}{\ell}e^a \qquad (2.23)$$

This implies

[3]As pointed out by Witten [19], in $3+1$ dimensions the rewriting of the Einstein-Hilbert action in terms of the vielbein and spin connection gives something of the form $\int A\wedge A\wedge(\mathrm{d}A + A^2)$, where A denotes collectively e and ω, which are treated as gauge fields. As it turns out, this action is not invariant under $iso(3,1)$ (Poincaré), $so(3,2)$ (AdS) or $so(4,1)$ (dS). Actually, it can be shown that no action can be built in four dimensions which is invariant under some of the aforementioned gravity groups. See [26, 27] for some further comments and a detailed discussion on the gauge realization of gravity in the case of odd dimensions. See also [28] for some recent progress in the attempts to build a Chern-Simons gravity action in four dimensions.

$$2e^a d\omega_a = \frac{\ell}{2}\left(A^a dA^a - \bar{A}^a d\bar{A}_a - d(A^a \bar{A}_a)\right) \tag{2.24}$$

where the last term, a total derivative, can be dropped.

On the other hand,

$$\begin{aligned} &\epsilon_{abc}\, e^a \wedge \left(\omega^b \wedge \omega^c + \frac{1}{3\ell^2} e^b \wedge e^c\right) \\ &\quad = \frac{\ell\epsilon_{abc}}{8}\left(A^a - \bar{A}^a\right) \wedge \left((A^b + \bar{A}^b) \wedge (A^c + \bar{A}^c) + \frac{1}{3}(A^b - \bar{A}^b) \wedge (A^c - \bar{A}^c)\right) \\ &\quad = \frac{\ell\epsilon_{abc}}{6}(A^a \wedge A^b \wedge A^c - \bar{A}^a \wedge \bar{A}^b \wedge \bar{A}^c) \end{aligned} \tag{2.25}$$

Thus, combining these two last expressions we get

$$\boxed{S_{EH} = \frac{\ell}{32\pi G_N}\int d^3x \left(\left(A_a dA^a + \frac{1}{3}\epsilon_{abc}A^a \wedge A^b \wedge A^c\right) - \left(\bar{A}_a d\bar{A}^a + \frac{1}{3}\epsilon_{abc}\bar{A}^a \wedge \bar{A}^b \wedge \bar{A}^c\right)\right)} \tag{2.26}$$

We have then managed to recast the Einstein-Hilbert action as something that has the form $\int (AdA + A \wedge A \wedge A)$, which we said to be an action appearing in the context of gauge theory. Actions of this form have been particularly useful in the study of geometric invariants[4], as well as in various problems of interest in condensed matter theory [20]. More precisely, the action appearing in these contexts – known as the Chern-Simons action – is defined as

$$S_{CS}[A] = \frac{k}{4\pi}\int \mathrm{Tr}\left(AdA + \frac{2}{3}A \wedge A \wedge A\right). \tag{2.27}$$

where k is a constant known as the Chern-Simons level.

To see that the terms of (2.26) are indeed of the form (2.27), one must think of A^a as the components of a gauge field A expanded in the basis of the generators of the algebra, i.e. $A = A^a J_a$ (and similarly for $\bar{A}$). As we will review later on, the Lie algebra associated with AdS_3 is $so(2,2) = sl(2,\mathbb{R}) \times sl(2,\mathbb{R})$, implying that both A and $\bar{A}$ are elements of $sl(2,\mathbb{R})$. For reasons that will become clear as we go on, we will work in the following representation of $sl(2,\mathbb{R})$

$$J_0 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad J_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad J_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.28}$$

[4]See, for example, the seminal paper [22] by Witten, where it is shown that Chern-Simons theory in $2+1$ dimensions gives a natural framework for understanding some aspects of knot theory.

where one has

$$\mathrm{tr}(J_a J_b) = \frac{1}{2}\begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\eta_{ab}. \tag{2.29}$$

The generators then satisfy

$$[J_a, J_b] = \epsilon_{abc}\eta^{cd}J_d, \qquad \mathrm{tr}(J_a J_b) = \frac{1}{2}\eta_{ab}, \qquad \mathrm{tr}(J_a J_b J_c) = \frac{1}{4}\epsilon_{abc}, \tag{2.30}$$

where 'tr' denotes trace over the *matrix indices*. The Tr of (2.27) is in general defined by the matrix trace in some irreducible representation multiplied by a constant c, so as to provide an inner product given by the Killing form η_{ab} appearing in (2.30). That is,

$$\begin{aligned} \mathrm{Tr}(J_a J_b) &\equiv \eta_{ab} = 2\mathrm{tr}(J_a J_b) \\ \mathrm{Tr}(J_a J_b J_c) &= 2\,\mathrm{tr}(J_a J_b J_c) = \frac{1}{2}\epsilon_{abc} \end{aligned}$$

and so in our case $c = 2$[5]. Thus, for example, the quadratic part of (2.27) then becomes

$$\int \mathrm{Tr}(A^a J_a \wedge d(A^b J_b)) = \mathrm{Tr}(J_a J_b)\int (A^a \wedge dA^b) = \eta_{ab}\int (A^a \wedge dA^b). \tag{2.31}$$

All this in turn implies that the terms of (2.27) have the following component representation:

$$A \wedge dA = \frac{1}{3!}A^a_{[\mu}\partial_\nu A^b_{\rho]}\, J_a J_b\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho, \tag{2.32}$$

$$A \wedge A \wedge A = \frac{1}{3!}A^a_{[\mu}A^b_\nu A^c_{\rho]}\, J_a J_b J_c\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho. \tag{2.33}$$

It is now easy to see that (2.23) is recovered from the combination[6]

[5] This constant depends on the representation because different inequivalent representations will give invariant bilinear forms differing by a multiplicative factor. See [29] for different examples of this. Note also that the metric on the Lie algebra should be non-degenerate so that the Chern-Simons action contains a kinetic energy for all components of the gauge fields.

[6] While it seems that by writing S_{EH} in terms of the Chern-Simons connection one is not doing anything more than a mere translation between different languages, there is an important subtlety to consider. Recall that the vielbeins were defined by the relation $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$. Since in general relativity the metric is supposed to be non-degenerate, this relation implies that the vielbeins should be invertible. However, if we take the Chern-Simons-like action that we got for S_{EH} and forget all reference to its metric version, there is not *a priori* any reason to require the invertibility of the vielbein. This extension turns out to have very important implications in the analysis of the renormalizability of three-dimensional gravity, as pointed out by Witten [19].

$$\boxed{S_{EH} = S_{CS}[A] - S_{CS}[\bar{A}]} \tag{2.34}$$

provided we make the identification $k = \frac{\ell}{4G_N}$.

It is important to note that the values of the Chern-Simons level k are constrained to be integers by gauge invariance of the quantum amplitude e^{iS}, which implies that ℓ/G_N is also quantized. To see how this remarkable feature comes out, let us perform the gauge transformation

$$A \to g^{-1}(A + d)g, \tag{2.35}$$

where g is an element of the gauge group. This gives:

$$\begin{aligned} S_{CS} \to & \frac{k}{4\pi} \int \mathrm{Tr} \left(g^{-1}(A + d)g \wedge d(g^{-1}(A + d)g) + \frac{2}{3} g^{-1}(A + d)g \wedge g^{-1}(A + d)g \wedge g^{-1}(A + d)g \right) \\ &= \frac{k}{4\pi} \int \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A - g^{-1} A\, dg g^{-1} dg + dg g^{-1} dA - \frac{1}{3} g^{-1} dg g^{-1} dg\, g^{-1} dg \right) \\ &= S_{CS} - \frac{k}{4\pi} \int \mathrm{Tr} \left(d(g^{-1} A dg) + \frac{1}{3} g^{-1} dg\, g^{-1} dg g^{-1} dg \right). \end{aligned}$$

The gauge invariance is then spoiled by the two last terms in the last equation. We can get rid of the first of these terms –a total derivative– by fixing appropriate boundary conditions on the field A. The case for the second term is a bit more subtle, but it involves the winding number

$$w(g) \equiv \frac{1}{24\pi^2} \mathrm{Tr}(g^{-1} dg\, g^{-1} dg g^{-1} dg), \tag{2.36}$$

whose integral can be shown to be an integer under suitable boundary conditions [20]. We conclude that the action is not gauge invariant, but changes by $2\pi k w(g)$. However, the only extra terms arising from a gauge transformation that can be tolerated are integer factors of 2π, which implies that k must be an integer. This is so because it is only in that scenario that the amplitude e^{iS} remains gauge invariant, which must be the case in order for quantities like $\langle \mathcal{O} \rangle$ to be well defined, as is clear, for example, from the functional representation $\langle \mathcal{O} \rangle = Z^{-1} \int \mathcal{D}A \mathcal{O}(A) \exp(iS(A))$ [23].

2.2.1 Isometries of AdS$_3$

In the previous section we rewrote the Einstein-Hilbert action in the language of the vielbein formalism and recognized a Chern-Simons-like form in the expression thus obtained. It is clear, however, that if our Chern-Simons theory is to be interpreted as gravity, it must retain the structure of the latter under diffeomorphisms and Lorentz transformations[7]. Luckily, to ensure this, all we need to do to is to choose the right Lie algebra for the Chern-Simons connections A and $\bar{A}$. We already saw that (2.34) could be obtained when the algebra is $sl(2,\mathbb{R})$. To understand why this is the case, let us develop a bit more.

Anti-de Sitter space is the maximally symmetric solution of Einstein's equation with a negative cosmological constant. It can be realized as a hyperboloid embedded in a 4-dimensional geometry [31] :

$$-v^2 - u^2 + x^2 + y^2 = -\ell^2, \tag{2.37}$$

with metric:

$$ds^2 = -du^2 - dv^2 + dx^2 + dy^2 \tag{2.38}$$

Here the radius ℓ is the same appearing in (2.18), which was said to be related to the cosmological constant by $\Lambda = -1/\ell^2$, as can be seen by finding the induced metric on the hyperboloid and plugging it into Einstein's equations. To find intrinsic coordinates on AdS_3, one just needs to solve (2.37). This can be done by setting

$$u = \ell \cosh\rho \sin\lambda$$
$$v = \ell \cosh\rho \cos\lambda$$
$$\ell \sinh\rho = \sqrt{x^2 + y^2}$$

Going to polar coordinates ($x = \ell \sinh\rho \cos\phi, \quad y = \ell \sinh\rho \sin\phi$) we obtain:

$$ds^2 = \ell^2(-\cosh^2\rho\, d\lambda^2 + d\rho^2 + \sinh^2\rho d\theta^2) \tag{2.39}$$

The signature of λ indicates that it acts as a timelike coordinate. However, λ is clearly periodic, so when we refer to AdS_3 we will actually mean the space in which that coordinate is unwrapped, i.e. $\lambda \in (-\infty, \infty)$, which the hyperboloid's universal covering (otherwise we would have a closed timelike curve). It is convenient to set $\lambda = \frac{t}{\ell}$ and introduce the coordinate $r = \ell \sinh\rho$. This leads us to what is conventionally referred to as AdS_3:

[7]See section 2.1 of [19] for detailed discussion on this subject.

$$ds^2 = -\left(\frac{r^2}{\ell^2}+1\right)dt^2 + \left(\frac{r^2}{\ell^2}+1\right)^{-1} dr^2 + r^2 d\theta^2 \tag{2.40}$$

Another set of coordinates commonly used to write Anti-de Sitter spacetime are the Poincaré coordinates, defined as

$$z = \frac{\ell}{u+x}, \quad \beta = \frac{y}{u+x}, \quad \gamma = \frac{-v}{u+x}, \tag{2.41}$$

in terms of which the line element reads:

$$ds^2 = \frac{\ell^2}{z^2}\left(dz^2 + d\beta^2 - d\gamma^2\right). \tag{2.42}$$

Now to the isometries. The group of transformations that leave the metric (2.38) invariant is denoted as $SO(2,2)$. Here, $(2,2)$ stands for the signature of the metric and SO stands for 'special orthonormal', meaning that its matrix representations are of determinant 1. That $SO(2,2)$ is the isometry group of AdS_3 can also be explicitly seen by finding the six Killing vectors of AdS and checking that three of them obey the $sl(2,\mathbb{R})$ algebra under Lie brackets, and that the other three form the same algebra and commute with the first three Killing vectors (see [31] for details). This means that the isometry group is indeed $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) = SO(2,2)$. Note, however, that this split is a special feature of AdS_3 since the isometry group of AdS_D is $SO(D-1,2)$, which splits only for $D=3$.

2.2.2 Equations of motion

Since we will be studying gravity in the Chern-Simons language we just described, let us see explicitly what form the equations of motion take in this new language. Clearly, these equations will correspond to Einstein field equations as we have done nothing but a rewriting of the Einstein-Hilbert action.

For small variations δA, the corresponding variation of the Chern Simons functional is

$$\begin{aligned}\delta S_{CS}[A] &= \frac{k}{4\pi}\int_M \mathrm{Tr}\left(\delta A \wedge dA\right) + \frac{k}{4\pi}\int_M \mathrm{Tr}\left(A\wedge d\delta A\right) + \frac{2k}{4\pi}\int_M \mathrm{Tr}\left(\delta A\wedge A\wedge A\right)\\ &= \frac{k}{4\pi}\int_M d\mathrm{Tr}\left(A\wedge\delta A\right) + \frac{2k}{4\pi}\int_M \mathrm{Tr}\left(\delta A\wedge\left(dA + A\wedge A\right)\right)\\ &= \frac{k}{4\pi}\int_{\partial M}\mathrm{Tr}\left(A\wedge\delta A\right) + \frac{2k}{4\pi}\int_M \mathrm{Tr}\left(\delta A\wedge\left(dA+A\wedge A\right)\right),\end{aligned} \tag{2.43}$$

where in the last step the Stokes formula has been used.

Imposing the appropriate boundary conditions, one obtains

$$\begin{aligned}\delta S_{CS}[A] &= \frac{k}{4\pi}2\int \mathrm{Tr}\left(\delta A\wedge(dA+A\wedge A)\right)\\ &= \frac{k}{4\pi}2\int \mathrm{dx}^3\epsilon^{\rho\mu\nu}\mathrm{Tr}\left\{\delta A_\rho\left[\frac{1}{2}\left(\partial_\mu A_\nu-\partial_\nu A_\mu\right)+\frac{1}{2}\left[A_\mu,A_\nu\right]\right]\right\}\end{aligned} \tag{2.44}$$

So,

$$\frac{\delta S_{CS}[A]}{\delta A_\rho} = \frac{k}{4\pi}\int \mathrm{dx}^3\epsilon^{\rho\mu\nu}F_{\mu\nu}, \tag{2.45}$$

and analogously,

$$\frac{\delta S_{CS}[\bar{A}]}{\delta \bar{A}_\rho} = \frac{k}{4\pi}\int \mathrm{dx}^3\epsilon^{\rho\mu\nu}\bar{F}_{\mu\nu}, \tag{2.46}$$

with[8]

$$F_{\mu\nu}\equiv\left(\partial_\mu A_\nu-\partial_\nu A_\mu\right)+\left[A_\mu,A_\nu\right], \qquad \bar{F}_{\mu\nu}\equiv\left(\partial_\mu \bar{A}_\nu-\partial_\nu \bar{A}_\mu\right)+\left[\bar{A}_\mu,\bar{A}_\nu\right] \tag{2.47}$$

The equations of motion are then

$$\boxed{F=0, \quad \bar{F}=0} \tag{2.48}$$

When building our *dynamical* solution, these equations will be slightly modified to include a source (we will take into account the backreaction of particles following geodesics). For the moment, let us see what are some of the already known solutions in AdS_3.

2.3 Some solutions in 2 + 1-dimensional gravity

2.3.1 The BTZ black hole

In [17] it was pointed out that the metric

$$ds^2 = -(N^\perp)^2dt^2+(N^\perp)-2dr^2+r^2(d\phi+N^\phi dt)^2, \tag{2.49}$$

[8] F and $\bar{F}$ are nothing but non-Abelian versions of the field strength tensor familiar from electrodynamics.

with

$$N^{\perp} = \left(-M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}\right)^{1/2}, \qquad N^{\phi} = -\frac{J}{2r^2} \tag{2.50}$$

is an exact solution of the Einstein field equations with negative cosmological constant $\Lambda = -1/\ell^2$. When $M > 0$ and $|J| \leq M\ell$, this solution has an outer event horizon[9] at $r = r_+$ and an inner horizon at $r = r_-$, where

$$r_{\pm} = \frac{M\ell^2}{2}\left\{1 \pm \left[1 - \left(\frac{J}{M\ell}\right)^2\right]^{1/2}\right\}, \tag{2.51}$$

which implies that

$$M = \frac{r_+^2 + r_-^2}{\ell^2}, \qquad J = \frac{2r_+ r_-}{\ell}, \tag{2.52}$$

which can be shown to correspond to the mass and angular momentum of the black hole [31].

The BTZ black hole shares many features with the "standard" black hole solutions in four dimension, such as the existence of an event horizon and some thermodynamic properties. However, this black hole presents the particular feature of not having a curvature singularity at the origin. Instead, the BTZ black hole posesses at that point what is called a 'causal singularity', in the sense that continuing past $r = 0$ would bring in closed timelike lines (see [31] for details). On the other hand, given that it has constant negative curvature, this black hole must be isometric to AdS_3, at least locally. This can be seen, for the region $r > r_+$, by performing the simple coordinate change

$$\begin{aligned}
x &= \left(\frac{r^2 - r_+^2}{r^2 - r_-^2}\right)^{1/2} \cosh\left(\frac{r_+}{\ell^2}t - \frac{r_-}{\ell}\phi\right) \exp\left(\frac{r_+}{\ell}\phi - \frac{r_-}{\ell^2}t\right) \\
y &= \left(\frac{r^2 - r_+^2}{r^2 - r_-^2}\right)^{1/2} \sinh\left(\frac{r_+}{\ell^2}t - \frac{r_-}{\ell}\phi\right) \exp\left(\frac{r_+}{\ell}\phi - \frac{r_-}{\ell^2}t\right) \\
z &= \left(\frac{r^2 - r_+^2}{r^2 - r_-^2}\right)^{1/2} \exp\left(\frac{r_+}{\ell}\phi - \frac{r_-}{\ell^2}t\right),
\end{aligned}$$

for which the metric becomes

$$ds^2 = \frac{\ell^2}{z^2}(dx^2 - dy^2 + dz^2) \tag{2.53}$$

[9] Note that the radius of curvature ℓ provides the length scale necessary to have a horizon in a theory in which the mass is dimensionless [17]

Eq. (2.53) is nothing but the standard Poincaré metric for anti-de Sitter space that we presented before in (2.42).

Using (2.52), the Euclidean version of BTZ can be written as

$$ds^2 = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2 \ell^2} dt_E^2 + \frac{\ell^2 r^2}{(r^2 - r_+^2)(r^2 - r_-^2)} dr^2 + r^2 \left(d\phi + \frac{r_+ |r_-|}{\ell^2 r^2} dt_E \right)^2 , \tag{2.54}$$

where t_E is the Euclidean time obtained by Wick-rotating $t \to -it_E$.

Another common form for this metric is

$$\boxed{ds^2 = d\rho^2 + 8\pi G \ell (\mathcal{L} dw^2 + \bar{\mathcal{L}} d\bar{w}^2) + (\ell^2 e^{2\rho/\ell} + (8\pi G)^2 \mathcal{L} \bar{\mathcal{L}} e^{-2\rho/\ell}) dw d\bar{w}} \tag{2.55}$$

where $w = \phi + it/\ell$, $\bar{w} = \phi - it/\ell$ are complex coordinates on the boundary, located at $e^{\rho/\ell} =: \ell r \to \infty$. The BTZ is recovered for $\mathcal{L} = \frac{1}{2\pi} \frac{(r_+ - r_-)^2}{16 G \ell}$, $\bar{\mathcal{L}} = \frac{1}{2\pi} \frac{(r_+ + r_-)^2}{16 G \ell}$. However, the interesting bit of this last form is that one can see that it solves the Einstein field equations for any $\mathcal{L} = \mathcal{L}(w)$ and $\bar{\mathcal{L}} = \bar{\mathcal{L}}(\bar{w})$.

We can of course write the (Euclidean) BTZ black hole in the Chern-Simons language. For this, let us choose an explicit basis for the $sl(2, \mathbb{R})$ algebra. We take

$$L_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad L_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L_0 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \tag{2.56}$$

which satisfy

$$[L_i, L_j] = (i - j) L_{i+j} \tag{2.57}$$

These are related to the more standard generators (2.30) through

$$J_0 = \frac{1}{2}(L_1 + L_{-1}), \quad J_1 = \frac{1}{2}(L_1 - L_{-1}), \quad J_2 = L_0 \tag{2.58}$$

It can then be checked that, in units $\ell = 1$, the BTZ metric is reproduced starting from the connections

$$\begin{aligned} A &= (e^{\rho} L_1 - \frac{2\pi \mathcal{L}}{k} e^{-\rho} L_{-1}) dw + L_0 d\rho \\ \bar{A} &= (e^{\rho} L_{-1} - \frac{2\pi \bar{\mathcal{L}}}{k} e^{-\rho} L_1) d\bar{w} - L_0 d\rho. \end{aligned} \tag{2.59}$$

We mentioned before that a pair of connections in which $\mathcal{L} = \mathcal{L}(w)$ and $\bar{\mathcal{L}} = \bar{\mathcal{L}}(\bar{w})$ can also be shown to solve the Einstein field equations. These coefficient functions turn out to be components of the boundary stress tensor,

$$T_{ww} = \mathcal{L}(w), \quad T_{\bar{w}\bar{w}} = \bar{\mathcal{L}}(\bar{w}). \tag{2.60}$$

This fact can be understood by analyzing asymptotic properties of generic solutions of pure gravity with negative cosmological constant with a boundary term. Imagine that we write the line element of such a solution in a form analogous to (2.55):

$$ds^2 = d\rho^2 + g_{ij}(x^k, \rho)dx^i dx^j, \quad i, j = 1, 2. \tag{2.61}$$

If we demand that g_{ij} takes the Fefferman-Graham [7] form as $\rho \to \infty$,

$$g_{ij}(x^k, \rho)dx^i dx^j = e^{\rho/\ell} g^{(0)}_{ij}(x^k) + g^{(2)}_{ij}(x^k) + \cdots , \tag{2.62}$$

then $g^{(0)}_{ij}$ defines the conformal boundary metric. In the context of the AdS/CFT correspondence, the CFT is said to live on a space with this metric. The boundary stress tensor works out to be

$$T_{ij} = \frac{1}{8\pi G\ell}\left(g^{(2)}_{ij} - \mathrm{Tr}(g^{(2)})g^{(0)}_{ij}\right) \tag{2.63}$$

where the trace is taken with $g^{(0)}_{ij}$.

From (2.55) we read $g^{(0)}_{ww} = 0 = g^{(0)}_{\bar{w}\bar{w}}$ and $g_{w\bar{w}} = 1/2$. The asymptotic symmetry group (i.e. the symmetry group of the CFT on the boundary) is obtained by considering coordinate transformation that leave the form of $g^{(0)}_{ij}$ unchanged. This is the case for the infinitesimal transformations [37]

$$\begin{aligned} w &\to w + \epsilon(w) - \frac{\ell^2}{2} e^{-2\rho/\ell} \partial^2_{\bar{w}} \bar{\epsilon}(\bar{w}) \\ \bar{w} &\to \bar{w} + \bar{\epsilon}(w) - \frac{\ell^2}{2} e^{-2\rho/\ell} \partial^2_{w} \bar{\epsilon}(w) \\ \rho &\to \rho - \frac{\ell}{2}\left(\partial_w \epsilon(w) + \partial_{\bar{w}} \bar{\epsilon}(\bar{w})\right) \end{aligned} \tag{2.64}$$

where $\epsilon(w), \bar{\epsilon}(\bar{w})$ are arbitrary functions. These transformations act nontrivially in $g^{(2)}_{ij}$, and so the stress tensor transforms as [35]

$$T_{ww} \to T_{ww} + 2\partial_w \epsilon(w) T_{ww} + \epsilon(w)\partial_w T_{ww} - \frac{c}{24\pi}\partial_w^3 \epsilon(w) \tag{2.65}$$

and analogously for $T_{\bar{w}\bar{w}}$. This is the transformation law for a stress tensor in a two-dimensional conformal field theory, with $c = 3\ell/2G$ the central charge. If one decomposes the stress tensor into modes, $L_n = \oint dw\, e^{-inw} T_{ww}$, it follows that the generators obey the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n}. \tag{2.66}$$

To see that $\mathcal{L}, \bar{\mathcal{L}}$ are the components of the boundary stress tensor, we then need to verify that they transform as in (2.65). Let us check this for the coefficient $\mathcal{L}$ in the connection A of (2.59). Infinitesimal gauge transformations act on this connection as

$$\delta A = d\Lambda + [A, \Lambda]. \tag{2.67}$$

The form of A is preserved if we take

$$\Lambda = b^{-1}\left[\epsilon(z)L_1 - \partial_z \epsilon(z) L_0 + \left(\frac{1}{2}\partial_z^2 \epsilon(z) - \frac{2\pi}{k}\mathcal{L}\epsilon(z)\right) L_{-1}\right] b, \qquad b = e^{\rho L_0}, \tag{2.68}$$

which acts as

$$\delta \mathcal{L} = \epsilon(z)\partial_z \mathcal{L} + 2\partial_z \epsilon(z)\mathcal{L} - \frac{k}{4\pi}\partial_z^3 \epsilon(z). \tag{2.69}$$

Comparing with (2.65), we see that $\mathcal{L}$ is indeed transforming as a stress tensor under conformal transformations.

2.3.2 Higher spin solutions

We have emphasized that the advantage of writing gravity solutions in terms of Chern-Simons connections is the possibility of a straightforward generalization to the case of higher-spins. Let us see how such a generalization actually takes place.

The main difference that arises when considering fields with spin higher than 2 is that now the connections are expanded in terms of generators of the corresponding $sl(N, \mathbb{R})$ algebra. Explicitly, this means that, in a given representation, we now have

$$A = A^a J_a + A^{a_1 \cdots a_{s-1}} T_{a_1 \cdots a_{s-1}} \tag{2.70}$$

$$\bar{A} = \bar{A}^a J_a + \bar{A}^{a_1 \cdots a_{s-1}} T_{a_1 \cdots a_{s-1}} \tag{2.71}$$

where the $T_{a_1 \cdots a_{s-1}}$ generate the higher-spin algebras.

For concreteness, let us consider the example of the spin-3 case. Here one can introduce a basis of generators J_a and T_{ab} which obey the following non-Abelian algebra [40]

$$\begin{aligned} [J_a, J_b] &= \epsilon_{abc} \eta^{cd} J_d, \\ [J_a, T_{bc}] &= \eta^{mn} \epsilon_{na(b} T_{c)m}, \\ [T_{ab}, T_{cd}] &= -\left(\eta_{a(c} \epsilon_{d)bm} + \eta_{b(c} \epsilon_{d)am} \right) \eta^{mn} J_n, \end{aligned} \tag{2.72}$$

where the T_{ab} are traceless and symmetric in a, b. Clearly, the generators J_a in the above equation obey the $sl(2,\mathbb{R})$ algebra that we encountered before in (2.30). The $sl(2,\mathbb{R})$ is then said to be embedded in the $sl(3,\mathbb{R})$ algebra (2.72). Other embeddings can of course be conceived in which different combinations of three of the generators solve (2.30). The particular embedding of (2.72) is known as the *principal embedding*[10].

The representation (2.72) can be built by defining the T_{ab} generators as

$$T_{ab} = \left(J_{(a} J_{b)} - \frac{2}{3} \eta_{ab} J_c J^c \right), \tag{2.73}$$

as fixed by tracelessness and symmetry (see e.g. [32]). The J_a appearing in this last equation are the generators of the $sl(2,\mathbb{R})$ algebra in the three-dimensional representation, since the generators T_{ab} are three by three matrices in the fundamental representation. The particular conventions and matrix representation of $sl(3,\mathbb{R})$ that we will be using are summarized in Appendix A.

Following (2.70), the $sl(3,\mathbb{R})$ connections can then be expanded as

$$A = A^{(2)} + A^{(3)} = A^a J_a + A^{ab} T_{ab} \tag{2.74}$$

$$\bar{A} = \bar{A}^{(2)} + \bar{A}^{(3)} = \bar{A}^a J_a + \bar{A}^{ab} T_{ab}. \tag{2.75}$$

[10]This will be the embedding we will be considering throughout our discussions. For examples of uses of a different embedding in the context of higher-spin black holes, see [38].

An obvious way of extending (2.59) to a higher-spin version would consist in adding terms of the form $A^{ab}T_{ab}$. However, we must make sure that the terms we add actually give rise to non-zero spin-3 currents. To understand why there might be this kind of subtlety, consider the connections of the BTZ black hole (2.59). Since the BTZ black hole is a solution of pure gravity (spin-2), it does not carry any higher-spin charge. This is reflected by the fact that $A_{BTZ}, \bar{A}_{BTZ}$ are expanded in terms of J_a, J_b, which are generators of $sl(2,\mathbb{R})$. However, these generators can be thought of as the generators of the $sl(2,\mathbb{R})$ subalgebra embedded in $sl(3,\mathbb{R})$ in a given representation (as in (2.72)). This means that we can think of A_{BTZ} and $\bar{A}_{BTZ}$ as elements of $sl(3,\mathbb{R})$ (or $sl(N,\mathbb{R})$, for that purpose) in which the higher-spin components are zero, i.e.

$$A_{BTZ} = A^{(2)}_{BTZ} + \underbrace{A^{(3)}_{BTZ}}_{=0} \tag{2.76}$$

and similarly for $\bar{A}_{BTZ}$. Seen as elements element of $sl(3,\mathbb{R})$, A_{BTZ} and $\bar{A}_{BTZ}$ are then higher-spin gravity solutions. Indeed, in a $sl(3,\mathbb{R})$ theory, the connections A_{BTZ} and $\bar{A}_{BTZ}$ will be subject to $sl(3,\mathbb{R})$ gauge transformations which will leave them globally untouched but which will mix the spin-2 part with the spin-3. That is,

$$A_{BTZ} \to A'_{BTZ} = \underbrace{A^{(2)'}_{BTZ}}_{\neq A^{(2)}_{BTZ}} + \underbrace{A^{(3)'}_{BTZ}}_{\neq 0} = A_{BTZ}. \tag{2.77}$$

It seems then that full-fledged higher-spin gravity solutions can be constructed by simply taking embeddings of $sl(2,\mathbb{R})$ in higher-rank algebra. It should be obvious, however, that this is a rather artificial way of building a higher-spin solution as we are not adding any new physics whatsoever. Indeed, as we will see, all we have done is to construct a higher-spin solution which does not carry higher-spin current[11]. The question then arises as to how one may build solutions carrying non-zero higher-spin charge.

The answer to the above question resides in the kind of asymptotic analysis that led us to the conclusion that the coefficient functions $\mathcal{L}, \bar{\mathcal{L}}$ in (2.59) are actually components of the boundary stress tensor. If we perform an analysis of that kind and identify the kind of

[11]This claim can also be understood if one considers the eingenvalues of the Casimirs of the representations, which encode the mass, the spin, and the higher-spin charges of the particles sourcing the solutions. The eigenvalues of the Casimirs are invariant under gauge transformations. This implies that a solution with no higher-spin will remain uncharged, no matter whether the gauge we pick is such that the connections have (as in (2.77)) or have not (as in (2.76)) higher-spin components. We will have some further comments on this later in this thesis.

higher-spin component that leads to non-zero spin-3 current in the boundary CFT, then we would be in conditions to construct a full-fledged higher-spin solution. This is precisely what was done in [36], where it was pointed out that the connections

$$
\begin{aligned}
A &= \left(e^{\rho}L_1 - \frac{2\pi\mathcal{L}}{k}e^{-\rho}L_{-1} - \frac{\pi}{2k}e^{-2\rho}\mathcal{W}(w)W_{-2}\right)dw + L_0 d\rho \\
\bar{A} &= \left(e^{\rho}L_{-1} - \frac{2\pi\bar{\mathcal{L}}}{k}e^{-\rho}L_1 - \frac{\pi}{2k}e^{-2\rho}\bar{\mathcal{W}}(\bar{w})W_{-2}\right)d\bar{w} - L_0 d\rho
\end{aligned}
\tag{2.78}
$$

yield to a charged higher-spin black hole[12]. These connections reduce to the BTZ ones for $\mathcal{W}, \bar{\mathcal{W}} = 0$. The new coefficient functions $\mathcal{W}, \bar{\mathcal{W}}$ can be shown to correspond to spin-3 currents. This is done by first identifying the asymptotic symmetry group, i.e. by identifying the most general gauge transformation preserving the asymptotic form of the connections, as in (2.64) or (2.68), for example. Such transformations do not change the form of the connections but act non-trivially on the functions $\mathcal{L}, \bar{\mathcal{L}}, \mathcal{W}, \bar{\mathcal{W}}$. These variations can be compared to what is expected for stress tensors and spin-3 operators in the CFT language[13], which allows to conclude that $\mathcal{W}$ is indeed a spin-3 operator. The authors of [36] extended this analysis so as to include chemical potentials $\mu, \bar{\mu}$ for the spin-3 operators in the connections (2.78):

$$
\begin{aligned}
A &= \left(L_1 - \frac{2\pi\mathcal{L}}{k}L_{-1} - \frac{\pi}{2k}\mathcal{W}(w)W_{-2}\right)dw \\
&\quad - \mu\left(W_2 - \frac{4\pi\mathcal{L}}{k}W_0 + \frac{4\pi^2\mathcal{L}^2}{k^2}W_{-2} + \frac{4\pi\mathcal{W}}{k}L_{-1}\right)d\bar{w} \\
\bar{A} &= \left(L_{-1} - \frac{2\pi\bar{\mathcal{L}}}{k}L_1 - \frac{\pi}{2k}\bar{\mathcal{W}}(\bar{w})W_{-2}\right)d\bar{w} \\
&\quad - \bar{\mu}\left(W_{-2} - \frac{4\pi\bar{\mathcal{L}}}{k}W_0 + \frac{4\pi^2\bar{\mathcal{L}}^2}{k^2}W_2 + \frac{4\pi\bar{\mathcal{W}}}{k}L_1\right)dw.
\end{aligned}
\tag{2.79}
$$

Here the following basis is being used that allows to recognize the asymptotic symmetry group straightforwardly:

[12] The relation between the generators $L_1, L_{-1}, W_{-2}, \ldots$ and the generators in (2.72) will be specified below.

[13] This is done by translating the variations of these functions into what is known as the operator product expansion (OPE) for the symmetry currents. From this OPE one can read the dimension of the corresponding operators, and thus the spin (see e.g. [34] for details).

$$
\begin{aligned}
[L_i, L_j] &= (i-j)L_{i+j} \\
[L_i, W_m] &= (2i-m)W_{i+m} \\
[W_m, W_n] &= -\frac{1}{3}(m-n)(2m^2+2n^2-mn-8)L_{m+n}
\end{aligned}
$$

where $-1 \leq i, j \leq 1$ and $-2 \leq m, n \leq 2$. This can be related to our basis in (2.72) via the isomorphism

$$
J_0 = \frac{1}{2}(L_1 + L_{-1}), \quad J_1 = \frac{1}{2}(L_1 - L_{-1}), \quad J_2 = L_0, \tag{2.80}
$$

and

$$
\begin{aligned}
T_{00} &= \frac{1}{4}(W_2 + W_{-2} + 2W_0), & T_{01} &= \frac{1}{4}(W_2 - W_{-2}), \\
T_{11} &= \frac{1}{4}(W_2 + W_{-2} - 2W_0), & T_{02} &= \frac{1}{2}(W_1 + W_{-1}), \\
T_{22} &= W_0, & T_{12} &= \frac{1}{2}(W_1 - W_{-1}).
\end{aligned}
$$

The considerations that led us to (2.79) constitute the standard approach to the construction of higher-spin solutions. The approach we will follow will differ in several respects. Since we are interested in dynamical solutions, we will have to lay aside pure gravity, which is topological. This implies that we will be dealing with an action that includes sources coupled to gravity. Based on [41] and [25], we will now present a framework which will allow us not only to achieve this coupling, but also to have control on the value of the spin charges by means of constraints on the values of the Casimir operators.

Chapter 3

A dynamical setup

In the previous chapter we presented a Chern-Simons-like formulation of pure three-dimensional gravity. We also saw that the gauge theory language of the new formulation allowed us to easily extend the pure gravity scenario to one involving gravity coupled to a *finite* tower of higher spin fields. This coupling was achieved by means of a change of the gauge group from $SL(2,\mathbb{R})$ (pure gravity) to $SL(N,\mathbb{R})$ (gravity plus higher-spin fields), a procedure with no obvious analogue in the metric language. While the higher-spin fields modify the theory in a non-trivial way, in particular by preventing strictly geometric notions to be gauge invariant (under a $SL(N,\mathbb{R})$ transformation, the higher-spin fields mix with the spin-2 field, i.e., with the metric), in all these cases the variation of the action led to the flatness condition

$$F = 0, \quad \bar{F} = 0, \tag{3.1}$$

meaning that no actual sources are added in this way. In this chapter we will see how to extend our treatment in order to consistently include sources in the right hand side of the equations (3.1). To do this, we will make use of a formulation due to Jorjadze, O'Raifeartaigh and Tsutsui [41] in which the action of a free particle is rewritten in an explicit group-theoretical fashion. As we mentioned in the introduction, this formulation played a crucial role in the recent work by Ammon, Castro and Iqbal [25], where a gauge theory version (i.e. involving quantities defined using the gauge connections rather than, say, geodesics or minimal surfaces) of the Ryu-Takayanagi formula (1.4) was proposed (see Eq. (1.6) and successive comments). As we saw, a gauge theory version of a geometric quantity has the advantage of allowing possible generalizations to higher-spins by means of a simple extension of the gauge group. Our immediate goal is then to build an action for the sources which is written in the appropriate language of gauge theories.

Before building our group theoretical action, however, let us understand in the first place why we want to include sources in (3.1). The main reason why we need to do this is that pure three-dimensional gravity is topological, as was implied by the simple counting argument we presented in the introduction. This means that, if we are to build dynamical (time-dependent) solutions, we will need to add propagating degrees of freedom (which would in turn source a time-dependent energy momentum tensor). The role of the sources is then clearly to provide us with such degrees of freedom. To see explicitly how one may obtain a dynamical solution from the coupling just described, let us consider the example of the Vaidya metric.

3.1 The Vaidya metric

The Vaidya metric is a simple non-static generalization of the Schwarzschild metric. In $3+1$ dimensions, it describes the non-empty external spacetime of a spherically symmetric and non-rotating star which is either emitting or absorbing null dust, and thus serves as a model to describe black hole formation by gravitational collapse, for example[1]. To see this, recall that the Schwarzschild metric is the most general spherically symmetric vacuum solution of the Einstein field equations, which for negative cosmological constant are given by

$$G^{\mu\nu} - \frac{1}{\ell^2} g^{\mu\nu} = 0 \tag{3.2}$$

In three dimensions, the Schwarzschild metric, solution of (3.2), can be written in spherical coordinates as

$$ds^2 = -\left(1 + \frac{r^2}{\ell^2} - M\right) dt^2 + \left(1 + \frac{r^2}{\ell^2} - M\right)^{-1} dr^2 + r^2 d\varphi^2, \tag{3.3}$$

where M is a positive constant. The singularity at the Schwarzschild radius $r_s = \ell\sqrt{M-1}$ can be removed by coordinate change, but it signals the fact that the Schwarzschild metric can describe black holes, since r_s actually happens to be the position of an event horizon, as is well known [8]. Thanks to this property, the Schwarzschild metric can be used to describe the empty exterior of both stars and black holes of mass M, according to whether

[1] See [8] for an extensive review on properties of Vaidya-like metrics. A detailed discussion on gravitational collapse can be found in [42].

the radius of the object is smaller or grater than r_s. However, the fact that this metric is static prevents us from treating dynamical scenarios such as the black holes formed by matter that collapses to below the Schwarzschild radius. To overcome this drawback, let us first rewrite our Schwarzschild metric in a form suited for the description of ingoing geodesics – and therefore for the description of an infalling shell, as in gravitational collapse.

From (3.3) we see that radial (i.e., $d\varphi = 0$) null geodesics are given by

$$0 = ds^2 = -\left(1 + \frac{r^2}{\ell^2} - M\right) dt^2 + \left(1 + \frac{r^2}{\ell^2} - M\right)^{-1} dr^2$$
$$\Rightarrow \left(\frac{dr}{dt}\right)^2 = \left(1 + \frac{r^2}{\ell^2} - M\right)^2 .$$

The positive root of the last equation will represent outgoing null geodesics, whereas the negative root will represent ingoing null geodesics. A form adapted to the description of ingoing geodesics can be obtained by substituting the time t with a coordinate v defined as

$$dv = dt + \frac{dr}{\left(1 + \frac{r^2}{\ell^2} - M\right)}, \tag{3.4}$$

since in this case ingoing null geodesics would simply correspond to $v =$ const. In terms of the coordinate v, known as the ingoing Eddington-Finkelstein coordinate, the Schwarzshild metric reads

$$ds^2 = -\left(1 + \frac{r^2}{\ell^2} - M\right) dv^2 + 2dvdr + r^2 d\varphi^2. \tag{3.5}$$

A sensible non-static generalization of this metric might be obtained by making $M = M(v)$, which, for example, reflects the fact that the mass of a radiating star is not constant. The metric thus obtained

$$ds^2 = -F(r,v)dv^2 + 2drdv + r^2 d\varphi^2, \tag{3.6}$$

with $F(r,v) = \left(1 + \frac{r^2}{\ell^2} - M(v)\right)$, is what is known as the Vaidya metric, after the name of the author who first proposed it [43]. It is important to note that by modifying the Schwarzschild metric we have obtained a metric that is no longer a solution of the vacuum Einstein equations. Rather, the metric (4.1) will be sourced by a non-zero energy momentum tensor which manifests the presence of the shell.

To understand why this simple generalization might describe properly the kind of dynamical solution we would like to obtain, let us consider gravity in three dimensions with a negative cosmological constant coupled to a single particle moving along a trajectory denoted by $X^\mu(s)$. The action of this system is

$$S = S_{EH} + S_p = \frac{1}{16\pi G_N}\int dx^3\sqrt{g}\left(R + \frac{2}{\ell^2}\right) + \int dx^3\sqrt{g}\int ds\left(-\frac{1}{4\lambda(s)}g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - \lambda(s)m^2\right)\frac{\delta^{(3)}(x^\mu - X^\mu(s))}{\sqrt{g}} \tag{3.7}$$

The variation of this action gives

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} = 0 = -\frac{1}{16\pi G_N}(G^{\mu\nu} - \frac{1}{\ell^2}g^{\mu\nu}) - \frac{1}{4\sqrt{-g}}\int ds\lambda^{-1}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\delta^{(3)}(x^\mu - X^\mu(s)), \tag{3.8}$$

$$\frac{\delta S}{\delta\lambda} = 0 = \frac{1}{4\lambda^2}g_{\mu\nu}(X)\frac{dX^\mu}{ds}\frac{dX^\nu}{ds} - m^2 \tag{3.9}$$

Note that the more familiar form $S_p = -m\int ds\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$ can be recovered by plugging Eq. (3.9) – the equation of motion of the Lagrange multiplier λ(s) – back into (3.7). The introduction of a Lagrange multiplier is in this case a standard practice to get rid of the square root in $S_p = -m\int ds\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$ and deal instead with the more comfortable quadratic form of S_p appearing in (3.7).

The reparametrization symmetry of the worldline s allows us to set $\lambda(s)$ in (3.8) to a constant[2]. If we set $\lambda(s) = 1$, Eq. (3.8) implies

$$\frac{1}{16\pi G_N}\left(G^{\mu\nu} - \frac{1}{\ell^2}g^{\mu\nu}\right) = -\frac{1}{4\sqrt{|g|}}\int ds\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\delta^{(3)}(x^\mu - X^\mu(s)). \tag{3.10}$$

Since we are interested in the Vaidya solution, let us take our metric *Ansatz* to be of the form (4.1). From this we read

$$G^{\mu\nu} - \frac{1}{\ell^2} = \frac{1}{2r}M'(v)\delta^{r\mu}\delta^{r\nu} \tag{3.11}$$

[2]This reparametrization symmetry is of course nothing else than manifestation of the fact that a geodesic can be parametrized by any monotonic function that maps points on the geodesic to unique values of the parameter. Consider the geodesic equation $\frac{d^2x^b}{ds^2} + \Gamma^b_{cd}\frac{dx^c}{ds}\frac{dx^d}{ds} = 0$. Suppose now that we change the parameter in an arbitrary way $v = v(s)$. The geodesic equation then becomes $\frac{d^2x^b}{dv^2} + \Gamma^b_{cd}\frac{dx^c}{dv}\frac{dx^d}{dv} = \frac{d^2v/ds^2}{(dv/ds)^2}\frac{dx^b}{dv}$. An affine parameter is that for which the right hand side of the last equation is zero. The parameter v is then said to be affinely related to s if $\sigma(s) = \frac{d^2v/du^2}{(dv/du)^2}$ is zero. We stress, however, that non-affine parameters can also be chosen.

which is also the energy-momentum tensor up to a constant factor, as follows from Einstein equations.

In the massless case, the constraint (3.9) requires

$$-\left(1+\left(\frac{X^r}{\ell}\right)^2+M(X^v)\right)\left(\frac{dX^v}{ds}\right)^2+2\frac{dX^r}{ds}\frac{dX^v}{ds}+\left(X^r\frac{dX^\varphi}{ds}\right)^2=0 \tag{3.12}$$

This is solved by the radial null path $X^v(s)=v_0$, $X^\varphi(s)=\varphi_0$, with v_0,φ_0 constants. With this at hand, the geodesic equation implies

$$\frac{dX^r}{ds}+\Gamma^r_{\rho\sigma}(X)\frac{dX^\rho}{ds}\frac{dX^\sigma}{ds}=\frac{dX^r}{ds}+\Gamma^r_{rr}(X)\frac{dX^r}{ds}\frac{dX^r}{ds}=0. \tag{3.13}$$

Furthermore, from the expression $\Gamma^r_{rr}=\frac{1}{2}g^{r\sigma}(\partial_r g_{\sigma r}+\partial_v g_{\sigma r}-\partial_\sigma g_{rr})$ it immediately follows that $\Gamma^r_{rr}=0$. Our null radial geodesic is then given by

$$\boxed{v(s)=v_0,\quad r(s)=r_0+c_0 s,\quad \varphi(s)=\varphi_0} \tag{3.14}$$

with v_0, r_0, c_0 and φ_0 constants.

Plugging the trajectory (3.14) in (3.10) gives

$$G^{\mu\nu}-\frac{1}{\ell^2}g^{\mu\nu}=-\frac{16\pi G_N}{4r}\int ds\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\delta(v-v_0)\delta(\varphi-\varphi_0)\delta(r-(r_0+c_0 s)), \tag{3.15}$$

and equating this to (3.11) results in

$$\begin{aligned}\frac{1}{2r}M'(v)=-\frac{16\pi G_N}{4r}\int ds\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\delta(v-v_0)\delta(\varphi-\varphi_0)\delta(r-(r_0+c_0 s))\\ \Rightarrow M'(v)=-8\pi G_N|r_0|\delta(v-v_0)\delta(\varphi-\varphi_0).\end{aligned}$$

In order to source the Vaidya solution, it remains to take a shell of particles instead of a single particle. This can be achieved by smearing the particle over the sphere, i.e. by taking

$$M'(v)=-8\pi G_N\delta(v-v_0)\sum_{j=1}^{N}|c_j|\delta(\varphi-\varphi_j). \tag{3.16}$$

Distributing uniformly over the sphere: $|c_j|=|C|\frac{2\pi}{N}$ and $\varphi=\frac{2\pi j}{N}$, and taking the continuous limit $N\to\infty$ we finally obtain

$$\boxed{M'(v)=-8\pi G_N\delta(v-v_0)|C|\int_0^{2\pi}d\phi\,\delta(\varphi-\phi)=-8\pi|C|G_N\delta(v-v_0)} \tag{3.17}$$

Our collapsing shell then sources an energy momentum tensor of the form $T^{\mu\nu} \propto \frac{1}{2r}M'(v)\delta^{r\mu}\delta^{r\nu} \propto \frac{\delta(v-v_0)}{r}\delta^{r\mu}\delta^{r\nu}$, which certainly presents dynamics. Note in passing that the overall constants of integration condensed in $|C|$ should be determined by the total mass of the black hole.

Returning to our problem of sourcing (3.1) so as to get gauge connections which would correspond to an appropriate dynamical setup, it is now clear that we should proceed in analogy to the considerations that led us to the Vaidya metric with mass profile (3.17). To do this, we will need to rewrite the action (3.7) in an explicit group theoretical manner. In the previous chapter we learned how to write the Einstein-Hilbert action in the language of gauge theory. It then remains to write S_p in this language. For this, it will be convenient to recast the action S_p as follows:

$$\boxed{S_p = \int ds \left(p_\mu \frac{dx^\mu}{ds} + \lambda(s) \left(p_\mu p_\nu g^{\mu\nu}(x) - m^2 \right) \right)} \tag{3.18}$$

where p_μ is the momentum conjugate to x^μ. It is easy to check that if we integrate out p_μ, we get back our by now familiar action

$$S_p = \int ds \left(-\frac{1}{4\lambda} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu - \lambda m^2 \right). \tag{3.19}$$

The form (3.18) was used in [41] as the starting point towards the quantization of a relativistic particle on the $SL(2,\mathbb{R})$ manifold via the method of Hamiltonian reduction. The key feature of this method is precisely that one constructs a system out of a much simpler Hamiltonian system by a reduction using constraints. In (3.18) the constraint is the mass-shell condition $p^2 = m^2$, as is clear by varying the action with respect to λ. Note that the Hamiltonian that corresponds to the action (3.19) is given by[3]

$$H = -\lambda(g^{\mu\nu}p_\mu p_\nu - m^2), \tag{3.20}$$

and so our constraint is equivalent to demanding that H be zero[4].

As we will see, from the form (3.18) it is possible to find a straightforward generalization that will allow us to consider particles carrying higher-spin charges. This generalization was first proposed and played a crucial role in [25], whose presentation we now partially follow.

[3]Recall that the Hamiltonian can be obtained from a Lagrangian $L(q,\dot{q})$ by means of a Legendre transform: $H = \dot{q}\frac{\partial L}{\partial \dot{q}} - L$.

[4]See [44] for details regarding he Hamiltonian reduction method, as well as its uses in the study of constrained systems.

Before obtaining such a generalization, however, we still need to find the appropriate group theory formulation of (3.18) that would allow us to couple it to (2.34). We will now show how this can be achieved by making some simple considerations in the context of group manifolds.

3.2 The action of a relativistic particle in the language of group manifolds

To understand how we can eventually rewrite (3.18) in the framework of group manifolds, let us first come back to our considerations on the vielbein formalism. In Section 2.1 we saw that the vielbeins e_μ^a are matrices which take us from a coordinate basis ∂_μ to a locally defined basis through the relation

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}, \tag{3.21}$$

where η_{ab} is a flat metric in its canonical form. This means that if we pick a local coordinate system y^a for the new basis, then the matrices $e_\mu{}^a$ will be given by

$$e_\mu{}^a = \frac{\partial y^a}{\partial x^\mu}, \tag{3.22}$$

which is of course the transformation matrix coming from the usual coordinate transformation $\partial_\mu = \frac{\partial y^a}{\partial x^\mu}\partial_a$.

If $g_{\mu\nu}$ is the metric defined on a manifold of a Lie group G, then the local metric η_{ab} of (3.21) will be precisely the same "metric" appearing in the commutation relations that define the algebra $\mathcal{G}$ associated with G. For instance, in the commutation relation

$$[J_a, J_b] = \epsilon_{abc}\eta^{cd}J_d, \tag{3.23}$$

which defines the algebra $sl(2,\mathbb{R})$, η plays the role of the local metric (in the sense of (3.21)) of the $sl(2,\mathbb{R})$ *manifold*, and the generators J_a expand a vector basis[5]. This is more clearly seen if we consider an explicit example of an element $U \in G$ parametrized in terms of the exponential map

$$U = e^{y^a J_a} \tag{3.24}$$

[5] An introductory exposition on the subject of Lie groups seen as manifolds can be found in [46].

As an element of a Lie group, U depends in a continuous and differentiable way on the set of real parameters y^a – this is actually the reason why a Lie group is at the same time a group and a differentiable manifold. The parameters y^a can then be interpreted as the components of a vector in an expansion in terms of the local vector basis J_a, i.e. as the parameters y^a appearing in (3.22). Now, from (3.24) we see that

$$U^{-1}\partial_\mu U = \frac{\partial y^a}{\partial x^\mu} J_a \underset{(3.22)}{=} e_\mu{}^a J_a \tag{3.25}$$

which means that we can rewrite (3.21) as follows:

$$g_{\mu\nu} = \eta_{ab} e_\mu{}^a e_\nu{}^b \equiv \mathrm{Tr}(e_\mu{}^a J_a e_\nu{}^b J_b) = \mathrm{Tr}(U^{-1}\partial_\mu U U^{-1}\partial_\nu U), \tag{3.26}$$

where Tr denotes contraction with the metric η_{ab} (thus, for example, for an element $P \in \mathcal{G}$, $\mathrm{Tr}P^2 = \eta_{ab}P^a P^b$).

Using (3.26), the action (3.18) can now be rewritten as

$$S[U, P, \lambda] = \int ds \left(\mathrm{Tr}(PU^{-1}\dot{U}) + \lambda(\mathrm{Tr}P^2 - c_2)\right). \tag{3.27}$$

Here $P \in \mathcal{G}$ is the equivalent of the conjugate momentum p_μ in (3.18), and, as indicated in [25], $c_2 = m^2$ turns out to be the value of the quadratic Casimir characterizing the infinite dimensional representation that naturally corresponds to our massive particle[6].

This action has a global symmetry group $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ that acts as

$$U(s) \to LU(s)R, \qquad P \to R^{-1}P(s)R, \tag{3.28}$$

with $L, R \in SL(2,\mathbb{R})$.

The equations of motion are

$$U^{-1}\frac{dU}{ds} + 2\lambda P = 0, \quad \frac{dP}{ds} = 0, \quad \mathrm{Tr}P^2 = c_2 \tag{3.29}$$

[6]This is exactly analogous to representations of the Poincaré group using as a basis the Hilbert space of a free particle. In this case, the states of a free particle with momentum $\mathbf{p}$ are denoted as $|\mathbf{p}, s\rangle$, where s labels all the quantum numbers. Since $\mathbf{p}$ is a continuous and unbounded variable, this base space is infinite-dimensional. By Wigner's theorem, the generators J^i, K^i, P^i and H^i are represented by Hermitian operators. A simple operator commuting with all the generators is $P_\mu P^\mu$, which has the value m^2 on a one-particle state. Therefore, the m appearing in our relation $c_2 = m^2$ is indeed the mass of the particle as it appears in (3.18). See sections 2.4 and 2.5 of [47] for more details of on representations of the Poincaré group on one-particle states.

3.2.1 Coupling

With a gauge theory formulation of both S_{EH} and S_p at hand, all we need to do now is to perform the minimal coupling between the fields $A, \bar{A}$ and U by promoting the global symmetry (3.28) to a local symmetry. We then require the action to be invariant under

$$U(s) \to L(x^\mu(s))U(s)R(x^\mu(s)), \qquad P \to R^{-1}(x^\mu(s))P(s)R(x^\mu(s)). \tag{3.30}$$

This can be done by substituting $\frac{d}{ds}U$ in the action with

$$D_s U = \frac{d}{ds}U + A_s U - U\bar{A}_s, \qquad A_s \equiv A_\mu \frac{dx^\mu}{ds}. \tag{3.31}$$

where $A_s \equiv A_\mu \dot{x}^\mu$

The coupled action is then given by

$$\begin{aligned} S(U,P;A,\bar{A}) &= \int ds(\mathrm{Tr}(PU^{-1}D_sU) + \lambda(s)(\mathrm{Tr}(P^2) - c_2)) \\ &= \int ds \left(\mathrm{Tr}PU^{-1}\dot{U} + \mathrm{Tr}PU^{-1}A_sU - \mathrm{Tr}PP\bar{A}_s + \lambda(\mathrm{Tr}P^2 - c_2)\right), \end{aligned} \tag{3.32}$$

whose variation with respect to U and P leads to

$$\boxed{\frac{d}{ds}P + \left[\bar{A}_s, P\right] = 0, \qquad U^{-1}D_sU + 2\lambda P = 0} \tag{3.33}$$

where P is subject to the constraint $\mathrm{Tr}(P^2) = c_2$, as can be immediately seen by varying with respect to λ.

The variation with respect to A gives:

$$\begin{aligned} \frac{\delta S_{probe}}{\delta A_\rho} =& \frac{\delta}{\delta A_\rho}\int ds \left(\mathrm{Tr}\left(PU^{-1}A_\lambda \frac{dx^\lambda}{ds}U\right)\right) \\ =& \int ds \frac{dx^\lambda}{ds}\frac{\delta}{\delta(A_\rho)^i{}_j}\left(P^a_b(U^{-1})^b{}_c(A_\lambda)^c{}_dU^d{}_a\right) \\ =& \int ds \frac{dx^\rho}{ds}\left(P^a_b(U^{-1})^b{}_c\,\delta^c{}_i\,\delta^j{}_dU^d{}_a\right) \\ =& \int ds \frac{dx^\rho}{ds}\left(U^j{}_aP^a_b(U^{-1})^b{}_i\right) \\ =& \int ds \frac{dx^\rho}{ds}UPU^{-1} \end{aligned} \tag{3.34}$$

The same procedure leads to the following result for the variation with respect to $\bar{A}$:

$$\frac{\delta S_{probe}}{\delta \bar{A}_\rho} = \int ds \frac{dx^\rho}{ds} P \tag{3.35}$$

Thus, the variation of the total action

$$S = S_{CS}[A] - S_{CS}[\bar{A}] + S(U, P; A, \bar{A}) \tag{3.36}$$

with respect to A leads to the equation of motion:

$$\begin{aligned} \frac{\delta S}{\delta A_\rho} =& \frac{k}{4\pi} \int d^3x\, \epsilon^{\rho\mu\nu} F_{\mu\nu} + \int ds \frac{dx^\rho}{ds} U P U^{-1} \\ =& \int d^3x \left(\frac{k}{4\pi} \epsilon^{\mu\nu\rho} F_{\mu\nu}(x) + \int ds \frac{dx^\rho}{ds} \delta^{(3)}(x - x(s)) U P U^{-1} \right) = 0, \end{aligned} \tag{3.37}$$

where we have used (2.45) and (2.46) combined with the results for the variation of the action of the probe. Multiplying the expression we just obtained by $\epsilon_{\alpha\beta\rho}$ gives:

$$\frac{k}{2\pi} F_{\alpha\beta} = -\int ds \frac{dx^\rho}{ds} \epsilon_{\alpha\beta\rho} \delta^{(3)}(x - x(x)) U P U^{-1}, \tag{3.38}$$

i.e.

$$\boxed{\frac{k}{2\pi} \left((\partial_\mu A_\nu - \partial_\nu A_\mu) + [A_\mu, A_\nu] \right) = -\int ds \frac{dx^\rho}{ds} \epsilon_{\mu\nu\rho} \delta^{(3)}(x - x(x)) U P U^{-1}} \tag{3.39}$$

and similarly for $\bar{A}$:

$$\boxed{\frac{k}{2\pi} \left(\left(\partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu \right) + [\bar{A}_\mu, \bar{A}_\nu] \right) = -\int ds \frac{dx^\rho}{ds} \epsilon_{\mu\nu\rho} \delta^{(3)}(x - x(x)) P} \tag{3.40}$$

These two last equations, together with (3.33), the constraint $\mathrm{Tr}(P^2) = c_2$ and the trajectory (4.21) define the setup we will dealing with. In general, this set of equations is highly non-trivial, and its precise form depends strongly on the choice of the path $x^\mu(s)$. However, from the point of view of the equations of motion, U acts as a gauge transformation on A[7], and so, as realized in [25], one may attempt to simplify a bit by picking the "gauge" $U = 1$. In this case, the first of the equations of motion (3.33) reads

$$\frac{d}{ds}(A - \bar{A})_\mu \frac{dx^\mu}{ds} = [A_\mu, \bar{A}_\nu] \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0, \tag{3.41}$$

[7]Indeed, the term $U^{-1} D_s U$ that appears repeatedly in our equations is nothing but $U^{-1} \frac{dU}{ds} U + U^{-1} A_s U - \bar{A}_s$

where we have substituted the value of P coming from the second of the equations (3.33). For a given pair of connections $A, \bar{A}$, Eq. (3.41) appears as a differential equation for the path. To see this more explicitly, let us rewrite the connections in terms of the vielbein and the spin connection. Eq. (3.41) then becomes

$$\begin{aligned}
\frac{d}{ds}(2e_\mu \frac{dx^\mu}{ds}) &= (\omega_\mu^a + e_\mu^a)(\omega_\nu^b - e_\nu^b)[J_a, J_b]\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \\
&= (\omega_\mu^a + e_\mu^a)(\omega_\nu^b - e_\nu^b)\epsilon_{abc}J^c\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \\
&= -2\omega_\mu^a e_\nu^b \epsilon_{abc}J^c\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \\
&= -2\omega_{\mu cb}e_\nu^b J^c\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}
\end{aligned} \tag{3.42}$$

If we use (2.10), our equation now reads

$$\begin{aligned}
\frac{d}{dt}\left(e_\mu \frac{dx^\mu}{dt}\right) &= -e_\rho \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} + \frac{d}{ds}e_\nu \frac{dx^\nu}{ds} \\
&= -e_\rho \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} + \frac{d}{ds}\left(e_\nu \frac{dx^\nu}{ds}\right) - e_\nu \frac{d^2x^\nu}{ds^2},
\end{aligned}$$

which is actually the familiar geodesic equation

$$\frac{d^2x^\rho}{ds^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0. \tag{3.43}$$

This means that as long as we choose geodesic paths, we can consistently set $U = 1$. Since we are interested in sources following the path (3.14), which is a geodesic, this fact will simplify enormously our task of solving the set of equations we found. Note, however, that there is a subtlety when we consider general $sl(N, \mathbb{R})$ algebras instead of $sl(2, \mathbb{R})$. The main difference comes from the fact that $sl(2, \mathbb{R})$ has three generators, whereas the higher-rank algebras will have in general more than three. While this seems to be irrelevant in (3.42), it is actually crucial that there be only three generators. This is so because, for given connections $A, \bar{A}$, the Eq. (3.42) is meant to solve for the three unknowns $x^\mu(s)$, but this equation actually condenses as many independent equations as there are generators. If there are more than three generators, our system of equations is then overconstrained, implying that in this case it is not consistent to consider geodesic paths and set $U = 1$ at the same time [25].

Chapter 4

Vaidya Chern-Simons

In Chapter 3 we presented a setup that would in principle allow us to build dynamical higher-spin gravity solutions in three dimensions. The key feature of our framework was the possibility of deriving equations of motion in the language of gauge theory, thus permitting extensions of the gauge group in a very natural and simple way. To have actual dynamics, we saw that it was necessary to couple particles to gravity, and we studied an explicit way of doing this using the example of the Vaidya metric. Now we would like to use our gauge theory equations to find the equivalent of the Vaidya metric in the Chern-Simons language. If we manage to extend the connections thus obtained to the higher-spin case, we would be achieving our goal of building dynamical higher-spin gravity solutions.

4.1 Vaidya-like connections

In Chapter 3 we saw that the Vaidya metric is a non-static generalization of the Schwarzschild metric. In terms of the ingoing Eddington-Finkelstein coordinate v, the Vaidya metric is of the form

$$ds^2 = -F(r,v)dv^2 + 2drdv + r^2 d\varphi^2, \tag{4.1}$$

where we now leave $F(r,v)$ unspecified.

From our discussion in section 2.1, it is clear the the vielbeins corresponding to the Vaidya metric can be obtained from

$$ds^2 = \eta_{ab} e^a \otimes e^b. \tag{4.2}$$

The form of the metric (4.1) suggests that we use the following flat metric[1]

$$\eta_{ab} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.3}$$

From (4.1) and (4.3) one reads[2]

$$\begin{aligned} e^0 &= F^{1/2}dv \\ e^1 &= F^{-1/2}dr \\ e^2 &= rd\varphi. \end{aligned} \tag{4.4}$$

At the end of section 3.1 we claimed the torsion-free condition

$$\omega^a{}_b \wedge e^b = -\mathrm{d}e^a \tag{4.5}$$

would allow us to find straightforwardly the spin connections from the vielbeins. For this we need to compute the exterior derivatives of the latter. They are given by

$$\begin{aligned} \mathrm{d}e^0 &= \tfrac{1}{2}F^{-1/2}\left[\tfrac{\partial F}{\partial r}dr + \tfrac{\partial F}{\partial v}dv\right] \wedge dv \\ &= \tfrac{1}{2}F^{-1/2}\tfrac{\partial F}{\partial r}dr \wedge dv = -\tfrac{1}{2}F^{-1/2}\tfrac{\partial F}{\partial r}e^0 \wedge e^1 \\ \mathrm{d}e^1 &= -\tfrac{1}{2}F^{-3/2}\left[\tfrac{\partial F}{\partial r}dr + \tfrac{\partial F}{\partial v}dv\right] \wedge dr \\ &= -\tfrac{1}{2}F^{-3/2}\tfrac{\partial F}{\partial v}dv \wedge dr = -\tfrac{1}{2}F^{-3/2}\tfrac{\partial F}{\partial v}e^0 \wedge e^1 \\ \mathrm{d}e^2 &= dr \wedge d\phi = \tfrac{F^{1/2}}{r}e^1 \wedge e^2 \end{aligned} \tag{4.6}$$

The computation of the antisymmetric spin connections ω_{ab} from Eq. (4.5) is carried out explicitly in appendix B. It yields

$$\begin{aligned} \omega^0 &= F^{1/2}\,d\varphi \\ \omega^1 &= 0 \\ \omega^2 &= -\frac{1}{2}\left(F^{-1}\frac{\partial F}{\partial v} - \frac{\partial F}{\partial r}\right)dv - \frac{1}{2}F^{-1}\frac{\partial F}{\partial r}dr \end{aligned} \tag{4.7}$$

[1]Since our spacetime is Lorentzian, this metric is obviously the Minkowski metric written in a non-standard basis.

[2]Notice that the choice $e^0 = -F^{1/2}dv$, $e^1 = -F^{1/2}dr$ is also possible. Our conclusions won't be affected in any way by switching to this other possibility.

where $\omega^a \equiv \frac{1}{2}\epsilon^a{}_{bc}\omega^{bc}$.

The reason why we have computed the vielbeins and the spin connection is of course that we wish to find the $SL(2,\mathbb{R})$ connections A and $\bar{A}$ corresponding the metric (4.1). They are given by

$$A = \left(\omega^a + \frac{1}{\ell}e^a\right)J_a, \qquad \bar{A} = \left(\omega^a - \frac{1}{\ell}e^a\right)J_a \tag{4.8}$$

where J_a are generators of the $sl(2,\mathbb{R})$ algebra. In section 2.2 we introduced a representation of this algebra and we alleged that that particular representation would prove useful for later considerations. Let us reproduce here that representation

$$J_0 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad J_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad J_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.9}$$

which satisfy

$$\mathrm{tr}(J_aJ_b) = \frac{1}{2}\begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\eta_{ab}. \tag{4.10}$$

It is now clear that the reason why we picked this representation is that its Killing form $\eta_{ab} = 2tr(J_aJ_b)$ coincides with the local frame metric (4.3). It is important that this be the case so as to raise and lower indices conveniently in expansions of the form A^aJ_a. Recall that A^a was built from vielbeins and spin connections that were constructed using the local-frame metric (4.3), and so its index is raised or lowered using that metric. On the other hand, the index of J_a is naturally raised or lowered by the Killing form, and so $A^aJ_a = A_aJ^a$ when the latter coincides with the local metric (4.3).

From (4.4) and (B.9) we finally obtain the following for the connections:

$$\begin{aligned} A = &\left(F^{1/2}\, d\varphi + \frac{1}{\ell}F^{1/2}dv\right)J_0 + \left(\frac{1}{\ell}F^{1/2}\right)J_1 \\ &+ \left[\frac{1}{2}\left(\frac{\partial F}{\partial r} - F^{-1}\frac{\partial F}{\partial v}\right)dv - \frac{1}{2}F^{-1}\frac{\partial F}{\partial r}dr + \frac{1}{\ell}rd\varphi\right]J_2, \end{aligned} \tag{4.11}$$

$$\begin{aligned} \bar{A} = &\left(F^{1/2}\, d\varphi - \frac{1}{\ell}F^{1/2}dv\right)J_0 - \left(\frac{1}{\ell}F^{1/2}\right)J_1 \\ &+ \left[\frac{1}{2}\left(\frac{\partial F}{\partial r} - F^{-1}\frac{\partial F}{\partial v}\right)dv - \frac{1}{2}F^{-1}\frac{\partial F}{\partial r}dr - \frac{1}{\ell}rd\varphi\right]J_2\,. \end{aligned} \tag{4.12}$$

These connections can be written as $A = A_\mu dx^\mu$ and $\bar{A} = \bar{A}_\mu dx^\mu$. From (4.11) and (4.12) one reads:

$$A_v = \frac{1}{\ell} F^{1/2} J_0 + \frac{1}{2}\left(\frac{\partial F}{\partial r} - F^{-1}\frac{\partial F}{\partial v}\right) J_2, \tag{4.13}$$

$$A_r = \frac{1}{\ell} F^{-1/2} J_1 - \frac{1}{2} F^{-1}\frac{\partial F}{\partial r} J_2, \tag{4.14}$$

$$A_\varphi = F^{1/2} J_0 + \frac{1}{\ell} r J_2, \tag{4.15}$$

and

$$\bar{A}_v = -\frac{1}{\ell} F^{1/2} J_0 + \frac{1}{2}\left(\frac{\partial F}{\partial r} - F^{-1}\frac{\partial F}{\partial v}\right) J_2, \tag{4.16}$$

$$\bar{A}_r = -\frac{1}{\ell} F^{-1/2} J_1 - \frac{1}{2} F^{-1}\frac{\partial F}{\partial r} J_2, \tag{4.17}$$

$$\bar{A}_\varphi = F^{1/2} J_0 - \frac{1}{\ell} r J_2. \tag{4.18}$$

We have then managed to find the gauge connections corresponding to (4.1). Our next step will be to substitute these connections in the equations of motions we found in the previous chapter and solve for $F(r, v)$. We will do this following the setup that led us to determine mass profile (3.17), corresponding to a thin null shell that sources a Vaidya-like metric.

4.2 The spin-2 case

We have already seen that the variation with respect to A and $\bar{A}$ of the total action (3.36) leads to

$$\frac{k}{2\pi}\left[(\partial_\mu A_\nu - \partial_\nu A_\mu) + [A_\mu, A_\nu]\right] = -\int ds \frac{dx^\rho}{ds}\epsilon_{\mu\nu\rho}\delta^{(3)}(x - x(s)) U P(s) U^{-1}, \tag{4.19}$$

and

$$\frac{k}{2\pi}\left[(\partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu) + \left[\bar{A}_\mu, \bar{A}_\nu\right]\right] = -\int ds \frac{dx^\rho}{ds}\epsilon_{\mu\nu\rho}\delta^{(3)}(x - x(s)) P(s), \tag{4.20}$$

where $x(s)$ is the trajectory of the probe particle. Since we are interested in dust shells that follow null geodesics, we choose the following trajectory parametrized by s:

$$v(s) = v_0, \quad r(s) = r_0 + c_0 s, \quad \varphi(s) = \varphi_0 \tag{4.21}$$

which we showed to be the trajectory followed by a probe particle in the massless limit in the AdS background (see (3.14)).

According to our discussion at the end of Chapter 3, such a choice of trajectory (a geodesic) allows us to set, in the spin-2 case, $U = 1$. Making use of this simplification and substituting the geodesic trajectory, Eq. (4.19) becomes

$$\frac{k}{2\pi}\left[(\partial_r A_\varphi - \partial_\varphi A_r) + [A_r, A_\varphi]\right] = 0 \tag{4.22}$$

$$\frac{k}{2\pi}\left[(\partial_v A_r - \partial_r A_v) + [A_v, A_r]\right] = 0 \tag{4.23}$$

$$\frac{k}{2\pi}\left[(\partial_v A_\varphi - \partial_\varphi A_v) + [A_v, A_\varphi]\right] = -\int ds \frac{dr}{ds} \epsilon_{v\varphi r} \delta^{(3)}(x - x(s)) P(s) \tag{4.24}$$

In Appendix C we have worked out the explicit expressions of the left-hand sides of these equations. The final results are

$$(\partial_r A_\varphi - \partial_r A_\varphi) + [A_r, A_\varphi] = F^{-1/2}\left(\frac{1}{\ell^2} r - \frac{1}{2}\frac{\partial F}{\partial r}\right) J_1 \tag{4.25}$$

$$(\partial_v A_r - \partial_r A_v) + [A_v, A_r] = \left(\frac{1}{\ell^2} - \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\right) J_2 \tag{4.26}$$

$$\begin{aligned}(\partial_v A_\varphi - \partial_\varphi A_v) + [A_v, A_\varphi] = &\left(-\frac{1}{\ell^2} F^{1/2} r + \frac{1}{2} F^{1/2} \frac{\partial F}{\partial r} - \frac{1}{2} F^{-1/2} \frac{\partial F}{\partial v} + \frac{1}{2} F^{-1/2} \frac{\partial F}{\partial v}\right) J_0 \\ &+ \left(-\frac{1}{\ell^2} F^{1/2} r + \frac{1}{2} F^{1/2} \frac{\partial F}{\partial r} - \frac{1}{2} F^{-1/2} \frac{\partial F}{\partial v}\right) J_1.\end{aligned} \tag{4.27}$$

Applying the equations of motion (4.22) and (4.23) in Eqs. (4.25) and (4.26) implies:

$$F(v, r, \varphi) = c + \frac{r^2}{\ell^2} + f(v), \tag{4.28}$$

where c is a constant. We see that the r-dependence is already as in the Vaidya metric. The function $f(v)$ is to be determined using Eqs. (4.24) and (4.27), the last of which now reads:

$$(\partial_v A_\varphi - \partial_\varphi A_v) + [A_v, A_\varphi] = -\frac{1}{2} F^{-1/2} \frac{df}{dv} J_1. \tag{4.29}$$

So,

$$\boxed{\frac{k}{2\pi}\frac{1}{2} F^{-1/2} \frac{df}{dv} J_1 = \int ds \frac{dr}{ds} \epsilon_{v\varphi r} \delta^{(3)}(x - x(s)) P(s)} \tag{4.30}$$

with

$$\delta^{(3)}(x-x(s)) = \frac{\delta(v-v(s))\delta(r-r(s))\delta(\varphi-\varphi(s))}{\sqrt{|g|}} \tag{4.31}$$

and

$$\epsilon_{\mu\nu\rho} = \sqrt{|g|}\,[\mu,\nu,\rho], \tag{4.32}$$

where $|g| = r^2$ denotes the determinant of the metric (4.1), and

$$[\mu,\nu,\rho] = \begin{cases} 1 & \text{if the argument is an even permutation of } vr\varphi \\ -1 & \text{if the argument is an odd permutation of } vr\varphi \\ 0 & \text{if two or more arguments are equal} \end{cases} .$$

Eq. (4.30) then reduces to

$$\begin{aligned} &\frac{k}{2\pi}\frac{1}{2}F^{-1/2}\frac{df}{dv}J_1 = \delta(v-v(s))\delta(\varphi-\varphi(s))\int dr\delta(r-r(s))P(s) \\ &\Rightarrow kF^{-1/2}\frac{df}{dv}J_1 = 2\delta(v-v(s))\int dr\delta(r-r(s))P(s). \end{aligned} \tag{4.33}$$

where we have "smoothed" the angular variable as in (3.17), i.e. we are considering a shell of particles instead of a single one.

In order to find $P(s)$ one must solve the equations of motion (3.33). Recall that P plays the role of the momentum of the particle, and so we expect that its equation encodes information about the mass of the particle. More precisely, we expect that the Casimir $\mathrm{tr}(P^2)$ "knows" something about our choice of trajectory, namely a null geodesic, which is followed uniquely by massless particles. For $U=1$ the first of equations (3.33) reads

$$A_\mu\frac{dx^\mu}{ds} - \bar{A}_\mu\frac{dx^\mu}{ds} + 2\lambda P = 0 \tag{4.34}$$

For radial geodesics ($\dot{\varphi}=0$) this equation reduces to

$$P = \frac{-1}{2\lambda}\left((A_v-\bar{A}_v)\frac{dv}{ds} + (A_r-\bar{A}_r)\frac{dr}{ds}\right) = -\frac{1}{\lambda\ell}\left(F^{1/2}J_0\frac{dv}{ds} + F^{-1/2}J_1\frac{dr}{ds}\right), \tag{4.35}$$

where we have used eqs. (4.13) to (4.17) for the expressions of the components of the connections. In particular, for an ingoing null geodesic ($\dot{v}=0$, $\dot{r}=c_0$) this equation implies

$$\boxed{P = -\frac{c_0}{\lambda\ell}F^{-1/2}J_1 = c_1 F^{-1/2}J_1} \tag{4.36}$$

where $c_1 \equiv -\frac{c_0}{\lambda\ell}$ is a constant[3].

With the trajectory being that of an ingoing null geodesic, the first of the equations of motion (3.33) gives:

$$\frac{d}{ds}P + (\bar{A}_r P - P\bar{A}_r)\frac{dr}{ds} = \frac{d}{ds}P - \left[\frac{1}{\ell}F^{-1/2}J_1 + \frac{1}{2}F^{-1}\frac{\partial F}{\partial r}J_2, P\right]c_0 = 0. \tag{4.37}$$

If $P = P^1 J_1$ eq. (4.37) can be recast in the form:

$$\frac{d}{ds}P^1 J_1 - \frac{c_0}{2}F^{-1}\frac{\partial F}{\partial r}P^1[J_2, J_1] \underset{(C.3)}{=} \frac{d}{ds}P^1 J_1 + \frac{c_0}{2}F^{-1}\frac{\partial F}{\partial r}P^1 J_1 \tag{4.38}$$

$$= \left(\frac{\partial P^1}{\partial r}\cancelto{c_0}{\frac{dr}{ds}} + \frac{\partial P^1}{\partial v}\cancelto{0}{\frac{dv}{ds}}\right)J_1 + \frac{c_0}{2}F^{-1}\frac{\partial F}{\partial r}P^1 J_1 \tag{4.39}$$

$$= c_0\frac{\partial P^1}{\partial r}J_1 + \frac{c_0}{2}F^{-1}\frac{\partial F}{\partial r}P^1 J_1 \tag{4.40}$$

which is solved by $P^1 = c_1 F^{-1/2}$, meaning that our solution is consistent with both equations of motion, as it should be.

It only remains to see what the implications of the constraint $\mathrm{Tr}P^2 = c_2$ are. We get:

$$\mathrm{Tr}P^2 = P^a P^b \eta_{ab} = -(P^0)^2 + 2P^0P^1 + (P^2)^2 = 0 = c_2 \tag{4.41}$$

i.e. *the probe is massless*, as is expected from the fact that it follows ingoing *null* geodesics.

Plugging (4.36) in (4.33) gives

$$kF^{-1/2}\frac{df}{dv}J_1 = 2\,c_1\delta(v - v(s))J_1 \int dr\delta(r - r(s))F^{-1/2}, \tag{4.42}$$

which implies that

$$\boxed{\frac{df}{dv} = \frac{2c_1}{k}\delta(v - v(s)) = -\frac{8G_N c_0}{\ell^2\lambda}\delta(v - v(s))} \tag{4.43}$$

Comparing with (3.17), we see that (4.43) is precisely what we wanted to obtain. The connections (4.11) and (4.12) are then the connections corresponding to the Vaidya metric such as we introduced it in 3.1.

[3]While λ can be a function of s, the residual reparametrization symmetry of the world-line coordinate s allows us to set $\lambda(s)$ to a constant. See footnote 2.

Integrated, the mass profile is of the form

$$f(v) = m\Theta(v - v_0) \tag{4.44}$$

where m is a constant to be determined by the total mass of the black hole[4]

In view of this positive realization, we will now attempt to use our setup in a higher-spin scenario, namely the $SL(3,\mathbb{R})$ case.

4.3 The higher-spin case

In Section 2.3.2 we saw that the main difference that arises when considering fields with spin higher than 2 is that the connections are expanded in terms of generators of the corresponding $sl(N,\mathbb{R})$ algebra. In the $sl(3,\mathbb{R})$, we introduced a base such that the connections are expanded as follows

$$\begin{aligned} A = A^{(2)} + A^{(3)} &= A^a J_a + A^{ab} T_{ab} \\ &= A^0 J_0 + A^1 J_1 + A^2 J_2 \\ &\qquad + A^{00} T_{00} + A^{01} T_{01} + A^{02} T_{02} + A^{11} T_{11} + A^{12} T_{12} \end{aligned} \tag{4.45}$$

$$\begin{aligned} \bar{A} = \bar{A}^{(2)} + \bar{A}^{(3)} &= \bar{A}^a J_a + \bar{A}^{ab} T_{ab} \\ &= A^0 J_0 + A^1 J_1 + A^2 J_2 \\ &\qquad + A^{00} T_{00} + A^{01} T_{01} + A^{02} T_{02} + A^{11} T_{11} + A^{12} T_{12} \end{aligned} \tag{4.46}$$

where the J_a generate the $sl(2,\mathbb{R})$ subalgebra and the five generators T_{ab} (recall that they are symmetric and traceless in the indices a, b) are the higher-spin generators. The explicit form of this representation can be found in Appendix A. The first thing to note is that now the Killing form is given by (A.2)[5], and so, for example,

[4] Presumably in terms of the ADM formalism[45].

[5] Note that now we have $\mathrm{tr}(J_a J_b) = 2\,\eta_{ab}$. So, compared to (2.30), we now have a factor 2 where before there used to be a factor 1/2. Of course, there is nothing wrong with this, as it just reflects the fact that the constant c in $\mathrm{tr}(J_a J_b) = c\mathrm{Tr}(J_a J_b)$ depends on the specific representation, as we said before.

$$c_2 = \mathrm{Tr}(P^2) = -\,(P^0)^2 + 2P^0P^1 + (P^2)^2 + \frac{1}{3}\left(4(P^{00})^2 - 8P^{00}P^{01} + (P^{01})^2 - 3(P^{02})^2 + 12P^{00}P^{11} + 6P^{02}P^{12}\right). \tag{4.47}$$

That is, the higher spin components of P also source the mass. Since now we are in a higher-spin algebra, we can ask ourselves what the equivalent of this would be for the higher-spin charge. As we have emphasized, the spin charges are related to the values of the Casimirs. The spin-3 charge is related to the value of the cubic Casimir. The latter can be built in a similar way as the quadratic one, namely by finding a fully symmetric invariant trilinear form (see [25] for details):

$$h_{lmn} \propto \mathrm{tr}(T_{(l}T_mT_{n)}) \tag{4.48}$$

where the indices l, m, n run from 1 to 8, and the T_m denote collectively the eight generators $\{J_0, J_1, J_2, T_{00}, T_{01}, T_{02}, T_{11}, T_{12}\}$ in the order implied. The parentheses in $T_{(l}T_mT_{n)}$ denote symmetrization.

The value of the cubic Casimir is then given by[6]

$$c_3 = \mathrm{Tr}P^3 = h_{lmn}P^lP^mP^n. \tag{4.49}$$

The first terms of the expression above are

$$c_3 = \frac{32}{3}(P^{00})^3 + \frac{4}{3}(P^{01})^3 + 12P^{01}(P^{02})^2 - 36(P^{02})^2P^{11} + \cdots . \tag{4.50}$$

The full expression can be found in Appendix A. The key point is that, as was to be expected, the spin-2 components of P do not source the spin-3 charge. On the other hand, not all combinations of spin-3 components will lead to a non-zero spin-3 charge. To see this, let us come back to our discussion of 2.3.2. There we pointed out that there is a very easy way of adding higher-spin components "by hand". The example we used was the BTZ black hole, whose connections are $sl(2,\mathbb{R})$-valued. We said that if we considered these connections to be actually $sl(3,\mathbb{R})$-valued connections in a particular gauge in which the spin-3 component happen to be zero, then we could add a non-zero spin-3 component just by picking a different

[6] In the last equality there should be the proportionality constant between Tr and tr but we have omitted it since it is not crucial for our considerations.

gauge. This is precisely what we wanted to illustrate in Eqs. (2.76) and (2.77), after which we commented that this was an artificial way of building a higher-spin solution and that no actual physics was added by doing this. At this stage that claim simply means that c_3 is gauge invariant (recall that P transforms according to (3.30)). Even if from (4.50) it seems that we just need some appropriate higher-spin components to make c_3 non-zero, the fact is that gauge transformations leave the value of c_3 invariant, and so zero remains zero. In particular, it would be impossible to go from a gauge where all the higher-spin components of P are zero to one where its only non-zero higher spin component is P^{00}, say.

Having discarded the "easy" way of finding higher-spin solutions, the question in then how do we find solutions with c_3 non-zero. The answer to this question was found by Ammon, Castro and Iqbal [25]. They proposed that for this one should generalize the action (3.32) to the form

$$S[U,P;A,\bar{A}] = \int ds(\mathrm{Tr}(PU^{-1}D_sU) + \lambda_2(\mathrm{Tr}(P^2) - c_2) + \lambda_3(\mathrm{Tr}(P^3) - c_3)), \qquad (4.51)$$

where λ_2 and λ_3 are Lagrange multipliers. The possibility of adding higher-spin charges by means of this simple generalization is actually the whole point of working with the constrained system that led us to (3.32). One can of course add as many higher-spin charges as one wants just by introducing the corresponding Lagrange multipliers. Now that we have gotten the appropriate action, let us see what the equations of motion look like.

The variation of (4.51) with respect to P and u gives

$$\boxed{\frac{d}{ds}P + [\bar{A}_s, P] = 0} \qquad (4.52)$$

$$\boxed{U^{-1}D_sU + 2\lambda_2 P + 3\lambda_3(P \times P) = 0} \qquad (4.53)$$

where $P \times P \equiv h_{abc}T^aP^bP^c$ and P is subject to the constraints $\mathrm{Tr}P^2 = c_2$ and $\mathrm{Tr}P^3 = c_3$. The variation with respect to A and $\bar{A}$ leads to the same equations as before, namely

$$\frac{k}{2\pi}\left((\partial_\mu A_\nu - \partial_\nu A_\mu) + [A_\mu, A_\nu]\right) = -\int ds \frac{dx^\rho}{ds}\epsilon_{\mu\nu\rho}\delta^{(3)}(x - x(x))UPU^{-1} \qquad (4.54)$$

and

$$\frac{k}{2\pi}\left(\left(\partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu\right) + [\bar{A}_\mu, \bar{A}_\nu]\right) = -\int ds \frac{dx^\rho}{ds}\epsilon_{\mu\nu\rho}\delta^{(3)}(x - x(x))P. \qquad (4.55)$$

We commented at the end of Chapter 3 that in the higher-spin case it is not consistent to set $U = 1$ and choose a geodesic path at the same time. This means that the field U must now reappear in all our equations, and we shall see that this will constitute a great complication of our task.

Let us write Eq. (4.54) with the connections expanded in the basis of $sl(3,\mathbb{R})$ generators:

$$\begin{aligned}&\frac{k}{2\pi}\left[(\partial_\mu A_\nu^{(2)}-\partial_\nu A_\mu^{(2)})+\left[A_\mu^{(2)},A_\nu^{(2)}\right]\right]+\\&\frac{k}{2\pi}\left[(\partial_\mu A_\nu^{(3)}-\partial_\nu A_\mu^{(3)})+\left[A_\mu^{(2)},A_\nu^{(3)}\right]+\left[A_\mu^{(3)},A_\nu^{(2)}\right]+\left[A_\mu^{(3)},A_\nu^{(3)}\right]\right]\\&=-\int ds\frac{dx^\rho}{ds}\epsilon_{\mu\nu\rho}\delta^{(3)}(x-x(s))UP(s)U^{-1}\end{aligned}$$

For $\rho = v, \varphi$ the right-hand side of this last equation is zero when one chooses the trajectory (4.21). Furthermore, if we choose the spin-2 component to match exactly what we found in the $sl(2,\mathbb{R})$ case[7], then the first line also vanishes for $\rho = v, \varphi$ (see (4.22),(4.23)). One is then left with the equations:

$$\frac{k}{2\pi}\left[(\partial_r A_\varphi^{(3)}-\partial_\varphi A_r^{(3)})+\left[A_r^{(2)},A_\varphi^{(3)}\right]+\left[A_r^{(3)},A_\varphi^{(2)}\right]+\left[A_r^{(3)},A_\varphi^{(3)}\right]\right]=0 \tag{4.56}$$

$$\frac{k}{2\pi}\left[(\partial_v A_r^{(3)}-\partial_r A_v^{(3)})+\left[A_v^{(2)},A_r^{(3)}\right]+\left[A_v^{(3)},A_r^{(2)}\right]+\left[A_v^{(3)},A_r^{(3)}\right]\right]=0 \tag{4.57}$$

$$\begin{aligned}&\frac{k}{2\pi}\left[(\partial_v A_\varphi^{(2)}-\partial_\varphi A_v^{(2)})+\left[A_v^{(2)},A_\varphi^{(2)}\right]\right]+\\&+\frac{k}{2\pi}\left[(\partial_v A_\varphi^{(3)}-\partial_\varphi A_v^{(3)})+\left[A_v^{(2)},A_\varphi^{(3)}\right]+\left[A_v^{(3)},A_\varphi^{(2)}\right]+\left[A_v^{(3)},A_\varphi^{(3)}\right]\right]\\&\qquad=-\int ds\frac{dr}{ds}\epsilon_{v\varphi r}\delta^{(3)}(x-x(s))UP(s)U^{-1}\\&\qquad=\delta(v-v_0)\delta(\varphi-\varphi(s))\int dr\delta(r-r(s))\mathbb{P}(s)\end{aligned} \tag{4.58}$$

where $\mathbb{P}(s)\equiv UP(s)U^{-1}$. The first three terms of the left-hand side of Eqs. (4.56) and (4.57) are proportional to the generators T_{ab}. The algebra (2.72) indicates that that cannot be the

[7] In principle, this need not be the case. However, it seems natural to impose that when we turn off the higher-spin components of the connections, one recovers the spin-2 connections (4.11) and (4.12). That is for example the case for the higher-spin black hole that we presented in (2.79), which reduces to BTZ for $\mathcal{W}=0$ and $\mu=0$

case for the last term of the left-hand side of these equalities. This means that one must deal with the independent equations

$$\bullet\ (\partial_r A_\varphi^{(3)} - \partial_\varphi A_r^{(3)}) + \left[A_r^{(2)}, A_\varphi^{(3)}\right] + \left[A_r^{(3)}, A_\varphi^{(2)}\right] = 0 \tag{4.59a}$$

$$\bullet\ \left[A_r^{(3)}, A_\varphi^{(3)}\right] = 0 \tag{4.59b}$$

$$\bullet\ (\partial_v A_r^{(3)} - \partial_r A_v^{(3)}) + \left[A_v^{(2)}, A_r^{(3)}\right] + \left[A_v^{(3)}, A_r^{(2)}\right] = 0 \tag{4.60a}$$

$$\bullet\ \left[A_v^{(3)}, A_r^{(3)}\right] = 0 \tag{4.60b}$$

On the other hand, Eq. (4.58) implies:

$$\frac{k}{2\pi}\left[(\partial_v A_\varphi^{(2)} - \partial_\varphi A_v^{(2)}) + \left[A_v^{(2)}, A_\varphi^{(2)}\right] + \left[A_v^{(3)}, A_\varphi^{(3)}\right]\right] = -\int ds \frac{dr}{ds}\epsilon_{v\varphi r}\delta^{(3)}(x - x(s))\mathbb{P}^a J_a \tag{4.61}$$

and

$$\begin{aligned}\frac{k}{2\pi}\left[(\partial_v A_\varphi^{(3)} - \partial_\varphi A_v^{(3)}) + \left[A_v^{(2)}, A_\varphi^{(3)}\right] + \left[A_v^{(3)}, A_\varphi^{(2)}\right]\right]& \\ = \delta(v - v_0)\delta(\varphi - \varphi(s)) \int dr \delta(r - r(s))\mathbb{P}^{ab}T_{ab}.&\end{aligned} \tag{4.62}$$

A similar argument holds for (4.20), which leads to the following equations:

$$\bullet\ (\partial_r \bar{A}_\varphi^{(3)} - \partial_\varphi \bar{A}_r^{(3)}) + \left[\bar{A}_r^{(2)}, \bar{A}_\varphi^{(3)}\right] + \left[\bar{A}_r^{(3)}, \bar{A}_\varphi^{(2)}\right] = 0 \tag{4.63a}$$

$$\bullet\ \left[\bar{A}_r^{(3)}, \bar{A}_\varphi^{(3)}\right] = 0 \tag{4.63b}$$

$$\bullet\ (\partial_v \bar{A}_r^{(3)} - \partial_r \bar{A}_v^{(3)}) + \left[\bar{A}_v^{(2)}, \bar{A}_r^{(3)}\right] + \left[\bar{A}_v^{(3)}, \bar{A}_r^{(2)}\right] = 0 \tag{4.64a}$$

$$\bullet\ \left[\bar{A}_v^{(3)}, \bar{A}_r^{(3)}\right] = 0 \tag{4.64b}$$

and for the v, φ components:

$$\frac{k}{2\pi}\left[(\partial_v \bar{A}^{(2)}_\varphi - \partial_\varphi \bar{A}^{(2)}_v) + \left[\bar{A}^{(2)}_v, \bar{A}^{(2)}_\varphi\right] + \left[\bar{A}^{(3)}_v, \bar{A}^{(3)}_\varphi\right]\right] = -\int ds \frac{dr}{ds}\epsilon_{v\varphi r}\delta^{(3)}(x - x(s))P^a J_a \tag{4.65}$$

$$\begin{aligned}\frac{k}{2\pi}&\left[(\partial_v \bar{A}^{(3)}_\varphi - \partial_\varphi \bar{A}^{(3)}_v) + \left[\bar{A}^{(2)}_v, \bar{A}^{(3)}_\varphi\right] + \left[\bar{A}^{(3)}_v, \bar{A}^{(2)}_\varphi\right]\right] \\ &= \delta(v - v_0)\delta(\varphi - \varphi(s))\int dr \delta(r - r(s))P^{ab}T_{ab}.\end{aligned} \tag{4.66}$$

We see that, compared to the equations for A, these last equations are much simpler in the sense that it is P who appears rather than $\mathbb{P}$. Looking back at (4.52), we see that we can solve for P without having to solve for U, whose equation of motion (4.53) is much more complicated (and requires to first solve for P, in any case.) Note that if we solve for P, then we could hope to find $\bar{A}$, since the two differ just by an integration over a delta function (see (4.55)). Let us then try and solve the equation (4.52).

4.3.1 Solving for the momentum

Split into its spin−2 and spin−3 components, Eq. (4.52) implies

$$\frac{d}{ds}P^{(2)} + c_0[\bar{A}^{(2)}_r, P^{(2)}] + c_0[\bar{A}^{(3)}_r, P^{(3)}] = 0 \tag{4.67a}$$

$$\frac{d}{ds}P^{(3)} + c_0[\bar{A}^{(2)}_r, P^{(3)}] + c_0[\bar{A}^{(3)}_r, P^{(2)}] = 0 \tag{4.67b}$$

We want to find connections whose spin−2 part coincides with (4.11), (4.12), and that are sourced by a non-zero higher-spin charge. This means that we must find a P such that $c_3 = \mathrm{Tr}P^3 \neq 0$. Many different *Ansätze* for P would lead to a non-zero higher-spin charge. For example, for the choice $P = P^{00}T_{00}$ one has $c_2 = \frac{4P^{00\,2}}{3}$ and $c_3 = \frac{32P^{00\,3}}{3}$[8]. However, this *Ansatz* is inappropriate for several reasons; in particular, such a choice would lead to more than one independent equation coming from (4.67b) for the one unknown P^{00}. One can see that, in general, the system of equations for the components of P will be overconstrained for simple *Ansätze* of P such as the aforementioned.

[8]While it may seem strange to even consider a solution with $c_2 \neq 0$ for the trajectory we have been using, it is not inconsistent to do so. The reason is that the interpretation of our trajectory as a geodesic is no longer valid in the presence of the higher-spin field. Massive probes can then follow such a trajectory just as massless probes can be "forced" to follow non-null trajectories by applying a $SL(3,\mathbb{R})$ gauge transformation.

In order to find a consistent P, let us use the gauge freedom to simplify Eqs. (4.67a) and (4.67a) by assuming that we have fixed the gauge in such a way that $\bar{A}_r^{(3)} = 0$. This choice is not as arbitrary as it may seem. Indeed, recall that (4.63b), (4.64b) must hold, and so $\bar{A}_r^{(3)}$ cannot have any value. With this choice, Eq. (4.67a) becomes

$$\frac{d}{ds}P^{(2)} + c_0[\bar{A}_r^{(2)}, P^{(2)}] = 0 \tag{4.68}$$

which is nothing but Eq. (4.37), whose solution was found to be

$$P^{(2)} = P^1 J_1, \tag{4.69}$$

with $P^1 = c_1 F^{-1/2}$. That the spin-2 part of P is precisely what was found in the $SL(2,\mathbb{R})$ case is of course consistent with the fact that we are demanding the spin-2 part of the $SL(3,\mathbb{R})$ $\bar{A}$ connection to coincide with what was found before. Note, however, that this last requirement does not lead necessarily to the identification we just mentioned. Indeed, if $\bar{A}_r^{(3)}$ were not zero, we would have gotten a completely different solution for $P^{(2)}$.

Eq. (4.67b) now reads

$$\frac{d}{ds}P^{(3)} + c_0[\bar{A}_r^{(2)}, P^{(2)}] = 0. \tag{4.70}$$

Reading $\bar{A}_r^{(2)} = -\frac{1}{\ell}F^{-1/2}J_1 - \frac{1}{2}F^{-1}\frac{\partial F}{\partial r}J_2$ from (4.17), this equation implies:

$$\frac{dP^{00}}{ds} + 2c_0\bar{A}_r^2 P^{00} = 0 \tag{4.71}$$

$$\frac{dP^{01}}{ds} + 3c_0\bar{A}_r^1 P^{02} + 2c_0\bar{A}_r^2 P^{00} = 0 \tag{4.72}$$

$$\frac{dP^{02}}{ds} - 2c_0\bar{A}_r^1 P^{00} + c_0\bar{A}_r^2 P^{02} = 0 \tag{4.73}$$

$$\frac{dP^{11}}{ds} + c_0\bar{A}_r^1 P^{02} + c_0\bar{A}_r^1 P^{12} + c_0\bar{A}_r^2 P^{01} - 2c_0\bar{A}_r^2 P^{11} = 0 \tag{4.74}$$

$$\frac{dP^{12}}{ds} - c_0\bar{A}_r^1 P^{01} - c_0\bar{A}_r^2 P^{12} = 0 \tag{4.75}$$

When solving these equations one must eventually integrate over s. However, since $r = r_0 + c_0 s$, the variable s can be changed to r. For example,

$$\int_0^s ds'\bar{A}_r^2 = \frac{1}{c_0}\int_{r_0}^r dr'\bar{A}_r^2(r') = -\frac{1}{2c_0}\log(\ell^2 F)\Big|_{r_0}^r = -\frac{1}{2c_0}\log\left(\frac{F}{F_{r_0}}\right) \tag{4.76}$$

with $F_{r_0} = F|_{r=r_0} = c + \frac{r_0^2}{\ell^2} + m\Theta(v - v_0)$.

The solutions are:

$$P^{00} = \frac{\mathcal{P}^{00}}{F_{r_0}} F \tag{4.77}$$

$$P^{01} = \mathcal{P}^{01} + \frac{3\mathcal{P}^{02}}{\ell F_{r_0}^{1/2}} r + \frac{4\mathcal{P}^{00}}{3\ell^3 F_{r_0}^{3/2}} r^2 \tag{4.78}$$

$$P^{02} = \left(\frac{F}{F_{r_0}}\right)^{1/2} \left(\mathcal{P}^{02} + \frac{2\mathcal{P}^{00}}{\ell F_{r_0}^{1/2}} r\right) \tag{4.79}$$

$$P^{11} = \left(\frac{F}{F_{r_0}}\right)^{-1} \left(\mathcal{P}^{11} + \frac{1}{\ell F_{r_0}^{1/2}} (\mathcal{P}^{12} + \frac{\mathcal{P}^{02}}{F_{r_0}} (c + m\Theta(v - v_0)))r + \frac{\mathcal{P}^{00}}{\ell^2 F_{r_0}^2} (c + m\Theta(v - v_0))r^2\right) + \left(\frac{F}{F_{r_0}}\right)^{-1} \left(\frac{5\mathcal{P}^{02}}{6\ell^3 F_{r_0}^{3/2}} r^3 + \frac{7\mathcal{P}^{00}}{\ell^4 F_{r_0}^2} r^4\right) \tag{4.80}$$

$$P^{12} = \left(\frac{F}{F_{r_0}}\right)^{-1/2} \left(\mathcal{P}^{12} - \frac{\mathcal{P}^{01}}{\ell F_{r_0}^{1/2}} r - \frac{3\mathcal{P}^{02}}{2\ell^2 F_{r_0}} r^2 - \frac{4\mathcal{P}^{00}}{3\ell^3 F_{r_0}^{3/2}}\right) \tag{4.81}$$

where the $\mathcal{P}$'s are integration constants. The connections $\bar{A}$ can be obtained following (4.65) and (4.66). As we mentioned, the situation for the connection A is more complicated as we would have to solve for U in (4.53). While some simplifications of the type we did for $\bar{A}$ are conceivable, finding a solution to this equation requires great computational power; for the moment, we leave this as an open problem. The major complications come from the number of unknowns we have to deal with. Indeed, treated as a matrix, U (or any of the fields) is a three by three matrix and so it has 9 elements. Treated as an element of the group written in the exponential map, the expression for U is even more complicated for computational purposes. In this case U is of the general form $e^{U^0 J_0 + U^1 J_1 + \cdots + U^1 1 T_{11} + U^{12} T_{12}}$, and even taking the exponential can prove computationally very demanding. A wiser approach than brute force at solving Eq. (4.53) and eventually Eq. (4.54) must be taken. In the next, final chapter we will further comment on this.

Chapter 5

Conclusions and perspectives

Inspired by the kind of setups that leads to the Vaidya metric, the simplest non-static generalization of the Schwarzschild solution, we have established a framework that allows us to look for dynamical higher-spin gravity solutions in three dimensions from a given set of equations of motion. These equations of motion were derived from an action corresponding to a particle carrying higher-spin charge coupled to gravity. The key feature of this action, first presented in [41] and further studied in [25], is that it is written in an explicit group-theoretical language, which allows for generalizations to higher-spin by means of an extension of the gauge group. In the specific spin-3 case that we treated in this thesis, that action is of the form

$$S[U, P; A, \bar{A}] = \int ds(\mathrm{Tr}(PU^{-1}D_sU) + \lambda_2(\mathrm{Tr}(P^2) - c_2) + \lambda_3(\mathrm{Tr}(P^3) - c_3)). \tag{5.1}$$

As we saw, the Langrange multipliers λ_2 and λ_3 serve to impose constraints on the mass and spin-3 charge of the particle, making them tunable parameters. When both c_2 and c_3 are zero, the particle follows ingoing geodesics, and we used this fact to form a collapsing shell composed of an infinite number of these particles. This exactly mirrors the Vaidya setup, and so one might expect that the connections obtained from our set of equations are actually the connections corresponding to the Vaidya metric. A comparison between the two metrics (the Vaidya metric and the one obtained from our connections), allowed us to see that this intuition is indeed realized, and so the claim follows that

$$A = \left(F^{1/2}\, d\varphi + \frac{1}{\ell}F^{1/2} dv\right) J_0 + \left(\frac{1}{\ell}F^{1/2}\right) J_1 \\ + \left[\frac{1}{2}\left(\frac{\partial F}{\partial r} - F^{-1}\frac{\partial F}{\partial v}\right) dv - \frac{1}{2}F^{-1}\frac{\partial F}{\partial r}dr + \frac{1}{\ell}r d\varphi\right] J_2, \tag{5.2}$$

$$\bar{A} = \left(F^{1/2}\, d\varphi - \frac{1}{\ell}F^{1/2} dv\right) J_0 - \left(\frac{1}{\ell}F^{1/2}\right) J_1 \\ + \left[\frac{1}{2}\left(\frac{\partial F}{\partial r} - F^{-1}\frac{\partial F}{\partial v}\right) dv - \frac{1}{2}F^{-1}\frac{\partial F}{\partial r}dr - \frac{1}{\ell}r d\varphi\right] J_2, \tag{5.3}$$

with $F(r,v) = \left(1 + \frac{r^2}{\ell^2} + m\Theta(v - v_0)\right)$, are the Vaidya connections (up to some integration constants).

However, the comparison between the two metrics is only available in the spin-2 scenario, since it is in that case that there is a direct map between the metric and the connections via the definitions of vielbein e and spin connection ω. For our purposes, the more interesting scenarios were those involving higher-spin, were such a map does not exist (at least not obviously). This means that one can no longer recur to the metric/gauge connection map to check whether or not there is dynamics. Nevertheless, there exists a natural way to ensure that one is constructing dynamical higher-spin solutions, namely by demanding that, if we turn off the higher-spin charges, they reduce to (5.2), (5.3), which are certainly dynamical. This is the kind of generalization we considered. We saw that, provided one picks the appropriate gauges, in this case some of the equations of motion are perfectly tractable. For example, we were able to find the momentum P of a solution carrying higher-spin charge (see Eqs. (4.77) to (4.81)). Moreover, integrating these equations over a delta function allows us to find straightforwardly the connections $\bar{A}$.

Notwithstanding, there is one equation of motion which poses more difficulties – that of U. As we commented, the equations for U should not be treated brutally, since they require great computational power. One possible way to overcome these difficulties would be to make the most of the gauge invariance of the action. Indeed, it is conceivable that, for some specific gauges, our equations could take a simple, tractable form. In fact, as we stressed above, this is precisely how we eventually solved for the momentum P. The question remains whether or not the same can be done for U. We will now explore some ideas and partial results in this direction.

A particular feature of our setup that could be used to simplify the equations is the fact that the shell is localized. As it turns out, this implies that the stress tensor is of the form

$T^{\mu\nu} \propto \frac{\delta(v-v_0)}{r}\delta^{r\mu}\delta^{r\nu}$, which is non-zero only along the shell. That the shell is localized is also reflected by the Dirac delta appearing in our equations:

$$\frac{k}{2\pi}\left((\partial_\mu A_\nu - \partial_\nu A_\mu) + [A_\mu, A_\nu]\right) = -\int ds \frac{dx^\rho}{ds}\epsilon_{\mu\nu\rho}\delta^{(3)}(x - x(x))UPU^{-1} \tag{5.4}$$

and

$$\frac{k}{2\pi}\left(\left(\partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu\right) + [\bar{A}_\mu, \bar{A}_\nu]\right) = -\int ds \frac{dx^\rho}{ds}\epsilon_{\mu\nu\rho}\delta^{(3)}(x - x(x))P. \tag{5.5}$$

Thus, if we consider the spacetime outside or inside the shell, we are in a region where the connections are flat. It might seem surprising that we even consider flat connections at all. After all, part of the point of introducing the shell was the need for non-flat connections. However, what we are contemplating here is not a mere stepping back. Indeed, it is conceivable that appropriate matching conditions would eventually allow us to connect the solutions in the flat regions so as to account for the shell[1]. Let us then see in what sense flat connections could simplify our task.

Looking back at (2.35), we see that flat connections are in general related to $A = \bar{A} = 0$ via

$$A = L\,dL^{-1}, \qquad \bar{A} = R^{-1}dR, \tag{5.6}$$

with L, R elements of the algebra in consideration. For U and P this reads:

$$U(s) = L(s)U_0R(s), \qquad P(s) = R^{-1}(s)P_0R(s), \tag{5.7}$$

where U_0 and P_0 are the solution of (4.53), (4.52) for $A = \bar{A} = 0$. To see how this simplifies everything, consider the equivalent of these equations in the spin-2 case, Eq. (3.33). Because in that case we could set $U = 1$[2], we never needed to worry about solving the equation for U. The fact is, however, that even without such a simplification we could found all the same have any solution corresponding to *flat* gauge connections. All that would have been needed is to use (5.7), where U_0 and P_0 are solutions of (3.33) for $A = \bar{A} = 0$, i.e

$$U_0(s) = u_0 e^{-2P_0\int ds\lambda(s)}, \qquad P_0 = \text{const}. \tag{5.8}$$

[1] This is actually a very standard practice when treating shells in general relativity. See e.g. [50] for details.

[2] See discussion at the end of Chapter 3.

where u_0 is a constant. This procedure of finding U and P for flat connections by relating them to their values when $A = \bar{A} = 0$ is what in [25] was called the 'nothingness trick'. By extending this trick to the spin-3 case, one could then hope to find flat connections satisfying the appropriate junction conditions along the shell, thus potentially allowing us to find the full, non-flat connections. To further develop these ideas will be our guiding goal in the immediate future.

Finally, it should be pointed out that a recent work by Perlmutter [12] casts some doubt on the ultimate physical consistency of the putative CFT duals of the $SL(N)$ theories at finite N we have discussed. The argument builds on the fact that they seem to violate a bound on chaos [48] that is intimately related to causality [49]. It would be nice if these claims could be explicitly tested using a higher-spin solution. Such a test would naturally require solutions that are dynamical. Our work could then provide a starting point in this direction.

Appendices

Appendix A

SL(3,R) conventions

Here we present our conventions for the algebra (2.72). We work in the following representation:

$$J_0 = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}, \qquad J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$
$$T_{00} = \begin{pmatrix} \frac{1}{3} & 0 & 2 \\ 0 & -\frac{2}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{3} \end{pmatrix}, \qquad T_{01} = \begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ -1 & 0 & -\frac{1}{3} \end{pmatrix}, \qquad T_{02} = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & -1 \\ 0 & \frac{1}{2} & 0 \end{pmatrix},$$
$$T_{11} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \qquad T_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \tag{A.1}$$

The quadratic Killing form is

$$\eta_{mn} = \frac{1}{2}\mathrm{tr}(T_m T_n) = \frac{1}{2} \begin{pmatrix} -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{8}{3} & -\frac{8}{3} & 0 & 4 & 0 \\ 0 & 0 & 0 & -\frac{8}{3} & \frac{2}{3} & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \end{pmatrix} \tag{A.2}$$

where the T_m denote collectively the generators $\{J_0, J_1, J_2, T_{00}, \ldots, T_{12}\}$.

The values of the Casimirs used in the main text are obtained in this representation through the definitions

$$\begin{aligned} c_2 \equiv \mathrm{Tr}(P^2) = & -(P^0)^2 + 2P^0P^1 + (P^2)^2 \\ & + \frac{1}{3}\Big(4(P^{00})^2 - 8P^{00}P^{01} + (P^{01})^2 - 3(P^{02})^2 + 12P^{00}P^{11} + 6P^{02}P^{12}\Big) \end{aligned}$$

and

$$\begin{aligned} \frac{3}{4}c_3 = & 8(P^{00})^3 + (P^{01})^3 + 9P^{01}(P^{02})^2 - 27(P^{02})^2P^{11} \\ & + 9(P^{00})^2(2P^{00} - 2P^{01} + 3P^{11}) + (P^{00})^2(-24P^{01} + 36P^{11}) + 9P^{01}P^{02}P^{12} \\ & + 27P^{02}P^1P^2 - 9P^{01}(P^2)^2 - 9P^0(4P^{00}P^1 - P^{01}P^1 + 3P^{02}P^2 - 3P^{12}P^2) \\ & + 3P^{00}(5(P^{01})^2 - 3(P^{02})^2 + 9(P^1)^2 - 12P^{01}P^{11} + 6P^{02}P^{12} - 9(P^{12})^2 + 3(P^2)^2) \end{aligned}$$

as can be explicitly checked.

Appendix B

The Vaidya spin connections

In this appendix we use (4.4) and (4.6), together with equation (4.5) to solve for the spin connections that enter in the gauge connection we build for the Vaidya metric. The results of the computation developed here are reported in Eq. (4.7) in the main text.

Let

$$\omega_{ab} = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix}, \tag{B.1}$$

where α, β and γ are one-forms to be determined, then

$$\omega^{a}{}_{b} = \eta^{ac}\omega_{cb} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} = \begin{pmatrix} -\alpha & 0 & \gamma \\ -\alpha & \alpha & \beta + \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} \tag{B.2}$$

Substituting this into (4.5) leads to:

$$
\begin{aligned}
\omega^0{}_b \wedge e^b =& \omega^0{}_0 \wedge e^0 + \omega^0{}_1 \wedge e^1 + \omega^0{}_2 \wedge e^2 = \frac{1}{2} F^{-1/2} \frac{\partial F}{\partial r} e^0 \wedge e^1 \\
&\Rightarrow \begin{cases} \omega^0{}_0 = -\alpha = -\alpha_0 e^0 + \underbrace{(-\frac{1}{2} F^{-1/2} \frac{\partial F}{\partial r})}_{-\alpha_1} e^1 - \cancelto{0}{\alpha_2} e^2 \\ \omega^0{}_2 = \gamma \propto e^2 \Rightarrow \gamma_0, \gamma_1 = 0 \end{cases}
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
\omega^1{}_b \wedge e^b =& \omega^1{}_0 \wedge e^0 + \omega^1{}_1 \wedge e^1 + \omega^1{}_2 \wedge e^2 = \frac{1}{2} F^{-3/2} \frac{\partial F}{\partial v} e^0 \wedge e^1 \\
&\Rightarrow \begin{cases} \alpha_1 \, e^0 \wedge e^1 + \alpha_0 \, e^0 \wedge e^1 = \frac{1}{2} F^{-3/2} \frac{\partial F}{\partial v} e^0 \wedge e^1 \Rightarrow \alpha_0 = \frac{1}{2} F^{-3/2} \frac{\partial F}{\partial v} - \frac{1}{2} F^{-1/2} \frac{\partial F}{\partial r} \\ \omega^1{}_2 = \beta + \gamma \underset{(B.3)}{=} \beta + \gamma_2 e^2 \propto e^2 \Rightarrow \beta_0, \beta_1 = 0 \end{cases}
\end{aligned}
\tag{B.4}
$$

$$
\begin{aligned}
\omega^2{}_b \wedge e^b =& \omega^2{}_0 \wedge e^0 + \omega^2{}_1 \wedge e^1 + \omega^2{}_2 \wedge e^2 = -\frac{F^{1/2}}{r} e^1 \wedge e^2 \\
&\Rightarrow \begin{cases} \omega^2{}_0 = -\beta \propto e^0 \Rightarrow \beta_1, \beta_2 = 0 \underset{(B.4)}{\Rightarrow} \beta = 0 \\ \omega^2{}_1 = -\gamma \underset{(B.3)}{=} -\gamma_2 e^2 = \frac{F^{1/2}}{r} e^2 \end{cases}
\end{aligned}
\tag{B.5}
$$

So we have

$$
\omega^a{}_b = \begin{pmatrix} -\alpha & 0 & \gamma \\ -\alpha & \alpha & \gamma \\ 0 & -\gamma & 0 \end{pmatrix} \Rightarrow \omega^{ab} = \begin{pmatrix} 0 & -\alpha & \gamma \\ \alpha & 0 & \gamma \\ -\gamma & -\gamma & 0 \end{pmatrix}, \tag{B.6}
$$

with

$$
\begin{aligned}
\alpha =& \left(\frac{1}{2} F^{-3/2} \frac{\partial F}{\partial v} - \frac{1}{2} F^{-1/2} \frac{\partial F}{\partial r} \right) e^0 + \frac{1}{2} F^{-1/2} \frac{\partial F}{\partial r} e^1, \qquad (B.7) \\
=& \frac{1}{2} \left(F^{-1} \frac{\partial F}{\partial v} - \frac{\partial F}{\partial r} \right) dv + \frac{1}{2} F^{-1} \frac{\partial F}{\partial r} dr \\
\gamma =& -\frac{F^{1/2}}{r} e^2 = -F^{1/2} \, d\varphi. \qquad (B.8)
\end{aligned}
$$

What enters in the Chern-Simons connections is $\omega^a \equiv \frac{1}{2} \epsilon^a{}_{bc} \omega^{bc}$, with $\epsilon^a{}_{bc} = \eta^{ae} \epsilon_{ebc}$. Using the expression for the inverse of (4.3) it is easy to see that

$$\epsilon^0{}_{01} = -\epsilon^0{}_{10} = \eta^{02}\epsilon_{201} = 0$$
$$\epsilon^0{}_{02} = -\epsilon^0{}_{20} = \eta^{01}\epsilon_{102} = -1$$
$$\epsilon^0{}_{12} = -\epsilon^0{}_{21} = \eta^{00}\epsilon_{012} = 0$$

$$\epsilon^1{}_{01} = -\epsilon^1{}_{10} = \eta^{12}\epsilon_{201} = 0$$
$$\epsilon^1{}_{02} = -\epsilon^1{}_{20} = \eta^{11}\epsilon_{102} = -1$$
$$\epsilon^1{}_{12} = -\epsilon^1{}_{21} = \eta^{10}\epsilon_{012} = 1$$

$$\epsilon^2{}_{01} = -\epsilon^2{}_{10} = \eta^{22}\epsilon_{201} = 1$$
$$\epsilon^2{}_{02} = -\epsilon^2{}_{20} = \eta^{21}\epsilon_{102} = 0$$
$$\epsilon^2{}_{12} = -\epsilon^2{}_{21} = \eta^{20}\epsilon_{012} = 0$$

With this we get

$$\begin{aligned} \omega^0 &= \frac{1}{2}\epsilon^0{}_{bc}\omega^{bc} = \epsilon^0{}_{02}\omega^{02} = -\gamma = F^{1/2}\,d\varphi \\ \omega^1 &= \frac{1}{2}\epsilon^1{}_{bc}\omega^{bc} = \epsilon^1{}_{02}\omega^{02} + \epsilon^1{}_{12}\omega^{12} = 0 \\ \omega^2 &= \frac{1}{2}\epsilon^2{}_{bc}\omega^{bc} = \epsilon^2{}_{01}\omega^{01} = -\alpha = -\frac{1}{2}\left(F^{-1}\frac{\partial F}{\partial v} - \frac{\partial F}{\partial r}\right)dv - \frac{1}{2}F^{-1}\frac{\partial F}{\partial r}dr \end{aligned} \tag{B.9}$$

Appendix C

Equations of motion for the Vaidya A connection

In this appendix we work out the explicit expressions of the left-hand sides of (4.22), (4.23) and (4.24).

Let us first differentiate the components (4.14) to (4.18):

$$\partial_r A_\varphi = \frac{1}{2} F^{-1/2} \frac{\partial F}{\partial r} J_0 + \frac{1}{\ell} J_2 \tag{C.1a}$$

$$\partial_\varphi A_r = 0 \tag{C.1b}$$

$$\partial_v A_r = -\frac{1}{2\ell} F^{-3/2} \frac{\partial F}{\partial v} J_1 + \frac{1}{2} F^{-2} \frac{\partial F}{\partial r} \frac{\partial F}{\partial v} J_2 \tag{C.1c}$$

$$\partial_r A_v = \frac{1}{2\ell} F^{-1/2} \frac{\partial F}{\partial r} J_0 + \frac{1}{2} \left(\frac{\partial^2 F}{\partial r^2} + F^{-2} \frac{\partial F}{\partial r} \frac{\partial F}{\partial v} \right) J_2 \tag{C.1d}$$

$$\partial_v A_\varphi = \frac{1}{2} F^{-1/2} \frac{\partial F}{\partial v} J_0 \tag{C.1e}$$

$$\partial_\varphi A_v = 0 \tag{C.1f}$$

For the commutators one finds:

$$
\begin{aligned}
[A_r, A_\varphi] &= \frac{1}{\ell}[J_1, J_0] + \frac{1}{\ell^2}F^{-1/2} r [J_1, J_2] - \frac{1}{2}F^{-1/2}\frac{\partial F}{\partial r}[J_2, J_0] \qquad \text{(C.2a)}\\
&= -\frac{1}{\ell}J_2 + \frac{1}{\ell^2}F^{-1/2} r J_1 - \frac{1}{2}F^{-1/2}\frac{\partial F}{\partial r}(J_0 + J_1),\\
[A_v, A_r] &= \frac{1}{\ell^2}[J_0, J_1] - \frac{1}{2\ell}F^{-1/2}\frac{\partial F}{\partial r}[J_0, J_2] + \frac{1}{2\ell}F^{-1/2}\left(\frac{\partial F}{\partial r} - F^{-1}\frac{\partial F}{\partial v}\right)[J_2, J_1] \qquad \text{(C.2b)}\\
&= \frac{1}{\ell^2}J_2 + \frac{1}{2\ell}F^{-1/2}\frac{\partial F}{\partial r}(J_0 + J_1) - \frac{1}{2\ell}F^{-1/2}\left(\frac{\partial F}{\partial r} - F^{-1}\frac{\partial F}{\partial v}\right)J_1\\
[A_v, A_\varphi] &= \frac{1}{\ell^2}F^{1/2} r [J_0, J_2] + \frac{1}{2}F^{1/2}\left(\frac{\partial F}{\partial r} - F^{-1}\frac{\partial F}{\partial v}\right)[J_2, J_0] \qquad \text{(C.2c)}\\
&= \frac{1}{\ell^2}F^{1/2} r (J_0 + J_1) + \frac{1}{2}F^{1/2}\left(\frac{\partial F}{\partial r} - F^{-1}\frac{\partial F}{\partial v}\right)(J_0 + J_1)\\
&= -\frac{1}{2}F^{-1/2}\frac{\partial F}{\partial v}(J_0 + J_1),
\end{aligned}
$$

where Eq. (2.30) has been used to compute the commutators of the corresponding generators:

$$
\begin{aligned}
[J_0, J_1] &= -[J_1, J_0] = \epsilon_{012}\left(\eta^{20}J_0 + \eta^{21}J_1 + \eta^{22}J_2\right) = J_2\\
[J_0, J_2] &= -[J_2, J_0] = \epsilon_{021}\left(\eta^{10}J_0 + \eta^{11}J_1 + \eta^{12}J_2\right) = -(J_0 + J_1) \qquad \text{(C.3)}\\
[J_1, J_2] &= -[J_2, J_1] = \epsilon_{120}\left(\eta^{00}J_0 + \eta^{01}J_1 + \eta^{02}J_2\right) = J_1,
\end{aligned}
$$

with $\epsilon_{012} \equiv 1$.

Combining (C.1) with (C.2) leads to:

$$
\begin{aligned}
(\partial_r A_\varphi - \partial_r A_\varphi) + [A_r, A_\varphi] &= \left(\frac{1}{2}F^{-1/2}\frac{\partial F}{\partial r} - \frac{1}{2}F^{-1/2}\frac{\partial F}{\partial r}\right) J_0 \\
&+ \left(\frac{1}{\ell^2}F^{-1/2}r - \frac{1}{2}F^{-1/2}\frac{\partial F}{\partial r}\right) J_1 + \left(\frac{1}{\ell} - \frac{1}{\ell}\right) J_2 \\
&= F^{-1/2}\left(\frac{1}{\ell^2}r - \frac{1}{2}\frac{\partial F}{\partial r}\right) J_1
\end{aligned} \tag{C.4}
$$

$$
\begin{aligned}
(\partial_v A_r - \partial_r A_v) + [A_v, A_r] &= \left(-\frac{1}{2\ell}F^{-1/2}\frac{\partial F}{\partial r} + \frac{1}{2\ell}F^{-1/2}\frac{\partial F}{\partial r}\right) J_0 \\
&+ \left(-\frac{1}{2\ell}F^{-3/2}\frac{\partial F}{\partial v} + \frac{1}{2\ell}F^{-1/2}\frac{\partial F}{\partial r} - \frac{1}{2\ell}F^{-1/2}\frac{\partial F}{\partial r} + \frac{1}{2\ell}F^{-3/2}\frac{\partial F}{\partial v}\right) J_1 \\
&+ \left(\frac{1}{\ell^2} - \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\right) J_2 \\
&= \left(\frac{1}{\ell^2} - \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\right) J_2
\end{aligned} \tag{C.5}
$$

$$
\begin{aligned}
(\partial_v A_\varphi - \partial_\varphi A_v) + [A_v, A_\varphi] &= \left(-\frac{1}{\ell^2}F^{1/2}r + \frac{1}{2}F^{1/2}\frac{\partial F}{\partial r} - \frac{1}{2}F^{-1/2}\frac{\partial F}{\partial v} + \frac{1}{2}F^{-1/2}\frac{\partial F}{\partial v}\right) J_0 \\
&+ \left(-\frac{1}{\ell^2}F^{1/2}r + \frac{1}{2}F^{1/2}\frac{\partial F}{\partial r} - \frac{1}{2}F^{-1/2}\frac{\partial F}{\partial v}\right) J_1
\end{aligned} \tag{C.6}
$$

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