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PhD thesis

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Holographic Three point Functions

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`We are like dwarves standing on the shoulders of giants, and so able to see more and see farther than the ancients. (Bernard of Chartres)`

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To Filippo and Tommaso
To my grandfather Natalino

Introduction

$\mathcal{N}$=4 Super Yang Mills is an interacting four dimensional non abelian gauge theory with a large amount of symmetries. The fact that it is a Yang Mills theory puts it in contact with the well known particle physics theories as Quantum ChromoDynamics or the Standard Model itself. On the other hand the rich symmetry structure, which is peculiar of this theory, simplifies dramatically the computations giving the hope to eventually exactly solve the theory. Beyond Poincaré invariance, $\mathcal{N}$=4 SYM has two substantial symmetries, namely conformal symmetry and supersymmetry.

Indeed $\mathcal{N}$=4 SYM possesses the maximal amount of supersymmetries in four dimensions. The combination of conformal symmetry and supersymmetry makes this theory the perfect candidate to interplay with gravity theories. The best example of this connection is the AdS/CFT correspondence [1–3] which relates a string theory defined on a specific space-time to a quantum field theory with conformal invariance (CFT), without gravity, defined on the boundary of this space-time. The first explicit formulation of the correspondence [1] states the duality between type IIB string theory defined on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM in four dimensions with gauge group $SU(N)$ [1]. The parameters of the theories are on one side the gauge coupling g_{CFT} and the rank of the gauge group N and on the other side the radius of the AdS_5 and the S^5 spaces and the string coupling constant g_s. They are related through the 't Hooft coupling constant λ in the following way

$$\lambda = g_{CFT}^2 N = 4\pi N g_s \quad \sqrt{\lambda} = \frac{R^2}{\alpha'}. \tag{1}$$

In fact this is a weak-strong duality, since the parameter λ is such that in the region where it is very small, one description (the CFT) is weakly coupled and the

[1]Another form of the correspondence which has been widely studied is the AdS_4/CFT_3 correspondence, which is a duality between type IIA string theory on $AdS_4 \times \mathbb{C}P^3$ and a $\mathcal{N} = 6$ superconformal Chern-Simons theory in three dimensions [4]

other (AdS) is strongly coupled, while the opposite is true when this parameter is large. Obviously a consequence of this statement is that it is very hard to check the validity of the correspondence but makes it more reasonable since the behaviour of the two theories is very different in the same λ regime.

One of the most successful attempts to make the correspondence operative and explicit has been to provide a map between observables computed in both theories [2, 3]. The first insight is that for each field in the five dimensional bulk there is a corresponding operator in the boundary field theory. This identification is not trivial at all in general cases but, exploiting the symmetries of the two theories for some specific cases, this becomes possible. For instance the graviton is associated to the stress energy tensor operator in the boundary. More interestingly it is possible to relate the gravity partition function Z, subject to appropriate boundary conditions, to the generating functional of the connected correlation functions in the CFT side, more precisely

$$Z\left[\phi\left(\vec{x},z\right)\Big|_{z=0}=\phi_0\left(\vec{x}\right)\right]=\langle e^{\int d^4x\phi_0(\vec{x})\mathcal{O}(x)}\rangle. \quad (2)$$

Thus to obtain an n- point correlation function one has to functionally differentiate n-times with respect to ϕ_0 the relation above and set ϕ_0 to zero. From this expression it is easy to see that operators of the gauge theory are mapped to on shell bulk fields on the gravity side. Eq. (2) becomes very useful when the gravity theory is weakly coupled since we can approximate the gravity partition function by the value of the classical action. This prescription has led to the computation of several examples of two and three point correlation functions for different kinds of operators using the assumption that (2) holds at the level of the supergravity approximation, namely for λ large the string theory becomes weakly coupled and the AdS curvature is so small that the supergravity approximation to string theory is trustable. If one would like to compute correlation functions of massive string modes, thus moving away from the supergravity approximation, this method is not suitable due to the lack of knowledge about the construction of the vertex operators for the full string theory in $AdS_5 \times S_5$.

Conformal symmetry, which survives also at the quantum level, strictly constrains the spatial dependence of the sets of two and three point correlation functions of local operators $\mathcal{O}$ to have the form

$$\langle \mathcal{O}_A(x_1)\,\mathcal{O}_B(x_2)\rangle = (\mathcal{N}_A\mathcal{N}_B)^{\frac{1}{2}}\,\frac{\delta_{AB}}{|x_1 - x_2|^{2\Delta_A}} \tag{3}$$

and

$$\langle \mathcal{O}_A(x_1)\,\mathcal{O}_B(x_2)\,\mathcal{O}_C(x_3)\rangle = \frac{(\mathcal{N}_A\mathcal{N}_B\mathcal{N}_C)^{\frac{1}{2}}\,c_{ABC}}{|x_1 - x_2|^{\Delta_A+\Delta_B-\Delta_C}|x_1 - x_3|^{\Delta_A+\Delta_C-\Delta_B}|x_2 - x_3|^{\Delta_B+\Delta_C-\Delta_A}} \tag{4}$$

where $\mathcal{N}_i$ with $i = A, B, C$ are normalization constants, Δ_i are the conformal dimensions of the operators and c_{ABC} are the structure constants. Moreover the dynamical information of the theory is all encoded in the conformal dimensions and in the structure constants of local operators. This is because in the operator product expansion for conformal field theories the coefficients of the expansion coincide with the structure constants. Thus, in principle, correlation functions of any number of primary operators can be constructed from the knowledge of the conformal dimensions and the structure constant coefficients.

The conformal dimensions of the operators, as all the physical quantities, admit an expansion in the 't Hooft coupling constant

$$\Delta(\lambda) = \Delta_0 + \lambda\,\Delta_1 + \lambda^2\,\Delta_2 + \dots \tag{5}$$

where Δ_0 is the classical dimension while the other terms are labeled by the superscript which denotes the loop order. From (3) it seems that in order to obtain the conformal dimensions of local operators one needs to know the two point functions of the corresponding operators, which can in principle be done order by order in perturbation theory in the sense of (5). Actually the approach used to compute the anomalous dimensions is different, in fact they are considered as eigenvalues of the dilatation operator, which is one of the generators of the conformal group. In this way, one has to diagonalise the dilatation operator and take into account the quantum correction to all order in λ and the mixing effects which can arise when considering quantum correction. A breakthrough has been the identification of the one loop dilatation operator with the Hamiltonian of an integrable spin chain [5]. More precisely it has been found that the one loop planar dilatation generator for the $SO(6)$ sector of scalar fields is isomorphic to the Hamiltonian of an integrable spin chain. The planar limit means that we take $N \to \infty$, this is reflected in the fact that the dilatation generator does not mix fields in different traces, thus we can just consider single trace operators. The dilatation generator acts on local operators and its eigenvalues are the anomalous dimensions.

When restricting to the $SU(2)$ sector we can consider only four scalar fields, out of the six present in $\mathcal{N}=4$ SYM, which can be grouped into two complex scalars Z and X. Thus gauge invariant operators with a definite one loop anomalous dimension are written as a linear combination of single trace operators involving Z and X. In the $SU(2)$ subsector the one loop dilatation operator can be seen as the $XXX_{\frac{1}{2}}$ Heisenberg Hamiltonian, thus the picture is more intuitive and one can think of Z being a spin up while X as spin down. The fact that the model is integrable is highly non trivial and allows to implement all the techniques used to exactly solve integrable models. In the last ten years this program has been extended to the full $PSU(2,2|4)$ group, which us the full symmetry group of $\mathcal{N}=4$ SYM, and at any loop order, thus in principle we have all the information to compute all the set of the conformal dimensions of the operators. For a recent review on these developments see [6].

Moving to three point function is not straightforward and it is still an open problem. To simplify the analysis let us roughly classify the operators in three groups: heavy, medium and light. Heavy operators are dual to classical string states with large angular momenta and the energies scale as $\lambda^{\frac{1}{2}}$, they have a non zero anomalous dimension. They can be represented as single trace operators involving a large number of fields with a large number of excitations [2]. Medium operators are the lightest massive states (short string states), they also have non trivial anomalous dimension but their energy scales as $\lambda^{\frac{1}{4}}$. They are single trace operators where the number of fields and the number of excitations is of order 1. Finally light operators are BPS operators, thus with vanishing anomalous dimensions at any loop order. They are dual to supergravity modes. Note that they are expressed by single traces operators where the excitations have zero momentum.

Three point functions of light operators have been deeply studied both from the gauge theory side with free field theory techniques and in the string side in the supergravity approximation with the methods explained above. These correlation functions do not acquire any quantum corrections and indeed the computations on both sides of the correspondence have found to agree. Going beyond this class of three point functions is quite involved. On the string theory side the problem is the

[2]Note that heavy operators can also be BPS in same cases.

identification of the appropriate string vertex operator corresponding to a given gauge invariant operator in the boundary theory. In fact even the computation of two point functions for non BPS states was not clear since [7–9]. On the gauge theory side in principle the problem can be posed, we need to take into account all the possible Wick contractions among the operators. It is obvious that the combinatorial problem arising is involved, even at tree level. In fact, differently form the case of two point functions, one needs to know the full wave function.

However in the last three years, a lot of progress has achieved in computing three point structure constants. In [10, 11] the case of three point functions of two heavy and one light operators has been studied. The idea is to insert the light and BPS operator as a perturbation into the two point function of the heavy operators, thus ignoring the back reaction. Using this method many examples involving specific examples for heavy operators and have been considered [12–33]. Quite remarkably in the same regime it is also possible to compare the string results with the gauge theory ones. One example is when the three operators are BPS, thus the results from the string and gauge theory side are expected to agree [34–36].

Another comparison can be made in the so called Frolov-Tseytlin limit [37, 38]. This limit was already used to study two point functions of non-BPS operators and it was crucial to establish comparisons between gauge and string theory computations. In the case of a string moving on S^5 with angular momentum J, the energy of the string is expanded in a limit of large J around a BPS solution with the expansion parameter $\lambda' = \frac{\lambda}{J^2}$. This expansion can be compared with the loop expansion in the gauge side. In this regime it is possible to directly compare the sigma-model action for semiclassical operators on the gauge theory side to the classical sigma model action on the string theory[3]. This approach has been extended to the case of heavy-heavy-light three point functions [21]. Also in this case an agreement between the gauge and string theory computation for planar non extremal tree level three point functions, properly normalised, for operators belonging to the $SU(3)$ sector of $\mathcal{N} = 4$ SYM has been found. The same matching has been observed for tree level three point functions of operators in the $SU(2) \times SU(2)$ sector of ABJM [39].

It is found that also the structure constants can be written in a λ expansion, namely

$$c_{123} = c_{123}^{(0)} + \lambda\, c_{123}^{(1)} + \lambda^2\, c_{123}^{(2)} + \dots \tag{6}$$

[3]This agreement breaks down at three loops, for more details see Ch. 1

An interesting question is how to compute the one loop correction c^1_{123}. This has been done in [40] in the same regime as in [21]. In this case the matching between the gauge theory and the string theory computations has not been found at order λ. This might be due to subtleties on the gauge theory computation or to corrections to the coherent states, that describe the heavy operators. Until now it is not fully clear if they contribute at this order in λ and possibly how to include them.

Another matching has been found for the case of three BMN operators with two impurities, comparing a perturbative computation in the gauge theory side with the string result obtained using the 3-string vertex matrix elements [41]. The matching still holds at one loop level [42]. In [43] the strong coupling description of euclidean BMN strings as saddle points of the path integral for three-point correlators has been considered. In [44] the case of operators dual to short string states is studied. The last two are examples of medium medium medium correlators.

The problem of three point functions of heavy operators from the string theory perspective has been addressed in a series of recent works, [45–48]. In [45, 46] the authors obtained the full solution for the three point functions of the large spin limit of the Gubser-Klebanov-Polyakov strings in AdS_3 spacetime. In [47] it is computed the AdS contribution to the three point functions of heavy operators which are dual to strings moving only in the S^5 and in [48] the same class of three point functions has been studied also providing some examples for the S^5 contribution.

A natural question that can be asked is if integrability, which has played an essential role in solving the spectral problem, can help also in the computation of structure constants. Indeed the answer is positive. The idea of translating the computation of three point functions into a spin chain language goes back to [49–51]. More recently in [52] the computation of planar, non extremal and tree level structure constants of operators belonging to the $SU(2)$ sector of $\mathcal{N} = 4$ SYM have been recast into the computation of a series of scalar products, which can be written in terms of the solutions of the Bethe equations (rapidities) of the three operators. This technique strongly relies on the usage of integrability techniques to map the operators to spin chains and to rewrite the Wick contraction operation in terms of spin chain quantities [4]. Remarkably these structure constants can also

[4]Using a spin chain approach also three point functions of two light and one heavy operators have computed [53].

be expressed in terms of quantities coming from the so called six vertex model, which can be written as determinants [54] [5]. In [56] the functional formalism has been used to factorize the scalar product of two Bethe states. With this method it has been possible to take two important limits: a classical limit, when the three operators can be considered as classical, and a BPS limit which reproduces the known results for three BPS operators, obtained by sending all the rapidities to infinity. This procedures can be extended to compute also the one loop contribution to the same class of three point functions [57]. Another direction is to extend this methods to broader sectors than $SU(2)$. This has been done for the $SU(3)$ case in [58]. In [59] the generalization for the scalar products to operators in the $SO(6)$ sector has been discussed. In [60] the case of operators belonging to the $SU(3)$ sector has been considered, using the connection with the six vertex model. Another generalisation of the method proposed in [54] has been considered in [39] for the case of operators in the $SU(2) \times SU(2)$ sector of the ABJM theory.

The structure of this thesis is the following

- Chapter 1 contains the introductory material. In particularly in 1.1 the AdS_5/CFT_4 correspondence and the holographic method for computing correlation functions are discussed. Then we the structure of two and three point functions of fields in AdS and operators in $\mathcal{N} = 4$ SYM. Sec. 1.2 is devoted to the spectral problem and to the Algebraic Bethe ansatz techniques, which are used to find eigenvectors and eigenvalues of the spin chain Hamiltonian. This is followed by a review of the six vertex model techniques, especially focused on how to rewrite the building block of the Algebraic Bethe Ansatz. Section 1.3 contains a review of a particular regime, the Frolov-Tseytlin limit. Finally in Sec. 1.4 the main features of the AdS_4/CFT_3 correspondence are reported, with remarks about the integrability in the $SU(2) \times SU(2)$ sector of the ABJM theory, also in its six vertex model counterpart.

- Chapter 2 is devoted to discuss the three point functions of two heavy operators and one light. Three different examples will be discussed. The first one is based on [34]. In Sec. 2.3 it is presented the comparison between two computations. On the string theory side it has been computed three point functions of two giant gravitons, moving in an $S^3 \subset S^5$ and $S^3 \subset AdS_5$ respectively, and one point like graviton and on the gauge side a three point

[5] The same method has been applied for operators composed of self dual components of the field strength tensor in planar QCD [55].

function of two Schur polynomials and a single trace chiral primary. The second example, in section 2.4 is the one loop correction to the three point function of two semiclassical operators and one light chiral primary operator in the $SU(2)$ sector of $\mathcal{N} = 4$ SYM [40]. On the gauge theory side it is used a coherent state approach for the heavy operators and it is compared this to the corresponding correction on the string theory side using semiclassical computations. Finally in Sec. 2.5 it is discussed the matching between a gauge theory and a string theory computation for the planar, tree level and non extremal three point functions of operators belonging to the $SU(2) \times SU(2)$ sector of ABJM [39].

- In chapter 3 integrability inspired techniques to compute three point functions in the gauge theory side are discussed. Firstly in Sec 3.1 the methods of [52] and [54] are reviewed. In section 3.2 the case of the generalisation of the scalar product to the $SO(6)$ sector is discussed [59]. Then in Sec. 3.3 it is presented the computation of three point functions for operators in the $SU(2) \times SU(2)$ sector of ABJM in a determinant form, obtaining an expression which is, up to a normalisation factor, the product of two $SU(2)$ three point functions [39].

- In 3.3 there are some conclusions and open problems.

- Appendix A contains the proof that the restricted scalar product in the $SU(2) \times SU(2)$ sector of ABJM , the one which factorizes, is still a scalar product meaning that it satisfies all the characterizing properties.

- Appendix B contains a brief review of the Jack polynomial basis, which is a basis of the symmetric group. The Schur polynomials are particular cases of the Jack functions.

- Appendix C provides the details of the $SU(2) \times SU(2)$ sigma model, which is necessary for the computation of the three point functions in the case when two operators are heavy and one is light in the Frolov-Tseytlin limit.

Chapter 1

Introductory material

Contents

1.1 The $\mathbf{AdS_5/CFT_4}$ correspondence

The AdS/CFT correspondence: an intuitive formulation The AdS/CFT correspondence is an equality between a theory of gravity (bulk theory) defined on a d dimensional space-time and a quantum field theory (boundary theory) living on the $d-1$ dimensional boundary of this space-time. In its initial formulation [1] it conjectures the equivalence between type IIB superstring theory defined on $AdS_5 \times S^5$ and four dimensional $\mathcal{N}$=4 super Yang-Mills theory (SYM) with gauge group $SU(N)$. Even if these two theories are completely different they are both peculiar. The latter contains the maximum amount of supercharges in four dimensions which completely constrains its lagrangian. The field content of $\mathcal{N}$=4 SYM is given by a gluon field, six scalars and four fermions (gluinos) transforming under the adjoint representation of the gauge group. It is Lorentz invariant and, less obviously, it is conformal and possesses an R-symmetry which rotates the scalar fields (and the fermions) among themselves, hence the bosonic symmetry group is $SO(2,4) \times SO(6)$. On the other side Anti de Sitter spaces in d dimensions AdS_d are maximally symmetric solutions of Einstein's equations with negative cosmological constant which have isometry group $SO(2,d-1)$. Therefore AdS_5 allocates an $SO(2,4)$ isometry group while S^5 is invariant under $SO(6)$ rotations. The fact that the full space is ten dimensional is clearly not accidental being the only dimensionality in which superstring theories are consistent. The global symmetries of the field theory precisely match the isometries group of the gravity theory and also the amount of supersymmetries of the two theories is the same. Even if this far to be a proof for the AdS/CFT correspondence it gives an hope that these two theories can describe the same physical system.

Parameters In its strongest formulation the AdS_5/CFT_4 correspondence relates at any value of the couplings ten dimensional type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ with coupling constant g_s and tension T, where the 5-form flux generated by the massless closed string modes through S^5 is an integer $N = \int_{S^5} F_5$, and $\mathcal{N}$=4 SYM with N colors and coupling constant g_{CFT}F by means of two relations involving the parameters of the theories

$$4\pi g_s = g_{CFT}^2 \qquad\qquad T = \frac{1}{2\pi}\sqrt{g_{CFT}^2 N} = \frac{1}{2\pi}\sqrt{\lambda} \tag{1.1}$$

where the tension $T = \frac{R^2}{2\pi\alpha'}$, R is the radius of both the AdS_5 and S^5 spaces and $\lambda = g_{CFT}^2 N$ is the 't Hooft coupling constant. Despite the fact that this is the most intriguing version of the correspondence because it is supposed to hold for any value of N and λ, it is very hard to make it concrete essentially because it is not known how to fully control string theory for generic values of the coupling. However there are some (very interesting) limits in which the correspondence is less general but easier implemented. One of those is the so called 't Hooft limit which consists in keeping λ fixed while sending $N \to \infty$. In this regime the Yang Mills theory is controlled only by planar diagrams which have the topology of a sphere, while the string theory is effectively a free theory. However the background in which the string theory lives is not trivial, in particular if λ is large the $AdS_5 \times S^5$ is weakly curved and the equations of motion reduce to the one of type IIB supergravity which are well understood and the dual gauge theory is strongly coupled. On the other hand when λ is small, SYM becomes weakly coupled and so can be treated perturbatively while the geometry becomes strongly curved. Still the string theory is free but the quantum corrections, controlled by the tension, are important. This is why the AdS/CFT correspondence is a weak-strong duality. Moreover the correspondence identifies the energy eigenstates of the $AdS_5 \times S^5$ string theory with gauge theory operators made of elementary fields of $\mathcal{N}$=4 SYM as well as the eigenvalue E, energy of the string state, equates the scaling dimension Δ of the specific dual operator in the gauge theory. This statement can be formulated in terms of the parameters of the theories in this way

$$E(\lambda, N) = \Delta(\lambda, N) \tag{1.2}$$

where (1.1) has to be kept in mind.

1.1.1 Holographic prescription for correlation functions

In any quantum field theory correlation functions of fields are the natural objects to study because they contain a big piece of information about the dynamics of the theory that we are considering. One way of obtaining correlation functions is by considering

$$\langle O(x_1)O(x_2)\dots O(x_n)\rangle = \frac{\delta}{\delta J(x_1)}\dots\frac{\delta}{\delta J(x_n)}Z[J]|_{J=0} \tag{1.3}$$

where $Z[J]$ is generating functional defined, in the euclidean signature, as

$$Z[J] = \frac{\int \mathcal{D}\, O e^{-S+\int d^dx O(x)J(x)}}{\int \mathcal{D} O e^{-S}}. \tag{1.4}$$

Here $\mathcal{D}\, O$ is the measure of the path integral and $O(x)$ is a field which is dynamical, differently to the source field J. In this formalism taking functional derivatives of $Z[J]$ with respect to the sources J and setting the source to gives the vacuum expectation values of the time ordered products of operators, which is precisely the definition of correlation functions in quantum field theories. In analogy to the definition of the free energy in statistical mechanics we can introduce another functional $W[J]$ as

$$W[J] = \log Z[J] \tag{1.5}$$

which is the generating functional of connected correlation functions. It is also possible to define correlation functions for operators, this can be done by adding additional sources in the Lagrangian of the theory. In this way the coupling constant depends on the spacetime and it becomes a source as well.

In 1998 Gubser, Klebanov and Polyakov [2] and Witten [3] proposed a prescription to compute n-point correlation functions in the context of the AdS/CFT correspondence which makes the correspondence itself much more operating and concrete because allows the comparison between correlation functions computed both in the gauge and string theory side. The general idea is to match the generating functional of the gauge theory to the full partition function of type IIB string theory in $AdS_5 \times S^5$ background with some specific boundary conditions, more formally

$$\langle e^{\int d^4x \phi_0(\mathbf{x})\mathcal{O}(\mathbf{x})}\rangle_{\texttt{CFT}} = \mathcal{Z}_{\texttt{string}}\left[\phi(\mathbf{x},z)|_{z=0} = \phi_0(\mathbf{x})\right] \tag{1.6}$$

where the l.h.s. is meant to be the same object that we defined in (1.4), $\mathbf{x}$ are the bulk coordinates and $z = 0$ is the location of the boundary [1]. This last statement is not obvious: the l.h.s. is the generating functional of the correlation functions for $\mathcal{O}(\mathbf{x})$ where, comparing with (1.4), $\phi_0(\mathbf{x})$ acts as a source (J), hence it is equivalent to functionally differentiate the l.h.s. and the r.h.s. of (1.6) with respect to ϕ_0. Note that the source lives on the boundary. This is the essence

[1]We are considering Poincaré coordinates for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ space namely $ds^2 = R^2\frac{dz^2+dx_\mu^2}{z^2} + R^2 d\Omega_5^2$, with $\mu = 1,\ldots,4$ and identifying x_μ with $\mathbf{x}$ which parametrizes the four dimensional boundary of AdS_5, the sphere part has zero size on the boundary.

of the holographic prescription: we can obtain correlation functions for operators in the gauge theory living on the boundary from the partition function of the superstring in the bulk. Therefore since the dynamics of quantum field theories is specified by correlation functions, the gauge theory living on the boundary is completely specified by the bulk gravitational theory. Actually there are limits in applying (1.6). This also stresses the fact that there should be an operator $\mathcal{O}_i$ for eve field ϕ_i in AdS. The first one is that the full partition function of type IIB superstring theory on $AdS_5 \times S^5$ is not known, secondly while taking functional derivatives of l.h.s. (r.h.s.) (1.6) one runs into UV (IR) divergences which need to be regularized [61]. The latter problem is solved by performing the so called holographic renormalization [62], we are not going to face with this in this thesis, for more details see the review [63] and references therein. The former is still an open problem, however in the limit where we do not have quantum corrections (as we mention above $g_s, l_s \to 0$ with $g_s N \geq 1$) the supergravity approximation can be used and the partition function on the r.h.s. of (1.6) in this limit takes the form of

$$\mathcal{Z}_{\texttt{string}}\left[\phi_0(\mathbf{x})\right] = \exp\left[-S_{\texttt{sugra}}\left[\phi_0(\mathbf{x})\right]\right] \tag{1.7}$$

and this equality has to be seen as a saddle point approximation where the $S_{\texttt{sugra}}$ is the on shell classical action. Also in this case one expects to encounter divergences but it is possible to use the holographic renormalization as well to handle with them.

1.1.2 The AdS side

AdS space AdS_{d+1} spaces are solutions of the Einstein's equations in $d+1$ dimensions with negative cosmological constant Λ

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R} = \Lambda g_{\mu\nu} \tag{1.8}$$

with $\Lambda = -\frac{d(d-1)}{R^2}$. We also have that

$$\mathcal{R} = -\frac{2d\,(d+1)}{R^2} \tag{1.9}$$

which implies that the Ricci tensor is proportional to the metric $\mathcal{R}_{\mu\nu} = -\frac{2d}{R^2} g_{\mu\nu}$ and therefore AdS spaces are Einstein spaces. In general in Minkowski space solutions of the Einstein's equation with constant can be realized as the set of solutions of a quadratic equation in a $d+2$ dimensional flat space, in fact if we consider a flat space time with the line element

$$ds^2 = -du^2 - dv^2 + dx_1^2 + dx_2^2 + \cdots + dx_d^2 \tag{1.10}$$

AdS_{d+1} spaces are hyperboloids defined by

$$-R^2 = -u^2 - v^2 + x_1^2 + x_2^2 + \cdots + x_d^2. \tag{1.11}$$

From (1.10) and (1.11) it is clear that the AdS_{d+1} has the isometry group $SO(2,d)$. Let us define two sets of coordinates that we will use in the following: global and Poincarè coordinates.

- Global coordinates are defined as

$$\begin{aligned} u &= R\cosh\rho\cos\tau \qquad (1.12)\\ v &= R\cosh\rho\sin\tau \\ x_i &= R\,\eta_i\sinh\rho \end{aligned}$$

  here $\sum_{i=1}^{d}\eta_i^2 = 1$ parametrize the unit S^{d-1} sphere, $\rho \in \mathbb{R}^+$ and $\tau \in [0, 2\pi]$. The associated line element is

$$ds^2 = R^2\left(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_{d-1}^2\right). \tag{1.13}$$

  This set of coordinates covers the Minkowskian hyperboloid exactly once, hence the name global. Since the time coordinate τ is periodic, the AdS space is not simply connected because there are timelike curves. This can be avoided by allowing $\tau \in \mathbb{R}$. Time translations and rotations on the sphere, $SO(2)\times SO(d-1)$ are manifest in this set of coordinates and constitute a compact subgroup of the full isometry group $SO(2,d)$.

- Poincaré coordinates are defined as

$$\begin{aligned} u &= \frac{1}{2}\left(\frac{1}{z} + z\left(R^2 + \vec{y}^2 - t^2\right)\right) \qquad (1.14)\\ v &= R\,z\,t \\ x_i &= R\,z\,y^i \\ x_n &= \frac{1}{2}\left(\frac{1}{z} + z\left(-R^2 + \vec{y}^2 - t^2\right)\right) \qquad (1.15) \end{aligned}$$

where $i = 1, \ldots, d-1$, $t \in \mathbb{R}$, $z > 0$ and $\vec{y} \in \mathbb{R}^{d-1}$. These coordinates cover only a part of the hyperboloid and the metric contains slices isomorphic to the Minkowskian space time. The line element can be written as

$$ds^2 = R^2 \left(z^2 \left(-dt^2 + d\vec{y}^2 \right) + \frac{dz^2}{z^2} \right). \tag{1.16}$$

While $z = \infty$ can be interpreted as a conformal boundary, $z = 0$ is a coordinate singularity where $\frac{\partial}{\partial t}$ becomes null. In this coordinate the explicit isometry group is $SO(1, d-1) \times SO(1, 1)$. Note that if we allow $z \to 1/z$ the metric becomes

$$ds^2 = \frac{R^2}{z^2} \left(-dt^2 + d\vec{y}^2 + dz^2 \right). \tag{1.17}$$

Fields in AdS In the following we want to compute two and three point functions of operators dual to scalar fields in AdS, in the limit where the supergravity approximation is trustable. Let us start by considering the Klein Gordon Lagrangian for a scalar field ϕ in AdS_5:

$$S = \frac{1}{2} \int d^{d+1}x \sqrt{g} \left(\partial^\mu \phi \partial_\mu \phi + m^2 \phi^2 \right) \tag{1.18}$$

$$= \frac{1}{2} \int d^d x dz z^{1-d} \left[(\partial_z \phi)^2 + (\partial_i \phi)^2 + \frac{m^2}{z^2} \phi^2 \right] \tag{1.19}$$

where to go from the first to the second line we insert the explicit form of the AdS metric in Poincaré coordinates (with R=1) $ds^2 = \frac{dz^2 + dx^i dx^i}{z^2}$. The equation of motion associated to this action are

$$- z^{d-1} \partial z \left(z^{1-d} \partial_z \phi \right) - \partial_i^2 \phi + \frac{m^2}{z^2} \phi = 0. \tag{1.20}$$

In order to specify non trivial boundary conditions for this scalar field we need to exploit the limit $z \to 0$ which allows to neglect the term $\partial_i^2 \phi$ in (1.20) [2] obtaining

$$\phi (z, \mathbf{x}) \sim \phi_1 (\mathbf{x}) z^{\Delta_-} + \phi_0 (\mathbf{x}) z^{\Delta_+} \tag{1.21}$$

where $\Delta_\pm = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + m^2}$ or, inverted, $\Delta (\Delta - d) = m^2$ where we use the convention that $\Delta = \Delta_+$ such that $d - \Delta < \Delta$. If $m^2 \geq -\frac{d^2}{4}$ the two solutions are real, this is the so called Breitenlohner-Freedman bound, meaning that the

[2] We can neglect such a term because if we substitute a plane wave ansatz of the form $\phi(z, x) = \phi(z) e^{ik\dot{x}}$ the k^2 term is negligible.

curvature of the space allows the mass squared to be also negative. Considering $\Delta_\pm$ with m satisfying the Breitenlohner-Freedman bound, if $\Delta > \frac{d}{2}$ the mode z^{Δ_+} in (1.21) is the renormalizable solution while z^{Δ_-} is the non renormalizable one. This is because the action can be modified by adding appropriate boundary terms and so the solution is normalizable with respect to this modified action for $\Delta \geq \frac{d}{2} - 1$ (unitary bound for a scalar field). Therefore there is a certain mass regime where both solutions are normalizable. Normalizable modes correspond to bulk excitations which decay at the boundary $z = 0$ while the non normalizable mode determines the boundary behavior of the solution

$$\phi_0\left(\mathbf{x}\right) = \lim_{z \to 0} z^{-\Delta_-} \phi\left(z, \mathbf{x}\right). \tag{1.22}$$

Note that this is precisely the boundary value of the field that we already encountered in (1.6).

However it is possible to reconstruct the solution of the equations of motion (1.20) for the whole euclidean space with the appropriate boundary conditions by using the kernel $K_\Delta\left(z, \mathbf{x}, \mathbf{x}'\right)$, the so called bulk to boundary propagator (Fig. 1.1),

$$K_\Delta\left(z, \mathbf{x}, \mathbf{x}'\right) = C_\Delta \left(\frac{z}{z^2 + \left(\mathbf{x} - \mathbf{x}'\right)}\right)^\Delta \tag{1.23}$$

where $C_\Delta = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma\left(\Delta - \frac{d}{2}\right)}$ is a constant which is fixed by requiring that

$$\phi_0\left(\mathbf{x}\right) = \lim_{z \to 0} z^{\Delta - d} \phi\left(z, \mathbf{x}\right). \tag{1.24}$$

Then the solution of (1.20) $\phi(z, \mathbf{x})$ is

$$\phi\left(z, \mathbf{x}\right) = \int_{\partial AdS_{d+1}} d^d\mathbf{x}' K_\Delta\left(z, \mathbf{x}, \mathbf{x}'\right)\phi_0(z, \mathbf{x}') \tag{1.25}$$

where we denote by ∂AdS_{d+1} the boundary of the euclidean AdS_{d+1} space. Note that for $z \to 0$ (1.25) reduces to (1.21) and

$$\phi_1\left(\mathbf{x}\right) = C_\Delta \int d^d\mathbf{x}' \frac{\phi_0(z, \mathbf{x}')}{\left(\mathbf{x} - \mathbf{x}'\right)^{2\Delta}} \tag{1.26}$$

which shows that the normalizable mode is entirely expressible in terms of boundary data.

Correlation functions in AdS Let us come back to our main goal: the computation of correlation functions. To obtain the expression of correlation functions

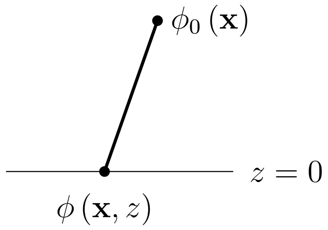


FIGURE 1.1: The bulk to boundary propagator.

in AdS we use the holographic prescription (1.6), more precisely

$$\langle \mathcal{O}_1 \mathcal{O}_2 \ldots \mathcal{O}_n \rangle = (-1)^{n+1} \frac{\delta^n S}{\delta\phi_0^1\, \delta\phi_0^2 \ldots \delta\phi_0^n}\Big|_{\phi_0^i=0} \tag{1.27}$$

where the action is meant to be on shell. Since only bulk interactions with less than n fields contribute non trivially to these derivatives, we are interested in terms in the action with at most n fields. We know how to specify the boundary value of the fields from the previous paragraph then the main ingredient left is the form of the on shell action. To do so let us substitute (1.21) into (1.19) and, using the equation of motion (1.20), we obtain the on shell action

$$S_0 = \frac{1}{2}\int_{\partial AdS_{d+1}} d^d\mathbf{x} \left(d\phi_0\phi_1 + (d-\Delta)\,\phi_0^2 z^{d-2\Delta} + \ldots \right). \tag{1.28}$$

The term proportional to $z^{d-2\Delta}$ is divergent because $\Delta > \frac{d}{2}$, so a counterterm has to be added to control this divergence. The regularized action is

$$S = (d-2\Delta)\int_{\partial AdS_{d+1}} d^d\mathbf{x}\, \phi_0(\mathbf{x})\, \phi_1(\mathbf{x})\,. \tag{1.29}$$

The one point function is given by varying the action with respect to the boundary value of the field $\phi_0(\mathbf{x})$

$$\langle \mathcal{O}(x) \rangle = \frac{\delta S}{\delta\phi_0(\mathbf{x})}\Big|_{\phi_0=0} \sim \phi_1(\mathbf{x})\Big|_{\phi_0=0} = 0, \tag{1.30}$$

where we use (1.26) in the last equality. What we have shown it is somehow trivial noticing that the vacuum expectation value of an operator should vanish if we want to preserve conformal symmetry but on the other hand this specifies that ϕ_1 is proportional to the vacuum expectation value of the dual operator.
If we replace the explicit expression for ϕ_1 (1.26) in (1.29) it is straightforward to

see that

$$S = (d - 2\Delta)\, C_\Delta \int_{\partial AdS_{d+1}} d^d\mathbf{x} d^d\mathbf{x}' \frac{\phi_0(\mathbf{x})\,\phi_0(\mathbf{x}')}{(\mathbf{x}-\mathbf{x}')^{2\Delta}} \tag{1.31}$$

and the two point function is

$$\langle \mathcal{O}(x)\mathcal{O}(x')\rangle = \frac{\delta^2 S}{\delta\phi_0(\mathbf{x})\,\delta\phi_0(\mathbf{x}')}\Big|_{\phi_0=0} = (d-2\Delta)\, C_\Delta\, (\mathbf{x}-\mathbf{x}')^{-2\Delta}\,. \tag{1.32}$$

To go to higher point functions we need to take into account interaction terms in the action which can be written as

$$S_{AdS_5} = \int d^5x \left(\frac{1}{2}\sum_i (\partial\phi_i)^2 + \frac{m_i^2}{2}\phi_i^2 + \sum_{k=3}^{n} \mathcal{C}_{i_1\ldots i_k}\phi_{i_1}\ldots\phi_{i_k} \right) \tag{1.33}$$

where for simplicity we restrict to the $d = 4 \to \mathrm{AdS}_5$ case and we consider no higher derivatives interactions and negligible couplings to other gauge and gravity fields. The equations of motion associated to this action differ from (1.20) for higher order terms and, more than that, it is not possible to get an exact solution. The Klein Gordon equation with source terms can be solved in a series expansion in λ using standard techniques involving Green functions, namely the kernels (1.23) and the bulk to bulk propagator which is another spatial kernel related to the bulk to boundary propagator via

$$K_\Delta(z,\mathbf{x},\mathbf{x}') = \lim_{z`\to\epsilon} \frac{\epsilon^\Delta}{2\Delta - d} K_\Delta(z, z`, \mathbf{x}, \mathbf{x}') \tag{1.34}$$

where $\epsilon \ll 1$. The key observation is that the iterative procedure stops after a finite number of steps because, as we mentioned above, when ϕ_0 appears more than n times it would not contribute to (1.27) which is our final goal. In this way we end up constructing a perturbative series and so a set of Feynman-like graph, the so called Witten diagrams, which we describe in 1.2. The simplest case is the computation of three point functions of scalar fields ϕ_1, ϕ_2 and ϕ_3 is easy because the only interaction term in the action contributing to the derivative is the cubic one ($\lambda\phi_1\phi_2\phi_3$) and the relevant diagram is Fig. 1.3 giving [3]

$$\begin{aligned}\langle \mathcal{O}(\mathbf{x_1})\mathcal{O}(\mathbf{x_2})\mathcal{O}(\mathbf{x_3})\rangle &= -\lambda \int d^{d+1}x K_{\Delta_1}(x,\mathbf{x_1})\, K_{\Delta_2}(x,\mathbf{x_2})\, K_{\Delta_3}(x,\mathbf{x_3}) \\ &= \frac{\lambda a_1}{|\mathbf{x_1}-\mathbf{x_2}|^{\Delta_1+\Delta_2-\Delta_3}|\mathbf{x_2}-\mathbf{x_3}|^{\Delta_2+\Delta_3-\Delta_1}|\mathbf{x_1}-\mathbf{x_3}|^{\Delta_1+\Delta_3-\Delta_2}}\end{aligned} \tag{1.35}$$

[3]The fields ϕ_i are dual to operators $\mathcal{O}_i$ in the field theory.

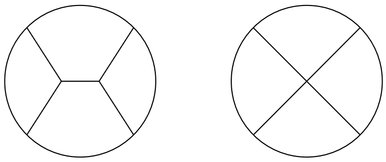


FIGURE 1.2: Example of Witten diagram for four point functions. The circumference denotes the boundary while the inside is the bulk of AdS. On the left: the 4 points on the boundary are connected by boundary to bulk propagators to the 2 points in the bulk where we insert vertices of the Lagrangian; the 2 vertices in the bulk are connected to each other by the bulk to bulk propagator and we integrate on their position as in Feynman rules. On the right: the 4 points on the boundary are connected by bulk to boundary propagators.

where $a_1 = -\frac{\Gamma(\Delta-2)}{2\pi^4}\prod_{i=1}^{3}\frac{\Gamma(\Delta-\Delta_i)}{\Gamma(\Delta_i-2)}$ and $\Delta = \frac{1}{2}(\Delta_1+\Delta_2+\Delta_3)$. Note that for com-

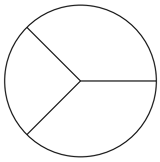


FIGURE 1.3: The diagram contributing for three point function .

puting correlation functions involving $n \leq 3$ scalar fields we do not need the bulk to bulk propagator which instead would enter in the higher point functions. We can anticipate some results that we are going to show in 1.1.3 namely that the spatial behavior of two and three point functions is completely fixed by conformal symmetry to be the same as (1.32) and (1.35) and that the spatial dependence of higher point correlation functions can be written as function of conformal ratios and reconstructed by means of the operator product expansion. We can see already the analogy with the computations reviewed in this paragraph, the fact that the spatial dependence of (1.32) and (1.35) is precisely the same as expected in any conformal field theory is not completely trivial because although there is a shared symmetry group there are no evident reasons a priori to expect that the AdS geometry describes the conformal field theory using the dynamical equivalence (1.6). Another remark is that what we reviewed is valid for scalar fields but it is generalizable to spinor, vector and tensor fields, technically it would be more involved but conceptually it is the same. A crucial point to obtain the correct normalization (essentially C_Δ which is encoded in the correct expression for the bulk to boundary propagator) for both (1.32) and (1.35) is the correct regularization of the action, namely the fact that we add the proper counterterms, this is one of the problem solved by the procedure called holographic renormalization mentioned above.

1.1.3 $\mathcal{N} = 4$ Super Yang-Mills

We have already stressed the role of this specific CFT in the context of the AdS/CFT correspondence. In the following we will review how to classify the operators and how conformal symmetry constraints the form of correlation functions. As we emphasized before the lagrangian depends on two parameters, namely the rank of the gauge group N and the coupling constant g_{CFT}, and it can be written as

$$\mathcal{L} = \frac{1}{g_{CFT}^2}\mathrm{Tr}\left[\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi_i)^2 - \frac{1}{2}[\phi_i, \phi_j]^2 + \bar{\lambda}\not{D}\lambda - i\Gamma^i[\phi_i, \lambda]\right] \tag{1.36}$$

where A_μ (with Lorentz vector index $\mu = 1, \ldots, 4$) is the gluonic gauge field, ϕ_i are the scalar fields ($i = 1, \ldots, 6$), λ are gluinos [4], Γ are ten dimensional Dirac gamma matrices and $D_\mu = \partial_\mu - i[A_\mu, \ ]$ is the non abelian covariant derivative. Supersymmetry ensures that all the fields are $N \times N$ matrices transforming under the same (adjoint) representation of the gauge group. There is also an alternative way of thinking to $\mathcal{N}$=4 SYM, namely by using superspace formalism. In this language the field content of $\mathcal{N}$=4 SYM can be reconstructed by noticing that in four dimensions and with 4 supersymmetries it exists only one vector multiplet, which can be seen as an $\mathcal{N}$= 2 vector and an $\mathcal{N} = 2$ hypermultiplet. The former multiplet splits into an $\mathcal{N}$= 1 vector multiplet and a chiral multiplet while the $\mathcal{N} = 2$ hypermultiplet is simply composed of two chiral multiplets. From this construction one can easily recover the field content that we already listed.

Symmetry group $\mathcal{N}$=4 SYM posses superconformal symmetry, combination of conformal symmetry and supersymmety, denoted as $PSU(2, 2|4)$ [5] which includes the conformal group $SO(2, 4)$, with 15 generators coming from translations P_μ, Lorentz transformations $L_{\mu\nu}$, dilatations D and special conformal transformations K^μ, supersymmetry generators (superpartners of translations) Q^a_α and $\bar{Q}_{\dot{\alpha}a}$ with $a = 1, \ldots, 4$, special superconformal generators(superpartners of special conformal transformations) $S_{\alpha a}$ and $\bar{S}^{\bar{a}}_{\dot{\alpha}}$ and SO(6) R-symmetry generators T^A with $A =$

[4]We represent these fermions as 16 component Majorana spinors in 10 dimensions.

[5]This is a consequence of the fact that the central charge in $\mathcal{N}$=4 SYM vanishes, otherwise naturally we would have had $SU(2, 2|4)$.

$1, \ldots, 15$ [6]. These generators form a superalgebra which has schematically the following commutation relations

$$\begin{array}{ccc} [D, P_\mu] = -i\, P_\mu & [D, K_\mu] = i\, K_\mu & [D, L_{\mu,\nu}] = 0 \\ [L, P] \sim P & [L, K] \sim K & [P, K] \sim L - D \\ \{Q, \bar{Q}\} \sim P & \{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0 & \{Q, S\} \sim L - T - D \\ [L, Q] \sim Q & [K, Q] \sim S & [P, S] \sim \bar{Q} \\ [L, S] \sim \bar{S} & [D, Q] = -\frac{i}{2} Q & [D, S] = \frac{i}{2} S. \end{array} \tag{1.37}$$

Hence the dimension of the generators is

$$\begin{array}{ccc} [D] = [L] = [T] = 0 & [P] = 1 & [K] = -1 \\ [Q] = \frac{1}{2} & [S] = -\frac{1}{2}. & \end{array} \tag{1.38}$$

The presence of supersymmetry obviously implies that some of the generators should be fermionic, in fact we can rearrange the generators in a matrix form

$$\begin{pmatrix} L, \bar{L}, P, K, D & Q, \bar{S} \\ \bar{Q}, S & T \end{pmatrix} \tag{1.39}$$

where the diagonal components are bosonic and the off diagonal are fermionic. An important feature of $\mathcal{N}$=4 SYM is that the superconformal symmetry is unbroken at the quantum level. One of the consequences of this fact is that the mass dimension of the fields is protected from quantum corrections and the total dimension of a product of fields is the sum of the mass dimensions of the single constituents. Now we want to specify the players of the game: gauge invariant operators. The simplest way to construct an operator is to have in general linear combinations of products of fields of any type, namely scalars, fermions and covariant derivatives [7]. In order to ensure gauge invariance it is necessary to consider traces of these products. Despite the unrenormalizability of the single fields, operators represented by traces are renormalizable.

Operators Superconformal symmetry is also useful to classify operators. We want to work in a specific representation of the conformal group in which our

[6] We use 2-component spinor notation so $\alpha = 1, 2$, the same stands for $\dot{\alpha}$.

[7] Note that out of coviariant derivatives one can built field strengths. We do not have gauge fields because they do not transform covariantly under gauge transformations.

operators have a well defined dimension Δ, namely they can be thought as eigenvectors of the dilatation operator D. So if we consider a generic local operator in the conformal field theory $\mathcal{O}(x)$ it will transform as $\mathcal{O}'(x) = \lambda^{-\Delta}\mathcal{O}(\lambda x)$ under dilatations $x \to \lambda x$. This means that

$$[D, \mathcal{O}(x)] = (\Delta + x^\mu \partial_\mu)\mathcal{O}(x). \tag{1.40}$$

Looking back to the first two commutation relations of (1.37) combined with (1.38), we notice that K_μ and P_μ act as respectively lowering and raising operators for $\mathcal{O}(x)$, meaning that by acting with K_μ (or P_μ) we obtain a new operator with dimension Δ decreased (or increased) by 1. Any unitary field theory has a lower bound for the dimension Δ of the fields, this means that there are operators annihilated by special conformal generators K_μ, these are called primary operators. From the last two relations of (1.37) we can see, in the same way of above, that among primary operators there is a subclass annihilated by conformal supercharges S which are the superconformal primaries. In this case by acting with S (Q) the dimension is lowered (raised) by $\frac{1}{2}$. Superconformal primaries are also lowest dimensional operators in superconformal multiplets, it is possible to generate all the other states of the multiplet (called descendants) with the action of Q. Note that it is impossible to obtain a superconformal primary as a result of the action of Q to any operator in the multiplet. Superconformal primaries that commute with at least one of the supercharges are called chiral primaries. They are also called BPS operators because they belong to shortened representations and their dimension is protected. The bosonic subgroup of the $PSU(2,2|4)$ is $SO(2,4) \times SO(6)$ which provides a unitary representation under which operators transform. States are labeled by the six Cartan eigenvalues of the representations, namely the dilatation eigenvalue Δ, two Lorentz spin S_1 and S_2 and Dynkin labels of an SU(4) representation J_1, J_2 and J_3 coming from the R-symmetry. It is convenient to group the six scalar fields in three complex fields in the following way

$$X = \phi_1 + i\,\phi_2 \qquad Y = \phi_3 + i\,\phi_4 \qquad Z = \phi_5 + i\,\phi_6 \quad + \text{c.c.}. \tag{1.41}$$

Correlation functions in $\mathcal{N}$=4 It is well known that conformal symmetry fixes completely the spatial dependence of two and three point correlation functions of scalar primary operators. This comes from the counting of the possible conformal invariants, i.e. number of conformal cross-ratios, that one can build.

Let us consider a n-point correlation function of scalar primary operators

$$C(x_1,\ldots,x_n)=\langle\phi_1(x_1)\phi_2(x_2)\ldots\phi_n(x_n)\rangle. \tag{1.42}$$

Our aim is to see what happens to this object under conformal transformations. Let us recall some basic properties. Conformal transformations are coordinate transformations which leave the metric invariant up to a scale factor, namely

$$g'_{\mu\nu}(x')=\Omega(x)g_{\mu\nu}(x), \tag{1.43}$$

in particular

- for transformations belonging to the Poincaré group $x'=x+a$: $\Omega(x)=1$,
- for dilatations $x'=\lambda\,x$: $\Omega(x)=\lambda^{-2}$,
- for special conformal transformations $x'=\frac{x+bx^2}{1+2b\cdot x+b^2x^2}$: $\Omega(x)=(1+2b\cdot x+b^2x^2)^2$.

The jacobian of the transformation obviously depends on Ω and it is given by $|\frac{\partial x'}{\partial x}|=\Omega^{-\frac{d}{2}}$ where d is the number of dimensions. Acting with a conformal transformation to (1.42) we obtain

$$C(x_1,\ldots,x_n)=\left(\prod_{i=1}^{n}\Omega_i^{-\frac{\Delta_i}{2}}\right)C(x'_1,\ldots,x'_n) \tag{1.44}$$

where $\Omega_i=\Omega(x_i)$ and the conformal invariance of the vacuum have been used. Translation invariance constraints n point function to depend on the differences of the spatial coordinates (x_i-x_j) while rotational invariance gives a more strict constraint namely that they can only depend on the magnitude $x_{ij}=|x_i-x_j|$. Scale invariance combined with special conformal transformations allow the dependence only on the so called conformal ratios or cross-ratios defined as $\frac{x_{ij}x_{kl}}{x_{ik}x_{jl}}$. Furthermore by counting all the possible quantities that we can form following these constraints, the number of all possible cross ratios is $\frac{n(n-3)}{2}$. This relation shows that there are no cross ratios for $n=2$ and $n=3$ [8] .

Two and three point functions and OPE In [65] John Cardy presents an

[8]There is also an alternative way of getting the dependence on cross ratios of correlation functions by using conformal Ward identities. We refer to [64] for details.

elegant and alternative[9] way of deriving the spatial dependence of two point correlation functions. The idea is the following

- consider a two point function $\langle\phi_1(x_1)\phi_2(x_2)\rangle$,
- apply a conformal transformation which maps $\{x_1, x_2\} \to \{x'_1, x'_2\}$ such that $\Omega(x_1)^{-\Delta_1}\Omega(x_2)^{-\Delta_2}\langle\phi_1(x_1)\phi_2(x_2)\rangle = \langle\phi_1(x'_1)\phi_2(x'_2)\rangle$,
- $\langle\phi_2(x_1)\phi_1(x_2)\rangle = \langle\phi_1(x_1)\phi_2(x_2)\rangle$ since the former is just a rotation of 180 degrees of the latter,
- applying the same conformal transformation as above we have that $\Omega(x_1)^{-\Delta_1}\Omega(x_2)^{-\Delta_2} = \Omega(x_1)^{-\Delta_2}\Omega(x_2)^{-\Delta_1}$ which is different from zero only if $\Delta_1 = \Delta_2$,
- $$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \frac{\delta_{12}}{x_{12}^{2\Delta}} \tag{1.45}$$
  where we are assuming that the proportionality coefficient is symmetric and real, because the operators are hermitian. This is what Cardy calls orthogonality of scaling operators. Note that we include the normalization in the operators.

As well as for two point also the spatial dependence of three point functions is completely fixed. In this case the special conformal transformations add an extra constraint which determines the exponents of the difference between spatial coordinates, thus we obtain

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{c_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}}. \tag{1.46}$$

Since x_{ij} is symmetric under the exchange of the indices also the coefficient c_{123} is so. Note that the expression (1.46) is valid if we unit normalize the two point function, as in (1.45). In quantum field theories, due to Wilson [66] and Zimmermann [67] [68], it is possible to write a short distance expansion for a product of two operators in terms of a complete basis of renormalized operators. This expansion is commonly called operator product expansion (OPE). Schematically we can

[9] It is alternative to the usual way of taking into account the effect of all the transformations belonging to the conformal group, similarly to what we used in the previous paragraph.

write

$$O_A(x)O_B(y) = \sum_i \mathcal{C}(x-y)O_i(y) \tag{1.47}$$

where O_A and O_B are generic local fields, O_i is the set of renormalized operators and $\mathcal{C}$ are numerical coefficients which may be singular when $x - y \to 0$. If the initial operators have a well defined canonical dimension then the basis of operators around which one expands has an increasing canonical dimension. Let us now focus on conformal field theories. In this case primary and descendant operators form a complete set under the OPE, this means that we can expand any product of primary fields in this way

$$\mathcal{O}_A(x)\mathcal{O}_B(y) = \sum_D \mathcal{C}_{ABD}(x-y)^{\Delta_D-\Delta_A-\Delta_B} \sum_n \beta^{(n)}_{ABD}(x-y)^{|n|}\mathcal{O}^{(n)}_D(y). \tag{1.48}$$

where we label with n the descendant level, $\mathcal{O}^{(0)}_D$ are the primary operators and $\beta^{(0)}_{ABD} = 1$. Using Ward identities for conformal field theories, the coefficients β_{ABD} can be reconstructed in terms of the conformal dimension of the operators while $\mathcal{C}_{ABD}$ characterize the CFT. The crucial point is that $\mathcal{C}_{ABD}$ is precisely the three point coefficient that we have denoted with c_{123} in (1.46). In fact if we substitute the OPE for two of the scalar primary operators in (1.46) and use the form for the unit normalized two point function (1.45) we easily obtain that the OPE coefficients are precisely the three point function structure constants [10]. This is one of the biggest advantages of conformal field theories because, using iteratively the OPE, the only ingredients needed to express all the n point correlation functions are conformal dimensions and structure constants.

1.2 Basics of integrability

1.2.1 Anomalous dimension

In $\mathcal{N} = 4$ SYM the gauge coupling g_{CFT} is not renormalized, nevertheless gauge invariant local operators are in general renormalizable. At the classical level the

[10] This is completely true when considering primary operators, in the case we are interested in descendants the OPE coefficients are proportional to the structure constants up to factors depending on the conformal dimensions and on the spatial separation between the operator insertions.

scaling dimension of gauge invariant operators is simply the sum of the dimensions of the constituent fields, but at the quantum level it acquires a so-called anomalous dimension. As we have previously reviewed the two point function of gauge invariant operators is completely fixed by conformal symmetry to be (1.45), where the conformal dimension appears in the exponent of the spatial part. We can write Δ as a function of the coupling constant $\Delta(g_{CFT}) = \Delta_0 + \gamma(g_{CFT})$ to distinguish the classical dimension Δ_0 to the anomalous dimension $\gamma(g_{CFT})$. For generic operators the two point function (1.45) admits a perturbative expansion in the coupling constant which, as in any quantum field theory, depends on the UV cutoff. Hence two conformal fields that at classical level have the same quantum numbers may not share this property once quantum correction are included, since they can have different anomalous dimensions. Thus the two point function of the two renormalized operators must be zero, though this may not be the case simply taking into account classical quantum numbers. This is essentially the so called mixing problem which can be analyzed by finding the correct renormalized operators and their respective anomalous dimensions. In fact if we consider

$$\mathcal{O}_R^A(x) = Z_B^A \mathcal{O}^B(x) \tag{1.49}$$

being $\mathcal{O}_R$ the renormalized operator, $\mathcal{O}$ the bare operator and Z_B^A is the renormalization matrix, and apply standard renormalization group transformation we obtain that the mixing matrix Γ_B^A is related to the wave function renormalization [11] through

$$\Gamma_B^A = \frac{\partial Z_C^A}{\partial \ln\mu}\left(Z^{-1}\right)_B^C \tag{1.50}$$

where μ is the usual renormalization scale and the anomalous dimension can be read from

$$\Gamma\mathcal{O} = \gamma\mathcal{O} \tag{1.51}$$

where $\mathcal{O}$ is meant to represent a basis of operators. Even if the conformal dimension can be read off from the two point correlation functions of the respective operators, it is most convenient to consider $\Delta(g_{CFT})$ as the eigenvalue of the mixing matrix, namely

$$D\mathcal{O}(x) = \Delta(g_{CFT})\,\mathcal{O}(x)\,. \tag{1.52}$$

[11] Note that the only quantity which is renormalized is the wave function because the beta function vanishes.

In the 't Hooft limit described previously the anomalous dimension, as all the other physical quantities, can be written in a topological expansion

$$\gamma\left(\lambda, \frac{1}{N}\right) = \sum_{g=0}^{\infty} \frac{1}{N^{2g}} \sum_{l=1} \lambda^l \gamma_{l,g} \tag{1.53}$$

where λ is the 't Hooft coupling, g is the genus of the surface spanned by the diagram and l is the loop order. In this thesis we only focus on the planar limit, which consists in considering $N \to \infty$ and $g_{CFT} \to 0$ with λ fixed. In this regime the dominant contributions to (1.53) are given by planar diagrams, namely by setting $g = 0$. In this limit big simplifications occur

- multi trace operators decouple, we consider only single trace operators;
- the dilatation operator can be regarded as the Hamiltonian of a spin chain (this is the place where integrability plays its central role);
- the string theory is non interacting, which amounts of saying that the scaling dimensions of the planar gauge theory should be identified with the energies of the free string theory.

In the following we develop on these points, especially reviewing how to solve integrable models.

1.2.2 Dilatation operator as spin chain Hamiltonian

In 2003 Minahan and Zarembo [5] established a crucial equivalence between the $SO(6)$ one loop dilatation operator and the Heisenberg spin chain Hamiltonian. Let us review why and how this works and the consequences brought by this relation.

- Let us consider renormalizable operators in the $SO(6)$ scalar sector, which is closed at one loop, of $\mathcal{N} = 4$ SYM

$$\mathcal{O}(x) = \mathcal{C}^{i_1 \dots i_L} : \mathrm{Tr}\left(\phi_{i_1} \dots \phi_{i_L}\right) : \tag{1.54}$$

  where : denotes the normal ordering.

- We can see operators of the form (1.54) as tensors which span a finite dimensional Hilbert space with dimension 6^L. This Hilbert space is interpreted as a spin chain of length L where the direction of the spin sitting at site l is meant to be the $SO(6)$ flavor ϕ_{i_l}. Thus our operator (1.54) can be regarded as

$$|\uparrow_{i_1}\dots\uparrow_{i_L}\rangle. \tag{1.55}$$

- The key observation is that the Hamiltonian associated to this spin chain is equivalent to the $SO(6)$ dilatation operator at one loop which can be computed giving

$$\Gamma^{(1)}=\frac{\lambda}{16\pi^2}\sum_{l=1}^{L}\left(2I_{l,l+1}-2P_{l,l+1}+K_{l,l+1}\right) \tag{1.56}$$

where the subscript (1) refers to the fact that it is the one loop dilatation operator and the identity operator I, the permutation operator P and the trace operator K act on the spin chain sites as

$$I|\dots\uparrow_a\uparrow_b\dots\rangle = |\dots\uparrow_a\uparrow_b\dots\rangle \tag{1.57}$$

$$P|\dots\uparrow_a\uparrow_b\dots\rangle = |\dots\uparrow_b\uparrow_a\dots\rangle \tag{1.58}$$

$$K|\dots\uparrow_a\uparrow_b\dots\rangle = \delta_{ab}\sum_{c=1}^{6}|\dots\uparrow_c\uparrow_c\dots\rangle. \tag{1.59}$$

Note that we only have nearest neighbour interactions and we have to consider the cyclity of the trace in the spin chain description by imposing that the spin chain state should be invariant under an index translation.

- The $SO(6)$ spin chain is integrable as shown in [69]

A very interesting subsector of $SO(6)$ is the $SU(2)$, which instead is closed at any loop order, which can be seen as formed by two of the three complex scalars (1.41). Let us choose for instance X and Z. Now the correspondence between the dilatation operator and the spin chain becomes more clear in the sense that the spin chain in this sector is the $XXX_{\frac{1}{2}}$ Heisenberg spin chain. The three X denote that we are dealing with identical coupling constants of the spin-spin interactions in the three spatial directions and the $\frac{1}{2}$ refers to the fact that the sites of the chain have spin $\frac{1}{2}$ degrees of freedom. Here the interpretation is simpler because we can think to the complex field Z being a spin up while X a spin down, in this way a generic operator is a closed spin chain [12] formed by a collection of a certain number of spin up and down

$$\mathcal{O} = \mathrm{Tr}\,(ZXXZXXXZ\ldots Z) \quad \rightarrow \quad |\uparrow\downarrow\downarrow\uparrow\downarrow\downarrow\downarrow\uparrow\ldots\uparrow\rangle. \tag{1.60}$$

In this specific case the trace operator in (1.56) acts trivially and so the dilatation operator simplifies to

$$\Gamma = \frac{\lambda}{8\pi^2}\sum_{l=1}^{L}\left(I_{l,l+1} - P_{l,l+1}\right) \tag{1.61}$$

where, up to the factor in front, can be easily identified with the $XXX_{\frac{1}{2}}$ spin chain Hamiltonian simply expressing the permutation operator in terms of the Pauli matrices associated to each spin chain site

$$P_{l,l+1} = \frac{1}{2}\left(\sigma_l\sigma_{l+1} + I_{n,n+1}\right). \tag{1.62}$$

1.2.3 Algebraic Bethe Ansatz

In the last paragraph we discuss the identification of the one loop dilatation operator to the spin chain Hamiltonian. The next step would be to compute the spectrum of the dilatation operator which, in the spin chain picture, is equivalent to diagonalize the Hamiltonian finding its eigenvalues and eigenvectors. This problem is not trivial to solve with numerical methods essentially because we may be interested in the case when the number of sites is infinite. In that case also numerics breaks down and even for finite site number the form of the eigenvectors is highly not trivial (if not impossible) to get. However in 1931 Hans Bethe [70] found a method to exactly solve the problem of diagonalizing the one dimensional $XXX_{\frac{1}{2}}$ spin chain Hamiltonian. Another breakthrough was the classical inverse scattering method, proposed firstly by Gardner, Green, Kruskal and Miura in [71] and extended by Faddeev, Zakharov [72] and Gardner [73], which consists in solving an initial value problem for certain classes of nonlinear partial differential equations by introducing the Lax connection. The quantum version of the inverse scattering method was studied in 1979 by Faddeev, Sklyanin and Takhtajanin [74]

[12] In this context when we write closed spin chain we mean a spin chain with closed and periodic boundary conditions. The former imply that the Lth site interacts with the $L-1$th as well as the 1st and the latter means that the lth site can be identified with the $l + aL$th site with $a \in \mathbb{Z}$.

and it gives a new formulation of the Bethe anstaz, which is nowadays called Algebraic Bethe ansatz and which will be the core of this section. This method is very powerful, mainly for two reasons:

1. It is a constructive method, it allows to construct the eigenvectors and the eigenvalues of integrable Hamiltonians;
2. It allows to prove the integrability of the model.

The Hilbert space of a spin chain with L sites can be written as a tensor product of the Hilbert spaces associated to each site

$$\mathcal{H} = h_1 \otimes h_2 \otimes \cdots \otimes h_L \tag{1.63}$$

and the dimensionality of the spaces h_i depends on the spin chain symmetry group, more specifically for $SO(N)$ it is $h = \mathbb{C}^N$. An important point is that we need to introduce an additional vector space [13], isomorphic to the physical one, which is called auxiliary space. The enlargement of the physical Hilbert space is peculiar and it serves to mediate the interaction among the physical degrees of freedom which in this way interact only with the auxiliary space. We need to introduce three objects which are the building blocks of this formalism:

1. An R matrix $(R_{ab}(c))$, which acts on the tensor product $\mathcal{V}_a \otimes \mathcal{V}_b$ where $\mathcal{V} = \mathbb{C}^N$ and c is a complex number called spectral parameter. The crucial point is that the R matrix is supposed to be a solution of the Yang-Baxter equations

$$R_{12}(u-v)\, R_{13}(u)\, R_{23}(v) = R_{23}(v)\, R_{13}(u)\, R_{12}(u-v)\,. \tag{1.64}$$

2. A monodromy matrix

$$M_a(u) = R_{a1} R_{a2} \dots R_{aL} \tag{1.65}$$

   where a refers to the auxiliary space. Note that the monodromy matrix acts on the product of the enlarged Hilbert space, the auxiliary and physical space (the tensor product of $L+1$ spaces). It satisfies the same Yang Baxter equation than the R-matrix which is depicted in Fig. 1.4.

[13] Actually it is needed to add more than one auxiliary space to prove some of the relations we will use later, but the idea is still the same.

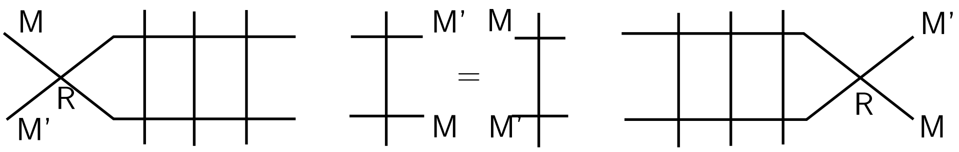


FIGURE 1.4: Yang Baxter relations for the monodromy matrix.

3. A transfer matrix is obtained by tracing the monodromy matrix over the auxiliary space
$$T(u) = \mathrm{tr}_a M_a(u) \tag{1.66}$$
and it acts on the full physical space, the tensor product of L spaces.

One drawback of this method is that it is based on the knowledge of the R matrix, which in general is not immediately inferable for a system. The general idea to find the expression for the R matrix is to consider a combination of all the possible invariant structures in the specific symmetry group. This is not a unique definition, since there are arbitrary functions of the spectral parameter which can be fixed (up to a normalization) using the Yang Baxter equations for the R matrix (1.64). In the following we analyze more in details the $SU(2)$ and the $SO(6)$ case.
$SU(2)$ **case** In the $SU(N)$ case the only tensorial structures we can think of are the permutation and the identity operator defined in the first two lines of (1.59). Specializing to the $SU(2)$ case we have that

$$R = \begin{pmatrix} a(u,z) & 0 & 0 & 0 \\ 0 & b(u,z) & c(u,z) & 0 \\ 0 & c(u,z) & b(u,z) & 0 \\ 0 & 0 & 0 & a(u,z) \end{pmatrix} \tag{1.67}$$

being

$$a(u,z) = \frac{u-z+\eta}{u-z} \qquad b(u,z) = 1 \qquad c(u,z) = \frac{\eta}{u-z} \tag{1.68}$$

where z is the set of quantum rapidities and η is the shift [14]. We construct a monodromy matrix as in (1.65) which we write in a convenient matrix form

$$M_a(u,z) = \begin{pmatrix} A(u,z) & B(u,z) \\ C(u,z) & D(u,z) \end{pmatrix}. \tag{1.69}$$

[14] In the homogeneous spin chain, the case we are interested in, we set $\eta = -z\frac{i}{2}$.

Thus the transfer matrix takes the form

$$T\left(u,z\right)=A\left(u,z\right)+D\left(u,z\right). \tag{1.70}$$

Introducing another auxiliary space, which we denote by a', we can construct

$$R_{aa'}\left(u-v\right)M_{a}\left(u\right)M'_{a}\left(v\right)=M'_{a}\left(v\right)M_{a}\left(u\right)R_{aa'}\left(u-v\right) \tag{1.71}$$

where we neglect from now the dependence on z to streamline the notation. Multiplying by $R^{-1}_{aa'}\left(u-v\right)$ from the left and taking the trace over the tensor product of the two auxiliary spaces we can prove that the terms that survive give rise to

$$T\left(u\right)T\left(v\right)=T\left(v\right)T\left(u\right)\quad\rightarrow\quad\left[T\left(u\right),T\left(v\right)\right]=0 \tag{1.72}$$

which is fulfilled for any u and v. Note again that the remarkable relation (1.72) is a consequence only of the Yang Baxter relation (1.71). We can Taylor expand $T\left(u\right)$ around u and (1.72) implies that

$$\left[Q_n,Q_{n'}\right]=0 \tag{1.73}$$

where the expansion coefficients Q_n are the set of conserved charges. For the moment this name could seem arbitrary but we will see that if we choose a specific series expansion Q_2 is the Hamiltonian and, more specifically, the Heisenberg $XXX_{\frac{1}{2}}$ spin chain Hamiltonian.

The key observation is that we can write the the Heisenberg $XXX_{\frac{1}{2}}$ spin chain Hamiltonian as a derivative of the transfer matrix

$$H_{XXX_{\frac{1}{2}}}=\eta\frac{d}{du}\mathrm{Log}\,T\left(u\right)\Big|_{u=\frac{\eta}{2}}. \tag{1.74}$$

where here $T\left(u\right)$ is the homogeneous transfer matrix namely $T\left(u,z\right)\Big|_{z=\frac{\eta}{2}}$. Essentially with (1.72) and (1.74) we have proven the integrability of the system. The identification (1.74) shines a light on this method because it shows that diagonalizing the Hamiltonian is equivalent to diagonalizing the transfer matrix. Hence the next step is to show how to construct the eigenvectors of $T\left(u\right)$. Let us step back to the Yang Baxter equation (1.71), if we plug in the explicit expression of the monodromy matrix (1.69) we obtain some relations between the different components of the matrix namely

$$B(u)\,B(v) = B(v)\,B(u) \tag{1.75}$$

$$C(u)\,C(v) = C(v)\,C(u) \tag{1.76}$$

$$(u-v+\eta)\,B(u)\,A(v) = \eta B(v)\,A(u) + (u-v)\,A(v)\,B(u) \tag{1.77}$$

$$\eta B(u)\,D(v) + (u-v)\,D(u)\,B(v) = (u-v+\eta)\,B(v)\,D(u)\,. \tag{1.78}$$

The transfer matrix in terms of the element of the monodromy matrix is given in (1.70) from which it is clear that at least $A(u)+D(u)$ should be diagonal on the states that we want to construct and $C(u)$ or $B(u)$ have to be the creation or annihilation operators. Let us choose C to be the annihilation operator in such a way that $C(u)\,|L^{\wedge}\rangle = 0$ where $|L^{\wedge}\rangle$ is a reference state, the highest weight state. Summarizing the action of the monodromy matrix to the highest weight state is

$$M_a(u,z)\,|L^{\wedge}\rangle = \begin{pmatrix} a(u,z) & \textit{not relevant} \\ 0 & d(u,z) \end{pmatrix} |L^{\wedge}\rangle \tag{1.79}$$

where a is defined in (1.68) and $d=1$ and *not relevant* means that we do not need the precise form of that operator in our dissertation. Note that we can do the same reasoning for the ket $\langle L^{\wedge}|$ which is a reference state as well. All the other states are created by acting with the creation operator $B(u)$ to the reference state $|L^{\wedge}\rangle$

$$B(u_1,z)\,B(u_2,z)\ldots B(u_l,z)\,|L^{\wedge}\rangle = |l\rangle \tag{1.80}$$

where for the moment $|l\rangle$ is a generic state. Note that since the B's commute among themselves the order in which they act on the reference state is irrelevant. It turns out that the state $|l\rangle$ is an eigenstate of the transfer matrix if and only if the set of spectral parameters $\{l\}$ satisfies the following relations

$$\prod_{j=1}^{L} \frac{u_i - z_j + \eta}{u_i - z_j} = \prod_{j\neq i}^{N} \frac{u_j - u_i - \eta}{u_j - u_i + \eta} \tag{1.81}$$

which are the so called Bethe equations and where N is the number of rapidities contained in the set $\{l\}$. As remarked earlier the eigenstates of the transfer matrix are also eigenstates of the spin chain Hamiltonian which is the spin chain version of the one loop dilatation operator in $\mathcal{N}=4$ SYM.

$SO(6)$ **case** In the following we restrict to the homogeneous spin chain and we review the nested Bethe ansatz which is used to find the solution for this higher rank spin chain. The diagonalisation of the anomalous dimension matrix (the dilatation operator) is equivalent to the diagonalisation of the integrable spin-chain Hamiltonian (1.56), so that the Bethe ansatz method can be implemented. In general the Bethe equations can be written in the form

$$\left(\frac{u_{\alpha,\beta}+i\frac{V_\alpha}{2}}{u_{\alpha,\beta}-i\frac{V_\alpha}{2}}\right)^L=\prod_{\substack{j=1,\dots,k_\beta\\ \beta=1,\dots,7}}^{(\alpha,k)\neq(\beta,j)}\frac{u_{\alpha,k}-u_{\beta,j}+i\frac{M_{\alpha,\beta}}{2}}{u_{\alpha,k}-u_{\beta,j}-i\frac{M_{\alpha,\beta}}{2}} \tag{1.82}$$

where $M_{\alpha,\beta}$ is the Cartan matrix and V_α is the vector of the Dynkin labels of the highest weight representation. For the $SO(6)$ case the rank of the Cartan algebra is three and there are three simple roots which can be denoted as

$$\alpha_1=(1,-1,0)\qquad \alpha_2=(0,1,-1)\qquad \alpha_3=(0,1,1) \tag{1.83}$$

thus the Cartan matrix takes the form $M_{\alpha\beta}=\alpha_\alpha\cdot\alpha_\beta$ and the highest weight state is given by $\vec{V}=(1,0,0)$

$$M_{\alpha\beta}=\begin{pmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{pmatrix}\quad V_\alpha=\begin{pmatrix}0\\1\\0\end{pmatrix}. \tag{1.84}$$

The Dynkin diagram of $SO(6)$ is in Fig. 1.5 and the explicit expression of the Bethe equations is

$$\begin{aligned}
e_1\left(u_{1,k}\right)^L&=\prod_{\substack{j=1\\ j\neq k}}^{M_1}e_1\left(u_{1,k}-u_{1,j}\right)\prod_{j=1}^{M_2}e_{-1}\left(u_{1,k}-u_{2,j}\right)\prod_{j=1}^{M_3}e_{-1}\left(u_{1,k}-u_{3,j}\right)\\
1&=\prod_{\substack{j=1\\ j\neq k}}^{M_2}e_2\left(u_{2,k}-u_{2,j}\right)\prod_{j=1}^{M_1}e_{-1}\left(u_{2,k}-u_{1,j}\right)\\
1&=\prod_{\substack{j=1\\ j\neq k}}^{M_3}e_2\left(u_{3,k}-u_{3,j}\right)\prod_{j=1}^{M_1}e_{-1}\left(u_{3,k}-u_{1,j}\right)
\end{aligned} \tag{1.85}$$

where $e_n\left(u\right)=\frac{u+i\frac{n}{2}}{u-i\frac{n}{2}}$. The R matrix for the $SO(6)$ case can be constructed by noticing that in this example there are three tensor structures allowed, namely I, P and K defined as (1.59) [15],and again Yang Baxter equations fix the ratio of the

[15]There is a bit of simplification in this sentence, in fact the three tensor structures allowed are combination of I, P and K, but it is possible to express the R matrix only in terms of these three operators.

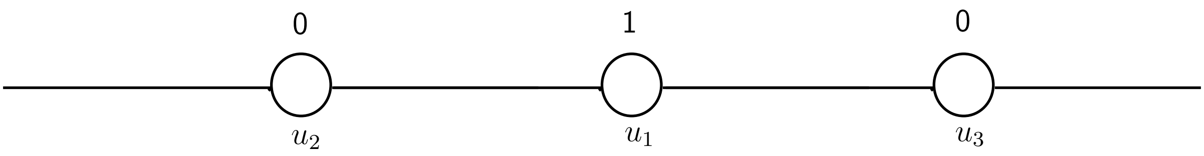


FIGURE 1.5: Dynkin diagram

functions with which the tensors appear, giving

$$R(u) = \frac{1}{2}\left[uK + (u-2) - (u-2)P\right] \tag{1.86}$$

acting on $\mathcal{V}^6 \otimes \mathcal{V}^6$ in the vector representation. It is possible to generalize the algebraic Bethe ansatz, construct the transfer matrix which is related through (1.74) to the Hamiltonian of the spin chain (1.56) since also in this case $R(0)$ is proportional to the permutation operator.

Integrability of the full $PSU(2,2|4)$ and at any loop order The integrability of the $SU(2)$ scalar sector of $\mathcal{N}=4$ is not confined to the one loop order, it has been proven its integrability up to three loops in [75]. In the limit of long spin chain an all loop asymptotic Bethe Ansatz for the $SU(2)$ sector has been proposed in [76] and generalized to the full $PSU(2,2|4)$ spin chain [77]. Actually beyond one loop the action of the dilatation operator cannot be written in terms of nearest neighbour spin chains because it involves multisite length-changing interactions. When the length of the spin chain is finite this method breaks down and wrapping corrections have to be taken into account. Notably there are three ways of considering finite size corrections: Lüscher corrections [78–80], Thermodynamic Bethe ansatz [81–85] and the Y-system [86–88].

1.2.4 Six vertex model

Even if the Algebraic Bethe ansatz is very powerful to solve integrable models, it is not very intuitive. In this section we will review the six vertex model techniques which will give a more physical intuition. These two approaches are closely related because any one dimensional quantum model is equivalent to a classical two dimensional system, which we chose to be the six vertex model (or ice model). This is an example of a lattice model in statistical mechanics which was originally introduced by Pauling [89] to provide a description of the two dimensional ice. Each vertex (oxygen atom) of the lattice is connected by an edge to four nearest neighbors (hydrogen ions). The states of the model are configurations of arrows on a finite grid such that the number of arrows pointing to the vertex is 2 (two hydrogens are closer to the oxygen than the other two). This restriction is called ice rule and it limits the number of allowed vertices to 6 (out of $2^4 = 16$) , hence the name of six vertex model [16]. The weights are numbers associated to each vertex depending on the states of the adjacent edges. The model is called homogeneous if the weight assigned to a vertex depends only on the configuration of arrows on adjacent edges and not on the vertex itself. An important quantity is the weight of a configuration which is defined as the product of the weights of all the vertices in the grid and intuitively it gives the relative probability of the occurrence of the given configuration.

Partition function Let us consider an $M \times N$ rectangular lattice with periodic boundary conditions which means that we can think of the edges in the lattice as forming a circle, where the arrows at the top must point in the same direction as the corresponding arrows on the other side of the lattice (the same for left and right). The net arrow flux in the vertical (or horizontal) direction is conserved from row to row (column to column) as a consequence of the ice rule. From the flux conservation, the six-vertex model can be represented by non-crossing paths going north-east. One can interpret these paths as trajectories of particles where time evolution is along the vertical direction. There can be several arrangements of horizontal bonds in between to rows of vertical edges. The partition function associated to the model is given by the sum over all possible configurations of the weights of the configuration

$$Z = \sum_{l \in \mathcal{C}} \prod_{i=1}^{l} \prod_{j=1}^{L} \omega^l (i, j) \tag{1.87}$$

where with $\mathcal{C}$ we denote all the possible configurations and $\omega^l_{i,j}$ is the weight of the vertex at the site (i, j) for a given configuration $l \in \mathcal{C}$. In general each vertex allowed by the ice rule has a different weight, so we should have six different weight. It turns out that if we use the fact that we use periodic boundary conditions and the system is invariant under arrow reversing (zero field condition), they are equal two by two and more precisely they are defined as

$$\omega^l (i, j) = \langle \alpha^j_i \mu^j_i | R | \alpha^j_{i+1} \mu^{j+1}_i \rangle \tag{1.88}$$

[16]There other ice-type models, for example the 8 vertex model where the ice rule is broken by adding two additional vertices. This model is connected to the XYZ spin chain and to the XY model.

where α and μ refer respectively to the horizontal and vertical edges and R is the R matrix associated to this system which is the same as (1.67) with the functions (1.68). Note that in this R matrix there are only three non zero entries which

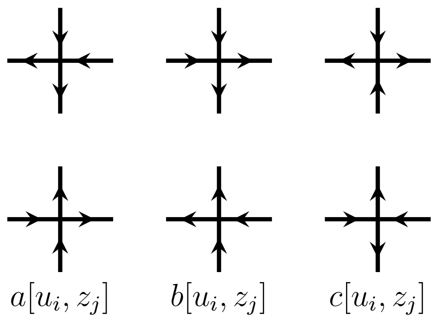


FIGURE 1.6: Six vertices with non zero weights, $a[u_i, z_j]$, $b[u_i, z_j]$ and $c[u_i, z_j]$ are defined in (1.68).

reflects what we discussed above. Thus we have six possible vertices with non zero weights as depicted in Fig.1.6. The transfer matrix T is obtained after summing over all these possible arrangements of horizontal arrows. Then T encodes this vertical evolution. In fact it is a linear operator which, acting on a row (a 2L dimensional vector) generates the next one. In the following we will see how the partition function is related to the row to row transfer matrix. The partition function (1.87) can be written explicitly for our $M \times N$ lattice in terms of the weights as

$$Z = \sum_{r=1}^{M} \sum_{m_l^r} a^{m_1+m_2} b^{m_3+m_4} c^{m_5+m_6} \tag{1.89}$$

where m_l^r is the number of vertices of type l in the r-th row. The row to row transfer matrix T should encode how each configuration at a fixed r evolves to the configuration $r+1$ so formally as $T_{\mathcal{C}_r}^{\mathcal{C}_{r+1}} = \sum a^{m_1+m_2} b^{m_3+m_4} c^{m_5+m_6}$ where the sum runs over all the vertices allowed by the orientation of the arrows in line r and $r+1$ (contribution of N terms equal to the number of columns). Thus the partition function (1.89) is

$$Z = \sum_{\mathcal{C}_1} \cdots \sum_{\mathcal{C}_M} T_{\mathcal{C}_2}^{\mathcal{C}_1} \ldots T_{\mathcal{C}_M}^{\mathcal{C}_{M-1}} T_{\mathcal{C}_1}^{\mathcal{C}_M} = \operatorname{Tr} T^M. \tag{1.90}$$

The crucial point is that since we impose periodic boundary conditions the number of arrows in a specific orientation is conserved from one row to the consecutive. As a consequence of this fact the transfer matrix has a block diagonal form, precisely as in (1.79). This allows the construction of all the states by acting with

B-operators that, in the six vertex model language are lines, to a reference state (see Fig. 1.7). The connection with the spin chain approach is that a six vertex

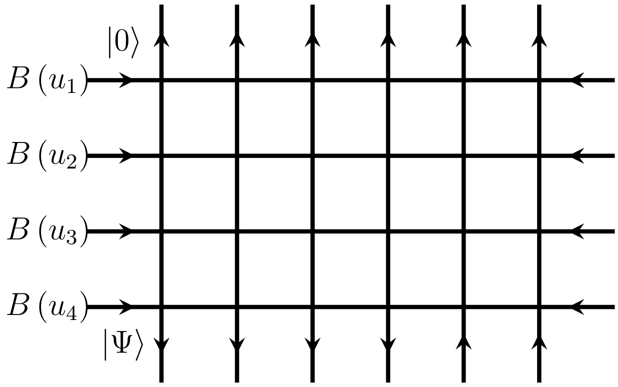


FIGURE 1.7: Action of B lines. The state $|0\rangle$ is an initial reference state. To obtain a final state $|\Psi\rangle$ we have to act with B operators. If u_i with $i = 1, \ldots, 4$ are solutions of the Bethe equations $|\Psi\rangle$ is an eigenstate.

model lattice should represent a transition from an initial to a final configuration by acting with a series of B and/or C lines which change the net spin by 1. Analogously to the spin chain case, we define a reference state $|\uparrow_{z_L}\rangle$, which we conventionally chose to be a state with all spin up and a generic state $|\uparrow_{z_{a\setminus L}}\rangle$ which denotes a state with a spin up and $L-a$ spin down.

Domain wall partition function Let us consider a square lattice $N \times N$ with domain wall boundary condition which means that the arrows on the upper and lower boundaries point into the square, and the ones on the left and right boundaries point out, see Fig. 1.8. From the graphical representation the definition in terms of B operators is clear

$$Z_N\left(\{u_N\},\{z_N\}\right) = \langle\downarrow_N| \prod_{i=1}^{N} B\left(u_i,\{z_N\}\right) |\uparrow_N\rangle \tag{1.91}$$

where it is worth noticing that the rapidities are not constrained to be solutions of the Bethe equations.

It was proven by Korepin [90] that the domain wall partition function $Z_N\left(\{u_N\},\{z_N\}\right)$ is uniquely determined by four conditions:

1. $Z_N\left(\{u_N\},\{z_N\}\right)$ is a symmetric function of the sets $\{u_N\}$ and $\{z_N\}$;
2. $Z_N\left(\{u_N\},\{z_N\}\right) = e^{-(N-1)u_i} P_{N-1}\left(e^{2u_i}\right)$ where P_{N-1} is a polynomial of order $N-1$;

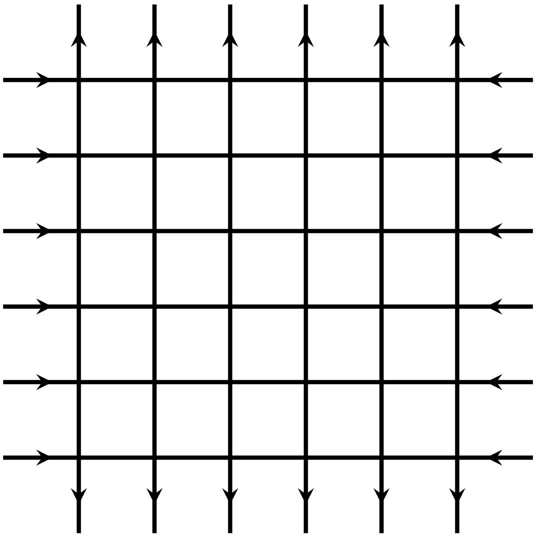


FIGURE 1.8: Domain wall partition function with $N = 6$.

3. If $u_N = z_N - \eta$ then Z_N can be written recursively as

$$Z_N\Big|_{u_N=z_N-\eta} = [\eta]\prod_{i=1}^{N-1}[u_i - z_N]\,[z_N - z_i - \eta]\,Z_{N-1} \tag{1.92}$$

where Z_{N-1} is the partition function of a lattice of size $L-1 \times L-1$;

4. $Z_1 = \gamma$ [17].

Note that the partition function is invariant under reversing all the arrows, the reversed configuration, which is made of N C-lines, is called dual. Thus the domain wall partition function is equal to the one pertaining to the dual configuration.
In 1987 [91] Izergin found the expression of the partition function for an $N \times N$ lattice with domain wall boundary conditions which satisfies all the conditions listed above:

$$Z_N\left(\{u_N\},\{z_N\}\right) = \frac{\det\left((\eta)\prod_{k\neq j}^{N}(u_k - z_j + \eta)(u_k - z_j)\right)_{1\le i,j\le N}}{\prod_{1\le i<j\le N}(u_i - u_j)(z_j - z_i)}. \tag{1.93}$$

The important feature of (1.93) is that the partition function is written as a determinant. Actually the spin chains, and the respective six vertex models, we are interested in are homogeneous, thus an interesting, which is not completely trivial,

[17] This condition is simply understood if we look at Fig.1.8 because for a 1×1 lattice only one vertex is present and it should have weight equal to $c = \eta$.

limit is when all the quantum rapidities are set to be equal, $z = z_1 = \cdots = z_N$. In this limit (1.93) becomes

$$Z_N^h\left(\{u\}_N, \{z\}\right) = \prod_{i=1}^{N}\left(\frac{u_i - z + \eta}{u_i - u_j}\right)(u_i - z)^N \det\left(\phi^{(j-1)}(u_i, z)\right)_{1\leq i,j\leq N} \quad (1.94)$$

where

$$\phi^j(u_i, z) = \frac{1}{j!}\partial_z^j\left(\frac{\eta}{(u_i - z + \eta)(u_i - z)}\right). \quad (1.95)$$

Scalar products Let us consider a lattice of size $N \times M$ with two sets of physical rapidities $\{u_N\}$ and $\{v_N\}$ and a set of quantum rapidities $\{z_M\}$. The scalar product represents how an initial state with all spins in one orientation evolves to a final state with the same spin configurations and is defined as

$$S_N\left(\{u_N\}, \{v_N\}, \{z_M\}\right) = \langle\uparrow_M|\prod_{i=1}^{N} C\left(u_i, , \{z_M\}\right)\prod_{j=1}^{N} B\left(v_j, \{z_M\}\right)|\uparrow_M\rangle \quad (1.96)$$

with graphical representation given in Fig. 1.9, we denote this configuration with N-B lines and N-C lines as $[M, N, N]$. Note that in this definition one set of

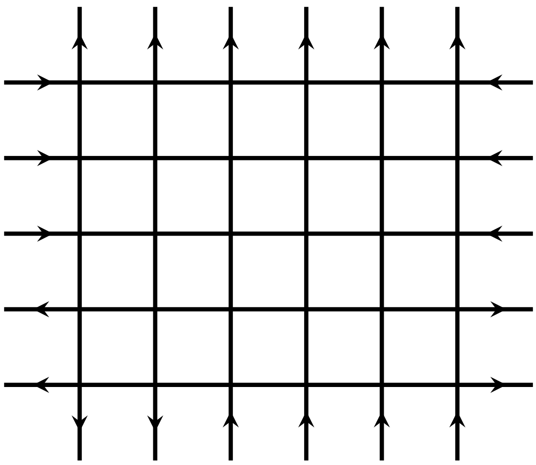


FIGURE 1.9: Scalar product S_n. In this particular example $N = 3$ (number of horizontal B-lines), $n = 2$ (number of horizontal C-lines), $\tilde{M} = 2$ and $M = 6$.

physical rapidities, f.i. $\{v_N\}$, are meant to satisfy the Bethe equations while the other set is free. We can think also to a slightly different object, denoted by S_n, which should describe the evolution of a state with all spin up $|\uparrow_M\rangle$ to a final state with $\tilde{M}$ spin down and $M - \tilde{M}$ spin up $\langle\downarrow_{\tilde{M}\setminus M}|$. By construction S_0 should be closely related to the partition function. If we look back to (1.96) the definition of S_n is straightforward

$$S_n\left(\{u_n\},\{v_N\},\{z_M\}\right) = \langle \downarrow_{\tilde{M}\setminus M} | \prod_{i=1}^{n} C\left(u_i,,\{z_M\}\right) \prod_{j=1}^{N} B\left(v_j,\{z_M\}\right) | \uparrow_M \rangle \quad (1.97)$$

where $\{u_n\}$ is a subset of $\{u_N\}$. As well as the domain wall partition function also the scalar product $S_n\left(\{u_n\},\{v_N\},\{z_M\}\right)$ is uniquely specified by four properties

1. S_n is symmetric in $\left\{z_{\tilde{M}+1},\ldots,z_M\right\}$;
2. S_n is a polynomial of degree $M-1$ with zeros at $u_n = z_i - \eta$ for $1 \le i \le \tilde{M}$;
3. if $u_n = u_{\tilde{M}+1}$, S_n is recursively given by

$$S_n\Big|_{u_n = z_{\tilde{M}+1}} = \prod_{i=1}^{M} \frac{\left(z_{\tilde{M}+1} - z_i + \eta\right)}{\left(z_{\tilde{M}+1} - z_i\right)} S_{n-1}\left(\{u_{n-1}\},\{v_N\},\{z_M\}\right) \quad (1.98)$$

4. For $n = 0$ S_0 is the domain wall partition function.

In the same way as before this is an $[M, N, n]$ configuration.

A determinant expression for S_n exists and it is given by

$$S_n = \frac{\det \mathcal{S}_n\left(\{u\}_n,\{v\}_N,\{z\}_M\right)}{\prod_{i=1}^{N}\prod_{j=1}^{\tilde{M}}\left[v_i - z_j\right]\prod_{1\le i<j\le n}\left[u_i - u_j\right]\prod_{1\le i<j\le N}\left[v_i - v_j\right]\prod_{1\le i<j\le \tilde{M}}\left[z_j - z_i\right]} \quad (1.99)$$

where

$$\mathcal{S}_n = \begin{pmatrix} f_1(z_1) & \ldots & f_1(z_{\tilde{M}}) & g_1(u_n) & \ldots & g_1(u_1) \\ \vdots & & & & & \vdots \\ f_N(z_1) & \ldots & f_N(z_{\tilde{M}}) & g_N(u_n) & \ldots & g_1(u_N) \end{pmatrix} \quad (1.100)$$

with

$$f_i(z_j) = \frac{[\eta]}{\left[u_i - z_j + \eta\right]\left[u_i - z_j\right]} \prod_{k\neq i}^{N} \frac{1}{v_k - z_j} \quad (1.101)$$

$$g_i(v_j) = \frac{[\eta]}{\left[u_i - v_j\right]}\left(\prod_{k\neq i}^{N}\frac{\left[v_j - z_k + \eta\right]}{\left[v_j - z_k + \eta\right]}\prod_{k=1}^{M}\left[u_k - v_j + \eta\right] - \prod_{k\neq i}^{N}\left[u_k - v_j - \eta\right]\prod_{k=1}^{M}\left[u - z_k\right]\right).$$

It turns out that if we start with a generic $[L, N_1, N_1]$ configuration, namely a lattice with an initial reference state with all spin up and N_1 -B lines and N_1 C lines,

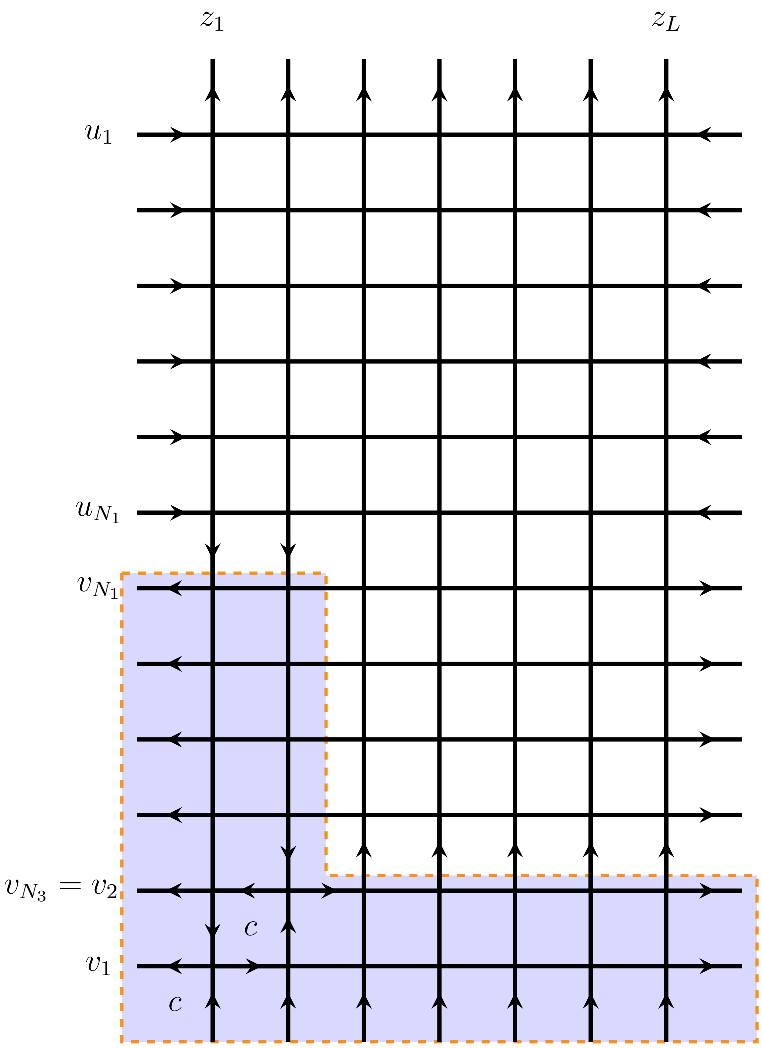


FIGURE 1.10: Freezing procedure. We start from a configuration with N_1 B-lines and N_1 C-lines. It is clear that the vertex on the left bottom can be either b or c type. If we impose the orientation in the picture and identify $v_1 = z_1$ we force that vertex to be c type. This forces all the other vertices in the bottom line to be a type. Now we can do the same with the vertex in the same diagonal, namely forcing $v_2 = z_2$. Again this vertex is c type and all the other in this line are a type. This procedure, which is called freezing, can be iterated N_3 times, with $N_3 = N_1 - N_2$. In the figure $N_1 = 6$, $N_2 = 4$ and $N_3 = 2$. By doing this, all the vertices in the light blue area are fixed to be a type. We can remove the N_3 bottom lines and we obtain another configuration which has a left corner also completely frozen.

and we fix $N_3 = N_1 - N_2$ vertices to be of c-type the partition function associated to this configuration is precisely S_n for the respective $[L, N_1, N_2]$ configuration [54, 92], see Fig. 1.10.

We need to consider two limits of (1.99):

(A) Homogeneous limit: $z = z_i$ for $i = 1, \ldots, M$

$$S_n^h = \frac{\prod_{i=1}^{N} [u_i - z + \eta]^M \det \mathcal{S}_n^h \left(\{u\}_n , \{v\}_N , \{z\}\right)}{\prod_{1\le i<j\le n} [u_j - u_i] \prod_{1\le i<j\le N} [v_i - v_j]} \tag{1.102}$$

where

$$\mathcal{S}_n^h = \begin{pmatrix} \tilde{f}_1^{(0)}(z) & \ldots & \tilde{f}_1^{(\tilde{M}-1)}(z) & g_1^h(u_n) & \ldots & g_1^h(u_1) \\ \vdots & & & & & \vdots \\ \tilde{f}_N^{(0)}(z) & \ldots & \tilde{f}_N^{(\tilde{M}-1)}(z) & g_N^h(u_n) & \ldots & g_1^h(u_N) \end{pmatrix} \tag{1.103}$$

and

$$\tilde{f}_j(z) = \frac{1}{j!}\partial_z^{(j)} f_i(z) \tag{1.104}$$

$$g_i^h(v_j) = \frac{[\eta]}{[u_i - v_j]} \left(\left(\frac{v_j - z + \eta}{v_j - z}\right)^M \prod_{k\neq i}^{N} [u_k - v_j + \eta] - \prod_{k\neq i}^{N} [u_k - v_j - \eta] \right);$$

(B) Gaudin norm: $N = n$ and $\{v\}_N = \{u\}_n$

$$\mathcal{N}(\{u_n\}, z) = \eta^n \prod_{k\neq i} \frac{u_k - u_j + \eta}{u_k - u_j} \det \Phi' \{u\}_n \tag{1.105}$$

where

$$\Phi_{ij} = -\partial_{u_j} \mathrm{Log} \left(\left(\frac{u_i - z + \eta}{u_i - z}\right)^M \prod_{\substack{k=1\\k\neq i}} \frac{u_k - u_i + \eta}{u_k - u_i - \eta} \right). \tag{1.106}$$

1.3 The large J limit

The identification of the energy of the string states with the conformal dimension of the operators in the gauge theory side beyond the supergravity approximation provides one of the non trivial test of the AdS/CFT correspondence and it allows to compare the two different regimes in situations far from the BPS sector. In the case when the string states have some large quantum numbers then they can be approximate semiclassically and they can be related to the dimensions of operators in the $\mathcal{N}$=4 SYM in non BPS sectors. In particular the energy of semiclassical string states with a large total momentum on the sphere J can be expanded in terms a parameter $\lambda' = \frac{\lambda}{J^2}$ and the coefficients of this expansion can be compared with the loop expansion of the anomalous dimension at weak coupling for operators with large J, this is the so called Frolov-Tseytlin limit [37, 38]. The expansion coefficients match the gauge theory side up to and including the second order in the expansion parameter, meaning two-loops on the gauge theory side, but the matching breaks down at three-loops [93, 94]. In [95] it is shown that the match at one-loop is not a coincidence but instead a result of the quantum corrections to the string being suppressed near the BPS point, enabling one to consider a regime where the classical action of the string is large even if one approaches weak 't Hooft coupling. The understanding of the Frolov-Tseytlin limit was further enhanced with the work of Kruczenski [38]. There it is shown how for semi-classical operators on the gauge theory side one can use a coherent state description thus enabling one to write down an effective sigma-model description (Landau Lifshitz sigma model). Hence, one can directly compare the sigma-model action for semi-classical operators on the gauge theory side to the classical sigma-model action on the string theory side in the Frolov-Tseytlin limit. In the following we briefly review the semiclassical strings and then, following [40], we describe how to compare the Landau Lifshiz sigma model emerging from the gauge theory description to the classical sigma model of the string theory.

1.3.1 Semiclassical strings

The bosonic part of the Polyakov action of the $\mathrm{AdS}_5 \times S^5$ space written in conformal gauge reads

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \, \partial^\alpha Y^A \partial_\alpha Y_A + \Lambda \left(Y^A Y^B \eta_{AB} + 1\right) + \partial^\alpha X^M \partial_\alpha X_M + \tilde{\Lambda} \left(X^M X^N \delta_{MN} - 1\right) \tag{1.107}$$

where Y^A are the embedding coordinates of $R^{4,2}$ and X^M are the embedding coordinates of R^6, Λ and $\tilde{\Lambda}$ are the Lagrange multipliers for τ and σ respectively [18]. The action (1.107) is supplemented by Virasoro constraints which, in the conformal gauge, take the form

$$\dot{Y}^A \dot{Y}_A + \dot{X}^M \dot{X}_M + Y'^A Y'_A + X'^M X'_M = 0 \tag{1.108}$$
$$\dot{Y}^A Y'_A + \dot{X}^M X'_M = 0.$$

[18] The signature of the metrics are meant to be: $\eta_{AB} = \mathrm{diag}\,(- - + \cdots +)$ and $\delta_{MN} = (+ \cdots +)$.

The six conserved charges can be written as

$$M_{AB} = -\frac{\sqrt{\lambda}}{4\pi}\int d\sigma\left(Y_A\dot{Y}_B - Y_B\dot{Y}_A\right) \tag{1.109}$$

$$\tilde{M}_{MN} = -\frac{\sqrt{\lambda}}{4\pi}\int d\sigma\left(X_M\dot{X}_N - X_N\dot{X}_M\right) \tag{1.110}$$

where the energy $E = M_{00}$, the two AdS spins are $(S_1, S_2) = (M_{12}, M_{34})$ and the three S^5 spins (J_1, J_2, J_3) are $\left(\tilde{M}_{12}, \tilde{M}_{34}, \tilde{M}_{56}\right)$. Single trace operators in the $SU(2)$ sector are dual to string states which sits in the center of AdS and are described by two spins in S^5.

1.3.2 Coherent state approach

Semiclassical operators in the $SU(2)$ sector of $\mathcal{N} = 4$ SYM theory can be described using the Landau-Lifshitz sigma model [38]. We already stressed the fact that in the planar limit the one-loop contribution to the dilatation operator of $\mathcal{N} = 4$ SYM theory can be regarded as the Heisenberg spin chain Hamiltonian which can be described by a discrete sigma model. Since quantum mechanics have been formalized coherent states have been deeply studied, see for instance [96]. The spin is essentially a quantum object having a discrete nature. On the other hand, in the classical limit a spin system is described by the classical Hamiltonian function in which the role of natural variables is played by two angles (for each spin), and these variables change continuously. The coherent state approach guarantees a continuous representation of a spin and the completeness of the set of states [19]. Let us review how to introduce this formalism. We consider gauge theory operators in the $SU(2)$ sector of $\mathcal{N} = 4$ SYM on $\mathcal{R}\times S^3$. To obtain a sigma-model description of single trace operators we introduce a coherent state $|\vec{n}\rangle$ for each site of the trace such that

$$\langle\vec{n}|\vec{\sigma}|\vec{n}\rangle = \vec{n} \tag{1.111}$$

where $\vec{\sigma}$ are the two by two Pauli matrices and $\vec{n}$ is a unit vector pointing to a point on the two-sphere parameterized as

$$\vec{n} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta) \tag{1.112}$$

[19] Actually it is even more than that since the coherent states form an overcomplete set, meaning that one of the state can be taken out and still the set remains complete.

In the limit $J \to \infty$ the Lagrangian of the Landau-Lifshitz model up to two loops reads [97]

$$\mathcal{L}_{LL} = \frac{1}{2}\sin\theta\dot{\varphi} - \frac{\lambda'}{8}(\vec{n}')^2 + \frac{\lambda'^2}{32}\left[(\vec{n}'')^2 - \frac{3}{4}(\vec{n}')^4\right] + \mathcal{O}(\lambda'^3) \tag{1.113}$$

where prime denotes derivatives with respect to the continuos variable σ which can be introduced to describe the trace in the limit $J \to \infty$. σ is periodic with period 2π therefore we map the k'th site to $\sigma = 2\pi k/J$ and we consider the field $\vec{n}(t,\sigma)$. Accordingly the discrete sum over the sites of the single trace operators is mapped to the integral $\frac{J}{2\pi}\int_0^{2\pi} d\sigma$. Moreover, in deriving (1.113) one also uses that

$$\vec{n}_{k+1} - \vec{n}_k = \exp\left(\frac{2\pi}{J}\partial_\sigma\right)\vec{n} - \vec{n} \tag{1.114}$$

It is important to note that the two-loop Lagrangian (1.113) is derived by including the effect of spin-flipped coherent state [97]. We will discuss this effect in Sec. 2.4.1.

1.3.3 String theory sigma model

Our starting point is the sigma-model for type IIB string theory on $\mathrm{AdS}_5 \times S^5$ in the regime in which it is described by the Landau-Lifshitz sigma-model [38]. We use in the following that

$$R^4 = \lambda(\alpha')^2. \tag{1.115}$$

This relates the string parameters R and α' to the 't Hooft coupling λ of $\mathcal{N} = 4$ SYM. Using this we can formulate the string theory result in terms of gauge theory variables. In the following we show how to compute the sigma-model Lagrangian up to the order λ^3. From now on we set $\alpha' = 1$ for simplicity. The metric for type IIB string theory on $\mathrm{AdS}_5 \times S^5$ in global coordinate can be written as

$$ds^2 = R^2\left[-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\,(d\Omega_3')^2 + d\zeta^2 + \sin^2\zeta\, d\alpha^2 + \cos^2\zeta\,(d\Omega_3)^2\right] \tag{1.116}$$

where the sphere is parametrized by

$$(d\Omega_3)^2 = d\psi^2 + \cos^2\psi d\phi_1^2 + \sin^2\psi d\phi_2^2 = d\psi^2 + d\phi_-^2 + d\phi_+^2 + 2\cos(2\psi)d\phi_- d\phi_+ \tag{1.117}$$

where $2\phi_\pm = \phi_1 \pm \phi_2$. The 5-form Ramond-Ramond field strength is

$$F_{(5)} = 2R^4 \left[\cosh\rho \, \sinh^3\rho \, dt \, d\rho \, d\Omega_3' + \sin\zeta \, \cos^3\zeta \, d\zeta \, d\alpha \, d\Omega_3\right] . \tag{1.118}$$

We want to consider a string which is point like in AdS and is moving non trivially on a 3-sphere contained in S^5, namely we consider the classical sigma-model on $\mathbb{R} \times S^3$. Thus we have three non trivial conserved charges, E, J_1 and J_2, which have the form

$$E = i\partial_t \ , \quad J \equiv J_1 + J_2 = -i\partial_{\phi_+} \tag{1.119}$$

and we can restrict to the region $\rho = \zeta = 0$. With these restriction the metric becomes

$$ds^2 = R^2[-dt^2 + (d\Omega_3)^2]. \tag{1.120}$$

Introducing the new angles

$$\theta \equiv 2\psi - \frac{\pi}{2} \ , \quad \varphi \equiv 2\phi_- \tag{1.121}$$

allows to write the metric for the 3-sphere in a different form namely

$$(d\Omega_3)^2 = \frac{1}{4}(d\Omega_2)^2 + \left(d\phi_+ + \frac{1}{2}\sin\theta d\varphi\right)^2 \tag{1.122}$$

where $(d\Omega_2)^2 = d\theta^2 + \cos^2\theta d\varphi^2$. It is convenient to introduce the coordinates

$$x^+ = \lambda' t \ , \quad x^- = \phi_+ - t \tag{1.123}$$

where $\lambda' = \lambda/J^2$ and we are considering the limit $\lambda' \to 0$ as in the gauge theory case. In these coordinates the expression for the charges is

$$i\partial_+ = H = \frac{E-J}{\lambda'} \ , \quad -i\partial_- = J \tag{1.124}$$

and the metric then takes the form

$$ds^2 = R^2\Big[\frac{1}{4}(d\Omega_2)^2 + \Big(2\frac{dx^+}{\lambda'} + dx^- + \omega\Big)(dx^- + \omega)\Big] \tag{1.125}$$

with $\omega = \frac{1}{2}\sin\theta d\varphi$.

The bosonic sigma-model Lagrangian and the Virasoro constraints are respectively

$$\mathcal{L} = -\frac{1}{2}h^{\alpha\beta}G_{\mu\nu}\partial_\alpha x^\mu \partial_\beta x^\nu \tag{1.126}$$

$$G_{\mu\nu}(\partial_\alpha x^\mu \partial_\beta x^\nu - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma x^\mu \partial_\delta x^\nu) = 0 \tag{1.127}$$

where $h^{\alpha\beta} = \sqrt{-\det\gamma}\gamma^{\alpha\beta}$ with $\gamma_{\alpha\beta}$ being the world-sheet metric. We define for convenience

$$A \equiv -h^{00} \ , \quad B \equiv h^{01} \ , \quad S_{\alpha\beta} \equiv G_{\mu\nu}\partial_\alpha x^\mu \partial_\beta x^\nu \tag{1.128}$$

where we used that $h^{\alpha\beta}$ has only two independent components since $\det h = -1$, thus $h^{11} = (1 - B^2)/A$. The Lagrangian and Virasoro constraints can now be written as

$$\mathcal{L} = \frac{A}{2} S_{00} - B S_{01} - \frac{1 - B^2}{2A} S_{11} \tag{1.129}$$

$$\begin{aligned} (1 + B^2) S_{00} + \frac{2B(1 - B^2)}{A} S_{01} + \frac{(1 - B^2)^2}{A^2} S_{11} = 0 \\ AB S_{00} + 2(1 - B^2) S_{01} - \frac{B(1 - B^2)}{A} S_{11} = 0. \end{aligned} \tag{1.130}$$

We make the following gauge choice

$$x^+ = \kappa\tau \tag{1.131}$$

$$2\pi p_- = \frac{\partial\mathcal{L}}{\partial\partial_\tau x^-} = \text{const.} \ , \quad \frac{\partial\mathcal{L}}{\partial\partial_\sigma x^-} = 0 \tag{1.132}$$

where κ is a constant. From (1.124) we see that τ does not give the right energy scale on the world-sheet. Therefore we introduce $\tilde{\tau} = \kappa\tau$ and use the notation

$$\dot{x}^\mu = \frac{\partial x^\mu}{\partial\tilde{\tau}} \ , \quad (x^\mu)' = \frac{\partial x^\mu}{\partial\sigma}. \tag{1.133}$$

We moreover make the following expansions of the quantities A and B

$$A = 1 + \kappa^2 A_1 + \kappa^4 A_2 + \cdots \ , \quad B = \kappa^3 B_1 + \kappa^5 B_2 + \cdots \tag{1.134}$$

This is consistent with the fact that to leading order we have that $A = 1$ and $B = 0$. We can then determine the constant κ from (1.132) and to leading order in λ' we find

$$J = \int_0^{2\pi} d\sigma p_- = \frac{R^2\kappa}{\lambda'} \tag{1.135}$$

Therefore, using (1.115), we have that $\kappa = \sqrt{\lambda'}$. We see thus that $\kappa \to 0$. We can now solve the gauge conditions as

$$\dot{x}^- = -\frac{1}{2}\sin\theta\dot{\varphi} - A_1 + \kappa^2(A_1^2 - A_2) + \mathcal{O}(\kappa^4) \ , \quad x^{-\prime} = -\frac{1}{2}\sin\theta\varphi' - \kappa^2 B_1 + \mathcal{O}(\kappa^4). \tag{1.136}$$

Inserting this in the Virasoro constraints we can now find the solution for A_1, A_2 and B_1

$$A_1 = \frac{1}{8}(\theta'^2 + \cos^2\theta\varphi'^2)\ , \quad B_1 = \frac{1}{4}(\dot\theta\theta' + \cos^2\theta\dot\varphi\varphi') \tag{1.137}$$

$$A_2 = \frac{1}{8}(\dot\theta^2 + \cos^2\theta\dot\varphi^2) - \frac{1}{128}(\theta'^2 + \cos^2\theta\varphi'^2)^2. \tag{1.138}$$

To write the gauge fixed Lagrangian

$$\mathcal{L}_g = \mathcal{L} - 2\pi\kappa p_-\dot x^- \tag{1.139}$$

we plug in $\dot x^-$, $x^{-\prime}$, A and B from (1.136) and (1.137)-(1.138). This gives an expansion in powers of λ'

$$\mathcal{L}_g = \mathcal{L}_0 + \lambda'\mathcal{L}_1 + \cdots \tag{1.140}$$

with

$$\frac{1}{R^2}\mathcal{L}_0 = \frac{1}{2}\sin\theta\dot\varphi - \frac{1}{8}(\theta'^2 + \cos^2\theta\varphi'^2) \tag{1.141}$$

$$\frac{1}{R^2}\mathcal{L}_1 = \frac{1}{8}(\dot\theta^2 + \cos^2\theta\dot\varphi^2) + \frac{1}{128}(\theta'^2 + \cos^2\theta\varphi'^2)^2 \tag{1.142}$$

From $\mathcal{L}_0$, one gets the energy at the order λ' as can be seen from (1.124). From $\mathcal{L}_1$ therefore one obtains the energy at order λ'^2 and so on. Here we only showed explicitly how to solve the sigma-model up to the order λ'^2, corresponding to $\mathcal{L}_1$, but the computation can be easily extended to the next order. However, it is convenient to introduce a more suitable notation in terms of the following parameterization

$$\vec n = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta) \tag{1.143}$$

where $\vec n$ is a unit vector pointing to a point in the two-sphere. Using this notation we can rewrite the expressions (1.141) and (1.142) as

$$\frac{1}{R^2}\mathcal{L}_0 = \frac{1}{2}\sin\theta\dot\varphi - \frac{1}{8}(\vec n')^2\ , \quad \frac{1}{R^2}\mathcal{L}_1 = \frac{1}{8}\dot{\vec n}^2 + \frac{1}{128}(\vec n')^4. \tag{1.144}$$

One should now make a field redefinition that removes the time derivatives in the Lagrangian. It has been shown [97] that, at the order we are working, this field redefinition corresponds to evaluating $\mathcal{L}_1$ on-shell, *i.e.* to substitute in the solution of the EOMs from $\mathcal{L}_0$ to get rid of the time derivatives. From $\mathcal{L}_0$ we find the EOMs

$$2\vec n \times \dot{\vec n} = -\vec n'' - \vec n(\vec n')^2 \tag{1.145}$$

We compute from this

$$4\dot{\vec{n}}^2 = (\vec{n}'')^2 - (\vec{n}')^4 \tag{1.146}$$

Thus the on-shell $\mathcal{L}_1$ is

$$(\mathcal{L}_1)_{\text{on}} = \frac{1}{32}(\vec{n}'')^2 - \frac{3}{128}(\vec{n}')^4 \tag{1.147}$$

and the field redefined gauge fixed Lagrangian is

$$\mathcal{L}_g = \mathcal{L}_0 + \lambda'(\mathcal{L}_1)_{\text{on}} + \cdots \tag{1.148}$$

giving

$$\frac{1}{R^2}\mathcal{L}_g = \frac{1}{2}\sin\theta\dot{\varphi} - \frac{1}{8}(\vec{n}')^2 + \frac{\lambda'}{32}\Big[(\vec{n}'')^2 - \frac{3}{4}(\vec{n}')^4\Big] + \mathcal{O}(\lambda'^2). \tag{1.149}$$

We can now proceed in the same way and include the next order in the computation. Also in this case we should perform a field redefinition to remove the time derivative from the λ'^2 correction to the Lagrangian [97, 98]. The final result is

$$\begin{aligned}\frac{1}{R^2}\mathcal{L}_g &= \frac{1}{2}\sin\theta\dot{\varphi} - \frac{1}{8}(\vec{n}')^2 + \frac{\lambda'}{32}\Big[(\vec{n}'')^2 - \frac{3}{4}(\vec{n}')^4\Big] \\ &- \frac{\lambda'^2}{64}\Big[(\vec{n}''')^2 - \frac{7}{4}(\vec{n}')^2(\vec{n}'')^2 - \frac{25}{2}(\vec{n}'\vec{n}'')^2 + \frac{13}{16}(\vec{n}')^6\Big] + \mathcal{O}(\lambda'^3)\end{aligned} \tag{1.150}$$

Note that, as we advertise at the beginning, the first two terms of (1.150) precisely coincide with the leading terms of (1.113). This is the main point that we want to stress.

1.4 The $\text{AdS}_4/\text{CFT}_3$ correspondence

In this section we will move to another expression of the duality: the $\text{AdS}_4/\text{CFT}_3$ correspondence. This formulation of the correspondence, put forth by Aharony, Bergman, Jafferis and Maldacena in 2008, states an equivalence between type IIA string theory on $\text{AdS}_4 \times \mathbb{C}P^3$ and a $\mathcal{N} = 6$ superconformal Chern-Simons theory in three dimensions [4]. In the following we will briefly review this form of the correspondence pointing out the differences with respect to the $\text{AdS}_5/\text{CFT}_4$ oldest sibling, especially focusing on its integrability.

1.4.1 ABJM theory

The theory ABJM theory is a three-dimensional superconformal Chern-Simons matter theory with gauge group $U(N)_k \times U(N)_{-k}$ where $\pm k$ are the levels of the gauge groups pertaining the two Chern-Simons theories. The field content of ABJM consists on four complex scalars $\mathcal{Z}^a$, four Dirac fermions Ψ_a and two gauge fields, associated to the two $U(N)$ group, A_μ and $\hat{A}_\mu$ where a is an $SU(4)$ symmetry index which runs from 1 to 4. The scalar and fermion fields and their adjoints transform in the fundamental and antifundamental representation of $SU(4)$ respectively. The R symmetry group is $SU(4)$, thus the scalar fields can be grouped as

$$\mathcal{Z}^a = (Z_1, Z_2, \bar{W}_1, \bar{W}_2) \ , \quad \bar{\mathcal{Z}}_a = (\bar{Z}_1, \bar{Z}_2, W_1, W_2) \tag{1.151}$$

where Z_1, Z_2 transform in the $N \times \bar{N}$ representation and W_1, W_2 in the $\bar{N} \times N$ representation of the gauge group. The conformal dimension of all the scalars is $\Delta = 1/2$. The covariant derivatives can be introduced in the following way

$$D_\mu Z = \partial_\mu Z + iA_\mu Z - iZ\hat{A}_\mu \tag{1.152}$$

$$D_\mu W = \partial_\mu W + i\hat{A}_\mu W - iW\bar{A}_\mu . \tag{1.153}$$

with scaling dimension +1.

Symmetries The ABJM theory is both conformal and supersymmetric, the full symmetry global group is the orthosymplectic supergroup $OSp(6|4)$ and it has $U(N)_k \times U(N)_{-k}$ gauge symmetry. When $k = 1, 2$, the supersymmetry is enhanced to $\mathcal{N} = 8$ but we will see that this is not problematic because we are interested only in large k. The bosonic sector of $OSp(6|4)$ is $SO(3,2) \times SU(4)$ where $SO(3,2)$ is the conformal group in three dimensions while $SU(4)$ is the R symmetry group and, as we have seen for the AdS$_5$/CFT$_4$ correspondence, gauge invariant primary operators are classified according to the quantum numbers associated to this bosonic sector namely

$$\underbrace{E, S,}_{SO(2)\times SO(3)\subset SO(3,2)} \qquad \underbrace{J_1, J_2, J_3}_{\text{Cartan generators of } SU(4)} \ . \tag{1.154}$$

The fermionic sub sector is formed by the $\mathcal{N} = 6$ transformation.

Let us stress one of the main differences with the AdS$_5$/CFT$_4$ correspondence. The latter is maximally supersymmetric and it has the maximal number of 32 supercharges while the former admits only 24 supercharges and thus it is not maximally supersymmetric.

Parameters The ABJM theory, as the $\mathcal{N}$=4 SYM, is described by two parameters: the rank of the gauge group N and the Chern-Simons level k. They are both integers but they are combined in the 't Hooft coupling $\lambda = \frac{N}{k}$ which, for large k and N, is continuous and the theory is weakly coupled when $\lambda \ll 1$. Since the ABJM theory admits an expansion in $\frac{1}{N}$ at fixed λ it is meaningful to define also a planar limit in analogy to the AdS$_5$/CFT$_4$ case.

1.4.2 Type IIA string theory on $AdS_4 \times \mathbb{C}P^3$

In general the ABJM theory is the world-volume theory of a stack of Nk $M2$ branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ singularity. Its near horizon geometry is M-theory compactified on AdS$_4 \times S^7/\mathbb{Z}_k$ with N units of four form flux through AdS$_4$. When k is large quite remarkably this gravitational theory is well approximated by type IIA string theory on AdS$_4 \times \mathbb{C}P^3$. We will analyze these aspects in details in the following. In this case the string coupling constant and the string tension are related to N and k by

$$g_s = \left(\frac{2^5\pi^2 N}{k^5}\right)^{1/4} \qquad \frac{R^2}{l_s^2} = 4\pi\sqrt{2\lambda} \tag{1.155}$$

where $R = R_{\mathbb{C}P^3} = 2R_{AdS}$. Again the 't Hooft coupling discriminates the regime in which the string theory is weakly or strongly coupled and when quantum corrections become important. Thus in the limit when $\lambda \gg 1$ and $k^5 \gg N$ we can safely state that the ABJM theory is dual to type IIA string theory on $AdS_4 \times \mathbb{C}P^3$.

Symmetries As we have seen for the AdS$_5$/CFT$_4$ correspondence also in this case the global symmetries match between the gauge and the string theory side. In fact the AdS$_4 \times \mathbb{C}P^3$ isometry group is $SO(3,2) \times SU(4)$. Once again we have that the charges associated are precisely (1.154) and the first two refer to the AdS space, being E and S associated to the time translation and the spin in the AdS$_4$, while J_1, J_2 and J_3 come from the $\mathbb{C}P^3$ and they are three independent Cartan generators of the $\mathbb{C}P^3$ momenta .

1.4.3 Spin chain approach

In this section we review the same identification that we exploit for the $\mathcal{N}$=4 SYM case between the dilatation operator and the spin chain Hamiltonian, as well as the

Bethe ansatz techniques which allow findings the spectrum of the ABJM theory. We will work mostly on the $SU(2) \times SU(2)$ sector reviewing how the algebraic Bethe ansatz works in this case.

Operators The matter fields in the ABJM theory transform under the bifundamental representation of the gauge group $U(N)\times U(N)$ thus gauge invariant single trace operators are constructed by taking the trace of even number of fields, alternating fields transforming under the $N\times\bar{N}$ representation and $\bar{N}\times N$. In complete analogy with the $\mathcal{N}$=4 SYM such operators can be interpreted as alternating spin chains [99]. We restrict to the case of operators made only of scalar fields which are in the form

$$\mathcal{O} = C^{b_1 b_2\cdots b_n}_{a_1 a_2\cdots a_n}\,\mathrm{tr}(\mathcal{Z}^{a_1}\bar{\mathcal{Z}}_{b_1}\cdots\mathcal{Z}^{a_n}\bar{\mathcal{Z}}_{b_n}). \tag{1.156}$$

The bare dimension of this operator is n. Chiral primary operators are operators for which the tensor $C^{b_1 b_2\cdots b_n}_{a_1 a_2\cdots a_n}$ is symmetric in upper as well as lower indices and, in addition, is traceless when tracing over one upper and one lower index. The two-loop planar dilatation operator in this sector is integrable and can be identified with the Hamiltonian of an alternating spin chain of length $2n$ with the spins in the odd sites transforming in the fundamental and the spins in the even sites in the anti-fundamental representation of $SU(4)$.

Algebraic Bethe ansatz Given the integrability of the model we can use the algebraic Bethe ansatz techniques to solve it. In this case we have a richer structure because we have to specify in which representation the vector spaces are, fundamental or antifundamental. Thus one needs introducing two sets of spectral parameters $\{u_o\}_L$ and $\{u_e\}_L$, associated respectively to the spin chain sitting in the odd and even sites and with cardinality L each. There are four R-matrices [99]

$$\begin{aligned}
R_{ab} &: V_a\otimes V_b \longrightarrow V_a\otimes V_b, & R_{ab}(u_o) &= u_o\, I_a\otimes I_b + \eta P_{ab,} \\
R_{\bar{a}\bar{b}} &: V_{\bar{a}}\otimes V_{\bar{b}} \longrightarrow V_{\bar{a}}\otimes V_{\bar{b}}, & R_{\bar{a}\bar{b}}(u_e) &= u_e\, I_{\bar{a}}\otimes I_{\bar{b}} + \eta P_{\bar{a}\bar{b}}, \\
R_{a\bar{b}} &: V_a\otimes V_{\bar{b}} \longrightarrow V_a\otimes V_{\bar{b}}, & R_{a\bar{b}}(u_o) &= u_o\, I_a\otimes I_{\bar{b}} + K_{a\bar{b}}, \\
R_{\bar{a}b} &: V_{\bar{a}}\otimes V_b \longrightarrow V_{\bar{a}}\otimes V_b, & R_{\bar{a}b}(u_e) &= u_e\, I_{\bar{a}}\otimes I_b + K_{\bar{a}b}.
\end{aligned} \tag{1.157}$$

where V_a and $V_{\bar{a}}$ are the vector spaces of the fundamental and anti-fundamental representation respectively. The operator I is the identity operator, P is the permutation, and K is the $SU(4)$ trace and η is the shift. From these R-matrices one constructs two monodromy matrices, one for sites of the fundamental representation and one for sites of the anti-fundamental representation

$$M_a(u_{a_o}) = R_{a1}(u_{a_o})R_{a\bar{1}}(u_{a_o})...R_{aJ}(u_{a_o})R_{a\bar{J}}(u_{a_o}), \tag{1.158}$$

$$M_{\bar{a}}(u_{a_e}) = R_{\bar{a}1}(u_{a_e})R_{\bar{a}\bar{1}}(u_{a_e})...R_{\bar{a}J}(u_{a_e})R_{\bar{a}\bar{J}}(u_{a_e}). \tag{1.159}$$

1.4.4 $SU(2) \times SU(2)$ sector

Minahan and Zarembo [99] have found that the $SU(2) \times SU(2)$ sector is a particular interesting sub sector of $SU(4)$ which is not mixed by anomalous dimension matrix. This sector is obtained by considering operators made only of two scalars transforming in two separate $SU(2)$ in each of the two $SU(4)$ multiplets in (1.151). We consider the scalars $Z_{1,2}$ in $\mathcal{Z}^a$ and $W_{1,2}$ in $\bar{\mathcal{Z}}_a$ thus (1.156) becomes

$$\mathcal{O} = C^{j_1 j_2 \cdots j_J}_{i_1 i_2 \cdots i_J} \mathrm{tr}(Z_{i_1} W_{j_1} \cdots Z_{i_J} W_{j_J}). \tag{1.160}$$

In this case the spin chain picture is simpler: if we think for instance to Z_1 and W_1 as spin up and Z_2 and W_2 as spin down it is clear that the alternating spin chain can be viewed as two decoupled $XXX_{1/2}$ Heisenberg spin chains, one on the odd and one on the even sites. The spin chains are only coupled by the momentum constraint which reflects the fact that our initial operator is described with one trace, invariant under cyclic permutation.

When restricting to the $SU(2) \times SU(2)$ sector, the action of the trace operator trivializes. Then (1.157) (in its inhomogeneous version) gives rise to R-matrices[20]

$$R_{ab}(u_o, z_o) = [u_o - z_o] \begin{pmatrix} \frac{[u_o-z_o+\eta]}{[u_o-z_o]} & 0 & 0 & 0 \\ 0 & 1 & \frac{[\eta]}{[u_o-z_o]} & 0 \\ 0 & \frac{[\eta]}{[u_o-z_o]} & 1 & 0 \\ 0 & 0 & 0 & \frac{[u_o-z_o+\eta]}{[u_o-z_o]} \end{pmatrix}_{ab} \equiv [u_o - z_o]\,\mathcal{R}_{ab}, \tag{1.161}$$

$$R_{\bar{a}\bar{b}}(u_e, z_e) = [u_e - z_e] \begin{pmatrix} \frac{[u_e-z_e+\eta]}{[u_e-z_e]} & 0 & 0 & 0 \\ 0 & 1 & \frac{[\eta]}{[u_e-z_e]} & 0 \\ 0 & \frac{[\eta]}{[u_e-z_e]} & 1 & 0 \\ 0 & 0 & 0 & \frac{[u_e-z_e+\eta]}{[u_e-z_e]} \end{pmatrix}_{\bar{a}\bar{b}} \equiv [u_e - z_e]\,\mathcal{R}_{\bar{a}\bar{b}}. \tag{1.162}$$

The remaining two are

$$R_{a\bar{b}}(u_o, z_e) = [u_o - z_e]\, I, \tag{1.163}$$

$$R_{\bar{a}b}(u_e, z_o) = [u_e - z_o]\, I. \tag{1.164}$$

where we introduce the auxiliary rapidities $\{z\}_{2L}$ which can be split for convenience in two sets $\{z_o\}_L$ and $\{z_e\}_L$ with obvious notation. Note that $\mathcal{R}_{ab}$ and $\mathcal{R}_{\bar{a}\bar{b}}$ are of the same form of (1.67), which mirrors the fact that we are dealing with two independent spin chains, one living in the odd and the other in the even sites. $R_{a\bar{b}}(u_o, z_e)$ and $R_{\bar{a}b}$ are proportional to the identity and make the decoupling of the two spin chains possible. With the same spirit we can construct the monodromy matrices

$$M_a(u_{a_o}, \{z_o, z_e\}_J) = \left(\prod_{i=1}^{J}[u_{a_o} - z_{i_o}][u_{a_o} - z_{i_e}]\right) \mathcal{R}_{a1}(u_{a_o}, z_{1_o}) \ldots \mathcal{R}_{aJ}(u_{a_o}, z_{J_o}), \tag{1.165}$$

$$M_{\bar{a}}(u_{a_e}, \{z_o, z_e\}_J) = \left(\prod_{i=1}^{J}[u_{a_e} - z_{i_o}][u_{a_e} - z_{i_e}]\right) \mathcal{R}_{\bar{a}\bar{1}}(u_{a_e}, z_{1_e}) \ldots \mathcal{R}_{\bar{a}\bar{J}}(u_{a_e}, z_{J_e}). \tag{1.166}$$

[20] Here R_{ab} is expressed in the basis $(|\uparrow_a\rangle \otimes |\uparrow_b\rangle, |\uparrow_a\rangle \otimes |\downarrow_b\rangle, |\downarrow_a\rangle \otimes |\uparrow_b\rangle, |\downarrow_a\rangle \otimes |\downarrow_b\rangle)$ and similarly for the other three.

Notice that (as usual) the indices a and $\overline{a}$ refer to auxiliary spaces. We see that up to trivial pre-factors we get one monodromy matrix which only involves $\mathcal{R}$-matrices with fundamental indices and one monodromy matrix which only involves $\mathcal{R}$-matrices with anti-fundamental indices. Let us write $M_a(u_{a_o}, \{z_o, z_e\}_J)$ in the following way

$$\begin{aligned} M_a(u_{a_o}, \{z_o, z_e\}_J) &= \begin{pmatrix} A_o(u_{a_o}, \{z_o, z_e\}_J) & B_o(u_{a_o}, \{z_o, z_e\}_J) \\ C_o(u_{a_o}, \{z_o, z_e\}_J) & D_o(u_{a_o}, \{z_o, z_e\}_J) \end{pmatrix}_a \\ &= \left(\prod_{i=1}^{J} [u_{a_o} - z_{i_o}][u_{a_o} - z_{i_e}]\right) \begin{pmatrix} \mathcal{A}_o(u_{a_o}, \{z_o, z_e\}_J) & \mathcal{B}_o(u_{a_o}, \{z_o, z_e\}_J) \\ \mathcal{C}_o(u_{a_o}, \{z_o, z_e\}_J) & \mathcal{D}_o(u_{a_o}, \{z_o, z_e\}_J) \end{pmatrix}_a \end{aligned} \tag{1.167}$$

and similarly for $M_{\overline{a}}(u_{a_e}, \{z_o, z_e\}_J)$. Then we define the reference state $|\uparrow_{z_N}\rangle$ as all spins up, i.e. $|\uparrow_{z_{2J}}\rangle = |\uparrow_{z_{1_o}}\rangle \otimes |\uparrow_{z_{1_e}}\rangle \otimes ... \otimes |\uparrow_{z_{J_o}}\rangle \otimes |\uparrow_{z_{J_e}}\rangle$ and from the usual constructions of the algebraic Bethe ansatz for the $SU(2)$ spin chain it follows that we can create an eigenstate with respectively j_1 spins at even sites flipped and j_2 spins at odd sites flipped as follows

$$\prod_{i=1}^{j_1} B_e(u_{i_e}, \{z_o, z_e\}_J) \prod_{i=1}^{j_2} B_o(u_{i_o}, \{z_o, z_e\}_J) |\uparrow_{z_{2J}}\rangle, \tag{1.168}$$

where we have used that B operators pertaining to even and odd sites and among themselves commute and where we have to require that $\{u_o\}$ and $\{u_e\}$ independently satisfy $SU(2)$ Bethe equations. Note that a similar expression holds for final states namely

$$\langle\uparrow_{z_{2J}} | \prod_{i=1}^{j_1} C_e(u_{i_e}, \{z_o, z_e\}_J) \prod_{i=1}^{j_2} C_o(u_{i_o}, \{z_o, z_e\}_J). \tag{1.169}$$

1.4.5 Six vertex model in ABJM

In analogy for what we have seen in the case of $\mathcal{N} = 4$ SYM, it is possible to have a description in terms of the six vertex model formalism for this case as well. The entries of the $\mathcal{R}$ matrices are the weights of the vertices in the six vertex model. In this case we have 20 different vertices that we depict in figure 1.11 with weights

$$a[u_i, z_j] = \frac{u_i - z_j + \eta}{u_i - z_j}, \qquad c[u_i, z_j] = \frac{\eta}{u_i - z_j}, \tag{1.170}$$

$$b[u_i, z_j] = d[u_{e_i}, z_{o_j}] = d'[u_{o_i}, z_{e_j}] = 1. \tag{1.171}$$

The most important point is that the weight of the mixed vertices, intersection of a blue and red line, is equal to 1. This fully reflects what we already stressed namely that we have two decoupled copies of $SU(2)$ six vertex model and indeed we have a set of red-red (blue-blue) vertices corresponding to the spin chain sitting in the odd (even) sites with weights equal to the ones that we have in the $SU(2)$ case of $\mathcal{N} = 4$ SYM. The normalized entries of the M matrix are precisely what we represented in Fig. 1.13, more precisely we depict the ones pertaining to the odd chain, if instead the horizontal line is blue than we refer to the entries of M_e.

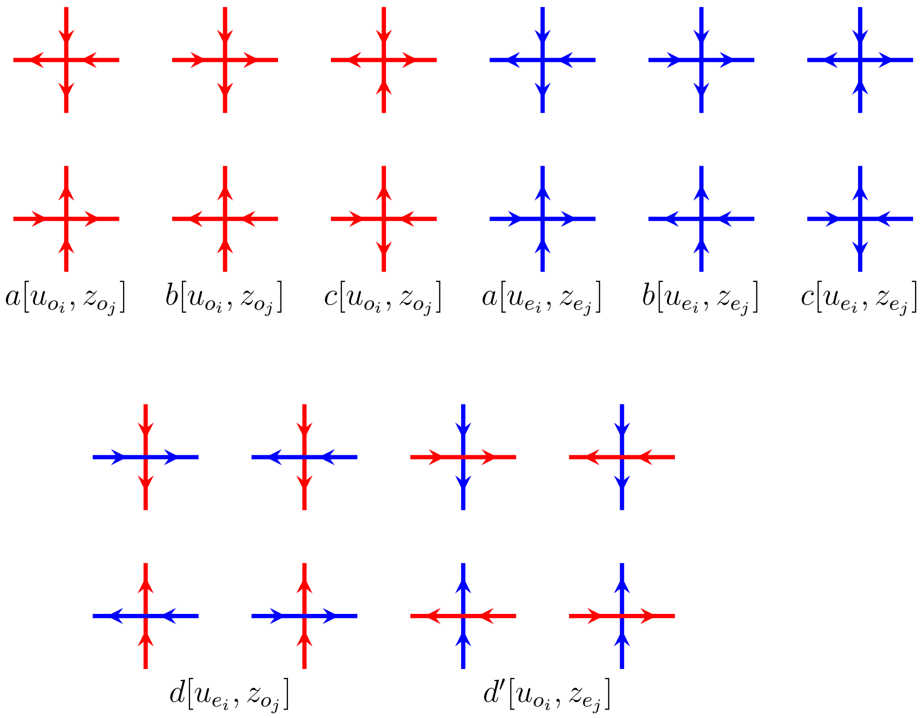


FIGURE 1.11: Possible vertices with non-zero weights

Thus we can define a transition apltitude in this way

$$Z_{2J}(\{u_o, u_e\}_J, \{z_o, z_e\}_J) = \langle \downarrow_{z_{2J}} | \prod_{i=1}^{J} \mathcal{B}_e(u_{i_e}, \{z_o, z_e\}_J) \prod_{i=1}^{J} \mathcal{B}_o(u_{i_o}, \{z_o, z_e\}_J) | \uparrow_{z_{2J}} \rangle. \tag{1.172}$$

This transition amplitude can be understood as a domain wall partition function for a vertex model as shown in figure 1.12.

Using the fact that the mixed vertices have unit weights and that the B operators commute among themselves we have that the partition function of the domain

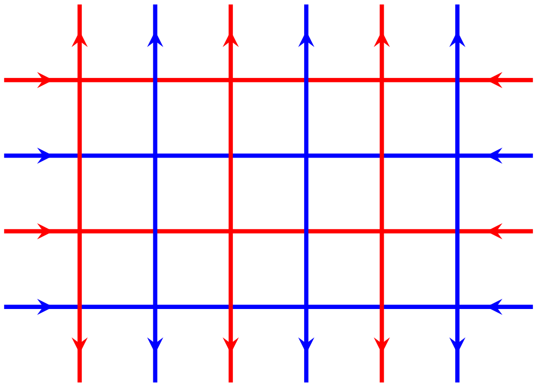


FIGURE 1.12: A domain wall partition function

wall configuration of the type shown in Fig. 1.12 can be written as

$$Z_{2J}(\{u_o, u_e\}_J, \{z_o, z_e\}_J) = Z_J(\{u_o\}_J, \{z_o\}_J) Z_J(\{u_e\}_J, \{z_e\}_J), \tag{1.173}$$

where $Z_J(\{u\}_J, \{z\}_J)$ is a domain wall partition function of the 6-vertex model on a lattice of size $J \times J$ connecting an initial state with all arrows pointing upwards to a final state with all arrows pointing down. Notice that obviously this domain wall partition function can be written in a determinant form.

In analogy with Sec. 1.2.4 we can introduce a scalar product defined as

$$\begin{aligned} &S[\{u_o, u_e\}_{N_1}, \{v_o, v_e\}_{N_2}, \{z_o, z_e\}_L] = \\ &= \langle \downarrow_{z_{N_3/L}} | \prod_{i=1}^{N_2} C_e(u_{i_e}, \{z_o, z_e\}_L) C_o(u_{i_o}, \{z_o, z_e\}_L) \prod_{j=1}^{N_1} B_e(v_{j_e}, \{z_o, z_e\}_L) B_o(v_{j_o}, \{z_o, z_e\}_L) | \uparrow_{z_L} \rangle \end{aligned} \tag{1.174}$$

where $N_3 = N_1 - N_2 > 0$ and

$$\langle \downarrow_{z_{N_3/L}} | = \langle \downarrow_{z_{o_1}} | \otimes \langle \downarrow_{z_{e_1}} | \otimes \cdots \otimes \langle \downarrow_{z_{o_{N_3}}} | \otimes \langle \downarrow_{z_{e_{N_3}}} | \otimes | \uparrow_{z_{0_{N_3+1}}} \rangle \otimes | \uparrow_{z_{e_{N_3+1}}} \rangle \otimes \cdots \otimes | \uparrow_{z_{o_L}} \rangle \otimes | \uparrow_{z_{e_L}} \rangle \tag{1.175}$$

which can be written as the product of two scalar product of a single $SU(2)$ spin chain as defined in (1.97). It is possible to prove that an object defined as (1.174) satisfy all the properties characterising a scalar product, see appendix A for details.

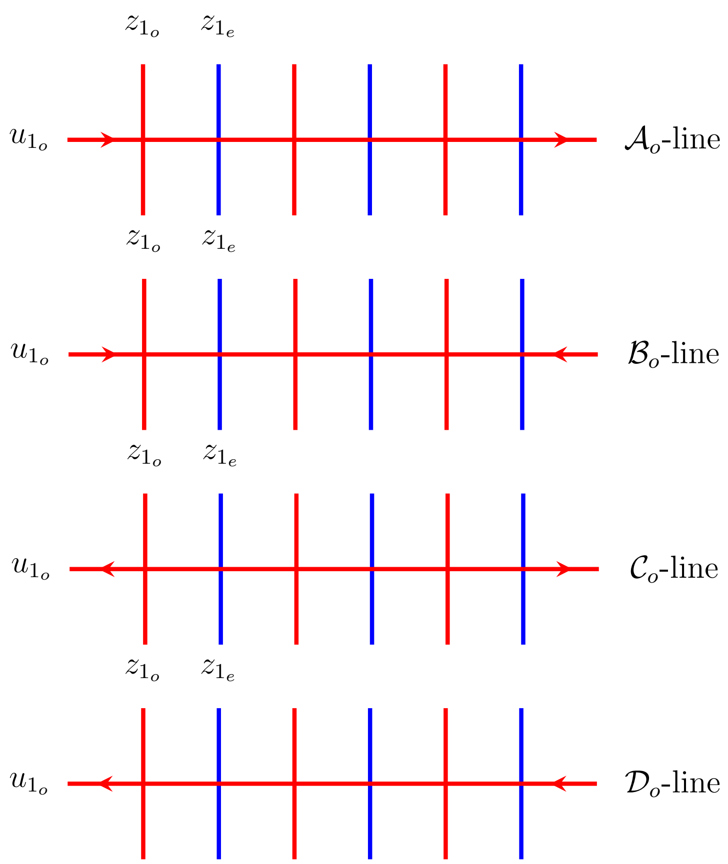


FIGURE 1.13: Horizontal lines.

Chapter 2

Three point functions with semiclassical methods

Contents

2.1 Introduction

The two building blocks containing all the dynamical information of a conformal field theory are the set of anomalous dimensions and the three point function coefficients. On the other hand conformal symmetry fixes completely the spatial dependence of the two and three point functions of primary operators. As we have seen the anomalous dimensions of primary operators are not directly read from the two point functions but they are obtained as eigenvalues of the dilatation operator or from the energy of the corresponding string states. Nowadays with the massive help of integrability techniques we can safely say that in principle we know the full set of anomalous dimensions of primary operators in $\mathcal{N} = 4$ SYM [82–84, 86] and in the ABJM theory [86, 100, 101]. Nevertheless the state of the art on the computation of three point structure constant is still far from a complete and general resolution. Let us give an overview for the case of the AdS_5/CFT_3 correspondence. Historically right after the Witten prescription [3] for the computation of correlation functions was put forth, three point functions of specific class of operators have been studied on both sides of the correspondence . We already reviewed the method for computing correlation functions of operators corresponding to scalar fields in AdS. The central points are that the partition function of the string theory on AdS with some specific boundary conditions can be identified with the generating functional of the boundary conformal field theory and that in the classical approximation the AdS partition function, obtained by inserting the solutions of the classical field equations, reduces to the one of the classical supergravity.

Three point functions of chiral primaries Three point functions of chiral primary operators have been computed in [2, 3, 102–105]. Note that some of these correlators have been computed also at weak coupling by free field theory calculation. The most interesting computation in this setting has been [106]. This paper deals with finding three point functions of chiral primary operators both on the gauge and on the gravity side in the planar limit. It turns out that this quantity is protected at any loop order, it does not depend on the coupling constant λ. The two computations of such three point functions are performed at different values of the coupling constant λ so in general there is no reason to expect that such computations should give the same answer. The fact that they did indeed agree, strongly suggests that these three point functions are actually independent of the coupling constant. In other words, that there should exist a non-renormalization theorem for three point functions of chiral primaries in superconformal field theories with sufficient amount of supersymmetry. A proof, which relies on the formalism of the analytic superspace, of such non-renormalization theorem is given for $\mathcal{N} = 4$ SYM in [107–112]. Note that only two and three point functions of such operators are protected, higher point functions do receive quantum corrections.

Three point functions of BMN Subsequently in [113] two and three point functions of protected BMN operators have been computed by matrix model techinques in the $\mathcal{N}$=4 SYM. The Berenstein, Maldacena and Nastase limit (BMN) [114] amounts of considering operators belonging to a sector of $\mathcal{N}$=4 SYM with R charge J which scale with N as

$$N \to \infty \quad J \to \infty \quad \text{with} \quad g_{CFT} \text{ and } \frac{J^2}{N} \text{ fixed.} \tag{2.1}$$

This limit can resemble the 't Hooft limit but actually it is different. In fact considering (2.1) one can see that λ is infinite and in general perturbative calculations, which come as a λ expansion, in the gauge theory side seem untreatable. However for BPS operators this is not true because they do not acquire quantum corrections, such operators in the large J limit are described by a trace of a long sequence of the same type of fields, for instance Z fields. The key point of [114] has been to slightly violate the BPS condition, namely insert in the long sequence of Z fields a small number of impurities. Two and three point functions of such operators have been computed [113] and it turns out that they receive quantum corrections in an effective loop counting parameter

$$\lambda' = \frac{\lambda}{J^2} \tag{2.2}$$

which is finite in the limit (2.1). Actually if one introduces a genus counting parameter

$$g_s = \frac{J^2}{N} \tag{2.3}$$

a new double scaling limit can be considered [113, 115] in which the two independent expansion parameters are λ' and g_s.

The simplest examples BMN operators are obviously single-trace operators built from a single complex scalar field Z, namely

$$\mathcal{O}^J = \operatorname{Tr} Z^J. \tag{2.4}$$

Using a Gaussian complex matrix model their two and three point functions have been calculated exactly [113] and read

$$\langle \mathrm{Tr}\, Z^J \;\mathrm{Tr}\, \bar{Z}^J \rangle = \frac{1}{J+1}\left\{\frac{\Gamma(N+J+1)}{\Gamma(N)} - \frac{\Gamma(N+1)}{\Gamma(N-J)}\right\} \tag{2.5}$$

$$= J\, N^J \left\{1 + \begin{pmatrix} J+1 \\ 4 \end{pmatrix} \frac{1}{N^2} + \dots\right\}. \tag{2.6}$$

$$\begin{aligned}
&\langle \mathrm{Tr}\, Z^J \;\mathrm{Tr}\, Z^K \,\mathrm{Tr}\, \bar{Z}^{J+K} \rangle \\
&= \frac{1}{J+K+1}\left\{\frac{\Gamma(N+J+K+1)}{\Gamma(N)} - \frac{\Gamma(N+J+1)}{\Gamma(N-K)}\right. \\
&\quad \left. + \frac{\Gamma(N+1)}{\Gamma(N-J-K)} - \frac{\Gamma(N+K+1)}{\Gamma(N-J)}\right\} \\
&= N^{J+K-1} J\, K\, (J+K) \\
&\quad \times \left\{1 + \frac{1}{3!N^2}\begin{pmatrix} K+J-1 \\ 2 \end{pmatrix}\left[\begin{pmatrix} K \\ 2 \end{pmatrix} + \begin{pmatrix} J \\ 2 \end{pmatrix} - 1\right] + \dots\right\}.
\end{aligned} \tag{2.7}$$

Here we have left out the trivial dependence on space-time coordinates and the 't Hooft coupling constant. Hence if we properly normalize the three point function we get for the structure constant

$$c_{J,K,K+J} \equiv \frac{\langle \mathcal{O}^J \mathcal{O}^K \bar{\mathcal{O}}^{J+K} \rangle}{\sqrt{\langle \mathcal{O}^J \bar{\mathcal{O}}^J \rangle \langle \mathcal{O}^K \bar{\mathcal{O}}^K \rangle \langle \mathcal{O}^{J+K} \bar{\mathcal{O}}^{J+K} \rangle}} \tag{2.8}$$

$$= \frac{1}{N}\sqrt{J\, K\, (J+K)}\left[1 + \mathcal{O}\left(\frac{1}{N^2}\right)\right]. \tag{2.9}$$

Note that $c_{J,K,K+J}$ is defined as (1.46). Eq. (2.9) is the well known expression for the three-point function of three chiral primaries, which was computed in [106]. This object can also be viewed as a two point function of a single trace operator and a double trace operator (since the contractions needed are the same in both cases) and we notice the well-known fact that single and multi-trace operators are orthogonal to the leading order in $\frac{1}{N}$ provided J is not too big. We will come back later on this point, also providing details about such matrix model computations in Sec. 2.3.

Spin chain approach In [49, 50, 116] a spin chain approach for the calculation of three point functions has been proposed. The idea has been to use the technology of integrable spin chains to the calculation of Yang-Mills correlation functions by expressing them in terms of matrix elements of spin operators on the corresponding spin chain, thus the structure constants for primary operators in the $SO(6)$ scalar sector of $\mathcal{N}=4$ SYM can be written in terms of the expectation values of some specific spin chain operators. This method is efficient because it simplifies the computation at tree level but most interestingly at loop order since it allows to relate the loop corrections to expectation values of some operators. There are subtleties related to the right renormalization group invariant quantity which characterizes the loop corrections which are widely analyzed in [116]. In [49] the method has been proposed and applied to the one loop correction. As we have already seen the three point function for renormalized primary operators O can be written as

$$\langle O_A(x_1)O_B(x_2)O_C(x_3)\rangle = \frac{\mathcal{N}_A\mathcal{N}_B\mathcal{N}_C c_{ABC}}{|x_{12}|^{\Delta_A+\Delta_B-\Delta_C}|x_{23}|^{\Delta_B+\Delta_C-\Delta_A}|x_{31}|^{\Delta_C+\Delta_A-\Delta_B}} \quad (2.10)$$

where $\mathcal{N}_A$, $\mathcal{N}_B$ and $\mathcal{N}_C$ denote the norm of the respective operators and we write the scaling dimension as $\Delta = \Delta_0 + \gamma$. Since we are interested in the one loop contribution to the three point structure constant we consider

$$c_{ABC} = c^0_{ABC}\left(1+\lambda c^1_{ABC} + O(\lambda^2)\right). \quad (2.11)$$

Using standard renormalization group techniques the renormalized operator can be written in terms of the bare operator $\mathcal{O}$

$$O_A = \mathcal{O}_A\left(1 - a_A\lambda + \gamma_A \ln\left|\frac{\Lambda}{\mu}\right| + O(\lambda^2)\right) \quad (2.12)$$

where μ is the renormalization group constant. Thus we can express (2.10) in terms of the bare operators obtaining

$$\begin{aligned}\langle \mathcal{O}_A(x_1)\mathcal{O}_B(x_2)\mathcal{O}_C(x_3)\rangle &= \frac{N_A N_B N_C c^0_{ABC}\left(1+\lambda c^1_{ABC}\right)\left(1+\lambda\left(a_A+a_B+a_C\right)\right)}{|x_{12}|^{\Delta_A+\Delta_B-\Delta_C}|x_{23}|^{\Delta_B+\Delta_C-\Delta_A}|x_{31}|^{\Delta_C+\Delta_A-\Delta_B}|\mu|^{\gamma_A+\gamma_B+\gamma_C}} \\ &= \frac{C^0}{|x_{12}|^{\Delta_{0A}+\Delta_{0B}-\Delta_{0C}}|x_{23}|^{\Delta_{0B}+\Delta_{0C}-\Delta_{0A}}|x_{31}|^{\Delta_{0C}+\Delta_{0A}-\Delta_{0B}}} \\ &\times\left(1+\lambda C^1 - \gamma_A \ln\left|\frac{x_{12}x_{13}\Lambda}{x_{23}}\right| - \gamma_B\ln\left|\frac{x_{12}x_{23}\Lambda}{x_{13}}\right| - \gamma_C\ln\left|\frac{x_{23}x_{13}\Lambda}{x_{12}}\right|\right)\end{aligned} \quad (2.13)$$

where $C^0 = \mathcal{N}_A\mathcal{N}_B\mathcal{N}_C c^0_{ABC}$ and $C^1 = c^1_{ABC} + a_A + a_B + a_C$. Note that even if singularly the a coefficients are scheme dependent, the combination c^1_{ABC} is scheme independent and that the planar three point function goes as N^{-1} as expected.

The main goal is to find an expression for this correction c^1_{ABC}, let us sketch how this can be written in terms of some combinatorial coefficient.

- At the planar level there are two classes of Feynman diagrams contributing: diagrams involving two operators (already present for two point functions) and diagrams acting on three operators.
- Using non renormalization theorems for BPS operators it is possible to constrain the contributions coming from these five diagrams in such a way that the one loop correction c^1_{ABC} can be written in terms of only two functions.
- Since c^1_{ABC} is scheme independent there is the freedom to work in any scheme.

All the observations above give as a result

$$\lambda\, c^1_{ABC} = \frac{\lambda}{32\pi^2}\left(f^1_{23} + f^2_{31} + f^3_{12}\right) \tag{2.14}$$

or equivalently

$$\lambda\, c^1_{ABC} = -\frac{1}{2}(\gamma_A + \gamma_B + \gamma_C) - \frac{\lambda}{16\pi^2}(b_{12} + b_{23} + b_{31}). \tag{2.15}$$

where we denote with b_{ij} and f^k_{ij} the operator dependent combination constants depending on the three operators. More precisely the coefficients b_{ij} refer to the two point Feynman diagram while f^k_{ij} to the three point ones. In general it is not easy to evaluate the combinatorial factors but in the $SO(6)$ sector at the planar level many simplifications occur and it is possible to get an expression in a close for the b_{ij} and f^k_{ij}. The operators can be written as usual in terms of traces as

$$\mathcal{O}[\psi_I] = \frac{1}{\lambda^{L/2} L}\tilde{\psi}_{i_1\ldots i_L}\,\mathrm{Tr}\;\phi^{i_1}\cdots\phi^{i_L} \tag{2.16}$$

where the indices are meant to be summed.
If we have three operators with length L_1, L_2 and L_3 we have that at tree level the three point function is

$$\langle\bar{\mathcal{O}}_1[\psi_I](x_1)\mathcal{O}_2[\psi_J](x_2)\mathcal{O}_3[\psi_K](x_3)\rangle_{\text{free}} = \frac{\tilde{\psi}_I\tilde{\psi}_J\tilde{\psi}_K\mathcal{I}_3}{N\,(8\pi^2)^{r+s+t}|x_{31}|^{2r}|x_{12}|^{2s}|x_{23}|^{2t}} \tag{2.17}$$

where we introduce

$$r = \frac{1}{2}\left(L_3 + L_1 - L_2\right)\,,\quad s = \frac{1}{2}\left(L_1 + L_2 - L_3\right)\,,\quad t = \frac{1}{2}\left(L_2 + L_3 - L_1\right) \tag{2.18}$$

which counts the number of contractions among $\mathcal{O}_3$ and $\mathcal{O}_1$, $\mathcal{O}_1$ and $\mathcal{O}_2$, $\mathcal{O}_2$ and $\mathcal{O}_3$ respectively, thus they have to be integer. Note that this works in the planar limit and indeed it scales as N^{-1}. Obviously we can face with two situations

- $\mathcal{O}_1$ is contracted with both $\mathcal{O}_2$ and $\mathcal{O}_3$ but $\mathcal{O}_2$ and $\mathcal{O}_3$ are not contracted among themselves which implies that $L_1 = L_2 + L_3$: extremal three point functions;
- $\mathcal{O}_1$ is contracted with $\mathcal{O}_2$ and $\mathcal{O}_3$ as well as $\mathcal{O}_2$ is contracted with $\mathcal{O}_3$ and $\mathcal{O}_1$ and so on, meaning that $L_1 > L_2 + L_3$: non extremal three point functions.

For our purposes we only review the case of non extremal three point functions. Thus the planar contribution to (2.13) for non extremal three point function can be recast as

$$\begin{aligned}\langle\bar{\mathcal{O}}_1[\psi_I](x_1)\Omega_2[\psi_J](x_2)\Omega_3[\psi_K](x_3)\rangle &= \frac{\tilde{C}^0}{N\,(8\pi^2)^{r+s+t}|x_{31}|^{2r}|x_{12}|^{2s}|x_{23}|^{2t}}\times\\ &\times\Big\{1+\big[b_{12}B(x_1,x_2)+b_{23}B(x_2,x_3)+b_{31}B(x_3,x_1)\big]\\ &+\big[f^1_{23}F(x_1;x_2,x_3)+f^2_{31}F(x_2;x_3,x_1)+f^3_{12}F(x_3;x_1,x_2)\big]\Big\}\end{aligned} \tag{2.19}$$

with

$$\begin{aligned} b_{12} &= \sum_{l=1}^{s-1}(2\mathcal{P}-2\mathcal{I}-\mathcal{K})^{j_{1\,l}j_{1\,l+1}}_{i_{2\,l}i_{2\,l+1}}\frac{\tilde{\psi}_I\tilde{\psi}_J\tilde{\psi}_K\mathcal{I}'_3}{\tilde{C}^0}\\ b_{23} &= \sum_{l=1}^{t-1}(2\mathcal{P}-2\mathcal{I}-\mathcal{K})^{k_{2\,l}k_{2\,l+1}}_{j_{3\,l}j_{3\,l+1}}\frac{\tilde{\psi}_I\tilde{\psi}_J\tilde{\psi}_K\mathcal{I}'_3}{\tilde{C}^0}\\ b_{31} &= \sum_{l=1}^{r-1}(2\mathcal{P}-2\mathcal{I}-\mathcal{K})^{i_{3\,l}i_{3\,l+1}}_{k_{1\,l}k_{1\,l+1}}\frac{\tilde{\psi}_I\tilde{\psi}_J\tilde{\psi}_K\mathcal{I}'_3}{\tilde{C}^0}\\ f^1_{23} &= \Big[(2\mathcal{P}-2\mathcal{I}-\mathcal{K})^{j_{11}k_{11}}_{i_{21}i_{31}}\mathcal{I}'_3+(2\mathcal{P}-2\mathcal{I}-\mathcal{K})^{k_{1r}j_{1s}}_{i_{3r}i_{2s}}\mathcal{I}'_3\Big]\frac{\tilde{\psi}_I\tilde{\psi}_J\tilde{\psi}_K}{\tilde{C}^0}\\ f^2_{31} &= \big[(2\mathcal{P}-2\mathcal{I}-\mathcal{K})^{k_{21}i_{21}}_{j_{31}j_{11}}\mathcal{I}'_3+(2\mathcal{P}-2\mathcal{I}-\mathcal{K})^{i_{2s}k_{2t}}_{j_{1s}j_{3t}}\mathcal{I}'_3\big]\frac{\tilde{\psi}_I\tilde{\psi}_J\tilde{\psi}_K}{\tilde{C}^0}\\ f^3_{12} &= \big[(2\mathcal{P}-2\mathcal{I}-\mathcal{K})^{i_{31}j_{31}}_{k_{11}k_{21}}\mathcal{I}'_3+(2\mathcal{P}-2\mathcal{I}-\mathcal{K})^{j_{3t}i_{3r}}_{k_{2t}k_{1r}}\mathcal{I}'_3\big]\frac{\tilde{\psi}_I\tilde{\psi}_J\tilde{\psi}_K}{\tilde{C}^0} \end{aligned} \tag{2.20}$$

where $\mathcal{P}$, $\mathcal{I}$, and $\mathcal{K}$ are the permutation, the identity and the trace operator respectively. The main result of this expression is that we are able to explicitly write down the one loop correction to the three point function using (2.14) and (2.15). It is with noticing again that there are two different contributions: one coming from

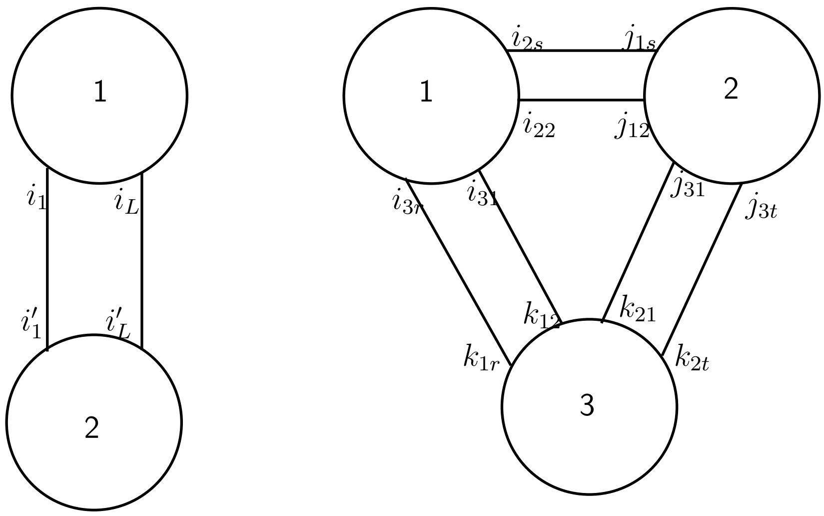


FIGURE 2.1: The indices i, j and k refer to $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ respectively and the subscripts f.i. i_{ab} means that the field at position b in $\mathcal{O}_1$ contracts with the respective field in the operator a.

two point Feynman diagrams contracting two of the three operators (for instance b_{12} represents the weight of the Feynman diagrams coming from the one loop correction to the two point function of $\mathcal{O}_1$ and $\mathcal{O}_2$) and the others belonging to three point Feynman diagrams with two contractions with one operators and one each of the other two operators (for instance f^1_{23} denotes the the weight of the Feynman diagrams with two legs in $\mathcal{O}_1$ and one each in $\mathcal{O}_2$ and $\mathcal{O}_3$). The notation is the same as [49] and it is explained in Fig. 2.1 and will become clearer in Sec. 2.4. It turns out that there is a spin chain interpretation for the coefficient b_{ij}, they can be identified with the open spin chain Hamiltonian, which is integrable. This comes form the fact that we can split a closed spin chain into two parts and since we loose the periodicity we obtaining two open spin chains. In this framework the f coefficient can be seen as the energy difference between a closed spin chain and the two associated open spin chains.

2.2 Semiclassical computation

Using Witten techniques for the computations of three point functions for our knowledge of the AdS/CFT correspondence confine to the case of operators dual to supergravity modes. In general if we would like to face with the problem of computing the three point correlation function for operators dual to massive string states we need to have a classical solution of the string equations of motion in the

Euclidean signature with the topology of a three punctured sphere with some particular boundary conditions on the punctures. Actually there is a simpler situation that can be studied, namely the case when two operators are heavy and one is light.

Two point functions of HH operators In 2010 the authors of [7] address the problem of computing two point correlation functions of local operators corresponding to classical string states. The aim of the paper has been to enstablish a mapping from classical spinning strings solution rotating in the center of the Minkowskian $AdS_5 \times S^5$ to certain solutions in the Poincarè patch which realize the two point functions for any choice of the positions of the operator insertions on the boundary. One would expect to obtain the proper Green function by considering a cylinder amplitude for the closed string and evaluate it by saddle point, with the condition that the string state corresponds to a classical state with appropriate angular momenta. Actually the procedure is not completely straightforward. The crucial observation is that when considering the time evolution of a certain semiclassical state, eigenfunction of the Hamiltonian, its wave function evolves according to its energy, which should be equal to its classical value. However, the semiclassical propagator for the system is governed only by the actions of classical trajectories. Thus in order to obtain the correct two point correlation function one has to do a classical computation, where the corresponding solution is just a geodesic in the AdS part and in the sphere part coincides with the usual spinning string solution. When evaluating the Polyakov action the effect of the wave function of the rotating string state with wavefunction contributions which makes the action integral into an Euclidean energy integral has to be included. Then one performs a saddle point evaluation of the integral over the modular parameter and ends up with the correct two point function. The same answer can be obtained also by the vertex operator approach [8, 9, 117].

Three point functions of HHL The insights on two point functions of massive states pose the attention on the problem of computing a class of three point functions in which two operators are heavy and dual to semiclassical string states and one is light dual to a supergravity mode [10, 11]. In this approach one treats the BPS state in the supergravity approximation while the massive string state is being treated in the first quantised string theory by summing over all classical trajectories with an appropriate action. This summation is a path integral. One considers $\mathcal{O}_I$ to be the chiral primary operator dual to a supergravity mode and $\mathcal{O}_A$, $\mathcal{O}_B$ two primary operators dual to massive string states. The approach of [10] is inspired by the method used in [118] to compute correlators between a Wilson loop and other operators, in the case of interest a chiral primary operator. Let us define this quantity

$$\langle \mathcal{O}_I(x) \rangle_{\mathcal{O}_A \mathcal{O}_B} = \frac{\langle \mathcal{O}_B(x_2) \mathcal{O}_A(x_1) \mathcal{O}_I(x) \rangle}{\langle \mathcal{O}_B(x_2) \mathcal{O}_A(x_1) \rangle} \tag{2.21}$$

where the left hand side is the generalisation of the Wilson loop method to the case when the Wilson loop operator is replaced by the product of two local operators. In the limit $|x| \to \infty$ it is possible to extract the structure constants from (2.21)

$$\langle \mathcal{O}_I(x) \rangle_{\mathcal{O}_A \mathcal{O}_B} = \frac{c_{ABI} |x_2 - x_3|^{\Delta_I}}{x^{2\Delta_I}} \tag{2.22}$$

where Δ_I is the conformal dimension of the chiral primary. Thus the correlator can be written as

$$\langle \mathcal{O}_I(x) \rangle_{\mathcal{O}_A \mathcal{O}_B} = \lim_{\epsilon \to 0} \frac{\pi}{\epsilon_I^{\Delta}} \sqrt{\frac{2}{\Delta_I - 1}} \frac{1}{\int \mathcal{D}\phi e^{-S_{sugra}[\phi]}} \int \mathcal{D}\phi \, \phi_I(x, \epsilon) \frac{1}{Z_{str}} \int \mathcal{D}X e^{S_{str}} \tag{2.23}$$

where ϕ are the supergravity fields and X are the embedding coordinates of the worldsheet in $AdS_5 \times S^5$. The string action depends on the supergravity fields and indeed the metric appearing in the action can be written as a sum of the $AdS_5 \times S^5$ metric and a piece that takes into account the perturbation induced by the insertion of the light field. In order to obtain the three point function one has to

- substitute the classical solution since the string path integral at strong coupling is dominated by the classical trajectory which is essentially the euclidean version of the classical geodesics;
- expand the string action with respect to the supergravity fields to the linear order in ϕ;
- perform the supergravity path integral. This gives a second order differential operator, which acts on the bulk to boundary propagator associated to the specific BPS operator that we insert, and depends on the string embedding coordinates;

- substitute the explicit expression for the differential operator, namely characterise the light operator [1].

Roughly speaking what one should do is to compute the two point function of the two heavy operators and then consider the perturbation induced by the insertion of the light operator by varying the string action with respect to the supergravity fluctuations. In this procedure one ignores the backreaction of the light state on the heavy one. Actually this would mean that the two heavy operators should be one the complex conjugate of the other but this cannot be fully true because it would lead to a vanishing three point function for R charge conservation. However the difference between the two operators is very small, for more details on this point see [21]. If we write the AdS_5 metric in Poincarè coordinates, parametrized by $z(\sigma)$ and $x^\mu(\sigma)$ with $\mu = 1, \dots, 4$ and the 5-sphere with the unit vector

$$\mathbf{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta\sin\alpha\cos\psi, \\ \cos\theta\sin\alpha\sin\psi, \cos\theta\cos\alpha\cos\beta, \cos\theta\cos\alpha\sin\beta) \quad (2.24)$$

the structure constant reads as

$$c_{ABI}|x_2 - x_3|^{\Delta_I} = \frac{2^{\frac{k}{2}-3}(k+1)\sqrt{k\lambda}}{\pi N} \int d^2\sigma Y_I(\mathbf{n})\, z^k \left[\frac{\partial_a X^\mu \partial^a X_\mu - \partial_a z \partial^a z}{z^2} - \partial_a \mathbf{n} \partial^a \mathbf{n}\right] \quad (2.25)$$

where the spherical harmonic function $Y_I(\mathbf{n})$ takes the form

$$Y_k = \left(\frac{n_1 + i n_2}{\sqrt{2}}\right)^k \quad (2.26)$$

when considering for instance a BPS operator of this kind

$$\mathcal{O}_k = \frac{1}{k}\left(\frac{4\pi^2}{\lambda}\right)^{\frac{k}{2}} \mathrm{Tr}\, Z^k. \quad (2.27)$$

The method can be applied in many cases, inserting specific geodesics corresponding to different heavy operators or/and considering different types of light operators [12–29] [2]. In the following we will apply this method in different cases, firstly we consider the situation when the two heavy operators are giant gravitons, so

[1]Note that the normalisation of the supergravity action and fields is peculiar to obtain the right result. We will see in 2.4 an explicit example.

[2]Three point functions of this kind, namely involving two heavy operators and one light chiral primary can be obtained also using the formalism of the vertex operator [30–33]. In this way also four point functions of two heavy and light operators have been computed [119].

we extend the method to D3 branes [34] and then we consider the one loop correction [40] to the three point function and we finally apply this procedure in the context of the AdS_4/CFT_3 correspondence [39].

2.3 Three point functions of two giant gravitons and one point like graviton

A natural question that can be asked is if it possible to generalize the procedure of [10] for studying correlation functions of two giant gravitons, which are solutions of the D3 brane action, and a point like graviton since we can treat giant gravitons as semiclassical heavy objects and the point like graviton as a light state with dimension $\Delta \ll \sqrt{N}$. Giant gravitons admit a dual gauge theory description in terms of Schur polynomials. Thus we can consider a correlation function of two giant graviton operators with a chiral primary which correspond on the gauge theory side to the correlation function of two Schur polynomials with a chiral primary operator. These quantities are expected to be protected from quantum corrections, owing to the shared BPS supersymmetry of the operators. More precisely in [34] we analyze respectively three point functions of

(A) two Schur polynomials in the antisymmetric representation and a single trace chiral primary

(B) two giant gravitons on S^5 and a point like graviton

and equivalentely

(A) two Schur polynomials in the symmetric representation and a single trace chiral primary

(B) two giant gravitons on AdS^5 and a point like graviton.

In the following we review how to describe giant gravitons and to characterize their dual operators in the configurations mentioned above and then we present the computation of the three point functions using Schur polynomial techinques on the gauge theory side and using semiclassical method on the string theory side. Finally we compare the two results.

Giant gravitons in $S^3 \subset S^5$ Giant gravitons are configurations of stable extended D3 branes wrapping a sphere S^3 moving in S^5 [120] and sitting at the center of AdS. The idea was that an initially point-like string, whose spherical harmonic modes on the S^5 part of the background are graviton modes, should with large enough angular momentum around a direction in S^5 blow up into a D3 brane wrapping a $S^3 \subset S^5$ supported against collapse by its interaction with the 4-form potential. In general the low energy effective action of a D3 brane can be written as

$$\begin{aligned} S_{D3} &= S_{DBI} + S_{WZ} = \\ &= -T_{D3} \int d^4\sigma \left(e^{-\phi} \sqrt{-\det\left(g_{ab} + B_{ab}\right) + 2\pi l_s^2 F_{ab}} \right) + T_{D3} \int d^4\sigma P\left[C_4\right] \end{aligned} \tag{2.28}$$

where T_{D3} is the tension of the D3 brane, F_{ab} is the field strength tensor for the electromagnetic fields living on the on the world volume and g_{ab} is the pullback of the metric, namely the metric induced on the world volume of the brane embedded into a higher dimensional space,

$$g_{ab} = \frac{\partial x^M}{\partial x^a} \frac{\partial x^N}{\partial x^b} g_{MN} \tag{2.29}$$

where we denote with $a, b = 0 \ldots, 3$ the worldvolume coordinates and with M, N the embedding coordinates. The same stands for the pullback of the Kalb-Ramond field B_{ab} and of the four form potential C_4 $P\left[C_4\right]$.
The first part of (2.28) is the Dirac Born Infeld action (DBI) [121] which takes into account the coupling to the massless NS string fields, the metric tensor g_{MN}, the Kalb Ramond field B_{MN} and the dilation ϕ, while the second part is the Wess Zumino term which instead refers to the coupling with the massless RR fields that for the D3 brane case is the four form potential C_4. Note that we are considering only the bosonic part of the action. Then we insert the background metric of $AdS_5 \times S^5$ which in global lorenzian coordinates reads

$$ds^2 = -\cosh\rho^2 dt^2 + d\rho^2 + \sinh\rho^2 d\widetilde{\Omega}_3^2 + d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\Omega_3^2. \tag{2.30}$$

where $\widetilde{\Omega}_3 \subset AdS_5$ is parametrized by $\widetilde{\chi}_1, \widetilde{\chi}_2, \widetilde{\chi}_3$ while $d\Omega_3 \subset S^5$ is parametrized by χ_1, χ_1, χ_3 and the radius of AdS is unitary. Thus the supergravity equations of motion fix a relation between the tension of the D3 brane and the number of units of the five form flux N through

$$T_{D3} = \frac{N}{\Omega_3} = \frac{N}{2\pi^2}. \tag{2.31}$$

The Kalb Ramond field B_{ab} vanishes in the $AdS_5 \times S^5$ and we are not considering any world volume gauge fields. The RR four-form potential C_4 which will be important for the giant graviton has its legs entirely in the S^5, it is proportional to the volume form [122]

$$C_{\phi\chi_1\chi_2\chi_3} = \cos^4\theta \, \mathrm{Vol}(\Omega_3) \tag{2.32}$$

where $\mathrm{Vol}(\Omega_3)$ indicates the volume element of the 3-sphere Ω_3. Since we want to describe a spherical symmetric configuration sitting at the center of AdS we take the ansatz

$$\rho = 0, \quad \sigma^0 = t, \quad \phi = \phi(t), \quad \sigma^i = \chi_i, \tag{2.33}$$

where obviously $\dot\phi$ is the angular velocity associated to the motion of the brane, with radius $\cos\theta$, in the S^3. Specifying (2.28) one gets

$$S = \int dt\, L = -N \int dt \left[\cos^3\theta \sqrt{1 - \dot\phi^2 \sin^2\theta} - \dot\phi \, \cos^4\theta\right]. \tag{2.34}$$

Independence of ϕ leads to a conserved angular momentum

$$k \equiv \frac{\delta L}{\delta\dot\phi} = \frac{N\dot\phi \sin^2\theta\cos^3\theta}{\sqrt{1-\dot\phi^2\sin^2\theta}} + N\cos^4\theta. \tag{2.35}$$

The action may be rewritten in terms of k, to give

$$S = N \int dt \, \frac{\cos^4\theta}{\sin\theta} \frac{l - \cos^2\theta}{\sqrt{(l-\cos^4\theta)^2 + \sin^2\theta\cos^6\theta}}, \tag{2.36}$$

where $l \equiv k/N$. One may also introduce an energy defined by

$$E \equiv \dot\phi k - L = \frac{N}{\sin\theta}\sqrt{(l-\cos^4\theta)^2 + \sin^2\theta\cos^6\theta}, \tag{2.37}$$

and which notably removes the WZ part of the action. Note that this a crucial point in order to obtain stable brane solutions. The energy is minimized by

$$\cos^2\theta = l, \qquad E_{\text{min.}} = k, \qquad S_{\text{min.}} = 0 \tag{2.38}$$

and by plugging this value in to (2.35), one finds that

$$\dot{\phi} = 1. \tag{2.39}$$

In (2.38) and (2.39) all the peculiar properties of the S^5 giant gravitons are summarized:

- the size of the brane increases with the angular momentum and the maximum radius that the brane can have is 1 which is equal to the radius of the S^5, this is a reflection of the stringy exclusion principle [123–125],
- when the brane has its maximum radius, its angular momentum is maximum, giant gravitons saturating this bound are called maximal,
- it exists a stable minimum and the energy is minimized by k which is the energy of a Kaluza Klein graviton with angular momentum k,
- the velocity of the center of mass of the brane cannot be bigger than the speed of light,
- giant gravitons are BPS states preserving 16 of the 32 supersymmetries, which carry the same quantum numbers of the point like gravitons.

Note that l cannot be grater than 1.
Giant gravitons in $S^3 \subset AdS_5$ It is also possible to have configurations of brane expanding on the AdS part of the space time [122, 126], with worldvolume $\mathbb{R} \times S^3 (\subset AdS_5)$. We refer to these configurations as dual giant gravitons. Again, we can go through an analogous analysis to find stable spherically symmetric brane solutions carrying the same quantum numbers of a point like graviton, following [122] and [34]. For our purposes, which will be clear in the following, we consider the action for the anti-D3-brane is

$$S_{D3} = -\frac{N}{2\pi^2} \int d^4\sigma \left(\sqrt{-g} + P[C_4]\right), \tag{2.40}$$

where $g_{ab} = \frac{\partial x^M}{\partial x^a} \frac{\partial x^N}{\partial x^b} g_{MN}$, where $a, b = 0, \ldots, 3$ label the worldvolume coordinates and M, N are the embedding coordinates. The four-form potential C_4 which will be important for this giant graviton has its legs entirely in the AdS_5, and may be taken as [122]

$$C_{t\widetilde{\chi}_1\widetilde{\chi}_2\widetilde{\chi}_3} = -\sinh^4 \rho \, \mathrm{Vol}(\widetilde{\Omega}_3) \tag{2.41}$$

where $\mathrm{Vol}(\widetilde{\Omega}_3)$ indicates the volume element of $S^3 \subset AdS_5$. Similarly to (2.33) we take the following anstaz

$$\rho = \text{const.}, \quad \sigma^0 = t, \quad \sigma^i = \widetilde{\chi}_i, \quad \phi = \phi(t), \quad \theta = \frac{\pi}{2}, \tag{2.42}$$

and obtains

$$S = \int dt\, L = -N \int dt \left[\sinh^3\rho\sqrt{\cosh^2\rho - \dot{\phi}^2} - \sinh^4\rho\right]. \tag{2.43}$$

Again the conserved angular momentum is

$$\widetilde{k} \equiv \frac{\delta L}{\delta \dot{\phi}} = \frac{N\dot{\phi}\sinh^3\rho}{\sqrt{\cosh^2\rho - \dot{\phi}^2}}. \tag{2.44}$$

The action may be rewritten in terms of $\widetilde{k}$, to give

$$S = -N \int dt \cosh\rho \sinh^4\rho \left[\sinh^2\rho\sqrt{\frac{1}{\sinh^6\rho + \widetilde{l}^2}} - 1\right] \tag{2.45}$$

where $\widetilde{l} \equiv \widetilde{k}/N$. One may also introduce an energy defined by

$$E \equiv \dot{\phi}\widetilde{k} - L = N\left[\cosh\rho\sqrt{\sinh^6\rho + \widetilde{l}^2} - \sinh^4\rho\right] \tag{2.46}$$

The energy is minimized by

$$\sinh^2\rho = \widetilde{l}, \qquad E_{\text{min.}} = \widetilde{k}, \qquad S_{\text{min.}} = 0, \tag{2.47}$$

and by plugging this value in to (2.44), one finds that

$$\dot{\phi} = 1. \tag{2.48}$$

Also this solution is dynamically stable and it preserves half of the supersymmetries. The most significant difference between giant gravitons and dual giant gravitons is that the latter do not admit any constraint on their size (or equivalently on the angular momentum), this is due to the fact the the AdS space is not compact.

Dual operators Initially it was presumed that the giant gravitons are described by single trace operators in the gauge theory side. This idea was motivated on the fact that in the large N limit a graviton with a small angular momentum is represented by a single trace operator thus a state with more than one graviton is dual to the product of single trace operators. Actually this is not the case when the angular momentum of the gravitons is of order N. In fact if one considers two multitrace operators which should be dual to two giant gravitons, namely they share the same conserved charges, the two states are not orthogonal as in principle they should be because the two states are distinguishable [127]. It can be shown that if we want to compute in the large N limit two point functions of half BPS operators, thus containing only one out of the three complex scalar fields, of order $\sqrt{N}$ or order 1 a large number of non planar interactions can be ignored since they are suppressed by a factor of $\frac{1}{N}$. But when the dimension of the operators is of order N we are no longer allowed to ignore the non-planar contributions, due to the large combinatoric factors. Therefore the trace number is not a good quantum number and we cannot identify the trace number on the gauge theory side with the particle number in the AdS Fock space. However it is possible to generalize the correspondence saying that $\frac{1}{2}$-BPS representation can be mapped to the space of Schur polynomials of $U(N)$ or equivalently to the space of Young diagrams characterizing representations of $U(N)$ [128]. This can be seen noticing that if we consider n Z fields to build the BPS operators we can form partitions of these n fields and there is a distinct operator for each partition of the n Z fields. For example if $n = 3$ we have three partitions and the corresponding operators are $\mathrm{Tr}\, Z^3$, $\mathrm{Tr}\, Z^2\, \mathrm{Tr}\, Z$, $\mathrm{Tr}^3\, Z$. Note that even in the large N limit these operators are not diagonal. There is a one to one correspondence between $\frac{1}{2}$-BPS operators representation of fixed n R charge and partitions of n. Remarkably a basis for this space is given by the Schur polynomials, which are a special subgroup of Jack functions, a basis for the symmetric group (for details see Appendix (B)). The Schur polynomial $\chi_R(Z)$ of a complex matrix Z is defined as

$$\chi_{R_n}(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_{R_n}(\sigma)\, Z_{i_1}^{i_{\sigma(1)}} \dots Z_{i_n}^{i_{\sigma(n)}}. \tag{2.49}$$

Here R_n denotes an irreducible representation of $U(N)$ described in terms of a Young tableau with n boxes. The sum is over all elements of the symmetric group S_n and $\chi_{R_n}(\sigma)$ is the character of the element σ in the representation R_n. The Schur polynomial basis exactly diagonalizes the two point functions to all orders in N in the free field theory. There is also a simple rule for evaluating two point functions in the zero coupling limit. For this reason it seems natural to consider this problem in the Schur polynomial basis. When we have a long column (with

$O(N)$ boxes) in the Young diagram, we associate this state with a giant graviton, growing in the S^5. The size of the giant is determined by the number of boxes in the Young diagram. A Young diagram with a large number of boxes in a column represents a giant graviton with a large amount of angular momentum and equivalently a large size. The Young diagram can only contain N boxes in a column. This corresponds to the cutoff associated with the stringy exclusion principle that we discussed previously. It is natural to associate the length of the column with the angular momentum, and therefore the size of the giant. If we consider two columns we can interpret them as a state with two giant gravitons. When considering a large number of boxes in a row, we can interpret the state as a giant graviton in the AdS_5. The Schur polynomials $\chi_{R_n}(Z)$ have the general structure

$$\chi_{R_n}(Z) = c_{0,n}\,\mathrm{Tr}\,Z^n + c_{1,n}\,\mathrm{Tr}\,Z\,\mathrm{Tr}\,Z^{n-1} + \ldots + c_{n,n}\,(\mathrm{Tr}\,Z)^n, \tag{2.50}$$

where the c's are constants independent of N and the sum is over all partitions. For instance

$$\begin{aligned}
\chi_{\yng(1)} &= \mathrm{Tr}\,Z \\
\chi_{\yng(1,1)} &= \frac{1}{2}\left(\mathrm{Tr}^2 Z - \mathrm{Tr}Z^2\right) \\
\chi_{\yng(1,1,1)} &= \frac{1}{6}\left(\mathrm{Tr}^3 Z + 3\mathrm{Tr}Z^2\mathrm{Tr}Z + 2\mathrm{Tr}Z^3\right) \\
\chi_{\yng(2)} &= \frac{1}{2}\left(\mathrm{Tr}^2 Z + \mathrm{Tr}Z^2\right) \\
\chi_{\yng(2,1)} &= \frac{1}{3}\left(\mathrm{Tr}^3 Z - \mathrm{Tr}Z^3\right)
\end{aligned}\quad . \tag{2.51}$$

It is clear that there is no limit in which the Schur polynomials reduce to chiral primary operators. It is also possible to extend this basis for near BPS operators containing more than one type of matrix field (Z and Y for example), individually or combined into matrix words, and in principle they include complex matrix fields, covariant derivatives and fermions. Notably also in this case we can introduce a basis made of the so called restricted Schur polynomials [129–134]. The dual description corresponds to excited giant gravitons which can be released by attaching open strings to giant gravitons.

Two and three point functions of Schur polynomials In [128] two and three point correlation functions of Schur polynomials in the free field theory have been computed. These correlation functions have been calculated exactly and read [128]

$$\langle \chi_R(Z)\chi_S(\bar{Z})\rangle = \delta_{R,S} f_R, \tag{2.52}$$

$$\langle \chi_R(Z)\chi_S(Z)\chi_T(\bar{Z})\rangle = g(R,S;T)\, f_T, \tag{2.53}$$

where $g(R,S,T)$ is the Littlewood-Richardson coefficient which counts the multiplicity with which the representation T appears in the tensor product of the representations R and S. The quantity f_R denotes the product of the weights of the boxes of the Young diagram labeling the Schur polynomial, note that these are different from the Dynkin weights. The weight of a box in the ith row and jth column is given by $N-i+j$ [3] thus

$$f_R = \prod_{i,j\in R} (N-i+j) \tag{2.54}$$

where the product runs over all boxes of the Young tableau of the representation R (and analogously for the representation T). To obtain the form for the three point functions one needs to use the so called product rule which intuitively plays the role of the OPE

$$\chi_R(Z)\chi_S(Z) = g(R,S,T)\chi_T(Z). \tag{2.55}$$

It is clear that by knowing two point functions and the product rules all the higher points correlation functions can be reconstructed, which precisely mirror what happens in any conformal field theory. Note that since they are $\frac{1}{2}$-BPS operators, the non renormalization theorems apply to this case as well so they do not get any quantum corrections. The cleanest examples are the Schur polynomials of the symmetric and the antisymmetric representations. When the number of boxes, k, in the Young tableau of the representation is large (i.e. $k \sim \mathcal{O}(N)$, with $N \to \infty$), the Schur polynomial of the symmetric representation is dual to a single giant graviton moving on S^5 with angular momentum k and wrapping an $S^3 \subset AdS_5$. For the antisymmetric case the giant graviton instead wraps an $S^3 \subset S^5$ [128]. If one wants to consider more than one giant graviton it is needed to add the same amount of rows or columns of the same length. Let us denote the Schur polynomial

[3]The motivation is very simple, the box at position: (1,1) has weight N, (1,2) has weight $N+1$, (2,1) has weight $N-1$ and so on. See Appendix (B) for more details.

for the symmetric representation with k boxes as $\chi_k^S(Z)$ and the Schur polynomial for the antisymmetric representation with k boxes as $\chi_k^A(Z)$. Then we find for the corresponding two and three-point functions

$$\langle \chi_k^S(\bar{Z})\chi_k^S(Z)\rangle = \prod_{j=1}^{k}(N-1+j), \tag{2.56}$$

$$\langle \chi_k^A(\bar{Z})\chi_k^A(Z)\rangle = \prod_{i=1}^{k}(N-i+1), \tag{2.57}$$

$$\langle \chi_k^S(\bar{Z})\chi_{k-J}^S(Z)\chi_J^S(Z)\rangle = \prod_{j=1}^{k}(N-1+j), \tag{2.58}$$

$$\langle \chi_k^A(\bar{Z})\chi_{k-J}^A(Z)\chi_J^A(Z)\rangle = \prod_{i=1}^{k}(N-i+1), \tag{2.59}$$

since for these cases $g(R,S;T)=1$.

2.3.1 Gauge theory side (A)

On the gauge theory side we want to compute the three point functions of two Schur polynomials in the symmetric or antisymmetric representation with a single trace chiral primary, thus the normalized structure constant is

$$c_{k,k-J,J}^{S/A} \equiv \frac{\langle \chi_k^{S/A}(\bar{Z})\chi_{k-J}^{S/A}(Z)\,\mathrm{Tr}\,Z^J\rangle}{\sqrt{\langle \chi_k^{S/A}(\bar{Z})\chi_k^{S/A}(Z)\rangle\langle \chi_{k-J}^{S/A}(\bar{Z})\chi_{k-J}^{S/A}(Z)\rangle\langle \mathrm{Tr}\,\bar{Z}^J\,\mathrm{Tr}\,Z^J\rangle}}, \tag{2.60}$$

where S and A refer respectively to the symmetric and the antisymmetric representation. Let us look at the numerator of (2.60), we need to compute

$$\langle \chi_k^S(\bar{Z})\chi_{k-J}^S(Z)\ \mathrm{Tr}\ Z^J\rangle \tag{2.61}$$

and

$$\langle \chi_k^A(\bar{Z})\chi_{k-J}^A(Z)\ \mathrm{Tr}\ Z^J\rangle. \tag{2.62}$$

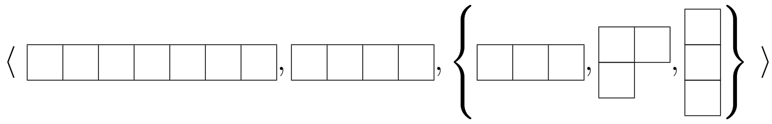
Even if it is not possible to obtain a chiral primary as a limit of a Schur polynomial, we can expand it in terms of the Schur polynomial basis, which is a complete and orthonormal basis for the symmetric group, in this way [135]

$$\mathrm{Tr}\,Z^J = \mathrm{Tr}(\sigma_0 Z), \tag{2.63}$$

where σ_0 is the cyclic permutation. Thus we have

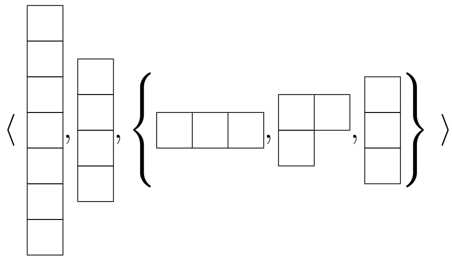
$$\text{Tr}\ Z^J = \sum_{R_J} \chi_{R_J}(\sigma_0)\chi_{R_J}(Z), \tag{2.64}$$

where the sum goes over all possible irreducible representations R_J corresponding to Young tableaux with J boxes. If we insert the expansion (2.64) for the single trace operator in (2.61) and (2.62) respectively we obtain for instance



(2.65)

and



(2.66)

for $k = 7$ and $J = 3$. It seems that even for small number of boxes the situation is slightly involved, but this is not the case. In fact looking at (2.53) it is clear that a simplification occurs, the only contribution in the single trace expansion that survives is the completely symmetric and antisymmetric representation respectively. The character $\chi_{R_J}(\sigma_0)$ can be written down in closed form for hook diagrams. Denoting the number of boxes in the first row of the hook diagram as $J - m$ it holds that $\chi^{hook}_{R_J}(\sigma_0) = (-1)^m$. Hence for the cases of interest to us we have

$$\chi^S_J(\sigma_0) = 1, \qquad \chi^A_J(\sigma_0) = (-1)^{J-1}. \tag{2.67}$$

This implies[4]

$$\begin{aligned} \langle \chi^S_k(\bar{Z})\chi^S_{k-J}(Z)\ \text{Tr}\ Z^J\rangle &= \prod_{j=1}^{k}(N-1+j), \\ \langle \chi^A_k(\bar{Z})\chi^A_{k-J}(Z)\ \text{Tr}\ Z^J\rangle &= (-1)^{J-1}\prod_{i=1}^{k}(N-i+1). \end{aligned} \tag{2.68}$$

[4]The $(-1)^J$ part of the prefactor in the antisymmetric case could be removed since one can equally well define the gauge theory dual of the antisymmetric giant graviton with angular momentum k to be $(-1)^k\chi^A_k(Z)$. However, in the following we will follow the usual definition in the Schur operator literature and keep the alternating sign.

Dividing with the relevant norms we hence find the structure constants

$$c^S_{k,k-J,J} = \frac{\sqrt{\prod_{p=k-J+1}^{k}(N+p-1)}}{\sqrt{JN^J(1+c(J)\,\frac{1}{N^2}+\ldots)}}, \tag{2.69}$$

$$c^A_{k,k-J,J} = (-1)^{(J-1)}\frac{\sqrt{\prod_{p=k-J+1}^{k}(N-p+1)}}{\sqrt{JN^J(1+c(J)\,\frac{1}{N^2}+\ldots)}}, \tag{2.70}$$

where the quantities in the denominators are nothing but $\sqrt{\langle \mathrm{Tr}\,\bar{Z}^J\,\mathrm{Tr}\,Z^J\rangle}$ which is given exactly in equation (2.5). In other words we have exact expressions for $c^S_{k,k-J,J}$ and $c^A_{k,k-J,J}$. Now, we are interested in the situation where the Schur polynomials correspond to large Young tableaux and where the chiral primary is a small operator, i.e. the limit

$$N \to \infty, \qquad k \to \infty, \qquad \frac{k}{N}\ \text{finite}, \qquad J \ll k, \tag{2.71}$$

and in particular $J \ll \sqrt{N}$. In this limit we find for the structure constants

$$c^S_{k,k-J,J} = \frac{1}{\sqrt{J}}\left(1+\frac{k}{N}\right)^{J/2}, \tag{2.72}$$

$$c^A_{k,k-J,J} = (-1)^{(J-1)}\frac{1}{\sqrt{J}}\left(1-\frac{k}{N}\right)^{J/2}. \tag{2.73}$$

Notice that this result does not reduce to the chiral primary result in any limit (in accordance with the fact that a chiral primary operator can not be obtained as a limit of a single Schur polynomial). Furthermore, we note that for the anti-symmetric representation we have the constraint $k \leq N$ while for the symmetric case k is unbounded.

Matrix model techniques The same results can be obtained using directly the matrix model techniques. In the following we sketch the essential steps to recover the results (2.68) and consequently (2.72) and (2.73). We can write our operators as

$$\mathcal{O}^J = \mathrm{Tr}\,Z^J, \quad \mathcal{O}_R = \mathrm{Tr}_R Z, \tag{2.74}$$

where R refers to a certain irreducible representation. The two point functions of these operators can be obtained by a zero dimensional complex matrix model, namely

$$\langle \mathcal{O}\bar{\mathcal{O}} \rangle = \frac{\int dZ d\bar{Z} e^{-\frac{2N}{\lambda}\,\mathrm{Tr}(Z\bar{Z})}\mathcal{O}\bar{\mathcal{O}}}{\int dZ d\bar{Z} e^{-\frac{2N}{\lambda}\,\mathrm{Tr}(Z\bar{Z})}}. \tag{2.75}$$

Following [136] one can write such expectation values in terms of integral over N diagonal degrees of freedom z_i, with $i = 1, \dots N$. In this formulation the operators (2.74) read

$$\mathcal{O}^J = \sum_{i=1}^{N} z_i^J\,, \tag{2.76}$$

$$\mathcal{O}_R = \frac{det(z_i^{h_j+N-j})}{det(z_i^{N-j})} = \frac{det(z_i^{h_j+N-j})}{\Delta(z)} \tag{2.77}$$

where $h_1 \geq h_2 \geq \dots h_N$ are the lengths of the rows of the Young diagram associated with the representation R and the two point function of operators of the form (2.77) become

$$\langle \mathcal{O}_R \bar{\mathcal{O}}'_R \rangle = \frac{\int \prod_{i=1}^{N} dz_i d\bar{z}_i e^{-\frac{2N}{\lambda}|z_i|^2} det(z_i^{h_j+N-j}) det(\bar{z}_i^{h'_j+N-j})}{\int \prod_{i=1}^{N} dz_i d\bar{z}_i e^{-\frac{2N}{\lambda}|z_i|^2} \mid \Delta(z) \mid^2}. \tag{2.78}$$

The integrals in (2.78) can be carried out by combinatorics, expanding the determinants. For the denominator we get

$$\begin{aligned}
&\int \prod_{i=1}^{N} dz_i d\bar{z}_i e^{-\frac{2N}{\lambda}|z_i|^2} \sum_{\sigma,\tau} (-1)^{\sigma+\tau} z_i^{N-\sigma(i)} \bar{z}_i^{N-\tau(i)} \\
&= \sum_{\sigma} \int \prod_{i=1}^{N} dz_i d\bar{z}_i e^{-\frac{2N}{\lambda}|z_i|^2} (z_i \bar{z}_i)^{N-\sigma(i)} \\
&= N! \prod_{j=1}^{N} \int_0^{2\pi} d\varphi \int_0^{\infty} dr e^{-\frac{2N}{\lambda}r^2} r^{2N-2j+1} \\
&= N! \prod_{j=1}^{N} \pi \frac{(N-j)!}{\left(\frac{2N}{\lambda}\right)^{N-j+1}}.
\end{aligned}$$

We only get a non-vanishing contribution in (2.79) if the powers of z_i and $\bar{z}_i$ are the same which reduces the double sum to a sum over a single permutation. Moreover, one concludes that all $N!$ permutation give the same result and the integrals are carried out by going to polar coordinates. Similarly, for the numerator we get [136]

$$\int dZ d\bar{Z} e^{-\frac{2N}{\lambda}\,\mathrm{Tr}(Z\bar{Z})} \mathrm{Tr}_R(Z) \mathrm{Tr}_{R'}(\bar{Z}) = \delta_{R,R'} N! \prod_{j=1}^{N} \pi \frac{(N+h_j-j)!}{\left(\frac{2N}{\lambda}\right)^{N+h_j-j+1}}. \tag{2.79}$$

In total we find for the norm of the operator $\mathcal{O}_R$ is

$$\langle \mathcal{O}_R \bar{\mathcal{O}}_R \rangle = \prod_{j=1}^{N} \left(\frac{\lambda}{2N}\right)^{h_j} \frac{(N+h_j-j)!}{(N-j)!}. \tag{2.80}$$

Let us first consider operators in the completely antisymmetric representation corresponding to a Young diagram with a single column of length $k < N$, i.e. operators for which $h_1 = h_2 = \cdots = h_k = 1$ and $h_{k+1} = h_{k+2} = \cdots = h_N = 0$. Let us denote these operators as $\mathcal{O}_k^A$. These operators have the norm

$$\langle \mathcal{O}_k^A \bar{\mathcal{O}}_k^A \rangle = \left(\frac{\lambda}{2N}\right)^k \prod_{j=1}^{N} (N-j+1)\,. \tag{2.81}$$

In particular, we note that for the representation consisting of a single box we have

$$\langle \mathcal{O}_1 \bar{\mathcal{O}}_1 \rangle = \langle \mathrm{Tr}\, Z\ \mathrm{Tr}\, \bar{Z} \rangle = \left(\frac{\lambda}{2N}\right) \cdot N. \tag{2.82}$$

This result coincides with the $J = 1$ limit of (2.9). Let us analyse the three point function in the form of (2.60). We can see that we already have the expressions of all the quantities in the denominator. For the numerator we have that

$$\begin{aligned}
\langle \mathcal{O}_k^A \bar{\mathcal{O}}_{k-J}^A \bar{\mathcal{O}}^J \rangle &\propto \int dZ d\bar{Z} e^{-\frac{2N}{\lambda} Tr(Z\bar{Z})} Tr_k Z\, Tr_{k-J} \bar{Z} Tr \bar{Z}^J \\
&\propto \int \prod_{i=1}^{N} dz_i d\bar{z}_i e^{-\frac{2N}{\lambda}|z_i|^2} det(z_i^{h_j+N-j}) det(\bar{z}_i^{\bar{h}_j+N-j}) \sum_{i=1}^{N} \bar{z}_1^J \\
&= N \sum_{\sigma,\tau} (-1)^{\sigma+\tau} \int dz_1 d\bar{z}_1 e^{-\frac{2N}{\lambda}|z_i|^2} z_1^{h(\tau_1)+N-\tau(1)} \bar{z}_1^{\bar{h}_{\sigma(1)}+N+J-\sigma(1)} \\
&\times \prod_{i=2}^{N} \int dz_1 d\bar{z}_1 e^{-\frac{2N}{\lambda}|z_i|^2} z_i^{h_{\tau(i)}+N-\tau(i)} \bar{z}_i^{\bar{h}_{\sigma(i)}+N-\sigma(i)}.
\end{aligned} \tag{2.83}$$

Here the first proportionality sign signifies that we are supposed to divide by the same factor as in (2.75) and the second one that we leave out some integrals over non-diagonal degrees of freedom which cancel out when the right denominator is introduced. The h and $\bar{h}$ variables are the weights of the Young tableaux and take the values

$$h_1 = h_2 = \cdots = h_k = 1, \qquad h_{k+1} = h_{k+2} = \cdots = h_N = 0, \tag{2.84}$$

$$\bar{h}_1 = \bar{h}_2 = \cdots = \bar{h}_{k-J} = 1, \qquad \bar{h}_{k-J+1} = \bar{h}_{k-J+2} = \cdots = \bar{h}_N = 0. \tag{2.85}$$

As usual we need to have the same number of z_i's and $\bar{z}_i$'s in the integrals. Hence we have the following constraints

$$\bar{h}_{\sigma(1)} - \sigma(1) + J = h_{\tau(1)} - \tau(1) \tag{2.86}$$

and

$$\bar{h}_{\sigma(i)} - \sigma(i) = h_{\tau(i)} - \tau(i) \quad \text{for} \quad i = 2, \ldots, N. \tag{2.87}$$

If we consider separately two cases, namely $\bar{h}_{\sigma(1)} = 1$ and $\bar{h}_{\sigma(1)} = 0$ we have three different situations

1. $\bar{h}_{\sigma(1)} = 1$ which does not contribute to (2.83);
2. $\bar{h}_{\sigma(1)} = 0$ and $h_{\tau(1)} = 0$ which does not contribute to (2.83);
3. $\bar{h}_{\sigma(1)} = 0$ and $h_{\tau(1)} = 1$ which does contribute.

From the analysis above it follows that $\sigma(1)$ and $\tau(1)$ are fixed so effectively that we should only sum over permutations of $(N-1)$ elements. Furthermore, it is obvious that if we fix the remaining part of the permutation is also fixed so the double sum reduces to a single sum. It is also clear that any allowed permutation leads to the same integrals so taking into account the four possible classes of contributions we can rewrite (2.83) as

$$\begin{aligned} \langle \mathcal{O}_k^A \bar{\mathcal{O}}_{k-J}^A \bar{\mathcal{O}}^J \rangle \propto N \sum_{\tau_{N-1}} (-1)^{\tau+\sigma(\tau)} \prod_{p=1}^{k-J} \pi \frac{(N-p+1)!}{\left(\frac{2N}{\lambda}\right)^{N-p+2}} \prod_{p=k-J+1}^{k-1} \pi \frac{(N-p)!}{\left(\frac{2N}{\lambda}\right)^{N-p+1}} \\ \pi \frac{(N-k+J)!}{\left(\frac{2N}{\lambda}\right)^{N-k+J+1}} \prod_{p=k+1}^{N} \pi \frac{(N-p)!}{\left(\frac{2N}{\lambda}\right)^{N-p+1}} \end{aligned} \tag{2.88}$$

which, divided by the proper normalisation, gives

$$\langle \mathcal{O}_k^A \bar{\mathcal{O}}_{k-J}^A \bar{\mathcal{O}}^J \rangle = \left(\frac{\lambda}{2N}\right)^k \frac{1}{(N-1)!} \sum_{\tau_{N-1}} (-1)^{\tau+\tau(\sigma)} \frac{(N-k+J)!}{(N-k)!} \prod_{p=1}^{k-J} (N-p+1)\,. \tag{2.89}$$

If we combine all the pieces we obtain precisely the same expression as in (2.70). Similarly it is possible to obtain (2.69) starting with operators $\mathrm{Tr}_R\, Z$ in which R is the completely symmetric representation.

2.3.2 String theory side (B)

In the following we will extend the semiclassical procedure introduced in [10] to D3 branes and more precisely to the case of giant gravitons. The idea is to use the fact that the holographic two point funcion of the giant gravitons are given by the D3 brane solution continued to the Euclidean Poincarè patch, in full analogy with the semiclassical spinning strings analized in [7, 10]. Then, in order to obtain the three point function we need to vary the Euclidean D3 brane (or anti D3 brane) action with respect to the supergravity fluctuations that correspond to the light operators inserted. Finally we have to evaluate those fluctuations on the Wick rotated giant graviton solutions described in Poincarè patch.

Coordinates We can map the global coordinates (2.30) that we discussed in the first part of this section into the Poincaré patch as follows. Take as a simplification AdS_3, for which the factor $d\widetilde{\Omega}_3^2 = d\psi^2$, then we have that

$$\begin{aligned} z &= \frac{1}{\cosh\rho\cos t - \sinh\rho\cos\psi}, \\ x^0 &= \frac{\cosh\rho\sin t}{\cosh\rho\cos t - \sinh\rho\cos\psi}, \qquad x^1 = \frac{\sinh\rho\sin\psi}{\cosh\rho\cos t - \sinh\rho\cos\psi}, \end{aligned} \tag{2.90}$$

where the metric of the Poincaré patch is

$$ds^2 = \frac{-\left(dx^0\right)^2 + \left(dx^1\right)^2 + dz^2}{z^2}. \tag{2.91}$$

On the path of the giant graviton we have $\rho = 0$. Continuing to Euclidean AdS, so that $t \to it$ and $x_0 \to -ix_0$ we have that

$$z = \frac{1}{\cosh t}, \quad x^0 = \tanh t, \quad x^1 = 0. \tag{2.92}$$

which does give the trajectory of [7], if we identify the Euclidean time direction in the Poincarè patch with the spatial direction in which the operators are separated on the boundary. In the case of the AdS giant graviton we will need the generalization to AdS_5 of the coordinate transformation

$$
\begin{aligned}
z &= \frac{1}{\cosh\rho\cos t - n_0\sinh\rho}, \\
x^0 &= \frac{\cosh\rho\sin t}{\cosh\rho\cos t - n_0\sinh\rho}, \qquad \vec{x} = \frac{\vec{n}\sinh\rho}{\cosh\rho\cos t - n_0\sinh\rho},
\end{aligned} \tag{2.93}
$$

where the $S^3 \subset AdS_5$ is given by the embedding coordinates $n_I = (n_0, \vec{n})$, $n_I n_I = 1$.

DBI part of the action The supergravity modes that we are interested in are fluctuations of the RR 5-form as well as the spacetime metric. They are by now very well known, and details can be found in [106, 118, 137–139]. The fluctuations are

$$
\begin{aligned}
\delta g_{\mu\nu} &= \left[-\frac{6\,\Delta}{5}\,g_{\mu\nu} + \frac{4}{\Delta+1}\,D_{(\mu}D_{\nu)}\right]\,s^{\Delta}(X)\,Y_{\Delta}(\Omega), \\
\delta g_{\alpha\beta} &= 2\,\Delta\,g_{\alpha\beta}\,s^{\Delta}(X)\,Y_{\Delta}(\Omega),
\end{aligned} \tag{2.94}
$$

where μ, ν are AdS_5 and α, β are S^5 indices. The symbol X indicates coordinates on AdS_5 and Ω coordinates on the S^5. The $D_{(\mu}D_{\nu)}$ represents the traceless symmetric double covariant derivative. The $Y_\Delta(\Omega)$ are the spherical harmonics on the five-sphere, while $s^\Delta(X)$ have arbitrary profile and represent a scalar field propagating on AdS_5 space with mass squared $= \Delta(\Delta-4)$, where Δ labels the representation of $SO(6)$ and must be an integer greater than or equal to 2.

The bulk-to-boundary propagator for s^Δ is given in [118], with normalization from [106]. It is

$$
\sqrt{\frac{\alpha_0}{B_\Delta}}\frac{z^\Delta}{((x-x_0)^2+z^2)^\Delta}, \tag{2.95}
$$

where,

$$
\alpha_0 = \frac{\Delta-1}{2\pi^2}, \qquad B_\Delta = \frac{2^{3-\Delta}N^2\Delta(\Delta-1)}{\pi^2(\Delta+1)^2}. \tag{2.96}
$$

The traceless symmetric double covariant derivative is

$$
D_{(\mu}D_{\nu)} \equiv \frac{1}{2}\,(D_\mu D_\nu + D_\nu D_\mu) - \frac{1}{5}g_{\mu\nu}\,g^{\rho\sigma}\,D_\rho D_\sigma. \tag{2.97}
$$

Using the giant graviton ansatz (2.33) and identifying $\phi = i\omega t$ the relevant part of the metric (2.30) is

$$g_{ab} = \begin{pmatrix} 1-\omega^2\sin^2\theta & 0 \\ 0 & \cos^2\theta\,\bar{g}_{ab} \end{pmatrix} \tag{2.98}$$

where the upper diagonal part is the tt component while the lower diagonal one $\bar{g}_{ab}$ is the metric on the unit 3-sphere Ω_3. Thus the fluctuations (2.94) can be written as

$$\delta g_{ab} = Y_\Delta(\Omega) \begin{pmatrix} h_{00} - 2\,\Delta\,s\,\omega^2\sin^2\theta & 0 \\ 0 & 2\,\Delta\,s\cos^2\theta\,\bar{g}_{ab} \end{pmatrix} \tag{2.99}$$

being h_{00} the tt-component of $\delta g_{\mu\nu}$ (up to a spherical harmonic function). We want to evaluate the fluctuation of the DBI part of the action (2.28). Recalling that $\delta\sqrt{g} = \frac{1}{2}\sqrt{g}\delta g^{ab}$ we have that

$$\delta\sqrt{g} = \frac{1}{2}\sqrt{g}Y_\Delta(\Omega)\left(\frac{h_{00} - 2\,\Delta\,s\,\omega^2\sin^2\theta}{1-\omega^2\sin^2\theta} + 6\,\Delta\,s\right). \tag{2.100}$$

Finally, substituting the expression for the DBI lagrangian (2.34) $N\cos^3\theta\left(1-\omega^2\ \sin^2\theta\right)^{\frac{1}{2}}$ and the explicit form of the fluctuations $\frac{h_{00}}{2} = \left(\frac{2}{\Delta+1}\partial_t^2 - \frac{\Delta(\Delta-1)}{\Delta+1}\right)s$, we obtain

$$\begin{aligned}\delta S_{DBI} =& \frac{1}{2}\int_{-\infty}^{+\infty} dt \frac{N\cos^3\theta\left(1-\omega^2\sin^2\theta\right)^{\frac{1}{2}} Y_\Delta\left(\Omega\right)}{1-\omega^2\sin^2\theta} \\ &\left(\frac{4}{\Delta+1}\partial_t^2 - \frac{2\Delta\left(\Delta-1\right)}{\Delta+1} - 8\Delta\,\omega^2\sin^2\theta + 6\Delta\right)s^\Delta.\end{aligned} \tag{2.101}$$

Replacing the field s^Δ with the bulk to boundary propagator (2.95), namely [5]

$$s^\Delta = \mathcal{R}\frac{\Delta+1}{N\Delta^{\frac{1}{2}}2^{2-\frac{\Delta}{2}}}z^\Delta \tag{2.102}$$

[5] We denote with $\mathcal{R} \equiv 1/x_B^2$ where x_B is the separation between the two bulk insertion points.

we get

$$\begin{aligned}\delta S_{DBI} &= \mathcal{R}\int_{-\infty}^{+\infty} dt \frac{\cos^3\theta}{\left(1-\omega^2\sin^2\theta\right)^{\frac{1}{2}}}\frac{(\Delta+1)\,\Delta^{\frac{1}{2}}}{2^{1-\frac{\Delta}{2}}}\frac{Y_\Delta(\Omega)}{\cosh^\Delta t}\left(1-2\omega^2\sin^2\theta+\tanh^2 t\right)\\ &= \frac{\cos^3\theta\,(\sin\theta)^\Delta\,(\Delta+1)\,\Delta^{\frac{1}{2}}}{2\left(1-\omega^2\sin^2\theta\right)^{\frac{1}{2}}}\frac{e^{\omega\Delta t}}{\cosh^\Delta t}\left(2-2\omega^2\sin^2\theta-\frac{1}{\cosh^2 t}\right)\end{aligned} \tag{2.103}$$

We specify $\omega = 1$ and so (2.103) becomes

$$\delta S_{DBI} = \mathcal{R}\int_{-\infty}^{+\infty} dt \frac{\cos^2\theta\,(\sin\theta)^\Delta\,(\Delta+1)\,\Delta^{\frac{1}{2}}}{2}\frac{e^{\Delta t}}{\cosh^\Delta t}\left(2\,\cos^2\theta-\frac{1}{\cosh^2 t}\right). \tag{2.104}$$

WZ part of the action We are left with the Wess-Zumino part of the action. We need to take into account how the supergravity fluctuations couple to the Wess Zumino potential. The fluctuations involving 4-form are

$$\begin{aligned}\delta a_{\mu_1\mu_2\mu_3\mu_4} &= 4\,\epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5}\nabla^{\mu_5} b\\ \delta a_{\alpha_1\alpha_2\alpha_3\alpha_4} &= -4\sum_I \epsilon_{\alpha\alpha_1\alpha_2\alpha_3\alpha_4} b^I \nabla^\alpha Y^I(\Omega)\end{aligned} \tag{2.105}$$

where $\mu_1 \ldots \mu_5$ are AdS_5 and $\alpha \ldots \alpha_4$ are S_5 indices. Because of all the legs of $C_{(4)}$ are in S_5 it is required only $\delta a_{\phi\chi_1\chi_2\chi_3}$:

$$\delta a_{\phi\chi_1\chi_2\chi_3} = -4\epsilon_{\theta\phi\chi_1\chi_2\chi_3} b^I \nabla^\theta Y^I(\Omega) = -2^{-\frac{\Delta}{2}+2}\epsilon_{\theta\phi\chi_1\chi_2\chi_3}\Delta b^I (\sin\theta)^{\Delta-1}\cos\theta e^{-\Delta\tau}. \tag{2.106}$$

Therefore the variation of the Wess Zumino part is

$$\begin{aligned}\delta S_{WZ} &= -2^{-\frac{\Delta}{2}+2} N\Delta e^{\Delta t}(\sin\theta)^\Delta\cos^4\theta s^\Delta =\\ &= -\cos^4\theta\,(\sin\theta)^\Delta\,(\Delta+1)\,\Delta^{\frac{1}{2}}\frac{e^{-\Delta\tau}}{\cosh^\Delta\tau}\end{aligned} \tag{2.107}$$

Antisymmetric giant gravitons and a point like graviton Let us make an important remark which is very important in order to fix the relative sign between the DBI and the WZ part of the Euclidean action. We remind that the Euclidean form of the D-brane action[6]

$$S^E_{D3} = \frac{N}{2\pi^2}\int d^4\sigma\left(\sqrt{g}-iP[C_4]\right), \tag{2.108}$$

[6]The anti-D-brane action has a flipped sign on the WZ part.

and note that the four-form potential with legs in the AdS_5 part of the geometry gains a $-i$ under the Wick rotation, $C_4^{AdS} \to -iC_4^{AdS}$, due to having a leg in the temporal direction; the potential on S^5 is unaffected. Plugging the Wick-rotated solutions into the Euclidean action always yields a real result, since the angle $\phi = -it$ compensates for the factor of i in the Wess-Zumino term for the giant graviton on S^5, whose four-form potential has a leg in the ϕ direction.

We can see that summing up (2.104) with (2.107) there are two terms that cancel against each other [34]. Actually this step is subtle as pointed out in [36] because these terms are of the form $0 \cdot \infty$ and in the extremal case the two terms do not cancel completely but it remains a finite part. Thus a regularisation procedure is needed as proposed in [36]. The idea to start with a non extremal correlator and take the extremal limit after performing the integration. Moving to the extremal case amounts of choosing a different light operator namely a BPS state like

$$\text{Tr}\left(Z^{J+l}\bar{Z}^l\right). \tag{2.109}$$

Thus we will have spherical harmonic functions and the integral over t depending on l. In fact if we consider the integral

$$\int_{-\infty}^{+\infty} dt \, \frac{e^{nt}}{\cosh^{\Delta+2n} t} = 2^{\Delta+2n-1} \frac{\Gamma(\frac{1}{2}(\Delta+n)+l)\Gamma(\frac{1}{2}(\Delta-j)+n)}{\Gamma(\Delta+2n)}. \tag{2.110}$$

we can see that for $n = 0$ this reduces to the integral that one has to compute for the extremal correlator, see (2.103). The spherical harmonics associated to the BPS operator (2.109) can be written as

$$Y_{\Delta,\Delta-2l} = \frac{\Gamma(J+l+1)\sqrt{(J+l+1)(l+1)}2^{-J/2}}{\Gamma(l+2)\Gamma(J+1)\sqrt{J+2l+1}\;2^l} \sin^J\theta e^{i(J\phi)} \,{}_2F_1(-l, J{+}l{+}2, J{+}1; \sin^2\theta). \tag{2.111}$$

where $Y_{\Delta,\Delta} = Y_\Delta$. Note that to get (2.107) we considered the θ derivative of the spherical harmonics. If we derive with respect to θ (2.111) and combine with the integral (2.110), where we substitute $\Delta = J$ and $\cos^2\theta = \frac{k}{N}$ to be in contact with the gauge theory computations, we have that the difference between the terms that were divergent is a finite piece. Adding all the pieces coming from the regularised (2.104) and (2.107) we obtain the final answer

$$c^A_{k,k-J,J} = \sqrt{J}\left(1 - \frac{k}{N}\right)^{J/2}. \tag{2.112}$$

Symmetric giant gravitons and a point like graviton In complete analogy with the antisymmetric case it is possible to study the case when the giant gravitons are spinning in AdS, [34]. We write the metric on $S^3 \subset AdS_5$ as

$$ds^2 = d\vartheta^2 + \cos^2\vartheta d\phi_1^2 + \sin^2\vartheta d\phi_2^2, \tag{2.113}$$

so that embedding coordinates are given by

$$n_I = (\cos\vartheta\sin\phi_1, \cos\vartheta\cos\phi_1, \sin\vartheta\sin\phi_2, \sin\vartheta\cos\phi_2). \tag{2.114}$$

The variation of the Lagrangian density is

$$\begin{aligned}\delta\mathcal{L} =& -\frac{N}{4\pi^2}\sinh^2\rho\cos\vartheta\sin\vartheta\left[-2\Delta s + h_{tt} + h_{\vartheta\vartheta} + \frac{h_{\phi_1\phi_1}}{\cos^2\vartheta} + \frac{h_{\phi_2\phi_2}}{\sin^2\vartheta}\right] \\ &+ \frac{2N}{\pi^2}\cosh\rho\sinh^3\rho\cos\vartheta\,\partial_\rho s.\end{aligned} \tag{2.115}$$

where the second line is the WZ part of the variation. We have that

$$h_{\mu\nu} = \frac{2}{\Delta+1}\Big[2\nabla_\mu\nabla_\nu - \Delta(\Delta-1)g_{\mu\nu}\Big]s, \tag{2.116}$$

while

$$\begin{aligned}\nabla_t\nabla_t s &= \partial_t^2 + \cosh\rho\sinh\rho\,\partial_\rho, \\ \nabla_\vartheta\nabla_\vartheta s &= \partial_\vartheta^2 + \cosh\rho\sinh\rho\,\partial_\rho, \\ \nabla_{\phi_1}\nabla_{\phi_1} s &= \partial_{\phi_1}^2 + \cos^2\vartheta\cosh\rho\sinh\rho\,\partial_\rho - \cos\vartheta\sin\vartheta\,\partial_\vartheta, \\ \nabla_{\phi_2}\nabla_{\phi_2} s &= \partial_{\phi_2}^2 + \sin^2\vartheta\cosh\rho\sinh\rho\,\partial_\rho + \cos\vartheta\sin\vartheta\,\partial_\vartheta.\end{aligned} \tag{2.117}$$

Now we may replace the field s with the bulk-to-boundary propagator

$$s \to \frac{\Delta+1}{2^{2-\Delta/2}\sqrt{\Delta}N}\frac{1}{(\cosh\rho\cos t - \cos\vartheta\sin\phi_1\sinh\rho)^\Delta}. \tag{2.118}$$

When adding together the variation of the DBI and the WZ term we get

$$\begin{aligned}\delta S =& \int_{-\infty}^{\infty} dt \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 \int_0^{\pi/2} d\vartheta \\ &\frac{2^{\Delta/2}\sqrt{\Delta}(\Delta+1)}{4\pi^2}\cos\vartheta\sin\vartheta\sinh^2\rho\frac{Y}{(\cosh\rho\cos t - \cos\vartheta\sin\phi_1\sinh\rho)^{\Delta+2}}\end{aligned} \tag{2.119}$$

We may re-cast the integral as follows

$$
\begin{aligned}
&\frac{2^{\Delta/2}\sqrt{\Delta}(\Delta+1)}{2\pi}\frac{\sinh^2\rho}{\cosh^{\Delta+2}\rho}\int_{-\infty}^{\infty}dt\int_0^{2\pi}d\phi_1\int_0^1 d\lambda\frac{Y_\Delta}{\cosh^{\Delta+2}t}\frac{\lambda}{\left[1-\frac{\lambda\sin\phi_1\tanh\rho}{\cosh t}\right]^{\Delta+2}}\\
&=\frac{\sqrt{\Delta}(\Delta+1)}{2\pi}\frac{\sinh^2\rho}{\cosh^{\Delta+2}\rho}\int_{-\infty}^{\infty}dt\int_0^{2\pi}d\phi_1\int_0^1 d\lambda\frac{Y_\Delta}{\cosh^{\Delta+2}t}\\
&\qquad\qquad\lambda\sum_{k=0}^{\infty}\left(\frac{\lambda\sin\phi_1\tanh\rho}{\cosh t}\right)^k\frac{\Gamma(\Delta+k+2)}{\Gamma(k+1)\Gamma(\Delta+2)}\\
&=\frac{2^{\Delta/2}\sqrt{\Delta}(\Delta+1)}{2\pi}\frac{\sinh^2\rho}{\cosh^{\Delta+2}\rho}\int_{-\infty}^{\infty}dt\int_0^{2\pi}d\phi_1\frac{Y_\Delta}{\cosh^{\Delta+2}t}\\
&\qquad\qquad\sum_{k=0}^{\infty}\frac{1}{k+2}\left(\frac{\sin\phi_1\tanh\rho}{\cosh t}\right)^k\frac{\Gamma(\Delta+k+2)}{\Gamma(k+1)\Gamma(\Delta+2)}\\
&=\frac{2^{\Delta/2}\sqrt{\Delta}(\Delta+1)}{2\sqrt{\pi}}\frac{\sinh^2\rho}{\cosh^{\Delta+2}\rho}\int_{-\infty}^{\infty}dt\frac{Y_\Delta}{\cosh^{\Delta+2}t}\\
&\qquad\qquad\sum_{k=0}^{\infty}\frac{\Gamma(k+1/2)}{\Gamma(k+2)}\left(\frac{\tanh\rho}{\cosh t}\right)^{2k}\frac{\Gamma(\Delta+2k+2)}{\Gamma(2k+1)\Gamma(\Delta+2)}
\end{aligned}
\tag{2.120}
$$

By doing the same regularisation that we have shown above [36] we find

$$c^S_{k,k-J,J}=\frac{1}{\sqrt{J}}\left(1+\frac{k}{N}\right)^{J/2} \tag{2.121}$$

where we have substituted $\sinh^2\rho=\frac{k}{N}$ and $\Delta=J$ to compare with the gauge theory results.

2.3.3 Comparison

We computed three point functions of BPS operators, which are protected, which means that the gauge theory and the string theory results should agree in the limit where they are supposed to be the description of the same object and indeed (2.121) agrees with the dual (2.72) and (2.112) agrees with (2.73). We already discussed the issues due to the fact that there are divergences coming out when considering extremal correlators. In [34] we do not consider this finite contribution and we ended up with a mismatch. Subsequently in [35] this disagreement has been tried to understand in two ways. Firstly studying the non extremal version of the three point functions presented above and non extremal three point functions using a fully back reacted bubbling geometry (LLM) which corresponds to Schur polynomials have been compared and indeed the two computations agree and they also equate the corresponding gauge theory results. Secondly the authors compute the extremal three point functions of two giant gravitons and a point like graviton using the bubble geometry analysis and these results coincide with the gauge theory side that we report in (2.73) and (2.72). The procedure of regularisation that we have discussed has been analysed in [36] and indeed it shows that we obtain agreement between the gauge and string theory computations.

2.4 Two heavy and one light operators at one loop in the AdS$_5$/CFT$_4$ correspondence

Another interesting comparison between a gauge theory computation and its string dual has been discussed in [21]. In this paper a weak/strong coupling match was found for the tree level part of a three point function with two very large operators and a small one. The gauge theory computation has been matched to the corresponding three point function on the string side, taking the Frolov-Tseytlin limit [37, 38] that we rewieved in Sec. 1.3. The operators considered in the three-point function are all in the $SU(3)$ sector of $\mathcal{N}=4$ SYM and they are chosen in such a way that the respective three point function is non extremal. An analogous computation for operators in the $SL(2)$ sector was considered in [27]. Also in this case it was found perfect agreement between the weak and strong coupling result. In particular, if we consider operators in the $SU(3)$ sector we find on both the gauge theory and string theory sides the energy (scaling dimension)

$$E = J + \frac{\lambda}{2J}\int_0^{2\pi}\frac{d\sigma}{2\pi}\partial_\sigma\bar{\mathbf{u}}\cdot\partial_\sigma\mathbf{u} + \mathcal{O}(\lambda^2/J^3) \tag{2.122}$$

with the non-linear sigma-model field $\mathbf{u}(\tau,\sigma)$ taking values in $\mathbb{C}^3$ and being a solution of the equations of motion (EOMs) following from using (2.122) as the Hamiltonian supplemented by the constraint $\mathbf{u}\cdot\partial_\sigma\bar{\mathbf{u}}=0$. The work of [21] can thus be seen as a natural extension of the work of [37, 38] to three point correlation functions, using the prescription of [10] for two semi-classical operators and a light chiral primary operator in the $SU(3)$ sector of $\mathcal{N}=4$ SYM theory. Amazingly, they found on both the gauge theory and the string theory side the same result

$$c_{123}^{(0)} = \frac{J}{N}\frac{(j_2+j_3)!}{j_2!j_3!}\sqrt{\frac{j_1!j_2!j_3!}{(j_1+j_2+j_3-1)!}}\int_0^{2\pi}\frac{d\sigma}{2\pi}\bar{u}_1^{j_1}u_2^{j_2}u_3^{j_3}. \tag{2.123}$$

The gauge theory operators are constructed from the three complex scalars Z, X and Y of $\mathcal{N}=4$ SYM theory and their complex conjugates $\bar{Z}$, $\bar{X}$ and $\bar{Y}$. The

Operator	field type	field type	field type
$\mathcal{O}_1$	$(J_1+j_1)\;\;\bar{Z}$	$(J_2-j_2)\;\;\bar{X}$	$(J_3-j_3)\;\;\bar{Y}$
$\mathcal{O}_2$	$(J_1)\;\;Z$	$(J_2)\;\;X$	$(J_3)\;\;Y$
$\mathcal{O}_3$	$j_1\;\;Z$	$j_2\;\;\bar{X}$	$j_3\;\;\bar{Y}$

Table 2.1: Operators in the $SU(3)$ sector of $\mathcal{N}=4$ SYM theory.

operators are described in table 2.1. We introduce the quantity $J=J_1+J_2+J_3$. Note that, by construction, this is a non-extremal three point function for $j_2+j_3\neq 0$. While $\mathcal{O}_3$ is taken to be a 1/2 BPS chiral primary operator, $\mathcal{O}_1$ and $\mathcal{O}_2$ are constructed as coherent states with corresponding sigma-model fields $\mathbf{u}(\tau,\sigma)$ and $\bar{\mathbf{u}}(\tau,\sigma)$, respectively. Here $\mathbf{u}=(u_1,u_2,u_3)$ is a solution of the EOMs following from the one-loop Hamiltonian (2.122). The coefficient (2.123) is then computed at tree-level by doing Wick contractions. On the string theory side, one considers the leading part of c_{123} in the Frolov-Tseytlin limit of the corresponding three point function using the prescription of [10]. In the following we explore whether the match of the three point correlation function coefficient (2.123) between the gauge theory and string theory sides can be extended beyond tree level on the gauge theory side to include the one-loop correction, corresponding to the first order in the Frolov-Tseytlin expansion parameter λ/J^2 on the string theory side [40] . We already discussed that the three point function coefficient admits an expansion in λ namely we have that[7]

$$c_{123} = c_{123}^{(0)} + \lambda c_{123}^{(1)} + \cdots \tag{2.124}$$

where c_{123} is the structure constant coefficient in (1.46). Note that we use the renormalized operators thus as we pointed out in 2.1 $c_{123}^{(1)}$ is scheme independent. In the following we compute c_{123}, on the gauge theory side we compute the one-loop correction to the tree level result including

[7]Note that we consider operators with unit normalised two point function, differently from [49] and that we introduce the superscript (0) to distinguish the tree level contribution from the one loop contribution which we label with the superscript (1).

- the contribution coming from requiring that the two heavy operators correspond to eigenstates of the two-loop correction of the dilation operator;
- the contribution coming from one loop diagrams for the three point function

while on the string theory side we use the approach of [10] and we obtain the expression for the three point function coefficient up to two loops. Note that to compute two loop correction to the three point function we need to solve the Landau Lifshitz sigma model to order λ'^3. At this order the wave function appearing in the correlation function admits a contribution of order λ'^2. We refer to Sec. 1.3 for details.

Operators We consider three operators in the $SU(2)$ sector of $\mathcal{N}=4$ SYM theory (note that obviously they are not in the same $SU(2)$ sector) and we report in table 2.2 the precise form of $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$. Note that in order to ensure that $\mathcal{O}_1$ and $\mathcal{O}_2$ can be considered semiclassical while $\mathcal{O}_3$ is a light operator we have that $J \gg j \gg 1$ where $J = J_1 + J_2$.

Operator	field type	field type
$\mathcal{O}_1$	$(J_1+j)\ \ \bar{Z}$	$(J_2-j)\ \ \bar{X}$
$\mathcal{O}_2$	$(J_1)\ \ Z$	$(J_2)\ \ X$
$\mathcal{O}_3$	$j\ \ Z$	$j\ \ \bar{X}$

Table 2.2: Operators in the $SU(2)$ sector of $\mathcal{N}=4$ SYM theory. Note that this gives a non-extremal three point function for any non zero j.

2.4.1 Gauge theory side

In the following we discuss how to represent the operators reported in table 2.2, we review how to compute the tree level three point function for such operators obtaining the result already present in [21] and then we proceed considering the one loop correction to the structure constant which involves two different contributions. Finally we mention the correction coming from the spin flipping in the coherent state description. Following [21] we use coherent states to approximate the two heavy operators, that we reviewed in 1.3 The three operators $\mathcal{O}_i(x_i)$, $i=1,2,3$, for which we compute the three point function are given as follows. All three operators are in the scalar sector of $\mathcal{N}=4$ SYM theory and we consider single trace operators made out of three complex scalars Z, X and Y. Moreover, each operator is in an $SU(2)$ sector of $\mathcal{N}=4$ SYM, see table 2.2 and thus is made of only two complex scalars that we chose to be Z and X. The $\mathcal{O}_1(x_1)$ and $\mathcal{O}_2(x_2)$ operators are semiclassical operators thus with $J \gg 1$ while $\mathcal{O}_3(x_3)$ is the BPS chiral primary operator. We can write

$$\mathcal{O}_1(x_1) = \mathcal{N}_1 \bar{\mathbf{u}}^{i_1}(\frac{k+1}{l})\,\bar{\mathbf{u}}^{i_2}(\frac{k+2}{l})\cdots\bar{\mathbf{u}}^{i_J}(\frac{k}{l}) : \mathrm{Tr}(\bar{W}_{i_1}\bar{W}_{i_2}\cdots\bar{W}_{i_J}) : (x_1) \quad (2.125)$$

$$\mathcal{O}_2(x_2) = \mathcal{N}_2 \mathbf{v}_{j_1}(\frac{k+1}{l})\,\mathbf{v}_{j_2}(\frac{k+2}{l})\cdots\mathbf{v}_{j_J}(\frac{k}{l}) : \mathrm{Tr}(W^{j_1}W^{j_2}\cdots W^{j_J}) : (x_2) \quad (2.126)$$

with

$$W^i = (Z, X)\ , \quad \bar{W}_i = (\bar{Z}, \bar{X})\ , \quad l \equiv \frac{J}{2\pi}. \quad (2.127)$$

Here $\mathbf{u}(\sigma)$ and $\mathbf{v}(\sigma)$ correspond for each site of the single trace operators to coherent states in the spin 1/2 representation of $SU(2)$. Specifically the k'th site is at $\sigma = k/l$ and the functions $\mathbf{u}(\sigma)$ and $\mathbf{v}(\sigma)$ are periodic in σ with period 2π and they take values in $\mathbb{C}^2$. Since the two operators are semiclassical also means that the functions $\mathbf{u}(\sigma)$ and $\mathbf{v}(\sigma)$ are slowly varying in σ. The third operator $\mathcal{O}_3(x_3)$ is a single trace chiral primary which can be written as

$$\mathcal{O}_3(x_3) = \mathcal{N}_3 : \mathrm{Tr}\,\mathrm{sym}(\bar{X}^j Z^j) : (x_3) \quad (2.128)$$

It is important to note that we did not include the corrections to the coherent state description of the operators (2.125)-(2.126) from the so-called spin-flipped coherent states [97]. We will discuss this point later. Introducing $\lambda' = \lambda/J^2$ we can arrange the expansion (2.124) as

$$c_{123} = c_{123}^{(0)} + \lambda' c_{123}^{(1)} + \mathcal{O}(\lambda'^2). \quad (2.129)$$

Tree level computation Let us briefly review the computation of the leading planar contribution to $\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle$ at tree-level [21]. We have two contributions, one coming from the planar contractions among the fields in $\mathcal{O}_3$ with $\mathcal{O}_1$ and $\mathcal{O}_2$ and another one from the contractions involving only fields in $\mathcal{O}_1$ with fields in $\mathcal{O}_2$, see Fig. 2.2. Disregarding propagators, combinatoric factors and such, the tree-level contractions give

$$A(k) = B \prod_{m=k+1}^{k+j} \frac{\bar{u}^1(\frac{m}{l})v_2(\frac{m}{l})}{\bar{\mathbf{u}}(\frac{m}{l})\cdot\mathbf{v}(\frac{m}{l})} \quad (2.130)$$

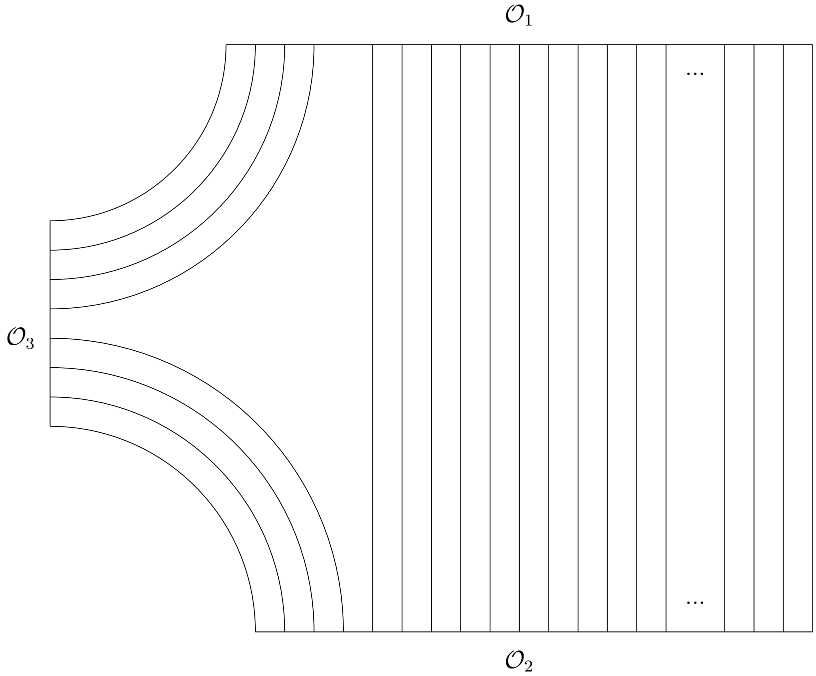


FIGURE 2.2: Tree level planar contractions between $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$.

where

$$B \equiv \prod_{m=1}^{J} \bar{\mathbf{u}}(\frac{m}{l}) \cdot \mathbf{v}(\frac{m}{l}). \tag{2.131}$$

Note that B depends on $\mathbf{u}(\sigma)$ and $\mathbf{v}(\sigma)$ but not on the choice of k. Including the sum over k, we have

$$\sum_k A(k) = B \sum_k \prod_{m=k+1}^{k+j} \frac{\bar{u}^1(\frac{m}{l})v_2(\frac{m}{l})}{\bar{\mathbf{u}}(\frac{m}{l}) \cdot \mathbf{v}(\frac{m}{l})}. \tag{2.132}$$

We can see from the equations above and from figure 2.2 that we have $J - j$ contractions involving fields in $\mathcal{O}_1$ and the respective complex conjugates in $\mathcal{O}_2$ and these fields can be either X or Z while there are j contractions between Z type fields of $\mathcal{O}_3$ with their conjugates in $\mathcal{O}_1$ and j contractions among $\bar{X}$ type fields of $\mathcal{O}_3$ with their conjugates in $\mathcal{O}_1$. Since $\mathcal{O}_1$ and $\mathcal{O}_2$ are semiclassical and $\mathcal{O}_3$ is a light chiral primary we assume that

- $\bar{\mathbf{u}}$ and $\mathbf{v}$ are slowly varying functions
- $j \ll J$
- $\mathbf{v}(\sigma) = \mathbf{u}(\sigma)$

giving

$$\sum_k A(k) \simeq B \sum_k \left(\frac{(\bar{u}^1 v_2)(\frac{k}{l})}{(\bar{\mathbf{u}} \cdot \mathbf{v})(\frac{k}{l})} \right)^j \simeq B\, J \int_0^{2\pi} \frac{d\sigma}{2\pi} \left(\frac{(\bar{u}^1 v_2)(\sigma)}{(\bar{\mathbf{u}} \cdot \mathbf{v})(\sigma)} \right)^j . \tag{2.133}$$

We now want to use the approximation in (2.133). We parameterize the difference between $\mathcal{O}_1$ and $\mathcal{O}_2$ as

$$\mathbf{v}(\sigma) = \mathbf{u}(\sigma) + \frac{j}{l} \delta \mathbf{u}(\sigma) \tag{2.134}$$

Using now (2.134) in (2.133) we find that $B = 1$ to leading order in j/J and hence

$$c_{123}^{(0)} = \frac{\mathcal{N}_3}{N} \sum_k A(k) = \frac{1}{N} \frac{j!J}{\sqrt{(2j-1)!}} \int_0^{2\pi} \frac{d\sigma}{2\pi} (\bar{u}^1 u_2)^j \tag{2.135}$$

up to finite size corrections in $1/J$, where we used that $\mathcal{N}_3 = \frac{j!}{\sqrt{(2j-1)!}}$. This is the same result already derived in [21], adapted to our operators.

One loop computation At one loop there are two types of corrections that one should take into account. The first type is due to the two loop contribution to the effective sigma model description which amounts to corrections of order λ to the external wave function. The second correction is due to the one loop diagrams with two legs in one of the operators and the other two legs in two different operators, as shown in Fig 2.3. These diagrams can be computed in the planar limit using the spin chain inspired methods of [49, 50, 116] that we reviewed in Sec. 2.1

1) Two loop correction to the eigenstates The first type of correction has been neglected in earlier studies of three point functions of gauge theory operators in $\mathcal{N} = 4$ SYM theory [49, 50, 116], as pointed out in [140–142]. While in general it is rather complicated to take into account this contribution, it actually becomes very easy for the particular set of operators that we are considering. This is due to the enormous simplification that one has by using a coherent state representation for the gauge theory operators. As already noticed in [21], this is also the reason that made the computation of the leading order contribution to c_{123} possible. In brief, to take into account this type of contribution we should simply use in our expressions the wave function which is solution of the EOMs up to two-loops that one derives from (1.113), with a change of notation from the vector $\vec{n}$ to $\mathbf{u}$. In fact writing

$$\mathbf{u} = \mathbf{u}^{(0)} + \lambda' \mathbf{u}^{(1)} + \mathcal{O}(\lambda'^2) \tag{2.136}$$

and substituting the full $\mathbf{u}$ in (2.135) one can compute these type of corrections order by order in λ'. We will implicitly compute these contributions by assuming

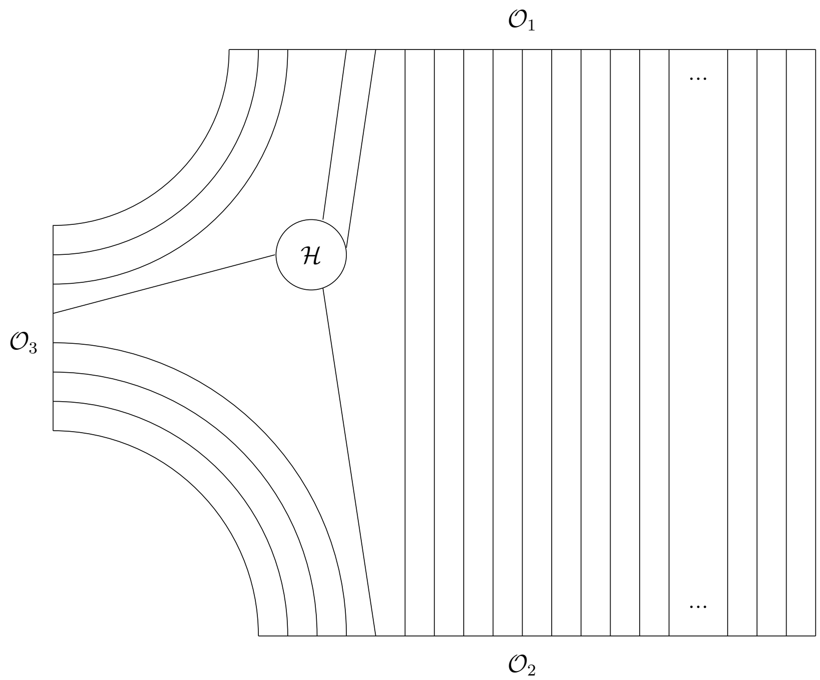


FIGURE 2.3: Example of a diagram contributing at one-loop with the insertion of the one-loop Hamiltonian with two legs in $\mathcal{O}_3$ and the other two legs in $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively.

that the function entering in the one-loop result for C_{123} is the one in (2.136). This procedure can be extended to include also higher orders in λ'. Note here that we assume that (2.134) holds also at order λ' since otherwise the difference between $\mathbf{u}$ and $\mathbf{v}$ would enter at order λ' when inserting (2.136) in (2.135).

2) One loop diagrams The other type of correction contributing at one-loop comes form the insertion of the one-loop Hamiltonian with two legs in one of the operators and the other two legs in two different operators (see Fig 2.3). We compute these corrections using the prescription given in [49, 50, 116] and more specifically we want to adapt to our setting (2.14) with the definitions of (2.20). Note that in this section we are using a different normalisation which amounts in considering operators which have a unit normalised two point function differently from [49]. Since we have three operators, there are three types of diagrams. From (2.14) we have that

$$c_{123}^{(1)} = \frac{1}{32\pi^2}\frac{J^2\mathcal{N}_3}{N}B\sum_{k=1}^{J}\left(\frac{(\bar{u}^1 v_2)(\frac{k}{l})}{(\bar{\mathbf{u}}\cdot\mathbf{v})(\frac{k}{l})}\right)^j\left(f_{23}^1(k)+f_{31}^2(k)+f_{12}^3(k)\right) \tag{2.137}$$

where B is given in (2.131), f_{23}^1 is the constant referring to the three point Feynman diagram with two contractions in $\mathcal{O}_1$ and one contraction each with $\mathcal{O}_2$ and $\mathcal{O}_3$

and so on. For a given k, (2.20) gives

$$f^1_{23}(k) = -\frac{\bar{u}^{i_1}(\frac{k+j+1}{l})\bar{u}^{i_2}(\frac{k+j}{l})v_{j_1}(\frac{k+j+1}{l})\delta^1_{j_2}}{(\bar{\mathbf{u}}\cdot\mathbf{v})(\frac{k+j+1}{l})\bar{u}^1(\frac{k+j}{l})}\mathcal{H}^{j_1j_2}_{i_1i_2} - \frac{\bar{u}^{i_1}(\frac{k+1}{l})\bar{u}^{i_2}(\frac{k}{l})\delta^1_{j_1}v_{j_2}(\frac{k}{l})}{\bar{u}^1(\frac{k+1}{l})(\bar{\mathbf{u}}\cdot\mathbf{v})(\frac{k}{l})}\mathcal{H}^{j_1j_2}_{i_1i_2} \tag{2.138}$$

$$f^2_{31}(k) = -\frac{\delta^{i_1}_2\bar{u}^{i_2}(\frac{k+j+1}{l})v_{j_1}(\frac{k+j}{l})v_{j_2}(\frac{k+j+1}{l})}{v_2(\frac{k+j}{l})(\bar{\mathbf{u}}\cdot\mathbf{v})(\frac{k+j+1}{l})}\mathcal{H}^{j_1j_2}_{i_1i_2} - \frac{\bar{u}^{i_1}(\frac{k}{l})\delta^{i_2}_2 v_{j_1}(\frac{k}{l})v_{j_2}(\frac{k+1}{l})}{(\bar{\mathbf{u}}\cdot\mathbf{v})(\frac{k}{l})v_2(\frac{k+1}{l})}\mathcal{H}^{j_1j_2}_{i_1i_2} \tag{2.139}$$

with

$$\mathcal{H}^{j_1j_2}_{i_1i_2} = 2(I-P)^{j_1,j_2}_{i_1,i_2}, \;,\quad I^{j_1j_2}_{i_1i_2} = \delta^{j_1}_{i_1}\delta^{j_2}_{i_2}\;,\quad P^{j_1j_2}_{i_1i_2} = \delta^{j_2}_{i_1}\delta^{j_1}_{i_2}. \tag{2.140}$$

From our choice of the operator $\mathcal{O}_3$, one can see that $f^3_{12} = 0$. Using that $\bar{\mathbf{u}}(\sigma)$ and $\mathbf{v}(\sigma)$ vary slowly we have that

$$\begin{aligned}
-\frac{1}{2}f^1_{23}(k) &= 2 - \frac{\bar{u}^1(\frac{k+j+1}{l})}{\bar{u}^1(\frac{k+j}{l})}\frac{\bar{\mathbf{u}}(\frac{k+j}{l})\cdot\mathbf{v}(\frac{k+j+1}{l})}{(\bar{\mathbf{u}}\cdot\mathbf{v})(\frac{k+j+1}{l})} - \frac{\bar{u}^1(\frac{k}{l})}{\bar{u}^1(\frac{k+1}{l})}\frac{\bar{\mathbf{u}}(\frac{k+1}{l})\cdot\mathbf{v}(\frac{k}{l})}{(\bar{\mathbf{u}}\cdot\mathbf{v})(\frac{k}{l})} \\
&= 2 - \left(1 + \frac{1}{l}\frac{\bar{u}^{1\prime}}{\bar{u}^1} + \frac{1}{2l^2}\frac{\bar{u}^{1\prime\prime}}{\bar{u}^1}\right)\bigg|_{\sigma=\frac{k+j}{l}}\left(1 - \frac{1}{l}\frac{\bar{\mathbf{u}}'\cdot\mathbf{v}}{\bar{\mathbf{u}}\cdot\mathbf{v}} + \frac{1}{2l^2}\frac{\bar{\mathbf{u}}''\cdot\mathbf{v}}{\bar{\mathbf{u}}\cdot\mathbf{v}}\right)\bigg|_{\sigma=\frac{k+j+1}{l}} \\
&\quad - \left(1 - \frac{1}{l}\frac{\bar{u}^{1\prime}}{\bar{u}^1} + \frac{1}{2l^2}\frac{\bar{u}^{1\prime\prime}}{\bar{u}^1}\right)\bigg|_{\sigma=\frac{k+1}{l}}\left(1 + \frac{1}{l}\frac{\bar{\mathbf{u}}'\cdot\mathbf{v}}{\bar{\mathbf{u}}\cdot\mathbf{v}} + \frac{1}{2l^2}\frac{\bar{\mathbf{u}}''\cdot\mathbf{v}}{\bar{\mathbf{u}}\cdot\mathbf{v}}\right)\bigg|_{\sigma=\frac{k}{l}} \\
&= \left\{2 - \left(1 + \frac{1}{l}\frac{\bar{u}^{1\prime}}{\bar{u}^1} + \frac{j}{l^2}\left(\frac{\bar{u}^{1\prime}}{\bar{u}^1}\right)' + \frac{1}{2l^2}\frac{\bar{u}^{1\prime\prime}}{\bar{u}^1}\right)\left(1 - \frac{j}{l^2}\bar{\mathbf{u}}'\cdot\delta\mathbf{u} + \frac{1}{2l^2}\bar{\mathbf{u}}''\cdot\mathbf{u}\right)\right. \\
&\quad \left. - \left(1 - \frac{1}{l}\frac{\bar{u}^{1\prime}}{\bar{u}^1} - \frac{1}{l^2}\left(\frac{\bar{u}^{1\prime}}{\bar{u}^1}\right)' + \frac{1}{2l^2}\frac{\bar{u}^{1\prime\prime}}{\bar{u}^1}\right)\left(1 + \frac{j}{l^2}\bar{\mathbf{u}}'\cdot\delta\mathbf{u} + \frac{1}{2l^2}\bar{\mathbf{u}}''\cdot\mathbf{u}\right)\right\}\bigg|_{\sigma=\frac{k}{l}} \\
&= \frac{1}{l^2}\left\{\bar{\mathbf{u}}'\cdot\mathbf{u}' - (j-1)\left(\frac{\bar{u}^{1\prime}}{\bar{u}^1}\right)' - \frac{\bar{u}^{1\prime\prime}}{\bar{u}^1}\right\}\bigg|_{\sigma=\frac{k}{l}}.
\end{aligned} \tag{2.141}$$

where we included terms up to order $1/J^2$. Similarly, we find

$$\begin{aligned}
-\frac{1}{2}f^3_{12}(k) &= 2 - \frac{v_2(\frac{k+j+1}{l})}{v_2(\frac{k+j}{l})}\frac{\bar{\mathbf{u}}(\frac{k+j+1}{l})\cdot\mathbf{v}(\frac{k+j}{l})}{(\bar{\mathbf{u}}\cdot\mathbf{v})(\frac{k+j+1}{l})} - \frac{v_2(\frac{k}{l})}{v_2(\frac{k+1}{l})}\frac{\bar{\mathbf{u}}(\frac{k}{l})\cdot\mathbf{v}(\frac{k+1}{l})}{(\bar{\mathbf{u}}\cdot\mathbf{v})(\frac{k}{l})} \\
&= \frac{1}{l^2}\left\{\bar{\mathbf{u}}'\cdot\mathbf{u}' - (j-1)\left(\frac{u_2{}'}{u_2}\right)' - \frac{u_2{}''}{u_2}\right\}\bigg|_{\sigma=\frac{k}{l}}
\end{aligned} \tag{2.142}$$

Inserting these results in (2.137) we obtain

$$c^{(1)}_{123} = -\frac{1}{2\,N}\frac{j!J}{\sqrt{(2j-1)!}}\int_0^{2\pi}\frac{d\sigma}{2\pi}(\bar{u}^1u_2)^j\left\{\bar{\mathbf{u}}'\cdot\mathbf{u}' - \frac{j-1}{2}\left(\frac{\bar{u}^{1\prime}}{\bar{u}^1} + \frac{u_2{}'}{u_2}\right)' - \frac{1}{2}\left(\frac{\bar{u}^{1\prime\prime}}{\bar{u}^1} + \frac{u_2{}''}{u_2}\right)\right\} \tag{2.143}$$

Combining this with the result for the leading order (2.135) with the wave function $\mathbf{u}$ solution of the EOMs up to two loops, we thus arrive at the final expression for the 3 point function

$$c_{123} = \frac{j!J}{N\sqrt{(2j-1)!}} \int_0^{2\pi} \frac{d\sigma}{2\pi} (\bar{u}^1 u_2)^j \left[1 - \frac{\lambda'}{2} \left\{ \bar{\mathbf{u}}' \cdot \mathbf{u}' + \frac{j^2-1}{2} \left(\frac{(\bar{u}^1 u_2)'}{\bar{u}^1 u_2} \right)^2 + \frac{\bar{u}^{1\prime} u_2'}{\bar{u}_1 u_2} \right\} \right] + \mathcal{O}(\lambda^2) \tag{2.144}$$

where we used partial integration to remove double derivatives. This is the quantity that we want to compare with the holographic counterpart.

Correction to coherent state description from spin-flipped coherent state

In the above, we computed the one-loop correction to the three point function for two heavy operators $\mathcal{O}_1(x_1)$ and $\mathcal{O}_2(x_2)$ and one light chiral primary operator $\mathcal{O}_3(x_3)$ using the coherent state description (2.125) and (2.126) for the two heavy operators. However, as found in [97], while at order λ gauge theory operators can be described in the long-wave length approximation using a coherent state, at order λ^2 one has to use a linear combination of a coherent state and a spin-flipped coherent state. This arises when integrating out the short scale degrees of freedom in the spin chain description. We consider below the effect of using the full linear combination, instead of only the coherent state part that we using in (2.125)-(2.126). Consider first the coherent state part. Note that we work in the $SU(2)$ sector in the following. We represent this by the state

$$|\psi_0\rangle = |\vec{n}_1\rangle \otimes |\vec{n}_2\rangle \otimes \cdots \otimes |\vec{n}_J\rangle \tag{2.145}$$

where for each site we write $|\vec{n}_k\rangle = R_k |\uparrow\rangle$ with R_k being a rotation matrix for the k'th site. The continuum description uses instead the function $\vec{n}(\frac{2\pi k}{J}) = \vec{n}_k$. The state (2.145) corresponds to the description of the $\mathcal{O}_1(x_1)$ and $\mathcal{O}_2(x_2)$ operators using Eqs. (2.125)-(2.126). Note also that we require $\vec{n}(\sigma)$ to solve the EOMs of the two-loop effective Lagrangian (1.113). However, as found in [97], the full gauge theory state at order λ^2 (and for large J) is given by

$$|\psi\rangle = \Big(1 - \frac{1}{2}\sum_{k,k'} |c_{k,k'}|^2\Big)|\psi_0\rangle + |\psi_1\rangle \ , \quad |\psi_1\rangle = \sum_{k,k'} c_{k,k'}|k,k'\rangle \tag{2.146}$$

where $|k,k'\rangle$ is built from the coherent states with two spin flips

$$|\downarrow_a\downarrow_b\rangle = R_1|\uparrow\rangle \otimes \cdots \otimes R_{a-1}|\uparrow\rangle \otimes R_a|\downarrow\rangle \otimes \cdots \otimes R_{b-1}|\uparrow\rangle \otimes R_b|\downarrow\rangle \otimes \cdots \otimes R_J|\uparrow\rangle \tag{2.147}$$

as follows

$$|k,k'\rangle = \frac{\sqrt{2}}{J} e^{-i(k+k')p} \sum_{b>a=1}^{J} e^{ika+ik'b} |\downarrow_a\downarrow_b\rangle \tag{2.148}$$

where p is a number giving an optional extra phase factor. Using the results and notation of [97] we can write

$$|\psi_1\rangle = \frac{J}{2\sqrt{2}} \sum_a \sum_{k,k'} \sum_{q=1}^{2} \lambda_q \frac{e^{i(k+k')(a-p)} e^{ik'q}}{\epsilon(k)+\epsilon(k')} A_{--}^{a,a+q} |{\downarrow_a \downarrow_{a+q}}\rangle \tag{2.149}$$

with

$$\lambda_1 = \frac{1}{4\pi^2} - \frac{\lambda}{16\pi^4} \ , \quad \lambda_2 = \frac{\lambda}{64\pi^4} \ , \quad \epsilon(k) = J^2[\lambda_1(1-\cos k) + \lambda_2(1-\cos 2k)] \tag{2.150}$$

where $\epsilon(k)$ is the energy for one spin flip, and for large J we have

$$A_{--}^{a,a+q} \simeq \frac{1}{2}\Big(\frac{2\pi q}{J}\Big)^2 B(\frac{2\pi a}{J}) \ , \quad B(\sigma) = -(\partial_\sigma \theta)^2 + \sin^2\theta (\partial_\sigma \varphi)^2 - 2i \sin\theta \partial_\sigma \theta \partial_\sigma \varphi \tag{2.151}$$

We now extract the part of this proportional to λ, discarding the terms which either give finite-size corrections at order λ^0 or terms of order λ^2. Then, for large J, we can write

$$|\psi_1\rangle = \frac{\lambda'}{4\sqrt{2}} \sum_a B(\frac{2\pi a}{J}) \sum_{q=1}^{2} F_q(a-p) |{\downarrow_a \downarrow_{a+q}}\rangle \tag{2.152}$$

$$F_1(a) = -\frac{1}{4J} \sum_{k,k'} \frac{e^{i(k+k')a} e^{ik'} (2-\cos 2k - \cos 2k')}{(2-\cos k - \cos k')^2} \ , \quad F_2(a) = \frac{1}{J} \sum_{k,k'} \frac{e^{i(k+k')a} e^{2ik'}}{2-\cos k - \cos k'} \tag{2.153}$$

Spin-flip correction and the three point function We now turn to the impact on the three point function computed in this paper. We can schematically write the two heavy operators as $\mathcal{O}_i(x_i) = \mathcal{O}_i^{(0)}(x_i) + \mathcal{O}_i^{(1)}(x_i)$, $i = 1, 2$, where $\mathcal{O}_i^{(0)}(x_i)$ are now the operators given in (2.125)-(2.126) using a coherent state description, and $\mathcal{O}_i^{(1)}(x_i)$ are the spin-flip corrections which can be inferred from (2.149). Note that to one-loop order we can approximate $|\psi\rangle = |\psi_0\rangle + |\psi_1\rangle$. We have

$$\begin{aligned} \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle &= \langle \mathcal{O}_1^{(0)}(x_1)\mathcal{O}_2^{(0)}(x_2)\mathcal{O}_3(x_3)\rangle + \langle \mathcal{O}_1^{(1)}(x_1)\mathcal{O}_2^{(0)}(x_2)\mathcal{O}_3(x_3)\rangle \\ &+ \langle \mathcal{O}_1^{(0)}(x_1)\mathcal{O}_2^{(1)}(x_2)\mathcal{O}_3(x_3)\rangle + \langle \mathcal{O}_1^{(1)}(x_1)\mathcal{O}_2^{(1)}(x_2)\mathcal{O}_3(x_3)\rangle \end{aligned} \tag{2.154}$$

We first remark that to one-loop order, we can only get a contribution from the spin-flip correction for the tree-level diagram since $|\psi_1\rangle$ in (2.152) is proportional to λ. Hence this also holds for $\mathcal{O}_{1,2}^{(1)}(x_i)$. From this it is also clear that the last term on the RHS of (2.154) is of order λ^2. Thus, at one-loop, the possible contributions from the spin-flipped coherent state corrections can come from computing the tree-level Wick contractions for the second and third terms on the RHS of Eq. (2.154) at tree-level. Consider the Wick contractions of the term $\langle \mathcal{O}_1^{(0)}(x_1)\mathcal{O}_2^{(1)}(x_2)\mathcal{O}_3(x_3)\rangle$. Thus, while $\mathcal{O}_1^{(0)}(x_1)$ is inferred from $|\psi_0\rangle$ the operator $\mathcal{O}_2^{(1)}(x_2)$ is inferred from $|\psi_1\rangle$. Consider now $|\psi_1\rangle$ of Eq. (2.152). Considering the tree-level Wick contractions we see that if the index a in (2.152) points to a site that contracts with $\mathcal{O}_1^{(0)}(x_1)$, the contribution is zero since $\langle \uparrow | \downarrow \rangle = 0$. Hence the non-zero contribution comes from values of a that point to sites that contracts with $\mathcal{O}_3(x_3)$ (and also such that either $a+1$ or $a+2$ contracts with $\mathcal{O}_3(x_3)$). Each of these sites contracts with an $\bar{X}$ in $\mathcal{O}_3(x_3)$. Due to the two spin flips, the contraction with the operator corresponding to the state $|\downarrow_a \downarrow_{a+q}\rangle$ picks up a factor $u_2^{j-2}u_1^2$ which combined with the Wick contractions between $\mathcal{O}_1(x_1)$ and $\mathcal{O}_3(x_3)$ gives a factor $(\bar{u}^1 u_2)^{j-2}(\bar{u}^1 u_1)^2$ factor. Combined with the other parts of (2.152) we pick up the contribution

$$\frac{\lambda'}{4\sqrt{2}} B(\sigma) \left[\sum_{a=1}^{j-1} F_1(a-p) + \sum_{a=1}^{j-2} F_2(a-p) \right] \tag{2.155}$$

One can find numerically that $F_1(a) + F_2(a)$ is of order $1/J$ for $a \neq 0$. However, taken separately $F_{1,2}(a)$ are of order J. Moreover, $F_{1,2}(a)$ peaks around $a = 0$. Indeed for large J one finds that $F_1(a) + F_2(a) \propto \delta_a$ (note that this result is consistent with using the approximation $\epsilon(k) + \epsilon(k') \simeq 2\epsilon(k')$ in [97]). Hence, the contribution (2.155) is highly sensitive to the value of p. Thus it is not completely clear how this kind of correction can be taken into account. However we can notice that

- From conservation of R-charge one could argue that the spin flip correction should end up being zero. Indeed, the expectation values of the R-charges changes when flipping the spins in the coherent state. This suggests that by R-charge conservation the second and third terms on the RHS (2.154) should be zero. However, this is not a precise argument since the coherent states are not eigenstates of the R-charges. Nevertheless one could speculate that the fact that the R-charges in $\langle \mathcal{O}_1^{(1)}(x_1)\mathcal{O}_2^{(0)}(x_2)\mathcal{O}_3(x_3)\rangle$ and $\langle \mathcal{O}_1^{(0)}(x_1)\mathcal{O}_2^{(1)}(x_2)\mathcal{O}_3(x_3)\rangle$ are not conserved on the level of expectation values should mean that their contributions are highly suppressed in the large J limit;

- a possible contribution from the spin-flip correction would seem to be proportional to the function $(\bar{u}^1 u_2)^{j-2}(\bar{u}^1 u_1 B(\sigma) + \bar{u}^2 u_2 \bar{B}(\sigma))$. This function is not proportional to any of the three terms at order λ in (2.144). Thus, if this contribution is non-zero it would seem that it introduces a new type of term in the three point function coefficient (2.144).

2.4.2 String theory side

In this section we describe the computation of the one and two loop correction to the holographic three-point function coefficient for the case of the two semiclassical operators and the small 1/2 BPS operator considered in the previous section. This is done following the work initiated in [10, 21]. The two large operators are described by semiclassical strings while the small BPS operator corresponds to a quantum string.

Three point function at order λ' Our starting point is the sigma-model for type IIB string theory on $\mathrm{AdS}_5 \times S^5$ in the regime in which it is described by the Landau-Lifshitz sigma-model [38]. We now are ready to use this result to compute the corrections to the holographic three point correlation function coefficient c_{123} for two semi-classical operators and a light chiral primary operator up to two-loops. If we rewrite (2.25) in our notation it gives

$$c_{123} = c_j \frac{\sqrt{\lambda}}{N} \int_{-\infty}^{+\infty} d\tau_e \int_0^{2\pi} \frac{d\sigma}{2\pi} \frac{(\bar{U}^1 U_2)^j}{\cosh^{2j}(\frac{\tau_e}{\kappa})} \left[\frac{2}{\kappa^2 \cosh^2(\frac{\tau_e}{\kappa})} - \frac{1}{\kappa^2} - \partial_a \bar{\mathbf{U}} \cdot \partial^a \mathbf{U} \right] \tag{2.156}$$

where τ_e is the Euclidean time and we already used the gauge choice (1.131) which allows to make contact with our gauge theory computation. c_j is a constant depending only on the parameter j which is associate to the supergravity mode dual to the chiral primary operator. In our case it is given by

$$c_j = \frac{(2j+1)!}{2^{2j+2} j! \sqrt{(2j-1)!}} \tag{2.157}$$

Here $\mathbf{U}(\tau, \sigma)$ is a complex vector that parametrizes the embedding of the type IIB string on S^5. We have that only two components of the complex vector are non zero because we chose our operators to be in $SU(2)$ sectors of $\mathcal{N}$=4 SYM

$$U_1 = \sin\psi e^{i\phi_1} \ , \quad U_2 = \cos\psi e^{i\phi_2} \ , \quad U_3 = 0 \tag{2.158}$$

and we work in the Frolov-Tseytlin limit [37, 38] which in our notation is

$$\kappa \to 0\ , \quad \frac{1}{\kappa}\partial_\tau \mathbf{U} \ \text{ fixed}\ , \quad \partial_\sigma \mathbf{U} \ \text{ fixed.} \tag{2.159}$$

We can then compute the term $\partial_a \bar{\mathbf{U}} \cdot \partial^a \mathbf{U}$ appearing in (2.156). In the limit (2.159) it becomes

$$\partial_a \bar{\mathbf{U}} \cdot \partial^a \mathbf{U} \quad = \quad -\frac{1}{\kappa^2} + \frac{1}{2}(\vec{n}')^2 - \frac{\kappa^2}{16}(\vec{n}')^4 + \mathcal{O}(\kappa^4) \tag{2.160}$$

Therefore up to terms second order in κ^2 (or equivalently λ') we get

$$c_{123} = c_j \frac{\sqrt{\lambda}}{N} \int_{-\infty}^{+\infty} d\tau_e \int_0^{2\pi} \frac{d\sigma}{2\pi} \frac{(e^{-i\varphi}\cos\theta)^j}{2^j \cosh^{2j}(\frac{\tau_e}{\kappa})} \left[\frac{2}{\kappa^2 \cosh^2(\frac{\tau_e}{\kappa})} - \frac{1}{2}(\vec{n}')^2 + \frac{\kappa^2}{16}(\vec{n}')^4 + \mathcal{O}(\kappa^4) \right] \tag{2.161}$$

We can now evaluate the integral over τ_e. It is clear that the integral over τ_e peaks around $\tau_e = 0$ in the $\kappa \to 0$ limit. However, we can get a possible contribution from expanding the integrand around $\tau_e = 0$. If we denote $G(\tau_e, \sigma) \equiv (e^{-i\varphi}\cos\theta)^j$ and expand we get

$$G(\tau_e, \sigma) = G(0, \sigma) + \tau_e \frac{\partial G}{\partial \tau_e}\Big|_{(0,\sigma)} + \frac{1}{2}\tau_e^2 \frac{\partial^2 G}{\partial \tau_e^2}\Big|_{(0,\sigma)} + \cdots \tag{2.162}$$

We end up having three different contribution:

1. $G(0, \sigma)$: this correction gives zero when integrated over τ_e since it is an odd function of τ_e;

2. $\frac{\partial G}{\partial \tau_e}$: this in general is a non-zero contribution but since $\tau_e \sim \kappa$ and because of the Frolov-Tseytlin limit, this correction is of order λ'^2. For this reason we see that no other part of the integrand will pick up a contribution in this way, since they are of higher order in λ' and we consider only terms up to order λ'^2;

3. $\frac{\partial^2 G}{\partial \tau_e^2}$: using the EOMs for the Landau-Lifshitz sigma model we obtain

$$\frac{\partial^2 G}{\partial \tau_e^2}\Big|_{(0,\sigma)} = \kappa^2 (e^{-i\varphi}\cos\theta)^j (K_1 + K_2) \tag{2.163}$$

with

$$K_1 = -\frac{j(j-1)}{4}\left[\frac{i\theta''}{\cos\theta} + \sin\theta(2\tan\theta\theta'\varphi' + i\varphi'^2 - \varphi'')\right]^2 \tag{2.164}$$

$$\begin{aligned} K_2 \quad &= -\frac{j}{16}\frac{1}{\cos\theta}\Big\{8i\sin\theta\,\theta'^3\varphi' - 4\cos\theta\,\sin^2\theta\,\varphi'^4 - 4\cos 3\theta\,\varphi''^2 + 4\theta'^2[\sin\theta\,\theta'' - 5i\cos\theta\,\varphi \\ &-15\cos\theta\,\sin^2\theta\,\varphi'^2] + \varphi'^2[(5\sin 3\theta - 19\sin\theta)\theta'' + i(7\cos 3\theta - 3\cos\theta)\varphi''] \\ &+4\theta'[\varphi'(\sin\theta(-4i\cos 2\theta\,\varphi'^2 + (11\cos 2\theta - 1)\varphi'' - 6i\cos\theta\,\theta'')) - 4i\sin\theta\,\varphi'''] \\ &+8\sin\theta\,\varphi'(\sin 2\theta\,\varphi''' - 2i\theta''') + 4\sin\theta(\theta'''' - 6i\theta''\varphi'') - 4\cos\theta(\theta''^2 - i\varphi''^2)\Big\} \qquad (2.165 \end{aligned}$$

Besides there are three types of integral to perform that we denote as I_0, I_1 and I_2 and they are given by

$$\begin{aligned} I_0 &= \frac{1}{\kappa}\int_{-\infty}^{+\infty}\frac{d\tau_e}{\cosh^{2j+2}(\frac{\tau_e}{\kappa})} = \frac{2^{2j+1}\,(j!)^2}{(2j+1)!} \\ I_1 &= \frac{1}{\kappa}\int_{-\infty}^{+\infty}\frac{d\tau_e}{\cosh^{2j}(\frac{\tau_e}{\kappa})} = I_0\frac{2j+1}{2j} \\ I_2 &= \frac{1}{\kappa^3}\int_{-\infty}^{+\infty}\frac{\tau_e^2 d\tau_e}{\cosh^{2j+2}(\frac{\tau_e}{\kappa})} = I_0\frac{1}{4}\Psi(1,1+j) \end{aligned} \tag{2.166}$$

where $\Psi(1,x) = \frac{d^2}{dx^2}\log\Gamma(x)$. Using this, our final result is

$$\begin{aligned} c_{123} \quad &= \quad \frac{J}{N}\frac{j!}{2^j\sqrt{(2j-1)!}}\int_0^{2\pi}\frac{d\sigma}{2\pi}(e^{-i\varphi}\cos\theta)^j\left[1 - \frac{(2j+1)}{4j}\left(\frac{\lambda'}{2}(\vec{n}')^2 - \frac{\lambda'^2}{16}(\vec{n}')^4\right)\right. \\ &\left. +\frac{\lambda'^2}{4}\Psi(1,1+j)(K_1+K_2) + \mathcal{O}(\lambda'^3)\right] \end{aligned} \tag{2.167}$$

where we used $\kappa^2 = \lambda'$.

2.4.3 Comparison

Having the result (2.167), we are ready to make the comparison with the gauge theory result (2.143). To this end, it is convenient to compute (2.156) in terms of a different set of coordinates. We do this in this section where we limit ourselves to consider only up to and including the one loop correction. We write the parametrization of the 3-sphere using the unitary vector $\mathbf{U}(\sigma,\tau) = e^{i\tau/\kappa}\mathbf{u}(\sigma,\tau)$, where $\mathbf{u}(\sigma,\tau) = (u_1(\sigma,\tau), u_2(\sigma,\tau), u_3(\sigma,\tau))$. The limit (2.159) then is

$$\kappa \to 0 \quad , \quad \frac{1}{\kappa}\partial_\tau \mathbf{u} \;\; \text{fixed}, \quad \partial_\sigma \mathbf{u} \;\; \text{fixed}. \tag{2.168}$$

In this limit the EOMs and Virasoro constraints reduce to $\frac{2i}{\kappa}\partial_\tau \mathbf{u} = \partial_\sigma^2 \mathbf{u} + 2\mathbf{u}\left(\partial_\sigma \bar{\mathbf{u}} \cdot \partial^\sigma \mathbf{u}\right)$ and $\bar{\mathbf{u}} \cdot \partial_\sigma \mathbf{u} = 0$. The holographic three point function coefficient can thus be computed from

$$c_{123} = c_j \frac{\sqrt{\lambda}}{N} \int_{-\infty}^{+\infty} d\tau_e \int_0^{2\pi} \frac{d\sigma}{2\pi} \frac{(\bar{u}^1 u_2)^j}{\cosh^{2j}(\frac{\tau_e}{\kappa})} \left[\frac{1}{\kappa^2 \cosh^2(\frac{\tau_e}{\kappa})} - \partial_\sigma \bar{\mathbf{u}} \cdot \partial_\sigma \mathbf{u} + \mathcal{O}(\kappa^2) \right]. \tag{2.169}$$

Note that this is the same expression used in [21] if one replaces κ with $1/\kappa$. Evaluating the integral over τ_e as before we obtain

$$c_{123} = \frac{J}{N} \frac{j!}{\sqrt{(2j-1)!}} \int_0^{2\pi} \frac{d\sigma}{2\pi} \left(\bar{u}^1 u_2\right)^j \left[1 - \lambda' \frac{2j+1}{2j} \partial_\sigma \bar{\mathbf{u}} \cdot \partial_\sigma \mathbf{u} + \mathcal{O}(\lambda'^2) \right]. \tag{2.170}$$

Comparing this result with the one for the dual gauge theory (2.144), it is evident that the leading terms in both sides perfectly match as already pointed out in [21] but there is a mismatch in the one loop correction. Actually it is not clear that one should have expected the gauge theory result (2.144) and the string theory result (2.170) to match[8]. This is true even for the leading order part corresponding to tree-level on the gauge theory side. However, the fact that the tree-level part does match the string side, certainly raises the hope that also the one-loop part should match. While our analysis seems to conclude that this is not the case, we should point out that there are a number of subtleties in the computations that may not be sufficiently well understood at present in the literature and could therefore possibly affect our results. First of all the possibility of a further contribution to the result (2.144) coming from the so-called spin-flipped coherent state [97], as discussed in Section 2.4.1. Moreover, among the other possible subtleties is the approximation $\mathcal{O}_1 \simeq \bar{\mathcal{O}}_2$. Indeed, in our computation we have assumed that (2.134) holds also at two-loop order. Moreover the gauge theory side of the computation, which is based on the prescription of [49, 50, 116], might still require some explicit tests on the line of the ones performed in [142]. Finally, with our current understanding of the AdS/CFT correspondence, it is not clear whether

[8] Note that it was conjectured in [31] that both the tree-level and one-loop contributions on the gauge theory side matches the zeroth and first order contributions in the Frolov-Tseytlin limit on the string theory side for three point function of this kind.

or not one should expect a matching of the two quantites. Nevertheless it has pointed out in [143] that in the limit when the number of imputities j in the small operator goes to infinity the matching in the Frolov-Tseytlin limit still holds. This comparison has been made using Integrability inspired techinques on the gauge theory, in this way one incorporates in the same structure the contributions coming from the mixing of the operators at two loop order and the ones coming from the one loop diagrams [57]. We will introduce these Integrability based techniques in the next chapter to compute tree level structure constants in the gauge theory side for non-BPS operators.

2.5 Two heavy and one light operators in the $SU(2) \times SU(2)$ sector of ABJM

In this section we want to extend the procedure of [21] in a further direction, namely we will compute the leading order contribution for the planar three point correlation function of two heavy operators and one light operator belonging to the $SU(2) \times SU(2)$ sector of ABJM theory [39]. The light operator is chosen to be a chiral primary and we work in the the Frolov-Tseytlin limit [37, 38]. Again we compare the results obtained in the gauge theory side using a coherent state approach for the heavy operators with the holographic counterpart using the prescription of [10]. In order to compare these results we are interested in computing the following quantity [9]

$$r = \frac{c^{\bullet\bullet\circ}}{c^{\circ\circ\circ}}. \tag{2.171}$$

We have used in (2.171) the same notation of [21], each circle in the superscript represents an operator appearing in the correlators and filled circles correspond to non-BPS operators while empty circles correspond to BPS ones. Note also that we put the subscripts $\lambda \ll 1$ and $\lambda \gg 1$ to distinguish the gauge theory computations from their holographic counterpart respectively. Considering the ratio (2.171) removes any dependence on normalization conventions [21] making possible the comparison.

Correlation functions in ABJM Unlike what is the case for the $AdS_5 \times S^5$

[9] Note that since we are only discussing the tree level contribution we remove the superscript (0) in denoting the tree level part of the three point function.

correspondence not much is known about three-point functions of its $AdS_4 \times \mathbb{C}P^3$ cousin. Planar three-point functions of scalar chiral primaries were calculated at strong coupling more than 10 years ago using M-theory on $AdS_4 \times S^7$ [144]. More recently, strong coupling results were obtained for the case of two giant gravitons and one tiny graviton, all BPS [145]. These three-point functions all show an explicit dependence on the 't Hooft coupling constant and hence are not protected like their $AdS_5 \times S^5$ counterparts [34, 106]. In [15] [146] certain three-point functions involving two (non-BPS) semi-classical string states and the dilaton field were presented. Perhaps for this reason, little effort has been put into studying the corresponding three-point functions at weak coupling. Weak coupling three-point functions only make sense for operators with well-defined conformal dimensions i.e. for operators which are eigenstates of the dilatation operator of the field theory. Scalar chiral primary operators belong to this category. Their two-point functions are protected. One can hence immediately proceed with the calculation of three-point functions of such operators. A number of tree-level results for three-point functions of scalar chiral primaries, including operators dual to giant gravitons, can be found in the references [145, 147]. Furthermore, it has been shown that the one-loop correction to any n-point function of scalar chiral primaries vanishes due to colour combinatorics [148] but apart from that there are no results on higher loop corrections to correlation functions neither of chiral primaries nor of more general operators.[10]

2.5.1 Gauge theory side

Operators We choose the operators $\mathcal{O}_1$ and $\mathcal{O}_2$ to be of the same length. This will also allow to avoid ambiguities arising from multiplying $\mathcal{O}_1$ and/or $\mathcal{O}_2$ by a phase [40]. All 3 operators are single trace operators and each of them belongs to an $SU(2) \times SU(2)$ sector of ABJM theory. We will consider non-extremal three point functions. Note that the coherent state functions that we use to describe the heavy operators are solutions of the Landau-Lifshits model up to two-loops (which is the first non trivial order in ABJM theory) [151, 152], see appendix C. In this section we wish to calculate a three point function of the type considered above in the limit where the two operators, $\mathcal{O}_1$ and $\mathcal{O}_2$, are much longer than the operator

[10] It is expected that n-point correlation functions of BPS operators involving space-time points with light like separation are related to n-sided light like polygonal Wilson loops and to scattering amplitudes [148] as it is the case in $\mathcal{N} = 4$ SYM [149]. Similar relations are argued to hold for more general classes of operators and for theories in general dimensions [150].

$\mathcal{O}_3$. In order to simplify the presentation we now restrict ourselves to the case described in Tab. 2.3, which is not the most general one. The most general case

Operator	field type	field type	field type	field type
$\mathcal{O}_1$	$(J-J_1)\ \ Z_1$	$(J_1)\ \ Z_2$	$(J-J_1)\ \ W_1$	$(J_1)\ \ W_2$
$\mathcal{O}_2$	$(J_1+j)\ \ \bar{Z}_2$	$J-J_1-j\ \ \bar{Z}_1$	$(J_1+j)\ \ \bar{W}_2$	$(J-J_1-j)\ \ \bar{W}_1$
$\mathcal{O}_3$	$j\ \ W_2$	$j\ \ \bar{Z}_1$	$j\ \ Z_2$	$j\ \ \bar{W}_1$

TABLE 2.3: Operators in the $SU(2)\times SU(2)$ sector of ABJM. The notation $(J-J_1)\ Z_1$ means that the number of Z_1 fields is $(J-J_1)$ and so on. We consider the case where $1\ll j\ll J_1, J$. Note that we do not specify which fields are vacua and which are excitations because it is irrelevant in this description.

is not conceptually more difficult, only the notation become cumbersome. From table 2.3 we see that the total length of $\mathcal{O}_1$ and $\mathcal{O}_2$ is $2J$ and that total length of $\mathcal{O}_3$ is $4j$ and its dimension is $2j$. We already took into account the momentum constraints. We take $J\gg j\gg 1$ and the operators $\mathcal{O}_1(x_1)$ and $\mathcal{O}_2(x_2)$ are semi-classical operators that we represent by coherent states as

$$\mathcal{O}_1=\dots(\mathbf{u}_o^{(2k-1)}\cdot\mathbf{Z})(\mathbf{u}_e^{(2k)}\cdot\mathbf{W})(\mathbf{u}_o^{(2k+1)}\cdot\mathbf{Z})(\mathbf{u}_e^{(2k+2)}\cdot\mathbf{W})\dots \tag{2.172}$$

$$\mathcal{O}_2=\dots(\bar{\mathbf{v}}_o^{(2k-1)}\cdot\bar{\mathbf{Z}})(\bar{\mathbf{v}}_e^{(2k)}\cdot\bar{\mathbf{W}})(\bar{\mathbf{v}}_o^{(2k+1)}\cdot\bar{\mathbf{Z}})(\bar{\mathbf{v}}_e^{(2k+2)}\cdot\bar{\mathbf{W}})\dots \tag{2.173}$$

where the notation is explained in Fig.2.4. The functions $\mathbf{u}_{(a)}(\sigma)$ and $\mathbf{v}_{(a)}(\sigma)$, where the subscript "a" is equal to "o" (for odd sites) and "e" (for even sites), correspond for each site of the single trace operators to coherent states in the spin 1/2 representation of $SU(2)$. They are periodic in σ with period 2π and satisfy the condition $\bar{\mathbf{u}}_{(a)}\cdot\mathbf{u}_{(a)}=1$. Moreover they obey the Landau-Lifshitz EOM, see appendix C. The operators $\mathcal{O}_1$ and $\mathcal{O}_2$ are semiclassical therefore the functions $\mathbf{u}_{(a)}(\sigma)$ and $\mathbf{v}_{(a)}(\sigma)$ are slowly varying in σ. The operator $\mathcal{O}_3(x_3)$ is given by

$$\mathcal{O}_3(x_3)=\mathcal{N}_3:\mathrm{Tr}(\mathrm{sym}((Z_1W_1)^j(\bar{W}_2\bar{Z}_1)^j)):(x_3) \tag{2.174}$$

where "sym" denotes all possible symmetrized states. Note that among all possible states in $\mathcal{O}_3$, the only one that contributes at the planar level to the Wick contractions in Fig. 2.4 for this type of three point function is

$$\mathcal{O}_3=\mathcal{N}_3\mathrm{tr}\left((Z_1W_1)^j(\bar{W}_2\bar{Z}_1)^j\right) \tag{2.175}$$

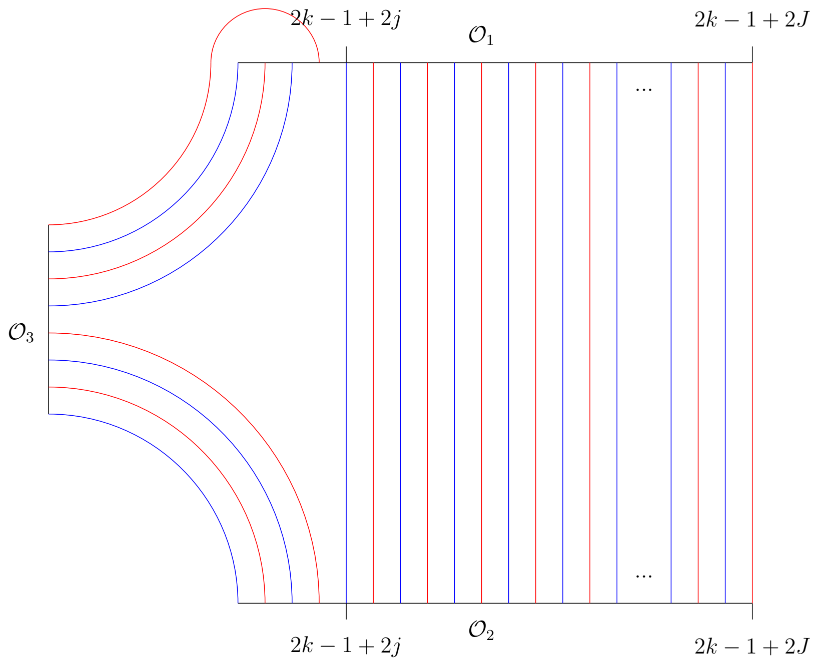


FIGURE 2.4: Tree level contractions between $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$. Note that the contractions among $\mathcal{O}_1$ and $\mathcal{O}_2$ involve fields at site number from $2k-1+2j$ to $2k-1+2J$ while the rest are among $\mathcal{O}_1$ and $\mathcal{O}_3$, $\mathcal{O}_2$ and $\mathcal{O}_3$. To consider all the contractions we have to sum over all possible k from 1 to J.

and the norm is $\mathcal{N}_3 = \frac{(j!)^2}{\sqrt{(2j)!(2j-1)!}}$, meaning that the respective two point function is unit normalized.

$c^{\bullet\bullet\circ}$ We can now compute the planar tree level contribution to $\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle$ at tree-level. In full analogy with what we have described in Sec. 2.4.1 we define

$$B \equiv \prod_{m=1}^{J} (\mathbf{u}_o^{(2m-1)} \cdot \bar{\mathbf{v}}_o^{(2m-1)})\,(\mathbf{u}_e^{(2m)} \cdot \bar{\mathbf{v}}_e^{(2m)})(\frac{p}{l}) \tag{2.176}$$

The quantity B takes into account the overlap between the large operators $\mathcal{O}_1$ and $\mathcal{O}_2$. In writing (2.176) we took into account that the field in the odd (even) sites of the operator $\mathcal{O}_1$ will contract with the fields in the odd (even) sites of the operator $\mathcal{O}_2$. Note that in (2.176) $l = J/2\pi$ and that B does not depend on the choice of the site k. Our convention for the tree-level three point diagram is that we contract the first $2j$ letters of $\mathcal{O}_1$ with $\mathcal{O}_3$ and the rest is then contracted with $\mathcal{O}_2$. Also, we contract the first $2j$ letters of $\mathcal{O}_2$ with $\mathcal{O}_3$ and the rest with $\mathcal{O}_1$ (see

Fig. 2.4). The tree-level contractions give

$$A(k)=B\prod_{m=k}^{k+j-1}u_o^1\left(\frac{(2m-1)\pi}{J}\right)u_e^1\left(\frac{2m\pi}{J}\right)\bar{v}_o^2\left(\frac{(2m-1)\pi}{J}\right)\bar{v}_e^2\left(\frac{2m\pi}{J}\right) \tag{2.177}$$

Summing over k, we have

$$\sum_{k=1}^{J}A(k)=\sum_{k=1}^{J}\prod_{m=k}^{k+j-1}\frac{u_o^1\left(\frac{(2m-1)\pi}{J}\right)u_e^1\left(\frac{2m\pi}{J}\right)\bar{v}_o^2\left(\frac{(2m-1)\pi}{J}\right)\bar{v}_e^2\left(\frac{2m\pi}{J}\right)}{\left(\mathbf{u}_o^{(2m-1)}\cdot\bar{\mathbf{v}}_o^{(2m-1)}\right)\left(\mathbf{u}_e^{(2m)}\cdot\bar{\mathbf{v}}_e^{(2m)}\right)}. \tag{2.178}$$

The meaning of this notation is the same of section 2.4 and it becomes clear from Fig. 2.4. Since $\mathbf{u}_{(a)}$ varies slowly and $j\ll J$, the difference for $\mathbf{u}_{(a)}$ at two different values of σ can be estimated using a Taylor expansion. Similarly can be done for $\mathbf{v}_{(a)}$. We find

$$\begin{aligned}\textstyle\sum_{k=1}^{J}A(k)\simeq B\sum_{k=1}^{J}&\left(\frac{u_o^1\left(\frac{(2k-1)\pi}{J}\right)u_e^1\left(\frac{2k\pi}{J}\right)\bar{v}_o^2\left(\frac{(2k-1)\pi}{J}\right)\bar{v}_e^2\left(\frac{2k\pi}{J}\right)}{\left(\mathbf{u}_o^{(2k-1)}\cdot\bar{\mathbf{v}}_o^{(2k-1)}\right)\left(\mathbf{u}_e^{(2k)}\cdot\bar{\mathbf{v}}_e^{(2k)}\right)}\right)^j\\ &\longrightarrow\mathcal{N}_3\,BJ\int_0^{2\pi}\frac{d\sigma}{2\pi}\left(\frac{u_o^1(\sigma)u_e^1(\sigma)\bar{v}_o^2(\sigma)\bar{v}_e^2(\sigma)}{\left(\mathbf{u}_o(\sigma)\cdot\bar{\mathbf{v}}_o(\sigma)\right)\left(\mathbf{u}_e(\sigma)\cdot\bar{\mathbf{v}}_e(\sigma)\right)}\right)^j.\end{aligned} \tag{2.179}$$

We now use the approximation $\mathbf{v}_{(a)}(\sigma)=\mathbf{u}_{(a)}(\sigma)$ in (2.179) more precisely that

$$\mathbf{v}_{(a)}(\sigma)=\mathbf{u}_{(a)}(\sigma)+\delta\mathbf{u}_{(a)}(\sigma) \tag{2.180}$$

where $\delta\mathbf{u}_{(a)}$ is of order j/J Using this relation we can easily see that $B=1$ to leading order in j/J and hence

$$c^{\bullet\bullet\circ}_{\lambda\ll1}=\frac{\mathcal{N}_3}{N}\sum_k A(k)=\frac{1}{N}\mathcal{N}_3\,J\int_0^{2\pi}\frac{d\sigma}{2\pi}\left(u_o^1(\sigma)u_e^1(\sigma)\bar{u}_o^2(\sigma)\bar{u}_e^2(\sigma)\right)^j \tag{2.181}$$

up to finite size corrections in $1/J$.

$c^{\circ\circ\circ}$ Now we want to compute $c^{\circ\circ\circ}$ which is the three-point correlation function coefficient for three chiral primaries with the same charges of the operators that we have considered above, see table 2.3. There are two ways for computing this quantity. The first one is described in [145]. From [145] we obtain to leading order and in our notation where $\mathcal{O}_1$ and $\mathcal{O}_2$ have the same dimension (length) J $(2J)$ and $\mathcal{O}_3$ has dimension (length) $2j$ $(4j)$, that

$$c^{\circ\circ\circ}_{\lambda\ll1}=\frac{J\sqrt{2j}}{N}\frac{(J-J_1+j)!J_1!((J-j)!)^2j!^2}{(J!)^2(J-J_1)!(J_1-j)!(2j)!}. \tag{2.182}$$

To get to this result we used that the quantity $< \mathcal{C}_1\mathcal{C}_2\mathcal{C}_3 >_{planar}$ that appears in [145] gives in this case

$$< \mathcal{C}_1\mathcal{C}_2\mathcal{C}_3 >_{planar} = \frac{(J-J_1+j)!J_1!((J-j)!)^2 j!^2}{(J!)^2(J-J_1)!(J_1-j)!(2j)!}. \tag{2.183}$$

Taking the limit $J, J_1 \to \infty$ keeping $J - J_1$ large, we have

$$c^{\circ\circ\circ} \sim \frac{\mathcal{N}_3}{N} J s^j, \tag{2.184}$$

where we have defined the quantity $s = \frac{J_1(J-J_1)}{J^2}$. The second way of computing (2.184) can be obtained using another result of [39] that we will describe in the next chapter. Using the result (2.181), we then compute

$$r_{\lambda \ll 1} = \frac{1}{s^j} \int_0^{2\pi} \frac{d\sigma}{2\pi} (u_o^1(\sigma) u_e^1(\sigma) \bar{u}_o^2(\sigma) \bar{u}_e^2(\sigma))^j. \tag{2.185}$$

We will show in the following that for the ratio $r_{\lambda \gg 1}$ at strong coupling we obtain the same result.

2.5.2 String theory side

Here we compute the holographic three-point function dual to the correlator of two heavy and one light operator considered in 2.5.1 using the prescription of [10]. The procedure outlined in [21, 27, 40] for type IIB string theory on $\mathrm{AdS}_5 \times S^5$ can be easily generalized to type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{C}P^3$ using the results of [144, 145, 153]. Our convention and notation for the $\mathrm{AdS}_4 \times \mathbb{C}P^3$ background for type IIA string theory are explained in appendix C.

In order to compute the holographic three point correlation function to leading order, we need to use the wave functions which are solution of the Landau-Lifshitz EOMs. This is because at each order in the expansion parameter, the wave function receives corrections coming from the next order contribution to the effective sigma model description. The wave functions that we will use in the next section to describe the heavy operators that appear in the three-point correlation function dual to the one computed on the gauge theory side, are solution of the EOMs that one derives from (C.15).
$c^{\bullet\bullet\circ}$ Here we compute the holographic three point function dual to the operators considered table 2.3 using the prescription of [10]. We start with the metric (C.1)

and parametrize the AdS$_4$ part as

$$ds^2_{\mathrm{AdS}_4} = \frac{dz^2 + d\mathbf{x}_\mu^2}{z^2}\;, \quad \mathbf{x}_\mu = (x_1, x_2, x_3) \tag{2.186}$$

and we parametrize the two two-spheres associated to the two $SU(2)$ sectors contained in $\mathbb{C}P^3$ we use two complex vectors $\mathbf{U}_e(\tau,\sigma) = (U_e^1, U_e^2)$ and $\mathbf{U}_o(\tau,\sigma) = (U_o^1, U_o^2)$. [11] With this parametrization the results of this section will be directly comparable with the ones of 2.5.1. We consider supergravity modes of dimension Δ represented by the following fluctuations [144, 145, 153]

$$h_{\mu\nu} = \frac{4}{\Delta+2}\left(\nabla_\mu\nabla_\nu + \frac{\Delta\,(1-\Delta)}{6} g_{\mu\nu}\right) s \tag{2.187}$$

$$h_{\alpha\beta} = \frac{\Delta}{3} g_{\alpha\beta} s. \tag{2.188}$$

where μ,ν refer to coordinate of AdS$_4$ and α,β to coordinate of CP^3. The bulk to boundary propagator s^A is given by

$$s = \frac{N_\Delta Y_\Delta z^{\frac{\Delta}{2}}}{\tilde{N}_3 x_B^{\Delta}} \tag{2.189}$$

where Y_Δ are the spherical harmonics on S^7, x_B defines the position of the chiral primary operator, $\tilde{N}_3 = \frac{(\frac{\Delta}{4})!^2}{(\frac{\Delta}{2})!}$ and

$$N_\Delta = \frac{\lambda^{1/4}}{N\,2^{1/4}\sqrt{\pi}} \frac{2^{\Delta/2}\,(\Delta+2)\,\sqrt{\Delta+1}}{\Delta}. \tag{2.190}$$

In the expression (2.189) there is a factor of $\tilde{N}_3$ of difference between the usual expression for the bulk to boundary propagator which relies on our choice for the chiral primary operator. Moreover we compute

$$\begin{aligned} h_{zz} &= \frac{4}{\Delta+2}\left[\left(\partial_z\partial_z - \Gamma^\lambda_{zz}\partial_\lambda\right) s + \frac{\Delta\,(1-\Delta)}{6z^2} s\right] \\ &= \frac{N_\Delta Y_\Delta}{x_B^\Delta}\frac{4}{\Delta+2}\frac{\Delta\,(\Delta+2)}{12} = \frac{N_\Delta Y_\Delta}{x_B^\Delta}\frac{\Delta}{3} \end{aligned} \tag{2.191}$$

[11] Note that in App. C we use a different parametrization for the two two-spheres. The two parametrizations are related by a coordinate transformation.

$$h_{xx} = \frac{4}{\Delta+2}\left[-\Gamma^{\lambda}_{xx}\partial_\lambda s + \frac{\Delta(1-\Delta)}{6z^2}s\right]$$
$$= -\frac{N_\Delta Y_\Delta}{x_B^\Delta}\frac{4}{\Delta+2}\frac{\Delta(\Delta+2)}{6} = -\frac{N_\Delta Y_\Delta}{x_B^\Delta}\frac{2\Delta}{.}3 \qquad (2.192)$$

With this ingredients we get that (2.25) gives in our case

$$c^{\bullet\bullet\circ} = a_j \lambda^{\frac{3}{4}} \int_{-\infty}^{\infty} d\tau_e \int_0^{2\pi} \frac{d\sigma}{2\pi} \frac{Y}{\cosh^{2j}\frac{\tau_e}{\kappa}} \left[\frac{3}{\kappa^2\cosh^2\frac{\tau_e}{\kappa}} - \frac{1}{\kappa^2} - \frac{(\partial_a \bar{\mathbf{U}}_e \cdot \partial^a \mathbf{U}_e + \partial_a \bar{\mathbf{U}}_o \cdot \partial^a \mathbf{U}_o)}{2}\right], \qquad (2.193)$$

where we already implemented the gauge choice (C.12), we introduced the Euclidean time τ_e and we defined

$$a_j = \frac{\sqrt{\pi(4j+1)}}{N}\frac{2^{\frac{1}{4}-2j}(2j+1)!}{j!^2}, \qquad Y = \left(U_e^1 \bar{U}_e^2 U_o^1 \bar{U}_o^2\right)^j. \qquad (2.194)$$

Note that we dropped the factor $x_B^{\Delta/2}$ and we used an explicit representation for the spherical harmonics. To compare with the gauge theory results we take the Frolov-Tseytlin limit [37, 38] which in our notation reads [40, 95, 151]

$$\kappa \to 0\,, \quad \frac{1}{\kappa}\partial_\tau \mathbf{U}_{e,o} \ \text{ fixed}\,, \quad \partial_\sigma \mathbf{U}_{e,o} \ \text{ fixed.} \qquad (2.195)$$

A subclass of solutions that can be mapped to coherent spin chain states at weak coupling is given by considering the parametrization $\mathbf{U}_{e,o}(\sigma,\tau) = e^{i\tau/\kappa}\mathbf{u}_{e,o}(\sigma,\tau)$ with the condition $\bar{\mathbf{u}}_e \cdot \mathbf{u}_e = 1$ and similarly for $\mathbf{u}_o$. The limit (2.195) becomes

$$\kappa \to 0\,, \quad \frac{1}{\kappa}\partial_\tau \mathbf{u}_{e,o} \ \text{ fixed}\,, \quad \partial_\sigma \mathbf{u}_{e,o} \ \text{ fixed.} \qquad (2.196)$$

The functions $\mathbf{u}_{e,o}$ are solutions of the Landau Lifshitz equations of motion derived from the action (C.15) and satisfy the Virasoro condition $\bar{\mathbf{u}}_e \cdot \partial_\sigma \mathbf{u}_e + \bar{\mathbf{u}}_o \cdot \partial_\sigma \mathbf{u}_o = 0$. Note that in our notation, the energy that one computes using the action (C.15) goes as $E - J \sim \mathcal{O}(\lambda/J^2)$. This is due to the rescaling of t in (C.7). This rescaling has the effect that the gauge constant $\kappa \sim \frac{\sqrt{\lambda}}{J}$. This implies that the expansion in powers of κ on the string side parallels the expansion in powers of λ/J^2 that one has on the gauge theory side. In the limit (2.196), (2.193), to leading order, gives

$$c^{\bullet\bullet\circ} = \frac{\lambda^{\frac{3}{4}}\sqrt{\pi(4j+1)}}{N}\frac{2^{\frac{1}{4}-2j}(2j+1)!}{j!^2}\int_{-\infty}^{\infty} d\tau_e \int_0^{2\pi}\frac{d\sigma}{2\pi}\left(u_e^1\bar{u}_e^2 u_o^1 \bar{u}_o^2\right)^j \frac{1}{\kappa^2\cosh^{2+2j}\frac{\tau_e}{\kappa}}. \qquad (2.197)$$

For $\kappa \to 0$, the integrand peaks around $\tau_e = 0$ and the τ-integral can thus be evaluated. The result reads

$$\int_{-\infty}^{+\infty} \frac{d\tau_e}{\kappa^2 \cosh^{2j+2}(\frac{\tau_e}{\kappa})} = \frac{2^{2j+1}\,(j!)^2}{\kappa\,(2j+1)!}. \tag{2.198}$$

Using that $\kappa = \frac{\sqrt{\lambda}}{J\pi\sqrt{2}}$ (see App. C) we obtain

$$c^{\bullet\bullet\circ} = J\frac{\lambda^{\frac{1}{4}}2^{\frac{3}{4}}}{\sqrt{\pi}N}\sqrt{4j+1}\int_0^{2\pi}\frac{d\sigma}{2\pi}(u_e^1\bar{u}_e^2u_o^1\bar{u}_o^2)^j. \tag{2.199}$$

$c^{\circ\circ\circ}$ The expression for the holographic three point for the chiral primaries with the same charges as the operators considered in Table2.3 can be computed using [145] where in our case $p = J - j$. We have that $n_6 = j$, $n_1 = n_2 = p = J - j$, $n_3 = j$. Note also that in our notation $\gamma_1 = \gamma_2 = 2j$, $\gamma_3 = 2J - 2j$ and $\gamma = 2J + 2j$ where we used that the relation between our notation and J_1, J_2 and J_3 in [145] is that $(J_1/2)_{\text{there}} = (J_2/2)_{\text{there}} = J_{\text{our}}$ and $(J_3/2)_{\text{there}} = 2j_{\text{our}}$. We get

$$C^{\circ\circ\circ}_{\lambda\gg 1} = \frac{\lambda^{\frac{1}{4}}2^{-\frac{1}{4}}}{N\sqrt{\pi}}\frac{(2J+1)J!^2}{(J+j)!(J-j)!}\frac{\sqrt{4j+1}(2j)!}{j!^2} < \mathcal{C}_1\mathcal{C}_2\mathcal{C}_3 > . \tag{2.200}$$

Using (2.183) to express $< \mathcal{C}_1\mathcal{C}_2\mathcal{C}_3 >$ we obtain

$$C^{\circ\circ\circ}_{\lambda\gg 1} = \frac{\lambda^{\frac{1}{4}}2^{-\frac{1}{4}}}{N\sqrt{\pi}}\sqrt{4j+1}\frac{(2J+1)(J-j)!}{(J+j)!}\frac{(J-J_1+j)!}{(J-J_1)!}\frac{J_1!}{(J_1-j)!}. \tag{2.201}$$

Note that this expression differs from (2.182) which is valid at weak coupling. In particular the dependence on the coupling is very different, meaning that the three point function for 3 chiral primaries in ABJM theory is not a protected quantity.

In the limit $J, J_1 \to \infty$ with $J - J_1$ large we have

$$c^{\circ\circ\circ} = \frac{\lambda^{\frac{1}{4}}2^{\frac{3}{4}}}{N\sqrt{\pi}}Js^j\sqrt{4j+1}. \tag{2.202}$$

We can now compute the ratio between (2.199) and(2.202) and compare it with the corresponding quantity (2.185) at weak coupling. We find

$$r_{\lambda\gg 1} = \left.\frac{c^{\bullet\bullet\circ}}{c^{\circ\circ\circ}}\right|_{\lambda\gg 1} = \frac{1}{s^j}\int_0^{2\pi}\frac{d\sigma}{2\pi}(u_e^1\bar{u}_e^2u_o^1\bar{u}_o^2)^j. \tag{2.203}$$

It is easy to see that to leading order (2.185) and (2.203) give the same result

$$r_{\lambda\gg 1} = r_{\lambda\ll 1}. \tag{2.204}$$

Note that we have that $r_{\lambda\gg 1} = r_{\lambda\ll 1}$ only in the limit $J, J_1 \to \infty$ which is the regime for which also the matching of [21] was observed. In the next chapter we consider another computation in the context of the AdS_4/CFT_3 correspondence to compute three point functions of non BPS operators from the gauge theory side.

Chapter 3

Integrability techniques

Contents

3.1 Introduction

The problem of computing three point functions on the gauge theory side amount of solving a combinatorial problem, which even at tree level is very hard to solve especially for long operators. We have already pointed out in Sec. 2.1 that using spin chain approaches one can identify an operator with a closed spin chain and split this closed spin chain into two parts ending up with two open spin chain states as the two segments of one closed spin chain [49] and use the Algebraic Bethe ansatz techniques, reviewed in Sec. 1.2, to diagonalize the spin chain Hamiltonian [50]. Recent developments in this stream have been achieved in a series of papers [52, 54] and subsequently [39, 55, 56, 58, 59, 154, 155] for tree level computations and [57] for one loop corrections. Let us review how to compute tree level structure constants in $\mathcal{N} = 4$ SYM. In this section we will mostly review [52] [54] where the authors have found an expression for the planar tree level, non extremal three point functions of operators belonging to the $SU(2)$ sector of $\mathcal{N}=4$ SYM in terms of the three sets of rapidities. In [52] the result is given as a specific sum over the possible ways of partitioning the rapidities of one operator in two subsets, while in [54] using six vertex model techniques, reviewed in Sec. 1.2.4the structure constants are given in a determinant form. The main advantage of these methods is that they depend only on the sets of rapidities of the three operators, meaning that they can in principle be applied to any operators belonging to the sector that we are considering. Then we discuss the generalization of these procedures in two directions, firstly [59] we peruse how to define a recursive relation for the scalar products of operators in the $SO(6)$ sector and secondly [39] we examine the case of three point functions of operators in the $SU(2)\times SU(2)$ sector of ABJM using the method proposed in [54].

3.1.1 From contractions to scalar products

Let us consider three operators $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ belonging to the $SU(2)$ sector of $\mathcal{N}=4$ SYM, thus made only of combinations of two complex scalars Z and X. In general there are is a huge degeneracy, many operators share the same scaling dimension. To avoid this problem we choose operators with well defined anomalous dimension, thus these operators are eigenstates of a $\text{XXX}_{\frac{1}{2}}$ spin chain Hamiltonian and the anomalous dimensions are the respective eigevalues. The most general operators which admit a non trivial non extremal and planar three point functions are of the form reported in table 3.1, their lengths are L_1, L_2 and L_3 and the number of impurities are N_1, N_2 and N_3 respectively. The planar contractions

Operator	Vacuum	Excitation
$\mathcal{O}_1$	$(L_1-N_1)\ \ Z$	$N_1\ \ X$
$\mathcal{O}_2$	$(L_2-N_2)\ \ \bar{Z}$	$N_2\ \ \bar{X}$
$\mathcal{O}_3$	$(L_3-N_3)\ \ Z$	$N_3\ \ \bar{X}$

Table 3.1: The three operators in the $SU(2)$ sector. We can consider a vacuum field as a spin up while an excitation as spin down.

between the operators are depicted in figure 3.4. Note that the contractions are among a scalar field and its complex conjugate and the number of contractions is fully determined by the length of the three operators namely

$$l_{ij} = \frac{1}{2}\left(L_i + L_j - L_k\right) \tag{3.1}$$

where l_{ij} denotes the number of contractions between $\mathcal{O}_i$ and $\mathcal{O}_j$ and $i \neq j \neq k = 1, 2, 3$. The main steps to obtain the three point function are

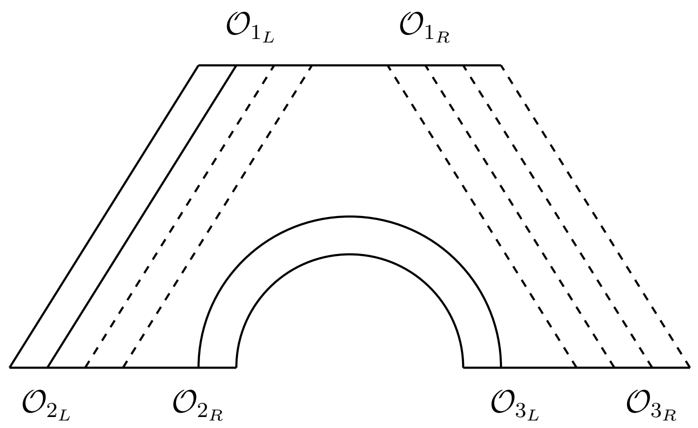


FIGURE 3.1: Tree level contractions between $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$. Solid lines represent vacua and dashed lines represent excitations.

- Map the operators to spin chain states:

$$\mathcal{O}_1 \quad \to |\mathcal{O}_1\rangle \tag{3.2}$$
$$\mathcal{O}_2 \quad \to |\mathcal{O}_2\rangle \tag{3.3}$$
$$\mathcal{O}_3 \quad \to |\mathcal{O}_3\rangle; \tag{3.4}$$

- cutting: split each closed spin chain into two sub-spin chains. $|\mathcal{O}_i\rangle_L$ denotes the left sub-spin chain and $|\mathcal{O}_i\rangle_R$ denotes the right sub-spin chain. In this way the original state can be written as $|\mathcal{O}_i\rangle = \sum H_{LR}|\mathcal{O}_i\rangle_L \otimes |\mathcal{O}_i\rangle_R$. H_{LR} is computed in [52]. The specific expression comes from the so called generalized two site model in which one can take two independent partitions of the Bethe roots and build a monodromy matrix. This matrix is given by the product of two commuting monodromy matrices. To obtain the precise form of H_{LR} we need to exploit that the vacuum is the tensor product of the two independent vacua and that the two partitions are completely independent.

- flipping: when we cut the operators we obtain two kets each operator. In order to translate in this formalism the Wick contraction operation it is needed to convert a ket into a bra. This is possible with the flipping operation. This amounts of changing the order of the fields into the state but not the charge, the latter property distinguishes this operation with the usual conjugation. Note a flipped Bethe state is still a Bethe state.

- sewing: finally the spin chain equivalent of the Wick contraction is to compute the scalar products, that we introduced in Sec.1.2.4. Remember that the scalar product simplifies when considering Bethe states.

Schematically we can write the three point function as

$$\tilde{c}_{123} = \sqrt{\frac{L_1 L_2 L_3}{\mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3}} \sum_{a,b,c} {}_{\mathrm{R}}\langle \mathcal{O}_{3_c} | \mathcal{O}_{1_a} \rangle_{L\,\mathrm{R}} \langle \mathcal{O}_{1_a} | \mathcal{O}_{2_b} \rangle_{L\,\mathrm{R}} \langle \mathcal{O}_{2_b} | \mathcal{O}_{3_c} \rangle_L \tag{3.5}$$

where $\mathcal{N}_i$ are the Gaudin norms and the sum runs over all the possible way of partitioning the sets of roots corresponding to each operators, namely all the possible way that we have to cut each operator and $\tilde{c}_{123} = N c_{123}$ is the planar part of the structure constant.
Let us look back to the form of the operators reported in table (3.1). Since we are interested in non extremal three point functions we can notice that $\mathcal{O}_3$ can contract with $\mathcal{O}_2$ only by contracting all its vacua, in fact in order to have a non trivial three point function we need

$$N_1 = N_2 + N_3. \tag{3.6}$$

This constraint has effect on (3.5) in two ways:

1. the contractions among $\mathcal{O}_2$ and $\mathcal{O}_3$ are only via vacua, thus the respective scalar product gives 1, while the contractions among $\mathcal{O}_1$ and $\mathcal{O}_3$ are only via excitations.

2. there is only one way to split each state into a left part and a right part.

Then in (3.5) it remains only one sum to evaluate and, following the notation of [54], we have that

$$\tilde{c}_{123} = \sqrt{\frac{L_1 L_2 L_3}{\mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3}} \mathcal{F}_1 \sum_{\alpha \cup \bar{\alpha} = \{u\}_{\beta N_1}} \mathcal{F}_2 \; {}_R\langle \mathcal{O}_3 | \mathcal{O}_1 \rangle_L \; {}_R\langle \mathcal{O}_1 | \mathcal{O}_2 \rangle_L \tag{3.7}$$

where the factors $\mathcal{F}_1$ and $\mathcal{F}_2$ are given explicitly in [52] and the sum runs over all the possible ways of partitioning the Bethe roots belonging to $\mathcal{O}_1$ and effectively the subscript β denotes the fact that $\mathcal{O}_1$ is meant to be a Bethe eigenstate. Note that these simplifications rely on the fact that our operators belong to the $SU(2)$ sector.

3.1.2 Scalar products and six vertex model

It is possible to rewrite the problem of computing the scalar products in (3.7) using six vertex model techniques. If we do not split $\mathcal{O}_1$ then we no longer have to sum over the possible way of partitioning its Bethe roots and thus we have

$$\tilde{c}_{123} = \sqrt{\frac{L_1 L_2 L_3}{\mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3}} \left({}_R\langle \mathcal{O}_3| \otimes {}_{\mathrm{L}}\langle \mathcal{O}_2| \right) |\mathcal{O}_1\rangle. \tag{3.8}$$

The notation in (3.8) means that we have two different scalar products involving

- the full initial state $|\mathcal{O}_1\rangle$ with the right final sub state ${}_R\langle \mathcal{O}_3|$
- the full initial state $|\mathcal{O}_1\rangle$ with the left final sub state ${}_{\mathrm{L}}\langle \mathcal{O}_2|$.

This can be described by a $[L, N_1, N_2]$ configuration but, if we look at figure 3.2, we note that a final reference state has to be taken into account, namely 3.2 the final reference state on the right in figure 3.2. In fact the whole expression (3.8) is represented by a $[L, N_1, N_2]$ configuration and an $N_3 \times N_3$ domain wall configuration, see figure 3.3. This allows to write the expression for the three point function in terms of the scalar product and a domain wall partition function

$$\tilde{c}_{123} = \sqrt{\frac{L_1 L_2 L_3}{\mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3}} Z^h_{N_3}(\{w\}_{N_3}) S^h[L, N_1, N_2](\{u_\beta\}_{N_1}, \{v\}_{N_2}) \tag{3.9}$$

in the homogenous limit. Note that we label with the subscript β the set of rapidities satisfying Bethe equations. The main advantage of this formulation is that either the scalar product and the domain wall partition function can be written in a determinant form. In this approach only one of the three states is a Bethe state, namely $\mathcal{O}_1$ in our convention, while in the approach that we reviewed in Sec. 3.1.1 all the three states are meant to be Bethe states.

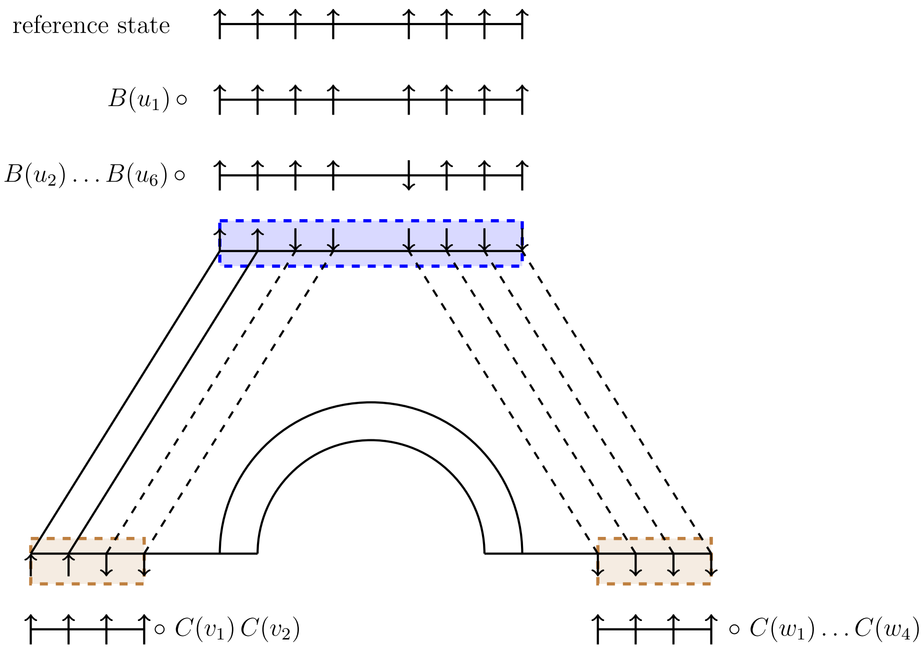


FIGURE 3.2: Operators as spin chain states. The operators $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ are described by their three sets of rapidities that we denote as $\{u_1, \dots, u_{N_1}\}$, $\{v_1, \dots, v_{N_2}\}$ and $\{w_1, \dots, w_{N_3}\}$ respectively. We consider $\mathcal{O}_1$ as an initial state, thus it is obtained by acting to an initial reference state N_1 times with B operators, while $\mathcal{O}_2$ and $\mathcal{O}_3$ are final states which are generated acting with N_2 and N_3 C operators respectively. In this case $N_1 = 6$, $N_2 = 2$ and $N_3 = 4$, the constraint (3.6) is trivially fulfilled.

3.2 $SO(6)$ scalar product

The results that we present in the last section rely heavily on the fact that we are considering operators belonging to the $SU(2)$ sector of $\mathcal{N} = 4$ SYM. An interesting problem to be addressed is the extension of these techniques to bigger sectors. In the following we present a conjecture for the generalisation of the scalar products to operators in the $SO(6)$ sector [59], thus generalizing the formulation of the three point function of [52]. We then calculate the three-point correlation function for three states which cannot be embedded into smaller sectors ($SU(2)$ or $SU(3)$) and show that this structure constant is identical to the one previously found from string field theory and perturbation theory independently[41]. Again we restrict ourselves to planar and non extremal three point functions.

In this section we will use a basis, the so called coordinate basis, which is different from the algebraic basis that we discuss so far. The coordinate basis arises naturally from the Coordinate Bethe ansatz which we did not review in this thesis,

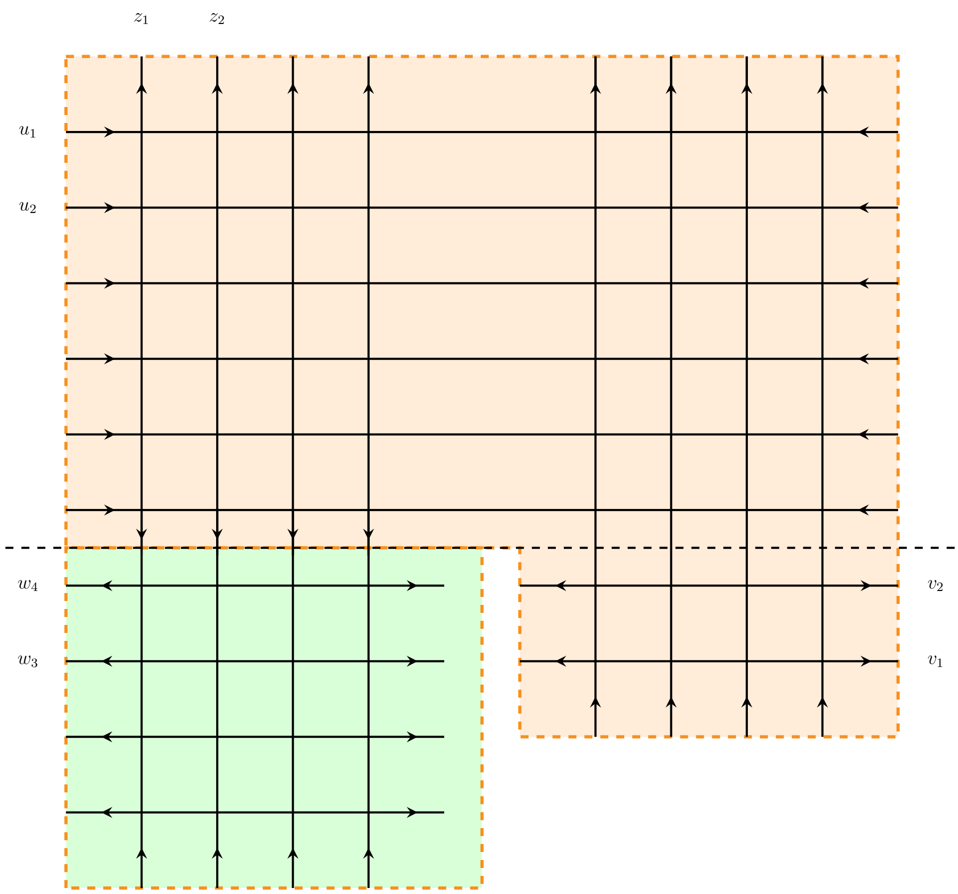


FIGURE 3.3: Six vertex model realisation of figure 3.2. Note that at the black dashed line there are six spin up and two down, which is identical to the final states in figure 3.2. The green part of the figure represent a domain wall configuration while the orange is a restricted $[8, 6, 2]$ configuration.

see for instance [156] for a recent review. However the two basis are related, we refer to section 2.3 of [52] for the precise transformation. We have to go through the three operation that we sketch before, namely the cutting, flipping and sewing procedures, which can be schematically written as

$$c_{123} = \sum_{\text{Root partitions}} \text{Cut } \times \text{Flip} \times \text{Norm} \times \text{Scalar products} \, . \tag{3.10}$$

The structure of the $SO(6)$ formula is conjectured by us to be analogous to (3.10) and directly generalizable from it apart from two subtle issues: the norms and the scalar products. We review how the Bethe ansatz techniques work in the $SO(6)$ sector and we introduce the notation that

$$u_i = \{\mathrm{u}_i, a_i\} \tag{3.11}$$

with which we represent the magnon u_i by a vector containing its rapidity u_i and its level index a_i, meaning that for each of the Bethe roots u_j we specify which of the simple roots is excited by a_j. a_j is the number of simple roots and runs from 1 to 3 for the $SO(6)$ sector. We show some of the commutation relations among the entries of the monodromy matrix, see (1.78). To be consistent with the notation of [52] we can introduce two functions f and g defined as

$$f(u) \quad = \quad 1 + \frac{i}{u} \tag{3.12}$$

$$g(u) \quad = \quad \frac{i}{u} \tag{3.13}$$

in such a way that all the commutation relations can be written in terms of these two functions. It has been shown in [52] that the final expression of the three point function is expressed in terms of these functions, and combinations of them. Let us consider that the generalisation to the $SO(6)$ sector is

$$\begin{aligned} f(u_i, u_j) &= 1 + \frac{iM_{a_i a_j}}{2(\mathrm{u}_i - \mathrm{u}_j)}, \\ g(u_i, u_j) &= \frac{iM_{a_i a_j}}{2(\mathrm{u}_i - \mathrm{u}_j)}. \end{aligned} \tag{3.14}$$

The indices a_i, a_j are exactly the level indices of the i^{th} magnon just defined above. The S-matrix everywhere remains defined as

$$S(u, v) = \frac{f(u, v)}{f(v, u)}. \tag{3.15}$$

The holonomy factors $a(u), d(u)$ retain their standard definitions for higher levels

$$\begin{aligned} a(u_j) &= \mathrm{u}_j + iV_{a_j}/2, \\ d(u_j) &= \mathrm{u}_j - iV_{a_j}/2, \\ e(u) &= \frac{a(u)}{d(u)} \end{aligned} \tag{3.16}$$

so that the Bethe equations have the form (1.85). Following [52] we introduce useful shorthand notation for products of functions: for an arbitrary function $F(u, v)$ of two variables and for arbitrary sets $\alpha, \bar{\alpha}$ of lengths $K, \bar{K}$, $\alpha = \{\alpha_i\}_K$, $\bar{\alpha} = \{\bar{\alpha}_i\}_{\bar{K}}$

$$
\begin{aligned}
F^{\alpha,\bar{\alpha}} &= \prod_{i,j} F(\alpha_i, \bar{\alpha}_j), \\
F^{\alpha,\alpha}_{<} &= \prod_{i<j} F(\alpha_i, \alpha_j), \\
F^{\alpha,\alpha}_{>} &= \prod_{i>j} F(\alpha_i, \alpha_j).
\end{aligned}
\tag{3.17}
$$

For functions $G(u)$ of a single variable let us define

$$
\begin{aligned}
&G^{\alpha} = \prod_j F(\alpha_j), \\
&G^{\alpha \pm i/2} = \prod_j F(\alpha_j \pm i/2).
\end{aligned}
\tag{3.18}
$$

We represent the operators $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ by three three Bethe vectors u, v, w of lengths L_1, L_2, L_3. Then we cut our operators in two gouts which we denote by α and $\bar{\alpha}$, β and $\bar{\beta}$, γ and $\bar{\gamma}$ such that

$$\alpha \cup \bar{\alpha} \;=\; u \tag{3.19}$$

$$\beta \cup \bar{\beta} \;=\; v \tag{3.20}$$

$$\gamma \cup \bar{\gamma} \;=\; w \tag{3.21}$$

The lengths $L_{\bar{\alpha}}, L_{\alpha}, L_{\bar{\beta}}, L_{\beta}, L_{\bar{\gamma}}, L_{\gamma}$ of these pieces are uniquely defined by the lengths of the three operators

$$
\begin{aligned}
L_{\alpha} &= L_{\bar{\beta}} = L_1 + L_2 - L_3, \\
L_{\beta} &= L_{\bar{\gamma}} = L_2 + L_3 - L_1, \\
L_{\gamma} &= L_{\bar{\alpha}} = L_3 + L_1 - L_2.
\end{aligned}
\tag{3.22}
$$

Thus we write (3.10) in terms of the sum over all possible partitions of the relative Bethe roots as

$$
\begin{aligned}
c_{123} = \sum_{\substack{\alpha \cup \bar{\alpha} = u \\ \beta \cup \bar{\beta} = v \\ \gamma \cup \bar{\gamma} = w}} & \sqrt{L_1 L_2 L_3}\; \texttt{Cut}(\alpha, \bar{\alpha}) \texttt{Cut}(\beta, \bar{\beta}) \texttt{Cut}(\gamma, \bar{\gamma}) \times \\
& \times \texttt{Flip}(\bar{\alpha}) \texttt{Flip}(\bar{\beta}) \texttt{Flip}(\bar{\gamma}) \times \\
& \times \frac{1}{\sqrt{\texttt{Norm}(u)\texttt{Norm}(v)\texttt{Norm}(w)}} \times \langle \alpha \bar{\beta} \rangle \langle \beta \bar{\gamma} \rangle \langle \gamma \bar{\alpha} \rangle \;.
\end{aligned}
\tag{3.23}
$$

The way of implementing these operations is summarised by

- `Cut`$(\alpha,\bar\alpha)$:
$$\texttt{Cut}(\alpha,\bar\alpha)=\left(\frac{a^{\bar\alpha}}{d^{\bar\alpha}}\right)^{L_1}\frac{f^{\alpha\bar\alpha}f_<^{\bar\alpha\bar\alpha}f_<^{\alpha\alpha}}{f_<^{uu}}, \tag{3.24}$$
and analogously `Cut`$(\beta,\bar\beta)$ and `Cut`$(\gamma,\bar\gamma)$;

- `Flip`$(\bar\alpha)$:
$$\texttt{Flip}(\bar\alpha)=(e^{\bar\alpha})^{L}_{\bar\alpha}\,\frac{g^{\bar\alpha-i/2}}{g^{\bar\alpha+i/2}}\frac{f_>^{\bar\alpha\bar\alpha}}{f_<^{\bar\alpha\bar\alpha}}, \tag{3.25}$$

- `Norm`(u):
$$\texttt{Norm}(u)=d^u a^u f_>^{uu}f_<^{uu}\frac{1}{g^{u+i/2}g^{u-i/2}}\det(\partial_j\phi_k)\ , \tag{3.26}$$
here $\partial_j=\frac{\partial}{\partial u_j}$ and the phases are the ratio of the left and right sides of the Bethe equations
$$e^{i\phi_j}=e(u_j)^{L_u}\prod_{k\neq j}S^{-1}(u_j,u_k), \tag{3.27}$$
with a,d,S satisfying the multi-level definitions above.

The generalisation of the scalar product is not trivial, in the sense that it cannot be obtained by just replacing the $SU(2)$ expressions with their $SO(6)$ counterpart, for more details see [59]. Note that we do not have the simplifications in the sums and any constraints on the number of Bethe roots as in the $SU(2)$ sector. However to circumvent this problem we formulate the $SO(6)$ norm conjecture via the recursive relation proposed in[52], eq.(A.5). This expression is completely regular and is formulated in terms of physically meaningful objects f,g,a,d,S, thus it makes full sense to conjecture that its validity extends towards a broader sector. The meaning of this formula goes beyond the original $SU(2)$ and is supposed to cover the full $SO(6)$

$$\langle v_1\dots v_N|u_1\dots u_N\rangle_N=\sum_n b_n\langle v_1\dots\hat v_n\dots v_N|\hat u_1\dots u_N\rangle_{N-1}-$$
$$-\sum_{n<m}c_{n,m}\langle u-1v_1\dots\hat v_n\dots\hat v_m\dots v_N|\hat u_1\dots u_N\rangle_{N-1}, \tag{3.28}$$

where

$$b_n=\frac{g(u_1,v_n)\left(\prod_{j\neq n}^N f(u_1,v_j)\prod_{j<n}^N S(v_j,v_n)-\frac{e(u_1)}{e(v_n)}\prod_{j\neq n}f(v_j,u_1)\prod_{j>n}S(v_n,v_j)\right)}{g(u_1+i/2)g(v_n-i/2)\prod_{j\neq 1}f(u_1,u_j)}, \tag{3.29}$$

and

$$c_{n,m} = \frac{e(u_1)g(u_1 - i/2)g(u_1, v_n)g(u_1, v_m)\prod_{j\neq n,m} f(v_j, u_1)}{g(u_1 + i/2)g(v_n - i/2)g(v_m - i/2)\prod_{j\neq 1} f(u_1, u_j)} \times$$
$$\times \left(\frac{S(v_m, v_n)}{e(v_n)} \prod_{j>n} S(v_n, v_j) \prod_{j<m} S(v_j, v_m) + \frac{d(v_m)}{a(v_n)} \prod_{j>m} S(v_m, v_j) \prod_{j<n} S(v_j, v_n) \right). \tag{3.30}$$

The whole derivation of this recursive relation can be found in [52] and even if it does not give a closed expression as in the $SU(2)$ case it is much more efficient than brute force computation and especially it is an attempt to go beyond the $SU(2)$ sector and to increase the examples of explicit computations of three point functions. We check this proposal with an explicit example, which has been already analysed both from the string theory and from the gauge theory side in [41] at tree level, and in [42] at one loop order. In [41] the authors computed the three point function of three two-impurity BMN operators with all non-zero momenta in field theory by using the perturbative expansion and in string theory by using the Dobashi-Yoneya 3-string vertex in the leading order of the Penrose expansion and they found an agreement, which holds also at one loop order [42], between these results. In order to reproduce this setting we introduce our states as Bethe states. We shall denote an N-root state as

$$\langle u| = \{\{\mathrm{u}_1, l_1\}, ...\{\mathrm{u}_N, l_N\}\} \tag{3.31}$$

where u_i denotes the value of the rapidity and l_i the level of Bethe Ansatz it belongs to. The states corresponding to those studied in[41, 42] are

$$\begin{aligned} \mathcal{O}_1 \sim \langle u| &= \{\{0,1\}, \{\tfrac{1}{2}\cot\tfrac{\pi n_1}{J_1+2}, 2\}, \{-\tfrac{1}{2}\cot\tfrac{\pi n_1}{J_1+2}, 2\}\}, \\ \mathcal{O}_2 \sim \langle v| &= \{\{0,3\}, \{\tfrac{1}{2}\cot\tfrac{\pi n_2}{J_2+2}, 2\}, \{-\tfrac{1}{2}\cot\tfrac{\pi n_2}{J_2+2}, 2\}\}, \\ \mathcal{O}_3 \sim \langle w| &= \{\{\tfrac{1}{2}\cot\tfrac{\pi n_3}{J+1}, 2\}, \{-\tfrac{1}{2}\cot\tfrac{\pi n_3}{J+1}, 2\}\}. \end{aligned} \tag{3.32}$$

The lengths of the states are $L_1 = J_1 + 2, L_2 = J_2 + 2, L_3 = J + 2$. The lengths of substates (or, alternatively, the number of contractions between each ith and jth states) are $L_{12} = 1,\ L_{23} = J_2 + 1,\ L_{31} = J_1 + 1$. To make contact with the notation of [41, 42] we introduce the parameter r: $J_1 = rJ, J_2 = (1 - r)J$.

Using the definitions of the $SO(6)$ a, d, f, g, S given above we find all the necessary factors. Expansion in $1/J$ is presumed everywhere below. We use one of the possible four choices of the partitions contributing at the leading order in $1/J$

$$\begin{aligned} &\alpha = \{\{0,1\},\{\tfrac{1}{2}\cot\tfrac{\pi n_1}{J_1+2},2\}\}, &&\bar{\alpha} = \{\{-\tfrac{1}{2}\cot\tfrac{\pi n_1}{J_1+2},2\}\}, \\ &\beta = \{\{-\tfrac{1}{2}\cot\tfrac{\pi n_2}{J_2+2},2\}\}, &&\bar{\beta} = \{\{0,3\},\{\tfrac{1}{2}\cot\tfrac{\pi n_2}{J_2+2},2\}\}, \\ &\gamma = \{\{-\tfrac{1}{2}\cot\tfrac{\pi n_3}{J+1},2\}\}, &&\bar{\gamma} = \{\{\tfrac{1}{2}\cot\tfrac{\pi n_3}{J+1},2\}\}, \end{aligned} \tag{3.33}$$

The flip and cut factors together are

$$\mathtt{Cut}(\alpha,\bar{\alpha})\mathtt{Cut}(\beta,\bar{\beta})\mathtt{Cut}(\gamma,\bar{\gamma}) \times \mathtt{Flip}(\bar{\alpha})\mathtt{Flip}(\bar{\beta})\mathtt{Flip}(\bar{\gamma}) = -1 \tag{3.34}$$

the norms yield

$$\mathtt{Norm}(u)\mathtt{Norm}(v)\mathtt{Norm}(w) = 4J^2 n_1^2 n_2^2 \pi^4 \tag{3.35}$$

and the scalar products read

$$\langle\alpha\bar{\beta}\rangle\langle\beta\bar{\gamma}\rangle\langle\gamma\bar{\alpha}\rangle = \frac{n_1 n_2 \sin^2(\pi n_3 r)}{2(n_1 - r n_3)(n_2 + (1-r)n_3)} \tag{3.36}$$

The other contributing partitions in the leading order are realized by simple transformations $n_1 \to -n_1, n_2 \to -n_2$. There are also partitions that contribute at higher orders in $1/J$, which we do not list here.

Taking all the pieces together we get

$$c_{123} = -\frac{n_3^2 J^{1/2}(r(1-r))^{3/2}\sin^2(\pi n_3 r)}{(n_2^2 - n_3^2(1-r)^2)(n_1^2 - n_3^2(1-r)^2)}, \tag{3.37}$$

which corresponds exactly to the results of [41]. This test reinforces the conjecture we made but in principle much more test have to be performed especially with more general operators.

A very interesting question is how to use the six vertex model techniques for this problem. Recently in [60, 157, 158] scalar products for states in the $SU(3)$ sector of $\mathcal{N} = 4$ SYM have been studied and also form factors [159], which in same special cases are related to three point functions. Quite remarkably three point functions of local single-trace operators in the scalar sector of $\mathcal{N} = 4$ SYM involving operators belonging to the $SU(3)$ sector have been studied in [58], for some cases they can be written in a determinant form. It would be good to check and validate this $SO(6)$ conjecture against these results. When reducing to the $SU(3)$ case indeed the results of [59] should agree with the ones obtained in [58, 60, 157, 158].

3.3 Three point functions in the $SU(2) \times SU(2)$ sector

Another direction to generalize these procedures is to analyse three point functions in the $SU(2) \times SU(2)$ sector of ABJM. We already discussed the case when two operators are heavy and one is light in 2.5 In the following we consider a more general case, where the operators can have any lengths and number of impurities, see table 3.2. With this choice of operators belonging to the $SU(2) \times SU(2)$ sector of ABJM we obtain a non trivial non extremal and planar three point function, it also exists another class of such three point functions which have trivial factorization properties [160].

Operator	Vacuum odd	Excitation odd	Vacuum even	Excitation even
$\mathcal{O}_1$	$(J-J_1)\ \ Z_1$	$J_1\ \ Z_2$	$(J-J_2)\ \ W_1$	$J_2\ \ W_2$
$\mathcal{O}_2$	$(J_1+j_2)\ \ \bar{Z}_2$	$(J-J_1-j_1)\ \ \bar{Z}_1$	$(J_2+j_2)\ \ \bar{W}_2$	$(J-J_2-j_1)\ \ \bar{W}_1$
$\mathcal{O}_3$	$j_2\ \ W_2$	$j_1\ \ \bar{Z}_1$	$j_2\ \ Z_2$	$j_1\ \ \bar{W}_1$

Table 3.2: The field content of our operators $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ of $SU(2) \times SU(2)$ type having a non-vanishing planar, non-extremal three-point function. The notation $J_1\ \ Z_2$ means that the number of Z_2-fields is J_1. It is understood that the number of fields of any type can not be negative.

Here we have indicated which fields are to be considered vacua and which are to be considered excitations in the interpretation of each operator as a state of two coupled $XXX_{1/2}$ spin chains. We have in mind the situation depicted in figure 3.4 with site number one being at the left end of each operator. When we contract the three operators at the planar level all vacuum fields from $\mathcal{O}_3$ are contracted with vacuum fields in $\mathcal{O}_2$ and all excitations of $\mathcal{O}_3$ are contracted with $\mathcal{O}_1$. This means that only a term in $\mathcal{O}_3$ for which all vacuum fields are to the left of all excitations can contribute to the three-point function. Notice also that for contractions involving $\mathcal{O}_1$ we connect even sites to even sites and odd sites to odd sites. For the contractions between $\mathcal{O}_2$ and $\mathcal{O}_3$, however, odd sites get connected to even sites and vice versa. We have illustrated the possible contractions in figure 3.4. Dashed lines are fields corresponding to excitations and solid lines are fields corresponding to vacua. We want to generalise the method proposed in [54]. In this picture the structure constant corresponding to the three-point function appearing in figure 3.4 can be written as the following inner product between Bethe states as in (3.9)

$$\tilde{c}_{123} = \mathcal{N}_{123} \left({}_R\langle \mathcal{O}_3 | \otimes {}_\mathrm{L}\langle \mathcal{O}_2 | \right) | \mathcal{O}_1 \rangle, \tag{3.38}$$

where the subscripts l and r refer to the left and right part respectively and where $\mathcal{N}_{123}$ is a normalization constant which we will see that it turns out to be different from the one present in (3.9) . In order to arrive at (3.38) we have exploited the fact that the inner product between two vacuum states is equal to one. Now $|\mathcal{O}_1\rangle$ is a Bethe eigenstate but ${}_r\langle \mathcal{O}_3 | \otimes {}_1\langle \mathcal{O}_2 |$ is not.

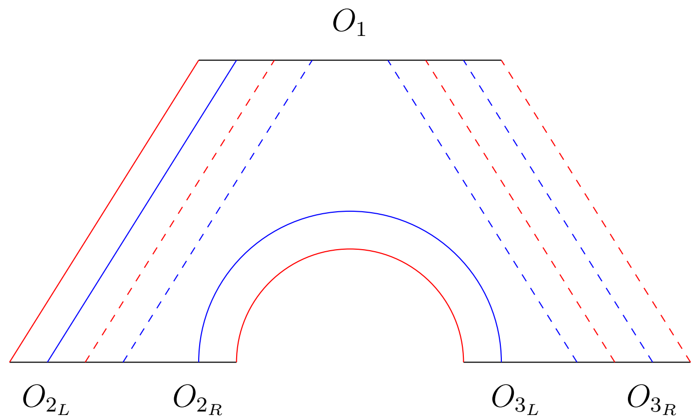


FIGURE 3.4: The possible contractions between $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$. The full lines represent vacua and the dashed lines represent excitations. The two different colours illustrate fields in the two different spin chains.

We want to generalize the construction of [54] to the case that we introduced, namely for operators in the $SU(2) \times SU(2)$ sector of ABJM theory. We already reviewed how the Algebraic Bethe ansatz works for the $SU(2) \times SU(2)$ sector of ABJM theory, the ingredients that we have to keep in mind are the following

- we start from the four R- matrices that we have in the $SU(4)$ spin chain [99] and form two monodromy matrices, one pertaining to the even sites of the spin chain and the other one to the odd sites of the spin chain. Thus one gets two sets of lowering operators B_e and B_o. In the six vertex model language this means that we have 20 different vertices, 6+6 coming from the non zero entries of two R-matrices and 4+4 form the remaining two R-matrices.

- When restricting to the $SU(2)\times SU(2)$ sub-sector, two of the four R-matrices trivialize and the remaining two become the R-matrices of two independent $SU(2)$ spin chains, one living on odd sites and one living on even sites. This is equivalent of saying that 8 vertices, the ones coming from the entries of the trivial R-matrices, have the weight equal to 1 and the remaining 12 vertices can be grouped in two and have separately the same weights as two copies of the usual 6 vertices in the $SU(2)$ case.

- The two monodromy matrices simply become the monodromy matrices of two independent $SU(2)$ spin chains and the corresponding lowering operators B_e and B_o become the usual $SU(2)$ spin flipping operators for even and odd sites respectively. Besides $\{u_o\}$ and $\{u_e\}$ independently satisfy $SU(2)$ Bethe equations. The fact that the two sets of B_e and B_o commute among themselves and also that $B_{e/o}$ commute for different values of the rapidities allow to write the partition function of a domain wall configuration as the product of two $SU(2)$ partition functions. The same can also be done for the scalar product, see figure 3.5.

It seems that in this way we have a full decoupled problem but actually this is not the case. There is a connection between the two sets of rapidities $\{u_o\}$ and $\{u_e\}$, they are related via the momentum constraint which says that the total momentum of all excitations should vanish and reflects the fact that the corresponding single trace operator of ABJM theory should be invariant when one or more pairs of fields are cyclically displaced. Thus, in full analogy with (3.9) we write the three point function as (see Fig. 3.5)

$$\begin{aligned}\tilde{c}_{123} = \quad \mathcal{N}_{123} \quad & Z_{j_1}\left(\{w_o\}\right) S[J,J_1,J-J_1-j_1](\{u_o\},\{v_o\}) \times \\ & Z_{j_1}\left(\{w_e\}\right)\, S[J,J_2,J-J_2-j_1](\{u_e\},\{v_e\}).\end{aligned} \tag{3.39}$$

Here the Z's are domain wall partition functions and the S's are Slavnov inner products. Both types of quantities can be expressed as determinants. The normalization constant $\mathcal{N}_{123}^{ABJM}$ takes the form

$$\mathcal{N}_{123} = \frac{\sqrt{J(j_1+j_2)(J+j_2-j_1)}}{\sqrt{\mathcal{N}_{1o}\mathcal{N}_{1e}\mathcal{N}_{2o}\mathcal{N}_{2e}\mathcal{N}_{3o}\mathcal{N}_{3e}}}. \tag{3.40}$$

Each term is equal to the partition function which one encounters when calculating three point functions of $\mathcal{N}=4$ SYM and which was already determined by

Foda who found that it could be written as a product of a Slavnov inner product and a domain wall partition function both evaluated in the homogeneous limit $z_{i_o}, z_{i_e} \to i/2$. The domain wall partition function comes from the lower left corner of the lattice while the remaining part constitutes a Slavnov scalar product. For simplicity we have depicted a case where we have the same number of excitations on the odd and the even lattice but the result holds in the general case as well. Again, it is a simple consequence of the decoupling of the two lattices. In order that the Bethe eigenstates which enter the three-point functions be normalized to unity we must divide the result by the Gaudin norm for each operator. In addition we must multiply by a factor which cures the fact that the presentation of our three-point function as in figure 3.5 fails to take into account the cyclicity of the ABJM operators. For this final factor one does not have a similar complete decoupling into a product of two factors. This is due to the alternating nature of the ABJM operators which implies that we only have cyclicity (in the horizontal direction) for the combined red-blue model and not for the red and blue model alone.

Let us make two comments:

- we mentioned in 2.5 that there is a method to express the structure constants of three chiral primaries as a limit of a more general formula. In [161] it was shown how to perform this limit for operators in $\mathcal{N}$=4 SYM theory, starting from the expression (3.9). This is the limit where all the rapidities of the three operators go to infinity and the nice insight of [161] has been to rewrite the Slavnov scalar product and the domain wall partition function using the so called functional formalism. Using the properties of the functionals, taking the BPS limit becomes treatable, as well as the classical limit. Thus we compute the three point functions of three chiral primaries by considering a limit of (3.39) where all the rapidities go to infinity. Adapting the procedure of [161] to our operators in the $SU(2) \times SU(2)$ of ABJM theory we find

$$C^{\circ\circ\circ} = J\sqrt{2j}\,\frac{(J-J_1+j)!J_1!((J-j)!)^2 j!^2}{(J!)^2(J-J_1)!(J_1-j)!(2j)!}. \tag{3.41}$$

  Note that, a part from a different normalization, this is precisely the square of the result of [161] for operators in the $SU(2)$ sector of $\mathcal{N}$=4 SYM theory. Note also that, as it should be, (3.41) and (2.182) are identical.

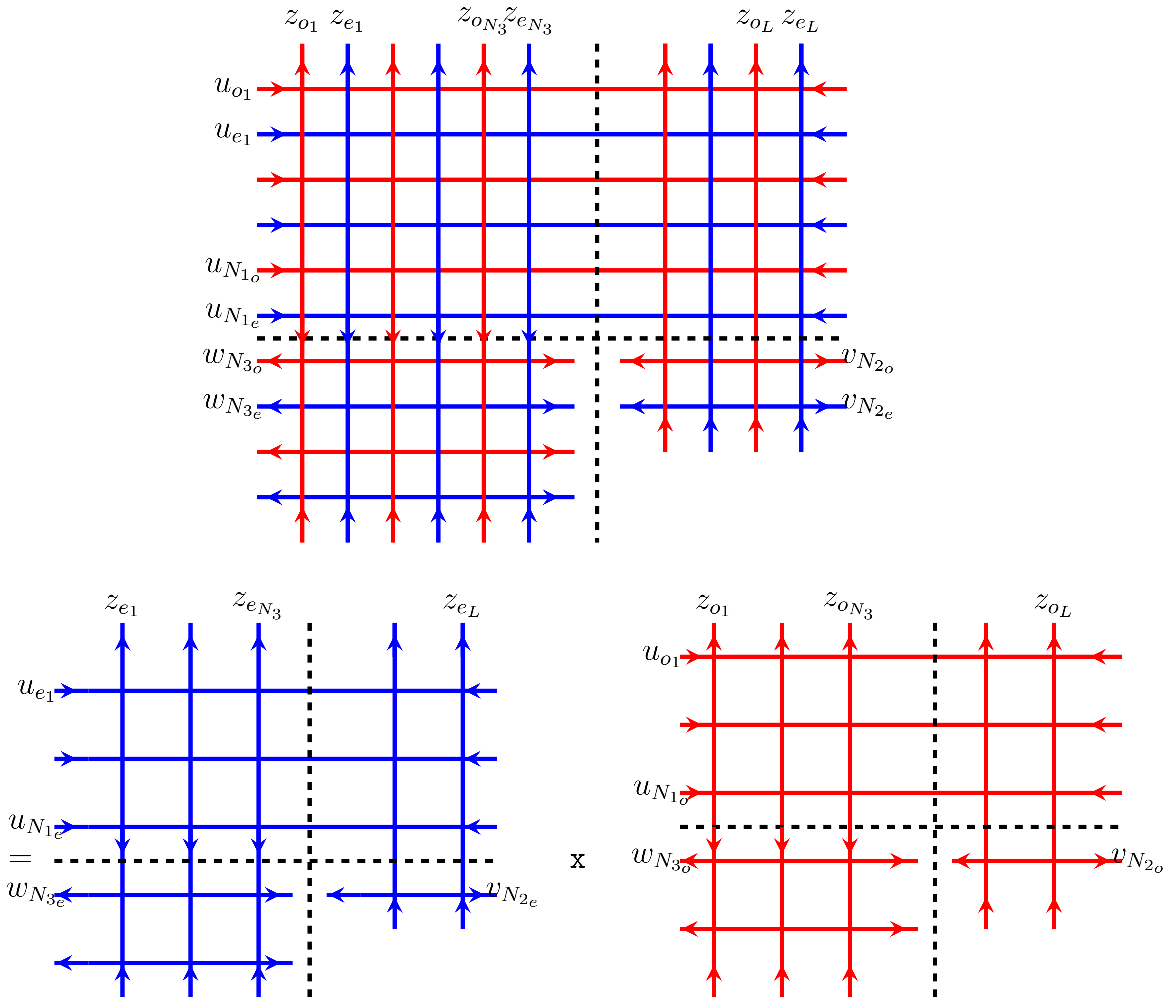


FIGURE 3.5: The decoupling of the three-point function into two parts.

- In 2.5 we compute the structure constants for an intermediate case, when two operators are non-BPS and the other one is a $\frac{1}{2}$-BPS. Our result, namely (2.185), does not show the almost complete factorzation that we notice in (3.39). This is due to the fact that the cyclicty properties enter in the two models in a complete different way. Actually in the six vertex model formalism the cyclity is not present explicitly. One of the most interesting future directions would be to find a procedure to obtain from (3.9) or equivalently (3.39) the gauge theory results in the intermediate case when two operators are much longer than the other one. This seems to be a difficult problem because in general one should be able to take a limit in which two of the three sets of rapidities are sent to zero while the remaining set goes to infinity.

Conclusions

In this thesis we have faced with the problem of computing three point correlation functions in the context of the AdS/CFT correspondence. We discussed in particular five different computations. Within the AdS_5/CFT_4 correspondence we compute

- tree level three point functions of two giant gravitons (wrapping an $S^3 \subset S^5$ and $S^3 \subset AdS_5$) and a point like graviton and its dual counterpart, namely two Schur polynomials (in the antisymmetric and symmetric representation) and a single trace chiral primary, section 2.3;
- one loop correction to planar, non extremal three point functions of two heavy and one light operators, both from the gauge and string side in the Frolov-Tseytlin regime, section 2.4;
- generalisation of the scalar product of two states belonging to the $SO(6)$ sector of $\mathcal{N} = 4$ SYM with implications on the construction of three point functions of 3 non-BPS operators from the gauge theory side, section 3.2 .

In the context of the AdS_4/CFT_3 we describe

- planar, non extremal three point functions of two heavy and one light operators, belonging to the $SU(2) \times SU(2)$ sector of ABJM, both in the gauge and string theory side, in the Frolov-Tseytlin limit, section 2.5;
- three point functions of operators belonging to the $SU(2) \times SU(2)$ sector of ABJM from the gauge theory side, obtaining a determinant representation, section 3.3.

We would like to conclude this thesis outlining some interesting open problems and future directions

- find a limit of the integrability inspired techniques on the gauge theory to reproduce the results of heavy-heavy-light correlators;

- extend the procedures discussed in chapter 3 to higher loops, and eventually all loops, and broader sectors. It would be extremely interesting to analyse for instance the $SL(2)$ sector to compare with string theory results exploiting the findings of [45];

- make concrete connections among correlation functions and scattering amplitudes, on the line of [162] [163] [149].

Appendix A

Scalar products in the $SU(2) \times SU(2)$ sector of ABJM

In this appendix we want to show that a scalar product defined as in section 1.4.5 satisfies the properties of the Slavnov scalar product and it is uniquely defined by them.

Lemma A.1. *1. $S[N_1, N_2, L]$ is symmetric in the variables $\{z_{o_{N_3+1}}, \ldots, z_{o_L}\}$ and $\{z_{e_{N_3+1}}, \ldots, z_{e_L}\}$*

2. *$S[N_1, N_2, L]$ is a fraction in which the numerator is a product of trigonometric polynomials of degree $L-1$ each in $v_{o_{N_2}}$ and $v_{e_{N_2}}$, with zeros occurring at the points $v_{o_{N_2}} = z_{o_i} - \eta$ and $v_{e_{N_2}} = z_{e_i} - \eta$ for all $1 \leq i \leq N_3$,and the denominator is a polynomial of degree L: $\prod_{i=1}^{L}(v_{o_{N_2}} - z_{o_i})(v_{e_{N_2}} - z_{e_i})$, with zeros occurring at the points $v_{o_{N_2}} = z_{o_i}$ and $v_{e_{N_2}} = z_{e_i}$ for all $1 \leq i \leq L$*

3. *Setting $v_{o_{N_2}} = z_{o_{N_3+1}}$ and $v_{e_{N_2}} = z_{e_{N_3+1}}$, $S[N_1, N_2, L]$ satisfies the recursion relation*

$$S[N_1, N_2, L]\Big|_{\substack{v_{o_{N_2}}=z_{o_{N_3+1}} \\ v_{e_{N_2}}=z_{e_{N_3+1}}}} = \prod_{i=1}^{L} \frac{z_{o_{N_3+1}} - z_{o_i} + \eta}{z_{o_{N_3+1}} - z_{o_i}} \prod_{j=1}^{L} \frac{z_{e_{N_3+1}} - z_{e_j} + \eta}{z_{e_{N_3+1}} - z_{e_j}} S[N_1, N_2-1, L] \tag{A.1}$$

4. *$S[N_1, N_2 = 0, L]$ is precisely the domain wall partition function*

$$S[N_1, 0, L] = Z_{N_1}(\{u_o, u_e\}_{N_1}, \{z_o, z_e\}_{N_1}) = Z_{N_1}(\{u_o\}_{N_1}, \{z_o\}_{N_1}) Z_{N_1}(\{u_e\}_{N_1}, \{z_e\}_{N_1}) \tag{A.2}$$

Proof

1. If we rotate the diagram of the scalar product by 90 degrees counterclockwise we can write it in terms of D lines like

$$\prod_{j=N_3+1}^{L} D_o(z_{o_j}\{u_o,u_e\}_{N_1}\cup\{v_o,v_e\}_{N_2})D_e(z_{e_j},\{u_o,u_e\}_{N_1}\cup\{v_o,v_e\}_{N_2}) \tag{A.3}$$

which are sandwiched together with $2N_3$ C lines between the initial and final state, from which it follows that $S[N_1,N_2,L]$ is symmetric in the variables $\{z_{o_{N_3+1}},\ldots,z_{o_L}\}$ and $\{z_{e_{N_3+1}},\ldots,z_{e_L}\}$, since all D_o and D_e commute among themselves and among each other.

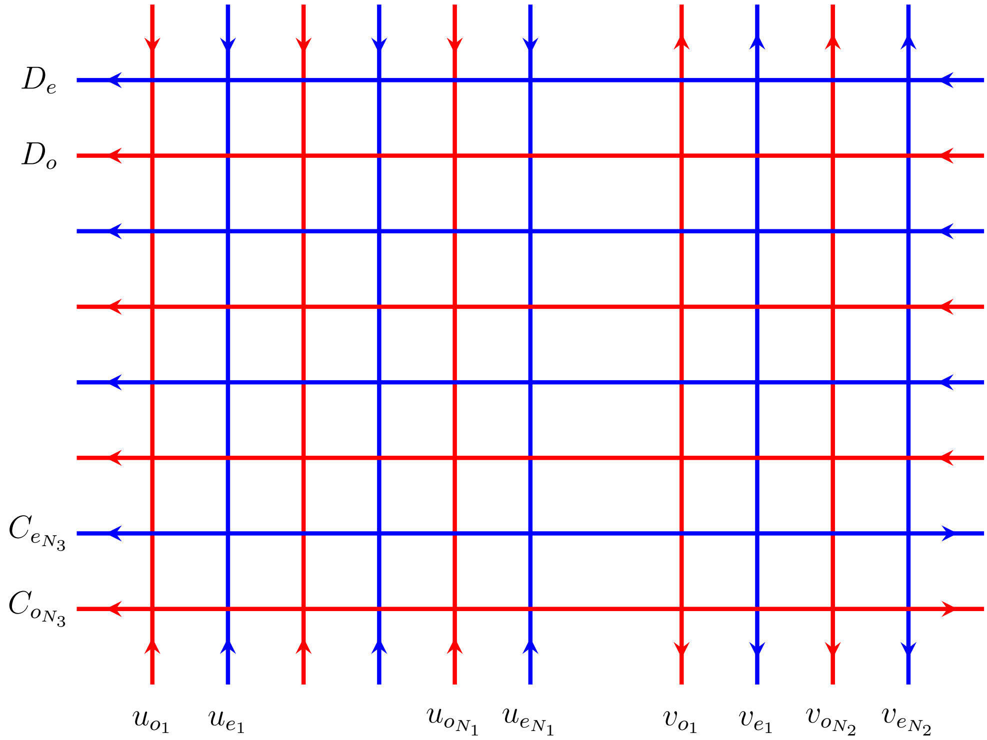


FIGURE A.1: Rotated grid.

2. Inserting the set of states

$$\sum_{m>N_3} \sigma_m^-|\downarrow_{z_{N_3/L}}\rangle\langle\downarrow_{z_{N_3/L}}|\sigma_m^+ \tag{A.4}$$

after $C_e(v_{e_{N_2}})$ and

$$\sum_{m'm>N_3} \sigma_m^-\sigma_{m'}^-|\downarrow_{z_{N_3/L}}\rangle\langle\downarrow_{z_{N_3/L}}|\sigma_m^+\sigma_{m'}^+$$

after $C_o(v_{o_{N_2}})$ we will have that

$$\begin{aligned}
&S[N_1,N_2,L] = \\
&\quad = \sum_{m'm>N_3} \langle \downarrow_{z_{N_3/L}} |C_e(v_{e_{N_2}})\sigma_m^-| \downarrow_{z_{N_3/L}}\rangle\langle\downarrow_{z_{N_3/L}} |\sigma_m^+ C_o(v_{o_{N_2}})\sigma_m^-\sigma_{m'}^-| \downarrow_{z_{N_3/L}}\rangle \\
&\quad \langle\downarrow_{z_{N_3/L}} |\sigma_m^+\sigma_{m'}^+ \prod_{i=1}^{N_2-1} C_e(v_{e_i})C_o(v_{o_i}) \prod_{j=1}^{N_1} B_e(u_{e_j})B_o(u_{o_j})| \uparrow_{z_L}\rangle = \\
&\quad = \sum_{m'm>N_3} \prod_{i=1}^{N_3} \frac{v_{o_{N_2}} - z_{o_i} + \eta}{v_{o_{N_2}} - z_{o_i}} \left(\prod_{i=N_3+1}^{m'-1} 1\right) \frac{\eta}{v_{o_{N_2}} - z_{o_{m'}}} \prod_{i=m'+1}^{L} \frac{v_{o_{N_2}} - z_{o_i} + \eta}{v_{o_{N_2}} - z_{o_i}} \\
&\quad \prod_{i=1}^{N_3} \frac{v_{e_{N_2}} - z_{e_i} + \eta}{v_{e_{N_2}} - z_{e_i}} \left(\prod_{i=N_3+1}^{m-1} 1\right) \frac{\eta}{v_{e_{N_2}} - z_{e_m}} \prod_{i=m+1}^{L} \frac{v_{e_{N_2}} - z_{e_i} + \eta}{v_{e_{N_2}} - z_{e_i}} \\
&\quad \langle\downarrow_{z_{N_3/L}} |\sigma_m^+\sigma_{m'}^+ \prod_{i=1}^{N_2-1} C_e(v_{e_i})C_o(v_{o_i}) \prod_{j=1}^{N_1} B_e(u_{e_j})B_o(u_{o_j})| \uparrow_{z_L}\rangle \qquad \text{(A.5)}
\end{aligned}$$

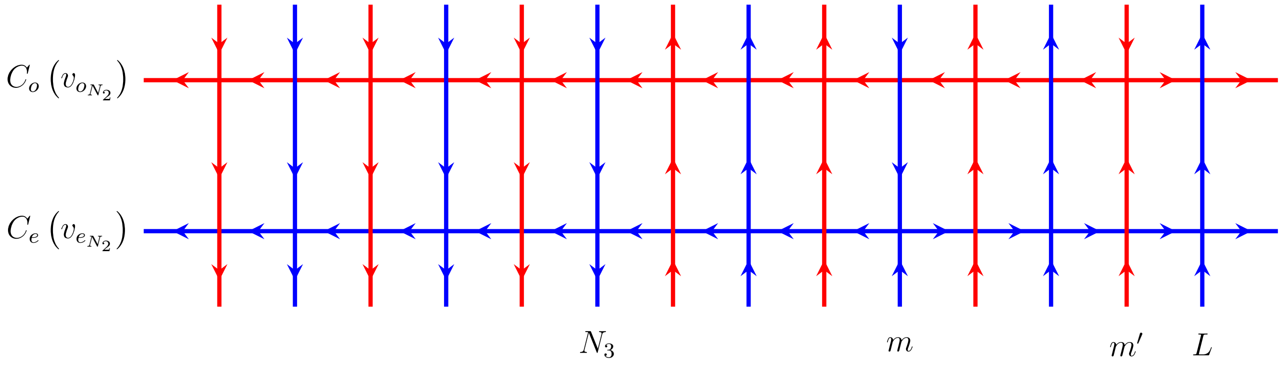


FIGURE A.2: C lines.

If we take out the factor $\prod_{i=1}^{L}(v_{o_{N_2}} - z_{o_i})(v_{e_{N_2}} - z_{e_i})$ from the denominator of (A.5) we will end up with a trigonometric polynomials of degree $L-1$ in $v_{o_{N_2}}$ and $v_{e_{N_2}}$ and, for each one, N_3 of the zeros of these polynomials in numerators are contained in the factors $\prod_{i=1}^{N_3}(v_{o_{N_2}} - z_{o_i} + \eta)$ and $\prod_{i=1}^{N_3}(v_{e_{N_2}} - z_{e_i} + \eta)$.

3. Setting $v_{o_{N_2}} = z_{o_{N_3+1}}$ and $v_{e_{N_2}} = z_{e_{N_3+1}}$ in the sum of (A.5), all terms in the sum can be neglected comparing the terms corresponding to $m' = (N_3+1)_o$ and $m = (N_3+1)_e$ for which we have infinities and we obtain

$$S[N_1,N_2,L]\Big|_{\substack{v_{o_{N_2}}=z_{o_{N_3+1}} \\ v_{e_{N_2}}=z_{e_{N_3+1}}}} = \prod_{i=1}^{L} \frac{z_{o_{N_3+1}} - z_{o_i} + \eta}{z_{o_{N_3+1}} - z_{o_i}} \prod_{j=1}^{L} \frac{z_{e_{N_3+1}} - z_{e_j} + \eta}{z_{e_{N_3+1}} - z_{e_j}} S[N_1,N_2-1,L] \qquad \text{(A.6)}$$

where we took into account the fact that

$$\langle \downarrow_{z_{N_3/L}} | \sigma^{+}_{(N_3+1)_e} \sigma^{+}_{(N_3+1)_o} = \langle \downarrow_{z_{N_3+1/L}} |. \tag{A.7}$$

4 In the case $N_2 = 0$ we have $N_3 = N_1$ and in $S[N_1, 0, L]$ all the vertices from row 1 to N_1 and from column $N_1 + 1$ to L are fixed to be of type b , d and d', which have all weight 1. What is left is precisely the domain wall partition function, see Fig.A.3.

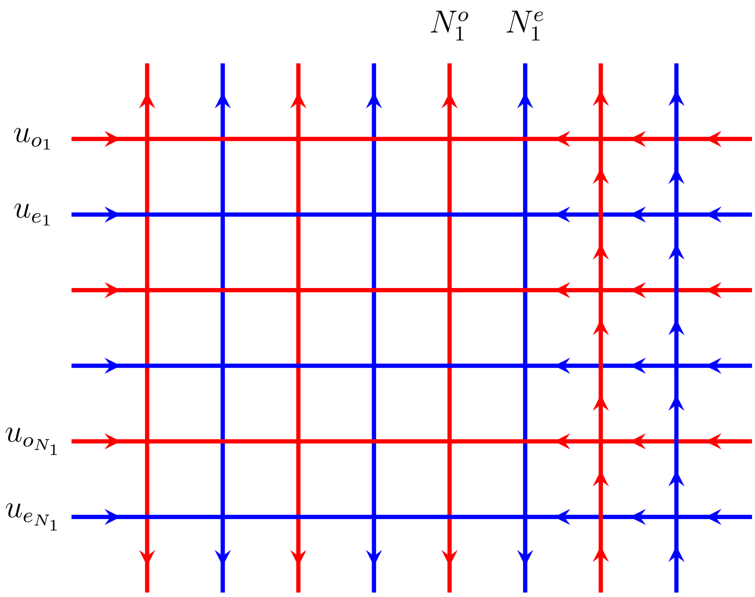


FIGURE A.3: Special case when the scalar product reduces to the partition function.

Lemma A.2. *The four conditions of A.1 determine the scalar product* $S[N_1, N_2, L]$ *uniquely.*

From condition 4. of A.1 on $S'[N_1, 0, L]$ and $S[N_1, 0, L]$ we know that $S'[N_1, 0, L] = S[N_1, 0, L] = Z_{N_1}(\{u_o, u_e\}_{N_1}, \{z_o, z_e\}_{N_1})$. Now let us assume that $S'[N_1, N_2 - 1, L] = S[N_1, N_2 - 1, L]$ for some $N_2 \geq 1$. Using this assumption together with condition 3. of A.1 on $S'[N_1, N_2, L]$ and $S[N_1, N_2, L]$ yields to

$$S'[N_1,N_2,L]\Big|_{\substack{v_{o_{N_2}}=z_{o_{N_3+1}}\\ v_{e_{N_2}}=z_{e_{N_3+1}}}} = \prod_{i=1}^{L}\frac{z_{o_{N_3+1}}-z_{o_i}+\eta}{z_{o_{N_3+1}}-z_{o_i}}\prod_{j=1}^{L}\frac{z_{e_{N_3+1}}-z_{e_j}+\eta}{z_{e_{N_3+1}}-z_{e_j}}S'[N_1,N_2-1,L] =$$
$$= \prod_{i=1}^{L}\frac{z_{o_{N_3+1}}-z_{o_i}+\eta}{z_{o_{N_3+1}}-z_{o_i}}\prod_{j=1}^{L}\frac{z_{e_{N_3+1}}-z_{e_j}+\eta}{z_{e_{N_3+1}}-z_{e_j}}S[N_1,N_2-1,L] =$$
$$= S[N_1,N_2,L]\Big|_{\substack{v_{o_{N_2}}=z_{o_{N_3+1}}\\ v_{e_{N_2}}=z_{e_{N_3+1}}}} \tag{A.8}$$

Condition 1. of of A.1 on S and S' states that both are symmetric in the variables $\{z_{o_{N_3+1}},\ldots,z_{o_L}\}$ and $\{z_{e_{N_3+1}},\ldots,z_{e_L}\}$. Using this fact in (A.8), we find that

$$S'[N_1,N_2,L]\Big|_{\substack{v_{o_{N_2}}=z_{o_i}\\ v_{e_{N_2}}=z_{e_i}}} = S[N_1,N_2,L]\Big|_{\substack{v_{o_{N_2}}=z_{o_i}\\ v_{e_{N_2}}=z_{e_i}}} \tag{A.9}$$

for all $N_3+1 \le i \le L$. Because of condition 2. of A.1 , S and S' are polynomials both in $v_{o_{N_2}}$ and $v_{e_{N_2}}$ of degree $L-1$ in the numerator and also S and S' have the same N_3 zeros occurring at the points $v_{o_{N_2}} = z_{o_i} - \eta$ and $v_{e_{N_2}} = z_{e_i} - \eta$ for all $1 \le i \le N_3$. Apart from the common zeros and the common expression in the denominator, the rest of the polynomial in the numerator is of degree $L - N_3 - 1$, but we have equality between S and S' in $L - N_3$ points,hence the equality is implied everywhere.
Now we define the following functions

$$f_i^o(z_{o_j}) = \frac{\eta}{(u_{o_i}-z_{o_j})(u_{o_i}-z_{o_j}+\eta)}\prod_{k=1}^{N_1}(u_{o_k}-z_{o_j}+\eta) \tag{A.10}$$

$$f_i^e(z_{e_j}) = \frac{\eta}{(u_{e_i}-z_{e_j})(u_{e_i}-z_{e_j}+\eta)}\prod_{k=1}^{N_1}(u_{e_k}-z_{e_j}+\eta) \tag{A.11}$$

$$g_i^o(v_{o_j}) = \left(\frac{\eta}{u_{o_i}-v_{o_j}}\right)\left(\left(\prod_{k\neq i}^{N_1}(u_{o_k}-v_{o_j}+\eta)\prod_{k=1}^{L}\frac{(v_{o_j}-z_{o_k}+\eta)}{(v_{o_j}-z_{o_k})}\right)-\prod_{k\neq i}^{N_1}(u_{o_k}-v_{o_j}-\eta)\right) \tag{A.12}$$

$$g_i^e(v_{e_j}) = \left(\frac{\eta}{u_{e_i}-v_{e_j}}\right)\left(\left(\prod_{k\neq i}^{N_1}(u_{e_k}-v_{e_j}+\eta)\prod_{k=1}^{L}\frac{(v_{e_j}-z_{e_k}+\eta)}{(v_{e_j}-z_{e_k})}\right)-\prod_{k\neq i}^{N_1}(u_{e_k}-v_{e_j}-\eta)\right) \tag{A.13}$$

Using this definitions, we construct the $N_1 \times N_1$ matrix

$$M[\{u_o, u_e\}_{N_1}, \{v_o, v_e\}_{N_2}, \{z_o, z_e\}_L] =$$
$$= \begin{pmatrix} f_1^o(z_{o_1}) & \cdots & f_1^o(z_{o_{N_3}}) & g_1^o(v_{o_1}) & \cdots & g_1^o(v_{o_{N_2}}) \\ \vdots & & \vdots & \vdots & & \vdots \\ f_{N_1}^o(z_{o_1}) & \cdots & f_{N_1}^o(z_{o_{N_3}}) & g_{N_1}^o(v_{o_1}) & \cdots & g_{N_1}^o(v_{o_{N_2}}) \end{pmatrix} \times$$
$$\times \begin{pmatrix} f_1^e(z_{e_1}) & \cdots & f_1^e(z_{e_{N_3}}) & g_1^e(v_{e_1}) & \cdots & g_1^e(v_{e_{N_2}}) \\ \vdots & & \vdots & \vdots & & \vdots \\ f_{N_1}^e(z_{e_1}) & \cdots & f_{N_1}^e(z_{e_{N_3}}) & g_{N_1}^e(v_{e_1}) & \cdots & g_{N_1}^e(v_{e_{N_2}}) \end{pmatrix} \tag{A.14}$$

We also define

$$D_S^o = \prod_{i=1}^{N_2}\prod_{j=1}^{N_3}(v_{o_i} - z_{o_j}) \prod_{1\le i<j\le N_2}(v_{o_i} - v_{o_j}) \prod_{1\le i<j\le N_1}(u_{o_i} - u_{o_j}) \prod_{1\le i<j\le N_3}(z_{o_j} - z_{o_i}) \tag{A.15}$$

$$D_S^e = \prod_{i=1}^{N_2}\prod_{j=1}^{N_3}(v_{e_i} - z_{e_j}) \prod_{1\le i<j\le N_2}(v_{e_i} - v_{e_j}) \prod_{1\le i<j\le N_1}(u_{e_i} - u_{e_j}) \prod_{1\le i<j\le N_3}(z_{e_j} - z_{e_i}) \tag{A.16}$$

Lemma A.3. *If the sets of rapidities $\{u_o\}_{N_1}$ and $\{u_e\}_{N_1}$ satisfy the Bethe equations, then*

$$S[N_1, N_2, L] = \frac{detM}{D_S^o D_S^e} \tag{A.17}$$

Proof One can show that the formula above satisfies all the conditions of A.1
1. All dependence on $\{z_{o_{N_3+1}}, \ldots, z_{o_L}\}$ and $\{z_{e_{N_3+1}}, \ldots, z_{e_L}\}$ of S defined above occurs in the functions $g_i^o(v_{o_j})$ and $g_i^e(v_{e_j})$ and apparently they are invariant under permutations $z_{o_i} \leftrightarrow z_{o_j}$ and $z_{e_i} \leftrightarrow z_{e_j}$ for all $i \neq j$
2. The dependence on $v_{o_{N_2}}$ and $v_{e_{N_2}}$ in the determinant occurs only in $g_i^o(v_{o_{N_2}})$ and $g_i^e(v_{e_{N_2}})$, from which we can easily take out $\prod_{i=1}^{L}(v_{o_{N_2}} - z_{o_i})(v_{e_{N_2}} - z_{e_i})$ as a denominator and since $\{u_o\}_{N_1}$ and $\{u_e\}_{N_1}$ satisfy the Bethe equations the numerators of $g_i^o(v_{o_{N_2}})$ and $g_i^e(v_{e_{N_2}})$ vanish in the limit $v_{o_{N_2}} \to u_{o_i}$ and $v_{e_{N_2}} \to u_{e_i}$. Then it follows that the pole in g is removable and therefore they are trigonometric polynomials of degree $L + N_1 - 2$ in $v_{o_{N_2}}$ and $v_{e_{N_2}}$ in the numerator. While D_S^o is a polynomial of degree $N_1 - 1$ in $v_{o_{N_2}}$ and D_S^e is a polynomial of degree $N_1 - 1$ in $v_{e_{N_2}}$ and they all are canceled by the zeros of the numerator. $N_2 - 1$ zeros $v_{o_{N_2}} = v_{o_j}$ and $v_{e_{N_2}} = v_{e_j}$ for $1 \le j \le N_2 - 1$ of D_S^o and D_S^e are canceled with the same zeros of the determinant. Since when we set $v_{o_{N_2}} = v_{o_j}$ and $v_{e_{N_2}} = v_{e_j}$ for

$1 \leq j \leq N_2 - 1$,it causes to columns of the determinant to become equal. The left N_3 zeros $v_{o_{N_2}} = z_{o_j}$ and $v_{e_{N_2}} = z_{e_j}$ for $1 \leq j \leq N_3$ of D_S^o and D_S^e are also canceled with the same zeros of det, since when we put $v_{o_{N_2}} = z_{o_j}$ and $v_{e_{N_2}} = z_{e_j}$ for $1 \leq j \leq N_3$ two columns of the determinants again become equal up to multiplicative factor

$$g_i^o(z_{o_j}) = \left(\frac{\eta}{u_{o_i} - z_{o_j}}\right)\left(\prod_{k\neq i}^{N_1}(u_{o_k} - z_{o_j} + \eta)\prod_{k=1}^{L}\frac{(z_{o_j} - z_{o_k} + \eta)}{(z_{o_j} - z_{o_k})}\right) = \prod_{k=1}^{L}\frac{(z_{o_j} - z_{o_k} + \eta)}{(z_{o_j} - z_{o_k})} f_i^o(z_{o_j}) \tag{A.18}$$

the second term in g is negligible comparing to the infinity of the first one.The same expression holds for $g_i^e(z_{e_j})$. So we see that the determinant expression is a polynomial of degree $L-1$ divided by $\prod_{i=1}^{L}(v_{o_{N_2}} - z_{o_i})(v_{e_{N_2}} - z_{e_i})$. And at last to get the N_3 zeros in order to satisfy the condition 2 of Lemma 1 and 2 we demonstrate

$$g_i^o(z_{o_j} - \eta) = \left(\frac{-\eta}{u_{o_i} - z_{o_j} + \eta}\right)\prod_{k\neq i}^{N_1}(u_{o_k} - z_{o_j}) = -\prod_{k=1}^{N_1}\frac{u_{o_k} - z_{o_j}}{u_{o_k} - z_{o_j} + \eta} f_i^o(z_{o_j}) \tag{A.19}$$

And again same kind of expression holds for $g_i^e(z_{e_j} - \eta)$ where two columns of determinants are equal up to a multiplicative factor hence producing the N_3 zeros
3. Using equation (A.18) and the definition of the matrix (A.14), it is clear that

$$\begin{aligned} &detM[\{u_o, u_e\}_{N_1}, \{v_o, v_e\}_{N_2}, \{z_o, z_e\}_L]\Big|_{\substack{v_{o_{N_2}} = z_{o_{N_3+1}} \\ v_{e_{N_2}} = z_{e_{N_3+1}}}} = \\ &= \prod_{k=1}^{L}\frac{(z_{o_{N_3+1}} - z_{o_k} + \eta)}{(z_{o_{N_3+1}} - z_{o_k})}\prod_{l=1}^{L}\frac{(z_{e_{N_3+1}} - z_{e_l} + \eta)}{(z_{e_{N_3+1}} - z_{e_l})} M[\{u_o, u_e\}_{N_1}, \{v_o, v_e\}_{N_2-1}, \{z_o, z_e\}_L] \end{aligned} \tag{A.20}$$

Furthermore we notice that for D_S^o and D_S^e

$$\left(\prod_{i=1}^{N_2}\prod_{j=1}^{N_3}(v_{o_i}-z_{o_j})\prod_{1\leq i<j\leq N_2}(v_{o_i}-v_{o_j})\prod_{1\leq i<j\leq N_1}(u_{o_i}-u_{o_j})\prod_{1\leq i<j\leq N_3}(z_{o_j}-z_{o_i})\right)\Bigg|_{v_{o_{N_2}}=z_{o_{N_3+1}}} =$$
$$= \left(\prod_{i=1}^{N_2-1}\prod_{j=1}^{N_3+1}(v_{o_i}-z_{o_j})\prod_{1\leq i<j\leq N_{2-1}}(v_{o_i}-v_{o_j})\prod_{1\leq i<j\leq N_1}(u_{o_i}-u_{o_j})\prod_{1\leq i<j\leq N_3+1}(z_{o_j}-z_{o_i})\right) \tag{A.21}$$

$$D_S^e[N_2,N_3]\Bigg|_{v_{e_{N_2}}=z_{e_{N_3+1}}} = D_S^e[N_2-1,N_3+1] \tag{A.22}$$

So we find that (A.17) satisfies the recursion relation of condition 3. of A.1 and A.2. **4.** Taking $N_2 = 0$ yields to

$$S[N_1,N_2=0,L] = \frac{detM[\{u_o,u_e\}_{N_1},\{z_o,z_e\}_L]}{\prod_{1\leq i<j\leq N_1}(u_{o_i}-u_{o_j})(z_{o_j}-z_{o_i})(u_{e_i}-u_{e_j})(z_{e_j}-z_{e_i})} \tag{A.23}$$

which is precisely the domain wall partition function.

Appendix B

Jack functions

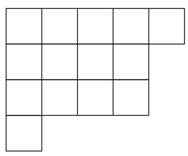
Definitions

- A partition [1] is any (finite or infinite) sequence

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots) \tag{B.1}$$

  of non negative integers in decreasing order $\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots \geq \lambda_r$ and containing only finitely many non-zero terms.

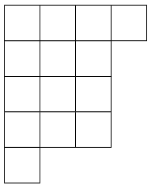
- The diagram of a partition λ may be formally defined as the set of points (i, j) such that $1 \leq j \leq \lambda_i$. In drawing such diagrams, as with matrices, the first coordinate i (row index) increases as goes downwards and the column index j increases as one goes from left to right. Usually one defines $s = (i, j)$. For example the diagram of the partition (5 4 4 1) is



- The conjugate of a partition λ is the partition λ' whose diagram is the transpose of the diagram λ (diagram obtained by reflection in the main diagonal).

[1] All the definitions and conventions are from [164]

For example the conjugate of (5 4 4 1) is



- One could specify the coordinates of the boxes in this way

1,1	1,2	1,3	1,4	1,5
2,1	2,2	2,3	2,4	
3,1	3,2	3,3	3,4	
4,1				

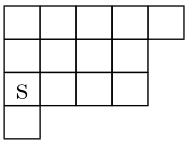
- For a partition λ the arm length is

$$a(s) = \lambda_i - j \tag{B.2}$$

and the leg length is

$$l(s) = \lambda_j^{'} - i \tag{B.3}$$

where λ_i is the number of boxes for a given i and $\lambda_j^{'}$ is the number of boxes in the conjugate partition for a given $j^{'}$.
Basically $a(s)$ counts the number of boxes to the right of s and $l(s)$ the number of boxes below s.
For example, consider the partition (5 4 4 1) [164]



In this case $a(s) = 3$, $l(s) = 1$.

The ring of symmetric function Consider the ring $\mathbb{Z}[x_1, \ldots, x_n]$ of polynomials in n independent variables $x_1, \ldots, x_n$ with rational integer coefficients. The symmetric group S_n acts on this ring by permuting the variables, and a polynomial is symmetric if it is invariant under this action. The symmetric polynomials form a subring $\Lambda_n = \mathbb{Z}[x_1, \ldots, x_n]^{S_n}$. If $f \in \Lambda_n$, we may write

$$f = \sum_{r \geq 0} f^{(r)} \tag{B.4}$$

where $f^{(r)}$ is the homogeneous component of f of degree r. Each $f^{(r)}$ is itself symmetric and so Λ_n is a graded ring. This is important because we could now adjoin another indeterminate x_{n+1} forming $\Lambda_{n+1} = \mathbb{Z}[x_1, \ldots, x_{n+1}]^{S_{n+1}}$. Therefore we have a surjective homomorphism $\Lambda_{n+1} \to \Lambda_n$ defined by setting x_{n+1}=0.

Monomial symmetric function Let λ be a partition. It determines a monomial $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \ldots$ The monomial symmetric function m_λ is the sum of all distinct monomials that can be obtained from x^λ by permutations of the x's. For example, $m_{(43)} = \sum x_i^4 x_j^3$ summed over all pairs such that $i \neq j$. The monomial symmetric functions form a basis of Λ_n.

Another basis for the symmetric functions is formed by the Schur functions. In general if one has two basis u_λ and v_λ of Λ_n it is possible to construct a non singular matrix of rational numbers called the transition matrix and denoted by $M(u, v)$. This matrix correlates the two basis in such a way that

$$u_\lambda = \sum_\mu M_{\lambda\mu} v_\mu \tag{B.5}$$

summed over all $\mu \leq \lambda$. It is possible to show that the transition matrix relating two $\mathbb{Q}$-basis of Λ_n is strictly upper triangular. This formalism could be applied to relate the Schur basis to the monomial one in such a way that

$$s_\lambda = m_\lambda + \sum_{\mu \leq \lambda} K_{\lambda\mu} m_\mu \tag{B.6}$$

where $K_{\lambda\mu}$ are suitable non negative coefficients called Kostka numbers.

Incidentally the relation (B.6) and the orthonormality condition ($\langle s_\lambda s_\mu \rangle = \delta_{\lambda\mu}$ for all partition λ and μ) provide a definition for the Schur functions.

Jack functions Jack symmetric function J_λ^α is a generalization of the Schur function and depends on a real parameter α. One of the many equivalent definitions of this function is the following [164].

Definition 1. The Jack symmetric function J_λ^α are orthogonal with respect to the inner product on power-sum functions namely

$$\langle p_\lambda p_\mu \rangle_\alpha = \delta_{\lambda\mu} \alpha^{l(s)} z_\lambda{}^{2} \tag{B.7}$$

[2] We define $p_\lambda = p_{\lambda_1} p_{\lambda_2} \ldots$ where $p_{\lambda_j} = \sum_i x_i^{\lambda_j}$.

where $z_\lambda = \prod_{i=1}^{l(\lambda)} a_i! i^{a_i}$, a_i being the number of occurrences of i in λ. In addition

$$J_\lambda^\alpha = m_\lambda + \sum_{\mu \leq \lambda} v_{\lambda\mu}^\alpha m_\mu \tag{B.8}$$

This definition is an extension of (B.6) and it is very useful because allows to write a recursion relation in order to deduce the explicit form (at least in principle) of Jack functions.
For a partition λ we define the upper hook length

$$h_\lambda^*(s) = l(s) + \alpha(1 + a(s)) \tag{B.9}$$

and the lower hook length

$$h_*^\lambda(s) = l(s) + 1 + \alpha\, a(s). \tag{B.10}$$

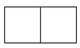
Then Jack polynomials could be written as [165]

$$J_\lambda^\alpha(x_1, x_2 \dots, x_n) = \sum_{\mu \leq \lambda} J_\mu^\alpha(x_1, x_2, \dots, x_{n-1}) x_n^{\lambda/\mu} \beta_{\lambda\mu} \tag{B.11}$$

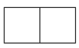
where the sum is over all $\mu \leq \lambda$ such that λ/μ is an horizontal strip and

$$\beta_{\lambda\mu} = \frac{\prod_{s\in\lambda} B_{\lambda\mu}^\lambda(s)}{\prod_{s\in\mu} B_{\lambda\mu}^\mu(s)} \quad \textit{where} \quad B_{\lambda\mu}^\nu(s) = \begin{cases} h_\nu^*(s) & \text{if } \lambda_j' = \mu_j' \\ h_*^\nu(s) & \text{otherwise.} \end{cases} \tag{B.12}$$

2 boxes symmetric It is possible to find the explicit expression of some simple Jack polynomials using eq. (B.11).
The simplest case is the partition λ whose diagram is



The only possibilities for the partition μ are

 and 

We are interested in $J_2^\alpha(x_1, x_2)$ and this is given by

$$J_\lambda^\alpha(x_1, x_2) = \sum_{\mu \leq \lambda} J_\mu^\alpha(x_1) x_2^{\lambda/\mu} \beta_{\lambda\mu} = J_{(1)}^\alpha(x_1) x_2 \beta_{21} + J_{(2)}^\alpha(x_1) \beta_{22}. \tag{B.13}$$

Using the relation (B.11), for a partition $\lambda = (\kappa)$ it is possible to see that $J^{\alpha}_{(\kappa)}(x_1) = x_1^{\kappa}(1+\alpha)(1+2\alpha)\ldots(1+(\kappa-1))\alpha)$. The task now is to compute the $\beta_{\lambda\mu}$:

$$\beta_{21} \quad = \quad h^{\lambda}_{*}(1,2) = 1 \tag{B.14}$$

while β_{22} is trivially 1 because λ=β and so we have the same factors on numerator and denominator. In conclusion we get

$$J^{\alpha}_{\lambda}(x_1, x_2) = x_1x_2 + (1+\alpha)x_1^2. \tag{B.15}$$

It is not so clear how to get the complete Jack polynomial but it seems reasonable to symmetrize this expression obtaining

$$J^{\alpha}_{\lambda}(x_1, x_2) = (1+\alpha)x_2^2 + 2x_1x_2 + (1+\alpha)x_1^2. \tag{B.16}$$

Following a more common and useful notation we could write Jack polynomials in terms of monomial function, namely

$$J^{\alpha}_{\lambda}(x_1, x_2) = (1+\alpha)m_{[2]} + 2m_{[11]}. \tag{B.17}$$

The Jack function reduces to the Schur polynomial for $\alpha = 1$.

3 boxes symmetric We compute now $J^{3}_{\lambda}(x_1, x_2, x_3)$ and using (B.11) it could be written as

$$J^{3}_{\lambda}(x_1, x_2, x_3) = \sum_{\mu \leq \lambda} J^{\alpha}_{\mu}(x_1x_2)x_3^{\lambda/\mu}\beta_{\lambda\mu} = J^{\alpha}_{(1)}(x_1x_2)x_3^2\beta_{31} + J^{\alpha}_{(2)}(x_1x_2)x_3\beta_{32} + J^{\alpha}_{(3)}(x_1x_2)\beta_{33}. \tag{B.18}$$

Using the usual recursion relation we are able to determine all the Jack polynomials in this expression

$$J^{\alpha}_{(1)}(x_1x_2) \quad = \quad x_1 \tag{B.19}$$

$$J^{\alpha}_{(3)}(x_1x_2) \quad = \quad x_1^3(1+\alpha)(1+2\alpha) + x_1^2(1+\alpha)x_2\beta_{32} + x_2^2(1+\alpha)x_1\beta_{31} \tag{B.20}$$

It is not so difficult to obtain the β's from the definition and we get $\beta_{32} = 1$ and $\beta_{31} = (1+\alpha)$. Putting all the information we have into (B.18) we finally get

$$\begin{aligned} J^{\alpha}_{3}(x_1, x_2, x_3) = {} & x_1\, x_3^2(1+\alpha) + x_2\, x_1^2(1+\alpha) + x_1\, x_2\, x_3 \\ & + x_1^3(1+\alpha)(1+2\alpha) + x_2\, x_1^2(1+\alpha) + x_1\, x_2^2(1+\alpha) \end{aligned} \tag{B.21}$$

After symmetrizing we write $J^\alpha_\lambda(x_1, x_2, x_3)$ in terms of monomial functions as

$$J^\alpha_3(x_1, x_2, x_3) = (1 + \alpha)(1 + 2\alpha)m_{[3]} + 3(1 + \alpha)m_{[21]} + 6m_{[111]} \tag{B.22}$$

4 boxes symmetric It is not so difficult to compute also $J^\alpha_\lambda(x_1, x_2, x_3, x_4)$. Using again (B.11) we could write

$$\begin{aligned} J^\alpha_4(x_1, x_2, x_3, x_4) &= \sum_{\mu\leq\lambda} J^\alpha_\mu(x_1x_2x_3)x_4^{\lambda/\mu}\beta_{\lambda\mu} \\ &= J^\alpha_{(1)}(x_1x_2x_3)x_4^3\beta_{41} + J^\alpha_{(2)}(x_1x_2x_3)x_4^2\beta_{42} \\ &\quad + J^\alpha_{(3)}(x_1x_2x_3)x_4\beta_{43} + J^\alpha_{(4)}(x_1x_2x_3)\beta_{44}. \end{aligned} \tag{B.23}$$

Let's list all the objects we need

$$\begin{aligned} J_1(x_1, x_2, x_3) &= 1 \\ J_2(x_1, x_2, x_3) &= (1 + \alpha)x_1^2 + x_1x_2 + x_1x_3 \\ J_4(x_1, x_2, x_3) &= (1 + \alpha)(1 + 2\alpha)(1 + 3\alpha)x_1^4 + x_1^3x_2(1 + \alpha)(1 + 2\alpha)\beta_{43} \\ &\quad + x_1^3x_3(1 + \alpha)(1 + 2\alpha)\beta_{43} + (1 + \alpha)x_1^2x_3^2\beta_{42} + x_1x_2x_3^2\beta_{41} \\ \beta_{41} &= 1 + 2\alpha \\ \beta_{42} &= 1 + \alpha \\ \beta_{43} &= 1 \end{aligned} \tag{B.24}$$

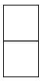
Putting everything together and symmetrizing (the coefficients are taken from MOPS [166], the Maple package) we have that

$$\begin{aligned} J^\alpha_4(x_1, x_2, x_3, x_4) = (1 + \alpha)(1 + 2\alpha)(1 + 3\alpha)m_{[4]} + 4(1 + \alpha)(1 + 2\alpha)m_{[31]} + \\ + 6(1 + \alpha)^2m_{[22]} + 12(1 + \alpha)m_{[211]} + 24m_{[1111]} \end{aligned} \tag{B.25}$$

From now on we change slightly the normalization to be consistent with the outcomes of MOPS, namely we divide for $(1 + \alpha)(1 + 2\alpha) \dots (1 + (n - 1)\alpha)$ which allows us to have 1 as a coefficient of $m_{[n]}$ instead of having it as the coefficient of $m_{[11\dots1]}$.

This makes also more clear that there is a chance to reduce Jack functions to the form of chiral primaries by setting α to be infinite. This fact holds for sure up to 12 boxes.

Antisymmetric case The simplest antisymmetric case is the partition λ whose diagram is



Eq. (B.11) allows us to have a partition μ composed only by

because it is the only part such that λ/μ is an horizontal strip. So $J_2^\alpha(x_1, x_2) = J_{(1)}^\alpha(x_1)x_2\beta_{21} = 2\, x_1\, x_2$. Note that the partition μ is this only one allowed regardless of the number of boxes present in the partition λ (in the complete antisymmetric representation). This means that we could guess the structure of $J_n^\alpha(x_1, x_2, \ldots, x_n) = n!\, x_1\, x_2 \ldots x_n$, this is reasonable because $\beta_{\lambda\mu}^{(1)} = \frac{h_*^\lambda(1,1)\ldots h_*^\lambda(n,1)}{1} = n!$.

Appendix C

Type IIA string theory on $AdS_4 \times \mathbb{C}P^3$ and its $SU(2) \times SU(2)$ sigma model limit

In this appendix it is described how to derive the sigma model lagrangian for the $SU(2)\times SU(2)$ sector of ABJM. This appendix is taken from [39]. The holographic dual of ABJM theory is given by type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{C}P^3$ [4] with metric

$$ds^2 = \frac{R^2}{4}\Big(- \cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\hat{\Omega}_2^2 \Big) + R^2 ds^2_{\mathbb{C}P^3}, \tag{C.1}$$

where for the moment we leave the $\mathbb{C}P^3$ part of the metric unspecified and where

$$\frac{R^2}{l_s^2} = \sqrt{2^5 \pi^2 \lambda}, \tag{C.2}$$

with $\lambda = N/k$ and with string coupling constant and Ramond-Ramond four-form field strength given by

$$g_s = \Big(\frac{2^5\pi^2 N}{k^5}\Big)^{\frac{1}{4}} \quad , \qquad F_{(4)} = \frac{3R^3}{8}\epsilon_{\mathrm{AdS}_4}. \tag{C.3}$$

In the regime $\lambda \gg 1$ and $N \ll k^5$, this is a valid background for type IIA string theory [4].

We are interested in zooming in to the $SU(2) \times SU(2)$ sector of type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{C}P^3$. This can be achieved by taking a limit of small momenta which was first found in [38] (see also [40, 95, 97, 151]). How to do this for type

IIA string theory on $\mathrm{AdS}_4 \times \mathbb{C}P^3$ is explained in detail in [151] and the relevant part of the metric becomes

$$ds^2 = -\frac{R^2}{4}dt^2 + R^2\Big[\frac{1}{8}d\Omega_2^2 + \frac{1}{8}d\Omega_2'^2 + (d\delta+\omega)^2\Big], \tag{C.4}$$

with R given in (C.2) and where

$$\begin{gathered} d\Omega_2^2 = d\theta_1^2 + \cos^2\theta_1 d\varphi_1^2 \ , \quad d\Omega_2'^2 = d\theta_2^2 + \cos^2\theta_2 d\varphi_2^2 \\ \omega = \tfrac{1}{4}(\sin\theta_1 d\varphi_1 + \sin\theta_2 d\varphi_2) \ , \quad \delta = \tfrac{1}{4}(\phi_1+\phi_2-\phi_3-\phi_4) \\ \varphi_1 = \phi_1 - \phi_2 \ , \quad \varphi_2 = \phi_4 - \phi_3 \end{gathered} \tag{C.5}$$

We see that the coordinates (θ_i, φ_i), $i = 1, 2$, parametrize two two-spheres corresponding to the two $SU(2)$ sectors. For later convenience, the two two-spheres can also be written in terms of two unit vectors fields $\vec{n}_{1,2}$ given by

$$\vec{n}_i = (\cos\theta_i\cos\varphi_i, \cos\theta_i\sin\varphi_i, \sin\theta_i)\,. \tag{C.6}$$

We now introduce the angular momenta L_1 and L_2 in one $SU(2)$ and L_3 and L_4 in the other $SU(2)$ with the condition $L_1 + L_2 + L_3 + L_4 = 0$. As explained in [151] the $SU(2) \times SU(2)$ sector is obtained by considering states for which $\Delta - L_1 - L_2$ is small, where Δ is the energy. This can be implemented as a sigma-model limit with the following coordinate transformation

$$\tilde{t} = \lambda' t \ , \quad \chi = \delta - \frac{1}{2}t, \tag{C.7}$$

where $\lambda' = \lambda/J^2$, $J \equiv L_1 + L_2$ and so that

$$\tilde{H} \equiv i\partial_{\tilde{t}} = \frac{(\Delta - J)}{\lambda'} \ , \quad 2J = -i\partial_\chi, \tag{C.8}$$

We see that sending $\lambda' \to 0$, one has that $\Delta - J \to 0$ which means that we keep the modes of the $SU(2) \times SU(2)$ sector dynamical, while the other modes become non-dynamical and decouple in this limit.

Using (C.7), the type IIA metric becomes

$$ds^2 = R^2\left[(\frac{1}{\lambda'}d\tilde{t} + d\chi + \omega)(d\chi+\omega) + \frac{1}{8}d\Omega_2^2 + \frac{1}{8}d\Omega_2'^2\right]. \tag{C.9}$$

The bosonic sigma-model Lagrangian and Virasoro constraints are

$$\mathcal{L} = -\frac{1}{2} G_{\mu\nu} h^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu, \tag{C.10}$$

$$G_{\mu\nu}(\partial_\alpha x^\mu \partial_\beta x^\nu - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma x^\mu \partial_\delta x^\nu) = 0, \tag{C.11}$$

with $G_{\mu\nu}$ being the metric (C.9). $h^{\alpha\beta} = \sqrt{-\det\gamma}\gamma^{\alpha\beta}$ with $\gamma_{\alpha\beta}$ being the world-sheet metric.

Our gauge choice is

$$\tilde{t} = \kappa\tau, \tag{C.12}$$

$$2\pi p_- = \frac{\partial\mathcal{L}}{\partial\partial_\tau x^-} = \text{const.}\ , \quad \frac{\partial\mathcal{L}}{\partial\partial_\sigma x^-} = 0. \tag{C.13}$$

Moreover, the constant κ can also be determined from

$$2J = P_\chi = \int_0^{2\pi} d\sigma p_\chi = \frac{R^2\kappa}{2\lambda'} = \frac{2\pi\sqrt{2\lambda}\kappa}{\lambda'}. \tag{C.14}$$

We see that $\kappa = \frac{\sqrt{\lambda'}}{\pi\sqrt{2}}$. Thus $\kappa \to 0$ for $\lambda' \to 0$. Moreover, from (C.8) we have that the right energy scale is given by $\tilde{\tau} = \kappa\tau$. This means that the quantity that we keep fixed in the limit $\kappa \to 0$ is $\dot{x}^\mu = \partial_{\tilde{\tau}} x^\mu$.

Proceeding as in [151], we can then solve the Virasoro constraints and the gauge conditions order by order in κ. This actually corresponds, on the gauge theory side, to an expansion in powers of λ'. Here we skip the various steps and report the final result for the action to leading order

$$I = \frac{J}{4\pi} \sum_{i=1}^{2} \int d\tilde{t} \int_0^{2\pi} d\sigma \Big[\sin\theta_i \dot{\varphi}_i - \pi^2 (\vec{n}_i)^2 \Big], \tag{C.15}$$

$$\sum_{i=1}^{2} \int_0^{2\pi} d\sigma \sin\theta_i \varphi_i' = 0, \tag{C.16}$$

where the last expression gives the momentum constraint.

We see that, up to the perturbative order we are interested in, by taking the $SU(2) \times SU(2)$ sigma-model limit we obtain two Landau-Lifshitz models added together (C.15), one for each $SU(2)$, which are related only through the momentum constraint (C.16) [151]. This is moreover consistent with results on the gauge theory side.

Bibliography

[1] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Adv.Theor.Math.Phys.* **2** (1998) 231–252, arXiv:hep-th/9711200 [hep-th].

[2] S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," *Phys.Lett.* **B428** (1998) 105–114, arXiv:hep-th/9802109 [hep-th].

[3] E. Witten, "Anti-de Sitter space and holography," *Adv.Theor.Math.Phys.* **2** (1998) 253–291, arXiv:hep-th/9802150 [hep-th].

[4] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, "$\mathcal{N} = 6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals," arXiv:0806.1218 [hep-th].

[5] J. A. Minahan and K. Zarembo, "The Bethe-ansatz for $\mathcal{N} = 4$ super Yang-Mills," *JHEP* **03** (2003) 013, hep-th/0212208.

[6] N. Beisert, C. Ahn, L. F. Alday, Z. Bajnok, J. M. Drummond, *et al.*, "Review of AdS/CFT Integrability: An Overview," *Lett.Math.Phys.* **99** (2012) 3–32, arXiv:1012.3982 [hep-th].

[7] R. A. Janik, P. Surowka, and A. Wereszczynski, "On correlation functions of operators dual to classical spinning string states," *JHEP* **1005** (2010) 030, arXiv:1002.4613 [hep-th].

[8] E. I. Buchbinder, "Energy-Spin Trajectories in $AdS_5 \times S^5$ from Semiclassical Vertex Operators," *JHEP* **1004** (2010) 107, arXiv:1002.1716 [hep-th].

[9] E. Buchbinder and A. Tseytlin, "On semiclassical approximation for correlators of closed string vertex operators in AdS/CFT," *JHEP* **1008** (2010) 057, arXiv:1005.4516 [hep-th].

[10] K. Zarembo, "Holographic three-point functions of semiclassical states," *JHEP* **1009** (2010) 030, arXiv:1008.1059 [hep-th].

[11] M. S. Costa, R. Monteiro, J. E. Santos, and D. Zoakos, "On three-point correlation functions in the gauge/gravity duality," *JHEP* **1011** (2010) 141, arXiv:1008.1070 [hep-th].

[12] R. Hernandez, "Three-point correlation functions from semiclassical circular strings," *J.Phys.* **A44** (2011) 085403, arXiv:1011.0408 [hep-th].

[13] S. Ryang, "Correlators of Vertex Operators for Circular Strings with Winding Numbers in
$AdS_5 \times S^5$," $JHEP$**1101**(2011)092, arXiv : 1011.3573[hep − th].

[14] G. Georgiou, "Two and three-point correlators of operators dual to folded string solutions at strong coupling," *JHEP* **1102** (2011) 046, arXiv:1011.5181 [hep-th].

[15] D. Arnaudov and R. Rashkov, "On semiclassical calculation of three-point functions in $AdS_4 \times CP^3$," *Phys.Rev.* **D83** (2011) 066011, arXiv:1011.4669 [hep-th].

[16] C. Park and B.-H. Lee, "Correlation functions of magnon and spike," *Phys.Rev.* **D83** (2011) 126004, arXiv:1012.3293 [hep-th].

[17] D. Bak, B. Chen, and J.-B. Wu, "Holographic Correlation Functions for Open Strings and Branes," *JHEP* **1106** (2011) 014, arXiv:1103.2024 [hep-th].

[18] D. Arnaudov, R. Rashkov, and T. Vetsov, "Three and four-point correlators of operators dual to folded string solutions in $AdS_5 \times S^5$," *Int.J.Mod.Phys.* **A26** (2011) 3403–3420, arXiv:1103.6145 [hep-th].

[19] R. Hernandez, "Three-point correlators for giant magnons," *JHEP* **1105** (2011) 123, arXiv:1104.1160 [hep-th].

[20] X. Bai, B.-H. Lee, and C. Park, "Correlation function of dyonic strings," *Phys.Rev.* **D84** (2011) 026009, arXiv:1104.1896 [hep-th].

[21] J. Escobedo, N. Gromov, A. Sever, and P. Vieira, "Tailoring Three-Point Functions and Integrability II. Weak/strong coupling match," *JHEP* **1109** (2011) 029, arXiv:1104.5501 [hep-th].

[22] C. Ahn and P. Bozhilov, "Three-point Correlation functions of Giant magnons with finite size," *Phys.Lett.* **B702** (2011) 286–290, arXiv:1105.3084 [hep-th].

[23] B.-H. Lee and C. Park, "Finite size effect on the magnon's correlation functions," *Phys.Rev.* **D84** (2011) 086005, arXiv:1105.3279 [hep-th].

[24] D. Arnaudov and R. Rashkov, "Quadratic corrections to three-point functions," *Fortsch.Phys.* **60** (2012) 217–223, arXiv:1106.0859 [hep-th].

[25] E. Alizadeh and C. M. Hirata, "Three-point correlators: examples from Lunin-Maldacena background," *Phys.Rev.* **D84** (2011) 086009, arXiv:1106.4298 [hep-th].

[26] C. Ahn and P. Bozhilov, "Three-point Correlation Function of Giant Magnons in the Lunin-Maldacena background," *Phys.Rev.* **D84** (2011) 126011, arXiv:1106.5656 [hep-th].

[27] G. Georgiou, "$SL(2)$ sector: weak/strong coupling agreement of three-point correlators," *JHEP* **1109** (2011) 132, arXiv:1107.1850 [hep-th].

[28] M. Michalcik, R. C. Rashkov, and M. Schimpf, "On semiclassical calculation of three-point functions in $AdS_5 \times T^(1,1)$," *Mod.Phys.Lett.* **A27** (2012) 1250091, arXiv:1107.5795 [hep-th].

[29] P. Bozhilov, "Leading finite-size effects on some three-point correlators in $AdS_5 \times S^5$," arXiv:1212.3485 [hep-th].

[30] R. Roiban and A. Tseytlin, "On semiclassical computation of 3-point functions of closed string vertex operators in $AdS_5 x S^5$," *Phys.Rev.* **D82** (2010) 106011, arXiv:1008.4921 [hep-th].

[31] J. Russo and A. Tseytlin, "Large spin expansion of semiclassical 3-point correlators in $AdS_5 x S^5$," *JHEP* **1102** (2011) 029, arXiv:1012.2760 [hep-th].

[32] L. F. Alday and A. A. Tseytlin, "On strong-coupling correlation functions of circular Wilson loops and local operators," *J.Phys.* **A44** (2011) 395401, arXiv:1105.1537 [hep-th].

[33] S. Ryang, "Extremal Correlator of Three Vertex Operators for Circular Winding Strings in AdS5xS5," *JHEP* **1111** (2011) 026, arXiv:1109.3242 [hep-th].

[34] A. Bissi, C. Kristjansen, D. Young, and K. Zoubos, "Holographic three-point functions of giant gravitons," *JHEP* **1106** (2011) 085, arXiv:1103.4079 [hep-th].

[35] P. Caputa, R. d. M. Koch, and K. Zoubos, "Extremal versus Non-Extremal Correlators with Giant Gravitons," *JHEP* **1208** (2012) 143, arXiv:1204.4172 [hep-th].

[36] H. Lin, "Giant gravitons and correlators," *JHEP* **1212** (2012) 011, arXiv:1209.6624 [hep-th].

[37] S. Frolov and A. A. Tseytlin, "Rotating string solutions: AdS/CFT duality in non- supersymmetric sectors," *Phys. Lett.* **B570** (2003) 96–104, hep-th/0306143.

[38] M. Kruczenski, "Spin chains and string theory," *Phys. Rev. Lett.* **93** (2004) 161602, hep-th/0311203.

[39] A. Bissi, C. Kristjansen, A. Martirosyan, and M. Orselli, "On Three-point Functions in the $AdS_4/CFT_3 Correspondence,'' JHEP \mathbf{1301} (2013) 137$, arXiv : 1211.1359[hep − th].

[40] A. Bissi, T. Harmark, and M. Orselli, "Holographic 3-Point Function at One Loop," *JHEP* **1202** (2012) 133, arXiv:1112.5075 [hep-th].

[41] G. Grignani and A. Zayakin, "Matching Three-point Functions of BMN Operators at Weak and Strong coupling," *JHEP* **1206** (2012) 142, arXiv:1204.3096 [hep-th].

[42] G. Grignani and A. Zayakin, "Three-point functions of BMN operators at weak and strong coupling II. One loop matching," *JHEP* **1209** (2012) 087, arXiv:1205.5279 [hep-th].

[43] T. Klose and T. McLoughlin, "A light-cone approach to three-point functions in $AdS_5 \times S^5$," *JHEP* **1204** (2012) 080, arXiv:1106.0495 [hep-th].

[44] J. A. Minahan, "Holographic three-point functions for short operators," *JHEP* **1207** (2012) 187, arXiv:1206.3129 [hep-th].

[45] Y. Kazama and S. Komatsu, "Wave functions and correlation functions for GKP strings from integrability," *JHEP* **1209** (2012) 022, arXiv:1205.6060 [hep-th].

[46] Y. Kazama and S. Komatsu, "On holographic three point functions for GKP strings from integrability," *JHEP* **1201** (2012) 110, arXiv:1110.3949 [hep-th].

[47] R. A. Janik and A. Wereszczynski, "Correlation functions of three heavy operators: The AdS contribution," *JHEP* **1112** (2011) 095, arXiv:1109.6262 [hep-th].

[48] E. Buchbinder and A. Tseytlin, "Semiclassical correlators of three states with large S^5 charges in string theory in $AdS_5 \times S^5$," *Phys.Rev.* **D85** (2012) 026001, arXiv:1110.5621 [hep-th].

[49] K. Okuyama and L.-S. Tseng, "Three-point functions in N = 4 SYM theory at one-loop," *JHEP* **0408** (2004) 055, arXiv:hep-th/0404190 [hep-th].

[50] R. Roiban and A. Volovich, "Yang-Mills correlation functions from integrable spin chains," *JHEP* **0409** (2004) 032, arXiv:hep-th/0407140 [hep-th].

[51] L. F. Alday, J. R. David, E. Gava, and K. S. Narain, "Towards a string bit formulation of $\mathcal{N} = 4$ super Yang-Mills," *JHEP* **04** (2006) 014, hep-th/0510264.

[52] J. Escobedo, N. Gromov, A. Sever, and P. Vieira, "Tailoring Three-Point Functions and Integrability," *JHEP* **1109** (2011) 028, arXiv:1012.2475 [hep-th].

[53] N. Gromov, A. Sever, and P. Vieira, "Tailoring Three-Point Functions and Integrability III. Classical Tunneling," arXiv:1111.2349 [hep-th].

[54] O. Foda, "N=4 SYM structure constants as determinants," *JHEP* **1203** (2012) 096, arXiv:1111.4663 [math-ph].

[55] C. Ahn, O. Foda, and R. I. Nepomechie, "OPE in planar QCD from integrability," *JHEP* **1206** (2012) 168, arXiv:1202.6553 [hep-th].

[56] I. Kostov, "Three-point function of semiclassical states at weak coupling," arXiv:1205.4412 [hep-th].

[57] N. Gromov and P. Vieira, "Tailoring Three-Point Functions and Integrability IV. Theta-morphism," arXiv:1205.5288 [hep-th].

[58] O. Foda, Y. Jiang, I. Kostov, and D. Serban, "A tree-level 3-point function in the su(3)-sector of planar N=4 SYM," arXiv:1302.3539 [hep-th].

[59] A. Bissi, G. Grignani, and A. Zayakin, "The SO(6) Scalar Product and Three-Point Functions from Integrability," arXiv:1208.0100 [hep-th].

[60] M. Wheeler, "Scalar products in generalized models with SU(3)-symmetry," arXiv:1204.2089 [math-ph].

[61] L. Susskind and E. Witten, "The Holographic bound in anti-de Sitter space," arXiv:hep-th/9805114 [hep-th].

[62] M. Bianchi, D. Z. Freedman, and K. Skenderis, "Holographic renormalization," *Nucl.Phys.* **B631** (2002) 159–194, arXiv:hep-th/0112119 [hep-th].

[63] K. Skenderis, "Lecture notes on holographic renormalization," *Class.Quant.Grav.* **19** (2002) 5849–5876, arXiv:hep-th/0209067 [hep-th].

[64] D. F. Philippe, M. Pierre, and D. S. (Author), "Conformal Field Theory," (1997) .

[65] J. Cardy, "Scaling and Renormalization in Statistical Physics," *Cambridge University Press* (1996) .

[66] K. G. Wilson, "On Products of Operators at Short Distance," *Cornell Report, Ithaca , New York, 1964 (unpublished)* (1964) .

[67] W. Zimmermann, "Local Operator Products and Renormalization in Quantum Field Theory," *Commun. math. Phys. 6, 161 (1967), 10, 325 (1968), Brandeis Lectures 1970, Vol.1* (1968) .

[68] K. G. Wilson and W. Zimmermann, "Operator Product Expansions and Composite Field Operators in the General Framework of Quantum Field Theory," *Commun. math. Phys. 24, 87—106 (1972)* (1971) .

[69] N. Reshetikhin, "Integrable Models of quantum one dimensional magnets with $O(n)$ and $Sp(2k)$ symmetry," *Theoret.Math.Phys.* **63** 555–569.

[70] H. Bethe, "Zur theorie der metalle," *Zeitschrift f́ur Physik* **71** (1931) 205–226. http://dx.doi.org/10.1007/BF01341708.

[71] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, "Method for solving the korteweg-devries equation," *Phys. Rev. Lett.* **19** (Nov, 1967) 1095–1097. http://link.aps.org/doi/10.1103/PhysRevLett.19.1095.

[72] V. E. Zakharov and L. D. Faddeev, " The Kortewegde Vries equation is a fully integrable Hamiltonian system," *Funktsional. Anal. i Prilozhen* **5:4** 18–27.

[73] C. Gardner, "Kortewegde Vries equation and generalizations. IV, The Kortewegde Vries equation as a Hamiltonian system," *J. Math. Phys* **12** 15481551.

[74] E. S. L.D. Faddeev and L. Takhtajan, "The quantum inverse problem," *Teor. Mat. Fiz.* **40** 15481551.

[75] N. Beisert, C. Kristjansen, and M. Staudacher, "The dilatation operator of $\mathcal{N} = 4$ super Yang-Mills theory," *Nucl. Phys.* **B664** (2003) 131–184, hep-th/0303060.

[76] N. Beisert, V. Dippel, and M. Staudacher, "A novel long range spin chain and planar $\mathcal{N} = 4$ super Yang-Mills," *JHEP* **07** (2004) 075, hep-th/0405001.

[77] N. Beisert and M. Staudacher, "Long-range $PSU(2,2|4)$ Bethe ansaetze for gauge theory and strings," *Nucl. Phys.* **B727** (2005) 1–62, hep-th/0504190.

[78] R. A. Janik and T. Lukowski, "Wrapping interactions at strong coupling: The Giant magnon," *Phys.Rev.* **D76** (2007) 126008, arXiv:0708.2208 [hep-th].

[79] Z. Bajnok and R. A. Janik, "Four-loop perturbative Konishi from strings and finite size effects for multiparticle states," *Nucl.Phys.* **B807** (2009) 625–650, arXiv:0807.0399 [hep-th].

[80] Z. Bajnok, A. Hegedus, R. A. Janik, and T. Lukowski, "Five loop Konishi from AdS/CFT," *Nucl.Phys.* **B827** (2010) 426–456, arXiv:0906.4062 [hep-th].

[81] J. Ambjorn, R. A. Janik, and C. Kristjansen, "Wrapping interactions and a new source of corrections to the spin-chain/string duality," *Nucl.Phys.* **B736** (2006) 288–301, arXiv:hep-th/0510171 [hep-th].

[82] N. Gromov, V. Kazakov, A. Kozak, and P. Vieira, "Exact Spectrum of Anomalous Dimensions of Planar N = 4 Supersymmetric Yang-Mills Theory: TBA and excited states," *Lett.Math.Phys.* **91** (2010) 265–287, arXiv:0902.4458 [hep-th].

[83] D. Bombardelli, D. Fioravanti, and R. Tateo, "Thermodynamic Bethe Ansatz for planar AdS/CFT: A Proposal," *J.Phys.* **A42** (2009) 375401, arXiv:0902.3930 [hep-th].

[84] G. Arutyunov and S. Frolov, "Thermodynamic Bethe Ansatz for the AdS(5) x S(5) Mirror Model," *JHEP* **0905** (2009) 068, arXiv:0903.0141 [hep-th].

[85] G. Arutyunov and S. Frolov, "Simplified TBA equations of the $AdS_5 \times S^5$ mirror model," *JHEP* **0911** (2009) 019, arXiv:0907.2647 [hep-th].

[86] N. Gromov, V. Kazakov, and P. Vieira, "Exact Spectrum of Anomalous Dimensions of Planar N=4 Supersymmetric Yang-Mills Theory," *Phys.Rev.Lett.* **103** (2009) 131601, arXiv:0901.3753 [hep-th].

[87] N. Gromov, V. Kazakov, and P. Vieira, "Exact Spectrum of Planar $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory: Konishi Dimension at Any Coupling," *Phys.Rev.Lett.* **104** (2010) 211601, arXiv:0906.4240 [hep-th].

[88] S. Leurent, D. Serban, and D. Volin, "Six-loop Konishi anomalous dimension from the Y-system," *Phys.Rev.Lett.* **109** (2012) 241601, arXiv:1209.0749 [hep-th].

[89] L. Pauling, "The structure and entropy of ice and of other crystals with some randomness of atomic arrangement," *Journal of the American Chemical Society* **57** (1935) no. 12, 2680–2684, http://pubs.acs.org/doi/pdf/10.1021/ja01315a102. http://pubs.acs.org/doi/abs/10.1021/ja01315a102.

[90] V. Korepin, "CALCULATION OF NORMS OF BETHE WAVE FUNCTIONS," *Commun.Math.Phys.* (1982) .

[91] A. G. Izergin, "Partition function of a six-vertex model in a finite volume (Russian)," *Dokl. Akad. Nauk SSSR* **297** (1987) 331333.

[92] M. Wheeler, "An Izergin-Korepin procedure for calculating scalar products in the six-vertex model," *Nuclear Physics B* **852** (Nov., 2011) 468–507, arXiv:1104.2113 [math-ph].

[93] J. Callan, Curtis G. *et al.*, "Quantizing string theory in $AdS_5 \times S^5$: Beyond the pp- wave," *Nucl. Phys.* **B673** (2003) 3–40, arXiv:hep-th/0307032.

[94] J. Callan, Curtis G., T. McLoughlin, and I. Swanson, "Holography beyond the Penrose limit," *Nucl. Phys.* **B694** (2004) 115–169, arXiv:hep-th/0404007.

[95] T. Harmark, K. R. Kristjansson, and M. Orselli, "Matching gauge theory and string theory in a decoupling limit of AdS/CFT," arXiv:0806.3370 [hep-th].

[96] A. M. Perelomov, "Generalized Coherent States and Their Applications," *Springer* (1986) 332.

[97] M. Kruczenski, A. V. Ryzhov, and A. A. Tseytlin, "Large spin limit of $AdS_5 \times S^5$ string theory and low energy expansion of ferromagnetic spin chains," *Nucl. Phys.* **B692** (2004) 3–49, hep-th/0403120.

[98] J. A. Minahan, A. Tirziu, and A. A. Tseytlin, "$1/J^2$ corrections to BMN energies from the quantum long range Landau-Lifshitz model," *JHEP* **11** (2005) 031, hep-th/0510080.

[99] J. A. Minahan and K. Zarembo, "The Bethe ansatz for superconformal Chern-Simons," *JHEP* **09** (2008) 040, arXiv:0806.3951 [hep-th].

[100] D. Bombardelli, D. Fioravanti, and R. Tateo, "TBA and Y-system for planar AdS(4)/CFT(3)," *Nucl.Phys.* **B834** (2010) 543–561, arXiv:0912.4715 [hep-th].

[101] N. Gromov and F. Levkovich-Maslyuk, "Y-system, TBA and Quasi-Classical strings in AdS(4) x CP3," *JHEP* **1006** (2010) 088, arXiv:0912.4911 [hep-th].

[102] W. Mueck and K. Viswanathan, "Conformal field theory correlators from classical scalar field theory on AdS(d+1)," *Phys.Rev.* **D58** (1998) 041901, arXiv:hep-th/9804035 [hep-th].

[103] D. Z. Freedman, S. D. Mathur, A. Matusis, and L. Rastelli, "Correlation functions in the CFT(d) / AdS(d+1) correspondence," *Nucl.Phys.* **B546** (1999) 96–118, arXiv:hep-th/9804058 [hep-th].

[104] H. Liu and A. A. Tseytlin, "D = 4 superYang-Mills, D = 5 gauged supergravity, and D = 4 conformal supergravity," *Nucl.Phys.* **B533** (1998) 88–108, arXiv:hep-th/9804083 [hep-th].

[105] M. Henningson and K. Sfetsos, "Spinors and the AdS / CFT correspondence," *Phys.Lett.* **B431** (1998) 63–68, arXiv:hep-th/9803251 [hep-th].

[106] S. Lee, S. Minwalla, M. Rangamani, and N. Seiberg, "Three point functions of chiral operators in D = 4, N=4 SYM at large N," *Adv.Theor.Math.Phys.* **2** (1998) 697–718, arXiv:hep-th/9806074 [hep-th].

[107] K. A. Intriligator, "Bonus symmetries of N=4 superYang-Mills correlation functions via AdS duality," *Nucl.Phys.* **B551** (1999) 575–600, arXiv:hep-th/9811047 [hep-th].

[108] K. A. Intriligator and W. Skiba, "Bonus symmetry and the operator product expansion of N=4 SuperYang-Mills," *Nucl.Phys.* **B559** (1999) 165–183, arXiv:hep-th/9905020 [hep-th].

[109] B. Eden, P. S. Howe, and P. C. West, "Nilpotent invariants in N=4 SYM," *Phys.Lett.* **B463** (1999) 19–26, arXiv:hep-th/9905085 [hep-th].

[110] P. S. Howe, C. Schubert, E. Sokatchev, and P. C. West, "Explicit construction of nilpotent covariants in N=4 SYM," *Nucl.Phys.* **B571** (2000) 71–90, arXiv:hep-th/9910011 [hep-th].

[111] A. Petkou and K. Skenderis, "A Nonrenormalization theorem for conformal anomalies," *Nucl.Phys.* **B561** (1999) 100–116, arXiv:hep-th/9906030 [hep-th].

[112] P. Heslop and P. S. Howe, "OPEs and three-point correlators of protected operators in N=4 SYM," *Nucl.Phys.* **B626** (2002) 265–286, arXiv:hep-th/0107212 [hep-th].

[113] C. Kristjansen, J. Plefka, G. W. Semenoff, and M. Staudacher, "A new double-scaling limit of N = 4 super Yang-Mills theory and PP-wave strings," *Nucl. Phys.* **B643** (2002) 3–30, arXiv:hep-th/0205033.

[114] D. Berenstein, J. M. Maldacena, and H. Nastase, "Strings in flat space and pp waves from $\mathcal{N} = 4$ super Yang Mills," *JHEP* **04** (2002) 013, hep-th/0202021.

[115] N. R. Constable *et al.*, "PP-wave string interactions from perturbative Yang-Mills theory," *JHEP* **07** (2002) 017, arXiv:hep-th/0205089.

[116] L. F. Alday, J. R. David, E. Gava, and K. Narain, "Structure constants of planar N = 4 Yang Mills at one loop," *JHEP* **0509** (2005) 070, arXiv:hep-th/0502186 [hep-th].

[117] A. A. Tseytlin, "On semiclassical approximation and spinning string vertex operators in $AdS(5) \times S**5$," *Nucl.Phys.* **B664** (2003) 247–275, arXiv:hep-th/0304139 [hep-th].

[118] D. E. Berenstein, R. Corrado, W. Fischler, and J. M. Maldacena, "The Operator product expansion for Wilson loops and surfaces in the large N limit," *Phys.Rev.* **D59** (1999) 105023, arXiv:hep-th/9809188 [hep-th].

[119] E. Buchbinder and A. Tseytlin, "Semiclassical four-point functions in $AdS_5 \times S^5$," *JHEP* **1102** (2011) 072, arXiv:1012.3740 [hep-th].

[120] J. McGreevy, L. Susskind, and N. Toumbas, "Invasion of the giant gravitons from anti-de Sitter space," *JHEP* **06** (2000) 008, hep-th/0003075.

[121] R. Leigh, "Dirac-Born-Infeld Action from Dirichlet Sigma Model," *Mod.Phys.Lett.* **A4** (1989) 2767.

[122] M. T. Grisaru, R. C. Myers, and O. Tafjord, "SUSY and goliath," *JHEP* **0008** (2000) 040, arXiv:hep-th/0008015 [hep-th].

[123] J. M. Maldacena and A. Strominger, "AdS(3) black holes and a stringy exclusion principle," *JHEP* **9812** (1998) 005, arXiv:hep-th/9804085 [hep-th].

[124] P.-M. Ho, S. Ramgoolam, and R. Tatar, "Quantum space-times and finite N effects in 4-D superYang-Mills theories," *Nucl.Phys.* **B573** (2000) 364–376, arXiv:hep-th/9907145 [hep-th].

[125] A. Jevicki and S. Ramgoolam, "Noncommutative gravity from the AdS / CFT correspondence," *JHEP* **9904** (1999) 032, arXiv:hep-th/9902059 [hep-th].

[126] A. Hashimoto, S. Hirano, and N. Itzhaki, "Large branes in AdS and their field theory dual," *JHEP* **08** (2000) 051, arXiv:hep-th/0008016.

[127] V. Balasubramanian, M. Berkooz, A. Naqvi, and M. J. Strassler, "Giant gravitons in conformal field theory," *JHEP* **0204** (2002) 034, arXiv:hep-th/0107119 [hep-th].

[128] S. Corley, A. Jevicki, and S. Ramgoolam, "Exact correlators of giant gravitons from dual N=4 SYM theory," *Adv.Theor.Math.Phys.* **5** (2002) 809–839, arXiv:hep-th/0111222 [hep-th].

[129] V. Balasubramanian, M.-x. Huang, T. S. Levi, and A. Naqvi, "Open strings from N=4 superYang-Mills," *JHEP* **0208** (2002) 037, arXiv:hep-th/0204196 [hep-th].

[130] O. Aharony, Y. E. Antebi, M. Berkooz, and R. Fishman, "'Holey sheets': Pfaffians and subdeterminants as D-brane operators in large N gauge theories," *JHEP* **0212** (2002) 069, arXiv:hep-th/0211152 [hep-th].

[131] D. Berenstein, "Shape and holography: Studies of dual operators to giant gravitons," *Nucl.Phys.* **B675** (2003) 179–204, arXiv:hep-th/0306090 [hep-th].

[132] V. Balasubramanian, D. Berenstein, B. Feng, and M.-x. Huang, "D-branes in Yang-Mills theory and emergent gauge symmetry," *JHEP* **0503** (2005) 006, arXiv:hep-th/0411205 [hep-th].

[133] R. de Mello Koch, J. Smolic, and M. Smolic, "Giant Gravitons - with Strings Attached (I)," *JHEP* **0706** (2007) 074, arXiv:hep-th/0701066 [hep-th].

[134] R. de Mello Koch, J. Smolic, and M. Smolic, "Giant Gravitons - with Strings Attached (II)," *JHEP* **0709** (2007) 049, arXiv:hep-th/0701067 [hep-th].

[135] S. Corley and S. Ramgoolam, "Finite factorization equations and sum rules for BPS correlators in N=4 SYM theory," *Nucl.Phys.* **B641** (2002) 131–187, arXiv:hep-th/0205221 [hep-th].

[136] K. Okuyama and G. W. Semenoff, "Wilson loops in N=4 SYM and fermion droplets," *JHEP* **0606** (2006) 057, arXiv:hep-th/0604209 [hep-th].

[137] H. J. Kim, L. J. Romans, and P. van Nieuwenhuizen, "Mass spectrum of chiral ten-dimensional n=2 supergravity on s^5," *Phys. Rev. D* **32** (Jul, 1985) 389–399.

[138] G. W. Semenoff and D. Young, "Wavy Wilson line and AdS / CFT," *Int.J.Mod.Phys.* **A20** (2005) 2833–2846, arXiv:hep-th/0405288 [hep-th].

[139] S. Giombi, R. Ricci, and D. Trancanelli, "Operator product expansion of higher rank Wilson loops from D-branes and matrix models," *JHEP* **0610** (2006) 045, arXiv:hep-th/0608077 [hep-th].

[140] N. Beisert, C. Kristjansen, J. Plefka, G. W. Semenoff, and M. Staudacher, "Bmn correlators and operator mixing in $\mathcal{N} = 4$ super yang-mills theory," *Nucl. Phys.* **B650** (2003) 125–161, hep-th/0208178.

[141] C. Kristjansen, "Review of AdS/CFT Integrability, Chapter IV.1: Aspects of Non-Planarity," arXiv:1012.3997 [hep-th].

[142] A. Grossardt and J. Plefka, "One-Loop Spectroscopy of Scalar Three-Point Functions in planar N=4 super Yang-Mills Theory," arXiv:1007.2356 [hep-th].

[143] N. Gromov and P. Vieira, "Quantum Integrability for Three-Point Functions," arXiv:1202.4103 [hep-th].

[144] F. Bastianelli and R. Zucchini, "Three point functions of chiral primary operators in d = 3, N=8 and d = 6, N=(2,0) SCFT at large N," *Phys.Lett.* **B467** (1999) 61–66, arXiv:hep-th/9907047 [hep-th].

[145] S. Hirano, C. Kristjansen, and D. Young, "Giant Gravitons on AdS_4xCP^3 and their Holographic Three-point Functions," arXiv:1205.1959 [hep-th].

[146] D. Arnaudov, "Three-point functions of semiclassical string states and conserved currents in $AdS_4 \times CP^3$," arXiv:1302.2119 [hep-th].

[147] T. K. Dey, "Exact Large R-charge Correlators in ABJM Theory," *JHEP* **1108** (2011) 066, arXiv:1105.0218 [hep-th].

[148] M. S. Bianchi, M. Leoni, A. Mauri, S. Penati, C. Ratti, *et al.*, "From Correlators to Wilson Loops in Chern-Simons Matter Theories," *JHEP* **1106** (2011) 118, arXiv:1103.3675 [hep-th].

[149] L. F. Alday, B. Eden, G. P. Korchemsky, J. Maldacena, and E. Sokatchev, "From correlation functions to Wilson loops," *JHEP* **1109** (2011) 123, arXiv:1007.3243 [hep-th].

[150] O. T. Engelund and R. Roiban, "On correlation functions of Wilson loops, local and non-local operators," *JHEP* **1205** (2012) 158, arXiv:1110.0758 [hep-th].

[151] G. Grignani, T. Harmark, and M. Orselli, "The SU(2) x SU(2) sector in the string dual of N=6 superconformal Chern-Simons theory," *Nucl. Phys.* **B810** (2009) 115–134, arXiv:0806.4959 [hep-th].

[152] D. Astolfi, V. G. M. Puletti, G. Grignani, T. Harmark, and M. Orselli, "Finite-size corrections in the SU(2) $\times$ SU(2) sector of type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{C}P^3$," *Nucl. Phys.* **B810** (2009) 150–173, arXiv:0807.1527 [hep-th].

[153] F. Bastianelli and R. Zucchini, "Bosonic quadratic actions for 11-D supergravity on AdS(7/4) x S(4/7)," *Class.Quant.Grav.* **16** (1999) 3673–3684, arXiv:hep-th/9903161 [hep-th].

[154] D. Serban, "A note on the eigenvectors of long-range spin chains and their scalar products," arXiv:1203.5842 [hep-th].

[155] D. Serban, "Eigenvectors and scalar products for long range interacting spin chains II: the finite size effects," arXiv:1302.3350 [hep-th].

[156] M. Staudacher, "Review of AdS/CFT Integrability, Chapter III.1: Bethe Ansátze and the R-Matrix Formalism," *Lett.Math.Phys.* **99** (2012) 191–208, arXiv:1012.3990 [hep-th].

[157] S. Belliard, S. Pakuliak, E. Ragoucy, N. Slavnov, S. Pakuliak, *et al.*, "Algebraic Bethe ansatz for scalar products in SU(3)-invariant integrable models," *J.Stat.Mech.* **1210** (2012) P10017, arXiv:1207.0956 [math-ph].

[158] S. Belliard, S. Pakuliak, E. Ragoucy, and N. Slavnov, "Bethe vectors of SU(3)-invariant integrable models," arXiv:1210.0768 [math-ph].

[159] S. Belliard, S. Pakuliak, E. Ragoucy, and N. Slavnov, "Form factors in SU(3)-invariant integrable models," arXiv:1211.3968 [math-ph].

[160] R. Pereira, "Structure constants in N=4 SYM and ABJM theory," *Master Thesis, The Perimeter Institute, unpublished* .

[161] I. Kostov, "Classical Limit of the Three-Point Function of N=4 Supersymmetric Yang-Mills Theory from Integrability," *Phys.Rev.Lett.* **108** (2012) 261604, arXiv:1203.6180 [hep-th].

[162] A. Sever, P. Vieira, and T. Wang, "From Polygon Wilson Loops to Spin Chains and Back," *JHEP* **1212** (2012) 065, arXiv:1208.0841 [hep-th].

[163] L. Ferro, T. Lukowski, C. Meneghelli, J. Plefka, and M. Staudacher, "Harmonic R-matrices for Scattering Amplitudes and Spectral Regularization," arXiv:1212.0850 [hep-th].

[164] I. Macdonald, *Symmetric functions and Hall polynomials.* Clarendon Press, 1979.

[165] J. Demmel and P. Koev, "Accurate and efficient evaluation of Schur and Jack functions," *Mathematics of Computation* **75** (2005) no. 253, 223–229.

[166] E. A. Dumitriu, I. and G. Shuman, "MOPS: Multivariate Orthogonal Polynomials (symbolically)," *ArXiv Mathematical Physics e-prints* (2004) .