On 8-dimensional, maximal, gauged supergravities and some higher order gravities

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En el año 2012 llegué a Madrid para dar inicio a el proyecto más grande que me había propuesto realizar hasta ese entonces. Un proyecto que involucraba trabajo académico y de exploración. Parte de ese proyecto se termina con esta tesis.

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List of Publications

The following articles, some unrelated to the present thesis, were published by the candidate during the realization of this work:

1 Oscar Lasso Andino, “RG-2 flow, mass and entropy”, arXiv:1806.10031[gr-qc]

2 Oscar Lasso Andino, Tomás Ortín , “On gauged maximal d=8 supergravities,” Class. Quant. Grav. **35** (2018) no.7, 075011. arXiv:1605.09629 [hep-th].

3 Oscar Lasso Andino, Tomás Ortín ,“The tensor hierarchy of 8-dimensional field theories,” JHEP **1610** (2016) 098. arXiv:1605.05882 [hep-th].

3 P. Bueno, P. A. Cano, A. O. Lasso and P. F. Ramírez, “f(Lovelock) theories of gravity”, JHEP 1604 (2016) 028. arXiv:1602.07310 [hep-th].

Abstract

This thesis focuses in various aspects of gauged 8-dimensional supergravity theories and on the study of certain higher order gravities. Supergravity theories are the limit, at low energies, of the String Theories. These theories were built in order to unify the 4 forces that exist in nature, under a unique and consistent formalism. Nowadays, scientist are trying to find experimentally the existence of supersymmetric particles. The existence of these particles may solve some fundamental problems like the nature of dark matter and dark energy. If discovered, they would automatically make supergravity the fundamental framework for describing the Universe where we live.

In this context, studying the most general supergravity theories in different dimensions is very important for describing Nature. Considering that the theories that describe the known interactions are gauge theories, we expect that the theory that unifies all interactions is also a gauge theory and, in the context of this thesis, a gauged supergravity theory.

The quantization of General Relativity does not give a renormalizable Quantum Field Theory. This fact does not allow us to know the behaviour of gravity at high energies and small scales. The effective action of any UV completion of General Relativity should contain terms with higher derivatives, involving contractions of the Riemann tensor and its covariant derivative. String Theories predict the appearance of higher order terms, which are corrections to the Einstein-Hilbert action. This provides a motivation to study theories of gravity of higher order in the curvature, known as higher-order gravities" or "modified gravities".

In this thesis we collect the results prsented in the publications [**?**, **?**, **?**].

In [**?**] we construct the tensor hierarchy of generic, linear, bosonic, 8-dimensional field theories. We first study the form of the most general 8-dimensional bosonic theory with Abelian gauge symmetries only and no massive deformations. Having constructed the most general Abelian theory, we study the most general gaugings of its global symmetries and the possible massive deformations using the *embedding tensor* formalism.

In [**?**] we study the gauging of maximal d=8 supergravity using the embedding tensor formalism and teh general results of the preceding paper. We focus on SO(3) gaugings, study all the possible choices of gauge fields and construct explicitly the bosonic actions for all these choices. We study the relation between the 8 dimensional supergravity buit by Salam and Sezgin in [**?**] by compactification of $d = 11$ supergravity and the theory constructed by Alonso-Alberca *et al.* in [**?**] by dimensional reduction of the so called "massive 11-dimensional supergravity" proposed by Meessen and Ortín in [**?**].

In [**?**] we study some aspects of $f(Lovelock)$ theories in d dimensions. These theories are generalizations of the $f(R)$ and Lovelock theories, where the gravitational action depends on an arbitrary function of the Euler densities in d dimensions. We show that these theories are equivalent to certain scalar-tensor theories, we study the linearized equations of the theory on general maximally symmetric backgrounds, and we find constraints on the couplings of a family of five-dimensional $f(Lovelock)$ theories using holographic entanglement entropy. Finally, we present some new black hole solutions in different dimensions.

Contents

1
Introduction

In this section we are going to introduce the concepts needed for a better understanding of the results obtained in the publications that have been collected in this thesis [?, ?, ?]. The main topics are gauged supergravity and higher order ("modified") gravities.

1.1 Gauged supergravity

Gauged supergravity is one of the main topics of this thesis. therefore, we are going to make a short introduction to the procedure of gauging (making local) global symmetries of a field theory. We are going to start by the simplest example of the gauging of an Abelian symmetry of the Dirac action, obtaining the Quantum Electrodynamics (QED)action and then we are going to generalize the technique to more than one Dirac spinor and non-Abelian symmetries. The result of this procedure for the symmetries considered are non-Abelian Yang-Mills (YM) gauge theories.

Next, we are going to introduce a slightly more general procedure for gauging global symmetries, known as the *Noether procedure* (or method). before we do that, we define the Noether current associated to a global symmetry of the action and show how to obtain it.

The Noether procedure, though, does not work for symmetries of the equations of motion which are not symmetries of the action, such as electric-magnetic duality symmetries because the standard procedure to obtain the Noether current does not work here.[1] Also, in 4 dimensions, one cannot use the dual (magnetic) vectors as gauge vectors. We are interested in finding the most general gaugings of a theory, including those that make use of the magnetic vectors as gauge vectors and those in which the gauge symmetry includes electric-magnetic duality rotations. Furthermore, we want to study the possibility of gauging different subgroup of the global symmetry group (of the equations of motion) using different combinations of the vectors of a given theory as gauge fields. A more general formalism is required.

The *embedding tensor formalism* allows one to do just that: studying the most general gaugings of a field theory with a given global symmetry group of the equations of motion and a given field content. This is, precisely, the situation of supergravity theories and, therefore, this formalism is specially well suited to explore the possible gauged supergravities that can be constructed from a given ungauged supergravity theory.

We are going to introduce this method, that we have applied to the gauging of maximal 8-dimensional supergravity in [?, ?] but,before we do that, we are going to review

[1] There is a generalization of the Noether current called Noether-Gaillard-Zumino current [?], though.

electric-magnetic duality symmetries in arbitrary dimensions.

Finally, we are going to give a brief introduction to supergravity theories in arbitrary dimensions, describing later some supergravity theories (ll-dimensional and 8-dimensional supergravities) that will be relevant for the results obtained.

1.1.1 Gauge theories

Today it is accepted by the scientific community that all the interactions in Nature can be described by four fundamental forces: electromagnetism, the weak force, the strong force and gravity.

The first three interactions are reasonably well understood at small scales and high energies: the actions that describe the three first forces are known and they correspond to those of Yang-Mills gauge theories (or *gauge theories*); when these theories are quantized, they are renormalizable Quantum Field Theories (QFTs) and one can compute scattering amplitudes unambiguously and make definite predictions about the outcomes of experiments which, so far, have been experimentally confirmed.

At large scales, of those three interactions, only electromagnetism is a relevant interaction and it is also well understood: the same action mentioned above gives rise to the Maxwell equations. At the same and larger (astronomical and cosmological) scales, gravity is also reasonably well understood, but the same action that gives rise to the Einstein equations, as different from the action that gives rise to the Maxwell equations, cannot be consistently quantized.

Many alternatives have been tried but, so far, none of them seems to be completely successful. Furthermore, we completely lack experimental hints on the quantum behaviour of gravity and, thus, gravity remains a mystery when studied at small scales.

Although we do not have a working theory of quantum gravity, there are several candidates for consistently describing the quantum nature of gravity. The most prominent candidate is String Theory [**?**, **?**, **?**, **?**, **?**] which also aims to describe the rest of the fundamental interactions in a unified way. It is well known that the low energy limit of different String Theories correspond to different gauged Supergravity theories [**?**, **?**]. In this context, the construction of gauged supergravity theories in different dimensions has been pursued with abundant results [**?**]. Studying the ungauged theories and all their possible gaugings in a systematic way and can be done with the help of the embedding tensor formalism. In order to do that, we first need to study the way in which a symmetry of a generic field theory can be gauged. In this section we present the simplest procedure for gauging a local symmetry. We study first the Abelian gaugings and later we end with the non-Abelian ones.

Gauging Abelian symmetries

Let us consider a the the Dirac Lagrangian

$$\mathcal{L}_D = -\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi, \qquad \partial\!\!\!/ = \gamma^{\mu}\partial_{\mu}, \tag{1.1}$$

where ψ is a complex Dirac spinor. This Lagrangian is invariant under the transformations

$$\psi(x) \rightarrow \psi'(x) = e^{iq\alpha}\psi(x), \tag{1.2}$$

where α and q are constant parameters. Such transformations are called *rigid* or *global* transformations because the parameters that describe them are constant. The global phase transformation (**??**) leaves the Lagrangian (**??**) invariant, and therefore, it is a symmetry of the theory. We say that the Lagrangian has a $U(1)$ global symmetry.

Suppose now that we want to have the freedom of choosing the phase of ψ at each point in spacetime independently. In that case, the theory in question should be invariant under

$$\psi(x) \to \psi'(x) = e^{iq\alpha(x)}\psi(x), \tag{1.3}$$

for arbitrary functions $\alpha(x)$. This kind of transformations whose "parameters" are arbitrary functions of the spacetime coordinates are called *local* or *gauge* transformations. If a theory is invariant under (**??**), then it will not depend on the phase of ψ. Thus, phase differences have no physical significance and are no longer determined by the dynamical equations of the theory: invariance under local transformations implies that some of the degrees of freedom are absent.

However, if we allow different phase rotations at different spacetime points i.e. $\alpha = \alpha(x)$ the transformation (**??**) is no longer a symmetry of the theory (**??**). Indeed, the derivative $\partial_\mu\psi$ transforms as

$$\partial_\mu\psi(x) \to \partial_\mu(e^{iq\alpha(x)}\psi) = e^{iq\alpha(x)}\partial_\mu(\psi(x)) + iq\partial_\mu(\alpha(x))e^{iq\alpha(x)}\psi. \tag{1.4}$$

Clearly, the term with $\partial_\mu\alpha$ spoils the possible invariance of the Lagrangian under the local transformation. The Lagrangian (**??**) does not have local $U(1)$ symmetry and if we insist in having local symmetry we should change something.

The definition of the derivative in the kinetic term is the problem: we should introduce a new *covariant* derivative D_μ such that it changes *covariantly* (*i.e.* with no $\partial_\mu\alpha$ term) under the transformation (**??**):

$$D_\mu\psi(x) \to e^{iq\alpha(x)}D_\mu\psi(x). \tag{1.5}$$

The only known way to construct a derivative with this property requires the introduction of a new vector field A_μ, that will be called a *gauge field*, with its own gauge transformation

$$A_\mu \to A_\mu(x) + \partial_\mu\Lambda, \tag{1.6}$$

where Λ is an arbitrary function of the spacetime coordinates. In order to satisfy (**??**) we must set[2]

$$\Lambda = \alpha. \tag{1.7}$$

The covariant derivative that we were looking for can be written as

$$D_\mu\psi = \partial_\mu\psi - iqA_\mu\psi = (\partial_\mu - iqA_\mu)\psi, \tag{1.8}$$

The two terms appearing in D_μ are related to infinitesimal transformations: the term ∂_μ can be interpreted as the result of an infinitesimal spacetime displacement on ψ, and the second term $-ieA_\mu\psi$ represents the result of an infinitesimal gauge transformation on ψ.

[2]In this simple setting this is a trivial statement. However, in more general settings, the identification between gauge parameters can be done in many different ways parametrized by the embedding tensor and it is important to stress that we are making this identification.

Now, in order to make the Lagrangian manifestly gauge invariant we replace the new covariant derivative D_μ in (**??**), obtaining

$$\mathcal{L} = -\bar{\psi}\not{D}\psi - m\bar{\psi}\psi = -\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi + iqA_\mu\bar{\psi}\gamma^\mu\psi. \tag{1.9}$$

If the gauge field is going to be a dynamical field of the theory, we need to add a kinetic term for it in this action that respects its gauge symmetry. Let us see how to construct it.

The covariant derivative satisfies the properties of a usual derivation, such as the Leibniz rule and distributive properties. Moreover, the commutator $[D_\mu D_\nu]$ is

$$[D_\mu, D_\nu]\psi = D_\mu(D_\nu\psi) - D_\nu(D_\mu\psi) = -iqF_{\mu\nu}\psi, \tag{1.10}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{1.11}$$

is called the field strength and it is gauge invariant. being first-order in derivatives of the gauge vector field and gauge invariant we can use it to construct the kinetic term we are looking for (see below).

The relation (**??**) is called Ricci identity, and since both sides have to be gauge covariant we will be using the commutator of covariant derivatives very often for building gauge covariant field strengths.

There is another identity satisfied by the covariant derivative, the Jacobi identity:

$$[D_\mu, [D_\nu, D_\rho]] + [D_\nu, [D_\rho, D_\mu]] + [D_\rho, [D_\mu, D_\nu]] = 0. \tag{1.12}$$

It implies the following relation for the field strength

$$D_\mu F_{\nu\rho} + D_\nu F_{\rho\nu} + D_\rho F_{\mu\nu} = 0. \tag{1.13}$$

Since, in the case at hands, the field strength is invariant under gauge transformations, after replacing the covariant derivative by the ordinary one, we can rewrite this identity in the equivalent form

$$\partial_\mu \star F^{\mu\nu} = 0. \tag{1.14}$$

The above identity is called the *Bianchi identity* and it is trivially satisfied by (**??**).

As mentioned above, we can use the field strength $F_{\mu\nu}$ to build a Lagrangian which, by construction, is going to be gauge invariant:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{1.15}$$

and, combining both Lagrangians (**??**) and (**??**) we have

$$\mathcal{L}_{QED} \quad = \mathcal{L} + \mathcal{L}_D \quad = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi + iqA_\mu\bar{\psi}\gamma^\mu\psi. \tag{1.16}$$

This Lagrangian represents the interacting theory of a fermion field ψ and a gauge vector field A_μ and it is the Lagrangian of the theory of Quantum Electrodynamics (QED). This is a prototypical example of a theory obtained by gauging a global symmetry. The theory has just two free parameters, the mass m of the fermion and its charge q.

Gauging non-Abelian symmetries

The symmetry transformation given in (**??**) can be written as

$$\psi(x) \to M\psi(x), \tag{1.17}$$

where $M \equiv e^{i\alpha}$ is a 1×1 unitary ($U(1)$) matrix. QED is invariant under the $U(1)$ gauge group. This statement rises an immediate question: can we build a gauge theory invariant under a larger gauge group, like the group of $n \times n$ unitary matrices $U(n)$? The answer is yes.[3] We will see a simple example and then proceed to generalize the formalism to all non-Abelian theories.

Let us take n complex fields $\psi_j(x)$ and collect them in a n-component vector

$$\Psi(x) = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \vdots \\ \psi_n(x) \end{bmatrix}.$$

The Dirac Lagrangian for n Dirac spinors reads:

$$\mathcal{L} = \bar{\Psi}\left[(i\gamma^\mu\partial_\mu - m)1_{n\times n}\right]\Psi, \tag{1.18}$$

and it is invariant under the transformation

$$\begin{aligned} \Psi(x) &\to M\Psi(x), \\ \bar{\Psi}(x) &\to \bar{\Psi}(x)M^\dagger, \end{aligned} \tag{1.19}$$

where, now, $M \in U(n)$.

Again, we want to have the freedom to make a different $U(n)$ transformation at each spacetime point, *i.e.* we want the theory to be invariant under the same kind of transformations with M now an arbitrary $U(n)$-valued function $M(x)$. However, it can be seen that (**??**) transforms under (**??**) with $M = M(x)$ as

$$\mathcal{L} \to \mathcal{L} + \bar{\Psi}(x)M^\dagger i\gamma^\mu\partial_\mu(M(x))\Psi(x), \tag{1.20}$$

so that the Lagrangian (**??**) is not invariant under the transformation (**??**). The solution is, again, to replace the standard partial derivative by a covariant derivative, which requires the introduction of a matrix-valued gauge vector field A_μ transforming as

$$A_\mu(x) \to \Lambda A_\mu \Lambda^\dagger - q(\partial_\mu\Lambda(x))\Lambda^\dagger, \tag{1.21}$$

where Λ is a $U(n)$-valued function. With the identification

$$\Lambda = M(x), \tag{1.22}$$

[3]QED is an Abelian theory -two successive $U(1)$ transformations commute- on the $U(n)$ generalization, two transformations will not commute in general, this is why these theories are called non-Abelian gauge theories.

we can build a covariant derivative that solves the problem exactly as in the $U(1)$ case

$$D_\mu \Psi(x) = (\partial_\mu - iqA_\mu)\Psi(x). \tag{1.23}$$

Now that we have seen how the generalization to $U(n)$ can be done, we proceed with the most general case.

Let us assume that M is a generic $n \times n$ unitary n-dimensional matrix, describing unitary representations of a new G. This matrix can be written as $M = e^{iH}$, where H is a Hermitian matrix i.e. $H^\dagger = H$. Since we have a continuous group we can expand M in terms of a basis of Hermitian $n \times n$ matrices that we will denote T_a and some real parameters ξ^a. The $T_a{}'s$ are called the group generators.

Since the identity element $e^0 = 1$ is contained in every group, we can write any group element of the form

$$e^{i\xi^a T_a} = 1 + i\xi^a T_a ... \tag{1.24}$$

Consider two group elements parametrized by ξ_a and η_a. The product

$$e^{i\xi^a T_a} e^{i\eta^b T_b} e^{-i\xi^a T_a} e^{-i\eta^b T_b} = 1 - \xi^a \eta^b [T_a, T_b] + ... \tag{1.25}$$

Being a product of group elements, the result must be a group element as well, and since the T_a's form a basis, we can write

$$[T_a, T_b] = i f_{bc}{}^a T_a, \tag{1.26}$$

where $f_{bc}{}^a$ are called the structure constants. These constants are real for compact semisimple groups. A set of matrices satisfying (**??**) is called a Lie algebra. From (**??**) we can classify all the possible Lie algebras.[4]

The expression (**??**) also helps to determine the possible representations that each Lie algebra has. One important representation is defined using $T^{adj}_{a\,bc} \equiv -if_{abc}$. These generators satisfy (**??**) and they form the so called *adjoint* representation. It exists for any Lie algebra and its dimension is the same of the Lie algebra.

The vector A_μ can be written as $A_\mu = A^a_\mu T_a$, where each component A^a_μ is called the gauge field. A_μ transforms in the adjoint representation.

Consider a linear G transformation of the field $\psi(x)$

$$\psi(x) \to \psi'(x) \equiv e^{iq\xi^a T_a}\psi(x). \tag{1.27}$$

When ξ^a is constant(global transformation). The partial derivative $\partial_\mu \psi$ is invariant under (**??**), but if we promote the ξ^a's to arbitrary functions of the spacetime coordinates, then, $\partial_\mu \psi$ transforms as

$$\partial_\mu \psi \to (\partial_\mu \psi(x))' = e^{iq\xi^a(x)T_a}\left(\partial_\mu \psi(x) + iq\partial_\mu(\xi^a(x))T_a\psi(x)\right), \tag{1.28}$$

[4] There is a well known classification of all semisimple Lie algebras: the $\mathfrak{su}(n)$ associated to the group $SU(n)$, which is the group of unitary matrices with unit determinant, the $\mathfrak{so}(n)$ associated to the group $SO(n)$, which is the group of orthogonal matrices with unit determinant; and the $\mathfrak{sp}(2n)$ associated to the group $Sp(2N)$ which is the group of matrices satisfying $U^T \Omega U = \Omega$ with

$$\Omega = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

There are also 5 exceptional semisimple Lie algebras called G_2, F_4, E_6, E_7 and E_8.

and we need another derivative that transforms covariantly. In order to define a covariant derivative we have to introduce a gauge vector A^a_μ transforming as:

$$A'^a_\mu = A^a_\mu - \partial_\mu \Lambda^a - q f_{bc}{}^a \Lambda^b A^c_\mu. \tag{1.29}$$

Making the identification

$$\Lambda^a = \xi^a, \tag{1.30}$$

we can see that

$$D_\mu \psi = \partial_\mu \psi - A^a_\mu T_a, \tag{1.31}$$

transforms covariantly.

The commutator of two covariant derivatives is

$$[D_\mu, D_\nu]\psi = iqF_{\mu\nu}\psi, \qquad F_{\mu\nu} = F^a_{\mu\nu} T_a, \tag{1.32}$$

where $F^a_{\mu\nu}$, the field strength, is given by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - q f^a{}_{bc} A^b_\mu A^c_\mu, \tag{1.33}$$

transforms covariantly under gauge transformations. We can use it to build an invariant Lagrangian that has to be added to that of the matter field ψ:[5]

$$\mathcal{L}_{YM} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}. \tag{1.34}$$

This Lagrangian is the Yang-Mills Lagrangian and it is a generalization of the Maxwell one. A non-Abelian gauge theory automatically contains self-interactions of the gauge field. Comparing both Lagrangians (the QED and the YM) we can see that in QED the gauge field does not transform under a global U(1) transformation, so we can say that it is uncharged. In a non-Abelian gauge theory, the gauge fields transform in the adjoint representation under a global transformation, so it carries charge, and that is why it couples to itself.

1.1.2 Symmetries in field theories

In this section we present a study of the global symmetries of the action of a theory and their gauging. For a detailed account see [**?**, **?**].

Le us consider an action $S(\varphi)$ where φ is a generic field:

$$S(\phi) = \int_\Sigma d^d x \mathcal{L}(\varphi, \partial\varphi). \tag{1.35}$$

Usually, $\mathcal{L}$ is a scalar density. We consider the variation of the action under arbitrary infinitesimal variations of the field φ. Using integration by parts, Stokes' theorem ,and requiring $\delta S = 0$ with the appropriate boundary conditions ($\delta\varphi|_{\partial\Sigma} = 0$), the *Euler-Lagrange* equations are found:

$$\frac{\delta S}{\delta \varphi} = \frac{\partial \mathcal{L}}{\partial \varphi} - \partial^\mu \left(\frac{\partial \mathcal{L}}{\partial \partial^\mu \varphi} \right) = 0, \tag{1.36}$$

[5] We have assumed implicitly a normalization of the generators of the gauge group so that the Killing metric K_{ab}, which is gauge invariant by construction, is proportional to the identity.

The addition of a total derivative term to the Lagrangian does not change the equations of motion as long as the boundary conditions for $\delta\varphi$ make the new boundary term vanish.

Now, we consider the infinitesimal transformations of the coordinates $\tilde{\delta}x^\mu$ and the variation of the field $\tilde{\delta}\varphi$:

$$\tilde{\delta}x^\mu = x'^\mu - x^\mu, \qquad \tilde{\delta}\varphi(x) = \varphi'(x') - \varphi(x), \tag{1.37}$$

where x and x' are the coordinates of the same point in different coordinate systems. The invariance of the action under these transformations, up to total derivatives, can be stated as follows

$$\tilde{\delta}S = \int_\Sigma d^d x \partial_\mu \mathbf{s}^\mu(\tilde{\delta}). \tag{1.38}$$

It follows that

$$\int_\Sigma d^d x \left(\partial_\mu \mathbf{J}^\mu(\tilde{\delta}) + \frac{\delta S}{\delta\varphi}\delta\varphi \right) = 0, \tag{1.39}$$

where

$$\mathbf{J}^\mu(\tilde{\delta}) = -\mathbf{s}^\mu(\tilde{\delta}) + T^\mu_{\ \nu}\tilde{\delta}x^\nu - \frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu\varphi}\partial_\rho\varphi\partial_\nu\tilde{\delta}x^\rho + ... \tag{1.40}$$

Here, $T^\mu_{\ \nu}$ is the energy-momentum tensor:

$$T^\mu_{\ \nu} = \eta^\mu_{\ \nu}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu}\partial_\nu\varphi + \dots \tag{1.41}$$

If the equations of motion $\delta S/\delta\varphi = 0$ are satisfied then we get the conservation law

$$\partial_\mu \mathbf{J}^\mu(\tilde{\delta}) = 0. \tag{1.42}$$

$\mathbf{J}^\mu$ is the *Noether current.* If the transformations depend on constant parameters σ^A

$$\tilde{\delta}x^\mu \equiv \sigma^A\tilde{\delta}_A x^\mu, \quad \tilde{\delta}\varphi \equiv \sigma^A\tilde{\delta}_A\varphi, \tag{1.43}$$

where $\tilde{\delta}_A x^\mu$ and $\tilde{\delta}_A\varphi$ are functions of the coordinates and φ and the $\sigma^A, A =, 1, .., n$ are the transformation parameters, there are n independent Noether currents $\mathbf{J}^\mu_A$.

If the transformations are local $\sigma^A = \sigma^A(x)$ then, up to a total derivatives, we have

$$\int_\Sigma d^d x \left(\partial_\mu \mathbf{J}^\mu_2(\sigma) + \sigma^A D_A \frac{\delta S}{\delta\varphi} \right) = 0, \tag{1.44}$$

where D_A are operators generally containing derivatives acting on the equations of motion. If we choose parameters such that $\mathbf{J}^\mu_2(\sigma)$ vanishes at the boundary then we obtain n identities relating different equations of motion

$$D_A\frac{\delta S}{\delta\varphi} = 0. \tag{1.45}$$

These identities are called Noether identities. They are satisfied off-shell. Thus, since the Noether identities are identically true for arbitrary values of the parameters, we obtain an off-shell conservation law:

$$\partial_\mu \mathbf{J}^\mu_2(\sigma) = 0. \tag{1.46}$$

It can be shown that this current vanishes on-shell and, therefore, it is not useful for constructing conserved quantities.

1.1.3 Noether method

This is a method for gauging a global symmetry of the action based on a simple observation: assuming that the action of a theory is invariant under some global transformations of the fields $\delta\varphi$ with constant parameters σ^A, if we now use local parameters $\sigma^A(x)$ we should find that the variation of the action has the form

$$\delta S = \int d^dx \partial_\mu \sigma^A \mathbf{J}^\mu_A, \tag{1.47}$$

because, it has to vanish for constant σ^A. Here $\mathbf{J}^\mu_A$ are the Noether currents associated to each of the independent transformation parameters σ^A.

Comparing the above formula with the generic gauge transformation rule of as many Abelian vector fields as independent transformations A^A_μ

$$\delta A^A_\mu = \partial_\mu \Lambda^A\,, \tag{1.48}$$

it is clear that we can cancel the above term in the variation of the action by identifying

$$\Lambda^A = \sigma^A, \tag{1.49}$$

and adding a term

$$-\int d^dx A^A_\mu \mathbf{J}^\mu_A, \tag{1.50}$$

to the original action. This is the basis of the Noether method, but, it is clear that the new total action will not be exactly invariant, in general, because $\delta\mathbf{J}^\mu_A \neq 0$. However, it will be invariant up to terms of higher order in the coupling constant (that we have not written explicitly). These terms will have the same form, where the Noether current is modified because of the introduction of the above coupling in the action. We can repeat the procedure adding new terms to the action and, if necessary, to the gauge transformation rules of the fields to cancel them. In many interesting cases, this iterative procedure ends after a finite number of steps, yielding an exactly gauge-invariant action.

Let us see how this method works in a simple example.

Let us consider the following Lagrangian for a complex scalar field Φ

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi, \tag{1.51}$$

invariant under phase transformations, which infinitesimally can be written as

$$\delta\Phi = iq\sigma\Phi, \tag{1.52}$$

where σ is the parameter of the transformation. These transformations constitute a $U(1)$ symmetry group, and q labels the representation of $U(1)$ corresponding to Φ. If σ is a function of the spacetime coordinates, the Lagrangian is not invariant

$$\delta\mathcal{L}_0 = -q\mathbf{j}^\mu \partial_\mu \sigma, \qquad \mathbf{j}^\mu = -\frac{i}{2}\left(\Phi\partial^\mu\bar{\Phi} - \bar{\Phi}\partial^\mu\Phi\right), \tag{1.53}$$

where $\mathbf{j}^\mu$ is the on-shell conserved current associated with the global invariance of the Lagrangian $\mathcal{L}_0$.

The above variation can be cancelled by the introduction of an Abelian gauge vector field A_μ transforming as $\delta A_\mu = \partial_\mu \Lambda$, making the identification $\Lambda = \sigma$ and adding a new term in the Lagrangian

$$\mathcal{L}_1 = qA_\mu \mathbf{j}^\mu, \tag{1.54}$$

whose variation (after the identification) is

$$\delta\mathcal{L}_1 = q\partial_\mu\sigma\mathbf{j}^\mu + qA_\mu\delta\mathbf{j}^\mu. \tag{1.55}$$

The first in this variation cancels exactly the variation of $\mathcal{L}_0$ and

$$\delta(\mathcal{L}_0 + \mathcal{L}_1) = qA_\mu\delta\mathbf{j}^\mu = -q^2|\Phi|^2 A_\mu\partial^\mu\sigma. \tag{1.56}$$

Using the same reasoning we can cancel this term by introducing yet another term of the form

$$\mathcal{L}_2 = \frac{1}{2}q^2|\Phi|^2 A^2, \tag{1.57}$$

in the Lagrangian because

$$\delta\mathcal{L}_2 = q^2|\Phi|^2 A_\mu\partial^\mu\sigma. \tag{1.58}$$

In this case we have achieved exact invariance in two iterations. Adding the kinetic term for the gauge field, the total Lagrangian of the gauged theory can be written as

$$\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 = \frac{1}{2}\mathcal{D}_\mu\Phi\mathcal{D}^\mu\bar{\Phi} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{1.59}$$

where

$$\mathcal{D}_\mu\Phi = (\partial_\mu - iqA_\mu)\Phi, \tag{1.60}$$

is the gauge-covariant derivative, which in this method arises in a completely natural form.

As we have mentioned, in more complicated cases, the Noether procedure will require the addition of more corrections both to the field transformation rules and the Lagrangian. Note that we have added by hand to the original Lagrangian the exact number of vector fields needed to gauge the existing symmetry. However, in many cases, the field content of the theory cannot be changed and we have a fixed number of vector fields transforming in a given representation of the global symmetry group. For instance, in supergravity theories with high $\mathcal{N}$.

The possible gaugings of the global symmetry group are usually strongly constrained and a method to explore systematically all the possibilities is the *embedding tensor* formalism [**?**, **?**, **?**]. If we have a symmetry of the equations of motion that does not leave invariant the action, the Noether method is not suitable for building a conserved current. That is the case of the electric-magnetic rotations. We want to study the general symmetries of equations of motion finding the conditions that a group of transformations has to satisfy in order to be a symmetry of the equations of motion.

Electric-magnetic duality

Electric-magnetic duality is part of a bigger group of dualities, *S dualities.* In these dualities the coupling constant is inverted and perturbative (weak coupling) and non-perturbative (strong coupling) regimes are related.

Electric–magnetic duality is a symmetry of the equations of motion which is not a symmetry of the action. It can also be seen as a mapping between two theories that describe, in different ways, the same degrees of freedom. We are interested in the most general case, but we will start by reviewing the well known case of Maxwell equations in 4 dimensions. Then, we will present the general case in 4 dimensions and, finally, we will proceed to increase the number of fields and dimensions. For more details see [?] [?].

The sourceless Maxwell equations

$$\partial_\mu F^{\mu\nu} = 0, \tag{1.61}$$

and the Bianchi identities

$$\partial_\mu \star F^{\mu\nu} = 0 \tag{1.62}$$

are invariant under the replacement $F \to \star F$, $\star F \to -F$. In a given frame, this replacement corresponds to the interchange of electric and magnetic field

$$\vec{B}' = -\vec{E}, \qquad \vec{E}' = \vec{B}. \tag{1.63}$$

The electric-magnetic duality transformation squares to minus the identity and, therefore, it generates a $\mathbb{Z}_2$ electric-magnetic-duality group. The $\mathbb{Z}_2$ group can be extended to a continuous symmetry group of the equations of motion:

$$\tilde{F} = aF + b \star F, \quad \star\tilde{F} = -bF + a \star F, \quad a^2 + b^2 \neq 0, \tag{1.64}$$

is an invertible transformation that leaves the set of the two equations invariant. It is convenient to define the duality vector

$$\vec{F} = \begin{pmatrix} F \\ \star F \end{pmatrix},$$

which satisfies

$$\star\vec{F} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{F}.$$

When the lower component of $\vec{F}$ is considered as independent of the upper one, the above constraint is called the linear, twisted self-duality constraint. It is equivalent to the original relation between the upper and lower components of $\vec{F}$, which solves it, but its structure is much more convenient to study more general duality transformations.

The Bianchi identity together with the Maxwell equations can be written in the compact form

$$\partial_\mu \vec{F}^{\mu\nu} = 0, \tag{1.65}$$

transforming in the vector representation of the duality group

$$\tilde{\vec{F}} = P\vec{F}, \quad P = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \pm\lambda \begin{pmatrix} \cos(\theta) & sin(\theta) \\ -\sin(\theta) & cos(\theta) \end{pmatrix}. \tag{1.66}$$

We can see that the duality group of the Maxwell equations is made of rescalings and $O(2)$ rotations of $\vec{F}$.

The replacement of F by $\star F$ is not a symmetry of the Maxwell action.[6] We have defined the electric-magnetic duality transformation in terms of the field strength for the

[6]There is a change in the Maxwell action sign because the identity $(\star F)^2 = -F^2$.

sake of convenience but we must bear in mind that the variable in the action is the vector field A_μ and not its field strength. In terms of the vector potential A_μ electric-magnetic duality transformations are non-local.[7]

In order to study more general cases, we have to introduce more formal and general definitions that allow us to handle theories with more vector fields (or other higher-rank form fields), with non-linear couplings and with couplings to scalar fields. We are going to follow closely [**?**, **?**, **?**]. For a good review in the context of supergravity see [**?**].

In four dimensions we consider a functional $S(F(A))$ of a single Abelian field strength $F(A)$ with no couplings to any other fields. Since by definition $F = dA$, the 2-form F always satisfies the Bianchi identity $dF = 0$. The dual (magnetic) field strength G can be defined for any such action by

$$\star G^{\mu\nu} \equiv 2\frac{\delta S[F]}{\delta F_{\mu\nu}}. \tag{1.67}$$

The equations of motion of the 1-form (Maxwell equations) can always be written in terms of the dual field strength G in the form[8]

$$\partial_\mu \star G^{\mu\nu} = 0, \tag{1.68}$$

which is a Bianchi identity for G. For instance, for the Maxwell action eq. (**??**) the dual field strength is $G = \star F$ and equation (**??**) is the Maxwell equation. The pair, Bianchi identity plus Maxwell equation, is invariant under global linear transformations of F and G of the form

$$\begin{pmatrix} F' \\ G' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix}, \tag{1.69}$$

or, infinitesimally

$$\delta \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix}. \tag{1.70}$$

In general, the action $S(F)$ is not invariant under these transformations. Moreover, we know that G and F are not independent and, actually, $G = G(F)$ (in the Maxwell case $G(F) = \star F$). Thus, we have to impose $G' = G'(F')$ with the same functional dependence.

We want to know when it happens that

$$S'(F') = S(F'). \tag{1.71}$$

This property is called *self-duality*, and it is a property of the duality transformations and of the action $S(F)$.[9] In the Maxwell case, the self-duality property requires $A = D$ and $B = -C$. This group of transformations is $\mathbb{R}^+ \times SO(2)$[10].

[7] If we want to perform the duality transformation in the action we need an action whose variable is the field strength. In such an action, the above replacement can be performed and its form-invariance can be tested. This procedure is called "Poincaré dualization" (see *e.q.* [**?**]).

[8] It is important that the action depends only depends on the vector field through its field strength.

[9] The condition that an action, the dual vector field strengths, and the duality parameters must satisfy in a self-dual theory is:

$$\frac{\delta}{\delta F_{\mu\nu}}\left((d+a)S(F) + \frac{1}{4}\int d^4y\,(cF^{\mu\nu}\star F_{\mu\nu} - bG^{\mu\nu}\star G_{\mu\nu} - 2aF^{\mu\nu}\star G_{ab})\right) = 0, \tag{1.72}$$

the so-called *Noether-Gaillard-Zumino identity*.

[10] Only the second factor leaves invariant the energy momentum tensor(which is necessary to have a symmetry of the equations when we couple the theory to Einstein's gravity),and, therefore, the electric-magnetic duality group of Maxwells' theory is just $SO(2)$.

Now, let us consider n fundamental (electric) vector field strengths F^Λ, and m scalar fields φ^i. For each of the vector field strengths F^Λ we can define a dual (magnetic) vector field strength $G_\Lambda(F,\varphi)$:

$$\star\, G_\Lambda^{\mu\nu} \equiv 2\frac{\delta S[F,\varphi]}{\delta F^\Lambda_{\mu\nu}}. \tag{1.73}$$

The Maxwell equations for each A^Λ take the form of a Bianchi identity [11] for the dual field strengths

$$\partial_\mu \star G_\Lambda{}^{\mu\nu} = 0, \tag{1.74}$$

As before, we construct $2n$-component vectors of the fundamental and dual vector field strengths and the transformations

$$\begin{pmatrix} F' \\ G' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix}, \tag{1.75}$$

or, infinitesimally,

$$\Delta \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix}. \tag{1.76}$$

where A, B, C, D, a, b, c and d are $n \times n$ matrices. As we are going to see, for consistency, they have to be supplemented by transformations of the scalar fields

$$\varphi'^i = f^i(\varphi), \quad \Delta\varphi^i = \xi^i(\varphi). \tag{1.77}$$

The linear (in vector field strengths) theories in $d = 4$ we are interested in, have the form

$$\begin{aligned} S(F,\varphi) \;=\; & \int d^4x\sqrt{|g|}\{R + \mathcal{G}_{ij}\partial_\mu\varphi^i\partial^\mu\varphi^j \\ & +2\mathfrak{Im}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\mu\nu}F^\Sigma{}_{\mu\nu} - 2\mathfrak{Re}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\mu\nu}\star F^\Sigma{}_{\mu\nu}\}. \end{aligned} \tag{1.78}$$

where $\mathcal{G}_{ij} = \mathcal{G}_{ij}(\varphi)$ is a positive definite metric, and $\mathcal{N}_{\Lambda\Sigma}(\varphi)$ is the symmetric, complex, $n \times n$ *period matrix*, whose imaginary part must be negative definite. The bosonic sectors of all the four-dimensional ungauged supergravities can be written in the form of (**??**). Moreover, we can add a scalar potential whose only effect is to restrict the possible symmetries of the theory to those that leave it invariant. The dual vector field strengths are defined as in the single-vector case[12]

$$\star G_\Lambda{}^{\mu\nu} \equiv \frac{1}{4\sqrt{|g|}}\frac{\delta S}{\delta F^\Lambda_{\mu\nu}}, \tag{1.79}$$

and, therefore,

$$G_\Lambda = \mathfrak{Re}\mathcal{N}_{\Lambda\Sigma}F^\Sigma + \mathfrak{Im}\mathcal{N}_{\Lambda\Sigma}\star F^\Sigma \Rightarrow G^+_\Lambda = \mathcal{N}^*_{\Sigma\Lambda}F^{\Sigma+}. \tag{1.80}$$

After applying the transformation (**??**) the new field strength G'_Λ is related to the new field strength F'^Λ by

$$G'^+_\Lambda = \mathcal{N}'^*_{\Sigma\Lambda}F'^{\Sigma+}, \tag{1.81}$$

[11] It implies the existence (locally) of as many dual vector fields A_Λ such that $G_\Lambda = dA_\Lambda$.

[12] Note that by convenience we choose a normalization different to that of eq. (**??**).

and using the reality of the matrices A, B, C and D it can be shown that the transformation rule of the period matrix $\mathcal{N}$ is

$$\mathcal{N}' = (C + D\mathcal{N})(A + B\mathcal{N})^{-1}. \tag{1.82}$$

The previous transformations must preserve the properties of the period matrix: the negative definiteness of the imaginary part and the symmetry of the matrix $\mathcal{N}'$. Using the symmetry of the matrix $\mathcal{N}'$ we arrive at the conditions

$$C^T A - (A^T D - C^T B)\mathcal{N} + \mathcal{N}(D^T B)\mathcal{N} - \text{transposed} = 0, \tag{1.83}$$

which are solved by

$$C^T A = A^T C, \qquad B^T D = D^T B, \qquad A^T D - C^T B = \kappa \mathbb{1}_{\bar{n}\times\bar{n}}, \tag{1.84}$$

for some real constant κ. Then, it is possible to show that the imaginary part $\mathfrak{Im}\mathcal{N}$ of the period matrix $\mathcal{N}'$ transforms as

$$\mathfrak{Im}\mathcal{N}' = \kappa(A^T + \mathcal{N}^\dagger B^T)^{-1}\mathfrak{Im}\mathcal{N}(A + B\mathcal{N})^{-1}, \tag{1.85}$$

and it will remain negative-definite if $\kappa > 0$. If we want these transformations to preserve the energy-momentum tensor as well (which is necessary in theories in which the vector fields a minimally coupled to gravity), the only allowed value is $\kappa = +1$.

The conclusion is that the most general symmetry of the equations of motion of the theories (**??**) acts linearly on the vector fields as a subgroup of $Sp(2n, \mathbb{R})$.[13] At the same time, the scalars must be transformed in such a way that the functional dependence of the period matrix on them remains invariant. These transformations of the scalars must leave the scalar metric invariant, *i.e.* they must be isometries of $\mathcal{G}_{ij}$. Only those isometries of the metric which are associated to the above $Sp(2n, \mathbb{R})$ transformations will be true symmetries of the equations of motion.[14] In the cases where the scalars do not appear in the period matrix, the duality group will contain as a factor additional to the $Sp(2n, \mathbb{R})$, the group of isometries of the scalar metric that acts precisely on the scalars that are not in the period matrix.

There is another way to see the symplectic group arise in this problem. Let us define a $2n$-component vector

$$(\mathcal{F}^M) \equiv \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix}, \qquad \mathcal{F}'^M = S^M{}_N \mathcal{F}^N, \tag{1.87}$$

where we will use the symplectic metric

$$\Omega_{MN} = \begin{pmatrix} 0 & \mathbf{1}_{n\times n} \\ \mathbf{1}_{n\times n} & 0 \end{pmatrix}, \tag{1.88}$$

[13] $Sp(2n, \mathbb{R})$ is the group of transformations S such that

$$S^T \Omega S = \Omega, \quad \Omega \equiv \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}. \tag{1.86}$$

[14] This statement is based on the dependence of the period matrix of the scalars and it could happen that some scalars do not occur on the period matrix. Moreover, it could happen that $\mathfrak{Re}\mathcal{N} = 0$ then, again, the statement may not be true on these conditions.

for lower and raise indices: $\mathcal{F}^N = \mathcal{F}_M \Omega^{MN}$. the componets of the contravariant $\mathcal{F}^M$ are

$$(\mathcal{F}^M) = (G_\Lambda, -F^\Lambda). \tag{1.89}$$

This vector satisfies the linear, twisted, self-duality constraint

$$\mathcal{M}_{MN}(\mathcal{N})\mathcal{F}^N = -\Omega_{MN} \star \mathcal{F}^M, \tag{1.90}$$

that must be preserved by duality transformations.

Using this notation and the above constraint, the energy-momentum tensor corresponding to the vectors in the action (**??**) can be written in these two equivalent ways

$$T^{vect}_{\mu\nu} = -4\mathcal{M}_{MN}(\mathcal{N})\mathcal{F}^M{}_\mu{}^\rho \mathcal{F}^N{}_{\nu\rho} = -4\Omega_{MN} \star \mathcal{F}^M{}_\mu{}^\rho \mathcal{F}^N{}_{\nu\rho} \tag{1.91}$$

where $\mathcal{M}(\mathcal{N})$ is a $2n \times 2n$ symplectic matrix

$$\mathcal{M}(\mathcal{N}) \equiv \begin{pmatrix} I_{\Lambda\Sigma} + R_{\Lambda\Delta} I^{\Delta\Omega} R_{\Omega\Sigma} & -R_{\Lambda\Delta} I^{\Delta}\Sigma \\ -I^{\Lambda\Omega} R_{\Omega\Sigma} & I^{\Lambda\Sigma} \end{pmatrix}, \tag{1.92}$$

where

$$I_{\Lambda\Sigma} = \mathfrak{Im}\mathcal{N}_{\Lambda\Sigma}, \quad R_{\Lambda\Sigma} = \mathfrak{Re}R_{\Lambda\Sigma}. \tag{1.93}$$

The energy-momentum tensor (**??**) is going to be invariant under duality transformations $\mathcal{F}' = S\mathcal{F}$ if these transformations preserve the symplectic metric Ω_{MN} (*i.e.* if they belong to $Sp(2n, \mathbb{R})$) and if they also preserve the form of the matrix $\mathcal{M}_{MN}(\mathcal{N})$:

$$\mathcal{M}_{MN}(\mathcal{N}) = (S^{-1})^P{}_M \mathcal{M}_{PQ}(\mathcal{N})(S^{-1})^Q{}_N. \tag{1.94}$$

The period matrix has to satisfy

$$\mathcal{M}'_{MN}(\mathcal{N}) = \mathcal{M}_{MN}(\mathcal{N}') \tag{1.95}$$

where $\mathcal{M}'_{MN}(\mathcal{N})$ is given by (**??**) and $\mathcal{N}'$ is given by (**??**).

Higher dimensions and higher ranks

The generalization of (**??**) to higher dimensions is[15]

$$\begin{aligned} \mathcal{I}(g, A_{p+1}, \phi^i) \quad = \quad & \int d^dx\sqrt{|g|}\left(R + \mathcal{G}_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j + 4\frac{(-1)^p}{(p+2)!} I_{\Lambda\Sigma}(\phi) F^\Lambda_{(p+2)} \cdot F^\Sigma_{(p+2)} \right. \\ & \left. + 4\xi^2 \frac{(-1)^p}{(p+2)!} R_{\Lambda\Sigma}(\phi) F^\Lambda_{(p+2)} \cdot \star F^\Sigma_{(p+2)}\right), \end{aligned} \tag{1.97}$$

where the scalar fields ϕ^i couple through the matrices $I_{\Lambda\Sigma}(\phi)$ and $R_{\Lambda\Sigma}(\phi)$ to the n $(p+2)$-form filed strengths $F^\Lambda_{(p+2)}$ of as many as $(p+1)$-form potentials $A^\Lambda_{(p+1)}$. The field strengths are defined by

$$F^\Lambda_{(p+2)} \equiv dA^\Lambda_{(p+1)}, \quad . \tag{1.98}$$

[15]The term

$$4\xi^2 \frac{(-1)^p}{(p+2)!} R_{\Lambda\Sigma}(\phi) F^2_{(p+2)} \cdot \star F^\Sigma_{(p+2)}, \tag{1.96}$$

does not exist for any d or p, it can only be different from zero when $p = \tilde{p} = (d-4)/2$.

and satisfy the Bianchi identities

$$dF^{\Lambda}_{(p+2)} = 0 \tag{1.99}$$

We use the notation

$$F^{\Lambda}_{(p+2)} \cdot F^{\Sigma}_{(p+2)} \equiv F^{\Lambda}_{(p+2)\mu_1..\mu_{p+2}} F^{\Sigma}_{(p+2)}{}^{\mu_1...\mu_{p+2}}. \tag{1.100}$$

The matrix $I_{\Lambda\Sigma}$ is symmetric and negative definite. The matrix $R_{\Lambda\Sigma}$ is a new scalar-dependent matrix such that

$$R_{\Lambda\Sigma} = \xi^2 R_{\Sigma\Lambda}, \quad \text{where} \quad \xi^2 = -(-1)^{d/2} = (-1)^{p+1}. \tag{1.101}$$

Then, depending on the value of ξ ($+i$ or $+1$) the duality group will change.The normalization of the action is chosen in such a way that when $d = 4, p = 0$ we recover the action (**??**).

As before, we define the dual $(\tilde{p} + 2)$-form field strengths $G_{(\tilde{p}+2)\Lambda}$ as

$$G_{\tilde{p}+2\Lambda} \equiv R_{\Lambda\Sigma} F^{\Sigma}_{(p+2)} + I_{\Lambda\Sigma} \star F^{\Sigma}_{(p+2)}. \tag{1.102}$$

The equations of motion of the $(p + 1)$-form potentials are

$$dG_{(\tilde{p}+2)\Lambda} = 0. \tag{1.103}$$

We arrange the electric and magnetic field strengths in a vector of $2n$ components, taking into account that only in the case $p = \tilde{p}$ we can mix the upper and lower components:

$$\left(\mathcal{F}^M\right) = \begin{pmatrix} F^{\Lambda}_{(p+2)} \\ G_{(\tilde{p}+2)\Lambda} \end{pmatrix}. \tag{1.104}$$

With this definition we can write equations (**??**)and (**??**) as

$$d\mathcal{F}^M = 0. \tag{1.105}$$

Assuming that $\mathcal{F}^M$ transforms linearly as

$$\mathcal{F}'^M = S^M{}_N \mathcal{F}^N, \quad S \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad p \neq \tilde{p} \rightarrow B = C = 0. \tag{1.106}$$

and requiring the consistency between these and the definitions of the magnetic filed strengths forces the matrices I and R to transform according to transformation (**??**) where now $\mathcal{N}$ is defined as $\mathcal{N} = R + \xi I$.

The contribution of the $(p + 1)-$form potentials to the energy momentum tensor can be rewritten as

$$T^{A(p+1)}_{\mu\nu} = \frac{4(-1)^{p+1}}{(p+1)!} \mathcal{M}_{MN}(\mathcal{N}) \mathcal{F}^M{}_{\mu}{}^{\cdots} \mathcal{F}^N{}_{\nu\cdots}, \tag{1.107}$$

where we have introduced the symmetric matrix

$$\mathcal{M}_{MN}(\mathcal{N}) \equiv \begin{pmatrix} I - \xi^2 R I^{-1} R & \xi^2 R I^{-1} \\ & \\ -I^{-1} R & \xi^4 I^{-1} \end{pmatrix}. \tag{1.108}$$

Defining the metric

$$\Omega_{MN} \equiv \begin{pmatrix} 0 & \mathbb{1} \\ \xi^2 \mathbb{1} & 0 \end{pmatrix}, \tag{1.109}$$

the linear, twisted, self-duality constraint takes the form

$$\mathcal{M}_{MN}(\mathcal{N})\mathcal{F}^N = \xi^2 \Omega_{MN} \star \mathcal{F}^M, \tag{1.110}$$

and, the energy momentum tensor (**??**) can be written

$$T^{A(p+1)}_{\mu\nu} = 4(-1)^{p+1}\xi^4 \Omega_{MN} \star \mathcal{F}^M{}_\mu{}^{\cdots} \mathcal{F}^N{}_{\nu\cdots}. \tag{1.111}$$

Following the same reasoning as in the previous case, we can conclude that the transformations that leave this energy-momentum tensor invariant are those that leave Ω_{MN} invariant:

$$p = \tilde{p}, \;\; \xi^2 = +1 \qquad \to O(n,n). \tag{1.112}$$

$$p = \tilde{p}, \;\; \xi^2 = -1 \qquad \to Sp(2n,\mathbb{R}). \tag{1.113}$$

$$p \neq \tilde{p}, \qquad \to \text{ no constraint.} \tag{1.114}$$

1.1.4 The embedding tensor formalism and the tensor hierarchy

As we have advanced in previous sections, given a theory with prescribed field content and global symmetry group, the embedding tensor formalism helps us to study the most general gaugings of the theory and (as a bonus) the most general deformations of the theory compatible with gauge invariance. For a review see [**?**] [**?**].

Let us consider an ungauged field theory with a global invariance group G that contains the vector fields A^M_μ transforming in some representation V_A of the group G[16]. The generators of the corresponding algebra **g** are called T_A,[17] with $A = 1,...,dim(\mathbf{g})$, and they satisfy the commutation relation (**??**).

We define the embedding tensor $\vartheta_M{}^A$, which determines the subgroup of vectors that will be used as gauge vectors, in other words, it determines the combinations of vectors $A^A_\mu = A_\mu{}^M \vartheta_M{}^A$ that can be taken as a gauge fields associated to a set of generators of T_A. The embedding tensor is a map $\vartheta : V \to \mathbf{g}$ whose image defines the gauge group. Thus, the embedding tensor can be used to define covariant derivatives of the gauged theory:

$$D_\mu = \partial_\mu - A_\mu{}^M \vartheta_M{}^A T_A. \tag{1.115}$$

The consistency of the formalism demands that the embedding tensor is gauge invariant:

$$\delta_P \vartheta_M{}^A = \vartheta_P{}^B \left(T_B{}^N{}_M \vartheta_N{}^A + f_{BC}^A \vartheta_M{}^C \right) = 0. \tag{1.116}$$

If we define $X_N{}^Q{}_P = \vartheta_N{}^A T_A{}^Q{}_P$, eq. (**??**) implies $\delta_M X_N{}^Q{}_P = 0$. Then,

$$[X_M, X_N] = -X_{MN}{}^P X_P = -X_{[MN]}^P X_P. \tag{1.117}$$

[16] Here we must consider all the vector fields of the theory, electric and (if we are in $d = 4$, magnetic as well. This, as we have reviewed in the previous section, in $d = 4$ the group G must be a subgroup of the symplectic group.

[17] The matrices $T_A{}^M{}_N$ must belong to the algebra of the symplectic group in $d = 4$: $T_A{}^M{}_N \Omega_{MP} = T_A{}^M{}_P \Omega_{MN}$.

Writing ${X_{MN}}^P$ as

$$ {X_{MN}}^P = X_{[MN]}^{\ \ P} + {Z^P}_{MN}, \tag{1.118} $$

we find that

$$ {\vartheta_P}^A {Z^P}_{MN} = 0. \tag{1.119} $$

Thus, the antisymmetry of ${X_{MN}}^P$ holds only upon contraction with the embedding tensor. Eq. (??) is implicit in eq. (??) and it is called the *quadratic constraint*, this constraint has to be satisfied by ${\vartheta_M}^A$ for having a valid gauging.

The simplest deformations of an ungauged theory are associated to the gauging of the global symmetries and are parametrized by the embedding tensor. However, other deformations can appear, and they may be parametrized by objects that are independent of the embedding tensor.[18]

One of the important aspects of the embedding tensor formalism is that it gives rise to the *tensor hierarchy* [?] [?] which consists of a set of potentials of all degrees $(1, .., d)$ and their corresponding curvatures, which are related by Bianchi identities. The tensor hierarchy is required by consistency of the procedure and it has to be compatible with the original field content of the theory.

The higher rank form fields of the theory are associated with extended objects of the theory, therefore the embedding tensor formalism constitutes an indirect way of determining the branes of the theory [?].

In the following, we describe the procedure for constructing the tensor hierarchy with the help of the embedding tensor formalism. For a more detailed description of the technique see [?, ?].

In a generic field theory in dimension d, with $0 \leq p \leq [\frac{d-1}{2}]$, the bosonic degrees of freedom are given by a set of (electric) p-form potentials with $p \geq 0$, all of them satisfying a certain number of second order differential equations. As a general rule, a p-form couples electrically to an electric $(p-1)$ brane and the dual $(d-p-2)$ form potential (which does not contribute with new degrees of freedom [19]) couples electrically to a magnetic $(d-p-3)$ brane.

The curvatures of the electric and magnetic forms satisfy Bianchi identities and therefore the equations of motion (second order equations) can be derived as integrability conditions of the duality relations. Schematically:

$$ \text{Bianchi identities} + \text{duality relations} \Leftrightarrow \text{equations of motion} $$

We consider a theory with a given number of (electric) p-forms to which we add their dual $(d-p-2)$-forms. We also assume that the theory has a number of scalar fields ϕ^i, that parametrize a target space with metric $\mathcal{G}_{ij}$. We start with the lowest rank form fields: the vectors (1-forms) A^I, and the 2-forms B_x. As mentioned before, depending on the dimension, these fields will transform in different representations of the global symmetry group: in $d = 4$ dimensions, I will be a symplectic index labeling the electric and magnetic vector fields and x will be an adjoint index because in $d = 4$ dimensions 2-forms are dual to the Noether currents of the global symmetry group, which carry an adjoint index; in

[18] One example is the matter coupled $N = 1, d = 4$ supegravity [?]. In other cases, specially in maximal and half maximal supergravities all deformation parameters depend on the embedding tensor.

[19] The magnetic forms are related to the electric forms through a (first order) duality relation.

$d = 5$ vectors are dual to 2-forms and the indices I and x must be identified; in $d = 6$ x will be a $O(n, n)$ index labelling the electric and magnetic 2-forms etc.

By assumption, the equations of motion of the theory are invariant under a global symmetry group with generators $\{T_A\}$ satisfying the algebra given in eq.(**??**). The corresponding matrices that represent the generators are $T_A{}^I{}_J$ and $T_A{}^x{}_y$.., then

$$\begin{aligned} \delta_\alpha A^I = &\ \alpha^A T_A{}^I{}_J A^J = &\ \alpha^A \delta_A A^I, \qquad (1.120) \\ \delta_\alpha B_x = &\ -\alpha^A T_A{}^y{}_x B_y = &\ \alpha^A \delta_A B_x. \qquad (1.121) \end{aligned}$$

where α^A are constants.

The transformation rules for the scalars are isometries of $\mathcal{G}_{ij}(\phi)$ generated by the Killing vectors $k_A{}^i(\phi)$, they satisfy the algebra

$$[k_A, k_B] = -f_{AB}{}^C k_C, \qquad (1.122)$$

then

$$\delta_\alpha \phi^i = \alpha^A k_A^i = \alpha^A \delta_A \phi^i, \qquad (1.123)$$

We will assume that each of the different form fields transform under Abelian gauge transformations, namely

$$\begin{aligned} \delta_\sigma A^I &= d\sigma^I, \qquad (1.124) \\ \delta_\sigma B_x &= d\sigma_x, \quad \text{etc.} \qquad (1.125) \end{aligned}$$

where the σ^I are 0-forms, the σ_x are 1-forms, etc.

Now, if we want to gauge the theory promoting the global parameters α^A to local functions $\alpha^A(x)$ depending of the spacetime coordinates we must identify them with some of the introduced σ^I, or equivalently, some of the 1-forms A^I should be identified with the gauge fields A^A associated with the transformations α^A. The embedding tensor allows us to make these identifications explicit:

$$\alpha^A(x) = \vartheta_I^A \sigma^I(x), \quad A^A = \vartheta_I{}^A A^I. \qquad (1.126)$$

The quadratic constraint eq.(**??**) in this notation becomes

$$-X_I{}^K{}_J \vartheta_K{}^A + \vartheta_I{}^B \vartheta_J{}^C f_{AB}^{\ \ C} = 0, \qquad (1.127)$$

where

$$X_I{}^K{}_J = \vartheta_I{}^B T_B{}^K{}_J. \qquad (1.128)$$

If the quadratic constraint is satisfied, then,

$$[X_I, X_J] = X_{[I}{}^K{}_{J]} X_K \qquad (1.129)$$

When we introduce more deformation tensors we will get more constraints quadratic in the deformation tensors, involving the embedding tensor.

The transformations of the scalars are

$$\delta_\sigma \phi^i = \sigma^I \vartheta_I{}^A k_A{}^i. \qquad (1.130)$$

The covariant derivative for the scalars is

$$\mathcal{D}\phi^i = d\phi^i - A^I \vartheta_I{}^A k^i_A. \tag{1.131}$$

If the 1-forms transform as:

$$\delta_\sigma A^I = \mathcal{D}\sigma^I + \Delta A^I, \tag{1.132}$$

where

$$\mathcal{D}\sigma^I = d\sigma^I - A^J X_J{}^I{}_K \sigma^K, \tag{1.133}$$

and ΔA^I is a possible additional term which is annihilated by the embedding tensor $\vartheta_I{}^A \Delta A^I = 0$. Then, the covariant derivative given in (**??**) transforms covariantly

$$\delta_\sigma \mathcal{D}\phi^i = \sigma^I \vartheta_I{}^A \partial_j k_A{}^i \mathcal{D}\phi^j. \tag{1.134}$$

The next field strength to be considered are those of the 1-forms, and the most direct way of building them is through the use of the Ricci identities

$$\mathcal{D}^2\phi^i = -F^I \delta_I \phi^i. \tag{1.135}$$

Applying the covariant derivative to (**??**) we find

$$\mathcal{D}^2\phi^i = d\mathcal{D}\phi^i - A^I \delta_I \mathcal{D}\phi^i. \tag{1.136}$$

Comparing (**??**) and (**??**) we can see that

$$F^I = dA^I - \frac{1}{2} X_J{}^I{}_K A^J \wedge A^K + \Delta F^I, \tag{1.137}$$

where, as before, ΔF^I is a possible additional term that is annihilated by the embedding tensor $\vartheta_I{}^A \Delta F^I = 0$. We want F^I to be a gauge covariant field strength (up to a terms annihilated by $\vartheta_I{}^A$), but it transforms as follows:

$$\delta_\sigma F^I = \sigma^J X_J{}^I{}_K F^K - 2X_{(J}{}^I{}_{K)} \left(\sigma^J F^K - \frac{1}{2} A^J \wedge \delta_\sigma A^K \right) + \mathcal{D}\Delta A^I + \delta_\sigma \Delta F^I. \tag{1.138}$$

Thus, we need to add ΔF in such a way that its gauge transformation $\delta_\sigma \Delta F^I$ annihilates the unwanted terms in (**??**). ΔF^I has to be a linear combination of the 2-forms B_x

$$\Delta F^I = Z^{Ix} B_x, \tag{1.139}$$

where Z^{Ix} is a new deformation tensor annihilated by the embedding tensor

$$Q^{Ax} \equiv Z^{Ix} \vartheta_I{}^A = 0. \tag{1.140}$$

The Z^{Ix} tensor must also be gauge invariant. Therefore, a new quadratic constrain appears[20]

$$Q_I{}^{Jx} \equiv X_I{}^K{}_J Z^{Kx} + X_I{}^x{}_y Z^{Ky} = 0, \qquad X_I{}^x{}_y \equiv \vartheta_I{}^B T_B{}^x{}_y. \tag{1.141}$$

[20]In $d = 4$ dimensions, since x is an adjoint index, Z^{Ix} has essentially the same indices as the embedding tensor and the simplest possibility will be to identify them. In $d = 5$ dimensions the two indices I and x are of the same type and one has to take into account the symmetry of Z^{Ix} under the interchange of its two indices.

The relation (**??**) is an orthogonality constraint. All the deformation tensors satisfy a quadratic constraint associated to their being gauge-covariant, and some of them will satisfy orthogonality constraints as well. The terms in (**??**) that are annihilated by the embedding tensor must be proportional to the new deformation tensor Z^{Ix}. Since B_x has its own Abelian transformations with 1-form parameter σ_x, we set

$$\Delta A^I = -Z^{Ix}\sigma_x, \tag{1.142}$$

and we demand that a tensor d_{xJK} must exist satisfying

$$X_{(J}{}^{I}{}_{K)} = Z^{Ix}d_{xJK}. \tag{1.143}$$

The transformation for B_x is

$$\delta_\sigma B_x = \mathcal{D}\sigma_x + 2d_{xJK}\left(\sigma^J F^K - \frac{1}{2}A^J \wedge \delta_\sigma A^K\right) + \Delta B_x, \tag{1.144}$$

where ΔB_x is a possible new deformation annihilated by Z^{Ix}. The vector gauge transformations associated to the terms $Z^{Ix}\sigma_x$ are called *massive* or *Stückelberg* gauge transformations, and they can be used to fix to a expected value (sometimes eliminate) some of the 1-forms. The 1-forms being annihilated are called Stückelberg fields. The terms $Z^{Ix}B_x$ are the Stückelberg couplings and will become mass terms for the 2-forms. The tensors Z^{Ix} are called massive deformations. Depending on the dimensions, Z^{Ix} and d_{xIJ} will have different properties.

In order to find the covariant field strength of the 2-forms B_x we proceed in a similar way. If we apply de covariant derivative to the Ricci identities eq. (**??**), we find that a requirement for gauge invariance of $\mathcal{D}^2\phi$ is that the covariant derivative of F^I eq. (**??**) must be zero up to a term annihilated by $\vartheta_I{}^A$. Thus, the resulting term must be a 3-form that transforms covariantly, and it must be proportional to Z^{Ix}. We get the following Bianchi identity

$$\mathcal{D}F^I - Z^{Ix}H_x = 0, \tag{1.145}$$

where H_x is the 3-form field strength that we were looking for. After calculating explicitly $\mathcal{D}F^I$, we obtain H_x up to a term ΔH_x which is annihilated by Z^{Ix}, which, in its turn, has to be a linear combination $Z_{xa}C^a$ of 3-forms C^a where the new deformation tensor satisfies $Z^{Ix}Z_{xa} = 0$.

Again, depending on the dimension, the index a carried by the 3-forms will have different properties. In $d = 4$ it will be the same index carried by the deformation tensors, in $d = 5$ it will be an adjoint index (3-forms are dual to the Noetehr currents in $d = 5$ and in $d = 6$ it will be the same index as the vector fields, because they are dual.

The gauge transformation of C^a can be found by requiring gauge covariance of H_x up to terms annihilated by by the new deformation tensor. The Bianchi identity for H_x has the form

$$\mathcal{D}H_x + d_{xIJ}F^I \wedge F^J - Z_{xa}G^a = 0, \tag{1.146}$$

where G^a are the 4-forms field strengths.

As we continue building the higher rank field strengths we will be collecting constraints (linear and quadratic) that our new tensors have to satisfy. At some point, we will reach the highest rank potentials of the tensor hierarchy, and they will have to be approached in a different way.

Schematically, we can denote by $c^\sharp$ all deformation tensors, including the embedding tensor, where $\sharp$ denotes all the corresponding indices. The magnetic duals of the 1-forms will be the $(d-3)$-forms $\tilde{A}_I$. Those forms have $(d-2)$-form field strengths $\tilde{F}_I$. They will contain in general a Stückelberg coupling to a $(d-2)$-form, that we call C_A, and the coupling tensor is going to be the embedding tensor $\vartheta_I{}^A$:

$$\tilde{F}_I = \mathcal{D}\tilde{A}_I + ... + \vartheta_I{}^A C_A. \tag{1.147}$$

If we take the covariant derivative of the above expression, we find the field strength (a $(d-1)$-form) of C_A, which we call G_A, up to terms that are annihilated when contracted with $\vartheta_I{}^A$. These extra terms in G_A form Stückelberg couplings for the $(d-1)$-form potentials. The coupling tensors will vanish upon contraction of the adjoint index with the embedding tensor. We will construct them in the following way.
All the deformation tensors must be gauge-invariant tensors, and if their gauge transformations are written as

$$\delta_\Lambda c^\sharp = -\Lambda^I Q_I{}^\sharp, \tag{1.148}$$

where $\Lambda^I(x)$ are the 0-form gauge-transformation parameters of the 1-forms A^I. Then, the following constraint has to be satisfied

$$Q_I{}^\sharp \equiv \delta_I c^\sharp = 0, \tag{1.149}$$

by each of them. All the constraints are, by construction, proportional to the embedding tensor

$$\delta_I c^\sharp = \vartheta_I{}^A \delta_A c^\sharp, \tag{1.150}$$

and can be written in the form

$$Q_I{}^\sharp = -\vartheta_I{}^A Y_A{}^\sharp, \tag{1.151}$$

which gives us as many tensors $Y_A^\sharp$ as deformation tensors $c^\sharp$ we have . Thus, the $(d-1)$-form field strengths will have the form

$$G_A = \mathcal{D}C_A + ... + \sum_\sharp Y_A{}^\sharp D_\sharp, \tag{1.152}$$

where, again, we have introduced as many $(d-1)$- form potentials $D_\sharp$ as deformation tensors $c^\sharp$ we have, transforming in the representation conjugate to the representation in which the $c^\sharp$ transform. The deformation tensors $Y_A{}^\sharp$ are

$$Y_A{}^\sharp \equiv \delta_A c^\sharp, \qquad \mathcal{Q}_I{}^\sharp = -\vartheta_I^A Y_A{}^\sharp = 0 \tag{1.153}$$

The $(d-1)$-form potentials are dual to all deformation tensors $c^\sharp$. For consistency, to be included in the action, the deformation parameters $c^\sharp$ have to be promoted to fields $c^\sharp(x)$ constrained to be constant. The constancy of the deformation tensors $c^\sharp$ is implemented in the action by the term

$$\int \sum_\sharp dc^\sharp \wedge D_\sharp. \tag{1.154}$$

where the $(d-1)$-form potentials $D_\sharp$ play the role of Lagrange multipliers.

Now, the d-form field strengths $K_\sharp$ of the $(d-1)$-form potentials $D_\sharp$ will have Stückelberg couplings to d-form potentials $E_\flat$ and therefore

$$K_\sharp = \mathcal{D}D_\sharp + ... + \sum_\flat W_\sharp^\flat E_\flat. \tag{1.155}$$

where the deformation tensors $W^{\flat}_{\sharp}$ are annihilated by the $Y^{\sharp}_{A}$. The d-form potentials are associated to all the constraints $Q^{\flat}$, and can be understood as Lagrange multipliers enforcing them in the action via term of the form $Q^{\flat}E_{\flat}$. Calculating $W^{\flat}_{\sharp}$ is difficult. However, we can circumvent the problem assuming that the constraints can be satisfied and

$$\delta_A(\mathcal{Q}^{\flat}E_{\flat}) = \left(Y^{\sharp}_{A}\frac{\partial \mathcal{Q}^{\flat}}{\partial c^{\sharp}}\right)E_{\flat} = 0. \tag{1.156}$$

From the previous equation, and since the equation has to be satisfied for arbitrary $E_{\flat}$, we have

$$Y^{\sharp}_{A}\frac{\partial \mathcal{Q}^{\flat}}{\partial c^{\sharp}} = 0, \tag{1.157}$$

so that we can make the identification

$$W_{\sharp}{}^{\flat} \equiv \frac{\partial \mathcal{Q}^{\flat}}{\partial c^{\sharp}}. \tag{1.158}$$

Four-dimensional theories

As an example let us take another look to the 4-dimensional theories [?]. We take n electric vectors A^{Λ}_{μ} which can be combined with the magnetic vectors $A_{\Lambda\mu}$ in a symplectic contravariant vector A^{M}_{μ}, where M labels the fundamental representation of $Sp(2n,\mathbb{R})$. In this case, $x \to A$ and $I, J \to M, N$ with $T_A{}^P{}_N \in Sp(2n,\mathbb{R})$. The most economical choices, in the sense that they do not require the introduction of new independent tensors apart from $\vartheta_N{}^A$ and $T_A{}^M{}_N$, are:

$$Z^{MA} = \Omega^{NM}\vartheta_N^{A}, \qquad d_{AMN} = -\frac{1}{2}T_{AMN} = -\frac{1}{2}T_A{}^P{}_N\Omega_{MP}, \tag{1.159}$$

where Ω_{MN} is the symplectic metric

$$\Omega_{MN} = \begin{pmatrix} 0 & \mathbf{1}_{n\times n} \\ \mathbf{1}_{n\times n} & 0 \end{pmatrix}, \qquad \Omega^{MN}\Omega_{NP} = -\delta^{M}_{P}, \tag{1.160}$$

and its inverse defined by

$$\Omega^{MN}\Omega_{NP} = -\delta^{M}_{P}, \tag{1.161}$$

In total, there are three constraints that the embedding tensor must satisfy in order to guarantee the consistency of the theory. First, it has to satisfy the following quadratic constraint

$$Q^{AB} \equiv \frac{1}{4}\vartheta^{M[A}\vartheta^{B]}_{M} = 0, \tag{1.162}$$

which guarantees that the magnetic and electric gaugings are mutually local. Moreover, the constraint (**??**) together with the antisymmetry of Ω^{MN} implies

$$Z^{MA}\vartheta_M{}^B = \vartheta^{MA}\vartheta_M^{\ B} = 0, \tag{1.163}$$

which is a property of the 4 dimensional case. Second, the quadratic constraint which tell us that the embedding tensor has to be gauge invariant. (see eq.(**??**))

$$Q_{NM}^{A} \equiv -\vartheta_N^{A}\left(T_A{}^P{}_M\vartheta_P{}^A - \vartheta_M{}^B f_{AB}{}^A\right) = 0. \tag{1.164}$$

Finally, the third constraint applies to all 4-dimensional supergravities that are free of gauge anomalies and can be expressed

$$L_{MNP} \equiv X_{(MNP)} = 0. \tag{1.165}$$

These three constraints are related by

$$Q^{A}{}_{(MN)} - 3L_{MNP}Z^{PA} - 2Q^{AB}T_{BMN} = 0. \tag{1.166}$$

In chapter **??** we will present the construction of the most general 8-dimensional theory with gauge invariance, for any field content and duality group. We study the most general gaugings of its global symmetries and the possible massive deformations using the embedding tensor formalism, constructing the complete tensor hierarchy using the Bianchi identities. In chapter **??** we particularize it to the field content, d-tensors and duality group of the maximal 8-dimensional supergravity and we focus on a family of $SO(3)$ gaugings.

1.1.5 A gentle introduction to Supergravity

Supergravity theories are theories which contain bosonic (B) and fermionic fields (F) and are invariant under some local supersymmetry transformations, generated by the spinors ϵ, which take the generic form

$$\begin{aligned} \delta_\epsilon B &\approx \bar{\epsilon} F, \\ \delta_\epsilon F &\approx B\epsilon + \partial\epsilon. \end{aligned} \tag{1.167}$$

If the fermionic parameters can be arbitrary functions of the spacetime coordinates this is a local fermionic symmetry. We can have a theory invariant under similar transformations generated by fermions, but only when the algebra of these fermionic transformations closes on local bosonic transformations including diffeomorphisms we can say that there is *local supersymmetry*, and the theory that enjoys this invariance is really a *supergravity theory* (SUGRA)[21] [**?**, **?**].

Local supersymmetry demands the presence of a gauge field which has to be fermionic and carries the same ("adjoint") indices as the local supersymmetry parameter $\epsilon(x)$:, Ψ_μ, the gravitino field. Moreover, invariance under diffeomorphisms demands another "gauge field": the metric $g_{\mu\nu}$.

Supergravity theories are a generalization of the theory of General Relativity. In a classical field theory, the B fields transform under the tensorial representations of the Lorentz group $SO(1,3)$, but the F fields transform under spinorial representations of $Spin(1,3)$, the universal cover of $SO(1,3)$. Therefore, contrary to what happens with the bosonic fields, they cannot be identified with a section of either the tangent or the cotangent bundle of the spacetime manifold $\mathcal{M}$. The problem is solved by introducing a new mathematical structure: a spin bundle. The fermions are going to be sections of this bundle. Then, if $\mathcal{M}$ admits this structure, we can use the Weyl-Cartan-Sciama-Kibble formalism. In this formalism, instead of using the metric g as the dynamic field associated

[21] It differs from κ–symmetry which is a fermionic symmetry, and in most models where it occurs, a gauge field is not necessary for its realization. κ –symmetry is typically an on-shell symmetry and consequently, there is no simple way to construct higher order κ–invariants. κ–symmetric actions are basically sigma models in which the target space is a superspace [**?**].

to gravity we use the Vierbein e. Both fields are related by $g = \eta(e, e)$, where η is the flat metric [**?**, **?**, **?**].

With the use of the Vierbein, the theory can be written as manifestly invariant under local $Spin(1, 3)$ transformations. We also introduce the spin connection ω which is considered an auxiliary field of the theory (its equation will be a constraint to the other fields). Now, the kinetic terms of the spinors can be written using the covariant derivative $\mathcal{D} \approx \partial + \omega$.. The Einstein-Hilbert action can be written in the form

$$S_{EH}(e, \omega) = \int \star R(w) \wedge e \wedge e. \tag{1.168}$$

The fields of supergravity theories are combined in *supermultiplets*, and they transform as representations of the super-Poincaré algebra. There is no a unique super-Poincaré algebra in general (except in $d = 11$) and the different possibilities give rise to different supergravities. The simplest way to classify them is as follows:

- By the dimension (and the signature if we are interested in exotic possibilities) of the spacetime. This determines the spinorial representations available. In Lorentzian signature $d \leq 11$, because, otherwise, there could be more than one graviton or higher spin fields.

- By the kind of spinors used to construct the supersymmetry parameters. These can be composed by more than one of the smallest spinorial representations. In odd dimensions, this is just a number $\mathcal{N}$. In even dimensions, there are chiral (Weyl) spinors but not all theories treat both chiralities independently. Non-chiral theories are denoted by a number $\mathcal{N}$ while chiral theories are denoted by two numbers $(\mathcal{N}_+, \mathcal{N}_-)$. $\mathcal{N}$ is limited by the condition that there are no more than 32 independent components of $\epsilon(x)$, (There is no more 32 supercharges on the Poincaré super algebra.) For example, in $d = 11$ we have $\mathcal{N} = 1$, in $d = 4$ we have $\mathcal{N} \leq 8$.

- By the multiplets they contain. By definition, all of them contain a supermultiplet that contains the Vielbein and the gravitino, which is called *supergravity (super) multiplet*. The rest are called *matter supermultiplets* by analogy with gravity. The supergravity multiplet contains the bosonic and fermionic fields that could be considered as matter from the simple gravity point of view. Some supergravities admit supermultiplets containing additional gravitini, which can gauge additional supersymmetries. In some cases (depending on the other matter fields) the theory can be reformulated as a theory with higher $\mathcal{N}$, with all gravitini in the SUGRA multiplet, although the limit of independent supersymmetry transformations has to be respected. Thus, the coupling to additional gravitini can be ignored, if we are only interested in classifying different supergravity theories, and we can focus only in matter supermultiplets containing scalars (spin 0), fermions (spin 1/2), p-form fields (spin 1) with $p \leq d - 3$ in d dimensions for a given $\mathcal{N}$

- The same matter multiplets can be coupled in different ways and, in order to define completely a supergravity theory, we have to describe these couplings. In many cases, the most efficient way to do it is through the symmetries of the theory which are allowed by the couplings.

– Theories in which there are no fields charged with respect to gauge symmetries are called ungauged (Abelian vector fields are not charged with respect to their own gauge transformations).

– If there are fields charged under local transformations, the theory is said to be a gauged SUGRA.

Gauged SUGRAS can be obtained from ungauged SUGRAS by gauging, identifying the local gauge parameter with those of the gauge transformations of the vector fields (1-forms) which became gauge fields of the new local symmetry. A systematic way of studying all possible gaugings of a supergravity theory is the embedding tensor formalism presented in previous sections. In general (but not always) gauging a global symmetry while preserving supersymmetry demands the introduction of a *scalar potential* which is interesting for phenomenological reasons. Obviously, the presence of gauge symmetries is also most interesting from the phenomenological point of view.

The symmetries of Supergravity theories

The local symmetries of ungauged SUGRAS are general coordinate transformations, local supersymmetry transformations, local Lorentz transformations and the $U(1)$ gauge transformations of the p-form fields. The global symmetries are

- $\mathcal{R}$-symmetry: All SUGRA's with a given d and $\mathcal{N}$ have a global symmetry that only acts on the fermion fields. Furthemore, other symmetries of the theories act on fermions via induced $\mathcal{R}$- symmetry transformations. For instance, in $d = 4$, the smallest spinors are Weyl or Majorana. The $\mathcal{R}$-symmetry group $U(\mathcal{N})$ acts naturally in the fundamental representation over complex Weyl spinors.

- Field redefinitions of the scalar fields. Since scalar fields can be seen as embedding coordinates in some space M_ϕ, $x^\mu \to \phi^i(x) \in M_\phi$, these field redefinitions can be seen as coordinate transformations in M_ϕ ($\phi^{i'} = \phi'^i(\phi)$). The transformations of this kind that are symmetries of a SUGRA are those that preserve certain structures. The most basic of them, present in all SUGRAs, is the metric of the scalar manifold M_ϕ, $\mathcal{G}_{ij}$. This metric is used to construct the kinetic terms of the scalar fields. The metric will be preserved if and only if the transformations are isometries of $\mathcal{G}_{ij}$. For low $\mathcal{N}$ and d, these metrics need not have any isometries, but all of them must admit a sort of bundle structure in which the $\mathcal{R}$-symmetry group plays an important role. For $\mathcal{N} \geq 3$ in $d = 4$, for $\mathcal{N} \geq 2$ in $d = 5$ and for any $\mathcal{N}$ in $d \geq 6$ all the M_ϕ's are Riemannian symmetric spaces.

In the following, we introduce particular examples of supergravity theories in different dimensions which are of interest for the results of this thesis.

11-dimensional supergravity

In this section we describe a few aspects of the the 11-dimensional supergravity. In eleven dimensions the Dirac matrices have $n = 32 \times 32$ dimensions, then it is easily seen that the field theory content of the $d = 11$ theory is the following:

- The Elfbein[22] e^a_μ
- A Majorana fermion[23] Ψ^μ with spin 3/2
- A completely antisymmetric gauge tensor[24] $C_{\mu\nu\rho}$

The bosonic fields of $N = 1, d = 11$ supergravity are the Elfbein and a the 3-form potential[25]

$$\left\{ \hat{\hat{e}}_{\hat{\hat{\mu}}}{}^{\hat{\hat{a}}}, \hat{\hat{C}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}\hat{\hat{\rho}}} \right\} . \tag{1.169}$$

The field strength of the 3-form is

$$\hat{\hat{G}} = 4\,\partial\,\hat{\hat{C}}\,, \tag{1.170}$$

which is invariant under the gauge transformation

$$\delta\,\hat{\hat{C}} = 3\,\partial\,\hat{\hat{\chi}}\,, \tag{1.171}$$

where $\hat{\hat{\chi}}$ is a 2-form.

The action for these bosonic fields is

$$\hat{\hat{S}} = \int d^{11}\hat{\hat{x}}\sqrt{|\hat{\hat{g}}|}\left[\hat{\hat{R}} - \tfrac{1}{2\cdot 4!}\,\hat{\hat{G}}^2 - \tfrac{1}{6^4}\tfrac{1}{\sqrt{|\hat{\hat{g}}|}}\,\hat{\hat{\epsilon}}\partial\hat{\hat{C}}\partial\hat{\hat{C}}\,\hat{\hat{C}}\right] . \tag{1.172}$$

The action (**??**) is invariant under Abelian gauge transformations of the three form, which together with the requirement of the absence of terms with more than two derivatives implies that the action is polynomial in $\hat{\hat{C}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}\hat{\hat{\rho}}}$. We are interested in different types of compactifications of the 11-dimensional supergravity to eight dimensions.

The AAMO supergravity

We present the results of compactification, using standard dimensional reduction on T^3, of the 11-dimensional supergravity (**??**). We closely follow [**?**].
The KK Ansatz for the Elfbein is given by

$$\left(\hat{\hat{e}}_{\hat{\hat{\mu}}}{}^{\hat{\hat{a}}}\right) = \begin{pmatrix} e_\mu{}^a & e_m{}^i A^m{}_\mu \\ 0 & e_m{}^i \end{pmatrix} , \qquad \left(\hat{\hat{e}}_{\hat{\hat{a}}}{}^{\hat{\hat{\mu}}}\right) = \begin{pmatrix} e_a{}^\mu & -A^m{}_a \\ 0 & e_i{}^m \end{pmatrix} , \tag{1.173}$$

where $A^m{}_a = e_a{}^\mu A^m{}_\mu$. Moreover, the internal metric on T^3 is defined as:

$$G_{mn} = e_m{}^i e_{n\,j} = -e_m{}^i e_n{}^j \delta_{ij}\,. \tag{1.174}$$

For future convenience, we label the KK vector with an upper index 1, *i.e.* $A^{1\,m}{}_\mu$. The 11-dimensional 3-form is decomposed by identifying objects with flat 11- and 8-dimensional

[22]It represents $\frac{(d-2)(d-2)}{2} - 1 = 44$ degrees of freedom.
[23]It represents $\frac{32(d-3)}{2} = 128$ degrees of freedom.
[24]It has 84 components.
[25]Index conventions: $\hat{\hat{\mu}}$ ($\hat{\hat{a}}$) are curved (flat) 11-dimensional, μ (a) are curved (flat) 8-dimensional, and m (i) are curved (flat) 3-dimensional (compact space). The signature is $(+ - \cdots -)$.

flat indices (up to factors coming from the rescaling of the metric). Similarly, the 11-dimensional 4-form field strength can be decomposed into the above 8-dimensional field strengths. Using the above Elfbein Ansatz, one finds

$$
\begin{aligned}
S \;=\; & \int d^8x\sqrt{|g_E|}\,\Big\{R_E + \tfrac{1}{4}\mathrm{Tr}\left(\partial\mathcal{M}\mathcal{M}^{-1}\right)^2 + \tfrac{1}{4}\mathrm{Tr}\left(\partial\mathcal{W}\mathcal{W}^{-1}\right)^2 \\
& -\tfrac{1}{4}F^{i\,m}\mathcal{M}_{mn}\mathcal{W}_{ij}F^{j\,n} + \tfrac{1}{2\cdot 3!}H_m\mathcal{M}^{mn}H_n - \tfrac{1}{2\cdot 4!}e^{-\varphi}G^2\,, \\
& -\tfrac{1}{6^3\cdot 2^4}\frac{1}{\sqrt{|g_E|}}\,\epsilon\,\Big[GGa - 8GH_mA^{2\,m} + 12G(F^{2\,m} + aF^{1\,m})B_m \\
& -8\epsilon^{mnp}H_mH_nB_p - 8G\partial aC - 16H_m(F^{2\,m} + aF^{1\,m})C\Big]\Big\}\;.
\end{aligned}
\tag{1.175}
$$

where the symmetric $Sl(2,\mathbb{R})/SO(2)$ matrix is given by

$$
\mathcal{W} = \frac{1}{\Im\mathrm{m}(\tau)}\begin{pmatrix} |\tau|^2 & \Re\mathrm{e}(\tau) \\ \Re\mathrm{e}(\tau) & 1 \end{pmatrix}, \tag{1.176}
$$

where τ is the complex combination

$$
\tau = a + ie^{-\varphi}. \tag{1.177}
$$

where

$$
F^{1\,m} = 2\partial A^{1\,m}\,, \tag{1.178}
$$

and where we have rescaling the resulting 8-dimensional metric to the Einstein frame $g_{\mu\nu} = e^{\varphi/3}g_{E\;\mu\nu}$. The kinetic term for $\mathcal{M}$ is an $Sl(3,\mathbb{R})/SO(3)$ sigma model and the fields arising from $\hat{\hat{C}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}\hat{\hat{\rho}}}$ are $\{C_{\mu\nu\rho}, B_{\mu\nu\,m}, A^{2\,m}{}_{\mu}, a\}$. These field strengths inherit the following gauge transformations from 11-dimensional gauge and general coordinate transformations of $\hat{\hat{C}}$:

$$
\begin{aligned}
\delta C &= 3\partial\Lambda - 6A^{1\,m}\partial\Lambda_m + 3\epsilon_{mnp}A^{1\,m}A^{1\,n}\partial\Lambda^{2\,p}\,, \\
\delta B_m &= 2\partial\Lambda_m - 2\epsilon_{mnp}A^{1\,n}\partial\Lambda^{2\,p}\,, \\
\delta A^{2\,m} &= \partial\Lambda^{2\,m}\,.
\end{aligned}
\tag{1.179}
$$

The gauge-invariant field strengths of the above fields are

$$
\begin{aligned}
G &= 4\partial C + 6F^{1\,m}B_m\,, \\
H_m &= 3\partial B_m + 3\epsilon_{mnp}F^{1\,n}A^{2\,p}\,, \\
F^{2\,m} &= 2\partial A^{2\,m}\,,
\end{aligned}
\tag{1.180}
$$

and lead to the following non-trivial Bianchi identities:

$$
\begin{aligned}
\partial G &= 2F^{1\,m}H_m\,, \\
\partial H_m &= \tfrac{3}{2}\epsilon_{mnp}F^{1\,n}F^{2\,p}\,,
\end{aligned}
\tag{1.181}
$$

The kinetic terms (except for that of C) are explicitly invariant under $Sl(2,\mathbb{R})$ transformations

$$\mathcal{W}' = \Lambda\mathcal{W}\Lambda^T\,, \qquad F^{i\,m\,\prime} = F^{j\,m}\left(\Lambda^{-1}\right)_j{}^i\,, \qquad \Lambda\in Sl(2,\mathbb{R})\,, \tag{1.182}$$

and $Sl(3,\mathbb{R})$ transformations

$$\mathcal{M}' = K\mathcal{M}K^T\,, \quad F^{i\,m\,\prime} = F^{i\,n}\left(K^{-1}\right)_n{}^m\,, \quad H'_m = K_m{}^n H_n\,, \quad K\in Sl(3,\mathbb{R})\,. \tag{1.183}$$

The equation of motion for the kinetic term of C together with the Bianchi identity, can be written:

$$\partial G^i = 2F^{i\,m}H_m\,, \tag{1.184}$$

where

$$G^1 \equiv G\,, \qquad G^2 \equiv -e^{-\varphi\star}G - aG\,. \tag{1.185}$$

G^i transforms as a doublet under $Sl(2,\mathbb{R})$ (just like the doublet $F^{i\,m}$) and therefore, the above equation of motion is covariant under $Sl(2,\mathbb{R})$ electric-magnetic duality transformations. The remaining equations of motion are covariant under $Sl(2,\mathbb{R})$ transformations as well.[26] Thus, the 8-dimensional supergravity (AMMO) has the following bosonic fields:

$$\{g_{\mu\nu}, C, B_m, A^{1\,m}, A^{2\,m}, a, \varphi, \mathcal{M}_{mn}\}\,, \tag{1.186}$$

with field strengths given by eqs. (**??**), (**??**) and action given by eq. (**??**). The scalars parametrize $Sl(3,\mathbb{R})/SO(3)$ and $Sl(2,\mathbb{R})/SO(2)$ sigma models. The action has the global invariance group $Sl(3,\mathbb{R})$ but the equations of motion are also invariant under $Sl(2,\mathbb{R})$ electric-magnetic duality transformations.

The SS supergravity

Here we present what we have called the SS supergravity in eight dimensions. In [**?**] Salam and Sezgin compactified the 11-dimensional supergravity (**??**) to 8 dimensions using the Scherk and Schwarz's procedure on a $SU(2)$ internal manifold obtaining a $SU(2)$ gauged $d=8$ supergravity. The gauged theory does not exhibit all the duality symmetries of the ungauged one. In particular, gauging usually break all electric-magnetic dualities.

Using the vielbein ansatz

$$V_M{}^A = \begin{pmatrix} e^{-k\phi/3}e^a_\mu & 0 \\ 2e^{2k\phi/3}A^\alpha_\mu L^i_\alpha & e^{2k\phi/3}L^i_\alpha \end{pmatrix}\,, \qquad V_A{}^M = \begin{pmatrix} e^{k\phi/3}e^\mu_a & -2ke^{2k\phi/3}A^\alpha_\mu e^\mu_\alpha \\ 0 & e^{-2k\phi/3}L^\alpha_i \end{pmatrix}\,, \tag{1.187}$$

where $\mu, a = 0,1,...,7$ and $\alpha, i = 8,9,10$ and L^i_α is the unimodular matrix ($\det L^i_\alpha = 1$) that represents the five scalars of the theory. ϕ is the sixth scalar arising from the gravity sector. The objects defined in (**??**) have the following dependences in the coordinates

$$e^a_\mu = e^a_\mu(x), \tag{1.188}$$

$$A^\alpha_\mu(x,y) = (U^{-1})^\alpha(y)A^\beta_\mu(x), \tag{1.189}$$

$$L^i_\alpha(x,y) = U^\beta_\alpha L^i_\beta(x), \tag{1.190}$$

[26] The structures are very similar to those of $N=4, d=4$ supergravity), the obvious difference being that in four dimensions we dualize 2-form field strengths and in eight dimensions we dualize 4-form field strengths.

where x labels the $d = 8$ spacetime $(x_0, x_1, ..x_7)$, and y the remaining three coordinates (x_8, x_9, x_{10}). The matrix U_α^β is a 3×3 invertible matrix, this matrix is taken to be an $SU(2)$ group element. L_α^i can be considered as a representative of the $SL(3, \mathbb{R})/SO(3)$ coset. The bosonic action is

$$
\begin{aligned}
S = \int \frac{1}{4}R - \frac{1}{4}e^{2\phi}F^\alpha_{\mu\nu}F^{\mu\nu\beta}g_{\alpha\beta} - \frac{1}{4}P_{\mu ij}P^{\mu ij} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-4\phi}(\partial_\mu B)^2 \\
-\frac{1}{16}g^2 e^{-2\phi}(T_{ij}T^{ij} - \frac{1}{2}T^2) - \frac{1}{48}e^{2\phi}G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma} - \frac{1}{2}G_{\mu\nu\rho i}G^{\mu\nu\rho i} \\
-\frac{1}{4}e^{-2\phi}G_{\mu\nu i}G^{\mu\nu i} - \frac{1}{24.72}e^{-1}\epsilon^{\nu_1...\nu_8}\left(BG_{\nu_1....}G_{...\nu_8} - 8G_{\nu_1...}G_{..\nu_7 i}B_{\nu_8 i}\right. \\
-8\epsilon^{ijk}G_{\nu_1\nu_2\nu_3 i}G_{\nu_4\nu_5\nu_6 j}B_{\nu_7\nu_8 k} + 12G_{\nu_1..}G_{\nu_5\nu_6 i}B_{\nu_7\nu_8 i} \\
\left.-12G_{\nu_1\nu_2\nu_3 i}G_{\nu_4\nu_5 i}B_{\nu_6\nu_7\nu_8} + 8G_{\nu_1..\nu_4}B_{\nu_5\nu_6\nu_t}\partial_{\nu_8}B\right),
\end{aligned}
\tag{1.191}
$$

where $T^{ij} = L_\alpha^i L_\beta \delta^{\alpha\beta}$ $T = T^{ij}\delta_{ij}$, $e = \det e_\mu^a$. Moreover, $P_{\mu ij} = e_\mu^a P_{aij}$ and it satisfies

$$L_i^\alpha(\partial_\mu \delta_\alpha^\beta - g\epsilon_{\alpha\beta\gamma}A_\mu^\gamma)L_{\beta j} = P_{\mu ij} + Q_{\mu ij} \tag{1.192}$$

$P_{\mu ij}$ is the symmetric and traceless part of the left hand side of the previous equations. The field strengths are

$$F_{\mu\nu}^\alpha = \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha + g\epsilon_{\alpha\beta\gamma}A_\mu^\beta A_\nu^\gamma. \tag{1.193}$$

The Bianchi identities satisfied by the field strengths are

$$\mathcal{D}_{[\lambda}G_{\mu\nu\rho\sigma]} = 4kF^\alpha_{[\lambda\mu}G_{\nu\rho\sigma]\alpha}, \tag{1.194}$$

$$\mathcal{D}_{[\lambda}G_{\mu\nu\rho]\alpha} = 3k\epsilon_{\alpha\beta\gamma}F^\beta_{[\lambda\mu}G_{\nu\rho]\gamma}, \tag{1.195}$$

$$\mathcal{D}_{[\lambda}G_{\mu\nu]\alpha} = 2kF^\alpha_{[\lambda\mu}\partial_{\nu]}B - \frac{1}{3k}gG_{\lambda\mu\nu\alpha}. \tag{1.196}$$

Thus, the bosonic field content of the $\mathcal{N} = 2$, $d = 8$ is given by

$$(g_{\mu\nu}, A_{\mu\nu\rho}, 3A_{\mu\nu}6A_\mu, 7\phi), \tag{1.197}$$

being all bosonic fields real. The spinor field are pseudo Majorana. The automorphism group of the superalgebra is $SU(2)$. The scalars transform as $5 + 1 + 1$ under $SU(2)$.

1.2 Modified theories of gravity

In this section we introduce the theories usually called "modified theories of gravity". After a brief justification of the need to introduce corrections to General Relativity, we proceed to introduce the $f(R)$ theories of gravity. Then, we focus on Lovelock theories of gravity an their properties.

1.2.1 From General relativity to higher order gravities and beyond

A field theory can be described conveniently using a variational principle and Einstein's theory is not an exception. The only independent scalar (constructed from the metric and its derivatives) that is at most second order in the derivatives of the metric is the Ricci scalar.[27] Thus, the simplest candidate for an action is the Einstein-Hilbert action

$$S_{EH} = \int d^4x\sqrt{-g}R. \tag{1.198}$$

This action is quadratic on the first derivatives of the metric up to total derivatives. The equations of motion obtained from (**??**), are the Einstein equations in vacuum

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0 \tag{1.199}$$

The functional (**??**) can accommodate a matter term and a cosmological constant Λ, the resulting action is

$$S_{EH} = \int d^4x\sqrt{-g}(R + 2\Lambda + L_m). \tag{1.200}$$

where L_m is the matter Lagrangian. The equations of motion of the metric coming from this action are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{1.201}$$

where $T_{\mu\nu}$ is the energy-momentum tensor given by

$$T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_M)}{\delta g_{\mu\nu}}. \tag{1.202}$$

The tensors $g_{\mu\nu}$ and $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ are the only two-index tensors that are symmetric and divergence free that can be build from the metric tensor and its derivatives and which are, at most, of second order in those derivatives.[28]

Many of the predictions that follow from the cosmological Einstein Equations (**??**), have been tested experimentally to high accuracy and, therefore, any generalization must reproduce (**??**) in some limit. The best way to ensure this is by adding new terms to the Einstein-Hilbert action, that introduce only small corrections -possibly at high energy or big curvature- to the experiments.

The simplest modifications to GR come from the addition of invariants of higher order in the Riemann curvature tensor. These modifications do not require the introduction of extra fields. Moreover, the modified theories remain dynamical metric theories with matter minimally coupled to the metric, therefore satisfying the Weak Equivalent Principle (WEP). Superstring Theories predict an infinite number of higher-order corrections to the EH action. Their effective gravitational theories are higher-order gravity theories, although only the lowest order corrections are explicitly known. As expected, the AdS/CFT duals of these higher order gravities are more general than those corresponding to GR. The higher order terms modify the behaviour of the gravitational field when curvature is strong, and these corrections will be important when studying inflationary cosmology, spacetime singularities and black hole physics.

[27] The Kretschman scalar $R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$ is another scalar that we can construct from the Riemann tensor that is itself made from second derivatives of the metric.

[28] The formalization of this statement is the Lovelock theorem: In four dimensions any tensor $P^{\mu\nu}$ whose components are function of the metric tensor $g^{\mu\nu}$ and its first and second derivatives (although linear in the second derivatives), and also symmetric and divergence-free, then the only possible for of $P^{\mu\nu}$ is $P^{\mu\nu} = \alpha G^{\mu\nu} + \beta g^{\mu\nu}$ where $G^{\mu\nu}$ is the Einstein tensor and α, β are arbitrary constants.

Historically, only four years after of the appearance of GR, first Weyl [?] and later Eddington [?] proposed new theories by including higher order terms in the action. However it was not until the 60‘s that the scientific community started to focus its attention to extending GR. The reason behind this interest was a theoretical one: as a quantum filed theory, GR is not renormalizable and, therefore, it is not a consistent quantum filed theory. Utiyama and De Witt [?] showed that, in order to be renormalizable at one-loop, GR has to be supplemented by higher curvature terms. Later on, Stelle [?] proved that gravitational actions which include terms quadratic in the curvature tensor are renormalizable, but the price to pay is the loss of unitarity. Moreover, in the context of String Theory, the α' corrections appearing in the effective SUGRA equations of motion, are present in the form of higher-order terms in the Riemannian tensor. All those corrections should become relevant in very strong-coupling regimes and curvatures. Therefore, corrections to GR would become important only near to Planck scales.

On the other hand, a possibility to be explored is that the the dark matter and dark energy problems are solved by modifying GR instead of modifying the energy-momentum tensor (matter and energy content of the Universe).

It is clear that a modification of GR is needed or that, at the very least, exploring the possible modifications of GR is an interesting problem. However, it is very difficult to decide how to deform GR because there are many possible ways to do it and, therefore, many proposals. The higher order gravities we are going to focus on are purely metric theories of gravity. Their actions are of the general form

$$S = \int \sqrt{-g} d^4x f(g_{\mu\nu}, R_{\mu\nu\sigma\rho}, \nabla_{\alpha_1} R_{\mu\nu\sigma\rho}, ..., \nabla_{(\alpha_1 ...} \nabla_{\alpha_m)} R_{\mu\nu\sigma\rho}), \tag{1.203}$$

where f is a scalar function of the metric, the Riemann tensor and its covariant derivatives. Higher-curvature contributions usually will result in higher-derivative operators acting on $g_{\mu\nu}$ which usually lead to field equations of order higher than two.[29] We are going to start with the simplest case: the $f(R)$ theories. Here the effects of strong curvature appear only through the scalar curvature. It is one of the most popular theories for explaining the large-scale phenomena. $f(R)$ theories are easy to handle and ideally suited as a toy theories. There is also another important physical reason for considering them: they seem to be the only ones avoiding the Ostrogradsky instability [?].[30]

1.2.2 f(R) theories of gravity

Here we give a quick introduction to $f(R)$ theories of gravity [?, ?]. The action for these theories is

$$S = \int d^4x \sqrt{-g} f(R), \tag{1.204}$$

where $f(R)$ is some function of the curvature scalar. If we take a series expansion of f we have

$$f(R) = ... + \frac{a_1}{R} - 2\Lambda + R + b_2 R^2 + .. \tag{1.205}$$

where the coefficients a_i and b_j have the needed dimensions, clearly for a linear $f(R)$ we obtain GR.

[29] There are some exceptions to this statement.

[30] The Ostrogradski theorem can be stated in this form: If the higher order time derivative Lagrangian is non-degenerate, there is at least one linear instability in the Hamiltonian of this system. See [?] [?].

Adding a matter term S_M to the action (**??**) we obtain the new action

$$S = \int d^4x\sqrt{-g}f(R) + S_M(g_{\mu\nu}, \psi), \tag{1.206}$$

and (using the second order formalism) the equations of motion for the metric are[31] [**?**]:

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - [\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box]f'(R) = \kappa T_{\mu\nu}, \tag{1.207}$$

where the prime stands for differentiation with respect to the argument, ∇_μ denotes de covariant derivative compatible with the metric,and $\Box \equiv \nabla^\mu\nabla_\mu$. Since R already contains second derivatives of the metric, the e.o.m's (**??**) are going to be of fourth order partial differential equation of the metric.

Taking the trace of equation (**??**) we obtain

$$f'(R)R - 2f(R) + 3\Box f'(R) = \kappa T, \tag{1.208}$$

where $T = g^{\mu\nu}T_{\mu\nu}$. Contrary to what happens in GR where $R = -\kappa T$, here the scalar curvature R is related with T differentially. This implies that $f(R)$ theories will admit a larger variety of solutions than General Relativity. For instance, in $f(R)$ theories $T = 0$ does not imply $R = cte$. In the case of maximally symmetric spaces, the equation (**??**) reduces to

$$f'(R)R - 2f(R) = 0. \tag{1.209}$$

For a given f, the previous equation is an algebraic equation of R, and only if $R = 0$, is a root of (**??**) the equation (**??**) reduces to $R_{\mu\nu} = 0$, giving as a maximally symmetric solution the Minkowski spacetime. If we go a little further and set $R = \Lambda$, where Λ is a constant, then (**??**) reduces to $R_{\mu\nu} = g_{\mu\nu}\frac{\Lambda}{4}$ and the maximally symmetric solution is the so called de Sitter (anti- de Sitter) space, exactly the same solution of GR with cosmological constant.

In $f(R)$ theories matter is minimally coupled to the metric. Then, it can be shown that $T_{\mu\nu}$ is divergence-free on- shell. We can arrive to the same conclusion by checking that the left-hand side of (**??**) is divergence-free which in its turn implies that $\nabla_\mu T^{\mu\nu} = 0$ [**?**].

In 4 dimensions any higher order curvature invariant either gives a pure divergence term (not contributing to the e.o.m) or, adds higher order derivatives to the field equations. In higher dimensions this is no longer true. Lovelock's theorem that asserts the existence of theories containing higher order curvature invariants with second order equations of motion. These theories are called Lovelock theories and we will introduce them in the next section.

1.2.3 Lovelock theories of gravity

One desirable property of any physical theory is that the e.o.m. are of second order in the fundamental variables, and we would like to generalize the Einstein-Hilbert action as much

[31] Here the boundary terms that appear in the variation do not vanish. Contrary to what happens with the Einstein-Hilbert action they do not combine into a total variation and therefore we cannot add a term -the so called Gibbons-Gauss-York term to cancel them. Fortunately, the action includes higher order derivatives of the metric allowing to fix more boundary conditions than just on the metric. This choice of fixing will be relevant for the Hamiltonian formulation of the theory although the field equations will be unaffected.

as we can maintaining that property. In 1971, Lovelock found the most general theories having only the metric as a fundamental field, and with second order field equations [?]. In that sense, Lovelock's theories are the generalisation of GR in higher dimensions and they reduce to GR in four dimensions.

The most general action of the Lovelock theories is

$$S = \frac{1}{16\pi G}\int_{\mathcal{M}} d^D x\sqrt{|g|}\sum_{j=0}^{j_{max}}\lambda_j \mathcal{L}_j, \tag{1.210}$$

where the sum goes up to $j_{max} \leq \frac{D-1}{2}$, the $\sqrt{g}\mathcal{L}_j$'s are the D-dimensional Euler densities of order j given by

$$\mathcal{L}_j = \frac{1}{2^j}\delta^{\mu_1...\mu_{2j}}_{\nu_1...\nu_{2j}} R^{\nu_1\nu_2}_{\mu_1\mu_2}..R^{\nu_{2j-1}\nu_{2j}}_{\mu_{2j-1}\mu_{2j}}, \tag{1.211}$$

and the generalized Kronecker δ function is defined as

$$\delta^{\mu_1...\mu_j}_{\nu_1...\nu_j} = j!\delta^{[\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}...\delta^{\mu_j]}_{\nu_j}. \tag{1.212}$$

In particular

$$\begin{aligned}
\mathcal{L}_0 &= 1, && (1.213)\\
\mathcal{L}_1 &= R, && (1.214)\\
\mathcal{L}_2 &= R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, && (1.215)\\
\mathcal{L}_3 &= R^3 - 12RR_{\mu\nu}R^{\mu\nu} + 16R_{\mu\nu}R^{\mu}_{\sigma}R^{\nu\sigma} + 24R_{\mu\nu}R_{\rho\sigma}R^{\mu\nu\rho\sigma} \\
&\quad +3RR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 24R_{\mu\nu}R^{\mu}{}_{\rho\sigma\theta}R^{\nu\rho\sigma\theta} + 4R_{\mu\nu\rho\sigma}R^{\mu\nu\theta\omega}R^{\rho\sigma}{}_{\theta\omega} \\
&\quad -8R_{\mu\nu\rho}{}^{\sigma}R^{\mu\theta\rho\omega}R^{\nu}{}_{\theta\sigma\rho\omega}. && (1.216)
\end{aligned}$$

$\mathcal{L}_0$ is a cosmological term, $\mathcal{L}_1$ is the Einstein-Hilbert Lagrangian and $\mathcal{L}_2$ is the Gauss-Bonnet one[32]. $\mathcal{L}_j$ vanishes identically for $j > [D/2]$ (for $j > D/2$ when D is even and for $j > (D-1)/2$ for odd D). The coupling constants λ_j have dimensions $[L]^{2n-D}$. The equations of motion derived when varying the action (**??**) are

$$\sum_{j=0}^{[D/2]}\lambda_n\Lambda_o^{1-j}\mathcal{E}^j_{\mu\nu} = 0, \tag{1.217}$$

where

$$\mathcal{E}^j_{\mu\nu} = -\frac{1}{2^{j+1}}g_{\alpha\mu}\delta^{\alpha\mu_1....\mu_{2j}}_{\nu\nu_1...\nu_{2j}} R^{\nu_1\nu_2}_{\mu_1\mu_2}...R^{\nu_{2j-1}\nu_{2j}}_{\mu_{2j-1}\mu_{2j}}, \tag{1.218}$$

$\Lambda_o^{-1/2}$ is some length scale and where $\lambda_i' s$ are dimensionless couplings

The Lovelock theories admit a generalization of Birkhoff's theorem [?, ?]. The lapse function of the black hole solution can be determined up to a polynomial equation (the so called Wheeler's polynomial [?]), and each of the solutions of the polynomial has an asymptotic behaviour that matches one of the possible maximally symmetric vacua of the theory [?, ?]. The Lovelock theories [?, ?] coincide with Einstein theory in 3 and

[32] $\mathcal{L}_1 + \mathcal{L}_2$ is the Lagrangian for the Einstein-Gauss-Bonnet gravity. This theory is renormalizable at all orders in perturbation theory, although they have ghosts. This problem can be circumvented when we confine the loss of unitarity in such a way that our theory becomes an effective field theory.

4 dimensions [**?**]. In $D > 4$ the Lovelock action includes the Einstein Hilbert action, therefore the GR is a particular case of it.

Moreover, as we showed, Lovelock gravities contain the quadratic Gauss-Bonnet term. This theory presents a very good scenario to study the corrections at short distance to GR due to the presence of higher curvature terms.

2

The tensor hierarchy of 8-dimensional field theories

This chapter is based on

Oscar Lasso Andino, Tomás Ortín ,
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Over the last years, a great effort has been made to explore the most general field theories. This exploration has been motivated by two main reasons. First of all there is the need to search for viable candidates to describe the fundamental interactions known to us (specially gravity) and the universe at the cosmological scale, solving the theoretical problems encountered by the theories available today. The second reason is the desire to map the space of possible theories and the different relations and dualities existing between them.

In the String Theory context, the landscape of $\mathcal{N} = 1, d = 4$ vacua has focused most of the attention, but more general compactifications have also been studied. At the level of the effective field theories the exploration has been carried out within the space of supergravity theories. Most ungauged supergravity theories (excluding those of higher order in curvature) and some of the gauged ones have been constructed in the past century [?], but the space of possible gaugings and massive deformations (related to fluxes, symmetry enhancements etc. in String theory) has started to be studied in a systematic way more recently with the introduction of the embedding-tensor formalism in Refs. [?, ?, ?]. The formalism was developed in the context of the study of the gauging of $\mathcal{N} = 8, d = 4$ supergravity in Refs. [?, ?], but it has later been used in theories with less supersymmetry in different dimensions.[1]

The embedding-tensor formalism comes with a bonus: the tensor hierarchy [?, ?, ?, ?, ?]. Using electric and magnetic vector fields in $d = 4$ dimensions as gauge fields requires the introduction of 2-form-potentials in the theory, which would be dual to the scalars. In $d = 6$ dimensions certain gaugings require the introduction of magnetic 2-form and 3-form potentials [?]. But the addition of higher-rank potentials does not stop there: as a general rule, the construction of gauge-invariant field strengths for the new p-form fields requires the introduction of $(p+1)$-form fields with Stückelberg couplings. This leads to a tensor hierarchy that includes all the electric and magnetic fields of the theory and opens up the systematic construction of gauged theories: construct the hierarchy using gauge invariance as a principle expressed through the Bianchi identities and find the equations

[1]See, for instance, Chapter 2 in Ref. [?], which contains a pedagogical introduction to the formalism and references.

of motion by using the duality relations between electric and magnetic fields of ranks p and $d-p-2$.

This approach has been used in Refs. [?, ?] to construct the most general 4-, 5- and 6-dimensional field theories[2] with gauge invariance with at most two derivatives. In this paper we want to consider the 8-dimensional case and construct the most general 8-dimensional field theory with gauge invariance and of second order in derivatives in the action: tensor hierarchy, Bianchi identities, field strengths, duality relations and action.[3]

Our main motivation for considering this problem is to simplify and systematize the construction of a one-parameter family of inequivalent gaugings with the same SO(3) group of maximal 8-dimensional supergravity, whose existence was conjectured in Ref. [?]:[4] using Scherk-Schwarz's generalized dimensional reduction [?] Salam and Sezgin obtained from 11-dimensional reduction an 8-dimensional SO(3)-gauged maximal supergravity in which the 3 Kaluza-Klein vectors played the role of gauge fields [?].[5] The ungauged theory, though, has a second triplet of vector fields coming from the reduction of the 11-dimensional 3-form that can also be used as gauge fields and an $\mathrm{SL}(2,\mathbb{R})$ global symmetry that relates these two triplets of vectors, suggesting one could use as gauge fields any linear combination of these triplets.

The gauged theory in which the second tripet of vectors (those coming from the reduction of the 11-dimensional 3-form) played the role of gauge fields was obtained in Ref. [?] by dimensional reduction of a non-covariant deformation of 11-dimensional supergravity proposed in Ref. [?, ?].[6] This theory has different Chern-Simons terms and a different scalar potential and provides an early example of inequivalent gauging with the same gauge group of a given supergravity theory. However, for the reasons explained above, the existence of a full 1-parameter family of inequivalent SO(3) gaugings is expected and it would be interesting to construct it and compare it with the 1-parameter family of inequivalent[7] SO(8) gaugings of $\mathcal{N}=8, d=4$ supergravity obtained in Ref. [?] and consider the possible higher-dimensional origin of the new parameter.

The construction of that 1-parameter family interpolating between Salam-Sezgin's theory and that of Ref. [?] is a complicated problem that will be addressed in a forthcoming publication [?]. In this paper we want to consider the general deformations (gaugings and massive transformations) of generic 8-dimensional theories. This result paves the way for the constraction of the 1-parameter family of gauged $\mathcal{N}=2, d=8$ theories which is our

[2]Not only supergravities, since use the embedding-tensor formalism is not restricted to supergravity theories.

[3]The tensor hierarchy of maximal 8-dimensional supergravity has been constructed in Ref. [?] in the context if exceptional field theory.

[4]By inequivalent here we mean theories which have different interactions, including, in particular, different scalar potentials. A more restrictive definition of inequivalent theories (a more general concept of equivalence of theories) is often used in the literature (in Ref. [?], for instance): theories related by a field redefinition (including non-local field redefinitions such as electric-magnetic dualities) are not considered to be inequivalent. With this definition, the theories in the family we are talking about would not be considered to be inequivalent.

[5]Other, more general, gaugings can be obtained via Scherk-Schwarz reduction [?, ?], but it is always the Kaluza-Klein vectors that play the role of gauge fields.

[6]Many gauged supergravities whose 11-dimensional origin is unkonown or, in more modern parlance, they contain non-geometrical fluxes (like Roman's 10-dimensional massive supergravity or alternative, inequivalent gaugings of other theories) can be obtained systematically from this non-covariant deformation of 11-dimensional supergravity [?], which seems to encode many of these non-geometrical fluxes.

[7]Inequivalent in the more restrictive sense explained above.

ultimate goal. However, it is an interesting problem by itself whose solution will provide us with the most general theories with gauge symmetry in 8 dimensions up to two derivatives.

The construction of the most general 8-dimensional theory with gauge symmetry and at most two derivatives, and this paper, are organized as follows: first, in Section **??**, we study the structure and symmetries (including electric-magnetic dualities of the 3-form potentials) of generic (up to second order in derivatives) 8-dimensional theories with Abelian gauge symmetry and no Chern-Simons terms.

In Section **??**, we consider Abelian, massless deformations of those theories, which consist, essentially, in the introduction via some constant "d-tensors" of Chern-Simons terms in the field strengths and action. The new intereactions are required to preserve the Abelian gauge symmetries and, formally, the symplectic structure of the electric-magnetic duality transformations of the 3-form potentials. We determine explicitly the form of all the electric and magnetic field strengths up to the 7-form field strengths, and give the gauge-invariant action in terms of the electric potentials. This will be our starting point for the next stage.

In Section **??** we consider the most general gauging and massive deformations (Stückelberg couplings) of the Abelian theory constructed in the previous section using the embedding-tensor formalism. We proceed as in Refs. [**?**, **?**], finding Bianchi identities for field strengths from the identities satisfied by the Bianchi identities of the lower-rank field strengths and, then, solving them. We have found the Bianchi identities satisfied by all the field strengths and we have managed to find the explicit form of the field strengths up to the 6-form.

In this approach, the "d-tensors" that define the Chern-Simons terms will be treated in a different way as in Ref. [**?**]: they will not be treated as deformations of the theory to be gauged, but as part of its definition. Therefore, we will not associate to them any dual 7-form potentials.

In Section **??** we study the construction of an action for the theory. The equations of motion are related to the Bianchi identities by the duality relations between electric and magnetic field strengths, but, at least in this case, they are not directly equal to them. In general they can be combinations of the Bianchi identities. To find the right combinations we derive the Noether identities that the off-shell equations of motion of these theories should satisfy as a consistency condition that follows from gauge invariance. Then, we compare those Noether identities with the identities satisfied by the Bianchi identities. Once the equations of motion have been determined in this way, we proceed to the construction of the action, which we achieve up to terms that only contain 1-forms and their derivatives, whose form is too complicated.

Section **??** contains our conclusions and the main formulae (field strengths, Bianchi identities etc.) of the ungauged and gauged theories are collected in the appendices to simplify their use.

2.1 Ungauged $d = 8$ theories

In this section we are going to consider the construction of generic (bosonic) $d = 8$ theories coupled to gravity containing terms of second order or lower in derivatives of any given field[8]. The field content of a generic $d = 8$ theory are the metric $g_{\mu\nu}$, scalar fields ϕ^x, 1-form fields $A^I = A^I{}_\mu dx^\mu$, 2-form fields $B_m = \frac{1}{2}B_{m\,\mu\nu}dx^\mu \wedge dx^\nu$ and 3-form fields $C^a = \frac{1}{3!}C^a{}_{\mu\nu\rho}dx^\mu \wedge dx^\nu \wedge dx^\rho$. For the moment, we place no restrictions on the range of the indices labeling these fields nor on the symmetry groups that may act on them leaving the theory invariant.

We are going to start by the simplest theory one can construct with these fields to later gauge it and deform it in different ways.

The simplest field strengths one can construct for these fields are their exterior derivatives:

$$F^I \equiv dA^I\,, \qquad H_m \equiv dB_m\,, \qquad G^a \equiv dC^a\,. \tag{2.1}$$

They are invariant under the gauge transformations

$$\delta_\sigma A^I = d\sigma^I\,, \qquad \delta_\sigma B_m = d\sigma_m\,, \qquad \delta_\sigma C^a = d\sigma^a\,, \tag{2.2}$$

where the local parameters $\sigma^I, \sigma_m, \sigma^a$ are, respectively, 0-, 1-, and 2-forms.

The most general gauge-invariant action which one can write for these fields is

$$\begin{aligned} S \;=\; & \int \{- \star \mathbb{1} R + \tfrac{1}{2}\mathcal{G}_{xy}d\phi^x \wedge \star d\phi^y + \tfrac{1}{2}\mathcal{M}_{IJ}F^I \wedge \star F^J + \tfrac{1}{2}\mathcal{M}^{mn}H_m \wedge \star H_n \\ & -\tfrac{1}{2}\Im\mathfrak{m}\mathcal{N}_{ab}G^a \wedge \star G^b - \tfrac{1}{2}\Re\mathfrak{e}\mathcal{N}_{ab}G^a \wedge G^b\}\,, \end{aligned} \tag{2.3}$$

where the kinetic matrices $\mathcal{G}_{xy}, \mathcal{M}_{IJ}, \mathcal{M}^{mn}, \Im\mathfrak{m}\mathcal{N}_{ab}$ as well as the matrix $\Re\mathfrak{e}\mathcal{N}_{ab}$ are scalar-dependent[9]. One could add CS terms to this action, but this possibility will arise naturally in what follows.

The equations of motion of the 3-forms C^a can be written in the form[10]

$$\frac{\delta S}{\delta C^a} = -d\frac{\delta S}{\delta G^a} = 0\,, \qquad \frac{\delta S}{\delta G^a} = R_a \equiv -\Re\mathfrak{e}\mathcal{N}_{ab}G^b - \Im\mathfrak{m}\mathcal{N}_{ab} \star G^b\,. \tag{2.5}$$

These equations can be solved locally by introducing a set of dual 3-forms C_a implicitly defined through their field strengths G_a

$$R_a = G_a \equiv dC_a\,. \tag{2.6}$$

It is convenient to construct vectors containing the fundamental and dual 3-forms:

$$(C^i) \equiv \begin{pmatrix} C^a \\ C_a \end{pmatrix}\,, \qquad G^i \equiv dC^i\,, \tag{2.7}$$

[8]The Chern–Simons (CS) terms may have terms with more than two derivatives, but they do not act on the same field.

[9]If $\Re\mathfrak{e}\mathcal{N}_{ab}$ is constant, then the last term is a total derivative.

[10]The equation of motion of a p-form field, $\delta S/\delta\omega^{(p)}$, is an $(8-p) - form$ defined by

$$\delta S = +\frac{\delta S}{\delta \phi^x} \wedge \delta\phi^x + \frac{\delta S}{\delta A_I} \wedge \delta A^I + \frac{\delta S}{\delta B^m} \wedge \delta B^m + \frac{\delta S}{\delta C^a} \wedge \delta C^a\,. \tag{2.4}$$

With our conventions, when acting on p-forms, $\star^2 = (-1)^{p-1}$.

so that the equations of motion and the Bianchi identities for the fundamental field strengths take the simple form

$$dG^i = 0\,. \tag{2.8}$$

In other words: we have traded an equation of motion by a Bianchi identity and a duality relation. In what follows we will do the same for all the fields in the action so that, in the end, we will have only a set of Bianchi identities and a set of duality relations between magnetic and electric fields.

The vector of field strengths G^i satisfies the following *linear, twisted, self-duality constraint*

$$\star\, G^i = \Omega^{ij}\mathcal{W}_{jk}G^k\,, \tag{2.9}$$

where

$$(\Omega_{ij}) = (\Omega^{ij}) \equiv \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}\,, \tag{2.10}$$

is the symplectic *metric* and

$$(\mathcal{W}_{ij}(\mathcal{N})) \equiv -\begin{pmatrix} I_{ab} + R_{ac}I^{cd}R_{db} & R_{ac}I^{cb} \\ & \\ I^{ac}R_{cb} & I^{ab} \end{pmatrix}\,, \qquad \Omega\mathcal{W}\Omega^T = \mathcal{W}^{-1}\,, \tag{2.11}$$

is a symplectic symmetric matrix[11]. The equations (**??**) are formally invariant under arbitrary $\mathrm{GL}(2n_3,\mathbb{R})$ transformations (n_3 being the number of fundamental 3-forms) but, just as it happens for 1-forms in $d=4$, the self-duality constraint Eq. (**??**) is only preserved by $\mathrm{Sp}(2n_3,\mathbb{R})$. As usual, the only $\mathrm{Sp}(2n_3,\mathbb{R})$ transformations which are true symmetries of the equations of motion are those associated to the transformations of the scalars which are isometries of $\mathcal{G}_{xy}$ and which also induce linear transformations of the other kinetic matrices. We will discuss this point in more detail later on.

The dualization of the other fields does not lead to any further restrictions.

In what follows we are going to generalize the simple Abelian theory that we have constructed by *deforming* it, adding new couplings. We will use two guiding principles: preservation of gauge symmetry (even if it needs to be deformed as well) and preservation of the *formal* symplectic invariance that we have just discussed.

2.2 Abelian, massless deformations

The deformations that we are going to consider in this section consist, essentially, in the introduction of CS terms in the field strengths and in the action. Stückelberg coupling will

[11] Basically the same that occurs in $d=4$ theories, $\mathcal{M}(\mathcal{N})$ see *e.g.* Ref. [**?**]. We use a slightly different convention for the sake of convenience and $\mathcal{M}(\mathcal{N}) = \mathcal{M}(-\mathcal{N})$ due to the unconventional sign on the definition of G_a.

be considered later. Only the 3- and 4-form field strengths admit these massless Abelian deformations. It is convenient to start by considering this simple modification of G^a:[12]

$$G^a = dC^a + d^a{}_I{}^m F^I B_m \,, \tag{2.12}$$

where $d^a{}_I{}^m$ is a constant tensor. The gauge transformations need to be deformed accordingly:

$$\delta_\sigma A^I = d\sigma^I \,, \qquad \delta_\sigma B_m = d\sigma_m \,, \qquad \delta_\sigma C^a = d\sigma^a - d^a{}_I{}^m F^I \sigma_m \,. \tag{2.13}$$

The action Eq. (**??**) remains gauge-invariant but the formal symplectic invariance is broken: if we do not modify the action, the dual 4-form field strengths are just $G_a = dC_a$ and $\mathrm{Sp}(2n_3, \mathbb{R})$ cannot rotate these into G^a in Eq. (**??**). Furthermore, the 1-form and 2-form equations of motion do not have a symplectic-invariant form.

This problem can be solved by adding a CS term to the action:

$$S_{CS} = \int \{-d_{aI}{}^m dC^a F^I B_m\} \,, \tag{2.14}$$

that modifies the equations of motion of the 3-forms

$$- d\frac{\delta S}{\delta dC^a} = 0 \,, \qquad \frac{\delta S}{\delta dC^a} = R_a - d_{aI}{}^m F^I B_m \,. \tag{2.15}$$

The local solution is now

$$dC_a \equiv R_a - d_{aI}{}^m F^I B_m \,, \tag{2.16}$$

and, since R_a is gauge-invariant, the dual, gauge-invariant, field strength must be defined by

$$R_a = dC_a + d_{aI}{}^m F^I B_m \equiv G_a \,. \tag{2.17}$$

Again, $(C^i) = \binom{C^a}{C_a}$ transforms linearly as a symplectic vector if $(d^i{}_I{}^m) \equiv \binom{d^a{}_I{}^m}{d_{aI}{}^m}$ also does. Then, we can define the symplectic vector of 4-form field strengths

$$G^i = dC^i + d^i{}_I{}^m F^I B_m \,, \tag{2.18}$$

invariant under the deformed gauge transformations

$$\delta_\sigma A^I = d\sigma^I \,, \qquad \delta_\sigma B_m = d\sigma_m \,, \qquad \delta_\sigma C^i = d\sigma^i - d^i{}_I{}^m F^I \sigma_m \,. \tag{2.19}$$

However, the deformed gauge transformations do not leave invariant the CS term Eq. (**??**). The only solution[13] is to add another term of the form[14]

---

[12] We will often suppress the wedge product symbols $\wedge$ in order to simplify the expressions that involve differential forms.

[13] We have not found any other.

[14] We use the compact notation $A^{IJ\cdots} = A^I A^J \cdots$, $F^{IJ\cdots} = F^I F^J \cdots$, $B_{mn\ldots} = B_m B_n \cdots$ etc., where we have suppressed the wedge product symbols.

$$S_{CS} = \int \{-d_{aI}{}^m dC^a F^I B_m - \tfrac{1}{2} d_{aI}{}^m d^a{}_J{}^m F^{IJ} B_{mn}\}\,, \tag{2.20}$$

provided the following constraint holds:

$$d_{a(I}{}^{[m} d^a{}_{J)}{}^{n]} = 0\,, \quad \text{so} \quad d_{i(I}{}^{(m} d^i{}_{J)}{}^{n)} = 0\,. \tag{2.21}$$

Observe that we are just using *formal* symplectic invariance: the symplectic vector $d^i{}_I{}^m$ is *transformed* into a different one. Thus, in general, one gets $\mathrm{Sp}(2n_3, \mathbb{R})$ multiplets of theories, except when $d^i{}_I{}^m$ is a symplectic invariant *tensor*,[15] which requires, at least, one of the indices I or m to be a symplectic index. In most cases the part of the symmetry group of the theory acting on the 3-forms, while embedded in $\mathrm{Sp}(2n_3, \mathbb{R})$, will be a much smaller group and, then, full symplectic invariance of $d^i{}_I{}^m$ may not be required.

As a nice check of the formal symplectic invariance of the deformed theory, we can check this invariance on the dual field strengths of the remaining fields[16], which is tantamount to checking the invariance of the equations of motion of the fundamental fields.

Using the duality relation $R_a = G_a$ the equations of motion of the 1-forms can be written in the form

$$\frac{\delta S}{\delta A^I} = -d\left\{\mathcal{M}_{IJ} \star F^J - d_{iI}{}^m G^i B_m - \tfrac{1}{2} d_{iI}{}^m d^i{}_J{}^n F^J B_{mn}\right\} = 0\,, \tag{2.22}$$

and can be solved by identifying all the terms inside the brackets with $d\tilde{A}_I$, where $\tilde{A}_I$ is a set of 5-forms. Taking into account gauge invariance, the 6-form field strengths $\tilde{F}_I$ have the following definition, duality relation and Bianchi identities:

$$\tilde{F}_I \equiv d\tilde{A}_I + d_{iI}{}^m G^i B_m + \tfrac{1}{2} d_{iI}{}^m d^i{}_J{}^n F^J B_{mn}\,, \tag{2.23}$$

$$\tilde{F}_I = \mathcal{M}_{IJ} \star F^J\,, \tag{2.24}$$

$$d\tilde{F}_I = d_{iI}{}^m G^i H_m\,, \tag{2.25}$$

and the equations of motion are of the 1-forms given by the Bianchi identities of the dual 6-form field strengths up to duality relations:

$$\frac{\delta S}{\delta A^I} = -\left\{d\tilde{F}_I - d_{iI}{}^m G^i H_m\right\}\,. \tag{2.26}$$

Using the duality relation $R_a = G_a$ and following the same steps for the 2-forms , we find

[15] The only symplectic-invariant vector is 0.

[16] We leave aside the scalars for the moment.

$$\tilde{H}^m = d\tilde{B}^m + d^i{}_I{}^m F^I C_i \,, \tag{2.27}$$

$$\tilde{H}^m = \mathcal{M}^{mn} \star H_n \,, \tag{2.28}$$

$$d\tilde{H}^m = -d_{iI}{}^m G^i F^I \,, \tag{2.29}$$

and the equations of motion of the 2-forms are given by the Bianchi identities of the dual 5-form field strengths up to duality relations:

$$\frac{\delta S}{\delta B_m} = -\left\{ d\tilde{H}^m + d_{iI}{}^m G^i F^I \right\} \,. \tag{2.30}$$

This completes the first Abelian deformation. The second non-trivial deformation of G^a that one could consider is the addition of a CS 4-form term $\sim d^a{}_{IJK} A^I F^J A^K$. The gauge transformation of this term is not a total derivative and we cannot make G^a gauge-invariant by deforming the gauge transformation rule of C^a only: we must also deform that of B_m, which, in its turn, induces a deformation of H_m by addition of a CS 3-form term. Since the deformation of H_m is essentially unique, it is more convenient to start from this side and redefine

$$H_m = dB_m - d_{mIJ} F^I A^J \,, \tag{2.31}$$

where $d_{mIJ} = d_{mJI}$[17] which is invariant under the gauge transformations

$$\delta_\sigma A^I = d\sigma^I \,, \qquad \delta_\sigma B_m = d\sigma_m + d_{mIJ} F^I \sigma^J \,, \tag{2.32}$$

and satisfies the Bianchi identities

$$dH_m = -d_{mIJ} F^{IJ} \,. \tag{2.33}$$

Under these gauge transformations and a generic $\delta_\sigma C^a$

$$\delta_\sigma G^a = d\left(\delta_\sigma C^a + d^a{}_I{}^m F^I \sigma_m\right) + d^a{}_I{}^m d_{mJK} F^{IJ} \sigma^K \,. \tag{2.34}$$

Adding a CS 4-form term to G^a

$$G^a = dC^a + d^a{}_I{}^m F^I B_m - \alpha d^a{}_I{}^m d_{mJK} A^I F^J A^K \,. \tag{2.35}$$

we find

$$\begin{aligned} \delta_\sigma G^a &= d\left[\delta_\sigma C^a + d^a{}_I{}^m F^I \sigma_m - \alpha d^a{}_I{}^m d_{mJK} (\sigma^I F^J A^K - A^I F^J \sigma^K)\right] \\ &\quad + d^a{}_I{}^m d_{mJK} \left[\alpha \sigma^I F^{JK} + (1-\alpha) F^{IJ} \sigma^K\right] \,. \end{aligned} \tag{2.36}$$

The last term can be made to vanish by simply requiring

[17] The antisymmetric part is a total derivative that can be absorbed into a redefinition of B_m.

$$\alpha d^a{}_I{}^m d_{mJK} = (\alpha - 1) d^a{}_{(J|}{}^m d_{m|K)I} \,. \tag{2.37}$$

Symmetrizing both sides of this equation w.r.t. IJK we conclude that

$$d^a{}_{(I|}{}^m d_{m|JK)} = 0 \,, \tag{2.38}$$

and going back to the original (unsymmetrized) equation this implies that $\alpha = 1/3$. We arrive to the field strength, gauge transformation and Bianchi identities

$$G^a = dC^a + d^a{}_I{}^m F^I B_m - \tfrac{1}{3} d^a{}_I{}^m d_{mJK} A^I F^J A^K \,, \tag{2.39}$$

$$\delta_\sigma C^a = d\sigma^a - d^a{}_I{}^m F^I \sigma_m + \tfrac{1}{3} d^a{}_I{}^m d_{mJK} (\sigma^I F^J A^K - A^I F^J \sigma^K) \,, \tag{2.40}$$

$$dG^a = d^a{}_I{}^m F^I H_m \,. \tag{2.41}$$

If these deformations are going to preserve formal symplectic invariance, we expect that these results extend to the dual 3-forms and 4-forms field strengths, that is:

$$G^i = dC^i + d^i{}_I{}^m F^I B_m - \tfrac{1}{3} d^i{}_I{}^m d_{mJK} A^I F^J A^K \,, \tag{2.42}$$

$$\delta_\sigma C^i = d\sigma^i - d^i{}_I{}^m F^I \sigma_m + \tfrac{1}{3} d^i{}_I{}^m d_{mJK} (\sigma^I F^J A^K - A^I F^J \sigma^K) \,, \tag{2.43}$$

$$dG^i = d^i{}_I{}^m F^I H_m \,, \tag{2.44}$$

while the identity

$$d^i{}_{(I|}{}^m d_{m|JK)} = 0 \,. \tag{2.45}$$

This requires the introduction of new CS terms in the action. If we define the CS terms in the 4-form field strengths by ΔG^i ($G^i = dC^i + \Delta G^i$), then we expect the following terms to be present:

$$S_{CS} = -\int \{ dC^a \Delta G_a + \tfrac{1}{2} \Delta G^a \Delta G_a \} \,. \tag{2.46}$$

Instead of checking in detail the gauge-invariance of these terms, it is more convenient to take the formal exterior derivative and check whether it is entirely given in terms of the gauge-invariant field strengths found above. if it is not, it should fail only by a total derivative which we can compensate by adding the corresponding terms to the action.

We have found that one has to relate $d^i{}_{(I|}{}^m d_{i|J)}{}^n$ to the tensor d_{mIJ}. The relation can be established by introducing a new tensor $d^{mnp} = -d^{nmp}$ and is given by

$$d^i{}_{(I|}{}^m d_{i|J)}{}^n = -2 d^{mnp} d_{pIJ} \,. \tag{2.47}$$

Observe that $d^i{}_{[I|}{}^m d_{i|J]}{}^n$ does not necessarily vanish.

Using the above relation we find a result of the expected form[18]

$$
\begin{aligned}
d\{dC^a\Delta G_a + \tfrac{1}{2}\Delta G^a\Delta G_a\} \;=\;& d_{aI}{}^m G^a F^I H_m - \tfrac{1}{3}d^{mnp}H_{mnp} + d\left\{-\tfrac{1}{6}d^{mnp}B_m dB_n dB_p\right.\\
&\left.+\tfrac{1}{2}d^{mnp}B_m H_{np} + \tfrac{1}{24}d^i{}_I{}^m d_{iJ}{}^n A^{IJ}\Delta H_m dB_n\right\} ,
\end{aligned}
\tag{2.49}
$$

from which it follows that the gauge-invariant CS term in the action is given, up to total derivatives, by

$$
\begin{aligned}
S_{CS} \;=\;& \int \left\{-dC^a\Delta G_a - \tfrac{1}{2}\Delta G^a\Delta G_a - \tfrac{1}{6}d^{mnp}B_m dB_n dB_p + \tfrac{1}{2}d^{mnp}B_m H_{np}\right.\\
&\left.+\tfrac{1}{24}d^i{}_I{}^m d_{iJ}{}^n A^{IJ}\Delta H_m dB_n\right\} .
\end{aligned}
\tag{2.50}
$$

Observe that only the completely antisymmetric part of d^{mnp} enters the action, even though we have only assumed it to be antisymmetric in the first two indices. We will henceforth assume that d^{mnp} is completely antisymmetric.

Now, as a final check of the consistency of our results, we can compute the dual field strengths $\tilde{H}^m$ and $\tilde{F}_I$, which should be formally symplectic invariant if the theory is, and their Bianchi identities, which should be given entirely in terms of other field strengths if the theory is indeed gauge invariant.

We find

$$
\tilde{H}^m = d\tilde{B}^m + d^i{}_I{}^m C_i F^I + d^{mnp}B_n(H_p + \Delta H_p) + \tfrac{1}{12}d^i{}_I{}^m d_{iJ}{}^n A^{IJ}\Delta H_n\,, \tag{2.51}
$$

$$
d\tilde{H}^m = d^i{}_I{}^m G_i F^I + d^{mnp}H_{np}\,, \tag{2.52}
$$

$$
\begin{aligned}
\tilde{F}_I \;=\;& d\tilde{A}_I + 2d_{mIJ}A^J(\tilde{H}_m - \tfrac{1}{2}\Delta\tilde{H}_m) - \left(d^i{}_I{}^m B_m - \tfrac{1}{3}d^i{}_J{}^m d_{mIK}A^{JK}\right)(G_i - \tfrac{1}{2}\Delta G_i)\\
&-\tfrac{1}{3}\left(d^i{}_I{}^m d_{mJK} - d^i{}_K{}^m d_{mIJ}\right)F^J A^K C_i - d^{mnp}d_{mIJ}A^J B_n H_p\\
&+\tfrac{1}{24}\left(d^i{}_K{}^m d_{iL}{}^n d_{mIJ} + 2d^i{}_{[I|}{}^m d_{i|K]}{}^n d_{mJL}\right)F^J A^{KL}B_n + \tfrac{1}{24}d^i{}_J{}^m d^n_{iK}d_{mIL}A^{JKL}dB_n\\
&-\tfrac{1}{180}d^i{}_L{}^n d_{iQ}{}^m d_{mIJ}d_{nPK}A^{JKLQ}F^P\,,
\end{aligned}
\tag{2.53}
$$

$$
d\tilde{F}_I = 2d_{mIJ}F^J\tilde{H}^m + d_{iI}{}^m G^i H_m\,. \tag{2.54}
$$

The duality relations are the same as in the undeformed case.

[18] We use repeatedly the identity

$$
2d^i{}_I{}^m d_{iJ}{}^n F^I A^J\Delta H_n = -6d^{mnp}\Delta H_n\Delta H_p + d\{\tfrac{1}{2}d^i{}_I{}^m d_{iJ}{}^n A^{IJ}\Delta H_n\}\,. \tag{2.48}
$$

As a further check of this construction, taking the exterior derivative of the Bianchi identities of all the field strengths one finds consistent results upon use of the properties of the deformation tensors $d^{i}{}_{I}{}^{m}, d_{mIJ}, d^{mnp}$.

We will not compute the gauge transformations of the higher-rank form fields since they will not be necessary in what follows.

2.2.1 The 6-form potentials and their 7-form field strengths

On general grounds (see [?] and references therein) the 6-form potentials are expected to be the duals of the scalars. However, maintaining the manifest invariances of the theory in the dualization procedure requires the introduction of as many 6-forms D_A as generators of global transformations δ_A leaving the equations of motion (not just the action) invariant. Hence, the index A labels the adjoint representation of the duality group. The 7-form field strengths K_A are the Hodge duals of the piece $j_A^{(\sigma)}(\phi)$ of the Noether–Gaillard–Zumino (NGZ) conserved 1-form currents $j_A = j_A^{(\sigma)}(\phi) + \Delta j_A$ associated to those symmetries (or, better, dualities) [?] which only depend on the scalar fields[19]

$$K_A \equiv -\star j_A^{(\sigma)}\,, \tag{2.55}$$

and their Bianchi identities follow from the conservation law for those currents

$$dK_A = d\star j_A^{(\sigma)} = d\star(j_A^{NGZ} - \Delta j_A) = -d\star\Delta j_A\,, \tag{2.56}$$

where we have used the conservation of the NGZ current.

The simplest procedure to compute Δj_A is to contract the equations of motion of the scalars with the Killing vectors $k_A{}^x(\phi)$ of the σ-model metric $\mathcal{G}_{xy}(\phi)$, which is given by

$$\begin{aligned}\frac{\delta S}{\delta\phi^x} \;=\;& -d(\star\mathcal{G}_{xy}d\phi^y) + \tfrac{1}{2}\partial_x\mathcal{G}_{yz}d\phi^y\wedge\star d\phi^z\\ &+\tfrac{1}{2}\partial_x\mathcal{M}_{IJ}F^I\wedge\star F^J + \tfrac{1}{2}\partial_x\mathcal{M}^{mn}H_m\wedge\star H_n + G^a\partial_x G_a\,.\end{aligned} \tag{2.57}$$

Using the Killing equation, we get

$$k_A{}^x\frac{\delta S}{\delta\phi^x} = -d\star j_A^{(\sigma)} + \tfrac{1}{2}k_A{}^x\partial_x\mathcal{M}_{IJ}F^I\wedge\star F^J + \tfrac{1}{2}k_A{}^x\partial_x\mathcal{M}^{mn}H_m\wedge\star H_n + G^a k_A{}^x\partial_x G_a\,. \tag{2.58}$$

We must now use the fact that the isometry generated by k_A will only be a symmetry of the equations of motion if[20]

[19] This is the contribution of the σ-model to the Noether current. The symmetries of the equations of motion are necessarily symmetries of the σ-model, i.e. isometries of the σ-model metric $\mathcal{G}_{xy}(\phi)$ generated by Killing vectors $k_A{}^x$. The indices A, B, C label the symmetries of the theory and, therefore, run over the adjoint representation of the Lie algebra of that symmetry group G. The contribution of the σ-model to the NGZ 1-form is $j_A^{(\sigma)} = k_A{}^x\mathcal{G}_{xy}d\phi^y$.

[20] The transformation rule for the period matrix is unconventional because our definition of the lower component of the symplectic vector of 4-form field strengths, $G_a = R_a$ is unconventional (the sign is the oposite to the conventional one).

$$\begin{aligned} k_A{}^x \partial_x \mathcal{M}_{IJ} &= -2T_A{}^K{}_{(I}\mathcal{M}_{J)K}\,, \\ k_A{}^x \partial_x \mathcal{M}^{mn} &= 2T_A{}^{(m}{}_p\mathcal{M}^{n)p}\,, \\ k_A{}^x \partial_x \mathcal{N}_{ab} &= -T_{A\,ab} - \mathcal{N}_{ac}T_A{}^c{}_b + T_{A\,a}{}^c\mathcal{N}_{cb} + \mathcal{N}_{ac}T_A{}^{cd}\mathcal{N}_{db}\,, \end{aligned} \tag{2.59}$$

where the matrices $T_A{}^I{}_J, T_A{}^m{}_n$ and

$$\left(T_A{}^i{}_j\right) \equiv \begin{pmatrix} T_A{}^a{}_b & T_A{}^{ab} \\ T_{A\,ab} & T_{A\,a}{}^b \end{pmatrix}\,, \tag{2.60}$$

are generators of the symmetry group G in the representation in which the 1-forms, 2-forms and 3-forms transform

$$[T_A, T_B] = f_{AB}{}^C T_C\,, \qquad [k_A, k_B] = -f_{AB}{}^C k_C\,. \tag{2.61}$$

As we have discussed, this implies that the matrices $T_A{}^i{}_j$ are generators of the symplectic group

$$T_A{}^i{}_{[j}\Omega_{k]i} = 0\,. \tag{2.62}$$

Upon use of the duality relations between field strengths, we find that

$$-\,k_A{}^x \frac{\delta S}{\delta \phi^x} = d\star j_A^{(\sigma)} + T_A{}^J{}_I F^I \tilde{F}_J + T_A{}^m{}_n \tilde{H}^n H_m - \tfrac{1}{2}T_{Aij}G^{ij} = 0\,, \tag{2.63}$$

on shell. The exterior derivative of the whole expression vanishes due to the Bianchi identities of the field strengths and to the invariance of the deformation tensors $d_{mIJ}, d^i{}_I{}^m$ and d^{mnp} under the δ_A transformations:

$$\begin{aligned} \delta_A d_{mIJ} &= -T_A{}^n{}_m d_{nIJ} - 2T_A{}^K{}_{(I|}d_{n|J)K} = 0\,, \\ \delta_A d^i{}_I{}^m &= T_A{}^i{}_j d^j{}_I{}^m - T_A{}^J{}_I d^i{}_J{}^m + T_A{}^m{}_n d^i{}_I{}^n = 0\,, \\ \delta_A d^{mnp} &= 3T_{Aq}^{[m|}d^{q|np]} = 0\,. \end{aligned} \tag{2.64}$$

This means that we can rewrite that equation locally as the conservation of the NGZ current

$$d\star j_A^{NGZ} = 0\,, \qquad j_A^{NGZ} \equiv j_A^{(\sigma)} + \Delta j_A\,, \tag{2.65}$$

where Δj_A is a very long and complicated expression whose explicit form will not be useful for us. A local solution is provided by $\star[j_A^{(\sigma)} + \Delta j_A] = -dD_A$ for the 6-form potential D_A and we get the definition of the 7-form field strength

$$\star\, j_A^{(\sigma)} = -dD_A + \star\Delta j_A \equiv K_A\,. \tag{2.66}$$

Its Bianchi identity is given by

$$dK_A = -d \star j_A^{(\sigma)} = T_A{}^I{}_J F^J \tilde{F}_I + T_A{}^m{}_n \tilde{H}^n H_m + \tfrac{1}{2} T_{Aij} G^{ij} \,. \tag{2.67}$$

In the kind of theories that we are considering here there is no reason to include potentials of rank higher than 6, unless we introduce a scalar potential depending on new coupling constants: one can then introduce 7-form potentials dual to those coupling constants. Since the introduction of these parameters would be purely *ad hoc*, we will postpone the study of this duality to the next section in which we will be able to use in the definition of the scalar potential the embedding tensor and the massive deformation parameters, which have well-defined properties.

One can also generalize the theory by adding a scalar potential. This addition is associated to the introduction of new deformation parameters. In gauged supergravity, which is the main case of interest, these deformation parameters are the components of the embedding tensor and the scalar potential arises in the gauging procedure, associated to the fermion shifts in the fermion's supersymmetry transformations. Thus, it is natural to deal with the scalar potential in the next section too.

The results obtained in this and the previous Section are summarized in Appendix **??**.

2.3 Non-Abelian and massive deformations: the tensor hierarchy

The next step in the construction of the most general $d = 8$ field theory is the gauging of the global symmetries of the theory. The most general possibilities can be explored using the embedding tensor formalism[21] and in this section we are going to set it up for the Abelian theories we have just found.[22] For the sake of convenience we are going to reproduce some of the formulae obtained above.

The starting point is the assumption that the equations of motion of the theory are invariant under a global symmetry group with infinitesimal generators $\{T_A\}$ satisfying the algebra

$$[T_A, T_B] = f_{AB}{}^C T_C \,. \tag{2.68}$$

The group acts linearly on all the forms of rank ≥ 1, including the 3-forms if the electric and magnetic 3-forms C^a and C_a are combined into a single symplectic vector of 3-forms $(C^i) = \binom{C^a}{C_a}$ as explained above and codify electric-magnetic transformations involving the scalars. The matrices that represent the generators are denoted by $\{T_A{}^I{}_J\}, \{T_A{}^m{}_n\}, \{T_A{}^i{}_j\}$ and the adjoint generators are $T_A{}^B{}_C = f_{AC}{}^B$. The matrices $T_A{}^i{}_j$ are generators of the symplectic group

$$T_A{}^i{}_{[j}\Omega_{k]i} = 0 \,, \qquad (\Omega_{ij}) = (\Omega^{ij}) \equiv \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} . \tag{2.69}$$

[21] In this section we will follow Ref. [?], where the essential references on the embedding tensor formalism can be found. We will also use the same notation.

[22] Observer that, in general, the theories that we are considering are just the bosonic sector of a theory that also contains fermions and whose symmetry group may include symmetries that only act on them. The total symmetry group would, then, be larger and the embedding tensor should take this fact into account.

We have

$$
\begin{aligned}
\delta_\alpha A^I &= \alpha^A T_A{}^I{}_J A^J\,, \quad \delta_\alpha B_m = -\alpha^A T_A{}^n{}_m B_n\,, \quad \delta_\alpha C^i = \alpha^A T_A{}^i{}_j C^j\,, \\
\delta_\alpha \tilde{A}_I &= -\alpha^A T_A{}^J{}_I \tilde{A}_J\,, \quad \delta_\alpha \tilde{B}^m = \alpha^A T_A{}^m{}_n \tilde{B}^n\,,
\end{aligned}
\tag{2.70}
$$

(the dual potentials transform in the dual covariant-contravariant representation).

The kinetic matrices $\mathcal{M}_{IJ}, \mathcal{M}^{mn}, \mathcal{W}_{ij}(\mathcal{N})$ also transform linearly: if $\delta_\alpha \equiv \alpha^A \delta_A$

$$
\delta_A \mathcal{M}_{IJ} = -2T_A{}^K{}_{(I}\mathcal{M}_{J)K}\,, \quad \delta_A \mathcal{M}^{mn} = 2T_A{}^{(m}{}_p \mathcal{M}^{n)p}\,, \quad \delta_A \mathcal{W}_{ij} = -2T_A{}^k{}_{(i}\mathcal{W}_{j)k}\,,
\tag{2.71}
$$

but the period matrix undergoes fractional-linear transformations which, infinitesimally, take the form

$$
\delta_A \mathcal{N}_{ab} = -T_{A\,ab} - \mathcal{N}_{ac} T_A{}^c{}_b + T_{A\,a}{}^c \mathcal{N}_{cb} + \mathcal{N}_{ac} T_A{}^{cd} \mathcal{N}_{db}\,, \qquad \left(T_A{}^i{}_j\right) = \begin{pmatrix} T_A{}^a{}_b & T_A{}^{ab} \\ T_{A\,ab} & T_{A\,a}{}^b \end{pmatrix}.
\tag{2.72}
$$

The k-form field strengths will transform in the same representation as the corresponding $(k-1)$-form potential, but only if the d-tensors $d_{mIJ}, d^i{}_I{}^m$ and d^{mnp} are invariant under the global symmetry group, *i.e.* if they must satisfy

$$
\begin{aligned}
\delta_A d_{mIJ} &= -T_A{}^n{}_m d_{nIJ} - 2T_A{}^K{}_{(I|} d_{n|J)K} = 0\,, \\
\delta_A d^i{}_I{}^m &= T_A{}^i{}_j d^j{}_I{}^m - T_A{}^J{}_I d^i{}_J{}^m + T_A{}^m{}_n d^i{}_I{}^n = 0\,, \\
\delta_A d^{mnp} &= 3T_{Aq}^{[m|} d^{q|np]} = 0\,.
\end{aligned}
\tag{2.73}
$$

The theories we have constructed are invariant under Abelian gauge transformations with 0-, 1- and 2-form parameters $\sigma^I, \sigma_m, \sigma^i$:

$$
\delta_\sigma A^I \sim d\sigma^I\,, \qquad \delta_\sigma B_m \sim d\sigma_m\,, \qquad \delta_\sigma C^i \sim d\sigma^i\,.
\tag{2.74}
$$

In order to gauge the global symmetries, we promote the global parameters α^A to local ones $\alpha^A(x)$ and we identify them with some combinations of the gauge parameters of the 1-forms σ^I via the embedding tensor $\vartheta_I{}^A$ as follows:

$$
\alpha^A \equiv \sigma^I \vartheta_I{}^A\,.
\tag{2.75}
$$

Using this redefinition in the transformation of the kinetic matrices $\mathcal{M}_{IJ}, \mathcal{M}^{mn}, \mathcal{W}_{ij}$ one immediately finds their gauge transformations:

$$
\delta_\sigma \mathcal{M}_{IJ} = -2\sigma^L X_L{}^K{}_{(I}\mathcal{M}_{J)K}\,, \quad \delta_\sigma \mathcal{M}^{mn} = 2\sigma^I T_I{}^{(m}{}_p \mathcal{M}^{n)p}\,, \quad \delta_\sigma \mathcal{W}_{ij} = -2\sigma^I X_I{}^k{}_{(i}\mathcal{W}_{j)k}\,,
\tag{2.76}
$$

where we have defined the matrices

$$X_I{}^J{}_K \equiv \vartheta_I{}^A T_A{}^J{}_K \,, \qquad X_I{}^m{}_n \equiv \vartheta_I{}^A T_A{}^m{}_n \,. \qquad X_I{}^i{}_j \equiv \vartheta_I{}^A T_A{}^i{}_j \,. \tag{2.77}$$

The gauge fields for these symmetries are given by

$$A^A \equiv A^I \vartheta_I{}^A \,. \tag{2.78}$$

With them we can construct gauge-covariant derivatives, which we will then use to derive Bianchi identities.

Is is convenient to start by constructing the covariant derivatives of the kinetic matrices $\mathcal{M}_{IJ}, \mathcal{M}^{mn}, \mathcal{W}_{ij}(\mathcal{N})$ which transform linearly. According to the general rule, the covariant derivative of a field Φ transforming as $\delta_A \Phi$ is given by

$$\mathcal{D}\Phi \equiv d\Phi - A^A \delta_A \Phi \,. \tag{2.79}$$

Then, with the above definition of gauge fields

$$\mathcal{D}\mathcal{M}^{mn} = d\mathcal{M}^{mn} - 2A^I X_I{}^{(m}{}_p \mathcal{M}^{n)p} \,, \tag{2.80}$$

$$\mathcal{D}\mathcal{M}_{IJ} = d\mathcal{M}_{IJ} + 2A^L X_L{}^K{}_{(I} \mathcal{M}_{J)K} \,, \tag{2.81}$$

$$\mathcal{D}\mathcal{W}_{ij} = d\mathcal{W}_{ij} + 2A^I X_I{}^k{}_{(i} \mathcal{W}_{j)k} \,. \tag{2.82}$$

These derivatives transform covariantly under gauge transformations $\delta_\sigma = \sigma^I \vartheta_I{}^A \delta_A$ provided that the embedding tensor is gauge-invariant

$$\delta_\sigma \vartheta_I{}^A = 0 \,, \tag{2.83}$$

and provided that the 1-forms transform as

$$\delta_\sigma A^I = \mathcal{D}\sigma^I + \Delta_\sigma A^I \,, \quad \text{where} \quad \left\{ \begin{array}{rcl} \Delta_\sigma A^I \vartheta_I{}^A & = & 0 \,, \\ \mathcal{D}\sigma^I & = & d\sigma^I - A^J X_J{}^I{}_K \sigma^K \,, \end{array} \right. \tag{2.84}$$

The condition Eq. (**??**) leads to the so-called *quadratic constraint*

$$\vartheta_J{}^B \left[T_B{}^K{}_I \vartheta_K{}^A - f_{BC}{}^A \vartheta_I{}^C \right] = 0 \,. \tag{2.85}$$

To determine $\Delta_\sigma A^I$ we have to construct the gauge-covariant 2-form field strengths F^I.

2.3.1 2-form field strengths

The simplest way to find the 2-form field strengths F^I is through the Ricci identities. A straightforward calculation using the quadratic constraint Eq. (**??**) leads to

$$\mathcal{D}\mathcal{D}\mathcal{M}_{mn} = -F^I \vartheta_I{}^A \delta_A \mathcal{M}_{mn}\,, \tag{2.86}$$

and analogous equations for $\mathcal{M}_{IJ}$ and $\mathcal{W}_{ij}(\mathcal{N})$, with

$$F^I = dA^I - \tfrac{1}{2} X_J{}^I{}_K A^{JK} + \Delta F^I\,, \quad \text{where} \quad \Delta F^I \vartheta_I{}^A = 0\,. \tag{2.87}$$

Under gauge transformations,

$$\delta_\sigma F^I = \sigma^J X_J{}^I{}_K F^K\,, \tag{2.88}$$

provided that

$$\delta_\sigma \Delta F^I = -\mathcal{D}\Delta_\sigma A^I + 2X_{(J}{}^I{}_{K)} \left(F^J \sigma^K - \tfrac{1}{2} A^J \delta_\sigma A^K\right)\,. \tag{2.89}$$

Given the field content of the theory, the natural candidate to ΔF^I and $\Delta_\sigma A^I$ are

$$\Delta F^I = Z^{I\,m} B_m\,, \qquad \Delta_\sigma A^I = -Z^{I\,m} \sigma_m\,, \tag{2.90}$$

where the new tensor $Z^{I\,m}$ is gauge-invariant and orthogonal to the embedding tensor:

$$\delta_\sigma Z^{I\,m} = 0\,, \tag{2.91}$$

$$Z^{I\,m} \vartheta_I{}^A = 0\,. \tag{2.92}$$

Then, the consistency of Eq. (??) with the above choice requires

$$\vartheta_{(J|}{}^A T_A{}^I{}_{|K)} = Z^{I\,m} d_{mJK}\,, \tag{2.93}$$

for some tensor $d_{mJK} = d_{mKJ}$ which will turn out to coincide with the tensor we introduced as an Abelian deformation in the previous sections. Since we have assumed $\vartheta_I{}^A$ and d_{mJK} to be gauge-invariant, $Z^{I\,m}$ is automatically gauge-invariant and we have one constraint less.

We conclude that[23]

$$F^I = dA^I - \tfrac{1}{2} X_J{}^I{}_K A^{JK} + Z^{I\,m} B_m\,, \tag{2.94}$$

$$\delta_\sigma F^I = \sigma^J X_J{}^I{}_K F^K\,, \tag{2.95}$$

$$\delta_\sigma A^I = \mathcal{D}\sigma^I - Z^{I\,m} \sigma_m\,, \tag{2.96}$$

$$\delta_\sigma B_m = \mathcal{D}\sigma_m + 2d_{mJK}\left(F^J \sigma^K - \tfrac{1}{2} A^J \delta_\sigma A^K\right) + \Delta_\sigma B_m\,, \quad \text{with} \quad Z^{I\,m} \Delta_\sigma B_m = 0\,. \tag{2.97}$$

[23]On general grounds, we expect a term of the form $-\sigma^I X_I{}^n{}_m B_n$ in the gauge transformation rule of B_m. This term is indeed present, but in a disguised form.

In the ungauged limit $\vartheta_I{}^A = Z^{I\,m} = 0$ we get the Abelian gauge transformations of the 2-form Eq. (**??**) if we identify the above 1-form σ_m (relabeled $\sigma_{\mathrm{g}\,m}$) with $\sigma_m - d_{mIJ}A^I\sigma^J$,

$$\sigma_{\mathrm{g}\,m} = \sigma_m - d_{mIJ}A^I\sigma^J\,, \tag{2.98}$$

confirming the identification of the d-tensor. Using that variable makes the non-Abelian gauge transformations more complicated and, therefore, we will stick to the above σ_m.

2.3.2 3-form field strengths

Again, the shortest way to find ΔB_m and the gauge-covariant 3-form field strength H_m is through the Bianchi identities. Taking the covariant derivative of the 2-form field strength, and using the generalized Jacobi identity

$$X_{[I|}{}^K{}_M X_{|J}{}^M{}_{L]} = \tfrac{2}{3}Z^{K\,m}d_{mM[I}X_J{}^M{}_{L]}\,, \tag{2.99}$$

we get

$$\mathcal{D}F^I = Z^{I\,m}H_m\,, \tag{2.100}$$

where

$$H_m = \mathcal{D}B_m - d_{mIJ}dA^IA^J + \tfrac{1}{3}d_{mMI}X_J{}^M{}_KA^{IJK} + \Delta H_m\,, \quad \text{with} \quad Z^{I\,m}\Delta H_m = 0\,. \tag{2.101}$$

In the ungauged limit $\vartheta_I{}^A = Z^{I\,m} = 0$ we recover the Abelian 3-form field strength in Eq. (**??**). On the other hand, by construction, the above field strength is gauge-covariant up to terms annihilated by $Z^{I\,m}$ under Eqs. (**??**) and (**??**). To show this explicitly, we will need further identities between the tensors of the theory that are more easily discovered by computing first the 4-form field strengths.

2.3.3 4-form field strengths

From this moment, following Ref. [**?**], we will determine the general form of the field strengths using the Bianchi identities and their consistency relations. This procedure yields gauge-covariant field strengths and one can later find explicitly the gauge transformations of the fields that produce that result.

Thus, we take the covariant derivative of both sides of Eq. (**??**), use the Ricci identity Eq. (**??**) for the l.h.s. and the explicit form of H_m in Eq. (**??**) for the r.h.s., and we find the Bianchi identity for H_m to be

$$\mathcal{D}H_m = -d_{mIJ}F^{IJ} + \Delta\mathcal{D}H_m\,, \quad \text{where} \quad Z^{I\,m}\Delta\mathcal{D}H_m = 0\,. \tag{2.102}$$

$\Delta\mathcal{D}H_m$ has to be gauge-invariant and scalar-independent and the only possibility is a 4-form combination of field strengths. $F^I \wedge F^J$ has already been used and we must use G^i, whose explicit form will be determined by consistency. We need to introduce a new gauge-invariant tensor Z_{im} orthogonal to $Z^{I\,m}$

$$Z^{I\,m}Z_{jm} = 0\,, \tag{2.103}$$

and, then, we arrive to the Bianchi identity

$$\mathcal{D}H_m = -d_{mIJ}F^{IJ} + Z_{im}G^i\,. \tag{2.104}$$

A direct calculation of $\mathcal{D}H_m$ using the explicit expression of H_m in Eq. (**??**) with $\Delta H_m = Z_{im}C^i$ can only give a consistent result if we introduce a tensor $d^i{}_I{}^m$ such that

$$X_I{}^m{}_n + 2d_{nIJ}Z^{J\,m} = Z_{in}d^i{}_I{}^m\,. \tag{2.105}$$

The tensor $d^i{}_I{}^m$ coincides with the one we introduced as an Abelian deformation. Also, observe that this relation makes the condition of gauge invariance of Z_{in} redundant.

We get

$$G^i = \mathcal{D}C^i + d^i{}_I{}^n\left[F^I B_n - \tfrac{1}{2}Z^{I\,p}B_nB_p - \tfrac{1}{3}d_{nJK}dA^JA^{IK} + \tfrac{1}{12}d_{nMJ}X_K{}^M{}_LA^{IJKL}\right] + \Delta G^i\,, \tag{2.106}$$

with

$$Z_{im}\Delta G^i = 0\,. \tag{2.107}$$

These 4-form field strengths reduce exactly to the Abelian ones in Eq. (**??**).

Now we are ready to check explicitly using the identity/constraint Eq. (**??**) that H_m in Eq. (**??**) with $\Delta H_m = Z_{im}C^i$ is gauge covariant up to terms proportional to Z_{im}, which are automatically annihilated by $Z^{I\,m}$. We find that

$$\begin{aligned}
\delta_\sigma C^i &= \mathcal{D}\sigma^i - d^i{}_I{}^n\left[\sigma^I H_n + F^I\sigma_n + \delta_\sigma A^I B_n - \tfrac{1}{3}d_{nJK}\delta_\sigma A^J A^{IK}\right] + \Delta_\sigma C^i\,,\\
&\quad \text{with}\quad Z_{im}\Delta_\sigma C^i = 0\,,\\
\Delta_\sigma B_m &= -Z_{im}\sigma^i\,.
\end{aligned} \tag{2.108}$$

These gauge transformations reduce to the Abelian ones Eq. (**??**) upon use of the property of the d-tensors Eq. (**??**) and the identifications Eq. (**??**) and

$$\sigma^i_{\rm g} = \sigma^i + d^i{}_I{}^n(B_n\sigma^I - \tfrac{1}{3}d_{nJK}A^{IJ}\sigma^K)\,. \tag{2.109}$$

2.3.4 5-form field strengths

Taking, once again, the covariant derivatives of both sides of the Bianchi identity for H_m, Eq. (**??**), and using the Bianchi identity for F^I, Eq. (**??**) and the newly introduced tensor $d^i{}_I{}^m$, we find that

$$\mathcal{D}G^i = d^i{}_I{}^mF^IH_m - Z^i{}_m\tilde{H}^m\,, \tag{2.110}$$

where $Z^i{}_m$ is a new gauge-invariant tensor orthogonal to Z_{im}

$$Z^i{}_m Z_{in} = 0\,, \tag{2.111}$$

and where the sign of that term has been chosen so as to get the same signs as in the ungauged case. In principle these two tensors could be completely unrelated (except for the constraints). However, since, in the physical theory, G^i is self-dual and $\tilde{H}^m$ is the electric-magnetic dual of H_m, it is natural to expect that the same tensors appear in both field strengths. Thus, we are going to assume that $Z^i{}_m$ has been obtained from Z_{jm} by raising the index with the symplectic metric tensor Ω^{ij}, that is

$$Z^i{}_m \equiv \Omega^{ji} Z_{jm}\,. \tag{2.112}$$

Then, there is no new constraint associated to its gauge invariance and, we just have the constraint Eq. (**??**) analogous to a constraint satisfied by the embedding tensor in 4-dimensional field theories.

2.3.5 6-form field strengths

Taking the covariant derivative of both sides of the Bianchi identity for G^i, Eq. (**??**) and using the Bianchi identities for the field strengths of lower rank, we find that we need to introduce three new tensors $d_{iI}{}^m, d^{mnp}, d^m{}_{IJK}$ and demand that

$$d^i{}_I{}^m Z_{jm} + X_I{}^i{}_j = -Z^i{}_m d_{jI}{}^m\,, \tag{2.113}$$

$$d^i{}_I{}^{[m|} Z^{I\,|n]} = -Z^i{}_p d^{pmn}\,, \tag{2.114}$$

$$d^i{}_{(I|}{}^m d_{m|JK)} = -Z^i{}_p d^p{}_{IJK}\,. \tag{2.115}$$

Lowering the i indices in the first equation with ϵ_{ik} and taking into account that $X_{I[kj]} = 0$, we conclude that it is natural to identify

$$d_{iI}{}^m = \Omega_{ij} d^j{}_I{}^m\,, \tag{2.116}$$

and rewrite the constraint as

$$X_{Iij} = -2Z_{(i|m} d_{|j)I}{}^m\,. \tag{2.117}$$

Using these constraints and the same reasoning as in the previous cases we find the next Bianchi identity and we can also solve it[24]

[24]Actually, it is easier to find $\tilde{H}^M$ from the previous Bianchi identity Eq. (**??**) taking the covariant derivative of the 4-form field strengths G^i in Eq. (**??**) with $\Delta G^i = -Z^i{}_m \tilde{B}^m$.

$$\mathcal{D}\tilde{H}^m = -d_{iI}{}^m G^i F^I + d^{mnp} H_{np} + d^m{}_{IJK} F^{IJK} + Z^{I\,m} \tilde{F}_I \,, \tag{2.118}$$

$$\begin{aligned}\tilde{H}^m = {} & \mathcal{D}\tilde{B}^m - d_{iI}{}^m F^I C^i + 2d^{mnp} B_n \left(H_p - Z_{ip} C^i - \tfrac{1}{2}\mathcal{D}B_p\right) \\ & + d^m{}_{IJK} dA^I dA^J A^K \\ & + \left(\tfrac{1}{12} d_{iJ}{}^m d^i{}_K{}^n d_{nIL} - \tfrac{3}{4} d^m{}_{IJM} X_K{}^M{}_L\right) dA^I A^{JKL} \\ & + \left(\tfrac{3}{20} d^m{}_{NPM} X_I{}^N{}_J - \tfrac{1}{60} d_{iM}{}^m d_I^{i\,n} d_{nPJ}\right) X_K{}^P{}_L A^{IJKLM} \\ & + Z^{Im} \tilde{A}_I \,, \end{aligned} \tag{2.119}$$

2.3.6 7-form field strengths

Provided that we impose the additional constraint[25]

$$d^i{}_{(I}{}^m d_{iJ)}{}^n + 2d^{mnp} d_{pIJ} + 3d^m{}_{IJK} Z^{K\,n} = +3d^n{}_{IJK} Z^{K\,m} \,, \tag{2.120}$$

the covariant derivative of the Bianchi identity Eq. (**??**) leads to the Bianchi identity for the 6-form field strengths

$$\mathcal{D}\tilde{F}_I = 2d_{mIJ} F^J \tilde{H}^m + d_{iI}{}^m G^i H_m - 3d^m{}_{IJK} F^{JK} H_m - \vartheta_I{}^A K_A \,, \tag{2.121}$$

$$\begin{aligned}\tilde{F}_I = {} & \mathcal{D}\tilde{A}_I + 2d_{mIJ} \tilde{B}^m F^J + d_{iI}{}^m C^i (H_m - \tfrac{1}{2} Z_{jm} C^j) \\ & - 3d^m{}_{IJK} B_m (F^J - \tfrac{1}{2} Z^{Jn} B_n)(F^K - \tfrac{1}{2} Z^{Kp} B_p) - \tfrac{1}{4} d^m{}_{IJK} Z^{Jn} Z^{Kp} B_{mnp} \\ & + \tfrac{1}{2} d_{iI}{}^m d^i{}_J{}^n (F^J - \tfrac{2}{3} Z^{Jp} B_p) B_{mn} - d_{iI}{}^m B_m \Box G^i + \cdots \end{aligned} \tag{2.122}$$

where we are denoting by $\Box G^i$ the part of G^i that only contains 1-forms A^I and their derivatives dA^I.

2.3.7 8-form field strengths

Taking the covariant derivative of Eq. (**??**) and using several of the constraints imposed above, we find that

$$\vartheta_I{}^A \mathcal{D} K_A = X_I{}^K{}_J F^J \tilde{F}_K + X_I{}^m{}_n \tilde{H}^n H_m + \tfrac{1}{2} X_{Iij} G^{ij} + 5 d_{m(IJ} d^m{}_{KLM)} F^{JKLM} \,. \tag{2.123}$$

[25] This constraint reduces to Eq. (**??**) in the ungauged, massless limit.

According to the general arguments in Ref. [?] the last term must vanish. It cannot arise in the Bianchi identity of the dual Noether-Gaillard-Zumino current associated to the global symmetries of the theory. Thus, we impose

$$d_{m(IJ}d^{m}{}_{KLM)} = 0\,, \tag{2.124}$$

and, from the definition of the X tensors, we get

$$\mathcal{D}K_A = T_A{}^K{}_J F^J \tilde{F}_K + T_A{}^m{}_n \tilde{H}^n H_m - \tfrac{1}{2} T_{A\,ij} G^{ij} + Y_A{}^\sharp L_\sharp\,, \tag{2.125}$$

where $Y_A{}^\sharp$ is a tensor orthogonal to the embedding tensor

$$\vartheta_I{}^A Y_A{}^\sharp = 0\,, \tag{2.126}$$

and where the index $\sharp$ runs over all the deformation tensors introduced so far, that we are going to denote collectively by $c^\sharp$. As argued in Ref. [?], the natural candidates for the $Y_A{}^\sharp$ tensors are the variations of the deformation tensors $c^\sharp$ under the global symmetries of the theory

$$Y_A{}^\sharp = \delta_A c^\sharp\,, \tag{2.127}$$

where A runs over the whole Lie algebra of the global symmetry group, because all the deformation tensors are required to be gauge invariant

$$\vartheta_I{}^A \delta_A c^\sharp = \vartheta_I{}^A Y_A{}^\sharp \equiv \mathcal{Q}_I{}^\sharp = 0\,, \tag{2.128}$$

where we have defined the constraints $\mathcal{Q}_A{}^\sharp$.

At this point there are two possibilities:

1. We can consider that all the independent tensors[26] $\{\vartheta_I{}^A, Z^{Im}, Z_{im}, -d_{mIJ}, d^i{}_I{}^m\}$ are deformations of the original theory introduced at the same time as the gauging of the global symmetries of the original symmetry is carried out. In this case they only have to be invariant under the global symmetries that have been gauged and not the stronger condition

$$\delta_A c^\sharp = 0\,, \tag{2.129}$$

   for any of them.

2. We can consider only the tensors $\{\vartheta_I{}^A, Z^{Im}, Z_{im}\}$ are deformations of the original theory, whose definition includes the tensors $\{-d_{mIJ}, d^i{}_I{}^m\}$. In this case, the latter must be invariant under the whole global symmetry group by hypothesis. The corresponding $Y_A{}^\sharp$ tensors are assumed to vanish identically, before they are contracted with the embedding tensor. This is the point of view that we have adopted here and it implies that there are only three sets of 8-form field strengths $\{L_\sharp\} = \{L_A{}^I, L_{Im}, L^{im}\}$ and only three corresponding sets of 7-form potentials

[26] The tensors $d^{mnp}, d^m{}_{IJK}$ are related to these and their gauge invariance is not an independent condition.

$\{E_\sharp\} = \{E_A{}^I, E_{Im}, E^{im}\}$ which are dual to the deformation tensors $\{\vartheta_I{}^A, Z^{Im}, Z_{im}\}$. In an action in which these tensors are generalized to spacetime-dependent fields, these dual potentials appear as Lagrange multipliers enforcing their constancy [?, ?].

We, thus, have to consider three constraints associated to gauge invariance

$$\mathcal{Q}_{IJ}{}^A \equiv \vartheta_I{}^B Y_{BJ}{}^A\,, \qquad Y_{BJ}{}^A \equiv \delta_B \vartheta_J{}^A = -T_B{}^K{}_J \vartheta_K{}^A + T_B{}^A{}_C \vartheta_J{}^C\,, \tag{2.130}$$

$$\mathcal{Q}_I{}^{Jm} \equiv \vartheta_I{}^B Y_B{}^{Jm}\,, \qquad Y_B{}^{Jm} \equiv \delta_B Z^{Jm} = T_B{}^J{}_K Z^{Km} + T_B{}^m{}_n Z^{Jn}\,, \tag{2.131}$$

$$\mathcal{Q}_{Iim} \equiv \vartheta_I{}^B Y_{Bim}\,, \qquad Y_{Bim} \equiv \delta_B Z_{im} = -T_B{}^j{}_i Z_{jm} - T_B{}^n{}_m Z_{in}\,, \tag{2.132}$$

and two constraints associated to global invariance

$$\mathcal{Q}_{AmIJ} \equiv Y_{AmIJ} = -\delta_A d_{mIJ} = T_A{}^n{}_m d_{nIJ} + 2T_A^K{}_{(I} d_{|m|J)K}\,, \tag{2.133}$$

$$\mathcal{Q}_A{}^i{}_I{}^m \equiv Y_A{}^i{}_I{}^m = \delta_A d^i{}_I{}^m = T_A{}^i{}_j d^j{}_I{}^m - T_A{}^J{}_I d^i{}_J{}^m + T_A{}^m{}_n d^i{}_I{}^n\,, \tag{2.134}$$

and the final form of the Bianchi identity for the 7-form field strengths is

$$\mathcal{D}K_A = T_A{}^K{}_J F^J \tilde{F}_K + T_A{}^m{}_n \tilde{H}^n H_m - \tfrac{1}{2} T_{A\,ij} G^{ij} + Y_{AI}{}^B L_B{}^I + Y_A{}^{Im} L_{Im} + Y_{Aim} L^{im}\,. \tag{2.135}$$

The occurrence of these $Y_A{}^\sharp$ has to be confirmed by taking again the covariant derivative of this Bianchi identity.

2.3.8 9-form field strengths

Taking the covariant derivative of both sides of the Bianchi identity Eq. (??) we arrive to[27]

$$Y_{AI}{}^B \left[\mathcal{D}L_B{}^I + F^I K_B\right] + Y_A{}^{Im} \left[\mathcal{D}L_{Im} + \tilde{F}_I H_m\right] + Y_{Aim} \left[\mathcal{D}L^{im} + G^i \tilde{H}^m\right]$$
$$+\mathcal{Q}_{AmIJ} \tilde{H}^m F^{IJ} + \mathcal{Q}_A{}^i{}_I{}^m G_i F^I H_m = 0\,. \tag{2.136}$$

Since we have assumed[28] $\mathcal{Q}_{AmIJ} = \mathcal{Q}_A{}^i{}_I{}^m = 0$, we arrive to the Bianchi identities

[27] By direct computation we have not found any constraint or $Y_A{}^\sharp$ tensor associated to either $d^m{}_{IJK}$ or d^{mnp}.

[28] Observe that, the alternative assumption is equally valid and can be made to work by including the 8-form field strengths $L^{mIJ}, L_i{}^I{}_m$.

$$\mathcal{D}L_B{}^I = -F^I K_B - W_B{}^{I\beta} M_\beta \,, \tag{2.137}$$

$$\mathcal{D}L_{Im} = -\tilde{F}_I H_m - W_{Im}{}^\beta M_\beta \,, \tag{2.138}$$

$$\mathcal{D}L^{im} = -G^i \tilde{H}^m - W^{im\beta} M_\beta \,, \tag{2.139}$$

where the $W_\sharp{}^\beta$ tensors are invariant tensors annihilated by the $Y_A{}^\sharp$ ones

$$Y_A{}^\sharp W_\sharp{}^\beta = 0 \,. \tag{2.140}$$

As shown in Ref. [?] these tensors are nothing but the derivatives of all the constraints satisfied by the deformation tensors (labeled by β) with respect to the deformation tensors themselves. This means that there are as many 9-form field strengths M_β and corresponding 8-form potentials N_β as constraints $\mathcal{Q}^\beta = 0$. In a general action the top-form potentials N_β would occur as the Lagrange multipliers enforcing the constraints $\mathcal{Q}^\beta = 0$.

As usual, this can be confirmed by acting yet again with the covariant derivative on the above three Bianchi identities. Let us first list all the constraints we have met:

1. First of all we have the gauge-invariance constraints

$$\mathcal{Q}_{IJ}{}^A \,, \;\; \mathcal{Q}_I{}^{Jm} \,, \;\; \mathcal{Q}_{Iim} \,, \tag{2.141}$$

   defined in Eqs. (??)-(??).

2. Secondly, we have the global-invariance constraints

$$\mathcal{Q}_{AmIJ} \,, \;\; \mathcal{Q}_A{}^i{}_I{}^m \,, \tag{2.142}$$

   defined in Eqs. (??) and ((??)).

3. Thirdly we have the orthogonality constraints between the three deformation tensors

$$\mathcal{Q}^{mA} \equiv -Z^{Im} \vartheta_I{}^A \,, \tag{2.143}$$

$$\mathcal{Q}_i{}^I \equiv Z_{im} Z^{Im} \,, \tag{2.144}$$

$$\mathcal{Q}_{mn} \equiv Z_{im} Z^i{}_n \,. \tag{2.145}$$

4. Next, we have the constraints relating the gauge transformations to the d-tensors

$$\mathcal{Q}_I{}^J{}_K \equiv X_{(I}{}^J{}_{K)} - Z^{Jm} d_{mIK} \,, \tag{2.146}$$

$$\mathcal{Q}_I{}^m{}_n \equiv X_I{}^m{}_n + 2 d_{nIJ} Z^{Jm} + Z_{in} d^i{}_I{}^m \,, \tag{2.147}$$

$$\mathcal{Q}_{Iij} \equiv -X_{Iij} - 2 Z_{(i|m} d_{|j)I}{}^m \,, \tag{2.148}$$

5. Finally, we have the constraints that related the d-tensors amongst them via the massive deformations Z

$$\mathcal{Q}^{imn} \equiv d^{i}{}_{I}{}^{[m|}Z^{I|n]} + Z^{i}{}_{p}d^{pmn}\,, \tag{2.149}$$

$$\mathcal{Q}_{IJ}{}^{mn} \equiv \tfrac{1}{2}d^{i}{}_{(I|}{}^{m}d_{i|J)}{}^{n} + d^{mnp}d_{pIJ} + 3d^{[m|}{}_{IJK}Z^{K|n]}\,, \tag{2.150}$$

$$\mathcal{Q}_{iIJK} \equiv Z_{im}d^{m}{}_{IJK} - d_{i(I|}{}^{m}d_{m|JK)}\,. \tag{2.151}$$

From Eq. (**??**) we get

$$\begin{aligned}
&\frac{\partial \mathcal{Q}_{IJ}{}^{A}}{\partial \vartheta_{K}{}^{B}}\left[\mathcal{D}M^{IJ}{}_{A} + F^{I}L_{A}{}^{J}\right] + \frac{\partial \mathcal{Q}^{mA}}{\partial \vartheta_{K}{}^{B}}\left[\mathcal{D}M_{mA} + H_{m}K_{A}\right]\\
&+\frac{\partial \mathcal{Q}_{I}{}^{J}{}_{K}}{\partial \vartheta_{K}{}^{B}}\left[\mathcal{D}M^{I}{}_{J}{}^{K} + F^{IK}\tilde{F}_{J}\right] + \frac{\partial \mathcal{Q}_{I}{}^{m}{}_{n}}{\partial \vartheta_{K}{}^{B}}\left[\mathcal{D}M^{I}{}_{m}{}^{n} + F^{I}\tilde{H}^{m}H_{n}\right]\\
&+\frac{\partial \mathcal{Q}_{Iij}}{\partial \vartheta_{K}{}^{B}}\left[\mathcal{D}M^{Iij} + F^{I}G^{ij}\right] = 0\,.
\end{aligned} \tag{2.152}$$

From Eqs. (**??**) and (**??**) we get very similar equations which guarantee the consistency of the whole construction of the tensor hierarchy that we have carried out in this section.

2.4 Gauge-invariant action for the 1-, 2- and 3-forms

The Bianchi identities of the full tensor hierarchy give rise to the equations of motion of the electric fields of the theory upon use of the duality relations (*on-duality-shell*). For field strengths of the 6-,5-, 4-forms they are given by

$$K_{A} = -\star j^{(\sigma)}_{A}\,, \qquad \tilde{F}_{I} = \mathcal{M}_{IJ}\star F^{J}\,, \qquad \tilde{H}^{m} = \star\mathcal{M}^{mn}H_{n}\,. \tag{2.153}$$

For the field strengths of the magnetic 3-forms they are given by

$$G_{a} = R_{a}\,, \tag{2.154}$$

where R_a has been defined in Eq. (**??**). Finally, the field strength of the 7-forms is, according to Refs. [**?**, **?**], dual to the derivatives of the gauge-invariant scalar potential with respect to the deformation parameters, denoted collectively by $c^{\sharp}$

$$L_{\sharp} = \star\frac{\partial V}{\partial c^{\sharp}}\,. \tag{2.155}$$

This identity follows from the scalar equation of motion in presence of a scalar potential together with the condition

$$k_A{}^x \frac{\partial V}{\partial \phi^x} = Y_A{}^\sharp \frac{\partial V}{\partial c^\sharp}\,, \tag{2.156}$$

which implies, after multiplication by the embedding tensor $\vartheta_I{}^A$, the gauge-invariance of the scalar potential.

In general, the equations of motion are combinations of different Bianchi identities on-duality-shell. In order to determine the combinations that correspond to the equations of motion we have to examine which combinations of Bianchi identities satisfy the Noether identities associated to the gauge invariances of the theory.

To start with, we need to introduce some notation for the Bianchi identities. This has been done in Appendix **??**. These Bianchi identities are related by a hierarchy of identities that are obtained by taking the covariant derivative of those with lower rank, as we have shown. These identities of Bianchi identities are collected in Appendix **??**.

Now, let us assume that a standard gauge-invariant action for the 0-forms $\mathcal{M}$ (or ϕ^x), 1-forms A^I, 2-forms B_m and electric 3-forms C^a exists. This means that the Bianchi identities $\mathcal{B}(\mathcal{Q}^\beta), \mathcal{B}(c^\sharp)$ and $\mathcal{B}(\mathcal{D}\mathcal{M}), \mathcal{B}(F^I), \mathcal{B}(H_m), \mathcal{B}(G_a)$ are satisfied, at least up to duality relations. The kinetic terms of the electric fields are written in terms of the gauge-invariant field strengths and this implies that the magnetic fields $C_a, \tilde{B}^m$ must necessarily occur in the action, albeit not as dynamical fields: their equations of motion will be trivial on-duality-shell.

Under these assumptions, the identities satisfied by the non-trivial Bianchi identities (*i.e.* those involving the magnetic field strengths) take the simplified form[29]

$$\mathcal{D}\mathcal{B}(H_m) - Z^a{}_m G_a \;=\; 0\,, \tag{2.157}$$

$$\mathcal{D}\mathcal{B}(G_a) - Z_{am}\mathcal{B}(\tilde{H}^m) \;=\; 0\,, \tag{2.158}$$

$$\mathcal{D}\mathcal{B}(\tilde{H}^m) + d^a{}_I{}^m \mathcal{B}(G_a)F^I + Z^{Im}\mathcal{B}(\tilde{F}_I) \;=\; 0\,, \tag{2.159}$$

$$\mathcal{D}\mathcal{B}(\tilde{F}_I) + 2d_{mIJ}\mathcal{B}(\tilde{H}^m)F^J - d^a{}_I{}^m \mathcal{B}(G_a)H_m + \vartheta_I{}^A\mathcal{B}(K_A) \;=\; 0\,. \tag{2.160}$$

If such an action exists, its invariance with respect to gauge transformations with parameters $\sigma^m, \sigma^i, \sigma_m, \sigma^I$ will imply that the equations of motion satisfy, off-shell, associated Noether identities. Up to the field equations of $\tilde{B}^m$ and C_a which are assumed to be satisfied up to dualities, they take the form

[29]We have also ignored the identities whose rank, as differential forms, is higher than eight.

$$\mathcal{D}\frac{\delta S}{\delta \tilde{B}^m} - Z^a{}_m \frac{\delta S}{\delta C_a} = 0 \quad (2.161)$$

$$\mathcal{D}\frac{\delta S}{\delta C^a} - Z_{am} \frac{\delta S}{\delta B_m} = 0 \quad (2.162)$$

$$\mathcal{D}\frac{\delta S}{\delta B_m} + Z^{Im}\left[\frac{\delta S}{\delta A^I} - d_{nIJ}A^J\frac{\delta S}{\delta B_n} - \left(d^a{}_I{}^n B_n - \tfrac{1}{3}d^a{}_J{}^n d_{nIK}A^{JK}\right)\frac{\delta S}{\delta C^a}\right] + d^a{}_I{}^m F^I \frac{\delta S}{\delta C^a} = 0 \quad (2.163)$$

$$\mathcal{D}\left[\frac{\delta S}{\delta A^I} - d_{nIJ}A^J\frac{\delta S}{\delta B_n} - \left(d^a{}_I{}^n B_n - \tfrac{1}{3}d^a{}_J{}^n d_{nIK}A^{JK}\right)\frac{\delta S}{\delta C^a}\right] + 2d_{mIJ}F^J\frac{\delta S}{\delta B_m} + d^a{}_I{}^m H_m \frac{\delta S}{\delta C^a} + \vartheta_I{}^A k_A{}^x \frac{\delta S}{\delta \phi^x} = 0 \quad (2.164)$$

Comparing directly with the above identities satisfied by the Bianchi identities, we conclude that, up to dualities, the equations of motion of the electric fields are related to the Bianchi identities of the magnetic field strengths by

$$k_A{}^x \frac{\delta S}{\delta \phi^x} = \mathcal{B}(K_A)\,, \quad (2.165)$$

$$\frac{\delta S}{\delta A^I} = \mathcal{B}(\tilde{F}_I) + \left(d^a{}_I{}^m B_m - \tfrac{1}{3}d^a{}_J{}^m d_{mIK}A^{JK}\right)\mathcal{B}(G_a) + d_{mIJ}A^J\mathcal{B}(\tilde{H}^m)\,, \quad (2.166)$$

$$\frac{\delta S}{\delta B_m} = \mathcal{B}(\tilde{H}^m)\,, \quad (2.167)$$

$$\frac{\delta S}{\delta C^a} = \mathcal{B}(G_a)\,. \quad (2.168)$$

This identification determines completely the field theory. For instance, the equation of motion for the electric 3-forms C^a must be

$$\frac{\delta S}{\delta C^a} = \mathcal{D}\left(\Im\mathfrak{m}\mathcal{N}_{ab}\star G^b + \Re\mathfrak{e}\mathcal{N}_{ab}G^b\right) + d_{aI}{}^m F^I H_m - Z_{am}\mathcal{M}^{mn}\star H_n\,, \quad (2.169)$$

etc.

Can we write an action gauge-invariant action for the electric fields ϕ^x, A^I, B_m and C^a from which these equations of motion follow, up to duality relations? We can follow the step-by-step procedure used in Ref. [?] for the 5- and 6-dimensional cases. This procedure consists in considering first an action $S^{(0)}$ containing only the gauge-invariant kinetic terms for the all the electric potentials ϕ^x, A^I, B_m, C^a and start adding the necessary Chern-Simons terms $S^{(1)}, S^{(2)}, \ldots$ to obtain the equations of motion of all the potentials occurring in $S^{(0)}$ in order of decreasing rank: $\tilde{B}^m, C^a, C_a, B_m, A^I$. At the first step it will be necessary to introduce terms $S^{(1)}$ containing $\tilde{B}^m$ but no new terms containing this potential will be introduced in the following steps. At the second step we will introduce terms $S^{(2)}$ containing C^a (but no $\tilde{B}^m$) and in the following steps we will not introduce any more terms containing it and so on and so forth.

We will not carry this procedure to the end because in eight dimensions the number of Chern-Simons terms involving just 2- and 1-form potentials is huge and its structure is very complicated. Nevertheless we are going to check that everything works as expected for the potentials of highest rank $\tilde{B}^m, C^a, C_a$ and we are going to find that only under certain conditions the action we are looking for exists

Our starting point is, therefore, the action

$$
\begin{aligned}
S^{(0)} \;=\; \int \big\{ & -\star\,\mathbb{1}R + \tfrac{1}{2}\mathcal{G}_{xy}\mathcal{D}\phi^x \wedge \star\mathcal{D}\phi^y + \tfrac{1}{2}\mathcal{M}_{IJ}F^I \wedge \star F^J + \tfrac{1}{2}\mathcal{M}^{mn}H_m \wedge \star H_n \\
& + \tfrac{1}{2}G^a \wedge R_a - \star\mathbb{1}V(\phi) \big\}\,,
\end{aligned}
\tag{2.170}
$$

where we have added a scalar potential $V(\phi)$. This action gives

$$
\frac{\delta S^{(0)}}{\delta \tilde{B}^m} = -Z^a{}_m R_a\,. \tag{2.171}
$$

This equations should be trivial on-duality-shell and, therefore, we must add to the action $S^{(0)}$

$$
S^{(1)} = \int Z^a{}_m (G_a + \tfrac{1}{2}Z_{an}\tilde{B}^n)\tilde{B}^m\,, \tag{2.172}
$$

so that

$$
\frac{\delta (S^{(0)} + S^{(1)})}{\delta \tilde{B}^m} = -Z^a{}_m (R_a - G_a)\,. \tag{2.173}
$$

The equation for C^a that follows from $S^{(0)} + S^{(1)}$ is

$$
\frac{\delta (S^{(0)} + S^{(1)})}{\delta C^a} = -\mathcal{D}R_a - Z_{am}\mathcal{M}^{mn} \star H_n\,, \tag{2.174}
$$

and, comparing with Eq. (**??**), we see that the term $+d_{aI}{}^m F^I H_m$ is missing and we must add a term of the form

$$
S^{(2)} = \int d_{aI}{}^m F^I \left(H_m - \tfrac{1}{2}Z_{bm}C^b\right) C^a\,, \tag{2.175}
$$

Observe that $\tilde{B}^m$ does not appear in this term and its equation of motion is, therefore, not modified by it. However in this term or in any other similar term the only part of $d_{aI}{}^m Z_{bm}$ that can occur is the antisymmetric one $d_{[a|I}{}^m Z_{|b]m}$ while in the term $+d_{aI}{}^m F^I H_m$ both the antisymmetric and the symmetric parts occur. The only way in which we can get that term in the equations of motion is by requiring

$$
d_{(a|I}{}^m Z_{|b)m} = -\tfrac{1}{2}X_{I\,ab} = 0\,. \tag{2.176}
$$

Under this assumption, which will also prove crucial to obtain the equations of motion of other fields, the equation of motion of C^a is Eq. (**??**), as we wanted.

The equation of the magnetic potential C_a, which should be trivial on-duality-shell which follows from the action we have constructed is

$$\frac{\delta(S^{(0)}+S^{(1)}+S^{(2)})}{\delta C^a} = Z^a{}_m\left[\mathcal{M}^{mn}\star H_n - \mathcal{D}\tilde{B}^m + d_{bI}{}^m F^I C^b\right]\,. \tag{2.177}$$

The last two terms belong to the field strength $\tilde{H}^m$ and we need to add

$$S^{(3)} = \int\left\{-\tfrac{1}{2}d^b{}_I{}^m Z^a{}_m F^I C_b C_a - \left[2d^{mnp}B_n(H_p - 2Z_{ip}C^i - \tfrac{1}{2}\mathcal{D}B_p) + \Box\tilde{H}^m\right]Z^a{}_m C_a\right\}\,, \tag{2.178}$$

where $\Box\tilde{H}^m$ is the part of the field strength $\tilde{H}^m$ that only contains 1-form potentials and their exterior derivatives. Observe that neither $\tilde{B}^m$ nor C^a appear in this term and, therefore, their equations of motion are not modified. Observe also that we are facing here the same problem we faced in getting the equation of motion of C^a: only $d^{[b|}{}_I{}^m Z^{|a]}{}_m$ can enter the action while the equation of motion contains also the symmetric part. The solution to this problem is the same: we demand

$$d^{(a|}{}_I{}^m Z^{|b)}{}_m = -\tfrac{1}{2}X_I{}^{ab} = 0\,. \tag{2.179}$$

Using Eqs.(**??**) and (**??**) The equation of motion of B_m that follows from the action $S^{(0)}+\cdots+S^{(3)}$ can be put in the form

$$\begin{aligned}\frac{\delta(S^{(0)}+\cdots+S^{(3)})}{\delta B_m} \;=\;& -\big[\mathcal{D}(\mathcal{M}^{mn}\star H_n) + d_{aI}{}^m F^I G^a - d^a{}_I{}^m F^I R_a\\ &-d^{mnp}H_{np} - Z^{Im}\mathcal{M}_{IJ}\star F^J\big] - d^{mnp}B_n Z^a{}_p(R_a - G_a)\\ &-d^{mnp}Z^a{}_p d_{aI}{}^q B_n B_q(F^I - \tfrac{1}{2}Z^{Ir}B_r)\\ &+d_{aI}{}^m d^a{}_J{}^n F^I(F^I - \tfrac{1}{2}Z^{Ip}B_p)B_n\\ &+d_{aI}{}^m F^I\Box G^a + d^a{}_I{}^{[m|}Z^{I|n]}B_n\Box G_a\\ &-d^{mnp}(H_n - Z_{in}C^i)(H_p - Z_{jp}C^j)\,.\end{aligned} \tag{2.180}$$

The expression in brackets in the r.h.s. is identical to $\mathcal{B}(\tilde{H}^m)$ up to dualities and up to the term $d^m{}_{IJK}F^{IJK}$. The next term vanishes on-duality-shell and the remaining terms should be eliminated. Observe that in the terms that need to be eliminated and introduced neither $\tilde{B}^m$ nor C^i occur (they only depend on B_m, A^I and their derivatives) and, therefore, their equations of motion will not be modified.

2.5 Conclusions

Following the same procedure as in Refs. [?, ?], in this paper we have constructed the most general 8-dimensional theory with gauge symnmetries and with at most two derivatives: field strengths (up to 6-forms), all the Bianchi identities and duality relations satisfied by all the field strengths (up to the 9-forms[30]), and the equations of motion of the fundamental fields. We have shown that they are characterized by a small number of invariant tensors (d-tensor, embedding tensor ϑ and massive deformations Z) that satisfy certain constraints that relate them among themselves and to the structure constants and generators of the global symmetry group, which has to act on the n_3 3-form potentials of the theory as a subgroup of $\mathrm{Sp}(2n_3, \mathbb{R})$.

We have found that the Bianchi identities satisfied by the 7-form field strengths (dual to the generalized Noether-Gaillard-Zumino current) have the general form predicted in Ref. [?], although in this case it is very difficult to find the explicit form of the 7-form field strengths.

We have constructed an action from which one can derive all the equations of motion except for those of the 1-form potentials because identifying the terms that only contain 1-forms becomes extremely complicated and time-consuming.

This general result can be applied to any 8-dimensional theory with a given field content, d-tensors defining Chern-Simons interactions and global symmetry group, such as maximal $d = 8$ supergravity. In a forthcoming publication we will solve the constraints satisfied by the deformation tensors (d-tensor, embedding tensor ϑ and massive deformations Z) searching for a 1-parameter family of different SO(3) gaugings of that theory.

[30] These identities are, of course, just formal, but they encode the gauge transformations of the 8-form potentials.

3

On gauged maximal $d = 8$ supergravities

This chapter is based on

Oscar Lasso Andino, Tomás Ortín ,
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Gauged/massive supergravities have received a great deal of attention over the last few years because they almost always include a scalar potential that could fix the moduli or provide an interesting inflationary model. While gauging a given supergravity theory obtained, for instance, as the low-energy limit of some string theory model, is just a technical problem which we now know how to handle in general, the string-theory description of that gauged theory, its 11-dimensional origin and the meaning of the new constants that appear in it (coupling constants, mass parameters etc.), are not always known.

The gauging of maximal 8-dimensional supergravity offers a particularly interesting example. Salam and Sezgin obtained this theory with an $SO(3) \subset SL(3, \mathbb{R})$ gauging in Ref. [**?**] by performing a Scherk-Schwarz reduction [**?**] of 11-dimensional supergravity [**?**].[1] The gauge fields of this theory are the three Kaluza-Klein vectors. However, the theory has another $SL(3, \mathbb{R})$ triplet of vectors that can be used as gauge fields: the vectors that come from the 11-dimensional 3-form. This alternative SO(3) gauging can be carried out directly in 8 dimensions by the standard methods, but it is not known how to obtain this theory from the conventional 11-dimensional supergravity.

Actually, it is believed that it should be possible to obtain this second SO(3)-gauged theory by an SL(2) rotation of the Salam-Sezgin one. These transformations of gauged theories are no longer symmetries of their equations of motion. Rather, they are (very complicated) field redefinitions. Thus, at a classical level, and from the 8-dimensional point of view, these two theories should be equivalent.

From the 11-dimensional point of view, the situation is less clear: on the one hand,in principle one may use the 8-dimensional relation between the fields in the two theories to construct a very unnatural and non-local[2] alternative compactification Ansatz which would give the second SO(3) gauged theory instead of the Salam-Sezgin one. On the other hand, it is hard to say whether these two theories are equivalent from the 11-dimensional point of view.

[1] Other 3-dimensional groups can be obtained by the same procedure, as shown in Refs. [**?**, **?**]. We also remind the reader of the U-duality group of this theory: it is $SL(2, \mathbb{R}) \times SL(3, \mathbb{R})$

[2] The SL(2) transformation that should relate these two SO(3) gauged theories involves electric-magnetic rotations of the 3-form potential.

It is somewhat surprising that the second SO(3)-gauged maximal supergravity can be obtained with *exactly the same compactification Ansatz* as the Salam-Sezgin one from the so-called *massive 11-dimensional supergravity* [?]. This theory is a deformation of 11-dimensional supergravity proposed in Ref. [?] as a candidate to 11-dimensional origin of Romans' massive $\mathcal{N} = 2A, d = 10$ supergravity [?].[3] This theory does not have 11-dimensional covariance, as it depends explicitly on the (commuting) Killing vectors but, somewhat mysteriously, it turns out that it can account for the 11-dimensional origin of several gauged supergravity theories (apart from Romans' and the 8-dimensional one under discussion) which are not obtained, wit the conventional compactification Ansatz, by standard methods [?].

Our goal in this paper twofold: first, we want to show that the gauged theory obtained from the compactification of *massive 11-dimensional supergravity* (which will be referred to henceforth as AAMO) is indeed one of the SO(3)-gauged maximal supergravities that can be obtained using the embedding tensor method. Second, we want to show that, from the 8-dimensional point of view, it is related to the Salam-Sezgin one (from this moment, SS) by an $\mathrm{SL}(2,\mathbb{R})$ transformation. We will achieve both goals by constructing a 1-parameter family of SL(2)-related SO(3)-gauged supergravities that interpolates between the SS and AAMO theories.[4]

The best way to construct these gauged theories is through the use of the embedding tensor formalism [?, ?, ?, ?, ?].[5] This formalism has been used in several maximal and half-maximal supergravities Refs. [?, ?, ?, ?, ?, ?, ?]. In the 8-dimensional case it has been used in Ref. [?] to study the possible subgroups of the U-duality group that can be gauged, regardless of the vectors used as gauge fields, by solving the constraints satisfied by the embedding tensor. The existence of continuous families of gauged supergravities escapes this kind of analysis, though, and we are actually interested in the explicit construction of the theory. In a more recent paper [?] we have used the embedding-tensor formalism to construct the most general 8-dimensional gauge theory (including its tensor hierarchy), for any field content and duality group. This result can immediately be particularized to the field content, d-tensors and duality group of the maximal 8-dimensional supergravity and we just have to find a 1-parameter SO(3) solution of the constraints satisfied by the embedding tensor an other deformation parameters to have the complete tensor hierarchy of the theory we are after. To end the construction of the bosonic theory it only remains to find the scalar potential and the equations of motion. We will explain how to do that in this case. We will also explain how to construct the supersymmetry transformation rules.

This paper is organized as follows: in Section **??** we review the matter content and symmetries of the ungauged theory. We will introduce a new basis of fields with simpler transformation properties, as required by the embedding tensor formalism. In Section **??** we will discuss the gauging, using that formalism, of the theory, applying the general results of Ref. [?]. We will show that there is a 1-parameter family of embedding and other deformation tensors associated to SO(3) gaugings we are after. In Section **??** we proceed to the explicit construction of the theory. Our conclusions are described in Section **??** and, in the appendices, the explicit forms of the field strengths, Bianchi identities, identities of Bianchi identities and duality relations, are collected.

[3] The supersymmetry transformations of this theory were studied in Ref. [?].

[4] The existence of these duality-related family of gaugings has been noticed in Refs. [?, ?].

[5] For recent reviews see Refs. [?, ?, ?, ?].

3.1 Ungauged $\mathcal{N} = 2$, $d = 8$ Supergravity

In this section we are just going to describe the aspects of the ungauged theory that we need to know in order to construct the family of gauged supergravities we are after.

$\mathcal{N} = 2$, $d = 8$ supergravity can be obtained by direct dimensional reduction of 11-dimensional supergravity on T^3 [?]. The scalars of the theory parametrize the coset spaces $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ and $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$. The U-duality group of the theory is $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$ and its fields are either invariant or transform in the fundamental representations of both groups. We use the indices $i, j, k = 1, 2$ for $\mathrm{SL}(2,\mathbb{R})$ doublets and $m, n, p = 1, 2, 3$ for $\mathrm{SL}(3,\mathbb{R})$ triplets.

The bosonic fields are

$$g_{\mu\nu}, C, B_m, A^{i\,m}, a, \varphi, \mathcal{M}_{mn}, \tag{3.1}$$

where C is a 3-form, B_m a triplet of 2-forms, $A^{i\,m}$, a doublet of triplets of 1-forms (six in total), a and φ are the axion and dilaton fields which can be combined into the axidilaton field

$$\tau \equiv a + ie^{-\varphi}\,, \tag{3.2}$$

or into the $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ symmetric matrix

$$(\mathcal{W}_{ij}) \equiv e^{\varphi} \begin{pmatrix} |\tau|^2 & a \\ a & 1 \end{pmatrix}, \quad \text{with inverse} \quad (\mathcal{W}^{ij}) \equiv e^{\varphi} \begin{pmatrix} 1 & -a \\ -a & |\tau|^2 \end{pmatrix}, \tag{3.3}$$

and, finally, $\mathcal{M}_{mn}$ is an $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ symmetric matrix whose explicit parametrization in terms of five independent scalars will not concern us for the moment. The inverses of these matrices will be written with upper indices.

The bosonic action obtained in Ref. [?] by simple dimensional reduction is[6]

$$\begin{aligned} S \;=\; \int d^8x\sqrt{|g|}\, \Big\{ & R + \tfrac{1}{4}\mathrm{Tr}\left(\partial\mathcal{M}\mathcal{M}^{-1}\right)^2 + \tfrac{1}{4}\mathrm{Tr}\left(\partial\mathcal{W}\mathcal{W}^{-1}\right)^2 \\ & -\tfrac{1}{4}F^{i\,m}\mathcal{M}_{mn}\mathcal{W}_{ij}F^{j\,n} + \tfrac{1}{2\cdot 3!}H_m\mathcal{M}^{mn}H_n - \tfrac{1}{2\cdot 4!}e^{-\varphi}G^2\,, \\ & -\tfrac{1}{6^3\cdot 2^4}\tfrac{1}{\sqrt{|g|}}\,\epsilon\left[GGa - 8GH_mA^{2\,m} + 12G(F^{2\,m} + aF^{1\,m})B_m\right. \\ & \left.-8\epsilon^{mnp}H_mH_nB_p - 8G\partial aC - 16H_m(F^{2\,m} + aF^{1\,m})C\right]\Big\}\,, \end{aligned} \tag{3.4}$$

where the field strengths are given by[7]

[6] The relation between the 8- and 11-dimensional fields can be found there. As mentioned in Ref. [?], one of the coefficients in the Chern-Simons part of the action (which has been checked explicitly to be gauge-invariant) differs from the corresponding one in Ref. [?].

[7] In this notation, used in Ref. [?], all the lower indices, which are not shown, are antisymmetrized with weight one. The difference with differential-form notation is the normalization of the components of the differential forms: $\omega^{(p)} = \frac{1}{p!}\omega^{(p)}_{\mu_1\cdots\mu_p}dx^{\mu_1} \wedge \cdots dx^{\mu_p}$, so, for instance, $d\omega^{(p)} = (p+1)\partial\omega^{(p)}$.

$$
\begin{aligned}
F^{i\,m} &= 2\partial A^{i\,m}\,. \\
H_m &= 3\partial B_m + 3\epsilon_{mnp}F^{1\,n}A^{2\,p}\,, \\
G &= 4\partial C + 6F^{1\,m}B_m\,,
\end{aligned}
\tag{3.5}
$$

3.1.1 Rewriting the theory

In order to study the gaugings of this theory using the embedding-tensor formalism it is convenient to use differential-form language and, furthermore, use a different basis of forms with better transformation properties under the duality groups (in particular, under $\mathrm{SL}(2,\mathbb{R})$): for instance, if the 3-form field strengths H_m are invariant under $\mathrm{SL}(2,\mathbb{R})$ transformations, it is obvious that the 2-forms B_m can only be invariant under those $\mathrm{SL}(2,\mathbb{R})$ transformations up to 1-form gauge transformations because the Chern-Simons term $3\epsilon_{mnp}F^{1n}A^{2p}$ has that same behaviour. This implies, in its turn, that the 3-form C only transforms as the first component of an $\mathrm{SL}(2,\mathbb{R})$ doublet (something we expect to happen on general grounds) up to gauge transformations. The conclusion is that we are going to need to redefine the 2- and 3-form potentials B_m and C, which we also denote as C^1 when needed.

In differential-form language, the above field strengths take the form

$$
\begin{aligned}
F^{im} &= dA^{im}\,. \\
H_m &= dB_m + \epsilon_{mnp}F^{1n}\wedge A^{2p}\,, \\
G &= dC + F^{1m}\wedge B_m\,.
\end{aligned}
\tag{3.6}
$$

The redefinition of the potentials that gives the the required properties of transformation under the U-duality group is

$$
\begin{aligned}
B_m &\longrightarrow B_m - \tfrac{1}{2}\epsilon_{mnp}A^{1n}\wedge A^{2p}\,, \\
C &\longrightarrow C^1 + \tfrac{1}{2}\epsilon_{mnp}A^{1m}\wedge A^{1n}\wedge A^{2p}\,.
\end{aligned}
\tag{3.7}
$$

In terms of these new potentials, the field strengths take the form[8]

$$F^{im} = dA^{im}\,, \tag{3.8}$$

$$H_m = dB_m + \tfrac{1}{2}\epsilon_{ij}\epsilon_{mnp}F^{in}A^{jp}\,, \tag{3.9}$$

$$G^1 = dC^1 + F^{1m}B_m + \tfrac{1}{6}\epsilon_{ij}\epsilon_{mnp}A^{1m}F^{in}A^{jp}\,. \tag{3.10}$$

The gauge transformations that leave these field strengths invariant are

[8] Here and, very often in what follows, we suppress the wedge product symbols $\wedge$ in order to simplify the expressions. We will introduce further simplifications in the notation along the way.

$$
\begin{aligned}
\delta_\sigma A^{im} &= d\sigma^{im}\,. \\
\delta_\sigma B_m &= d\sigma_m - \epsilon_{ij}\epsilon_{mnp}\left(F^{in}\sigma^{jp} - \tfrac{1}{2}A^{in}\delta_\sigma A^{jp}\right)\,, \qquad (3.11)\\
\delta_\sigma C^1 &= d\sigma^1 - \left[\sigma^{1m}H_m + F^{1m}\sigma_m + \delta_\sigma A^{1m}B_m + \tfrac{1}{6}\epsilon_{jk}\epsilon_{mnp}\delta_\sigma A^{jn}A^{1m}A^{kp}\right]\,,
\end{aligned}
$$

and the gauge-invariant bosonic action can be written in the form

$$
\begin{aligned}
S \;=\; \int \Big\{ & -\star R + \tfrac{1}{4}\mathrm{Tr}\left(d\mathcal{M}\mathcal{M}^{-1}\wedge\star d\mathcal{M}\mathcal{M}^{-1}\right) + \tfrac{1}{4}\mathrm{Tr}\left(d\mathcal{W}\mathcal{W}^{-1}\wedge\star d\mathcal{W}\mathcal{W}^{-1}\right) \\
& +\tfrac{1}{2}\mathcal{W}_{ij}\mathcal{M}_{mn}F^{im}\wedge\star F^{jn} + \tfrac{1}{2}\mathcal{M}^{mn}H_m\wedge\star H_n + \tfrac{1}{2}e^{-\varphi}G^1\wedge\star G^1 - \tfrac{1}{2}aG^1G^1 \\
& +\tfrac{1}{3}G^1\left[H_mA^{2m} - B_mF^{2m} + \tfrac{1}{2}\epsilon_{mnp}F^{2m}A^{1n}A^{2p}\right] \\
& +\tfrac{1}{3}H_mF^{2m}\left[C^1 + \tfrac{1}{6}\epsilon_{mnp}A^{1m}A^{1n}A^{2p}\right] \\
& +\tfrac{1}{3!}\epsilon^{mnp}H_mH_n\left(B_p - \tfrac{1}{2}\epsilon_{pqr}A^{1q}A^{2r}\right)\Big\}\,.
\end{aligned}
\qquad (3.12)
$$

It is not difficult to check that the (formal[9]) exterior derivative of the Chern-Simons part of this action (the last three lines) is just a combination of gauge-invariant field strengths:

$$
d(\mathrm{Chern-Simons}) = -H_mF^{2m}G^1 - \tfrac{1}{3!}\epsilon^{mnp}H_mH_nH_p\,, \qquad (3.13)
$$

which ensures its gauge-invariance up to total derivatives under the transformations Eqs. (**??**).

Global $\mathrm{SL}(2,\mathbb{R})$ covariance requires the introduction of another 3-form C^2 so we can define a doublet of 4-form field strengths

$$
G^i \equiv dC^i + F^{im}B_m + \tfrac{1}{6}\epsilon_{jk}\epsilon_{mnp}A^{im}F^{jn}A^{kp}\,, \qquad (3.14)
$$

invariant under the gauge transformations $\delta_\sigma A^{im}$ and $\delta_\sigma B_m$ in Eq. (**??**) and

$$
\delta_\sigma C^i = d\sigma^i - \left[\sigma^{im}H_m + F^{im}\sigma_m + \delta_\sigma A^{im}B_m + \tfrac{1}{6}\epsilon_{jk}\epsilon_{mnp}\delta_\sigma A^{jn}A^{im}A^{kp}\right]\,. \qquad (3.15)
$$

This (magnetic, dual) field is related by electric-magnetic duality to the original (electric, fundamental) C so there are no new degrees of freedom on *duality shell*[10]

$$
G^2 = e^{-\varphi}\star G + aG \equiv \tilde{G}\,, \qquad (3.16)
$$

and the relation is such that, using it, the equation of motion of C that follows from the action Eq. (**??**)

[9] It is the total derivative of an 8-form in 8 dimensions.

[10] Observe that $\tilde{G}$ is a combination of the field strength of the electric 3-form G, its Hodge dual $\star G$ and the scalars, while G^2 is the field strength of the magnetic 3-form C^2.

$$-\frac{\delta S}{\delta C} = d\tilde{G} - F^{1m} H_m \,, \tag{3.17}$$

becomes the Bianchi identity for the field strength G^2.

Then, denoting with a Δ the part of a $(p+1)$-field strength that does not contain the derivative of the p-form potential, using the above definitions we can rewrite the action Eq. (**??**) in a more compact form that we will use later:

$$\begin{aligned} S = & \int \Big\{ -\star R + \tfrac{1}{4}\mathrm{Tr}\left(d\mathcal{M}\mathcal{M}^{-1} \wedge \star d\mathcal{M}\mathcal{M}^{-1}\right) + \tfrac{1}{4}\mathrm{Tr}\left(d\mathcal{W}\mathcal{W}^{-1} \wedge \star d\mathcal{W}\mathcal{W}^{-1}\right) \\ & + \tfrac{1}{2}\mathcal{W}_{ij}\mathcal{M}_{mn}F^{im} \wedge \star F^{jn} + \tfrac{1}{2}\mathcal{M}^{mn}H_m \wedge \star H_n + \tfrac{1}{2}G\tilde{G} - dC^1\Delta G^2 - \tfrac{1}{2}\Delta G^1 \Delta G^2 \\ & - \tfrac{1}{12}\epsilon^{mnp}B_m dB_n dB_p + \tfrac{1}{4}\epsilon^{mnp}B_m H_n H_p - \tfrac{1}{24}\epsilon_{ij}A^{im}A^{jn}\Delta H_m dB_n \Big\} \,. \end{aligned} \tag{3.18}$$

Potentials dual to the 2-forms (the 4-forms $\tilde{B}^m$), to the 1-forms A^{im} (the 5-forms $\tilde{A}_{im}$) and to the scalars (the 6-forms D_A, where the index A runs over the adjoint representation of the duality group $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$), and their gauge-invariant field strengths $(\tilde{H}^m, \tilde{F}_{im}, K_A)$ can also be defined by dualizing the equations of motion of the corresponding electric fields. We will not need them now, but they can be found in Ref. [**?**]. They can also be recovered by setting to zero the deformation parameters in the field strengths of the gauged theory that we are going to construct in the next section and which are listed in Appendix **??**.

3.2 SO(3) gaugings of $\mathcal{N} = 2$, $d = 8$ supergravity

The gaugings and massive deformations of general 8-dimensional field theories have been studied in depth using the embedding-tensor formalism in Ref. [**?**] using the notation of Ref. [**?**] and the general procedure used in Refs. [**?**,**?**] for the 4-,5- and 6-dimensional cases: finding identities for Bianchi identities, solving those identities for the Bianchi identities and then solving the Bianchi identities for the field strengths. In particular, the tensor hierarchy has been constructed and the form of most of the field strengths has been fully determined. The action was only determined up to terms containing 2-forms due to the very large number of complicated terms occurring in it.

In this section we are going to specialize the results of Ref. [**?**] to the particular case of $\mathcal{N} = 2$, $d = 8$ supergravity and, then, we are going to select the family of gaugings we are interested in[11]. Since the case we are going to study is far simpler than the general case, we are going to determine almost the bosonic action.

In order to particularize the results of Ref. [**?**] to $\mathcal{N} = 2$, $d = 8$ supergravity we have to particularize the generic field content, the d-tensors occurring in the Chern-Simons terms and the global symmetry group considered there.

Let us start by reviewing the U-duality group of the theory. The U-duality group

[11] A partial analysis of the possible gaugings (that is: the possible solutions to the constraints satisfied by the embedding tensor) was performed in Ref. [**?**].

of this theory is, exactly, $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(3,\mathbb{R})$[12] and we remind the reader of the group isomorphism $\mathrm{SL}(2,\mathbb{R}) \sim \mathrm{Sp}(2,\mathbb{R})$. The adjoint indices of the U-duality group are denoted collectively by $A, B, \ldots$. The adjoint indices of $\mathrm{SL}(2,\mathbb{R})$ are $\alpha, \beta, \ldots = 1,2,3$. The adjoint indices of $\mathrm{SL}(3,\mathbb{R})$ are $m, n, \ldots = 1,2,3$ for the SO(3) subgroup that we want to gauge and $a, b, \ldots = 1, \cdots, 5$ for the rest of the generators.

The only structure constants that we need to know explicitly are those of the SO(3) subgroup:[13]

$$[T_m, T_n] = f_{mn}{}^{p} T_p = -\epsilon_{mn}{}^{p} T_p \,, \tag{3.19}$$

so the SO(3) generators in the fundamental/adjoint representation are the matrices

$$T_m{}^{n}{}_{p} = \epsilon_m{}^{n}{}_{p} = -\epsilon_{mpn} \,. \tag{3.20}$$

We also need to know that the coset space $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ is a symmetric space and the structure constants with mixed indices $f_{ma}{}^{b}$ provide a representation of SO(3) acting on the $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ indices $a, b, \cdots$:

$$T_m{}^{a}{}_{b} = f_{mb}{}^{a} \,. \tag{3.21}$$

As for the generators of $\mathrm{SL}(2,\mathbb{R}) \sim \mathrm{Sp}(2,\mathbb{R})$ in the fundamental representation $T_\alpha{}^{i}{}_{j}$ we just need to know the property

$$T_\alpha{}^{k}{}_{[j}\epsilon_{i]k} \equiv T_{\alpha\,[ij]} = 0 \,, \tag{3.22}$$

Let us consider now the field content. In Ref. [?] the scalars were collectively denoted by ϕ^x. We are going to keep using that notation for the time being. The vector fields carried indices $I, J, \ldots$ and they must be replaced by composite indices im, jn etc. where $i, j, \ldots = 1,2$ and $m, n, \ldots = 1,2,3$ are indices in the fundamental representations of $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(3,\mathbb{R})$, respectively. The notation for the 2-forms is the same. In Ref. [?] the electric 3-forms carry an index a which is the upper component of a symplectic index denoted by $i, j, \ldots$. In the case at hands, a takes only one value: 1 (C^1) which will be sometimes omitted (C). The lower index 1 is equivalent to an upper index 2: $C_1 = \epsilon_{12}C^2 = C^2$ and, therefore $(C^i) = \begin{pmatrix} C^1 \\ C_1 \end{pmatrix} = \begin{pmatrix} C^1 \\ C^2 \end{pmatrix}$. On the other hand, $C_i \equiv \epsilon_{ij}C^j$.

Finally, in order to find the values of the d-tensors for this theory it is enough to compare the field strengths of this theory with those of the generic ungauged theory constructed in Ref. [?]. Comparing Eqs. (??),(??) and (??) with

$$F^I = dA^I \,. \tag{3.23}$$

$$H_m = dB_m - d_{mIJ} F^I A^J \,, \tag{3.24}$$

$$G^i = dC^i + d^i{}_I{}^m F^I B_m - \tfrac{1}{3} d^i{}_I{}^m d_{mJK} A^I F^J A^K \,. \tag{3.25}$$

[12] There is only one additional rescaling symmetry, but it acts on the spacetime metric and, therefore, we will not consider it here.

[13] SO(3) indices are raised and lowered with the unit metric and, therefore, there is no distinction between upper and lower SO(3) indices. We choose their position for the sake of convenience and esthetics.

we conclude that the d-tensors can be constructed entirely in terms of the U-duality invariant tensors $\delta^i{}_j, \epsilon_{ij}, \delta^m{}_n, \epsilon_{mnp}$:

$$\begin{aligned} d_{mIJ} &\rightarrow d_{m\,in\,jp} = -\tfrac{1}{2}\epsilon_{mnp}\epsilon_{ij}\,, \\ d^i{}_I{}^m &\rightarrow d^i{}_{jn}{}^m = \delta^i{}_j\delta^m{}_n\,. \end{aligned} \tag{3.26}$$

The tensor d^{mnp} is related to these by

$$d^i{}_{(I|}{}^m d_{i|J)}{}^n = -2d^{mnp}d_{pIJ}\,, \quad \Rightarrow \quad d^{mnp} = +\tfrac{1}{2}\epsilon^{mnp}\,. \tag{3.27}$$

We can immediately use the results of Ref. [?] to determine the form of the 5-form field strengths $\tilde{H}^m$ (dual to the H_m) and the 6-forms $\tilde{F}_{im}$ (dual to the 2-forms F^{im})[14]. We can also derive the the Bianchi identities satisfied by all of them and also by the 7-form field strengths K_A dual to the Noether current 1-forms of the scalar σ-model $j^{(\sigma)}_A$ where $A = m, a, \alpha$ runs in the adjoint of the U-duality group. The later are given by

$$j^{(\sigma)}_A \equiv \mathcal{G}_{xy}k_A{}^x d\phi^y\,, \tag{3.28}$$

where $\mathcal{G}_{xy}(\phi)$ is the σ-model metric and $k_A{}^x(\phi)$ is the Killing vector of that metric associated to the generator of the U-duality group T_A

$$[T_A, T_B] = f_{AB}{}^C T_C\,, \qquad [k_A, k_B] = -f_{AB}{}^C k_C\,. \tag{3.29}$$

We are, however, interested in the gauged theory. The most general gaugings can be found using the embedding-tensor formalism. In this theory, the embedding tensor has the form $\vartheta_{im}{}^A$. We know there are at least two possible $SO(3) \subset SL(3,\mathbb{R})$ gaugings of this theory:

1. Salam and Sezgin's [?], in which the 3 vector fields A^{1m} coming from the metric of 11-dimensional supergravity (that is, the 3 Kaluza-Klein (KK) vector fields) are used as gauge fields.

2. The AAMO [?] gauging in which the 3 gauge fields are the A^{2m} coming from the 3-form of 11-dimensional supergravity.

These two sets of gauge fields are related by the discreet electric-magnetic $SL(2,\mathbb{R})$ duality transformation $\tau \rightarrow -1/\tau$ before gauging. Correspondingly, the SS gauging corresponds to choosing an embedding tensor whose only non-vanishing components are $\vartheta_{im}{}^n = g\delta_i{}^1\delta_m{}^n$ where g is the coupling constant, and the AAMO gauging corresponds to the choice $\vartheta_{im}{}^n = g\delta_i{}^2\delta_m{}^n$.

From the 8-dimensional supergravity point of view, one could use any other $SL(2,\mathbb{R})$ transformed of the A^{1m} triplet as gauge fields. This suggests that a continuous family of equivalent $SO(3)$ gaugings should exist. The corresponding embedding tensor has the form

$$\vartheta_{im}{}^n = v_i\delta_m{}^n\,, \tag{3.30}$$

[14]The explicit expressions of the field strengths for a generic 8-dimensional theory is only given up to the 6-forms in Ref. [?].

where v_i is a 2-component vector transforming in the fundamental of the electric-magnetic $\mathrm{SL}(2,\mathbb{R})$ duality group and can describe a one-parameter family of equivalent SO(3) gaugings of the theory[15]. The SO(3) gauge fields are combinations of the two triplets of vector fields

$$\vartheta_{in}{}^{m} A^{in} = v_i A^{im}\,, \tag{3.31}$$

and include, as limiting cases, the Salam-Sezgin and the AAMO theories.

Our candidate to embedding tensor Eq. (**??**) must solve the quadratic constraint, which implies its own gauge invariance

$$\vartheta_{im}{}^{B} Y_{B\,jn}{}^{A} = 0\,, \tag{3.32}$$

where the Y tensor is

$$\begin{aligned} Y_{B\,jn}{}^{A} \equiv \delta_B \vartheta_{jn}{}^{A} &= -T_B{}^{k}{}_{j}\vartheta_{kn}{}^{A} - T_B{}^{p}{}_{n}\vartheta_{jp}{}^{A} + T_B{}^{A}{}_{C}\vartheta_{jn}{}^{C} \\ &= -T_B{}^{k}{}_{j}\vartheta_{kn}{}^{A} - T_B{}^{p}{}_{n}\vartheta_{jp}{}^{A} + f_{BC}{}^{A}\vartheta_{jn}{}^{C}\,. \end{aligned} \tag{3.33}$$

For the above embedding tensor Eq. (**??**), the only non-vanishing components of the $Y_{A\,im}{}^{B}$ tensor are

$$\begin{aligned} Y_{a\,im}{}^{n} &= -v_i T_a{}^{n}{}_{m}\,, \qquad Y_{a\,im}{}^{b} = -v_i f_{ma}{}^{b}\,, \\ Y_{\alpha\,im}{}^{n} &= -T_\alpha{}^{j}{}_{i} v_j \delta_m{}^{n}\,, \end{aligned} \tag{3.34}$$

and, therefore, the quadratic constraint is automatically satisfied and the embedding tensor is, in principle, admissible.

There are other parameters associated to deformations of the theory that must be considered together with the embedding tensor because they can be related. The d-tensors being defined already in the undeformed theory, the rest of the deformations of the theory are dictated by the Stückelberg mass parameters Z^{imn} and Z_{im}.

Z^{imn} is related to the embedding tensor through the defining relation[16]

$$\vartheta_{(im|}{}^{A} T_A{}^{jn}{}_{|kp)} = Z^{jnq} d_{q\,im\,jp}\,. \tag{3.35}$$

through the orthogonality relation

$$\vartheta_{im}{}^{A} Z^{imn} = 0\,, \tag{3.36}$$

and through the requirement of gauge invariance

$$\vartheta_{im}{}^{A} Y_A{}^{jnp} = 0\,, \quad \text{where} \quad Y_A{}^{jnp} \equiv \delta_A Z^{inp}\,. \tag{3.37}$$

It is not difficult to see that the only solution to these three constraints is

[15] One of the two degrees of freedom of v_i corresponds to the gauge coupling constant.

[16] In this equation the parenthesis indicates the simultaneous symmetrization of the pairs of indices im and kp.

$$Z^{imn} = v^i \delta^{mn}\,, \qquad (v^i = \epsilon^{ji} v_j)\,, \tag{3.38}$$

and, therefore, the only non-vanishing components of the tensor ${Y_a}^{imn}$ are

$${Y_a}^{imn} = v^i {T_a}^{(m}{}_q \delta^{n)q}\,, \qquad {Y_\alpha}^{imn} = {T_\alpha}^i{}_j v^j \delta^{mn}\,. \tag{3.39}$$

Z_{im} must be orthogonal to Z^{inm}

$$Z_{im} Z^{jnm} = 0\,, \tag{3.40}$$

which can only be satisfied by $Z_{im} = 0$. This solution is gauge-invariant and the corresponding Y tensor vanishes identically:

$$Y_{A\,im} = 0\,, \qquad A = m, a, \alpha\,. \tag{3.41}$$

There are five constraints more relating the three deformation tensors ${\vartheta_{im}}^A, Z^{imn}$ and Z_{im} among themselves and to the d-tensors [**?**]:

$$\begin{aligned}
{\vartheta_{im}}^A {T_A}^p{}_n + 2 d_{n\,im\,jq} Z^{jqp} + Z_{jn} d^j{}_{im}{}^p &= 0\,,\\
{\vartheta_{im}}^A T_{A\,jk} + 2 Z_{(j|n} d_{|k)im}{}^n &= 0\,,\\
d^i{}_{jp}{}^{[m|} Z^{jp|n]} + Z^i{}_p d^{pmn} &= 0\,,\\
\tfrac{1}{2} d^k{}_{(ip|}{}^m d_{k|jq)}{}^n + d^{mnp} d_{p\,ip\,jq} + 3 d^{[m|}{}_{ip\,jq\,lr} Z^{lr|n]} &= 0\,,\\
Z_{im} d^m{}_{jn\,kp\,lq} - d_{i(jn|}{}^m d_{m|kp\,lq)} &= 0\,,
\end{aligned} \tag{3.42}$$

where $d^m{}_{jn\,kp\,lq}$ is another d-tensor fully symmetric in the three lower (pairs of) indices. They are satisfied identically when this tensor vanishes.

The conclusion of this section is that we have found a set of deformation parameters

$${\vartheta_{im}}^n = v_i {\delta_m}^n\,, \quad Z^{imn} = v^i \delta^{mn}\,, \quad Z_{im} = 0 \tag{3.43}$$

that describe a one-parameter family of SO(3) gaugings of maximal 8-dimensional supergravity with the properties we were looking for.

In what follows we are going to construct explicitly this family of theories using the general results of Ref. [**?**].

3.3 Construction of the 1-parameter family of equivalent SO(3)-gauged $\mathcal{N} = 2$, $d = 8$ supergravities

The first step is the construction of the tensor hierarchy. Since this has been done in Ref. [**?**] for most of the fields in a generic 8-dimensional theory, we just have to replace the values of the d-tensors and the deformation tensors to get most of the field strengths and all the Bianchi identities and the identities relating all the Bianchi identities. They can be found in Appendices ??, ?? and ??, respectively. Nevertheless, we would like to remark the definitions of the first and second covariant derivatives of the scalars

$$\mathcal{D}\phi^x \equiv d\phi^x - A^{im} v_i k_m{}^x \,, \tag{3.44}$$

$$\mathcal{D}\star\mathcal{D}\phi^x \equiv d\star\mathcal{D}\phi^x + \Gamma_{yz}{}^x \mathcal{D}\phi^y \wedge \star\mathcal{D}\phi^z - A^{im} v_i \partial_y k_m{}^x \wedge \star\mathcal{D}\phi^y \,, \tag{3.45}$$

and the fact that the Noether current 1-forms defined in Eq. (??)[17] also need to be covariantized

$$j_A^{(\sigma)} \equiv k_A{}^x \mathcal{G}_{xy} \mathcal{D}\phi^y \,, \tag{3.46}$$

$$\mathcal{D}\star j_A^{(\sigma)} \equiv d\star j_A + f_{AB}{}^C A^B \wedge \star j_C \,. \tag{3.47}$$

As explained in Ref. [?], the Bianchi identities of the magnetic fields are related to the equations of motion of the electric ones upon the use of the duality relations between electric and magnetic field strength given in Appendix ?? and assuming that the Bianchi identities of the electric field strengths are satisfied. The precise relation can be found by studying the Noether identities associated to the gauge invariance of the action of the theory (whose existence we assume) and, adapted to this theory, is

$$k_A{}^x \frac{\delta S}{\delta \phi^x} = \mathcal{B}(K_A) \,, \qquad A = m, a, \alpha \,, \tag{3.48}$$

$$\frac{\delta S}{\delta A^{im}} = \mathcal{B}(\tilde{F}_{im}) + \left(\delta^1{}_i B_m - \tfrac{1}{6}\epsilon_{mnp} A^{1n} A^{jp}\right) \mathcal{B}(G^2) - \tfrac{1}{2}\epsilon_{mnp}\epsilon_{ij} A^{jn} \mathcal{B}(\tilde{H}^p) \,, \tag{3.49}$$

$$\frac{\delta S}{\delta B_m} = \mathcal{B}(\tilde{H}^m) \,, \tag{3.50}$$

$$\frac{\delta S}{\delta C^1} = \mathcal{B}(G^2) \,. \tag{3.51}$$

From these relations we find

[17] In absence of interactions between the scalars and other fields these Noether currents are conserved $d\star j_A^{(\sigma)} = 0$. After gauging, in general they are no longer conserved. Their covariant generalizations are covariantly conserved $\mathcal{D}\star j_A^{(\sigma)} = 0$, though. See Eq. (??) and its ungauged limit.

$$
\begin{aligned}
k_m{}^x \frac{\delta S}{\delta \phi^x} &= -\mathcal{D} \star j_m^{(\sigma)} - \epsilon_m{}^n{}_p \left[\mathcal{M}_{nq} \mathcal{W}_{ij} F^{ip} \wedge \star F^{jq} - \mathcal{M}^{pq} H_n \wedge \star H_q \right] , & (3.52) \\
k_a{}^x \frac{\delta S}{\delta \phi^x} &= -\mathcal{D} \star j_a^{(\sigma)} - T_a{}^n{}_p \left[\mathcal{M}_{nq} \mathcal{W}_{ij} F^{ip} \wedge \star F^{jq} - \mathcal{M}^{pq} H_n \wedge \star H_q \right. \\
&\quad \left. - v_i \star \frac{\partial V}{\partial \vartheta_{ip}{}^n} + v^i \star \frac{\partial V}{\partial Z_{ipn}} \right] , & (3.53) \\
k_\alpha{}^x \frac{\delta S}{\delta \phi^x} &= -d \star j_\alpha^{(\sigma)} - T_\alpha{}^i{}_j \left[\mathcal{M}_{mn} \mathcal{W}_{ik} F^{jm} \wedge \star F^{kn} + \tfrac{1}{2} \mathcal{W}_{ik} G^j \wedge \star G^k \right. \\
&\quad \left. - v_i \delta^m{}_n \star \frac{\partial V}{\partial \vartheta_{jn}{}^m} + v^j \delta^{mn} \star \frac{\partial V}{\partial Z_{imn}} \right] , & (3.54) \\
\frac{\delta S}{\delta A^{im}} &= -\mathcal{D}(\mathcal{W}_{ij} \mathcal{M}_{mn} \star F^{jn}) - \epsilon_{mnp} \epsilon_{ij} F^{jn} \mathcal{M}^{pq} \star H_q - (\delta_i{}^1 \tilde{G} - \delta_i{}^2 G) H_m \\
&\quad - v_i K_m + \left(\delta^1{}_i B_m - \tfrac{1}{6} \epsilon_{mnp} A^{1n} A^{jp} \right) \frac{\delta S}{\delta C} - \tfrac{1}{2} \epsilon_{mnp} \epsilon_{ij} A^{jn} \frac{\delta S}{\delta B_p} , & (3.55) \\
\frac{\delta S}{\delta B_m} &= -\mathcal{D}(\mathcal{M}^{mn} \star H_n) + F^{1m} \tilde{G} - F^{2m} G + \tfrac{1}{2} \epsilon^{mnp} H_n H_p + v^i \mathcal{W}_{ij} \mathcal{M}_{mn} \star F^{jn} , & (3.56) \\
\frac{\delta S}{\delta C} &= -d\tilde{G} + F^{2m} H_m \, . & (3.57)
\end{aligned}
$$

The scalar equations of motion can be recovered from the above three relations by using

1. The relation that expresses the gauge-invariance of the scalar potential

$$
k_A{}^x \frac{\partial V}{\partial \phi^x} = Y_A{}^\sharp \frac{\partial V}{\partial c^\sharp} \, , \tag{3.58}
$$

where the index $\sharp$ labels the deformations $c^\sharp$, which, in this case, are just $\vartheta_{im}{}^A$, Z^{imn} and Z_{im}. Using the values of the Y-tensors computed before and

$$
\frac{\partial V}{\partial Z_{im}} = 0 \, , \tag{3.59}
$$

we get the relations

$$
\begin{aligned}
k_m{}^x \frac{\partial V}{\partial \phi^x} &= 0 \, , & (3.60) \\
k_a{}^x \frac{\partial V}{\partial \phi^x} &= -v_i T_a{}^p{}_n \frac{\partial V}{\partial \vartheta_{in}{}^p} + v^i T_a{}^n{}_p \frac{\partial V}{\partial Z^{inp}} \, , & (3.61) \\
k_\alpha{}^x \frac{\partial V}{\partial \phi^x} &= -T_\alpha{}^j{}_i v_j \delta^p{}_n \frac{\partial V}{\partial \vartheta_{in}{}^p} + T_\alpha{}^i{}_j v^j \delta^{np} \frac{\partial V}{\partial Z_{inp}} \, . & (3.62)
\end{aligned}
$$

2. The invariance of the theory under the U-duality group implies that the kinetic matrices $\mathcal{M}_{mn}(\phi)$ and $\mathcal{W}_{ij}$ satisfy the following relations:

$$\begin{aligned} k_m{}^x\partial_x\mathcal{M}_{np} &= -2\epsilon_m{}^q{}_{(n}\mathcal{M}_{p)q}\,, \\ k_a{}^x\partial_x\mathcal{M}_{np} &= -2T_a{}^q{}_{(n}\mathcal{M}_{p)q}\,, \\ k_\alpha{}^x\partial_x\mathcal{W}_{ij} &= -2T_\alpha{}^K{}_{(i}\mathcal{W}_{j)k}\,. \end{aligned} \tag{3.63}$$

   The axidilaton field τ transforms non-linearly under $\mathrm{SL}(2,\mathbb{R})$ (fractional-linear transformations). Taking into account the (unconventional, by an overall sign) definition of the dual 4-form $\tilde{G}$ that constitutes the lower entry of the symplectic vector G^i[18], the infinitesimal $\mathrm{SL}(2,\mathbb{R})$ transformations of τ take the form

$$k_\alpha{}^x\partial_x\tau = -T_{\alpha\,11} + (T_{\alpha\,1}{}^1 - T_\alpha{}^1{}_1)\tau + T_\alpha{}^{11}\tau^2\,. \tag{3.64}$$

3. Finally, using the Killing equation it is not difficult to prove the following identity for the Killing vectors $k_A{}^x$ of a metric $\mathcal{G}_{xy}(\phi)$ and the associated covariantized Noether 1-form defined in Eq. (**??**)

$$k_{A\,x}\mathcal{D}\star\mathcal{D}\phi^y = \mathcal{D}\star j_A^{(\sigma)}\,. \tag{3.65}$$

   Then, the scalar equations of motion are

$$\begin{aligned} \frac{\delta S}{\delta\phi^y} = & -\mathcal{G}_{xy}\mathcal{D}\star\mathcal{D}\phi^y + \tfrac{1}{2}\partial_x\left\{\mathcal{W}_{ij}\mathcal{M}_{mn}F^{im}\wedge\star F^{jn} + \mathcal{M}^{mn}H_m\wedge\star H_n\right. \\ & \left. + e^{-\varphi}G\wedge\star G - aG\wedge G - V(\phi)\right\}\,. \end{aligned} \tag{3.66}$$

These equations can be split into those corresponding to the scalars in the coset spaces $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ and $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ in the obvious way.

We will discuss the form of the potential later.

The scalar equations of motion give us all the kinetic terms in the action:

$$\begin{aligned} S^{(0)} = \int & \left\{-\star R + \tfrac{1}{4}\mathrm{Tr}\left(\mathcal{D}\mathcal{M}\mathcal{M}^{-1}\wedge\star\mathcal{D}\mathcal{M}\mathcal{M}^{-1}\right) + \tfrac{1}{4}\mathrm{Tr}\left(d\mathcal{W}\mathcal{W}^{-1}\wedge\star d\mathcal{W}\mathcal{W}^{-1}\right)\right. \\ & \left. + \tfrac{1}{2}\mathcal{W}_{ij}\mathcal{M}_{mn}F^{im}\wedge\star F^{jn} + \tfrac{1}{2}\mathcal{M}^{mn}H_m\wedge\star H_n + \tfrac{1}{2}e^{-\varphi}G\wedge\star G - \tfrac{1}{2}aG\wedge G - V\right\}\,. \end{aligned} \tag{3.67}$$

(We have added the Hilbert-Einstein term, which, evidently, should be there). Now we have to add the Chern-Simons terms necessary to obtain the other equations of motion, starting by those of the higher-rank potentials (C). However, all the Chern-Simons terms of the ungauged theory must be present (since we must recover it in the $v^i = 0$ limit) and it makes sense to add to the above action the covariantization of those terms, namely

[18] It is this definition that brings us to the unconventional $\mathrm{SL}(2,\mathbb{R})$ matrix $\mathcal{W}$

$$
\begin{aligned}
S^{(1)} \;=\; & \int \big\{ -dC^1 \Delta G^2 - \tfrac{1}{2}\Delta G^1 \Delta G^2 - \tfrac{1}{12}\epsilon^{mnp} B_m \mathcal{D}B_n \mathcal{D}B_p + \tfrac{1}{4}\epsilon^{mnp} B_m H_n H_p \\
& -\tfrac{1}{24}\epsilon_{ij} A^{im} A^{jn} \Delta H_m \mathcal{D}B_n \big\} \,,
\end{aligned} \tag{3.68}
$$

where now the field strengths and derivatives are the covariant ones and

$$
\Delta H_m = H_m - \mathcal{D}B_m \,, \qquad \Delta G^i = G^i - dC^i \,. \tag{3.69}
$$

C only occurs in one place in this Chern-Simons term and, therefore, using $d\Delta G^2 = dG^2$ and the Bianchi identity $\mathcal{B}(G^2)$ in Eq. (**??**) we get

$$
\frac{\delta S^{(0)} + S^{(1)}}{\delta C^1} = -d\tilde{G} + d\Delta G^2 = -d\tilde{G} + F^{2m} H_m \,, \tag{3.70}
$$

in agreement with Eq. (**??**).

For the 2-forms we find

$$
\begin{aligned}
\frac{\delta S^{(0)} + S^{(1)}}{\delta B_m} \;=\; & \frac{\delta S}{\delta B_m} + \tfrac{1}{12} v_i F^{im} B_n B_n + \tfrac{1}{6} v_i F^{in} B_n B_m + \tfrac{1}{2} v_i \Box G^i B_m \\
& + \tfrac{1}{2}\epsilon_{ij} \Box G^i \Box F^{jm} - \tfrac{1}{4}\epsilon^{mnp} \Delta H_n \Delta H_p + \tfrac{1}{24}\mathcal{D}\left(\epsilon_{ij} A^{im} A^{in} \Delta H_n\right) \,,
\end{aligned} \tag{3.71}
$$

where $\delta S/\delta B_m$ is the expected equation of motion, given in Eq. (**??**), and where the boxes acting on field strengths denote the terms on that field strength that only depend on the 1-form fields. Thus, the terms in the second line only depend on the 1-form fields and it is very easy to add a term to the action, linear in B_m to cancel them. However, we must make sure, first, that those terms always depend on v^i, so they disappear in the ungauged limit. Indeed, expanding them we find that all the v-independent terms in them cancel. As for the unwanted terms in the first line (all of them v-dependent), they can be easily integrated. We conclude that we must add to the action a new correction:

$$
\begin{aligned}
S^{(2)} \;=\; & \int \big\{ -\tfrac{1}{12} v_i (F^{im} - v^i B_m) B_m B_n B_n + \tfrac{1}{4}\epsilon^{mnp} B_m \Delta H_n \Delta H_p - \tfrac{1}{2}\epsilon_{ij} \Box G^i \Box F^{jm} B_m \\
& + \tfrac{1}{24}\epsilon_{ij} A^{im} A^{in} \mathcal{D}B_m \Delta H_n \big\} \,.
\end{aligned} \tag{3.72}
$$

Varying $S^{(0)} + S^{(1)} + S^{(2)}$ with respect to C and B_m gives the expected equations of motion.

The terms that remain to be added only contain 1-forms and their derivatives and only contribute to the equations of motion of the 1-forms. They are of the form $(dA)^2 A^4$ and $(dA)A^6$ and their form is exceedingly complicated and we have not determined them.

3.3.1 The scalar potential

Finally, we have to find the scalar potential. The scalar potential must satisfy Eq. (**??**), but this equation does not fully determine it. In supergravity theories, the scalar potential is determined by supersymmetry, and is quadratic in the *fermion shifts*.[19]

There seem to be no general rules available in the literature to construct the fermion shifts of any gauged supergravity, although, based on the example of gauge $\mathcal{N} = 3, d = 4$ supergravity [**?**], it was suggested in Ref. [**?**] that they can be written in terms of the *dressed structure constants* of the gauge group.

Looking into Ref. [**?**], we can see that the fermion shifts of SO(3)-gauged $\mathcal{N} = 2, d = 8$ supergravity theory fit into this general rule and are written in terms of

$$f_{\mathbf{mn}}{}^{\mathbf{p}} \equiv L_{\mathbf{m}}{}^{m} L_{\mathbf{n}}{}^{n} L_{p}{}^{\mathbf{p}} f_{mn}{}^{p} \,, \tag{3.73}$$

where $f_{mn}{}^{p} = \epsilon_{mnp}$, the matrix $L_{\mathbf{m}}{}^{n}$ is the SL(3, $\mathbb{R}$)/SO(3) coset representative, and $L_{m}{}^{\mathbf{n}}$is its inverse.[20]

It is, however, well-known that in $\mathcal{N} = 1, 2, d = 4$ supergravities the fermion shifts are written in terms of the *momentum maps* $P_A{}^{\Sigma}$ associated to the symmetries being gauged: the index A runs over its Lie algebra and the index Σ runs over the Lie Algebra of the R-symmetry group. Thus, in this theory, they would have be $P_A{}^{m}$ with $A = m, a, \alpha$.

As discussed in Ref. [**?**], these two ways of writing fermion shifts are, actually, equivalent because the dressed structure constants can be rewritten in terms of the momentum maps. The momentum maps, though, can be combined with the embedding tensor in a natural way ($\vartheta_{im}{}^{n} P_n{}^{p}$) and more general gaugings can be considered. We will, therefore, use the momentum maps to write the fermion shifts of the theory at hands.

A problem one finds in trying to write fermion shifts with the right structure is that the structure of the fermion shifts and of the entire supersymmetry transformations given in Ref. [**?**] does not show the transformation properties of the spinors under the R-symmetry group SO(2)×SO(3) ∼U(1)×SU(2), because the fermions obtained in the dimensional reduction from 11 dimensions are not symplectic-Majorana. A symplectic-Majorana (pair) ϵ^I $I = 1, 2$ transforms as a doublet under SU(2) and as a singlet under U(1) in a natural way. Therefore, we are going to use symplectic-Majorana spinors in our proposal: gravitini $\psi_{\mu I}$, dilatini λ^I_m and supersymmetry parameters ϵ^I and we are going to define the fermion shifts $S_{IJ}, N_{\mathbf{m}}{}^{I}{}_{J}$

$$\begin{aligned} \delta_\epsilon \psi_{\mu I} &\sim \cdots + S_{IJ}\epsilon^J \,, \\ \delta_\epsilon \lambda^I_{\mathbf{m}} &\sim \cdots + N_{\mathbf{m}}{}^{I}{}_{J}\epsilon^J \,. \end{aligned} \tag{3.74}$$

Now, in order to construct S_{IJ} and $N_m{}^I{}_J$ it is necessary to introduce an object with properties similar to those of the symplectic sections of $\mathcal{N} = 2, d = 4$ theories and their generalizations to higher $\mathcal{N}$ denoted by $\mathcal{V}^M{}_{IJ}$ where the index M labels the vectors available in the theory (electric and magnetic in 4 dimensions) and the indices I, J are R-symmetry indices in the representation carried by the spinors (the fundamental of SU()$\mathcal{N}$). This generalization, must have the same structure, i.e. $\mathcal{V}^{im}{}_{IJ}$ and our proposal for this object is

[19] The exception is $\mathcal{N} = 1, d = 4$ supergravity, which, even in the ungauged case, admits a scalar potential entirely built from the superpotential, which is largely arbitrary.

[20] Here $m, n, p = 1, 2, 3$ are, as in the rest of this paper, indices of the fundamental (vector) representation of SL(3, $\mathbb{R}$) and $\mathbf{m}, \mathbf{n}, \mathbf{p} = 1, 2, 3$ are indices in the fundamental representation of SO(3)).

$$\mathcal{V}^{im}{}_{IJ} \equiv V^i L_{\mathbf{m}}{}^m \epsilon_{IK} \sigma^{\mathbf{m}\,K}{}_J\,, \quad \text{and} \quad \mathcal{V}^{im}{}_{\mathbf{m}} \equiv V^i L_{\mathbf{m}}{}^m\,, \tag{3.75}$$

where we have introduced

$$(V_i) \equiv e^{\varphi/2}(\tau \quad 1)\,, \tag{3.76}$$

which transforms linearly under $\mathrm{SL}(2,\mathbb{R})$ up to a U(1) phase.

Using these ingredients, the fermion shifts can be written in the form

$$S_{IJ} = \mathcal{V}^{im}{}_{[I|K} \vartheta_{im}{}^n P_n{}^{\mathbf{p}} (\sigma^{\mathbf{p}})^K{}_{|J]}\,, \tag{3.77}$$

$$N_{\mathbf{m}}{}^I{}_J = \mathcal{V}^{in}{}_{\mathbf{r}} \vartheta_{in}{}^p P_p{}^{\mathbf{s}} \left(\delta^{\mathbf{r}}{}_{\mathbf{m}} \delta^{\mathbf{q}}{}_{\mathbf{s}} - \tfrac{1}{2}\delta_{\mathbf{m}}{}^{\mathbf{q}} \delta^{\mathbf{r}}{}_{\mathbf{s}}\right) (\sigma^{\mathbf{q}})^I{}_J\,, \tag{3.78}$$

where the $(\sigma^{\mathbf{p}})$ are Pauli's sigma matrices. For the class of gaugings that we are considering, with embedding tensor $\vartheta_{im}{}^n = v_i \delta_m{}^n$

$$S_{IJ} = V^i v_i L_{\mathbf{n}}{}^m P_m{}^{\mathbf{n}} \epsilon_{IJ}\,, \tag{3.79}$$

$$N_{\mathbf{m}}{}^I{}_J = V^i v_i \left(L_{\mathbf{m}}{}^n P_n{}^{\mathbf{p}} - \tfrac{1}{2}\delta_{\mathbf{m}}{}^{\mathbf{p}} L_{\mathbf{q}}{}^n P_n{}^{\mathbf{q}}\right) (\sigma^{\mathbf{p}})^I{}_J\,. \tag{3.80}$$

Now we observe that the dressed structure constants can, in this case, be expressed in these two different ways:

$$f_{\mathbf{mn}}{}^{\mathbf{p}} = \begin{cases} L_{\mathbf{m}}{}^q \Gamma_{\mathrm{Adj}}(L^{-1})_q{}^A (T_A)_{\mathbf{n}}{}^{\mathbf{p}}\,, \\ \epsilon_{\mathbf{mnq}} T^{\mathbf{qp}}\,, \end{cases} \tag{3.81}$$

where we have defined

$$T^{\mathbf{mn}} \equiv L_p{}^{\mathbf{m}} L_p{}^{\mathbf{n}}\,. \tag{3.82}$$

Contracting both identities with $\epsilon^{\mathbf{npr}}$ we find[21]

$$2L_{\mathbf{m}}{}^p P_p{}^{\mathbf{n}} = -T^{\mathbf{mn}} + \delta^{\mathbf{mn}} T\,, \quad \text{where} \quad T \equiv \delta_{\mathbf{mn}} T^{\mathbf{mn}}\,, \tag{3.83}$$

which allows us to express the fermion shifts entirely in terms of $T^{\mathbf{mn}}$:

$$S_{IJ} = \epsilon_{IJ} V^i v_i T\,, \tag{3.84}$$

$$N_{\mathbf{m}}{}^I{}_J = V^i v_i \left(T^{\mathbf{mp}} - \tfrac{1}{2}\delta_{\mathbf{mp}} T\right) (\sigma^{\mathbf{p}})^I{}_J\,, \tag{3.85}$$

as they appear in Ref. [?].

[21] This projects the first identity over the SO(3) generators $A = m$ and we remind the reader our definition of momentum map $P_B{}^{\mathbf{m}} = \Gamma_{\mathrm{Adj}}(L^{-1})_B{}^{\mathbf{m}}$.

The combination of the fermion shifts that gives the scalar potential is

$$V = -\tfrac{1}{4}S_{IJ}S^{*\,IJ} + \tfrac{1}{8}\delta^{\mathbf{mn}}N_{\mathbf{m}}{}^{I}{}_{J}N^{*}_{\mathbf{n}I}{}^{J} = -\tfrac{1}{2}\mathcal{W}^{ij}v_i v_j\left[\mathrm{Tr}(\mathcal{M})^2 - 2\mathrm{Tr}(\mathcal{M}^2)\right]\,, \tag{3.86}$$

where $\mathcal{W}^{ij}$ is the $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ symmetric matrix defined in Eq. (**??**) and where we have used, to simplify the comparison with the results of Refs. [**?**, **?**]

$$\mathcal{M}_{mn} \equiv L_m{}^{\mathbf{p}}L_n{}^{\mathbf{p}}\,, \quad \text{so that} \quad T = \mathrm{Tr}(\mathcal{M})\,, \quad \text{and} \quad T^{\mathbf{mn}}T^{\mathbf{mn}} = \mathrm{Tr}(\mathcal{M}^2)\,. \tag{3.87}$$

The expression obtained is SO(3) invariant and, formally (because v^i is rotated) $\mathrm{SL}(2,\mathbb{R})$ invariant. For $v_i = g\delta_i{}^1$ one recovers the scalar potential of the Salam-Sezgin theory[22] and for $v_i = g\delta_i{}^2$ the scalar potential of the AAMO theory is recovered.

Since our main concern here was to find the scalar potential, in this section we have only studied the fermion shifts in the supersymmetry transformations of the fermions. It is worth, however, discussing the general form of all the supersymmetry transformations of the theory. We know that the supersymmetry transformations of the bosonic fields do not change under gauging, as a rule. The structure of the supersymmetry transformations of the fermions is, apart from the additional fermion shifts, the same as in the ungauged case with the field strengths replaced by the new ones and the derivatives replaced by gauge-covariant derivatives. Since the fermions only transform under the R-symmetry group, their covariant derivatives are different from the covariant derivatives of the bosonic fields, which transform in representations of the whole duality group. The construction of these covariant derivatives offers no particular problems and is discussed in detail in Ref. [**?**].

3.4 Conclusions

By applying the general results obtained in Ref. [**?**] we have constructed explicitly a 1-parameter family of $\mathrm{SL}(2,\mathbb{R})$-related, SO(3)-gauged maximal 8-dimensional supergravities that interpolates between Salam and Sezgin's [**?**] and AAMO's [**?**], realizing the possibilities noticed in Refs. [**?**, **?**]: for each value of that parameter a different combination of the two triplets of 1-forms (one coming from the 11-dimensional metric and the other coming from the 11-dimensional 3-form) plays the rôle of gauge vectors.

The existence of this family confirms the identification of AAMO with a honest SO(3)-gauged maximal 8-dimensional supergravity in spite of its very unconventional origin: the so-called *massive 11-dimensional supergravity* proposed in Ref. [**?**]. Furthermore, it proves its relation with the Salam-Sezgin theory by an $\mathrm{SL}(2,\mathbb{R})$ transformation, something that would have been very difficult to do directly. Thus, we have achieved the two goals stated in the introduction. At the same time, our result poses further questions: what is the 11-dimensional origin of all the theories in this family if we insist in using the same compactification Ansatz?

A key ingredient of the gauged supergravities we have constructed is the scalar potential. This is not determined by the tensor hierarchy, which only puts generic constraints on it. In a supergravity theory (different from $\mathcal{N} = 1, d = 4$) the scalar potential

[22] Beware of the different conventions for the dilaton field!

is a quadratic form on the fermion shifts. These have to be scalar-dependent expressions linear on the embedding tensor, but their general form is not known.[23] This is one of the main obstructions to find a general formulation of all gauged supergravities in all dimensions. We have proposed a general form of the fermion shifts for maximal 8-dimensional supergravity that reproduces the fermion shifts found by Salam and Sezgin and gives the expected (formally) duality-invariant form of the scalar. Interestingly enough, this form is similar to that of the fermion shifts occurring in 4-dimensional supergravities, where the scalar fields appear combined in an object (symplectic section and generalizations) related to part of the coset representative. We believe that this object should exist in any supergravity theory (if it can be gauged at all) and its identification and study should be the key for finding the general formulation of gauged supergravities we wish for.

[23] They are known in $\mathcal{N} = 2, d = 4, 5$ supergravity and in other $\mathcal{N} \geq 3, d = 4$ supergravities as well. See Ref. [**?**] and references therein.

4

f(Lovelock) theories of gravity

This chapter is based on

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Higher-derivative theories of gravity have been subject of intense study in recent years. The reasons for this interest are diverse. From a fundamental perspective, it is clear now that general relativity is an effective description which most certainly needs to be completed in the ultraviolet. A characteristic manifestation of the putative underlying theory would be the appearance of a series of higher-derivative terms, consisting of different contractions of the Riemann tensor and its covariant derivatives, which would correct the Einstein-Hilbert (EH) action at sufficiently high energies. This is of course the case of String Theory, which generically predicts an infinite series of such terms — see *e.g.,* [?,?,?].

While many explicit String Theory models giving rise to particular effective higher-derivative theories have been constructed, there exists a less fundamental but more practical approach which has been also vastly studied in the literature. Such approach consists in regarding certain higher-derivative theories as quantum gravity toy models. This is the case, for example, of *topologically massive gravity* [?] and *new massive gravity* [?] in three dimensions, or *critical gravity* [?] in four. In these theories, and others of the like, the EH action is supplemented by a few additional higher-derivative terms which improve some of the properties of the original theory — *e.g.,* by making it renormalizable [?, ?].

Constructions of this kind are also very useful in the holographic context [?,?,?]. Indeed, through the holographic dictionary, higher-derivative theories have been successfully used to unveil various properties of general strongly coupled systems in various dimensions — see *e.g.,* [?, ?, ?, ?, ?, ?]. In this context, the philosophy also consists in regarding these theories as computationally useful toy models: if a certain property holds for general strongly coupled conformal field theories (CFTs), it is reasonable to expect that these toy models are able to capture it — and that has been proven to be very often the case. A paradigmatic example of this class of theories is *quasi-topological gravity* [?, ?], which was precisely conceived as a multi-parameter holographic toy model of strongly coupled CFTs in various dimensions.

Most likely, the area of research in which higher-derivative gravities have appeared more often is cosmology. In that context, these terms are considered with the idea that general relativity might not be, after all, the right description of the gravitational interaction at cosmological scales. This is of course motivated by the puzzling existence of dark matter and dark energy, as well as by the need to construct a coherent picture — beyond the Λ-CDM model — of the universe evolution able to incorporate, in particular, an inflationary scenario compatible with the observations.[1]

Two of the higher-derivative theories which have received more attention within the areas explained above are Lovelock [?, ?] and $f(R)$ gravities — see *e.g.,* [?]. While there has been a large amount of papers studying different aspects of these higher-derivative gravities, remarkably little work has been done on the class of theories which most naturally incorporates both $f(R)$ and Lovelock in a common framework. We are talking, of course, about f(Lovelock) gravities, which are the subject of this paper.

The most general f(Lovelock) action can be written as

$$S_{f(\text{Lovelock})} = \frac{1}{16\pi G}\int_{\mathcal{M}} d^D x\,\sqrt{|g|}\, f(\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_{\lfloor D/2 \rfloor})\,, \tag{4.1}$$

where f is some differentiable function of the dimensionally extended Euler densities (ED)[2]

$$\mathcal{L}_j = \frac{1}{2^j}\delta^{\mu_1\ldots\mu_{2j}}_{\nu_1\ldots\nu_{2j}} R^{\nu_1\nu_2}_{\mu_1\mu_2}\ldots R^{\nu_{2j-1}\nu_{2j}}_{\mu_{2j-1}\mu_{2j}}\,. \tag{4.2}$$

In particular, *e.g.,* $\mathcal{L}_1 = R$, and $\mathcal{L}_2 = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, which are the usual EH and Gauss-Bonnet (GB) terms respectively. Note that $\mathcal{L}_j$ vanishes identically for $j > \lfloor D/2 \rfloor$, *i.e.,* for $j > D/2$ when D is even, and for $j > (D-1)/2$ for odd D.

Naturally, the above action (??) reduces to the usual Lovelock and $f(R)$ theories when we choose f to be a linear combination of ED and some arbitrary function of the Ricci scalar respectively,

$$f_{\text{Love.}} = \sum_{n=0}^{\lfloor D/2 \rfloor} \lambda_n\, \Lambda_0^{1-n}\, \mathcal{L}_n\,, \quad f_{f(R)} = f(R)\,, \tag{4.3}$$

where $\Lambda_0^{-1/2}$ is some length scale, and λ_j are dimensionless couplings.[3] When D is even, the combination '$\sqrt{|g|}\,\mathcal{L}_{D/2}$' is topological in the sense that its integral over a boundaryless manifold is proportional to the manifold's Euler characteristic.[4] The variation of each of these topological terms can be written as a boundary term,[5] and does not contribute to the equations of motion of the Lovelock theory. This is the case *e.g.,* of Einstein gravity in two dimensions, and GB in four. However, the situation changes when the action is no longer a linear combination of ED like in the general f(Lovelock) theory. For example, terms of the form '$\sqrt{|g|}\,R\cdot\mathcal{L}_2$' are not topological in four dimensions. Another distinctive feature of Lovelock gravities which is not inherited in the more general f(Lovelock) framework is the fact that the former have second-order equations of motion. In fact, Lovelock gravities are the most general theories of gravity involving arbitrary combinations of the Riemann

[1] The body of literature in this area is huge. See *e.g.,* [?, ?, ?, ?] for some nice reviews on higher-derivative gravities and cosmology.

[2] The alternate Kronecker symbol is defined as: $\delta^{\mu_1\mu_2\ldots\mu_r}_{\nu_1\nu_2\ldots\nu_r} = r!\delta^{[\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\ldots\delta^{\mu_r]}_{\nu_r}$.

[3] Note that explicit cosmological constant and EH terms can be trivially made appear in $f_{\text{Love.}}$ by setting: $\lambda_0 = -2$ and $\lambda_1 = 1$ respectively. Similarly, we could replace $f_{f(R)}$ in (??) by $f_{f(R)} = -2\Lambda_0 + R + f(R)$ to make those terms explicit in the $f(R)$ action.

[4] For manifolds with boundary, a boundary term needs to be added to the Lovelock action to produce the right Euler characteristic, see *e.g.,* [?]. We review such term in the next section.

[5] Indeed, locally, it is possible to write the terms '$\sqrt{|g|}\,\mathcal{L}_{D/2}$' themselves as total derivatives [?].

tensor which possess second-order equations of motion.[6] f(Lovelock) theories generically have fourth-order equations — see (??) and (??).

Certainly, the research area in which f(Lovelock) gravities have been considered more actively so far is cosmology — see *e.g.,* [?, ?, ?, ?, ?, ?, ?, ?, ?, ?], where, for example, they have been used to reproduce numerous features of the Λ-CDM model. In that context, the spacetime dimension is fixed to $D = 4$ for obvious reasons, the f(Lovelock) action becomes a function of the Ricci scalar and the GB terms alone, and these theories are better known as '$f(R, \mathcal{G})$' gravities.

An interesting theoretical development was carried out in [?]. In that paper, the following formula for the gravitational entropy in f(Lovelock) theories was proposed,

$$S_{\mathrm{SW}} = \frac{1}{4G}\int_m d^{(D-2)}x\sqrt{h_m}\sum_{p=1}^{\lfloor D/2\rfloor}\left[p\,\frac{\partial f}{\partial \mathcal{L}_p}\cdot {}^{(D-2)}\mathcal{L}_{p-1}\right]\,, \tag{4.4}$$

where m is the corresponding horizon, and ${}^{(D-2)}\mathcal{L}_{p-1}$ is the $(p-1)$-th ED associated to the pullback metric. This functional reduces to the well-known Jacobson-Myers functional (JM) for Lovelock gravities [?] and, as shown in [?], it satisfies and increase theorem for small perturbations of Killing horizons as well as a generalized version of the second law for minimally-coupled fields. Apart from its interest in black hole thermodynamics, (??) has also been used in the holographic context. In fact, it is known [?, ?] that the JM functional gives rise to the right universal terms when used to compute holographic entanglement entropy (HEE)[7] for these theories. This fact, along with the increase theorem already mentioned, was interpreted in [?, ?] as evidence for S_{SW} to be the right HEE functional for f(Lovelock) theories. The results found in those papers provide strong evidence that this is indeed the case.

The last two paragraphs summarize, to the best of our knowledge, the few aspects of general f(Lovelock) theories which have been so far studied in the literature. The goal of this paper is to develop several more.

4.0.1 Main results

Our main results, section by section, can be summarized as follows:

- In section ??, we generalize the Gibbons-Hawking-York (GHY) boundary term of general relativity [?, ?], and its extensions to Lovelock [?, ?] and $f(R)$ [?] gravities to general f(Lovelock) theories. This new term — see (??) below — reduces to these in the appropriate subcases, and makes the f(Lovelock) action differentiable. The construction of this boundary term allows us to determine the number of physical degrees of freedom of the theory, which turns out to be $D(D-3)/2 + r$, where r is the rank of the Hessian matrix $H_{nm} = \partial_n \partial_m f$.

- In section ??, we make this counting of degrees of freedom explicit by showing that f(Lovelock) theories are equivalent to scalar-Lovelock gravities containing r scalar fields.

[6] This statement is true for metric theories of gravity.

[7] See section ?? for more details on entanglement entropy.

- In section **??**, we linearize the f(Lovelock) equations on a maximally symmetric background (m.s.b.). Interestingly, we find that these theories do not propagate the usual ghost-like massive graviton characteristic of higher-derivative gravities. Furthermore, we show that certain non-trivial f(Lovelock) theories are also free of the — also characteristic — scalar mode, thus providing new examples of higher-derivative gravities which only propagate the usual physical graviton field on these backgrounds. For these theories, the equations of motion are second-order in any gauge, and the only effect of the higher-derivative terms appears in an overall factor whose effect is to change the normalization of the Newton constant. We provide examples of this class of theories in general dimensions.

- In section **??**, we consider holographic theories dual to some classes of f(Lovelock) theories and find constraints on the allowed values of their couplings. The first set of constraints is found by simply imposing the corresponding theory to admit an AdS_D solution. After that we consider the holographic entanglement entropy of various entangling regions in the boundary theory, and find additional constraints by imposing the holographic surfaces to close off smoothly in the bulk.

- Last, but not least, in section **??** we construct new black hole solutions for certain f(Lovelock) theories. In particular, we start by embedding all solutions of pure Lovelock theory — involving a single ED, $\mathcal{L}_n$, plus a cosmological constant — in $f(\mathcal{L}_n)$, with special focus on static and spherically symmetric black holes. In particular, we construct the $f(\mathcal{L}_n)$ generalizations of the Schwarzschild(-AdS/dS) and Reissner-Nordström(-AdS/dS) black holes. We also construct new solutions for theories satisfying $f(\mathcal{L}_n^0) = f'(\mathcal{L}_n^0) = 0$ for some constant $\mathcal{L}_n^0$. We go on to study under what conditions solutions of the general Lovelock theory can be embedded in f(Lovelock) theories depending on several ED. Finally, we construct a new static and spherically symmetric black hole solution of a particular $f(R, \mathcal{L}_2)$ theory in general dimensions.

- We comment on future directions in section **??**.

Let us get started.

4.1 Variational problem and boundary term

In this section we study the variational problem in f(Lovelock) theories. Our main result is a new boundary term which generalizes the well-known GHY one for Einstein gravity as well as its generalizations to Lovelock and $f(R)$ theories. As we will see, the addition of this term to the f(Lovelock) action makes the corresponding variational problem well-posed.

A physical theory is often defined through an action functional, which is a map from a normed vector space (usually a space of functions) to the real numbers. On general grounds, the dynamical variables of the theory are described by some fields ϕ^a. The action $S[\phi^a]$ consists in turn of a definite integral over a spacetime manifold $\mathcal{M}$, being the integrand a function of those fields and their derivatives, *i.e.*,

$$S[\phi^a] = \int_{\mathcal{M}} d^D x \sqrt{|g|} f(\phi^a, \nabla\phi^a, \dots)\,. \tag{4.5}$$

Now, by a *well-posed* variational problem we mean one for which the action functional is differentiable. That is, under small variations of the fields $\phi^a \to \phi^a + \delta\phi^a$ we must be able to write the variation of the functional as

$$S[\phi^a + \delta\phi^a] - S[\phi^a] = \delta S[\phi^a, \delta\phi^a] + \mathcal{O}\left((\delta\phi^a)^2\right) , \tag{4.6}$$

where $\delta S[\phi^a, \delta\phi^a]$ is linear on $\delta\phi^a$. If we perform this variation explicitly in (**??**), we find two terms, namely

$$\delta S = \int_{\mathcal{M}} d^D x \sqrt{|g|}\, \mathcal{E}_a\, \delta\phi^a + \int_{\partial\mathcal{M}} d^{D-1}x \sqrt{|h|}\, \theta\left(\phi^a, \nabla\phi^a, \delta\phi^a, \nabla\delta\phi^a, \dots\right) . \tag{4.7}$$

Here, $\mathcal{E}_a$ is a function of the fields and their derivatives, h is the determinant of the induced metric on the boundary $\partial\mathcal{M}$ and θ is some function of the fields and their derivatives. While the first term is linear on $\delta\phi^a$, the second is not necessarily of that form. Field perturbations need to respect the field (Dirichlet) boundary conditions, *i.e.,* they are required to satisfy $\delta\phi^a|_{\partial\mathcal{M}} = 0$. However in general $\nabla\delta\phi^a|_{\partial\mathcal{M}} \neq 0$ and in consequence this boundary term may not be trivially zero, making the action functional non-differentiable.

When this is the case, one can sometimes modify the original action by introducing an appropriate boundary term such that its variation cancels this contribution. When it can be constructed, this boundary term makes the functional differentiable and the variational problem becomes well-posed. After the addition of the boundary term, the complete action reads

$$S[\phi^a] = \int_{\mathcal{M}} d^D x \sqrt{|g|}\, f(\phi^a, \nabla\phi^a, \dots) + \int_{\partial\mathcal{M}} d^{D-1}x \sqrt{|h|}\, \psi\left(\phi^a, \nabla\phi^a, \dots\right) . \tag{4.8}$$

Now, its variation is simply given by

$$\delta S = \int_{\mathcal{M}} d^D x \sqrt{|g|}\, \mathcal{E}_a\, \delta\phi^a , \tag{4.9}$$

because ψ has been chosen in a way such that $(\theta + \delta\psi) = 0$.

The principle of least action asserts that a field configuration ϕ_0^a is a solution of the theory if it constitutes a stationary point of the action functional, *i.e.,* if $\delta S[\phi_0^a] = 0$. Hence, solutions of the theory satisfy the equations of motion $\mathcal{E}_a = 0$.

Before we go on, let us mention that, in general, the boundary term is not the only addition to the original action that needs to be made. In particular, extra counter-terms usually need to be included in order for the action to be finite when evaluated on configurations satisfying the equations of motion. We will not be concerned with that issue here.[8]

4.1.1 Equations of motion

Let us now see how the ideas sketched in the previous subsection apply to the f(Lovelock) theory, whose action is given by (**??**). If we vary this action with respect to the metric, we find

$$\delta S_{f(\text{Love.})} = \frac{1}{16\pi G} \int_{\mathcal{M}} d^D x \sqrt{|g|}\, \mathcal{E}_{\mu\nu} \delta g^{\mu\nu} + \frac{\varepsilon}{16\pi G} \int_{\partial\mathcal{M}} d^{D-1}x \sqrt{|h|} \sum_{n=1}^{\lfloor D/2 \rfloor} \delta v_n^{\mu} n_\mu \partial_n f\, , \tag{4.10}$$

[8]Let us parenthetically mention that such counter-terms where constructed for AdS$_D$ spacetimes in [**?**] and [**?**, **?**] for Einstein and Lovelock theories respectively.

where we have used the Stokes theorem in the second term. Here, n_μ is a vector orthonormal to the boundary $\partial\mathcal{M}$ with $n_\mu n^\mu = \varepsilon$ and $h_{\mu\nu} = g_{\mu\nu} - \varepsilon\, n_\mu n_\nu$ is the pullback metric. The quantities $\mathcal{E}_{\mu\nu}$ and δv_j^μ are given by

$$\mathcal{E}_{\mu\nu} = \sum_{n=1}^{\lfloor D/2 \rfloor} \left[\mathcal{E}_{\mu\nu}^{(n)} + \frac{1}{2} g_{\mu\nu} \mathcal{L}_n - 2P_{\alpha\nu\lambda\mu}^{(n)} \nabla^\alpha \nabla^\lambda \right] \partial_n f - \frac{1}{2} g_{\mu\nu} f \,, \tag{4.11}$$

and

$$\delta v_j^\mu = 2 g^{\beta\sigma} P_{\alpha\beta}^{(j)\mu\nu} \nabla^\alpha \delta g_{\nu\sigma} \,, \tag{4.12}$$

respectively. In these expressions we have defined the following tensors[9]

$$\mathcal{E}_{\mu\nu}^{(j)} = \frac{-1}{2^{j+1}} g_{\alpha\mu} \delta_{\nu\nu_1 \ldots \nu_{2j}}^{\alpha\mu_1 \ldots \mu_{2j}} R_{\mu_1\mu_2}^{\nu_1\nu_2} \cdots R_{\mu_{2j-1}\mu_{2j}}^{\nu_{2j-1}\nu_{2j}} \,, \quad P_{\alpha\beta}^{(j)\mu\nu} = \frac{-j}{2^j} \delta_{\alpha\beta\lambda_1 \ldots \lambda_{2j-2}}^{\mu\nu\sigma_1 \ldots \sigma_{2j-2}} R_{\sigma_1\sigma_2}^{\lambda_1\lambda_2} \cdots R_{\sigma_{2j-3}\sigma_{2j-2}}^{\lambda_{2j-3}\lambda_{2j-2}} \,. \tag{4.13}$$

Also, in (**??**) and (**??**) we have used the notation $\partial_n f = \partial f / \partial \mathcal{L}_n$, which will appear throughout the text. Now, if we forget the boundary contribution for a moment, we see that the equations of motion of the theory read

$$\mathcal{E}_{\mu\nu} = 0 \,, \tag{4.14}$$

whose trace is[10]

$$\sum_{n=1}^{\lfloor D/2 \rfloor} \left[n\mathcal{L}_n - 2n(D-2n+1) \mathcal{E}_{\mu\nu}^{(n-1)} \nabla^\mu \nabla^\nu \right] \partial_n f - \frac{D}{2} f = 0. \tag{4.15}$$

As expected, these reduce to the Lovelock and $f(R)$ equations of motion,

$$\sum_{n=0}^{\lfloor D/2 \rfloor} \lambda_n \Lambda_0^{1-n} \mathcal{E}_{\mu\nu}^{(n)} = 0 \,, \quad \mathcal{E}_{\mu\nu}^{f(R)} = f'(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} + \left(g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) f'(R) = 0 \,, \tag{4.16}$$

when we choose $f = f_{\text{Love.}}$ and $f = f_{f(R)}$ as in (**??**) respectively. In particular, observe that $\nabla^\lambda(\partial_j f) = 0 \ \forall j$ when f is a linear combination of ED — corresponding to the usual Lovelock theory — so the term contributing with fourth-order derivatives in (**??**) disappears in that case. In appendix **??**, we provide the explicit equations of motion corresponding to D-dimensional f(Lovelock) theories which are only functions of the Ricci scalar and the GB terms, *i.e.*, $f = f(R, \mathcal{L}_2)$. These are, in particular, the most general f(Lovelock) gravities in four dimensions, as the densities $\mathcal{L}_p$ identically vanish for all $p \geq 3$ in that case.

4.1.2 Generalized boundary term

Let us now see what happens with the boundary contribution to δS. As we explained at the beginning of this section, the variational problem for (**??**) cannot be well-posed because such an action is not differentiable, as is clear from the presence of the boundary term in (**??**).

[9] Both tensors are divergence-free in all indices, *i.e.*, $\nabla^\mu \mathcal{E}_{\mu\nu}^{(j)} = 0$, $\nabla^\alpha P_{\alpha\beta}^{(j)\mu\nu} = 0$.

[10] In order to get this result we used the relations: $\mathcal{E}_\alpha^{(n)\alpha} = (n - D/2)\mathcal{L}_n$ and $P_{\alpha\mu}^{(n)\lambda\mu} = n(D-2n+1)\mathcal{E}_\alpha^{(n-1)\lambda}$.

In the familiar case of Einstein gravity, the problem is solved through the introduction of the usual GHY term [?, ?]

$$S_{\rm EH} \to S_{\rm EH} + S_{\rm GHY}\,, \quad \text{where} \quad S_{\rm GHY} = \frac{\varepsilon}{8\pi G}\int_{\partial\mathcal{M}} d^{D-1}x\sqrt{|h|}K, \tag{4.17}$$

and K is the trace of the second fundamental form associated to the boundary normal n_μ, *i.e.,* $K = g^{\mu\nu}K_{\mu\nu}$, where $K_{\mu\nu} = h^\rho_\mu \nabla_\rho n_\nu$. It is a standard exercise to show that the variation of this corrected action does not contain additional boundary terms as long as the usual Dirichlet boundary condition[11]

$$\delta g_{\mu\nu}\Big|_{\partial\mathcal{M}} = 0\,, \tag{4.18}$$

is satisfied. Indeed, the variation of K produces a term which exactly cancels the original boundary contribution coming from the variation of the EH action, plus additional terms which vanish for configurations respecting (**??**). Hence, the corrected EH action is differentiable. Since the only condition we need to impose in order to find a solution to the theory is (**??**), *i.e.,* we only need to fix the metric at the boundary, we can obtain the number of classical degrees of freedom of Einstein gravity as the number of independent components of the boundary metric. This yields the well-known result: $n_{\rm dof} = D(D-3)/2$.

The problem becomes more involved in Lovelock and $f(R)$ gravities, for different reasons in each case. Lovelock theories possess second-order equations of motion, and the metric does not propagate additional degrees of freedom with respect to Einstein gravity. Therefore, the only boundary condition that one needs to impose is again given by (**??**). However, the boundary term that needs to be added to the usual Lovelock action — see (**??**) and (**??**) — in order to make it differentiable is considerably more involved than the GHY term. The full Lovelock action is given by[12]

$$S_{\rm Love.} \to S_{\rm Love.} + S_{\rm MTZ}\,, \quad \text{where} \quad S_{\rm MTZ} = \frac{\varepsilon}{16\pi G}\sum_{n=0}^{\lfloor D/2\rfloor} \lambda_n \Lambda_0^{1-n}\int_{\partial\mathcal{M}} d^{D-1}x\sqrt{|h|}Q_n\,, \tag{4.19}$$

and where

$$Q_n = 2n\int_0^1 dt\, \delta^{\mu_1\ldots\mu_{2n-1}}_{\nu_1\ldots\nu_{2n-1}} K^{\nu_1}_{\mu_1}\left[\frac{1}{2}R^{\nu_2\nu_3}_{\mu_2\mu_3} - t^2 K^{\nu_2}_{\mu_2}K^{\nu_3}_{\mu_3}\right]\cdots\left[\frac{1}{2}R^{\nu_{2n-2}\nu_{2n-1}}_{\mu_{2n-2}\mu_{2n-1}} - t^2 K^{\nu_{2n-2}}_{\mu_{2n-2}}K^{\nu_{2n-1}}_{\mu_{2n-1}}\right]. \tag{4.20}$$

Indeed, it is possible to prove that

$$\delta Q_n\Big|_{\delta g_{\mu\nu}|_{\partial\mathcal{M}}=0} = n_\mu \delta v^\mu_n\,, \tag{4.21}$$

i.e., the variation of this term exactly cancels the boundary contribution which appears from the variation of $S_{\rm Love.}$ as long as the boundary condition (**??**) is satisfied. Therefore, the addition of $S_{\rm MTZ}$ makes the Lovelock variational problem well-posed. Of course, $S_{\rm MTZ}$ reduces to $S_{\rm GHY}$ in the particular case of Einstein gravity.

[11] Clearly, boundary terms in general, and the GHY one in particular, are not unique. They are only unique up to contributions whose variations vanish when we impose Dirichlet boundary conditions.

[12] The 'MTZ' label here stands for Myers [?], Teitelboim and Zanelli [?] who independently first showed how to construct this boundary term. The equivalence of both approaches was proven in [?].

As opposed to Lovelock theories, general $f(R)$ gravities have fourth-order equations of motion. This means that the theory contains more degrees of freedom than Lovelock gravity and that besides (**??**), additional boundary conditions must be imposed. In fact, as we review in section **??**, $f(R)$ gravities with $f''(R) \neq 0$ are equivalent to Brans-Dicke theories, in which a scalar field ϕ related to the $f(R)$ metric through $\phi = f'(R)$ is coupled to the gravitational field. Hence, it is natural to expect that a condition of the form

$$\delta\phi|_{\partial\mathcal{M}} = \delta(f'(R))|_{\partial\mathcal{M}} = (f''(R)\delta R)|_{\partial\mathcal{M}} = 0 \to \delta R|_{\partial\mathcal{M}} = 0\,, \tag{4.22}$$

needs to be added in that case. On the other hand, we expect again the boundary term to reduce to the GHY one for $f(R) = R - 2\Lambda_0$. These observations turn out to be right, as the $f(R)$ variational problem can be made well-posed by considering the following action[13]

$$S_{f(R)} \to S_{f(R)} + S_{\text{MB}}\,, \quad \text{where} \quad S_{\text{MB}} = \frac{\varepsilon}{8\pi G}\int_{\partial\mathcal{M}} d^{D-1}x\sqrt{|h|}f'(R)K\,. \tag{4.23}$$

This trivially reduces to S_{GHY} for Einstein gravity. Besides, its variation precisely compensates the extra boundary term produced from the variation of $S_{f(R)}$. In particular, imposing (**??**) one finds

$$\delta S_{f(R)} + \delta S_{\text{MB}} = \frac{1}{16\pi G}\int_{\mathcal{M}} d^D x\sqrt{|g|}\,\mathcal{E}^{f(R)}_{\mu\nu}\,\delta g^{\mu\nu} + \frac{\varepsilon}{8\pi G}\int_{\partial\mathcal{M}} d^{D-1}x\sqrt{|h|}K\,\delta(f'(R))\,, \tag{4.24}$$

where $\mathcal{E}^{f(R)}_{\mu\nu}$ is given in (**??**). Hence, we observe that imposing the additional boundary condition (**??**) on $f'(R)$ — or equivalently, on R — makes the corrected action differentiable. While one might feel uncomfortable at first by imposing boundary conditions on functions that depend on derivatives of the metric like (**??**), let us stress that the introduction of S_{MB} is necessary to reproduce the correct ADM energy in the Hamiltonian formalism as well as the right black hole entropy — *i.e.,* one which matches the result obtained with Wald's formula — using the Euclidean semiclassical approach [**?**]. We observe that $f(R)$ theories with $f''(R) \neq 0$ have $D(D-3)/2+1$ degrees of freedom.

Let us finally turn to the general f(Lovelock) case. We propose the following boundary term

$$S_{f(\text{Love.})} \to S_{f(\text{Love.})} + \tilde{S}\,, \quad \text{where} \quad \tilde{S} = \frac{\varepsilon}{16\pi G}\int_{\partial\mathcal{M}} d^{D-1}x\sqrt{|h|}\sum_{n=1}^{\lfloor D/2\rfloor}\partial_n f(\mathcal{L})Q_n\,, \tag{4.25}$$

and where $S_{f(\text{Love.})}$ is given in (**??**). It is straightforward to check that this reduces to S_{MB}, S_{MTZ} and S_{GHY} in the particular cases of $f(R)$, Lovelock and Einstein gravity respectively. After imposing the Dirichlet condition (**??**) on the boundary metric, the variation of this corrected action becomes

$$\delta S_{f(\text{Love.})} + \delta\tilde{S} = \frac{1}{16\pi G}\int_{\mathcal{M}} d^D x\sqrt{|g|}\,\mathcal{E}_{\mu\nu}\,\delta g^{\mu\nu} + \frac{\varepsilon}{16\pi G}\int_{\partial\mathcal{M}} d^{D-1}x\sqrt{|h|}\sum_{n,m=1}^{\lfloor D/2\rfloor}\partial_m\partial_n f\,\delta\mathcal{L}_m Q_n\,, \tag{4.26}$$

[13]In this case, the first to have considered this boundary term seem to have been Madsen and Barrow in [**?**]. See also [**?**, **?**].

where we have used the relation

$$\delta(\partial_n f) = \sum_{m=1}^{\lfloor D/2 \rfloor} \partial_m \partial_n f \, \delta\mathcal{L}_m \, . \tag{4.27}$$

Equation (**??**) suggests that, in addition to the metric, we need to fix the Euler densities at the boundary, *i.e.,*

$$\delta\mathcal{L}_n \Big|_{\partial\mathcal{M}} = 0, \quad n = 1, \ldots, \lfloor D/2 \rfloor. \tag{4.28}$$

However, notice that it is enough to fix the derivatives of f,

$$\delta\left(\partial_n f\right) \Big|_{\partial\mathcal{M}} = 0, \quad n = 1, \ldots, \lfloor D/2 \rfloor, \tag{4.29}$$

which is a weaker condition in general. If the Hessian matrix, $H_{nm} = \partial_n \partial_m f$, is not singular, *i.e.,* if $\det H_{nm} \neq 0$, then the conditions (**??**) and (**??**) are equivalent. But if this determinant is zero, then not all the conditions in (**??**) are independent. In fact, if r is the rank of the Hessian matrix,

$$r = \operatorname{rank} H_{nm} \, , \tag{4.30}$$

then there are r independent conditions. Thus, only r quantities must be fixed at the boundary and the number of physical degrees of freedom in f(Lovelock) theory is given by:

$$n_{\text{dof}} = \frac{D(D-3)}{2} + r. \tag{4.31}$$

With respect to GR or Lovelock gravity there are r additional degrees of freedom. Depending on the function, r can take values from 0 to $\lfloor D/2 \rfloor$. In the next section we will see that these additional degrees of freedom can be interpreted as scalar fields in an equivalent scalar-Lovelock theory.

4.2 Equivalence with scalar-tensor theories

It is a well known fact that $f(R)$ gravity is equivalent to a scalar-tensor theory of the Brans-Dicke class — see *e.g.,* [**?**, **?**]. This can be easily seen by considering an action of the form

$$S = \frac{1}{16\pi G} \int_{\mathcal{M}} d^D x \sqrt{|g|} \, \left[f(\chi) + f'(\chi)(R - \chi) \right] \, . \tag{4.32}$$

The equation of motion for the auxiliary field χ, $f''(R)(R-\chi) = 0$, implies $\chi = R$ provided $f''(R) \neq 0$. Substituting this back in (**??**), we recover the $f(R)$ action. Now, assuming the field redefinition $\phi = f'(\chi)$ can be inverted,[14] we can rewrite (**??**) as

$$S_{\text{BD}} = \frac{1}{16\pi G} \int_{\mathcal{M}} d^D x \sqrt{|g|} \, \left[\phi \, R - V(\phi) \right] \, , \tag{4.33}$$

where $V(\phi) = \chi(\phi)\phi - f(\chi(\phi))$. This is the action of a Brans-Dicke theory with parameter $\omega_0 = 0$.

[14] A sufficient condition for this is $f''(R) \neq 0$.

The situation is slightly more sophisticated in the case of f(Lovelock) theories. In analogy with (**??**), let us consider the following action containing $\lfloor D/2 \rfloor$ auxiliary scalar fields $\chi_1, \ldots, \chi_{\lfloor D/2 \rfloor}$,

$$S = \frac{1}{16\pi G} \int_{\mathcal{M}} d^D x \sqrt{|g|} \left[f + \sum_{n=1}^{\lfloor D/2 \rfloor} \partial_n f \cdot (\mathcal{L}_n - \chi_n) \right] , \qquad (4.34)$$

where $f = f(\chi_1, \ldots, \chi_{\lfloor D/2 \rfloor})$. The equations of motion for the auxiliary fields are constraints which relate them to the dimensionally-extended Euler densities,

$$\sum_{n=1}^{\lfloor D/2 \rfloor} \partial_n \partial_m f \cdot (\mathcal{L}_n - \chi_n) = 0, \quad m = 1, \ldots, \lfloor D/2 \rfloor . \qquad (4.35)$$

Hence, we see that if we set

$$\mathcal{L}_n = \chi_n \, , \;\; n = 1, \ldots, \lfloor D/2 \rfloor \, , \qquad (4.36)$$

(**??**) is satisfied and (**??**) reduces to the f(Lovelock) action (**??**). In general, however, this will not be the only solution to (**??**). There are two possibilities that we explain in the following subsections.

4.2.1 Non-degenerate case

If the Hessian matrix H_{nm} is non-singular, *i.e.,* if $\det(H_{nm}) \neq 0$, (**??**) is indeed the only solution to the constraint equations (**??**), and the action (**??**) is equivalent to the original f(Lovelock) one (**??**).

In this situation, we can perform the invertible field redefinition

$$\phi_n = \partial_n f(\chi_1, \ldots, \chi_{\lfloor D/2 \rfloor}) \, , \;\; n = 1, \ldots, \lfloor D/2 \rfloor \, , \qquad (4.37)$$

which allows us to rewrite (**??**) as

$$S = \frac{1}{16\pi G} \int_{\mathcal{M}} d^D x \sqrt{|g|} \left[\sum_{n=1}^{\lfloor D/2 \rfloor} \phi_n \mathcal{L}_n - V \right] , \qquad (4.38)$$

where[15]

$$V = \sum_{n=1}^{\lfloor D/2 \rfloor} \chi_n(\phi) \phi_n - f(\chi(\phi)) \qquad (4.39)$$

is the Legendre transform of f. This form of the action, which clearly resembles — and generalizes — the $f(R)$ scalar-tensor action in (**??**), was first noted to be related to the f(Lovelock) action (**??**) in [**?**].

[15] We use the notation 'χ' and 'ϕ' to generically refer to the $\lfloor D/2 \rfloor$ scalars χ_n and the same number of ϕ_n. For example, $f(\chi(\phi))$ stands for $f(\chi_1(\phi_1, \ldots, \phi_{\lfloor D/2 \rfloor}), \ldots, \chi_{\lfloor D/2 \rfloor}(\phi_1, \ldots, \phi_{\lfloor D/2 \rfloor}))$.

4.2.2 Degenerate case

If the Hessian matrix is singular, *i.e.,* if $\det(H_{nm}) = 0$, the system of equations (**??**) is indeterminate. In particular, the space of solutions has dimension $\lfloor D/2 \rfloor - r$, where $r = \text{rank}(H_{mn})$. Hence, unless $r = \lfloor D/2 \rfloor$, which corresponds to the case studied in the previous subsection[16], there are infinite solutions to (**??**). This is nothing but a manifestation of the fact that we have included $\lfloor D/2 \rfloor - r$ too many scalars to account for the actual number of physical degrees of freedom of the corresponding f(Lovelock) theory. Let us see how we can reduce this number. It is clear that we cannot perform a Legendre transform this time, because the Hessian matrix is singular, which implies that the change of variables in (**??**) is not invertible. We can make, however, a *semi*–Legendre transform. This goes as follows: let us define the fields ϕ_n as before:

$$\phi_n = \partial_n f(\chi_1, \ldots, \chi_{\lfloor D/2 \rfloor}), \quad n = 1, \ldots, \lfloor D/2 \rfloor. \tag{4.40}$$

Then, there is a subset $I \subset \{1, \ldots, \lfloor D/2 \rfloor\}$ of r indices such that $\phi_I = \{\phi_i\}_{i \in I}$ are independent variables, in the sense that

$$\det\left[(\partial \phi_{i_1}/\partial \chi_{i_2})_{i_1, i_2 \in I}\right] = \det\left[(H_{i_1 i_2})_{i_1, i_2 \in I}\right] \neq 0. \tag{4.41}$$

Let J be the complementary set of indices, $J = \{1, \ldots, \lfloor D/2 \rfloor\} - I$. Now, since there must be only r independent fields, the rest of the fields, $\phi_j, j \in J$ must depend on the formers ϕ_I. Hence, there exist some functions g_j such that

$$\phi_j = g_j(\phi_I), \quad j \in J. \tag{4.42}$$

Then, we can consider $\phi_I \cup \chi_J$ as our set of *independent* variables.[17] We define the *semi*–Legendre transform of f as:

$$\tilde{V}(\phi_I) = \sum_{i \in I} \chi_i(\phi_I, \chi_J) \cdot \phi_i + \sum_{j \in J} \chi_j \cdot g_j(\phi_I) - f\left(\chi_I(\phi_I, \chi_J); \chi_J\right). \tag{4.43}$$

This seems to be a function of both the ϕ_I and the χ_J. However, it is easy to check that the derivative of $\tilde{V}$ with respect to any χ_J vanishes, $\partial_J \tilde{V} = 0$, which implies that $\tilde{V}$ is actually a function of the r fields ϕ_I alone. This allows us to rewrite the original action (**??**) as

$$S = \frac{1}{16\pi G} \int_M d^D x \sqrt{|g|} \left[\sum_{i \in I} \phi_i \mathcal{L}_i + \sum_{j \in J} g_j(\phi_I) \mathcal{L}_j - \tilde{V}(\phi_I)\right]. \tag{4.44}$$

This theory is equivalent to f(Lovelock), since we have eliminated the spurious degrees of freedom that appeared in the original action (**??**). The equations of motion for the scalar fields have now a unique solution given precisely by $\chi_n(\phi_I) = \mathcal{L}_n$.

We see that, on general grounds, f(Lovelock) gravity is equivalent to a scalar-Lovelock theory with r scalars, where r is the rank of the Hessian matrix of f, and whose action is given by (**??**). In the case of Lovelock gravity such an analogy does not exist: the Hessian is zero and so is the number of scalars. In appendix **??**, we explicitly construct the

[16]Indeed, if $r = \lfloor D/2 \rfloor$ then $\det(H_{nm}) \neq 0$ and the dimensionality of the space of solutions is 0, *i.e.*, there is a unique solution given by (**??**).

[17]The change of variables $(\chi_n) \to (\phi_I; \chi_J)$ is now invertible.

equivalent scalar-Lovelock theories for a pair of classes of f(Lovelock) theories including both degenerate and non-degenerate subcases.

Let us finally mention that Lovelock theories have been recently proposed to be effectively described by Einstein gravity coupled to certain p-form gauge fields [?].[18] We will not explore here how such relation might extend to the more general f(Lovelock) scenario.

4.3 Linearized equations of motion

In this section we study the linearized equations of motion of f(Lovelock) gravity on a general m.s.b., with particular emphasis on AdS$_D$. On general grounds, the linearized equations of motion of higher-derivative gravities on a m.s.b. are fourth-order in derivatives. From these equations it is possible to identify, in addition to the usual spin-2 massless graviton, a scalar field corresponding to the trace of the perturbation, as well as an additional massive spin-2 field, which generally presents an undesirable ghost-like behavior — see *e.g.,* [?] for a discussion. Remarkably, we find that for general f(Lovelock) gravities, this massive graviton is absent, and the linearized equations of motion are second-order. Further, we find that for certain non-trivial classes of theories, the extra spin-0 degree of freedom is also absent, hence providing examples of theories for which, just like for Einstein or Lovelock, the only dynamical perturbation on a m.s.b. is the usual massless graviton.

An interesting motivation for constructing higher-derivative theories without the extra spin-0 and spin-2 modes in AdS$_D$ backgrounds was made clear in [?], where the authors observed that a particular higher-derivative theory containing a non-trivial cubic term [?, ?] — and which is well-known now as quasi-topological gravity[19] — was also free of these extra fields.[20] The reason is that holographic calculations involving graviton propagators become easily doable for theories satisfying this property, while providing non-trivial information about the dual CFTs. In the case of quasi-topological gravity, these holographic studies were performed in [?].

Maximally symmetric solutions

Let us consider the following action,

$$S = \frac{1}{16\pi G}\int_{\mathcal{M}} d^D x\sqrt{|g|}\left[-2\Lambda_0 + R + \lambda f(\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_{\lfloor D/2\rfloor})\right] , \tag{4.45}$$

i.e., we make explicit the EH and cosmological constant terms for clarity reasons. The equation of motion is simply given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_0\, g_{\mu\nu} + \lambda\,\mathcal{E}_{\mu\nu} = 0\,, \tag{4.46}$$

where $\mathcal{E}_{\mu\nu}$ is given in (??). As anticipated, we assume our background to be maximally symmetric, with metric $\bar{g}_{\mu\nu}$. The Riemann tensor of such spacetime is given by

$$\bar{R}^{\mu\nu}_{\alpha\beta} = \Lambda\delta^{\mu\nu}_{\alpha\beta}, \tag{4.47}$$

[18] See also [?].

[19] Higher-derivative extensions of quasi-topological gravity were constructed in [?, ?].

[20] Indeed, the linearized equations of quasi-topological gravity are identical to those of Einstein gravity up to an overall factor [?].

for some real constant Λ which, for AdS$_D$, is related to the AdS radius L by $\Lambda = -1/L^2$. When $\lambda = 0$, *i.e.,* for Einstein gravity, the maximally symmetric spacetime satisfying (**??**) is a solution of the theory provided Λ is related to the cosmological constant Λ_0 through

$$\Lambda = \frac{2\Lambda_0}{(D-1)(D-2)} \,. \tag{4.48}$$

For non-vanishing λ, we find the following constraint equation

$$\frac{2\Lambda_0 - \lambda f\left(\bar{\mathcal{L}}\right)}{(D-1)(D-2)} = \Lambda - \sum_{n=1}^{\lfloor D/2 \rfloor} \frac{2\lambda\, n(D-3)!}{(D-2n)!} \Lambda^n\, \partial_n f\left(\bar{\mathcal{L}}\right) \,, \tag{4.49}$$

where the bars mean that the corresponding quantities are evaluated on the background metric, and where we have used the following expressions

$$\bar{\mathcal{L}}_n = \frac{D!}{(D-2n)!}\Lambda^n \,, \quad \bar{\mathcal{E}}^{(n)}_{\mu\nu} = -\frac{1}{2}\frac{(D-1)!}{(D-2n-1)!}\Lambda^n g_{\mu\nu} \,. \tag{4.50}$$

Given f and Λ_0, (**??**) is an algebraic equation for Λ, and its solutions determine the possible vacua of the theory. In general, some of these vacua will contain ghost-like gravitons and will be unstable [**?**]. Note that this can occur even if the theory propagates only a single graviton mode,[21] like in the case of Lovelock theories. This problem can be avoided by choosing the vacuum that reduces to the Einstein gravity one when the higher-order couplings vanish — *i.e.,* when $\lambda \to 0$ in the case considered here.

As we will see in section **??**, the embedding equation (**??**) can be used to constrain the space of allowed values for the f(Lovelock) couplings.

Linearized equations

Let us now consider a small perturbation of our background metric, $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, with $h_{\mu\nu} << 1$ for all $\mu, \nu = 0, \ldots, D$. At linear order in $h_{\mu\nu}$, the ED and the tensors $\mathcal{E}^{(n)}_{\mu\nu}$ read

$$\mathcal{L}_n = \bar{\mathcal{L}}_n + \frac{n(D-2)!}{(D-2n)!}\Lambda^{n-1} R^L \,, \quad \mathcal{E}^{(n)}_{\mu\nu} = \bar{\mathcal{E}}^{(n)}_{\mu\nu} + \frac{n(D-3)!}{(D-2n-1)!}\Lambda^{n-1} G^L_{\mu\nu} \,, \tag{4.51}$$

where R^L and $G^{L\mu}_{\nu}$ are the linearized Ricci scalar and Einstein tensor respectively.[22] The following result is also necessary:

$$\bar{P}^{(n)\mu\nu}_{\alpha\beta} = -\frac{n(D-2)!}{2(D-2n)!}\Lambda^{n-1}\delta^{\mu\nu}_{\alpha\beta}. \tag{4.55}$$

[21] We thank Rob Myers for clarifying this point to us.

[22] The linearized Einstein tensor, Ricci tensor and Ricci scalar are given respectively by

$$G^{L\alpha}_{\mu} = \bar{g}^{\alpha\nu} R^L_{\mu\nu} - \frac{1}{2}\delta^{\alpha}_{\mu} R^L - \Lambda(D-1) h^{\alpha}_{\mu} \,. \tag{4.52}$$

$$R^L_{\mu\nu} = \frac{1}{2}\bar{\nabla}_\mu \bar{\nabla}_\sigma h^\sigma_\nu + \frac{1}{2}\bar{\nabla}_\nu \bar{\nabla}_\sigma h^\sigma_\mu - \frac{1}{2}\bar{\nabla}_\nu \bar{\nabla}_\mu h - \frac{1}{2}\bar{\Box} h_{\mu\nu} + \Lambda D h_{\mu\nu} - \Lambda h \bar{g}_{\mu\nu} \,, \tag{4.53}$$

$$R^L = \bar{\nabla}_\mu \bar{\nabla}_\nu h^{\mu\nu} - \bar{\Box} h - \Lambda(D-1) h \,. \tag{4.54}$$

Using this information we can find the linearized version of (??). It reads

$$\alpha\, G^L_{\mu\nu} + \Lambda\,\beta\,\bar{g}_{\mu\nu} R^L + \frac{\beta}{(D-1)}(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\nu\bar{\nabla}_\mu)R^L = 0\,, \tag{4.56}$$

where α and β are the following constants

$$\alpha = 1 + \lambda \sum_{n=1}^{\lfloor D/2\rfloor} \partial_n f\left(\bar{\mathcal{L}}\right) \frac{n(D-3)!}{(D-2n-1)!}\,\Lambda^{n-1}, \tag{4.57}$$

$$\beta = \lambda \sum_{n,m=1}^{\lfloor D/2\rfloor} \partial_n\partial_m f\left(\bar{\mathcal{L}}\right) \frac{n\,m(D-2)!(D-1)!}{(D-2n)!(D-2m)!}\,\Lambda^{n+m-2}. \tag{4.58}$$

Note that these constants are the only signature of the function f in the linearized equations. In particular, observe that when $\Lambda = 0$, *i.e.,* for Minkowski spacetime, all sign from the higher-curvature terms of order cubic or higher disappears from the linearized equations.

For arbitrary values of Λ, the full linearized equation (??) in terms of $h_{\mu\nu}$ reads

$$\begin{aligned}&\alpha\left[\bar{\nabla}_{(\mu|}\bar{\nabla}_\sigma h^\sigma_{|\nu)} - \frac{1}{2}\bar{\nabla}_\nu\bar{\nabla}_\mu h - \frac{1}{2}\bar{\Box}h_{\mu\nu} + \Lambda h_{\mu\nu} - \Lambda h\bar{g}_{\mu\nu}\right] + \\ &+\left[\bar{g}_{\mu\nu}\left(\Lambda\beta - \frac{\alpha}{2}\right) + \frac{\beta}{D-1}\left(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu\right)\right]\left[\bar{\nabla}_\alpha\bar{\nabla}_\beta h^{\alpha\beta} - \bar{\Box}h - \Lambda(D-1)h\right] = 0\,,\end{aligned} \tag{4.59}$$

and its trace is given by

$$\begin{aligned}&[D\Lambda\beta - \alpha(D/2-1)]\left[\bar{\nabla}_\mu\bar{\nabla}_\nu h^{\mu\nu} - \bar{\Box}h - \Lambda(D-1)h\right] \\ &- \Lambda\beta(D-1)\bar{\Box}h + \beta\left[\bar{\Box}\bar{\nabla}_\mu\bar{\nabla}_\nu h^{\mu\nu} - \bar{\Box}^2 h\right] = 0.\end{aligned} \tag{4.60}$$

The two equations above are not particularly illuminating. However, it is already noticeable the absence of terms of the form $\bar{\Box}^2 h_{\mu\nu}$ in (??). Such terms indicate the presence of ghost-like massive spin-2 fields and its absence is a nice feature of this class of theories. In order to make this statement more clear, we can exploit the 'gauge' symmetry of the linearized equations under transformations of the form $\delta h_{\mu\nu} = \bar{\nabla}_\mu\xi_\nu + \bar{\nabla}_\nu\xi_\mu$. In particular, we choose the following (transverse) gauge

$$\bar{\nabla}_\mu h^{\mu\nu} = \bar{\nabla}^\nu h\,. \tag{4.61}$$

In this gauge, the linearized equation (??) and its trace (??) become

$$\begin{aligned}&\alpha\left[\frac{1}{2}\bar{\nabla}_\mu\bar{\nabla}_\nu h - \frac{1}{2}\bar{\Box}h_{\mu\nu} + \Lambda h_{\mu\nu}\right] + \Lambda\left[\alpha\frac{(D-3)}{2} - (D-1)\Lambda\beta\right]\bar{g}_{\mu\nu}h \\ &- \Lambda\beta\left[\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu\right]h = 0\,.\end{aligned} \tag{4.62}$$

$$\Lambda\left[\beta\bar{\Box} + D\Lambda\beta - \frac{\alpha(D-2)}{2}\right]h = 0. \tag{4.63}$$

These expressions reduce to those obtained in [?, ?] and [?] in the particular cases of R^2 and $f(R)$ gravities respectively. Observe that all quartic derivatives have disappeared from these equations. Let us now split $h_{\mu\nu}$ into its trace and traceless components as

$$h_{\mu\nu} = \hat{h}_{\mu\nu} + \frac{1}{D}\bar{g}_{\mu\nu}h. \tag{4.64}$$

If we rewrite (??) using this new tensor we are left with the following inhomogeneous equation

$$-\frac{\alpha}{2}\left[\bar{\Box}\hat{h}_{\mu\nu}-2\Lambda\hat{h}_{\mu\nu}\right]+\left[\frac{\alpha}{2}+\Lambda\beta\right]\left[\bar{\nabla}_\mu\bar{\nabla}_\nu h-\frac{\bar{g}_{\mu\nu}}{D}\bar{\Box}h\right]=0. \tag{4.65}$$

This is still not completely satisfactory, as it contains terms involving the trace. We can however find an homogeneous equation by defining a new traceless tensor[23]

$$t_{\mu\nu}=\hat{h}_{\mu\nu}-\frac{2\beta}{(D-2)\alpha}\left[\bar{\nabla}_\mu\bar{\nabla}_\nu h-\frac{\bar{g}_{\mu\nu}}{D}\bar{\Box}h\right]. \tag{4.66}$$

Indeed, by using (??) and (??), one finds that $t_{\mu\nu}$ satisfies the equation

$$-\frac{\alpha}{2}\left[\bar{\Box}t_{\mu\nu}-2\Lambda t_{\mu\nu}\right]=0. \tag{4.67}$$

This is the equation for a traceless and massless spin-2 field. Hence, in f(Lovelock) theories, $t_{\mu\nu}$ is the tensor that represents the usual graviton. The other physical propagating degree of freedom is of course h. Indeed, (??) is the equation of a scalar field of mass

$$M^2=\frac{(D-2)\alpha}{2\beta}-D\Lambda\,. \tag{4.68}$$

This is of course provided $\beta\neq 0$. In such a case, when the background is AdS_D, the holographic dictionary [?, ?, ?] tells us that h is dual to a scalar operator $\mathcal{O}_\Delta$ in the $(D-1)$-dimensional boundary CFT with scaling dimension

$$\Delta=\frac{(D-1)}{2}+\sqrt{\frac{(D+1)^2}{4}+\frac{\alpha(D-2)L^2}{2\beta}}\,, \tag{4.69}$$

where we wrote $\Lambda=-1/L^2$. When $\beta/(\alpha L^2)$ is small and positive, $\mathcal{O}_\Delta$ is a highly-irrelevant positive-norm operator with $\Delta\simeq\sqrt{\alpha(D-2)L^2/(2\beta)}$. On the other hand, if $\beta/(\alpha L^2)$ is small and negative, Δ becomes imaginary, and h is a ghost-like field with tachyonic mass exceeding the Breitenlohner-Freedman (BF) bound [?, ?]. In that case, our f(Lovelock) theory would automatically be unstable if we interpreted it as a complete description rather than as an effective low energy theory.

4.3.1 Theories without dynamical scalar

Something interesting happens when $\beta=0$. In that case, just like for Einstein, Lovelock or quasi-topological gravities, the scalar mode is absent, and (??) just tells us that the transverse gauge condition (??) imposes the trace to vanish, *i.e.,* $h=0$. In those cases, the only physical field is the massless graviton $t_{\mu\nu}$.

In fact, when $\beta=0$, the full equations of motion become second order for any gauge. Indeed, in that case (??) becomes

$$\alpha\,G^L_{\mu\nu}=8\pi G\,T_{\mu\nu}\,, \tag{4.70}$$

where we have included the stress tensor of some additional matter fields in the right-hand side in order to stress that the overall factor α is non-trivial, as it determines the normalization of Newton's constant: $G_{\text{eff}}=G/\alpha$. Hence, we observe that for these theories,

[23] We follow the procedure presented in [?].

the linearized equations are exactly the same as for Einstein gravity but with an effective Newton constant controlled by α.

Now, interestingly enough, $\beta = 0$ does not necessarily imply $\lambda = 0$, *i.e.,* there are non-trivial f(Lovelock) theories satisfying (**??**) and for which the only propagating degree of freedom is therefore the ususal graviton. These are characterized by the conditions

$$\sum_{n,m=1}^{\lfloor D/2 \rfloor} \partial_n \partial_m f\left(\bar{\mathcal{L}}\right) \frac{n\, m(D-2)!(D-1)!}{(D-2n)!(D-2m)!} \Lambda^{n+m-2} = 0\,, \quad \partial_n \partial_m f\left(\bar{\mathcal{L}}\right) \neq 0\,, \tag{4.71}$$

for some n, m. Of course, these are satisfied by infinitely many classes of f(Lovelock) gravities. For example, theories of the form

$$S = \frac{1}{16\pi G} \int_{\mathcal{M}} d^D x \sqrt{|g|} \left[-2\Lambda_0 + R + \lambda \left(R^v \mathcal{L}_2^w - \gamma R^{2w+v}\right)\right]\,, \tag{4.72}$$

where

$$\gamma = \frac{v^2 + 4(w-1)w + v(4w-1)}{(v+2w)(v+2w-1)} \frac{(D-2)^w (D-3)^w}{D^w (D-1)^w}\,, \tag{4.73}$$

for some $v, w \geq 0$, satisfy the requirements. If we choose $v = w = 1$, $D = 4$ in (**??**), we find

$$S = \frac{1}{16\pi G} \int_{\mathcal{M}} d^4 x \sqrt{|g|} \left[-2\Lambda_0 + R + \lambda \left(R\mathcal{L}_2 - \frac{1}{9} R^3\right)\right]\,. \tag{4.74}$$

This is an example of a non-trivial four-dimensional cubic-order theory of gravity with second-order linearized equations of motion and which therefore only propagates a single massless graviton around m.s.b. The action (**??**) is somewhat reminiscent of critical gravity [**?**], a four-dimensional quadratic-theory for which the scalar degree of freedom is also absent, and the extra spin-2 field is massless. We stress again that all f(Lovelock) gravities are free of such spin-2 fields, so all theories satisfying (**??**), like (**??**), have in this sense a better behavior than critical gravity: the scalar is also absent and there is no need to set the mass of the extra graviton to zero because there is no extra graviton at all either. Also, recall that quasi-topological gravity exists only for $D \geq 5$, and that all Lovelock theories but Einstein gravity are trivial — or topological in the case of Gauss-Bonnet — in four dimensions. This makes (**??**) — and the rest of $D = 4$ theories satisfying (**??**) — particularly interesting and worth further study in our opinion.

4.3.2 Comments on unitarity

The propagator of $h_{\mu\nu}$ in any perturbatively unitary higher-curvature gravity around a m.s.b. is equal to the propagator of a quadratic theory of the form

$$S = \frac{1}{16\pi G} \int_{\mathcal{M}} d^D x \sqrt{|g|} \left[-2\Lambda_0 + R + c_1 R^2 + c_2 \mathcal{L}_2\right]\,. \tag{4.75}$$

In particular, the parameters of the corresponding higher-curvature theory are related to G, Λ_0, c_1 and c_2 above. These parameters are in turn constrained to satisfy different inequalities in order for the theory to be unitary — essentially these come from imposing that the effective Newton constant is positive and that the mass of the scalar mode is positive for dS_D and greater than the BF bound for AdS_D. We refer to [?] for details — see also [?]. Observe that in all cases, the massive spin-2 graviton is absent, given that the R^2 and GB terms do not introduce it. This is unsurprising, given that such field is generically a ghost and spoils unitarity. Hence, in general, the massless graviton and the scalar are the only allowed degrees of freedom in a unitary theory. Whenever $c_1 = 0$, the scalar will also be absent, leaving us with the usual massless graviton, and nothing else. The theories considered in the previous subsection belong to this class. For these, the linearized equations are second-order in any gauge, as we already stressed.

The problem of classifying or identifying in full generality which higher-curvature theories share propagator with any of the curvature-square gravities in (**??**) — *i.e.,* which of them are unitary on m.s.b. provided the appropriate constraints are satisfied — is non-trivial in general. In [?], the authors carried out this classification for the most general gravity theory constructed from contractions of the metric and the Riemann tensor at cubic order in curvature and in general dimensions. Using their results, it is not difficult to check that the three cubic f(Lovelock) terms constructed as products of ED — namely R^3, $R\mathcal{L}_2$ and $\mathcal{L}_3$ — belong to this class of theories. More generally, we expect *all* f(Lovelock) theories to be perturbatively unitary around m.s.b. as long as the appropriate constraints on the couplings are satisfied. We leave a thorough exploration of this issue for future work.

4.4 Holographic constraints on the coupling values

Holography [?, ?, ?] has become one of the main motivations for the study of higher-derivative gravities. As mentioned in the introduction, these theories have been used to characterize various properties of general CFTs in various dimensions — see *e.g.,* [?,?,?,?, ?,?]. In order for a higher-derivative theory to admit a physically sensible dual description in the holographic context, it must satisfy certain requisites. Such requisites — which have been previously considered many times in the past, *e.g.,* [?,?,?,?] — generically translate into constraints on the allowed values of the gravitational couplings. The most obvious example is the requirement that the theory admits at least one AdS vacuum — otherwise one cannot even talk about any 'dual theory'! Other considerations which in general lead to constraints on the gravity couplings consist of asking the dual theory to respect causality, unitarity, or certain quantum information inequalities.

In this section, we will find constraints on the allowed values of the gravitational couplings of a particular class of f(Lovelock) theories. The first set of constraints will come from imposing AdS_D to be a solution of the corresponding theory. For the second, we will restrict ourselves to $D = 5$, and we will use holographic entanglement entropy (HEE) — see subsection **??** for details.

A particularly relevant subclass of f(Lovelock) theories which appears several times throughout this paper is the one consisting of linear combinations of arbitrary products of

ED. In general dimensions, we have the following possibilities at each other in curvature

$$\begin{aligned} &R, \\ &R^2,\, \mathcal{L}_2, \\ &R^3,\, R\mathcal{L}_2,\, \mathcal{L}_3, \\ &R^4,\, R^2\mathcal{L}_2,\, R\mathcal{L}_3,\, \mathcal{L}_2^2,\, \mathcal{L}_4, \\ &R^5,\, R^3\mathcal{L}_2,\, R^2\mathcal{L}_3,\, R\mathcal{L}_2^2,\, R\mathcal{L}_4,\, \mathcal{L}_2\mathcal{L}_3,\, \mathcal{L}_5, \\ &\ldots \end{aligned} \tag{4.76}$$

There are $1, 2, 3, 5, 7, 11, 15, 22, 30, 42, \ldots$ of these. Note that at p-th order in curvature, the number of terms is given by the so-called *Partition Function* $P(p)$, which counts the number of ways in which the integer p can be written as a sum of positive integers.[24] In four and five dimensions, the most general Lagrangian density of this kind corresponds to a linear combination of terms of the form '$R^v\mathcal{L}_2^w$', where $v\,, w \in \mathbb{N}$. It will be for this last class of theories that we will construct the constraints.

4.4.1 AdS$_D$ embedding

As we have just anticipated, let us for now focus on the following D-dimensional subclass of f(Lovelock) theories

$$S_{v,w} = \frac{1}{16\pi G}\int_{\mathcal{M}} d^Dx\,\sqrt{|g|}\left[R + \frac{(D-1)(D-2)}{\tilde{L}^2} + \tilde{L}^{(2v+4w-2)}\lambda_{v,w}R^v(\mathcal{L}_2)^w\right], \tag{4.77}$$

where we have chosen the cosmological constant to be negative, $\Lambda_0 = -(D-1)(D-2)/(2\tilde{L}^2)$, and where $\lambda_{v,w}$ is a dimensionless coupling. When $\lambda_{v,w} = 0$, the embedding equation for AdS$_D$, whose metric in Poincaré coordinates reads

$$ds^2 = \frac{L^2}{z^2}\left[-dt^2 + dz^2 + d\vec{x}^2_{(D-2)}\right], \tag{4.78}$$

simply imposes that the scale $\tilde{L}$ in the action is equal to the AdS radius L. Of course, as we include additional higher order terms, this is no longer true, and the relation between both scales depends on the new gravitational couplings. For general f(Lovelock) theories, the corresponding embedding equation is (**??**). Applying it to the particular case of (**??**), it becomes

$$1 - f_\infty - C_{v,w} f_\infty^{v+2w}\lambda_{v,w} = 0\,, \tag{4.79}$$

where

$$C_{v,w} = (-1)^{v-1}(D-1)^{w+v-1}D^{w+v-1}(D-2)^{w-1}(D-3)^w(D-2(v+2w))\,, \tag{4.80}$$

and where we have defined $f_\infty = \tilde{L}^2/L^2$. It is not possible to solve the above equation for f_∞ in full generality. However, it suffices for our purposes to obtain the set of values of $\lambda_{v,w}$ for which the above equation is satisfied in a physically sensible way. In particular,

[24] For example, for $p = 4$ we have $4 = 1+1+1+1 = 3+1 = 2+2 = 2+1+1 = 4$, so $P(4) = 5$. $P(p)$ also coincides with the number of conjugacy classes of the permutation group of order p.

we need to require f_∞ to be positive and tend to one as $\lambda_{v,w} \to 0$. Using (**??**), we can write $\lambda_{v,w}$ as a function of f_∞, *i.e.,*

$$\lambda_{v,w}(f_\infty) = \frac{1 - f_\infty}{C_{v,w}\; f_\infty^{v+2w}}\,. \tag{4.81}$$

Now, when $C_{v,w} > 0, (C_{v,w} < 0)$ the function $\lambda_{v,w}(f_\infty)$ has a global minimum (maximum) at

$$f_\infty^* = \frac{(2w + v)}{(2w + v - 1)}\,. \tag{4.82}$$

This directly implies the following constraint on the coupling constant $\lambda_{v,w}$,

$$\begin{aligned} \lambda_{v,w} &\geq \lambda_{v,w}(f_\infty^*)\,, \quad \text{when} \quad C_{v,w} > 0\,, \\ \lambda_{v,w} &\leq \lambda_{v,w}(f_\infty^*)\,, \quad \text{when} \quad C_{v,w} < 0\,, \end{aligned} \tag{4.83}$$

and where

$$\lambda_{v,w}(f_\infty^*) = -\frac{(2w + v - 1)^{2w+v-1}}{C_{v,w}\,(2w + v)^{2w+v}}\,. \tag{4.84}$$

For example, if we choose $v = 0$, $w = 1$, and define $\lambda_{0,1} = \lambda_{\rm GB}/((D-3)(D-4))$ — as it is customary — (**??**) becomes the usual GB theory, and (**??**) is nothing but the well-known constraint $\lambda_{\rm GB} \leq 1/4$, [**?**, **?**, **?**]. Another familiar example corresponds to $v = 2$ and $w = 0$, which is nothing but R^2 gravity. The constraint reads in that case $\lambda_{2,0} \leq (D-2)/(2(D-1)(D-4))$. Interestingly, this does not impose any constraint in four dimensions, which is a consequence of the fact that the couplings of general quadratic gravities do not enter into the embedding equation of AdS_4,[25] see *e.g.,* [**?**]. Actually, we observe that this phenomenon occurs whenever

$$D = 2(v + 2w)\,. \tag{4.85}$$

This means, for example, that AdS_6 is a solution of R^3 and $R\mathcal{L}_2$ gravities for arbitrary values of the corresponding gravitational couplings.

4.4.2 Holographic entanglement entropy

In this subsection we will use holographic entanglement entropy (HEE) to find additional constraints on $\lambda_{v,w}$ for five-dimensional theories. Before explaining our procedure, let us start with some essentials about entanglement entropy in the holographic context.

Consider a bipartition of the Hilbert space of some quantum system, $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$, and some state ρ. The entanglement entropy (EE) is defined with respect to the reduced state corresponding to one of the partitions, say A, obtained by tracing out the degrees of freedom in B, $\rho_A = \mathrm{Tr}_B\, \rho$. In particular, the EE is defined as the Von Neumann entropy of ρ_A, *i.e.,* $S_{\rm EE}(\rho_A) = -\,\mathrm{Tr}(\rho_A \log \rho_A)$. In the following, we will restrict ourselves to spatial bipartitions, meaning that A will always be a physical spatial region at a fixed time slice, and B its complement.

In the context of holography, EE is computed using the Ryu-Takayanagi (RT) prescription [**?**, **?**]. According to this, given an asymptotically AdS_D spacetime dual to some

[25] In other words, AdS_4 is a solution of general curvature-squared gravities — and R^2 in particular — as long as $L = \tilde{L}$, just like for Einstein gravity. We come back to this point in section **??**.

state in the boundary theory, the HEE for a region A in the corresponding CFT is obtained by extremizing the area functional of codimension-2 bulk surfaces m which are homologous to A in the boundary (and, in particular, $\partial m = \partial A$). More precisely,

$$S_{\rm RT}(A) = \underset{m\sim A}{\rm ext}\left[\frac{\mathcal{A}(m)}{4G}\right], \quad \mathcal{A}(m) = \int_m d^{D-2}x\,\sqrt{h_m}\,, \tag{4.86}$$

where G is the Newton constant and h_m is the determinant of the metric induced on m. Naturally, this prescription is only valid for theories dual to Einstein gravity in the bulk. In particular, when higher-derivative terms are introduced in the bulk theory, the area functional in (??) must be modified to something like

$$S(A) = \underset{m\sim A}{\rm ext}\, S_{\rm grav}(m)\,, \tag{4.87}$$

where $S_{\rm grav}(m)$ is a new bulk functional which depends on the particular higher-derivative theory, and which reduces to (??) for Einstein gravity. Much effort has been put into trying to identify the explicit form of $S_{\rm grav}(m)$ for different higher-derivative bulk theories, with remarkable success — see [?,?,?,?,?,?,?,?,?,?,?] for a non-exhaustive list of references. In particular, a new functional consisting of a Wald-like term [?,?,?] plus corrections involving extrinsic curvatures has been proposed to hold for general higher-derivative gravities [?]. While such proposal passes various consistency checks, certain subtleties arise [?, ?, ?, ?] when the theory is not Einstein, curvature-squared or Lovelock, which make it unclear how to use this prescription in general.[26]

As explained in the introduction, the authors of [?] proposed (??) to be the right formula for the gravitational entropy in f(Lovelock) theories. In that paper, the authors were able to show that this functional satisfies an increase theorem for linearized perturbations of Killing horizons. Besides, $S_{\rm SW}$ reduces to the well-known JM functional for Lovelock gravities [?] which, as already mentioned, gives rise to the right universal terms when used to compute HEE for these theories. In [?, ?], these two facts were interpreted as evidence that $S_{\rm SW}$ is in fact the right HEE functional for f(Lovelock) theories. The results found in those papers strongly support this claim.

Our plan is to use (??) to find new constraints on the coupling values $\lambda_{v,w}$. The idea [?] is to consider simple entangling regions for which the surface m can be parametrized as some function $g(z)$ of the holographic coordinate. While extremizing $S_{\rm SW}$ — *i.e.,* finding the explicit form of $g(z)$ — is an impracticable task in general, we do know that m must close off smoothly at some bulk point $z = z_h$. Hence, we can assume that $g(z)$ admits a series expansion around z_h,

$$g(z) = \sum_{i=0}^{\infty} c_i (z_h - z)^{\alpha+i}\,. \tag{4.88}$$

Besides, we need to impose that $g(z_h) = 0$ and $g'(z_h) = -\infty$, since the tangent to the surface will be perpendicular to the z direction at that point. These conditions imply the constraints $0 < \alpha < 1$ and $c_0 > 0$, which we will use to find bounds on $\lambda_{v,w}$.

Evaluating (??) for our f(Lovelock) theory (??), one finds

$$S_{v,w} = \frac{1}{4G}\int_m d^3x\sqrt{h_m}\,\left[A + BL^2\,\mathcal{R}_m\right]\,, \tag{4.89}$$

[26] We thank Rong-Xin Miao for useful comments about this point.

where we defined the constants[27]

$$A = 1 + \lambda_{v,w}\,(-1)^{v-1} 2^{2v+3w-2} 5^{v+w-1} 3^{w} v\, f_{\infty}^{(v+2w-1)}\,, \quad B = \frac{w(1-A)}{3v}\,, \tag{4.90}$$

and where $\mathcal{R}_m$ is the Ricci scalar associated to the induced metric on the holographic surface m. In particular, note that for $v = 0$, $w = 1$, $\lambda_{0,1} = \lambda_{\rm GB}/2$, (??) reduces to the JM functional for GB gravity, *i.e.*,

$$S_{\rm JM} = \frac{1}{4G}\int_m d^3x\sqrt{h_m}\,\left[1 + \lambda_{\rm GB}\tilde{L}^2\,\mathcal{R}_m\right]. \tag{4.91}$$

In [?], new constraints on $\lambda_{\rm GB}$ were obtained using this functional for some simple entangling regions following the procedure outlined above. Here we will generalize those results to arbitrary values of v and w. A quick look at (??) and (??) shows that no bounds can be found using this technique for theories with $w = 0$. The reason is that for those, the holographic extremal surface will be the same as in Einstein gravity, so it will not depend on the value of $\lambda_{v,0}$.

Let us start considering an entangling region consisting of a slab of width l defined by $x_1 \in [-l/2, \leq l/2]$, $x_2 \in (-\infty, +\infty)$, $x_3 \in (-\infty, +\infty)$. Now, using the obvious symmetry along the $x_{2,3}$ directions, we can parametrize the holographic surface m as $t_{\rm E} = 0$, $x_1 = g(z)$. The induced metric on this surface reads

$$ds_m^2 = \frac{L^2}{z^2}\left[(1+\dot{g}^2)dz^2 + dx_2^2 + dx_3^2\right], \tag{4.92}$$

where we used the notation $\dot{g} = dg(z)/dz$. The Euler-Lagrange equation for $g(z)$ obtained from (??), reads

$$-3(A-2B)\dot{g} - 6(A-B)\dot{g}^3 - 3A\dot{g}^5 + (A-2B)\ddot{g}z + (A+4B)z\dot{g}^2\ddot{g} = 0\,. \tag{4.93}$$

Inserting now the series expansion (??) in this equation, we find that the only value of α compatible with the smoothness requirements is $\alpha = 1/2$. Using this, we can find the value of c_0 by imposing the coefficient of the lowest order term in (??) to vanish. By doing so, we find

$$c_0 = \sqrt{\frac{2(A+4B)z_h}{3A}}\,. \tag{4.94}$$

This imposes $(A+4B)/A > 0$ which, after some careful calculations, gives rise to the following constraint on $\lambda_{v,w}$

$$\begin{aligned} \lambda_{v,w} &< \lambda_{v,w}^{(\rm s)}\,, \quad \text{when} \quad v \text{ even}\,, \\ \lambda_{v,w} &> \lambda_{v,w}^{(\rm s)}\,, \quad \text{when} \quad v \text{ odd}\,, \end{aligned} \tag{4.95}$$

where

$$\lambda_{v,w}^{(\rm s)} = \frac{(-1)^v(5v+4w-5)^{v+2w-1}}{v^{2w+v}2^{2v+3w-2}3^{v+3w-1}5^{v+w-1}}\,. \tag{4.96}$$

and which is valid whenever $v \geq 1$ and $w \geq 1$. The GB case, $v = 0$, $w = 1$, is a bit special, and one finds

$$\lambda_{\rm GB} = 2\lambda_{0,1} > -\frac{5}{16}\,, \tag{4.97}$$

[27] Note that these are related to the constant α defined in (??) through: $\alpha = A - 2B$.

in agreement with the result of [?]. For $v = 0$ and $w > 1$, no bounds on $\lambda_{0,w}$ are found. Also, as a check of our procedure, we have verified that no bounds appear when $w = 0$ — indeed, $(A + 4B)/A = 1$ in that case.

Let us now consider a cylindrical entangling surface. We write the AdS_5 metric as

$$ds^2 = \frac{L^2}{z^2}\left[dt_{\mathrm{E}}^2 + dz^2 + d\rho^2 + \rho^2 d\theta^2 + dx_3^2\right] , \tag{4.98}$$

and let the cylinder be defined as $t_{\mathrm{E}} = 0$, $\rho \in [0, R]$, $\theta \in [0, 2\pi)$, $x_3 \in (-\infty, +\infty)$. Again, the symmetry of the entangling surface allows us to parametrize the holographic surface as $t_{\mathrm{E}} = 0$, $\rho = g(z)$. The pullback metric on such surface reads

$$ds_m^2 = \frac{L^2}{z^2}\left[(1 + \dot{g}^2)dz^2 + g^2 d\theta^2 + dx_3^2\right] . \tag{4.99}$$

The corresponding Euler-Lagrange equation for $g(z)$ reads in this case

$$\begin{aligned} &- (1 + \dot{g}^2)\left[3g\dot{g}\left[A(1 + \dot{g}^2) - 2B\right] + z\left[A - 2B + (A + 4B)\dot{g}^2\right]\right] \\ &+ z\left[6Bz\dot{g} + g(A - 2B + (A + 4B)\dot{g}^2)\right]\ddot{g} = 0 . \end{aligned} \tag{4.100}$$

From this we find again $\alpha = 1/2$, and using this result we obtain the only allowed value of c_0 to be

$$c_0 = \sqrt{\frac{2z_h}{3A}\left[(A + 4B) \pm \sqrt{A^2 + 16B^2 - 10AB}\right]} , \tag{4.101}$$

so we are lead to impose $A^2 + 16B^2 - 10AB \geq 0$. A careful analysis shows that the condition that follows from this inequality reads

$$\begin{aligned} \lambda_{v,w} &\leq \lambda_{v,w}^{(\mathrm{c})} , \quad \text{when} \quad v \text{ even} , \\ \lambda_{v,w} &\geq \lambda_{v,w}^{(\mathrm{c})} , \quad \text{when} \quad v \text{ odd} , \end{aligned} \tag{4.102}$$

where

$$\lambda_{v,w}^{(\mathrm{c})} = \frac{(-1)^v(-5 + 5v + 12w)^{v+2w-1}}{3^{w-1}2^{3w+2v-2}5^{w+v-1}(3v + 8w)^{v+2w}} . \tag{4.103}$$

As opposed to the slab — for which the GB case was special — this condition is the same for all values $v \geq 0$ and $w \geq 1$. In particular, one finds

$$\lambda_{\mathrm{GB}} = 2\lambda_{0,1} \leq \frac{7}{64} , \tag{4.104}$$

again in agreement with the bound found in [?]. We have checked again that no bounds are found when $w = 0$, as expected.

In sum, for the family of theories (??) in five dimensions, we have found constraints from the AdS_5 embedding, and from imposing the holographic surface corresponding to a slab and a cylindrical entangling region to close off smoothly in the bulk. Combining all these constraints, we found the following bounds on $\lambda_{v,w}$,

$$\begin{aligned} \lambda_{v,w}(f_\infty^*) \leq \lambda_{v,w} &\leq \lambda_{v,w}^{(\mathrm{c})} , \quad \text{when} \quad v \text{ even} , \\ \lambda_{v,w}(f_\infty^*) \geq \lambda_{v,w} &\geq \lambda_{v,w}^{(\mathrm{c})} , \quad \text{when} \quad v \text{ odd} , \end{aligned} \tag{4.105}$$

which are valid for $v \geq 1$, $w \geq 1$ and $v = 0$, $w > 1$. Note that these are quite strong constraints in general. For example, one finds $\lambda_{1,1}(f_\infty^*) = 1/270 \simeq 0.0037$, $\lambda_{1,1}^{(\mathrm{c})} =$

$-18/6655 \simeq -0.0027$. Further, for larger values of v and w, the quantities $\lambda_{v,w}(f_\infty^*)$ and $\lambda_{v,w}^{(c)}$ become increasingly smaller. In the case of Gauss-Bonnet, the bounds read instead $-5/16 \leq \lambda_{\rm GB} \leq 7/64$. Finally, recall that when $w = 0$, we only have the bound from the AdS$_D$ embedding, *i.e.,* the one given in (**??**).

4.5 Black hole solutions

In this section we construct analytic solutions of several f(Lovelock) theories in various dimensions.

4.5.1 Pure f(Lovelock)

Let us start considering a f(Lovelock) theory consisting of a function of a single ED, $\mathcal{L}_n$, *i.e.,*

$$S = \frac{1}{16\pi G}\int_{\mathcal{M}} d^Dx\sqrt{|g|}\, f(\mathcal{L}_n)\,. \tag{4.106}$$

Of course, for $n = 1$, this reduces to $f(R)$ gravity, whose constant R solutions were first studied in [**?**]. Here we will generalize some of their results by constructing constant-$\mathcal{L}_n$ solutions to (**??**) for arbitrary values of n. The field equations of the theory read

$$f'(\mathcal{L}_n)\mathcal{E}_{\mu\nu}^{(n)} + \frac{1}{2}g_{\mu\nu}\left[\mathcal{L}_n f'(\mathcal{L}_n) - f(\mathcal{L}_n)\right] - 2P^{(n)}_{\alpha\nu\lambda\mu}\nabla^\alpha\nabla^\lambda f'(\mathcal{L}_n) = 0\,, \tag{4.107}$$

where the tensors $\mathcal{E}_{\mu\nu}^{(n)}$ and $P^{(n)}_{\alpha\nu\lambda\mu}$ were defined in (**??**). In particular, note that the equations of motion of a pure Lovelock theory consisting of a single ED of order n plus a cosmological constant term, *i.e.,*

$$f(\mathcal{L}_n) = -2\Lambda_0 + \Lambda_0^{1-n}\mathcal{L}_n\,, \tag{4.108}$$

satisfy

$$\mathcal{E}_{\mu\nu}^{(n)} = -\Lambda_0^n g_{\mu\nu}\,, \quad \text{which implies} \quad \mathcal{L}_n = \frac{2D}{(D-2n)}\Lambda_0^n\,. \tag{4.109}$$

In other words, all solutions of (**??**) have a constant $\mathcal{L}_n$ proportional to the n-th power of the cosmological constant, just like all solutions of general relativity in the absence of matter have a constant Ricci scalar proportional to Λ_0.

Now, let us see under what conditions spacetimes of constant $\mathcal{L}_n$ solve the general $f(\mathcal{L}_n)$ equations (**??**). If we assume $\mathcal{L}_n$ to be constant, the term with the covariant derivatives vanishes, and we are left with

$$f'(\mathcal{L}_n)\mathcal{E}_{\mu\nu}^{(n)} + \frac{1}{2}g_{\mu\nu}\left[\mathcal{L}_n f'(\mathcal{L}_n) - f(\mathcal{L}_n)\right] = 0\,, \tag{4.110}$$

whose trace reads

$$nf'(\mathcal{L}_n)\mathcal{L}_n - \frac{D}{2}f(\mathcal{L}_n) = 0\,. \tag{4.111}$$

Given a particular f, this is an algebraic equation which solutions of (**??**) are forced to satisfy. In particular, assuming it admits a solution, (**??**) fixes $\mathcal{L}_n$ to some constant value which we will denote $\mathcal{L}_n^0$. Now we have two possibilities, depending on whether the derivative of f vanishes when evaluated on $\mathcal{L}_n^0$. If $f'(\mathcal{L}_n^0) \neq 0$, we recover the pure Lovelock field equations, while if $f'(\mathcal{L}_n^0) = 0$, we do not need to impose any additional condition, because, in that case, all configurations satisfying $\mathcal{L}_n = \mathcal{L}_n^0$ are already extremal points of the action. Let us explain both cases in more detail.

Assume first that $f'(\mathcal{L}_n^0) \neq 0$. In that case, we can rewrite (**??**) as

$$\mathcal{E}_{\mu\nu}^{(n)} = -\Lambda_{0,\text{eff}}^n \, g_{\mu\nu} \,, \quad \text{where} \quad \Lambda_{0,\text{eff}}^n = \frac{(D-2n)}{2D} \mathcal{L}_n^0 \,, \tag{4.112}$$

which is nothing but the pure Lovelock equation of motion (**??**) with an effective cosmological constant $\Lambda_{0,\text{eff}}$ determined by the solution of (**??**). Hence, any solution of pure Lovelock plus cosmological constant, is also a constant-$\mathcal{L}_n$ solution of (**??**) provided $f'(\mathcal{L}_n^0) \neq 0$. This allows, in particular, to embed all Einstein gravity plus cosmological constant solutions in $f(R)$ whenever $f'(R^0) \neq 0$, as explained in [**?**].

Static black hole solutions of pure Lovelock gravities have been previously considered several times — see *e.g.,* [**?**, **?**, **?**, **?**, **?**]. In particular, for $D > 2n$, a theory of the form (**??**) admits the following interesting generalization of the Schwarzschild(-AdS/dS) black hole solution [**?**]

$$ds^2 = -g(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2 d\Omega_{(D-2)}^2 \,, \tag{4.113}$$

where

$$g(r) = h(r) = 1 \mp r^2 \left[\frac{a}{r^{D-1}} + \Lambda_0^n \frac{2(D-2n-1)!}{(D-1)!} \right]^{1/n} . \tag{4.114}$$

In this expression, a is an integration constant which can be related to the solution's mass, and the $+$ sign in front of the term in brackets is allowed only when n is even. According to our analysis above, this is also a solution of (**??**) for theories satisfying $f'(\mathcal{L}_n^0) \neq 0$. More precisely, using (**??**) we find that the solution to (**??**) can be written as (**??**) with

$$g(r) = h(r) = 1 \mp r^2 \left[\frac{a}{r^{D-1}} + \mathcal{L}_n^0 \frac{(D-2n)!}{D!} \right]^{1/n} , \tag{4.115}$$

where again $\mathcal{L}_n^0$ is a solution to (**??**). For $n = 1$, this reduces to the well-known $f(R)$ Schwarzschild(-AdS/dS) black hole [**?**]

$$g(r) = h(r) = 1 - \frac{a}{r^{D-3}} - \frac{R^0}{D(D-1)} r^2 \,. \tag{4.116}$$

For general values of n, (**??**) and (**??**) describe a f(Lovelock) generalization of the Schwarzschild(-AdS/dS) solution.

Note that if the dimension is the critical one, $D = 2n$, (**??**) is trivially satisfied because $\mathcal{E}_{\mu\nu}^{(D/2)} = 0$ identically. In that case, any solution of (**??**) — *i.e.,* any constant-$\mathcal{L}_n$ spacetime — is a solution of (**??**). For example, in $D = 4$, $f(\mathcal{L}_2)$ always allows for a solution with $\mathcal{L}_2 = \mathcal{L}_2^0$.

Before we turn to the $f'(\mathcal{L}_n^0) = 0$ case, let us make a further observation. Assuming $D > 2n$, let us consider a theory of the form

$$f(\mathcal{L}_n) = -2\Lambda_0 + \Lambda_0^{1-n} \mathcal{L}_n + \alpha \Lambda_0^{1-\frac{D}{2}} \mathcal{L}_n^{\frac{D}{2n}} \,, \tag{4.117}$$

for some dimensionless constant α. Interestingly, all solutions of the $\alpha = 0$ theory are also solutions of (??). This can be easily seen by imposing the $\alpha = 0$ equation of motion (??) in the equations (??) and (??) corresponding to (??). By doing so, we observe that the terms proportional to α exactly cancel each other out. This explains, in particular, why all solutions of Einstein gravity plus cosmological constant are also solutions of such theory with an additional R^2 term in four dimensions [?]. Hence, we observe that (??) with (??) is also a solution of (??). The reason for this general behavior can be traced back to the fact that $\mathcal{L}_n^{D/2n}$ is scale-invariant, *i.e.,* it is preserved by a rescaling of the metric. Now, a rescaling of the metric changes the scale of the theory. Hence, such scale cannot depend on α.

Let us now turn to the cases for which $f'(\mathcal{L}_n^0) = 0$. If this happens, (??) imposes also $f(\mathcal{L}_n^0) = 0$ and the equations of motion (??) are automatically satisfied. This means that spacetimes of constant-$\mathcal{L}_n$ are solutions of (??) when these two conditions are satisfied. Obviously, this happens because a configuration $\mathcal{L}_n^0$ satisfying

$$f'(\mathcal{L}_n^0) = f(\mathcal{L}_n^0) = 0\,, \tag{4.118}$$

is always an extremum of the action. The existence of this kind of configurations depends on the particular theory under consideration. The simplest example is probably $f(\mathcal{L}_n) = \mathcal{L}_n^2$, for which $\mathcal{L}_n^0 = 0$ is clearly an extremum of the action and therefore a solution. The lesson is that in order to find a solution for these theories, we only need to require $\mathcal{L}_n$ to be equal to the constant $\mathcal{L}_n^0$ for which (??) holds. Since this is a single scalar equation for the metric, the number of possible solutions is huge. In particular, for an ansatz of the form (??) with $h(r) = g(r)$, one gets

$$g(r) = 1 \mp r^2 \left[\frac{a}{r^{D-1}} + \frac{b}{r^D} + \mathcal{L}_n^0 \frac{(D-2n)!}{D!}\right]^{1/n}\,, \tag{4.119}$$

which has two integrations constants, a and b, instead of one. For $n = 1$, this reduces to

$$g(r) = 1 - \frac{a}{r^{D-3}} - \frac{b}{r^{D-2}} - \frac{R^0}{D(D-1)} r^2\,, \tag{4.120}$$

as observed in [?]. Note that for $D = 4$, this takes the familiar form of the Reissner-Nordström(-AdS/dS) solution, with b playing the role of the charge squared. Observe that this fact is accidental, and occurs only in four dimensions. In fact, it can be easily seen that the f(Lovelock)-Maxwell system

$$S = \int_{\mathcal{M}} d^D x \sqrt{|g|} \left[\frac{1}{16\pi G} f(\mathcal{L}_n) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right]\,, \tag{4.121}$$

admits the following generalization of the Reissner-Nordström(-AdS/dS) solution

$$g(r) = h(r) = 1 \mp r^2 \left[\frac{a}{r^{D-1}} - \frac{c}{r^{2(D-2)}} + \mathcal{L}_n^0 \frac{(D-2n)!}{D!}\right]^{1/n}\,, \tag{4.122}$$

when $f'(\mathcal{L}_n^0) \neq 0$, where $\mathcal{L}_n^0$ is again a solution of (??) and where c is a constant related to the electric charge. Comparing (??) with (??), we see that only when $D = 4$ the exponents of the terms proportional to b and c respectively are equal.

If one considers the general ansatz (??) with two unknown functions, the system is underdetermined, since we only have one equation. For example, let us consider $f(\mathcal{L}_2) = \mathcal{L}_2^2$ in $D = 4$. In this theory, a family of solutions is given by $\mathcal{L}_2 = 0$. Assuming the ansatz (??), we get the following equation for g and h

$$\frac{h(r)^{1/2}\,[h(r)-1]}{g(r)^{1/2}}\frac{dg(r)}{dr} = c_1 \tag{4.123}$$

where c_1 is an integration constant. Choosing one of the functions at will, one can find the other by solving the above equation. This approach was followed, *e.g.,* in [?] to construct solutions to pure R^2 gravity.[28]

4.5.2 General f(Lovelock)

In the previous subsection, we restricted ourselves to the case of f(Lovelock) theories consisting of functions of a single ED. Let us now explore what the situation is when one considers a function which depends on all the non-vanishing ED's, $f(\mathcal{L}_1, ..., \mathcal{L}_{\lfloor D/2\rfloor})$. Just like we were able to embed all solutions of the pure Lovelock Lagrangian (??) in $f(\mathcal{L}_n)$, we would like to embed solutions of the general Lovelock action (??) in the general f(Lovelock) one (??). A simple argument shows that, in general, f(Lovelock) theory does not contain all solutions of Lovelock. The reasoning goes as follows. Assume that all solutions of the Lovelock equations (??) are also solutions of f(Lovelock) theory (??). Then, the ED associated to these metrics satisfy the trace equation

$$\sum_{n=0}^{\lfloor D/2\rfloor} \lambda_n \Lambda_0^{1-n}\left(n - \frac{D}{2}\right)\mathcal{L}_n = 0\,, \tag{4.124}$$

but now, assume that these metrics also solve (??). Then, (??) should reduce to (??) whenever (??) is satisfied. This implies, in particular, that all the partial derivatives of f evaluated on the solution must be constant, which of course is not true for arbitrary functions $f(\mathcal{L}_1, ..., \mathcal{L}_{\lfloor D/2\rfloor})$. Hence, we observe that, on general grounds, solutions of Lovelock gravity are not embeddable in f(Lovelock) unless f is chosen in an appropriate way — like we did in the previous section by making it depend on a single ED.

We claim that the most general f(Lovelock) theory whose solutions include all the Lovelock theory ones is given by a function of the form

$$f(\mathcal{L}_1, ..., \mathcal{L}_{\lfloor D/2\rfloor}) = \alpha \sum_{n=0}^{\lfloor D/2\rfloor} \lambda_n \Lambda_0^{1-n}\mathcal{L}_n + \left[\sum_{n=0}^{\lfloor D/2\rfloor} \lambda_n \Lambda_0^{1-n}\left(n - \frac{D}{2}\right)\mathcal{L}_n\right]^2 \tilde{f}(\mathcal{L}_1, ..., \mathcal{L}_{\lfloor D/2\rfloor})\,, \tag{4.125}$$

where α is a constant and $\tilde{f}$ is an arbitrary function such that its derivatives are non-singular when the squared quantity is zero.

4.5.3 A critical black hole in $f(R, \mathcal{L}_2)$

As we have seen, finding black solutions to f(Lovelock) theories involving more than one ED seems to be a difficult challenge. An exception is, of course, the case in which f

[28] See also, *e.g.,* [?], where certain aspects of AdS black holes in pure curvature-squared gravities where considered.

is a linear combination of ED, corresponding to general Lovelock theories — see *e.g.,* [?,?,?,?,?,?]. A possible simplification that has been often considered in the literature for other higher-derivative gravities — see *e.g.,* [?,?], consists of fixing some of the couplings of the theory to particular values which allow for solutions which would not exist otherwise. We will explore this approach here. In particular, let us consider the following $f(R,\mathcal{L}_2)$ theory — whose general equations of motion are specified in appendix **??** — consisting of the standard Einstein-Hilbert action plus certain higher-order corrections

$$S=\int_{\mathcal{M}}\frac{d^Dx\sqrt{-g}}{16\pi G}\left[R+\frac{(D-1)(D-2)}{\tilde{L}^2}+\alpha\tilde{L}^2R^2+\beta\tilde{L}^2\mathcal{L}_2+\gamma\tilde{L}^4R\mathcal{L}_2+\delta\tilde{L}^6\mathcal{L}_2^2\right], \tag{4.126}$$

where α, β, γ and δ are dimensionless constants, and where we have chosen the cosmological constant to be negative and determined by some length scale $\tilde{L}$. If the coupling parameters satisfy

$$\alpha=\frac{1}{4(D-1)(D-2)}\,,\quad \beta=\frac{\lambda}{(D-2)(D-3)}\,,\quad \gamma=2\alpha\beta\,,\quad \delta=\alpha\beta^2\,, \tag{4.127}$$

then (**??**) allows for the following solution

$$ds^2=-g(r)dt^2+\frac{1}{g(r)}dr^2+r^2d\Omega^2_{(D-2)}\,, \tag{4.128}$$

with

$$g(r)=1+\frac{r^2}{2\lambda\tilde{L}^2}\left[1-\sqrt{1-4\lambda\left(\frac{2D-4}{D}-\frac{c_1}{r^{D-1}}+\frac{c_2}{r^D}\right)}\right], \tag{4.129}$$

where c_1 and c_2 are integration constants. This solution describes an asymptotically AdS$_D$ black hole as long as the constants are chosen so that $g(r)>0$ for all $r>r_h$ and $g(r_h)=0$ for some positive value of r.

The reason why the election of parameters in (**??**) allows for this solution is not very mysterious. In fact, (**??**) makes the Lagrangian in (**??**) become a perfect square,

$$f(R,\mathcal{L}_2)=\frac{(D-1)(D-2)}{\tilde{L}^2}\left[1+\frac{\tilde{L}^2R}{2(D-1)(D-2)}+\frac{\lambda\tilde{L}^4\mathcal{L}_2}{2(D-1)(D-2)^2(D-3)}\right]^2, \tag{4.130}$$

which implies that any configuration satisfying $f(R,\mathcal{L}_2)=0$ also satisfies $\partial_1 f(R,\mathcal{L}_2)=\partial_2 f(R,\mathcal{L}_2)=0$ and is therefore a solution of the corresponding equations of motion.

When $\lambda=0$, all the higher-derivative terms in (**??**) but the R^2 one disappear and (**??**) takes the form of (**??**), *i.e.,*

$$g(r)=1-\frac{a}{r^{D-3}}-\frac{b}{r^{D-2}}+\frac{r^2}{\tilde{L}^2}\,, \tag{4.131}$$

as observed in [?]. In [?,?], the thermodynamic properties of various black hole solutions of (**??**) with $\lambda=0$ were studied. As observed there, while some of the solutions correspond to regular black holes with finite horizons, amusingly enough, they always possess vanishing entropy and mass. The reason is that both the on-shell action — including boundary terms — and the Wald entropy involve factors of either $f(R)$ or $f'(R)$, which vanish for these configurations, as we have just seen. In the case $\lambda\neq 0$, the situation is exactly the same. In particular, the action, the boundary term we have proposed in (??) — necessary to compute the on-shell action — and the entropy functional of [?] — see (??) — vanish on-shell for configurations of this kind. The physical interpretation of these solutions is unclear to us.

4.6 Final comments and perspectives

In this paper we have developed several aspects of f(Lovelock) theories. A summary of our main findings was already provided in section ??, so we will not repeat it here. However, let us comment on some additional directions which would be worth exploring in the future.

Note that we have followed a metric approach to f(Lovelock) theories. However, higher-derivative gravities can in general be studied using other methods. This is the case, for example, of the Palatini and metric-affine formalisms, in which the connection and the metric are regarded as independent fields. These formulations have been explored in the cases of $f(R)$ and Lovelock gravities — see *e.g.,* [?, ?, ?, ?], and it would be natural to extend them to the more general f(Lovelock) framework.

Another basic aspect omitted in this paper that has been often considered in the cases of $f(R)$ [?,?,?,?] and Lovelock [?,?,?,?,?,?][29] and which could be studied for general f(Lovelock) theories is the Hamiltonian formulation.

We also think that it would be interesting to explore how our results on the absence of massive gravitons on m.s.b. for f(Lovelock) theories extend to less symmetric backgrounds. We already mentioned in the introduction that most of the previous studies on f(Lovelock) theories were performed in the context of cosmology. It has been in this area that such explorations have been already pursued for certain cosmological backgrounds in the case of $f(R, \mathcal{L}_2)$ theories [?, ?]. The theories studied in section ?? seem to be particularly relevant in this respect, as they only propagate the usual graviton on m.s.b. It should be possible to clarify whether this property extends to other backgrounds, and what the implications of these results are.

Obviously, constructing additional analytic solutions to these theories and studying their properties would also be a very interesting task, although a challenging one in general. Let us remark that some hope might exist for the class of theories constructed in subsection ??, for which the linearized equations of motion are second-order. In fact, the very same happens for quasi-topological gravity and in that case analytic black hole solutions were built in [?] in spite of the higher-derivative and non-topological character of the theory. A perhaps more doable task which we have somewhat overlooked here would consist in studying the regularity conditions and thermodynamic properties of the $f(\mathcal{L}_n)$ black holes constructed in section ??.

Let us finally mention that, as far as we know, f(Lovelock) theories have only been considered within the holographic context in [?, ?] — rather successfully in that case. It would be interesting to start considering them more often as holographic toy models. This is particularly so for the class of theories constructed in ??. For these, all holographic calculations involving the graviton propagator could be easily performed, given that the

[29] In fact, this is a subtle topic in the case of Lovelock gravity, the reason being that, in a standard approach, momenta are generically multivalued functions of the time derivatives of the metric, making the Hamiltonian approach ill-defined.

only effect of the higher-derivative terms is a change in the normalization of the Newton constant.[30] For instance, the coefficient characterizing the stress-tensor two-point function C_T — see *e.g.*, [?] for definitions — in holographic theories dual to f(Lovelock) gravities satisfying (??) would be given by

$$C_T = \left[1 + \lambda \sum_{n=1}^{\lfloor D/2 \rfloor} \partial_n f\left(\bar{\mathcal{L}}\right) \frac{n(D-3)!}{(D-2n-1)!} \Lambda^{n-1} \right] C_T^E \,, \tag{4.132}$$

where C_T^E is the central charge corresponding to Einstein gravity — see *e.g.*, [?]. We leave for future work to further develop the holographic aspects of these theories.

[30]We thank Rob Myers for this remark.

A

Summary of relations for the generic $d = 8$ theory

A.1 Ungauged, massless Abelian theory

Field strengths

$$
\begin{aligned}
F^I &= dA^I\,. && \text{(A.1)}\\
H_m &= dB_m - d_{mIJ}F^I A^J\,, && \text{(A.2)}\\
G^i &= dC^i + d^i{}_I{}^m F^I B_m - \tfrac{1}{3} d^i{}_I{}^m d_{mJK} A^I F^J A^K\,, && \text{(A.3)}\\
\tilde{H}^m &= d\tilde{B}^m + d^i{}_I{}^m C_i F^I + d^{mnp} B_n (H_p + \Delta H_p) + \tfrac{1}{12} d^i{}_I{}^m d_{iJ}{}^n A^{IJ} \Delta H_n\,, && \text{(A.4)}\\
\tilde{F}_I &= d\tilde{A}_I + 2 d_{mIJ} A^J (\tilde{H}_m - \tfrac{1}{2}\Delta\tilde{H}_m) - \left(d^i{}_I{}^m B_m - \tfrac{1}{3} d^i{}_J{}^m d_{mIK} A^{JK}\right)(G_i - \tfrac{1}{2}\Delta G_i) \\
&\quad -\tfrac{1}{3}\left(d^i{}_I{}^m d_{mJK} - d^i{}_K{}^m d_{mIJ}\right) F^J A^K C_i - d^{mnp} d_{mIJ} A^J B_n H_p \\
&\quad +\tfrac{1}{24}\left(d^i{}_K{}^m d_{iL}{}^n d_{mIJ} + 2 d^i{}_{[I|}{}^m d_{i|K]}{}^n d_{mJL}\right) F^J A^{KL} B_n + \tfrac{1}{24} d^i{}_J{}^m d_{iK}{}^n d_{mIL} A^{JKL} dB_n \\
&\quad -\tfrac{1}{180} d^i{}_L{}^n d_{iQ}{}^m d_{mIJ} d_{nPK} A^{JKLQ} F^P\,. && \text{(A.5)}
\end{aligned}
$$

Bianchi identities

$$
\begin{aligned}
dF^I &= 0\,, && \text{(A.6)}\\
dH_m &= -d_{mIJ}F^{IJ}\,, && \text{(A.7)}\\
dG^i &= d^i{}_I{}^m F^I H_m\,, && \text{(A.8)}\\
d\tilde{H}^m &= d^i{}_I{}^m G_i F^I + d^{mnp}H_{np}\,, && \text{(A.9)}\\
d\tilde{F}_I &= 2d_{mIJ}F^J\tilde{H}^m + d_{iI}{}^m G^i H_m\,, && \text{(A.10)}\\
dK_A &= T_A{}^I{}_J F^J\tilde{F}_I + T_A{}^m{}_n\tilde{H}^n H_m - \tfrac{1}{2}T_{Aij}G^{ij}\,. && \text{(A.11)}
\end{aligned}
$$

Duality relations

$$
\begin{aligned}
\star G^i &= \Omega^{ij}\mathcal{W}_{jk}G^k\,, \quad \text{or} \quad G_a^+ = -\mathcal{N}^*_{ab}G^{b\,+}\,, && \text{(A.12)}\\
\star\tilde{H}^m &= \mathcal{M}^{mn}H_n\,, && \text{(A.13)}\\
\star\tilde{F}_I &= -\mathcal{M}_{IJ}F^J\,, && \text{(A.14)}\\
\star K_A &= -j_A^{(\sigma)}\,, && \text{(A.15)}\\
\star L_\sharp &= -\frac{\partial V}{\partial c^\sharp}\,. && \text{(A.16)}
\end{aligned}
$$

A.2 Gauged theory

Field strengths

$$F^I = dA^I - \tfrac{1}{2}X_J{}^I{}_K A^{JK} + Z^{I\,m}B_m\,, \tag{A.17}$$

$$H_m = \mathcal{D}B_m - d_{mIJ}dA^I A^J + \tfrac{1}{3}d_{mMI}X_J{}^M{}_K A^{IJK} + Z_{im}C^i\,, \tag{A.18}$$

$$\begin{aligned} G^i = {} & \mathcal{D}C^i + d^i{}_I{}^n\left[F^I B_n - \tfrac{1}{2}Z^{I\,p}B_nB_p + \tfrac{1}{3}d_{nJK}dA^J A^{KI}\right. \\ & \left. +\tfrac{1}{12}d_{nMJ}X_K{}^M{}_L A^{IJKL}\right] - Z^i{}_m\tilde{H}^m\,, \end{aligned} \tag{A.19}$$

$$\begin{aligned} \tilde{H}^m = {} & \mathcal{D}\tilde{B}^m - d_{iI}{}^m F^I C^i + d^{mnp}B_n\left(H_p + \Delta H_p - 2Z_{ip}C^i\right) \\ & +d^m{}_{IJK}dA^I dA^J A^K \\ & +\left(\tfrac{1}{12}d_{iJ}{}^m d^i{}_K{}^n d_{nIL} - \tfrac{3}{4}d^m{}_{IJM}X_K{}^M{}_L\right)dA^I A^{JKL} \\ & +\left(\tfrac{3}{20}d^m{}_{NPM}X_I{}^N{}_J - \tfrac{1}{60}d_{iM}{}^m d^i_I{}^n d_{nPJ}\right)X_K{}^P{}_L A^{IJKLM} \\ & +Z^{Im}\tilde{A}_I\,. \end{aligned} \tag{A.20}$$

Bianchi identities

The Bianchi identities satisfied by the field strengths of the gauged theory are $\mathcal{B}(\cdot) = 0$ where

$$\mathcal{B}(L_A{}^I) = -\left[\mathcal{D}L_A{}^I + F^I K_A + W^I{}_A{}^\beta M_\beta\right], \tag{A.21}$$

$$\mathcal{B}(L_{Im}) = -\left[\mathcal{D}L_{Im} + \tilde{F}_I H_m + W_{Im}{}^\beta M_\beta\right], \tag{A.22}$$

$$\mathcal{B}(L^{im}) = -\left[\mathcal{D}L^{im} + G^i \tilde{H}^m + W^{im\,\beta} M_\beta\right], \tag{A.23}$$

$$\mathcal{B}(K_A) = \mathcal{D}K_A - T_A{}^I{}_J F^J \tilde{F}_I - T_A{}^m{}_n \tilde{H}^n H_m + \tfrac{1}{2} T_{A\,ij} G^{ij} - Y_A{}^\sharp L_\sharp\,, \tag{A.24}$$

$$\mathcal{B}(\tilde{F}_I) = -\left[\mathcal{D}\tilde{F}_I - 2d_{mIJ} F^J \tilde{H}^m - d_{iI}{}^m G^i H_m + 2d^m{}_{IJK} F^{JK} H_m + \vartheta_I{}^A K_A\right], \tag{A.25}$$

$$\mathcal{B}(\tilde{H}^m) = -\left[\mathcal{D}\tilde{H}^m + d_{iI}{}^m F^I G^i - d^{mnp} H_{np} - d^m{}_{IJK} F^{IJK} - Z^{Im} \tilde{F}_I\right], \tag{A.26}$$

$$\mathcal{B}(G_i) = -\left[\mathcal{D}G_i - d_{iI}{}^m F^I H_m + Z_{im} \tilde{H}^m\right], \tag{A.27}$$

$$\mathcal{B}(H_m) = -\left[\mathcal{D}H_m + d_{mIJ} F^{IJ} - Z_{im} G^i\right], \tag{A.28}$$

$$\mathcal{B}(F^I) = -\left[\mathcal{D}F^I - Z^{Im} H_m\right], \tag{A.29}$$

$$\mathcal{B}(\mathcal{D}\mathcal{M}) = -\left[\mathcal{D}\mathcal{D}\mathcal{M} + F^I \vartheta_I{}^A \delta_A \mathcal{M}\right], \tag{A.30}$$

$$\mathcal{B}(c^\sharp) = \mathcal{D}c^\sharp\,, \tag{A.31}$$

$$\mathcal{B}(\mathcal{Q}^\beta) = \mathcal{Q}^\beta\,. \tag{A.32}$$

Here $\sharp$ labels the deformation parameters and β the constraints, as discussed in Sections **??** and **??**.

Identities of Bianchi identities

$$\mathcal{D}\mathcal{B}(F^I) - Z^{Im}\mathcal{B}(H_m) = 0\,, \tag{A.33}$$

$$\mathcal{D}\mathcal{B}(H_m) - 2d_{mIJ}F^I\mathcal{B}(F^J) + Z_{im}G^i = 0\,, \tag{A.34}$$

$$\mathcal{D}\mathcal{B}(G_i) - d_{iI}{}^m\left[\mathcal{B}(H_m)F^I + H_m\mathcal{B}(F^I)\right] - Z_{im}\mathcal{B}(\tilde{H}^m) = 0\,, \tag{A.35}$$

$$\begin{aligned}\mathcal{D}\mathcal{B}(\tilde{H}^m) - d_{iI}{}^m\left[\mathcal{B}(G^i)F^I + G^i\mathcal{B}(F^I)\right] + 2d^{mnp}\mathcal{B}(H_n)H_p&\\ -3d^m{}_{IJK}\mathcal{B}(F^I)F^{JK} + Z^{Im}\mathcal{B}(\tilde{F}_I) &= 0\,,\end{aligned} \tag{A.36}$$

$$\begin{aligned}\mathcal{D}\mathcal{B}(\tilde{F}_I) + 2d_{mIJ}\left[\mathcal{B}(\tilde{H}^m)F^J + \tilde{H}^m\mathcal{B}(F^J)\right]&\\ +d_{iI}{}^m\left[\mathcal{B}(G^i)H_m + G^i\mathcal{B}(H_m)\right]&\\ -3d^m{}_{IJK}\left[2\mathcal{B}(F^J)F^K H_m + F^{JK}\mathcal{B}(H_m)\right] + \vartheta_I{}^A\mathcal{B}(K_A) &= 0\,,\end{aligned} \tag{A.37}$$

$$\begin{aligned}\mathcal{D}\mathcal{B}(K_A) + T_A{}^I{}_J\left[\mathcal{B}(F^J)\tilde{F}_I + F^J\mathcal{B}(\tilde{F}_I)\right]&\\ +T_A{}^m{}_n\left[\mathcal{B}(\tilde{H}^n)H_m + \tilde{H}^n\mathcal{B}(H_m)\right] + T_{A\,ij}G^i\mathcal{B}(G^j) + Y_A{}^\sharp\mathcal{B}(L_\sharp) &= 0\,.\end{aligned} \tag{A.38}$$

B

Summary of relations for the SO(3)-gauged $\mathcal{N} = 2, d = 8$ supergravity

B.1 Field strengths

$$\mathcal{D}\phi^x = d\phi^x - A^{im} v_i k_m{}^x \,, \tag{B.1}$$

$$F^{im} = dA^{im} + \tfrac{1}{2}\epsilon^{mnp} v_j A^{jn} A^{ip} + v^i B_m \,, \tag{B.2}$$

$$H_m = \mathcal{D}B_m + \tfrac{1}{2}\epsilon_{mnp}\epsilon_{ij} dA^{in} A^{jp} - \tfrac{1}{4} v_i A^{im} (\epsilon AA) \,, \tag{B.3}$$

$$G^i = dC^i + F^{im} B_m + \tfrac{1}{6}\epsilon_{mnp}\epsilon_{jk} dA^{jn} A^{im} A^{kp} - v^i \left[\tfrac{1}{2} B_m B_m - \tfrac{1}{32}(\epsilon AA)^2\right] \,, \tag{B.4}$$

$$\begin{aligned}\tilde{H}^m &= \mathcal{D}\tilde{B}^m + \epsilon_{ij} F^{im} C^j + \tfrac{1}{2}\epsilon^{mnp} B_n \left(H_p + \Delta H_p\right) \\ &\quad + \tfrac{1}{24}\epsilon_{pqr}\epsilon_{ij}\epsilon_{kl} dA^{ip} A^{jq} A^{kr} A^{lm} + \tfrac{1}{160} v_i A^{im} (\epsilon AA)^2 + v^i \tilde{A}_{im} \,,\end{aligned} \tag{B.5}$$

$$\begin{aligned}\tilde{F}_{im} &= \mathcal{D}\tilde{A}_{im} - \epsilon_{ij}\epsilon_{mnp} F^{jn} \tilde{B}_p - \epsilon_{ij} H_m C^j - v_i B_m B_n B_n - \tfrac{1}{2}\epsilon_{ij} \Delta F^{jn} B_m B_n \\ &\quad + \tfrac{1}{24}\epsilon_{il}\epsilon_{jj'}\epsilon_{kk'}\epsilon_{mnp}\epsilon_{qrs} dA^{jq} A^{j'r} A^{ks} A^{k'n} A^{lp} + \tfrac{1}{16}\epsilon_{ij} v_k \epsilon_{mnp} A^{jn} A^{kp} (\epsilon AA)^2 \\ &\quad - v_i D_m \,,\end{aligned} \tag{B.6}$$

$$K_m = \mathcal{D}D_m + \ldots \,, \tag{B.7}$$

$$K_a = \mathcal{D}D_a + \ldots \,, \tag{B.8}$$

$$K_\alpha = dD_\alpha + \ldots \,, \tag{B.9}$$

where the SO(3)-covariant derivatives that appear in these expressions are

$$\mathcal{D}B_m = dB_m + \epsilon_{mnp} v_i A^{in} B_p \,, \qquad \mathcal{D}\tilde{B}^m = d\tilde{B}^m + \epsilon^{mnp} v_i A^{in} \tilde{B}^p \,. \tag{B.10}$$

and where we have used the shorthand notation

$$\Delta H_m = H_m - \mathcal{D}B_m\,, \qquad (\epsilon AA) = \epsilon_{ij}A^{im}A^{jn}\,. \tag{B.11}$$

B.2 Bianchi identities

The Bianchi identities satisfied by the field strengths of the gauged theory are $\mathcal{B}(\cdot) = 0$ where

$$\mathcal{B}(L_n{}^{im}) = -\left[\mathcal{D}L_n{}^{im} + F^{im}K_n + W^{im}{}_n{}^{\beta}M_\beta\right]\,, \tag{B.12}$$

$$\mathcal{B}(L_a{}^{im}) = -\left[\mathcal{D}L_a{}^{im} + F^{im}K_a + W^{im}{}_n{}^{\beta}M_\beta\right]\,, \tag{B.13}$$

$$\mathcal{B}(L_{imn}) = -\left[\mathcal{D}L_{imn} + 2\tilde{F}_{i(m}H_{n)} + W_{imn}{}^{\beta}M_\beta\right]\,, \tag{B.14}$$

$$\mathcal{B}(K_m) = \mathcal{D}K_m - T_m{}^n{}_p\left[F^{ip}\tilde{F}_{in} + \tilde{H}^pH_n\right]\,, \tag{B.15}$$

$$\mathcal{B}(K_a) = \mathcal{D}K_a - T_a{}^n{}_p\left[F^{ip}\tilde{F}_{in} + \tilde{H}^pH_n - v_iL^{ip}{}_n + v^i\delta^{mp}L_{imn}\right]\,, \tag{B.16}$$

$$\mathcal{B}(K_\alpha) = dK_\alpha - T_\alpha{}^i{}_j\left[F^{jm}\tilde{F}_{im} + \tfrac{1}{2}G^jG_i - v_i\delta^m{}_nL^{jn}{}_m + v^j\delta^{mn}L_{imn}\right]\,, \tag{B.17}$$

$$\mathcal{B}(\tilde{F}_{im}) = -\left[\mathcal{D}\tilde{F}_{im} + \epsilon_{mnp}\epsilon_{ij}F^{jn}\tilde{H}^p + \epsilon_{ij}G^jH_m + v_iK_m\right]\,, \tag{B.18}$$

$$\mathcal{B}(\tilde{H}^m) = -\left[\mathcal{D}\tilde{H}^m - \epsilon_{ij}F^{im}G^j - \tfrac{1}{2}\epsilon^{mnp}H_nH_p - v^i\tilde{F}_{im}\right]\,, \tag{B.19}$$

$$\mathcal{B}(G^i) = -\left[dG^i - F^{im}H_m\right]\,, \tag{B.20}$$

$$\mathcal{B}(H_m) = -\left[\mathcal{D}H_m - \tfrac{1}{2}\epsilon_{mnp}\epsilon_{ij}F^{im}F^{jn}\right]\,, \tag{B.21}$$

$$\mathcal{B}(F^{im}) = -\left[\mathcal{D}F^{im} - v^iH_m\right]\,, \tag{B.22}$$

$$\mathcal{B}(\mathcal{D}\mathcal{M}_{mn}) = -\left[\mathcal{D}\mathcal{D}\mathcal{M}_{mn} + 2v_iF^{ip}\epsilon_{p(m}{}^q\mathcal{M}_{n)q}\right]\,, \tag{B.23}$$

$$\mathcal{B}(d\mathcal{W}_{ij}) = -dd\mathcal{W}_{ij}\,, \tag{B.24}$$

where the SO(3)-covariant derivatives with indices m are identical to those of B_m and $\tilde{B}^m$ in Eq. (**??**)

$$\mathcal{D}K_a = dK_a - v_iA^{im}f_{ma}{}^bK_b\,. \tag{B.25}$$

B.3 Identities of Bianchi identities

$$\mathcal{D}\mathcal{B}(F^{im}) - v^i\mathcal{B}(H_m) = 0\,, \quad \text{(B.26)}$$

$$\mathcal{D}\mathcal{B}(H_m) + \epsilon_{ij}\epsilon_{mnp}F^{in}\mathcal{B}(F^{jp}) = 0\,, \quad \text{(B.27)}$$

$$\mathcal{D}\mathcal{B}(G_i) - \epsilon_{ij}\left[\mathcal{B}(H_m)F^j + H_m\mathcal{B}(F^{jm})\right] = 0\,, \quad \text{(B.28)}$$

$$\mathcal{D}\mathcal{B}(\tilde{H}^m) - \epsilon_{ij}\left[\mathcal{B}(G^i)F^{jm} + G^i\mathcal{B}(F^{jm})\right] + \epsilon^{mnp}\mathcal{B}(H_n)H_p + v^i\mathcal{B}(\tilde{F}_{im}) = 0\,, \quad \text{(B.29)}$$

$$\begin{aligned}\mathcal{D}\mathcal{B}(\tilde{F}_{im}) + \epsilon_{ij}\epsilon_{mnp}\left[\mathcal{B}(\tilde{H}^n)F^{jp} + \tilde{H}^n\mathcal{B}(F^{jp})\right]& \\ +\epsilon_{ij}\left[\mathcal{B}(G^j)H_m + G^j\mathcal{B}(H_m)\right] + v_i\mathcal{B}(K_m) &= 0\,, \quad \text{(B.30)}\end{aligned}$$

$$\begin{aligned}\mathcal{D}\mathcal{B}(K_m) + \epsilon_{mnp}\left[\mathcal{B}(F^{in})\tilde{F}_{ip} + F^{in}\mathcal{B}(\tilde{F}_{ip})\right]& \\ -\epsilon_{mnp}\left[\mathcal{B}(\tilde{H}^n)H_p + \tilde{H}^n\mathcal{B}(H_p)\right] &= 0\,. \quad \text{(B.31)}\end{aligned}$$

$$\begin{aligned}\mathcal{D}\mathcal{B}(K_a) + T_A{}^m{}_n\left[\mathcal{B}(F^{in})\tilde{F}_{im} + F^{in}\mathcal{B}(\tilde{F}_{im})\right]& \\ +T_a{}^m{}_n\left[\mathcal{B}(\tilde{H}^n)H_m + \tilde{H}^n\mathcal{B}(H_m)\right]& \\ -v_iT_a{}^m{}_n\mathcal{B}(L_n{}^{im}) - v_if_{ma}{}^b\mathcal{B}(L_b{}^{im}) + v^iT_a{}^m{}_n\mathcal{B}(L_{imnn}) &= 0\,. \quad \text{(B.32)}\end{aligned}$$

$$\begin{aligned}\mathcal{D}\mathcal{B}(K_\alpha) + T_\alpha{}^i{}_j\left[\mathcal{B}(F^{jm})\tilde{F}_{im} + F^{jm}\mathcal{B}(\tilde{F}_{im})\right] + T_{\alpha\, ij}G^i\mathcal{B}(G^j)& \\ +T_{\alpha\, ij}v^j\mathcal{B}(L_m{}^{im}) + T_\alpha{}^i{}_jv^j\mathcal{B}(L_{imm}) &= 0\,. \quad \text{(B.33)}\end{aligned}$$

B.4 Duality relations

$$\star G^i = \epsilon^{ij}\mathcal{W}_{jk}G^k\,, \qquad (G^2 = \tilde{G} \equiv e^{-\varphi}\star G + aG)\,, \tag{B.34}$$

$$\tilde{H}^m = \mathcal{M}^{mn}\star H_n\,, \tag{B.35}$$

$$\tilde{F}_{im} = \mathcal{W}_{ij}\mathcal{M}_{mn}\star F^{jn}\,, \tag{B.36}$$

$$K_m = -\star j_m^{(\sigma)}\,, \tag{B.37}$$

$$K_a = -\star j_a^{(\sigma)}\,, \tag{B.38}$$

$$K_\alpha = -\star j_\alpha^{(\sigma)}\,, \tag{B.39}$$

$$L_n{}^{im} = \star\frac{\partial V}{\partial \vartheta_{im}{}^n}\,, \tag{B.40}$$

$$L_a{}^{im} = \star\frac{\partial V}{\partial \vartheta_{im}{}^a}\,, \tag{B.41}$$

$$L_{imn} = \star\frac{\partial V}{\partial Z^{imn}}\,. \tag{B.42}$$

C

$f(R,\mathcal{L}_2)$ equations of motion in general dimensions

In this appendix we write down the equations of motion of $f(R,\mathcal{L}_2)$ theories in general dimensions. Observe that this is the most general f(Lovelock) theory in four dimensions — the remaining ED identically vanish in that case. We need to compute the quantities appearing in (**??**). The tensor $\mathcal{E}^{(1)}_{\mu\nu}$ is just the Einstein tensor:

$$\mathcal{E}^{(1)}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\,, \tag{C.1}$$

while

$$P^{(1)\mu\nu}_{\alpha\beta} = -\frac{1}{2}\delta^{\mu\nu}_{\alpha\beta}\,. \tag{C.2}$$

Now, if $D=4$, $\mathcal{E}^{(2)}_{\mu\nu}=0$, while for $D\geq 5$,

$$\mathcal{E}^{(2)}_{\mu\nu} = 2RR_{\mu\nu} - 4R_{\mu\rho}R^{\rho}_{\nu} + 2R_{\alpha\beta\rho\mu}R^{\alpha\beta\rho}{}_{\nu} - 4R_{\mu\rho\nu\sigma}R^{\rho\sigma}\,. \tag{C.3}$$

Finally, we have

$$P^{(2)\mu\nu}_{\alpha\beta} = -\delta^{\mu\nu}_{\alpha\beta}R + 8\delta^{[\mu}_{[\alpha}R^{\nu]}_{\beta]} - 2R^{\mu\nu}_{\alpha\beta}\,, \tag{C.4}$$

which is also valid in four dimensions. Using this information we can write the equations of motion for $D=4$ and $D\geq 5$ theories respectively as

$$\begin{aligned}&\frac{\partial f}{\partial R}R_{\mu\nu} - \frac{1}{2}\left[f - \mathcal{L}_2\frac{\partial f}{\partial \mathcal{L}_2}\right]g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\frac{\partial f}{\partial R} - \\ &-4\left[G_{\mu\nu}\Box + \frac{1}{2}R\nabla_\mu\nabla_\nu - 2R_{\alpha(\mu}\nabla_{\nu)}\nabla^\alpha + (g_{\mu\nu}R_{\alpha\beta} + R_{\mu\alpha\beta\nu})\nabla^\alpha\nabla^\beta\right]\frac{\partial f}{\partial \mathcal{L}_2} = 0\,.\end{aligned} \tag{C.5}$$

$$\begin{aligned}&\frac{\partial f}{\partial R}R_{\mu\nu} + \frac{\partial f}{\partial \mathcal{L}_2}\left[2RR_{\mu\nu} - 4R_{\mu\rho}R^{\rho}_{\nu} + 2R_{\alpha\beta\rho\mu}R^{\alpha\beta\rho}{}_{\nu} - 4R_{\mu\rho\nu\sigma}R^{\rho\sigma}\right] \\ &-\frac{1}{2}\left[f - \mathcal{L}_2\frac{\partial f}{\partial \mathcal{L}_2}\right]g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\frac{\partial f}{\partial R} - \\ &-4\left[G_{\mu\nu}\Box + \frac{1}{2}R\nabla_\mu\nabla_\nu - 2R_{\alpha(\mu}\nabla_{\nu)}\nabla^\alpha + (g_{\mu\nu}R_{\alpha\beta} + R_{\mu\alpha\beta\nu})\nabla^\alpha\nabla^\beta\right]\frac{\partial f}{\partial \mathcal{L}_2} = 0\,.\end{aligned} \tag{C.6}$$

Observe that these equations reduce to the corresponding $f(R)$ equations of motion when $\partial_2 f = 0$. Notice also that a linear term in $\mathcal{L}_2$ gives no contribution in $D=4$ while it does for $D\geq 5$ theories, as expected. The trace of these equations can be written as

$$\frac{\partial f}{\partial R}R + 2\frac{\partial f}{\partial \mathcal{L}_2}\mathcal{L}_2 - \frac{D}{2}f + (D-1)\Box\frac{\partial f}{\partial R} + (D-3)(2R\Box - 4R_{\mu\nu}\nabla^\mu\nabla^\nu)\frac{\partial f}{\partial \mathcal{L}_2} = 0\,, \tag{C.7}$$

which is a valid expression for $D\geq 4$.

D

Examples of equivalent scalar-tensor theories

In this appendix we explicitly compute the equivalent scalar-tensor theories for a couple of classes of f(Lovelock) theories. The fist example consists of the most general sum of quadratic functions of ED in $D = 4$. In the second, we consider a single f(Lovelock) term consisting of a general product of ED in arbitrary dimensions.

Quadratic function

The most general f(Lovelock) action containing the usual EH and a negative cosmological constant terms plus quadratic linear combinations of ED in four dimensions reads

$$S = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{|g|}\, f(R,\mathcal{L}_2)\,, \tag{D.1}$$

where

$$f(R,\mathcal{L}_2) = \frac{6}{\tilde{L}^2} + R + \alpha\tilde{L}^2R^2 + \beta\tilde{L}^4R\mathcal{L}_2 + \gamma\tilde{L}^6\mathcal{L}_2^2\,, \tag{D.2}$$

and where α, β and γ are dimensionless constants. The Hessian matrix of $f(R,\mathcal{L}_2)$ reads in this case:

$$H(f) = \begin{pmatrix} 2\alpha\tilde{L}^2 & \beta\tilde{L}^4 \\ \beta\tilde{L}^4 & 2\gamma\tilde{L}^6 \end{pmatrix}\,. \tag{D.3}$$

Leaving the trivial case $\alpha = \beta = \gamma = 0$ aside, we see that:

$$\mathrm{rank}(H) = \begin{cases} 2 & \text{if } 4\alpha\gamma - \beta^2 \neq 0\,, \\ 1 & \text{if } 4\alpha\gamma - \beta^2 = 0\,. \end{cases} \tag{D.4}$$

Hence, according to our analysis in the main text, in the first case we need to introduce two scalars, while in the second it is enough with a single one. We have the function

$$f(\phi_1,\phi_2) = \frac{6}{\tilde{L}^2} + \phi_1 + \alpha\tilde{L}^2\phi_1^2 + \beta\tilde{L}^4\phi_1\phi_2 + \gamma\tilde{L}^6\phi_2^2\,. \tag{D.5}$$

Then we define

$$\varphi_1 = \frac{\partial f}{\partial\phi_1} = 1 + 2\alpha\tilde{L}^2\phi_1 + \beta\tilde{L}^4\phi_2\,, \tag{D.6}$$

$$\varphi_2 = \frac{\partial f}{\partial\phi_2} = \beta\tilde{L}^4\phi_1 + 2\gamma\tilde{L}^6\phi_2\,. \tag{D.7}$$

If $4\alpha\gamma - \beta^2 \neq 0$ then these fields are independent. The Legendre transform of f reads

$$\tilde{V}(\varphi_1, \varphi_2) = -\frac{6}{\tilde{L}^2} + \frac{1}{\tilde{L}^6(4\alpha\gamma - \beta^2)}\left(\gamma\tilde{L}^4(\varphi_1 - 1)^2 - \beta\tilde{L}^2(\varphi_1 - 1)\varphi_2 + \alpha\varphi_2^2\right) . \qquad \text{(D.8)}$$

Therefore, the equivalent scalar-tensor theory is

$$S' = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{|g|}\left[\frac{6}{\tilde{L}^2} + \varphi_1 R + \varphi_2\mathcal{L}_2 - \frac{\gamma\tilde{L}^4(\varphi_1 - 1)^2 - \beta\tilde{L}^2(\varphi_1 - 1)\varphi_2 + \alpha\varphi_2^2}{\tilde{L}^6(4\alpha\gamma - \beta^2)}\right] . \qquad \text{(D.9)}$$

Of course, this does not work if $4\alpha\gamma - \beta^2 = 0$. In that case, the quadratic term is a perfect square of the form

$$S = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{|g|}\left[\frac{6}{\tilde{L}^2} + R + \lambda\tilde{L}^2(R + c\tilde{L}^2\mathcal{L}_2)^2\right] , \qquad \text{(D.10)}$$

where c is some unimportant constant. We find that this is equivalent to the following scalar-tensor theory with one single scalar φ

$$S' = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{|g|}\left[\frac{6}{\tilde{L}^2} + \varphi R + \varphi c\tilde{L}^2\mathcal{L}_2 - \frac{(\varphi - 1)^2}{4\lambda\tilde{L}^2}\right] . \qquad \text{(D.11)}$$

General product of ED

Let us consider now the following action:

$$S = \frac{1}{16\pi G}\int_{\mathcal{M}} d^Dx\sqrt{|g|}\left[-2\Lambda_0 + R + \lambda\prod_{i=1}^{n}\mathcal{L}_{p_i}^{v_i}\right] , \qquad \text{(D.12)}$$

where $\{p_i\}_{i=1}^n \subset \{1, 2, \ldots, \lfloor D/2 \rfloor\}$ and $v_i \in \mathbb{Z} - \{0\}$ are non-zero exponents. This action contains a rather generic f(Lovelock) term, namely, one consisting of a product of ED. Let us also assume that $p_1 = 1$, so there is a power of R in the product. The Hessian matrix of $f(\mathcal{L}) = -2\Lambda_0 + R + \lambda\prod_{i=1}^n \mathcal{L}_{p_i}^{v_i}$ reads

$$H_{ij} = (v_i v_j - v_i\delta_{ij})\frac{\lambda\prod_{i=1}^n \mathcal{L}_{p_i}^{v_i}}{\mathcal{L}_{p_i}\mathcal{L}_{p_j}} , \qquad \text{(D.13)}$$

whose rank can be seen to be given by

$$r = \begin{cases} n & \text{if } \sum_{i=1}^n v_i \neq 1, \\ n - 1 & \text{if } \sum_{i=1}^n v_i = 1. \end{cases} \qquad \text{(D.14)}$$

The second case can happen if we allow the exponents to be non-integer or if some of them are negative. In the first case we can compute the Legendre transform of the function $f(\chi_1, \ldots, \chi_n) = -2\Lambda_0 + \chi_1 + \lambda\prod_{i=1}^n \chi_i^{v_i}$. The transformed fields are

$$\phi_1 = \partial_1 f(\chi) = 1 + \frac{v_1\lambda}{\chi_1}\prod_{j=1}^{n}\mathcal{L}_{p_j}^{v_j}, \quad \phi_i = \partial_i f(\chi) = \frac{v_i\lambda}{\chi_i}\prod_{j=1}^{n}\mathcal{L}_{p_j}^{v_j}, \quad i > 1, \qquad \text{(D.15)}$$

so we find:

$$\tilde{V}(\phi) = 2\Lambda_0 + (s-1)\lambda^{\frac{1}{1-s}} \left(\frac{\phi_1 - 1}{v_1}\right)^{\frac{v_1}{s-1}} \prod_{j=2}^{n} \left(\frac{\phi_j}{v_j}\right)^{\frac{v_j}{s-1}}, \tag{D.16}$$

where $s = \sum_{i=1}^{n} v_i$. Therefore, this theory is equivalent to

$$S' = \frac{1}{16\pi G}\int_{\mathcal{M}} d^D x\sqrt{|g|}\left[-2\Lambda_0 + \sum_{i=1}^{n}\phi_i\mathcal{L}_{p_i} - (s-1)\lambda^{\frac{1}{1-s}} \left(\frac{\phi_1 - 1}{v_1}\right)^{\frac{v_1}{s-1}} \prod_{j=2}^{n} \left(\frac{\phi_j}{v_j}\right)^{\frac{v_j}{s-1}}\right]. \tag{D.17}$$

If $s = 1$, the Legendre transform is constant $\tilde{V} = 2\Lambda_0$, and we have the constraint

$$\left(\frac{\phi_1 - 1}{v_1}\right)^{v_1} \prod_{i=2}^{n} \left(\frac{\phi_i}{v_i}\right)^{v_i} = \lambda, \tag{D.18}$$

from which we can extract, for example, ϕ_n as a function of the rest of the fields:

$$\phi_n = (1-s')\lambda^{\frac{1}{1-s'}} \left(\frac{\phi_1 - 1}{v_1}\right)^{\frac{v_1}{s'-1}} \prod_{j=2}^{n-1} \left(\frac{\phi_j}{v_j}\right)^{\frac{v_j}{s'-1}}, \tag{D.19}$$

where now, $s' = \sum_{i=1}^{n-1} v_i$. Hence, in the case in which $s = 1$ — which means that f is a homogeneous function of degree 1 — the theory is equivalent to the following scalar-Lovelock theory with $n-1$ scalar fields and without scalar potential

$$S' = \frac{1}{16\pi G}\int_{\mathcal{M}} d^D x\sqrt{|g|}\left[-2\Lambda_0 + \sum_{i=1}^{n-1}\phi_i\mathcal{L}_{p_i} + (1-s')\lambda^{\frac{1}{1-s'}} \left(\frac{\phi_1 - 1}{v_1}\right)^{\frac{v_1}{s'-1}} \prod_{j=2}^{n-1} \left(\frac{\phi_j}{v_j}\right)^{\frac{v_j}{s'-1}} \mathcal{L}_{p_n}\right]. \tag{D.20}$$

E
Resumen

Esta tesis se enfoca en el estudio de varios aspectos de las teorías de supergravedad gaugeada en 8 dimensiones y ciertas gravedades de orden superior. Las teorías de supergravedad corresponden a límites de baja energía de las teorías de cuerdas, las cuales se construyeron con el objetivo de unificar las cuatro fuerzas fundamentales de la Naturaleza bajo un único marco teórico. Actualmente se busca confirmar experimentalemente la existencia de partículas supersimétricas, la existencia de estas partículas resuelve algunos de los problemas de los modelos actuales. Su descubrimeinto, haría que alguna de las supergravedades sea una teoría válida para describir el universo en que vivimos.

En este contexto, estudiar las teorías de supergravedad más generales en diferentes dimensiones es de vital importancia para describir la Naturaleza. Considerando que las teorías que describen las interacciones que conocemos son teorías gauge, se esperaría que la teoría que unifica todas las interacciones también lo sea y, dentro del contexto de esta tesis, que sea una teoría de supergravedad gaugeada.

La cuantización de la Relatividad General no da como resultado una teoría cuántica renormalizable, lo que nos impide conocer cómo se comporta la gravedad a altas energías y cortas distancias. Sin importar cuál sea la teoría gravitatoria completa en el límite ultravioleta, la acción efectiva de la teoría deberá contener términos con derivadas superiores que involucran contracciones del tensor de Riemann y sus derivadas covariantes. La teoría de cuerdas predice la aparición de términos de este tipo, los cuales agregan correcciones a la acción de Einstein Hilbert.

En esta tesis se recopilan los resultados obtenidos en las publicaciones [**?**, **?**, **?**].

En [**?**] construimos la jerarquía tensorial de las teorías de campos bosónicas en 8 dimensiones más generales con simetría gauge. Primeramente estudiamos la forma de la teoría más general con simetría gauge abeliana. Una vez construida la teoría abeliana, procedemos a estudiar, usando el formalismo del *embedding tensor*, los gaugeos más generales de las simetrías de la teoría y sus posibles deformaciones masivas.

En [**?**] estudiamos la supergravedad maximal gaugeada en 8 dimensiones. Enfocándonos en los gaugeos $SO(3)$, analizamos todos los posibles campos gauge y construimos explícitamente las acciones bosónicas. Estudiamos la relación entre la supergravedad gaugeada en 8 dimensiones construida por Salam y Sezgin [**?**] a partir de la reducción dimensional generalizada de la supergravedad en 11 dimensiones, y la teoría construida en Alonso-Alberca *et al.* [**?**], a través de la reducción dimensional simple de la *supergravedad masiva en 11 dimensiones* propuesta por Meessen y Ortín en [**?**].

En [**?**] estudiamos algunos aspectos the las teorías f(Lovelock) en d dimensiones. Estas teorías son generalizaciones de las teorías f(R) y Lovelock, donde la acción gravitacional depende de alguna función arbitraria de las densidades de Euler en d dimensiones. Demostramos que estas teorías son equivalentes a ciertas teorías tipo escalar-tensor, estudiamos las ecuaciones linealizadas de la teoría en vacíos maximalmente simétricos, y estudiamos las restricciones de los acoplos de una familia de teorías f(Lovelock) en 5 dimensiones usando la entropía de entrelazamiento holográfica. Finalmente, se presentan algunas soluciones nuevas de agujeros negros en varias dimensiones.

F
Conclusiones

En el capítulo **??** hemos construido explícitamente, siguiendo el procedimiento detallado en [**?**] [**?**], la teoría más general en 8 dimensiones con simetrías gauge y con un máximo de dos derivadas. Hemos construido las intensidades de campo (hasta las 6-formas), todas las identidades de Bianchi, las relaciones de dualidad que satisfacen las intensidades de campo (hasta las 9-formas) y las ecuaciones de movimiento de los campos fundamentales. Hemos demostrado que están caracterizadas por un conjunto pequeño de tensores invariantes (el d-tensor, el tensor de embedding ϑ y las deformaciones masivas Z) que satisfacen ciertas restricciones. Tales restricciones son relaciones tanto entre ellos, como con las constantes de estructura y los generadores del grupo de simetría global. Hemos encontrado que las identidades de Bianchi que satisfacen las intensidades de campo de las 6-formas (duales a la corriente generalizada de Noether-Gaillard-Zumino) tienen la forma descrita en [**?**] aunque es muy complicado encontrar la forma explícita de las intensidades de campo de las 6-formas en términos de los potenciales resolviendo las identidades de Bianchi.

Hemos construido una acción de la cual se pueden derivar todas las ecuaciones de movimiento a excepción de las ecuaciones correspondientes a los potenciales (1-formas) debido a que identificar los numerosos términos que contienen únicamente a las 1-formas es extremadamente complicado y tedioso. Estos resultados se pueden aplicar a cualquier teoría en 8 dimensiones que contenga campos escalares y p-formas, d-tensores que definan las interacciones de Chern-Simons y un grupo global de simetría, como la supergravedad maximal en 8 dimensiones.

En el capítulo **??**, usando los resultados generales obtenidos en el capítulo **??** [**?**], hemos construido explícitamente una familia uniparamétrica de supergravedades maximales $SO(3)$ gaugeadas que interpolan entre la supergravedad de Salam y Sezgin [**?**] y la supergravedad AAMO [**?**], calculando las distintas posibilidades que se mencionan en [**?**] [**?**]: para cada valor de un parámetro una combinación diferente de los dos tripletes de 1-formas (el uno proveniente de la métrica 11-dimensional y el otro de la 3-forma 11-dimensional) juega el papel de triplete de vectores gauge. La existencia de esta familia confirma la identificación de la teoría AAMO con una supergravedad 8-dimensional "honesta" maximal, con grupo gauge $SO(3)$, a pesar de su origen no convencional: la supergravedad massiva 11-dimensional propuesta en [**?**]. Además, esto prueba su relación con la teoría de Salam-Sezgin a través de una transformación $\mathrm{SL}(2,\mathbb{R})$, algo que en principio hubiera sido muy complicado de calcular directamente.

Nuestros resultados dejan algunas incógnitas sin resolver: ¿cuál es el origen 11-dimensional de todas las teorías en esta familia si insistimos en usar el mismo "ansatz" de compactificación?

Un elemento importante en las supergravedades gaugeadas que hemos construido es el potencial escalar, el cual no está determinado por la jerarquía tensorial, que sólo le impone restricciones. En una teoría de supergravedad cualquiera (excepto la supergravedad $\mathcal{N} = 1, d = 4$) el potencial escalar es una forma cuadrática en los "shifts" de fermiones, los cuales tienen que ser dependientes de los escalares y lineales con respecto al *embedding tensor*, pero su forma general no es conocida.[1] Ésta es una de las principales obstrucciones para encontrar una formulación general de todas las supergravedades gaugeadas en todas las dimensiones. Hemos propuesto una forma general para los "shifts" de fermiones para la supergravedad maximal 8-dimensional que reproduce los "shifts" de fermiones propuestos por Salam y Sezgin, y también reproducen la forma invariante bajo la dualidad esperada para el potencial escalar. Esta forma es similar a la de los "shifts" de fermiones que aparecen en las supergravedades 4-dimensionales, donde los campos escalares aparecen combinados en un objeto relacionado con una parte del representante de coset. Consideramos que este objeto debe existir para cualquier teoría de supergravedad que pueda ser gaugeada. Una clara identificación debería ser la clave para encontrar la formulación general de las supergravedades gaugeadas.

En el capítulo **??** presentamos una generalización del término de frontera de Gibbons-Hawking-York para el caso general de las teorías $f(Lovelock)$. La construcción de este término de frontera nos permite determinar el número de grados de libertad físicos de la teoría, el cual es $D(D-3)/2+r$, donde r es el rango de la matriz Hessiana $H_{nm} = \partial_n \partial_m f$. Además, demostramos que las teorías $f(Lovelock)$ son equivalentes a las teorías escalar-Lovelock que contienen r campos escalares. Probamos que las teorías $f(Lovelock)$ no propagan el gravitón masivo fantasma típico de las gravedades con derivadas superiores. Además, existen ciertas teorías $f(Lovelock)$ no triviales que únicamente propagan un gravitón y cuyas ecuaciones son de segundo orden en cualquier gauge.

Considerando teorías holográficas duales a algunas tipos de teorías $f(Lovelock)$ encontramos restricciones para los valores de las constantes de acoplo. Finalmente, presentamos algunas soluciones de agujeros negros para algunas teorías $f(Lovelock)$ en diferentes dimensiones.

[1] Se conocen en la supergravedad $\mathcal{N} = 2, d = 4, 5$ así como en otras supergravedades $\mathcal{N} \geq 3, d = 4$. Para mas detalle véase [**?**].

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