

Isochronous classical systems and quantum systems with equally spaced spectra

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Abstract.

We study isoperiodic classical systems, what allows us to find the classical isochronous systems, i.e. having a period independent of the energy. The corresponding quantum analog, systems with an equally spaced spectrum are analysed by looking for possible creation-like differential operators. The harmonic oscillator and the isotonic oscillator are the two main essentially unique examples of such situation.

Keywords: Isochronous systems, ladder operators, equispaced spectra

1. Introduction

Inverse problems have been playing a very relevant rôle in the development of physical theories. The physical laws describing the different phenomena are constructed from experimental data, and their symmetries and regularities are used to discard possible laws or to pick up other laws among other mathematically possible formulations.

In this paper we review some results on two important inverse problems and report some recent progress on both. The first one is the characterisation of possible convex potentials giving rise to isochronous motions, i.e. systems for which the bounded motions have a period which does not depend on the energy. With this aim we first review in Section 2 the very classical problem of the determination of the potential from the energy dependence of the period, a problem which dates back to Abel [1], and therefore the characterisation of isoperiodic potentials, those giving rise to the same dependence. The harmonic oscillator is the prototype of an isochronous system and all other isochronous systems are isoperiodic with the harmonic oscillator. After a characterisation of such systems we report the recent approach used in [2] to recover a result by Chalykh and Veselov [3]: among rational potentials only the harmonic oscillator and the isotonic oscillator [4, 5] produce isochronous motions. There are however other isochronous systems described by non-rational potentials, for instance potentials for which the second derivative has a discontinuity.

A related problem in quantum mechanics is the determination of those potentials with an equispaced energy spectrum. Of course, only in the best conditions of regularity the solution of both problems can be related and this is only possible to order \hbar . Our approach here is based on the search for creation- and annihilation-like operators which are assumed to be differential

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operators of arbitrary order. Therefore we study the possibility of finding as ladder operators some differential operator of order k :

$$A_k = \sum_{j=0}^k v_{k-j}(x) \frac{d^j}{dx^j} = \sum_{j=0}^k v_{k-j}(x) \partial^j, \quad v_0(x) \equiv 1.$$

The problem is solved for $k = 1, 2$, and once again we find the harmonic oscillator and the isotonic oscillator and we also provide the equations for $k = 3$. As the solution of such a system of differential equations cannot be carried out explicitly we attack the problem from a new perspective and reduce the problem to the search of rational solutions of a high-order Korteweg de Vries equation.

2. Isochronous systems

The classical harmonic oscillator in one dimension, $\ddot{x} = -\omega^2 x$, is one of most studied models and enjoys the property of isochronicity: all the solutions of its dynamics are periodic with angular frequency ω , $q = q_0 \cos \omega t + (v_0/\omega) \sin \omega t = A \cos(\omega t + \varphi)$, while A and φ are arbitrary. Moreover, ω is fixed from the own equation and it does not depend on the energy E . This can be directly proved from the expression of the period. Recall that for a fixed energy E , the period $T(E)$ is given by

$$T(E) = \sqrt{2m} \int_{x_-(E)}^{x_+(E)} \frac{dx}{\sqrt{E - U(x)}}, \quad (1)$$

where $x_-(E)$ and $x_+(E)$ are the roots of the equation $U(x) = E$: $U(x_{\pm}(E)) = E$. Then, when $U(x) = (1/2) m \omega^2 x^2$ it turns out to be $T(E) = 2\pi/\omega$. This suggests us that we can consider as possible generalisations of the classical harmonic oscillator either systems for which the period does not depend on the energy but whose general solution is not the one given before, or simply systems for which the general solution is also $x = A \cos(\omega t + \varphi)$, but in which ω can depend on the energy (a nonlinear system). The last possibility has been analysed in [6] (see [7, 8] for the quantum version) and looks as a position-dependent mass oscillator. The two-dimensional version of this system has been shown to be super-integrable and its Hamilton–Jacobi equation is super-separable [6] and corresponds to the motion of a harmonic oscillator in a surface of constant curvature $-\lambda$.

The first possible generalisation was however to look for isochronous systems. Consider the particular case of a convex potential well with a minimum in $x = 0$, for which the potential function is

$$U(x) = \begin{cases} U_-(x) & \text{if } x < 0 \\ U_+(x) & \text{if } x > 0 \end{cases}$$

where $U_+(x)$ is an increasing function and $U_-(x)$ is a decreasing function, $x U'(x) > 0$ if $x \neq 0$, such that then we can invert x as a function of U :

$$x = \begin{cases} \phi_-(U) & \text{if } U < 0 \\ \phi_+(U) & \text{if } U > 0 \end{cases}$$

In this case, for each energy E there is a oscillatory motion between the two closest turning points $x_-(E)$ and $x_+(E)$, i.e. such that $\phi_-(E) = x_-(E)$ and $\phi_+(E) = x_+(E)$. The period $T(E)$ for a fixed energy E is given by (1), which under the change of variable $y = U(x)$, or the inverse expression $x = \phi(y)$, as $U(0) = 0$ and $U(x_+(E)) = E$, i.e. $\phi(0) = 0$ and $\phi(E) = x_+(E)$, becomes

$$T(E) = 2\sqrt{2m} \int_0^E \frac{\phi'(y)}{\sqrt{E-y}} dy = 2\sqrt{2mE} \int_0^1 \frac{\phi'(Ez)}{\sqrt{1-z}} dz, \quad ,$$

where $y = Ez$. The system is said to be isochronous if the period does not depend on the energy, $dT(E)/dE = 0$, which leads to

$$\frac{1}{2\sqrt{E}} \int_0^1 \frac{\phi'(Ez)}{\sqrt{1-z}} dz + \sqrt{E} \int_0^1 \frac{\phi''(Ez)}{\sqrt{1-z}} z dz = 0 .$$

Therefore the condition for the system to be isochronous is

$$\int_0^1 \frac{\zeta(Ez)}{\sqrt{1-z}} dz = 0 = \frac{1}{\sqrt{E}} \int_0^E \frac{\zeta(y)}{\sqrt{E-y}} dy , \quad \forall E > 0 ,$$

where $\zeta(z) = 2z\phi''(z) + \phi'(z)$. This is only possible when $\zeta(z) = 0$, and the solution of $2Ez\phi''(Ez) + \phi'(Ez) = 0$ is $\phi'(y) = C/\sqrt{y}$, and consequently, the solution for which $\phi(0) = 0$ is $\phi(y) = 2C\sqrt{y}$, with inverse function $U(x) = (4C^2)^{-1}x^2$, i.e. under the assumed regularity conditions only the harmonic oscillator is an isochronous system.

An interesting inverse problem is the determination of a function $U(x)$ giving rise to a given dependence function $T(E)$. As Abel proved in [1], when the potential is convex as indicated above, such expression can be inverted (see e.g. Chapter 2 of Landau's book [9]) giving rise to the following integral equation for $T(E)$:

$$\phi_+(U) - \phi_-(U) = \frac{1}{\pi\sqrt{2m}} \int_0^U \frac{T(E)}{\sqrt{U-E}} dE , \quad (2)$$

and, therefore, the knowledge of $T(E)$ only allows us to determine the difference $\phi_+(U) - \phi_-(U)$. In fact, we can split the r.h.s. of the integral (1) as a sum of two integrals between $x_-(E)$ and 0 and between 0 and $x_+(E)$, respectively, and using the same change of variable as before, we find that if $\Delta = \phi_+ - \phi_-$, then

$$T(E) = \sqrt{2m} \int_0^E \frac{\Delta'(y)}{\sqrt{E-y}} dy .$$

Recall that the convolution product $f_1 * f_2$ of two functions f_1 and f_2 is given by

$$(f_1 * f_2)(E) = \int_0^E f_1(E-z)f_2(z) dz ,$$

and then $T(E)$ is $\sqrt{2m}$ times the convolution product of $\psi(E) = 1/\sqrt{E}$ and $\Delta'(E)$. Using Laplace transformation $\mathcal{L}[f](s) = \int_0^\infty e^{-sE}f(E) dE$ in both sides, and taking into account that $\mathcal{L}[f_1 * f_2] = \mathcal{L}[f_1]\mathcal{L}[f_2]$, we find that $\mathcal{L}[T](s) = \sqrt{2m}\mathcal{L}[\psi](s)\mathcal{L}[\Delta'](s)$. But, as $\mathcal{L}[\psi](s) = \sqrt{\pi/s}$ and $\mathcal{L}[\Delta'](s) = s\mathcal{L}[\Delta](s)$ we can get $\mathcal{L}[\Delta](s)$ from $\mathcal{L}[T](s) = \sqrt{2m}\sqrt{\pi/s}s\mathcal{L}[\Delta](s)$, and we obtain

$$\mathcal{L}[\Delta](s) = \frac{1}{\sqrt{2\pi^2m}} \sqrt{\frac{\pi}{s}} \mathcal{L}[T](s) = \frac{1}{\sqrt{2\pi^2m}} \mathcal{L}[\psi](s) \mathcal{L}[T](s) ,$$

namely, $\mathcal{L}[\Delta](s) = (2\pi^2m)^{-1/2}\mathcal{L}[\psi * T](s)$, and we recover in this way the equation (2).

Note that for periodic motions with a period $\tau = 2\pi/\omega$ independent of the energy and $m = 1$,

$$\phi_+(U) - \phi_-(U) = \frac{1}{\pi\sqrt{2}} \int_0^U \frac{\tau}{\sqrt{U-E}} dE = \frac{2\tau}{\pi\sqrt{2}} \sqrt{U}$$

and therefore, as the turning points satisfy $U(x_{\pm}(E)) = E$, namely, $\phi_{\pm}(E) = x_{\pm}(E)$,

$$x_+(E) - x_-(E) = \frac{\sqrt{2}\tau}{\pi} \sqrt{E} = \frac{2\sqrt{2}}{\omega} \sqrt{E},$$

that in the particular case of a regular symmetric potential, for which $x_-(E) = -x_+(E)$, the isochronicity condition becomes

$$x_+(E) = \frac{\tau}{\sqrt{2}\pi} \sqrt{E} = \frac{\sqrt{2}}{\omega} \sqrt{E}, \quad (3)$$

a condition which in particular the harmonic oscillator holds.

The important point is that the general solution of (2) is of the form

$$\phi_+(U) = \phi_+^{(0)}(U) + f(U), \quad \phi_-(U) = \phi_-^{(0)}(U) + f(U),$$

where $\phi_+^{(0)}$ and $\phi_-^{(0)}$ are a particular solution of the problem and $f(U)$ is an arbitrary function of U for which $\phi_+(U)$ and $-\phi_-(U)$ be monotonous increasing functions. As an example, the choices $U(x) = \tanh^2(x)$ and $f(U) = 2 \operatorname{Artanh} \sqrt{U}$ lead to the Morse potential.

Note that if we choose $f(U) = -(1/2)(\phi_+^{(0)}(U) + \phi_-^{(0)}(U))$, we find a solution ϕ^s such that $\phi_-^s(U) = -\phi_+^s(-U)$ and corresponds to a potential that is symmetric under reflection. This potential is nothing but the Steiner symmetrisation of the potential [10]. Of course, the ambiguity of the choice for the function $f(U)$ disappears when we impose the additional condition that the curve $U = U(x)$ be symmetric with respect to the U -axis, i.e. $U(x) = U(-x)$, because in this case $\phi_+(U) = -\phi_-(U)$ and therefore,

$$\phi(U) = \frac{1}{2\pi\sqrt{2m}} \int_0^U \frac{T(E)}{\sqrt{U-E}} dE.$$

Note that a potential $U(x)$ and a shear transformed potential $U_f(x)$ defined by $U_f(x + f(U(x))) = U(x)$, for an arbitrary function f , i.e. $\phi_{f\pm}(U) = \phi_{\pm}(U) + f(U)$, are isoperiodic [2].

The general solution for an isochronous system with period T is given by

$$\phi_-(U) = -\frac{T}{\pi} \sqrt{\frac{U}{2}} + f(U), \quad \phi_+(U) = \frac{T}{\pi} \sqrt{\frac{U}{2}} + f(U),$$

and in particular, for $f(U) = a$ we find

$$\phi_-(U) = -\frac{T}{\pi} \sqrt{\frac{U}{2}} + a, \quad \phi_+(U) = \frac{T}{\pi} \sqrt{\frac{U}{2}} + a,$$

and we obtain the harmonic oscillator potential

$$U(x) = \frac{\omega^2}{2} (x - a)^2, \quad \omega = \frac{2\pi}{T},$$

while if the function f is chosen to be

$$f(U) = \alpha \frac{T}{\pi} \sqrt{\frac{U}{2}},$$

then

$$\phi_-(U) = (-1 + \alpha) \frac{T}{\pi} \sqrt{\frac{U}{2}}, \quad \phi_+(U) = (1 + \alpha) \frac{T}{\pi} \sqrt{\frac{U}{2}},$$

which for $|\alpha| \neq 1$, corresponds to the potential of two half-oscillators

$$U(x) = \begin{cases} \frac{1}{2} m \omega_1^2 x^2 & \text{if } x \leq 0 \\ \frac{1}{2} m \omega_2^2 x^2 & \text{if } x \geq 0 \end{cases}$$

with different couplings

$$\omega_1 = \frac{2\pi}{(1-\alpha)T}, \quad \omega_2 = \frac{2\pi}{(1+\alpha)T},$$

glued together at the origin. Note that $1/\omega_1 + 1/\omega_2 = 2/\omega_0$, where $\omega_0 = 2\pi/T$. Conversely, for such a potential with $\omega_1 \neq \omega_2$ it suffices to make the choices

$$\alpha = \frac{\omega_1 - \omega_2}{\omega_1 + \omega_2} \quad \text{and} \quad \omega_0 = \frac{2\omega_1\omega_2}{(\omega_1 + \omega_2)}.$$

The cases $\alpha = \pm 1$ correspond to a half-oscillator and its reflected one: either ω_1 or ω_2 vanishes and ω_0 is two times the other one.

Another very fundamental identity which characterises all potentials (related by a shear transformation) having a given dependency $T(E)$ of the period as a function of the energy is [2]:

$$U(x) = U(x + W_U(U(x))) \quad \text{where} \quad W_U(V) = \frac{1}{\pi\sqrt{2m}} \int_0^V \frac{T(E)}{\sqrt{V-E}} dE.$$

Using such relation it can be easily shown (see [2]) that a convex polynomial potential $U(x)$ is isochronous if and only if $U(x) = ax^2 + bx + c$. This results can be generalised for the case of meromorphic rational functions, for which a rational potential $U(x)$ which does not reduces to a polynomial is isochronous if and only if

$$U(x) = \left(\frac{ax^2 + bx + c}{x + d} \right)^2.$$

On the other side a slight modification of Joukowski transformation, $(J_\lambda(z) = z + \frac{\lambda}{z})$, with $\lambda \in \mathbb{R}$, which plays a relevant rôle in aerodynamics applications,

$$J_g(x) = \frac{x}{2} - \frac{2\alpha}{x}, \quad \alpha \geq 0,$$

may be used to prove that if $U(x)$ is a bounded below even convex potential with $\lim_{x \rightarrow \infty} U(x) = \infty$, then for any positive real number α the potential U_α given by

$$U_\alpha(x) = U(J_\alpha(x)) = U\left(\frac{x}{2} - \frac{2\alpha}{x}\right)$$

is isoperiodic with $U(x)$. Finally, a kind of converse property was also proved in [2]: Any non-trivial rational potential U_* which is isoperiodic to a given even convex polynomial potential U is either of the form $U_c = U(x + c)$ or $U_\alpha(x) = U(x/2 - 2\alpha/x)$, for any value of α .

Note that the potential corresponding to the isochronous harmonic oscillator $U_{h_0}(x) = (1/2)m\omega^2 x^2$ is, up to addition of a constant, of the form $U_g = (1/8)m\omega^2 x^2 + g/x^2$, with $g = 2m\alpha^2\omega^2$.

3. Quantum systems with equally spaced spectra

The creation and annihilation operators for the quantum harmonic oscillator were introduced by Dirac [11] in 1927 for the description of emission and absorption of radiation. They play a fundamental rôle in quantum mechanics and quantum field theory, and therefore, it is interesting to understand whether other quantum systems admit such creation and annihilation operators. In this section we analyse this problem for the particular case of a quantum one-dimensional systems with rational potentials with no more than one real pole. It appears that, under enough general conditions, such operators can only exist for two known systems, namely for standard and singular quantum oscillators. We guess that this statement is valid, probably, for a wider class of potentials, not only for rational ones, but this more general interesting problem is still open.

Let us first remind that the Hamiltonian of quantum oscillator is (we use the system of natural units with Planck constant $\hbar = 1$ and also choose $m = 1$):

$$H = \frac{1}{2}(p^2 + \omega^2 x^2), \quad p = -i \frac{d}{dx}.$$

The creation and annihilation operators a^+ and a are operators satisfying the equations $[H, a] = -\omega a$ and $[H, a^+] = \omega a^+$, and Dirac defined them by the formulae

$$a = \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{\omega}} \frac{d}{dx} + \sqrt{\omega} x \right), \quad a^+ = \frac{1}{\sqrt{2}} \left(-\frac{1}{\sqrt{\omega}} \frac{d}{dx} + \sqrt{\omega} x \right),$$

so that they satisfy the commutation relation $[a, a^+] = 1$. It looks more convenient to use similar operators but with another normalisation of such operators

$$A = \frac{1}{\sqrt{2}} \left(\frac{d}{dx} + \omega x \right), \quad A^+ = \frac{1}{\sqrt{2}} \left(-\frac{d}{dx} + \omega x \right),$$

so that this form is still valid for $\omega = 0$. Notice that $[A, A^+] = \omega$.

Moreover, the fundamental relations

$$[H, A] = -\omega A, \quad [H, A^+] = \omega A^+, \quad (4)$$

allow us to consider such operators as ladder operators, because if ψ is an eigenvector of H with eigenvalue E , then if $A^+\psi \neq 0$, $A^+\psi$ is an eigenvector of H corresponding to the eigenvalue $E + \omega$, while if $A\psi \neq 0$, $A\psi$ will be an eigenvector of H with eigenvalue $E - \omega$. This implies that at least a part of the spectrum of H is equispaced with a equispacing ω . Moreover, we will restrict ourselves to the case in which there exists a cyclic eigenstate ψ_0 , in the sense that the set of orthogonal vectors $\{(A^+)^k \psi_0 \mid k = 0, 1, \dots\}$ is a complete set. Then all the spectrum will be equispaced.

In this case of the harmonic oscillator $H' = H - (1/2)\omega = (1/2)A^+A$ essentially coincides with its partner $\tilde{H}' = (1/2)AA^+$ and, as the creation and annihilation operators are of first order, they are also intertwining operators between H' and its partner \tilde{H}' , i.e. they satisfy $AH' = \tilde{H}'A$ and therefore $H'A^+ = A^+\tilde{H}'$. The Hamiltonian H' factorised as before is shape-invariant, a remarkable property (see e.g. [12] and references therein).

Note however that, as we shall see later on, there may be higher-order intertwining operators between two Hamiltonians and that they give rise to relations among creation and annihilation operators for both systems when these exist [13], but they can only be formal creation and annihilation operators: they satisfy (4) but when applying such operators to an eigenfunction of one Hamiltonian we can obtain a non-normalisable function and therefore they are not giving rise to an eigenfunction of the partner Hamiltonian.

Another quantum system admitting creation and annihilation operators and giving rise to a constant separation ω among neighbour energy levels is the singular harmonic oscillator (also called isotonic oscillator) [4, 5], namely, the system describing by the Hamiltonian

$$H = \frac{1}{2} \left(p^2 + \frac{1}{4} \omega^2 x^2 \right) + \frac{g^2}{x^2}.$$

Here only positive values of x are allowed and the spectrum of this system is also equispaced and, as indicated in the preceding section, the corresponding classical system is isochronous. Moreover, as indicated above, Chalykh and Veselov recently [3] proved that these are the only two cases with a rational potential $U(x)$ for which the classical system is isochronous.

It is to be remarked that the creation and annihilation operators for the isotonic oscillator are not first-order but second-order differential operators. The limiting case $g \rightarrow 0$ is the so-called ‘half-oscillator’, i.e. a particle moving in the harmonic potential on the ‘half’ line

$$U(x) = \begin{cases} \frac{1}{8} \omega^2 x^2 & \text{for } x > 0, \\ \infty & \text{for } x < 0; \end{cases}$$

from all the eigenstates of the Hamiltonian for the harmonic oscillator on the full line only odd solutions will still be eigenstates in the ‘half’ oscillator case.

Let us remark that given a quantum system described by a Hamiltonian with a potential $U(x)$, any other potential obtained either by translation, i.e. $U_a(x) = U(x - a)$, or by reflection, $U_r(x) = U(-x)$, has the same spectrum as the given system. When we are interested in the existence of ladder operators such kind of potentials should be considered as equivalent and it is enough to determine one representative in each equivalence class. On the other hand, as not only regular potentials play a rôle but also singular ones, we will analyse the problem also for singular systems, admitting only the case, for simplicity, in which U is a rational function. If it has k real poles there will be $k + 1$ different quantum problems, one in each interval between two neighbour poles. Therefore we will restrict ourselves to the simpler cases in which U is either regular or it has a real pole of arbitrary multiplicity. In other words, we are restricting ourselves to the case in which either U is regular or it has a real pole assumed to be at $x = 0$. We are therefore interested in analysing whether, for such a given Hamiltonian, there is a realisation of such operators A_k^- and A_k^+ as differential operators of order k , i.e.

$$A_k^- = \sum_{j=0}^k v_{k-j}(x) \frac{d^j}{dx^j} = \sum_{j=0}^k v_{k-j}(x) \partial^j, \quad v_0(x) \equiv 1, \quad (5)$$

with $\partial = d/dx$. We shall denote A_k^+ the adjoint operator of A_k^- .

4. Ladder operators in one-dimensional quantum systems

Let us consider a quantum one-dimensional system described by a Hamiltonian

$$H = \frac{1}{2} p^2 + U(x) = -\frac{1}{2} \frac{d^2}{dx^2} + U(x) = -\frac{1}{2} \partial^2 + U(x) \quad (6)$$

where we assume that the potential $U(x)$ is given either by a rational function free of real poles (and the configuration space is the whole real line), or with exactly one real pole, assumed to be at $x = 0$, what is enough general because of the invariance of the problem under shift and reflection mentioned before. In this last case the configuration space is $(0, \infty)$. Our aim is to determine the explicit forms such a function $U(x)$ can take in order for the quantum system

to admit creation and annihilation operators and hence the Schrödinger equation for stationary states

$$H\psi = E\psi,$$

has, at least a part of, the discrete spectrum equispaced: $E_n = E_0 + n\omega$, $n = 0, 1, 2, \dots$

As mentioned before, a particularly well-known example is the harmonic oscillator, for which $U(x) = (1/2)\omega^2 x^2$, and $E_0 = \omega/2$, and another example is $U_1(x) = (1/8)\omega^2 x^2 + g^2/x^2$, defined in the interval $(0, \infty)$.

The question is when do ladder operators being differential operators of order k exist in the case under consideration, i.e. when differential operators of order k , A_k^\pm , satisfying

$$[H, A_k^\pm] = \pm\omega A_k^\pm, \tag{7}$$

exist, where, as indicated in (5),

$$A_k^- = \frac{d^k}{dx^k} + v_1(x)\frac{d^{k-1}}{dx^{k-1}} + \dots + v_k(x) = \partial^k + v_1(x)\partial^{(k-1)} + \dots + v_k(x),$$

and A_k^+ denotes the adjoint operator of A_k^- .

When substituting this expression in the previous equation (7) we obtain a system of differential equations for the unknown functions $U(x)$ and $v_j(x)$ for $j = 1, \dots, k$.

We start by analysing the two simplest cases:

1. $k = 1$. Here A_1^- is given by $A_1^- = d/dx + v_1(x) = \partial + v_1(x)$ and then the commutation condition $[H, A_1^-] = -\omega A_1^-$, having in mind that

$$\left[-\frac{1}{2}\partial^2, v_1(x)\right] = -\frac{1}{2}v_1'' - v_1'\partial,$$

leads to the system of differential equations

$$\begin{cases} v_1' = \omega, \\ \frac{1}{2}v_1'' + U' = \omega v_1, \end{cases}$$

from which we obtain $v_1(x) = \omega x + \alpha$ and then $A_1^- = \partial + \omega x + \alpha_1$ and $U(x) = \frac{1}{2}\omega^2 x^2 + \alpha_1 x + \alpha_2$, where α_1 and α_2 are arbitrary constants. This potential function can also be written, up to an additive constant, in the form

$$U(x) = \frac{1}{2}\omega^2(x - x_0)^2, \quad x_0 = -\frac{\alpha_1}{\omega^2}.$$

2. $k = 2$. In this case

$$A_2^- = \frac{d^2}{dx^2} + v_1(x)\frac{d}{dx} + v_2(x) = \partial^2 + v_1(x)\partial + v_2(x).$$

Taking into account that, for any function $F(x)$ we have

$$[\partial, F] = F', \quad [\partial^2, F] = F'' + 2F'\partial,$$

then, from $[-(1/2)\partial^2 + U, A_2^-] = -\omega A_2^-$ we arrive to

$$\left[-\frac{1}{2}\partial^2 + U(x), \partial^2 + v_1(x)\partial + v_2(x)\right] = -v_1'\partial^2 + \left(-\frac{1}{2}v_1'' - v_2' - 2U'\right)\partial - \frac{1}{2}v_2'' - v_1U' - U''.$$

Therefore, the commutation relation $[H, A_2^-] = -\omega A_2^-$ leads to the following system of differential equations

$$\begin{cases} v_1' = \omega, \\ \frac{1}{2} v_1'' + v_2' + 2U' = \omega v_1, \\ \frac{1}{2} v_2'' + v_1 U' + U'' = \omega v_2. \end{cases} \quad (8)$$

From the first equation we obtain that $v_1(x) = \omega x + \alpha_1$, and replacing this value for v_1 in the two last equations, they become

$$\begin{cases} v_2' + 2U' = \omega(\omega x + \alpha_1) \\ \frac{1}{2} v_2'' + (\omega x + \alpha_1)U' + U'' = \omega v_2. \end{cases}$$

The second equation can be rewritten, using the first one, as $(\omega x + \alpha_1)U' + \frac{1}{2}\omega^2 = \omega v_2$, and then we can take derivatives in this expression and we obtain $v_2' = U' + (1/\omega)(\omega x + \alpha_1)U''$, and when we put this in the first equation we arrive at

$$3U' + \frac{1}{\omega}(\omega x + \alpha_1)U'' = \omega(\omega x + \alpha_1).$$

Therefore, the function $w = U'$ satisfies the inhomogeneous linear first-order equation

$$\frac{1}{\omega}(\omega x + \alpha_1)w' + 3w = \omega(\omega x + \alpha_1).$$

The general solution of the associated homogeneous linear first-order equation is

$$w = \frac{C}{(\omega x + \alpha_1)^3},$$

while we can see that

$$w_1 = \frac{\omega^2}{4}x + \frac{1}{4}\omega\alpha_1$$

is a particular solution of the inhomogeneous equation. Therefore,

$$U'(x) = \frac{C}{(\omega x + \alpha_1)^3} + \frac{\omega^2}{4}x + \frac{1}{4}\omega\alpha_1 \implies U(x) = \frac{C_1}{(\omega x + \alpha_1)^2} + \frac{\omega^2}{8}x^2 + \frac{\omega\alpha_1}{4}x,$$

which can also be written, up to addition of a constant, and in the relevant case for which $C_1 > 0$, as

$$U(x) = \frac{g^2}{(x + \alpha)^2} + \frac{\omega^2}{8}(x + \alpha)^2.$$

where $\alpha = \alpha_1/\omega$ and $g^2 = C_1/\omega^2$.

Finally, the value of v_2 obtained when we replace U by the previous expression in the relation $v_2 = (x + \alpha)U' + \frac{1}{2}\omega$ is

$$v_2 = \frac{\omega}{2} - \frac{2g^2}{(x + \alpha)^2} + \frac{\omega^2}{4}(x + \alpha)^2.$$

Therefore, A_2^- is given by

$$A_2^- = \left(\frac{d}{dx} + \frac{\omega}{2}(x + \alpha) \right)^2 - \frac{2g^2}{(x + \alpha)^2}.$$

2. $k = 3$. In this case

$$A_3^- = \frac{d^3}{dx^3} + v_1(x) \frac{d^2}{dx^2} + v_2(x) \frac{d}{dx} + v_3(x) = \partial^3 + v_1(x) \partial^2 + v_2(x) \partial + v_3(x).$$

The commutator of H with A_3^- is given by

$$\begin{aligned} [H, A_3^-] &= -v_1' \partial^3 + \left[-v_2' - \frac{1}{2} v_1'' - 3U'\right] \partial^2 + \left[-v_3' - \frac{1}{2} v_2'' - 2v_1 U' - 3U''\right] \partial \\ &+ \left[-\frac{1}{2} v_3'' - v_2 U' - v_1 U'' - U'''\right] \end{aligned}$$

and when we assume that these two operators satisfy the commutation relation $[H, A_3^-] = -\omega A_3^-$, we obtain the following system of differential equations

$$\begin{cases} v_1' = \omega \\ v_2' + \frac{1}{2} v_1'' + 3U' = \omega v_1 \\ v_3' + \frac{1}{2} v_2'' + 2v_1 U' + 3U'' = \omega v_2 \\ \frac{1}{2} v_3'' + v_2 U' + v_1 U'' + U''' = \omega v_3. \end{cases} \quad (9)$$

Unfortunately, neither the solution of this system is an easy task nor the computation for the cases $k > 3$ is simple, and we should look the problem from a more general perspective. Our aim is to point out that the two cases we have studied seems to be the only possible cases.

We first remark that the properties of the corresponding classical problem are very useful for dealing with the quantum problem. Quantum systems with equispaced spectra are analogous to isochronous systems when the potentials are rational functions [14, 15], but there exist other such quantum systems whose analogous are not isochronous and only in the WKB approximation these classes of generalised harmonic oscillators coincide [14, 15, 16, 17, 18].

If a rational potential $U(x)$ is such that classical problem is isochronous with an angular frequency ω , then the asymptotic behaviour of $U(x)$ is given by

$$U(x) \sim \frac{1}{2} \omega^2 x^2, \quad \text{at } x \rightarrow \pm\infty,$$

if $U(x)$ has not singularities on the real axis, and

$$U(x) \sim \frac{1}{8} \omega^2 x^2, \quad \text{at } x \rightarrow \infty,$$

in $x > 0$ if $U(x)$ has a singularity on the real axis. This fact is a direct consequence of the result in [3]. We give here a simpler proof for this less general result, which furthermore shows the reason of factor $1/8$ instead of $1/2$. If the potential $U(x)$ has not singularities on the real axis, then the asymptotic behaviour of $U(x)$ at $x = \pm\infty$ should be of the form $U(x) \sim \alpha x^{2n}$, with $n \in \mathbb{N}_+$, because if the leading term is an odd power the motion at sufficiently high energy would be unbounded. Therefore we can assume that the asymptotic behaviour is given by an even function.

Recall that the expression of the period as a function of the energy in a one-dimensional bounded, and therefore periodic, motion of a particle of mass $m = 1$ under the action of a potential $U(x)$ is (1) which gives rise to (2) from which we derived the isochronicity condition

$$x_+(E) - x_-(E) = \frac{\sqrt{2} \tau}{\pi} \sqrt{E} = \frac{2\sqrt{2}}{\omega} \sqrt{E},$$

that in the particular case of a regular symmetric potential reduces to (3). When $T(E)$ is not constant but it is asymptotically constant for big enough E , i.e. if we assume that

$$T(E) = \tau \left(1 + \frac{\alpha_1}{E} + \frac{\alpha_2}{E^2} + \dots \right),$$

we obtain that

$$\phi_+(U) - \phi_-(U) = \frac{1}{\pi\sqrt{2}} \int_0^U \frac{T(E)}{\sqrt{U-E}} dE = \frac{2}{\pi\sqrt{2}} \sqrt{U} \left(\tau + \frac{\beta_1}{U} + \frac{\beta_2}{U^2} + \dots \right),$$

with $\beta_k = \alpha_k I_k$ where

$$I_k = \int_0^1 \frac{2 d\zeta}{(1-\zeta^2)^k}$$

and then

$$x_+(E) - x_-(E) = \frac{\sqrt{2}}{\pi} \sqrt{E} \left(\tau + \frac{\beta_1}{E} + \frac{\beta_2}{E^2} + \dots \right). \quad (10)$$

On the other side, given a rational potential with a real pole assumed to be at $x = 0$, the classical motion takes place in the open interval $(0, \infty)$ and then, as $\lim_{E \rightarrow \infty} x_-(E) = 0$, we see that for sufficiently high energy $x_+(E)$ behaves as $x_+(E) \sim (\sqrt{2}\tau/\pi)\sqrt{E}$.

In the case of a potential $U(x) \sim kx^{2n}$ the isochronicity condition for high energy leads to

$$k \left(\frac{2}{\omega} \right)^{2n} \left(\frac{E}{2} \right)^n = E \quad \text{if } U \text{ is regular}$$

and

$$k 2^n \left(\frac{2}{\omega} \right)^{2n} \left(\frac{E}{2} \right)^n = E \quad \text{if } U \text{ has a pole,}$$

with $\omega = 2\pi/\tau$, and therefore $n = 1$ and then either $k = \omega^2/2$, for regular U , or $k = \omega^2/8$, if U has a pole.

Of course in the asymptotic behaviour of the potential only the leading term is determined and therefore the above asymptotic dependence can be replaced by any second order polynomial in x with the same leading term as the above potential.

As a corollary, using the semi-classical approach to quantum mechanics, we can conclude that if the energy spectrum of the quantum Hamiltonian (6), with a rational function $U(x)$, is equispaced, with difference ω among neighbour eigenstates, then the asymptotic behaviour for $x \rightarrow \infty$ of the potential is $U(x) \sim \alpha\omega^2 x^2$, or any second order polynomial $P_2(x)$ of second degree with the same leading term, because the classical limit should be a periodic motion. Here $\alpha = 1/2$ when U is regular whereas $\alpha = 1/8$ when there is one real pole.

This suggests us to study the case of potentials of the form $U(x) = P_2(x) + U_1(x)$, or simply $U(x) = \alpha\omega^2 x^2 + U_1(x)$ with $U_1(x)$ decreasing at $x \rightarrow \pm\infty$.

Let assume that the Hamiltonian for such a potential admits ladder operators $A^-(\omega)$ and $A^+(\omega)$ for which $A^\pm(0)$ exist. Then the condition $[H(\omega), A^-(\omega)] = -\omega A^-(\omega)$ when particularised for $\omega = 0$ leads to the commutativity of $A(0)$ with $H(0) = H_1 = -(1/2)d^2/dx^2 + U_1(x)$.

In the fundamental papers by Burchnell and Chaundy, the theory of commuting differential operators has been developed [19]. More specifically, in the case we are considering there are two different possibilities depending on k being either even or odd. When k is even $k = 2m$, the only possibility is that the differential operator A_{2m} be a polynomial function of order m in H_1 . If, on the contrary, k is odd, then the results of Burchnell and Chaundy (see also [20, 21]) establish

that $U_1(x)$ should be a solution of a high-order Korteweg de Vries equation [22]. Therefore the potential function $U_1(x)$ takes the form [20, 21]:

$$U_1(x) = \sum_{j=0}^l \frac{m_j(m_j + 1)}{(x - x_j)^2} \quad (11)$$

where m_j are non-negative integers and x_j are complex numbers, i.e. $U_1(x)$ only can have (maybe complex) poles of second order at points x_j . Note, that as we have assumed that the potential is real, with each complex pole, its conjugate value is also a pole.

There are only two possibilities:

1. *There is no real pole but all poles x_j are complex numbers.*

Hence, there will be an even number of poles, $l = 2r$, and with each pole, its conjugate is also a pole.

The assumed non-existence of real poles implies that the potential function U_1 in the domain of the integral (2) is bounded and when there exist terms in (11) they will destroy the preceding isochronicity condition [1], and therefore in such a case the analogous quantum case cannot have an equispaced spectrum.

2. *There is one real pole at the point x_0 .*

As in this paper we consider to be equivalent two potentials obtained one from the other by means of shift and reflection and we are restricting ourselves to the case of potentials having at most one real pole, at the point x_0 , we can choose x_0 to be $x_0 = 0$. Consequently, the function $U_1(x)$ is a regular function on the semi-axis $0 < x < +\infty$, and once again as the potential is real, with each complex pole, its conjugate value is also a pole, i.e. there are $l = 2r + 1$ poles and only one, $x_0 = 0$, is real.

In this case the function $U(x)$ should have the following behaviour:

$$U(x) \sim \begin{cases} \frac{g^2}{x^2}, & x \rightarrow 0, \\ \frac{1}{8} \omega^2 x^2 + \frac{g_1^2}{x^2}, & x \rightarrow \infty, \end{cases}$$

where

$$g^2 = m_0(m_0 + 1), \quad g_1^2 = \sum_{j=0}^{2r+1} m_j(m_j + 1),$$

i.e., $g_1 \geq g > 0$. The equation $U(x) = E$ has two roots $x_-(E) < x_+(E)$ such that for big enough values of the energy are $E \rightarrow \infty$:

$$\begin{aligned} x_+(E) &\sim \sqrt{\frac{E}{\omega}} + \frac{g_1}{\sqrt{E}}, \\ x_-(E) &\sim \frac{g}{\sqrt{E}}. \end{aligned}$$

Therefore, the difference between both turning points as a function of the energy is

$$x_+(E) - x_-(E) \sim \sqrt{\frac{E}{\omega}} + \frac{g_1 - g}{\sqrt{E}},$$

and as indicated above, in order for the spectrum to be equidistant, and then the analogous classical system be isoperiodic, we should have $g_1 = g$. Hence, we can have only one pole (or all $m_j = 0$ except m_0). We arrive to the known case of the isotonic oscillator:

$$U(x) = \frac{1}{8} \omega^2 x^2 + \frac{g^2}{x^2}.$$

Note however that the derivation of this property is not fully rigorous because one can admit that the classical system be such that $T(E)$ is not constant but it is asymptotically constant.

So, we come to the following results:

If we consider the family of rational potentials $U(x)$ of the form

$$U(x) = \alpha \omega^2 x^2 + U_1(x), \quad \alpha = \frac{1}{2} \text{ or } \frac{1}{8},$$

where ω is a positive constant, and $u_1(x)$ is a rational function having at most a real pole and vanishing at $x \rightarrow \pm\infty$. Then, the quantum problem admits creation and annihilation operators (which include the eigenfunctions of the harmonic oscillator in their domains) and has an equispaced spectrum with distance ω only for the two cases considered before.

It is also worthy of mention that in multidimensional case we know some examples of quantum systems admitting creation and annihilation operators, B_k^+ and B_k , see papers [23, 24] and [25] for details. However, the analog of the previous result for multidimensional rational potentials remains still unknown.

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