Solutions in Gravity Theories Emerge from Thermodynamics –A Mini Review of a Recent Progress in Graviti-Thermodynamics

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In a recent work, we present a new point of view to the relation of gravity and thermodynamics. We derive some solutions in Einstein theory as well as in modified gravity theories by thermodynamics considerations with the aid of the Misner-Sharp mass in an adiabatic system. In this paper we make a concise review of this approach.

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1. Some Thoughts of the Misner-Sharp Mass

Contrary to intuitions, the mass (energy) of a gravitational field is a very intricate problem, which is still left open for modern physicists. Any efforts to ascertain the stress-energy tensor of the gravitational fields will be futile. The reason roots in the equivalence principle. Base on this principle, a free-falling observer senses a Minkowski spacetime locally, in which the stress-energy of the gravitational field, and thus the density of gravitational mass *should* be zero. According the requirements of diffeomorphism invariance, the stress-energy should vanish in frames of any observers. To evade this strong constraint from equivalence principle, people turn to stress-energy pseudo tensors, for example the Einstein-Tolman form, the Landau-Lifshitz form, and the mϕller form, etc. An obvious defect of these forms is that the pseudo tensor is coordinates-dependent. They must vanish for some observers, which contradicts to our basic concept to the stress energy. Generally the existence of a stress energy should not depend on the coordinates. On the other hand, the total mass of a gravitational field seems a little easier problem. We have well-defined ADM mass at spacelike infinity and Bondi-Sachs mass at null infinity on an asymptotic flat manifold. However, only the concept of total mass is not enough in the processes in astrophysics. In the observations of gravitational wave radiations from a compact star, we must quantitatively show how many gravitational energies are carried away by the gravitational waves from the compact star. In the gravitational collapse, we must detect how many energies of the collapsing matter are transferred to gravitational energies. In either case, it seems a concept of local gravitational mass is necessary. As we have mentioned, a covariant stress energy for gravity field is impossible. As an unavoidable concession, we consider the quasi-local form of gravitational mass, i.e., the mass inclosed a two-surface.

After several decades' studies, we now have many different forms of quasi-local forms of gravitational mass. In this article we concentrate on the Misner-Sharp mass. In fact, the Misner and Sharp find some clues of the form of the Misner-Sharp mass by exploring the transformation from matter to gravitational field in a collapsing model [1]. The explicit form of

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Misner-Sharp mass is presented in [2]. We review it in a logic way rather than its historic way. After collapsing, the mass of the baryons in a Newtonian star vanishes. Based on the energy conservation law, we can reasonably infer that the mass of the residue black hole spacetime is equal to the mass of the original star. This idea is used to determine the mass parameter in black hole solutions. This argument implies that we may define the mass of gravity field through the mass of the source matters. In a spacetime with stress energy T_{ab} and a timelike Killing vector ξ^b,

$$\nabla^a(T_{ab}\xi^b) = 0, \tag{1}$$

does not imply a conservation law of matters. We clarify this point by a simple example.

In a general spherically symmetric static spacetime in the area coordinates r,

$$ds^2 = -f(r)dt^2 + h^{-1}(r)dr^2 + r^2 d\Omega^2, \tag{2}$$

the mass of matter in a two-sphere $r = r_0$ reads,

$$M_0 = -4\pi \int_0^{r_0} dr\ T_0^0 h^{-1/2} r^2, \tag{3}$$

and

$$\xi^a = \sqrt{\frac{h}{f}} \left(\frac{\partial}{\partial t}\right)^a \tag{4}$$

while the conserved charge corresponding to the conserved current $T_{ab}\xi^b$ reads,

$$M = \int *(T_{ab}\xi^b) = -4\pi \int_0^{r_0} dr\ T_0^0 r^2. \tag{5}$$

The conserved charge M is different from the mass of matter inclosed in the two-surface $r = r_0$. The conserved charge M encodes not only the mass of the matters, but also the mass of the gravitational field. Thus a single M_0 in (3) does not obey the conservation law (1). One may be impulsive to define the gravitational mass by subtracting the matter mass M_0 from the total mass M,

$$M_{\text{gravity}} = M - M_0. \tag{6}$$

Unfortunately, M_{gravity} is neither positive definite, nor a component of a conserved current. To call it mass of gravity seems ineffectual.

The most useful concept in the above discussions is the total mass M, as a conserved charge of the spacetime. In Einstein gravity we have the relation between geometry and stress energy,

$$G_{ab} = 8\pi T_{ab}, \tag{7}$$

where we set the Newton constant $G = 1$. By using the Einstein equation (7), we rewrite (5) as follows,

$$M = -\frac{1}{2}\int_0^{r_0} dr\ G_0^0 r^2 = \frac{r}{2}(1-h). \tag{8}$$

Apparently, the above formula only depends on the metric, which is called Misner-Sharp mass as a quasi-local form of gravitational mass and labeled as M_{ms} henceforth.

We emphasize that M_{ms} is the total mass of matter and gravity from its definition. In this definition, gravity mass is always smuggled by matter mass. In a vacuum region, for example a shell dwelling at r_1 to r_2 ($0 < r_1 < r_2$) in the Schwarzschild spacetime, the Misner-Sharp mass is zero. The Misner-Sharp mass does not change when more vacuum regions (without singularities) are included. This is a critical property of the Misner-Sharp mass. Starting from this property, we can obtain fairly rich information of the spacetime metric in consideration. Furthermore, this property is inherited in the modified gravities. The reason is clear. In modified gravities with a timelike Killing vector, we still define,

$$M_{ms} = \int *(T_{ab}\xi^b). \tag{9}$$

Therefore in any vacuum region without singularity $M = 0$. The concept of Misner-Sharp mass also has fairly wide applications in dynamical spaces. Generally there is no conserved charge in a dynamical spacetime. However, in some special cases with high symmetries we can find a similar conserved charge. A typical case is the spacetime with spherically (spatially flat, pseudo spherically) symmetry where a Kodama vector $K^{\ a}$ is permitted [3]. The property of the Kodama vector is similar to the Killing vector,

$$\nabla_{(a}K_{b)} = 0, \tag{10}$$

which also yields a conservation law,

$$\nabla^a(T_{ab}K^b) = 0, \tag{11}$$

and the conserved charge,

$$M_{ms} = \int *(T_{ab}K^b). \tag{12}$$

Thus, a vacuum without singularity still implies a vanishing M_{ms}. We will see the significance of this property.

2. Derivation of the Schwarzschild Spacetime

The discussions in the above section is only to display some thoughts of the Misner-Sharp mass. Now, in a formal discussion, we try to include a little more general cases. First, we consider a static space with different topologies rather than the only spherically symmetric cases. A static metric which permits a two-dimensional maximally symmetric subspace with three types of sectional curvatures $k = 1,\ 0,\ -1$ reads,

$$ds^2 = -f(r)dt^2 + h^{-1}(r)dr^2 + r^2 d\Omega_2^2, \tag{13}$$

in which $f(r),\ h(r)$ are general functions of r, Ω_2 denotes a unit two-sphere, two-cube, or two-pseudo-sphere, depending on the sectional curvature $k = 1,\ 0,\ -1$, respectively. The corresponding Misner-Sharp mass reads,

$$M_{ms} = \int *(T_{ab}\xi^b) = -\frac{1}{2}\int_0^{r_0} dr\ G_0^0 r^2. \tag{14}$$

The result is similar to (8),

$$M_{ms} = \frac{r}{2}(k-h). \tag{15}$$

To explore the information hided in the Misner-Sharp mass, we take (13) and (15) as our starting point.

As we have stressed in the last section, in vacuum space the Misner-Sharp mass does not depend on the radius r in the metric (13). In thermodynamic language, this is equal to consider an adiabatic Misner-Sharp system,

$$\delta M_{ms} = 0. \tag{16}$$

Thus we obtain,

$$k - h - rh' = 0, \tag{17}$$

where a prime denotes the derivative with respect to r. Then immediately we reach,

$$h = \left(k - \frac{C}{r}\right), \tag{18}$$

where C is an integration constant. To determine C, we back substitute h into (15) to obtain,

$$C = 2M_{ms}. \tag{19}$$

Here, we obtain h with 3 kinds of sectional curvatures as a unity. One sees that only the property that vacuum implies invariance of Misner-Sharp mass is enough to determine a component of the static metric. If we can determine f, we obtain the complete vacuum solution.

We use surface gravity to determine $f(r)$. In spherically symmetric spacetime the geometric surface gravity is adapted to the unified first law [4]. The unified first law is a nice progress in gravi-thermodynamics. The black hole thermodynamics was set up on the global quantities of the spacetime, which always depends on asymptotic behaviour of a manifold. It is not very helpful to understand the realistic processes in astrophysics. While only a patch of a manifold is involved in the unified first law, which can be applied to a realistic processes in astrophysics when the knowledge of the whole manifold is lack. Correspondingly we only need the quasi-local quantities rather than global quantities. Therefore it is no essential difficulties to be used in dynamical spacetimes [5]. The energy adapted to unified first law is just the Misner-Sharp mass, which is critical to make our reasoning be self-consistent.

In four dimensional spacetime, the geometric surface gravity adapted to the unified first law is,

$$\kappa = \frac{M_{ms}}{r^2} - 4\pi r w, \tag{20}$$

where the work term,

$$w = -\frac{1}{2} I^{ab} T_{ab}, \tag{21}$$

and

$$I_{ab} = -f dt^2 + h^{-1} dr^2. \tag{22}$$

T denotes the stress energy of the matter fields. $T = 0$ in vacuum spacetime.

We see that the surface gravity κ in (20) is independent on the sectional curvature of k. As usual, we calculate the surface gravity as a product of the magnitude of the four-acceleration of a motionless particle at the static coordinates and $(-g_{00})^{1/2}$. In coordinates (13), the world line of a rest particle is,

$$X^\mu = (t, r_0, \theta_0, \phi_0), \tag{23}$$

where (θ, ϕ) are the coordinates of the two-dimensional inner space and a subscript 0 denotes that the corresponding quantity is a constant. The four-velocity for this particle U_μ

$$U^\mu = \frac{dX^\mu}{d\tau}, \tag{24}$$

where τ denotes the proper time of this particle. Then the 4-acceleration for this particle reads,

$$A^\mu = U^\nu \nabla_\nu U^\mu. \tag{25}$$

The magnitude of A^μ,

$$a = \sqrt{|g_{\mu\nu} A^\mu A^\nu|}. \tag{26}$$

Thus the surface gravity κ is calculated by

$$\kappa = a\sqrt{-g_{00}} = a\sqrt{f} = \frac{1}{2}(f/h)^{-1/2} f'. \tag{27}$$

The surface gravity obtained in this way, which we called "dynamical surface gravity", should be equal to the geometric surface gravity in (20),

$$\frac{1}{2}(f/h)^{-1/2} f' = \frac{M_{ms}}{r^2}. \tag{28}$$

This equation is easy to integrate using (18),

$$f = \left[\left(k - \frac{2M_{ms}}{r}\right)^{1/2} + D\right]^2, \tag{29}$$

where D is an integration constant. D can be derived by the Newtonian condition. A basic requirement of any reasonable metrics is to degenerate to the Newtonian gravity in the weak field limit. As usual, the Newtonian metric for a spherically metric ($k = 1$) reads,

$$ds^2 = -(1+2\phi)dt^2 + (1-2\phi)dr^2 + r^2 d\Omega_2^2, \tag{30}$$

where $\phi = -M/r$ denotes the Newtonian potential and M marks the central mass. An important property of the Misner-Sharp mass is that it is exactly the Newtonian mass M in a spherically symmetric case [4]. Expanding f at large r limit, we obtain

$$f = 1 - \frac{2M_{ms}(1+D)}{r} + 2D + D^2. \tag{31}$$

Then it is clear,

$$\phi = -\frac{2M_{ms}(1+D)}{r} + 2D + D^2. \tag{32}$$

The correct Newtonian limit requires $D = 0$ for the case $k = 1$. For the cases $k = 0,\ -1$ we have no corresponding Newtonian limits. Locally we treat the cases $k = 0,\ -1$ as analytic prolongations of the case $k = 1$. Thus they share a unified $D = 0$. We complete the derivation of Schwarzschild solution (including the three cases of different topologies) from the Misner-Sharp mass form by thermodynamic considerations.

3. Solutions with Matter Sources

The vacuum solutions with different topologies have been derived by a variation of the Misner-Sharp mass with respect to r in an adiabatic system. An improvement of this demonstration is to introduce the generalized forces at the right hand of (16) to include the effects of matter fields. The validity of this method is not obvious from the properties of the Misner-Sharp mass, as discussed in the Sec. 1. However, we will show that it is really valid in discussing the solutions with matter source. First, we discuss dS/AdS space. The metric assumption (13) and the Misner-Sharp mass (15) take the same in this case. When a pressure term is introduced, (16) becomes

$$\delta M_{ms} = -PdV, \tag{33}$$

where P =constant (dS/AdS depends on the sign of P) is the pressure and V is the volume in consideration. It is easy to obtain $h(r)$ for this asymptotical dS/AdS,

$$h = \left(k - \frac{C}{r} + \frac{8\pi P}{3} r^2\right). \tag{34}$$

According to the customs, we use the cosmological constant to replace the pressure,

$$\Lambda = -8\pi P. \tag{35}$$

We obtain the Minser-Sharp mass for dS/AdS by back instituting (34) into (15),

$$M_{msd} = \frac{C}{2} + \frac{\Lambda}{6} r^3. \tag{36}$$

Interestingly, this result does not depend on the sectional curvature k.

Now the corresponding work term appears,

$$w = \frac{1}{8\pi}\Lambda. \tag{37}$$

And the geometric surface gravity (20) becomes,

$$\kappa = \frac{C}{2r^2} - \frac{\Lambda r}{3}. \tag{38}$$

While the dynamical surface gravity (27) does not depend on the concrete forms of f and h. The equality of the geometric surface gravity and the dynamical surface gravity is written as,

$$\frac{1}{2}(f/h)^{-1/2} f' = \frac{C}{2r^2} - \frac{\Lambda r}{3}. \tag{39}$$

We obtain the form of f by direct integration,

$$f = \left[\left(1 - \frac{C}{r} - \frac{\Lambda}{3} r^2\right)^{1/2} + D_1\right]^2, \tag{40}$$

where D_1 labels an integration constant. Similar to the vacuum case, we obtain $D_1 = 0$ by using the Newtonian limit under $\Lambda = 0$ and large r approximation.

Then, we consider a system with an electromagnetic field. The first law in an adiabatic Misner-Sharp system is modified to

$$\delta M_{ms} = \Phi dq, \tag{41}$$

where $\Phi = q/r$ labels the electric potential and q represents the electric charge at $r = 0$. Then we reach,

$$h = \left(k - \frac{C}{r} + \frac{q^2}{r^2}\right). \tag{42}$$

We calculate the Misner-Sharp mass M_{msq} with an electromagnetic field,

$$M_{msq} = \frac{C}{2} - \frac{q^2}{2r}, \tag{43}$$

and the work term becomes,

$$w = \frac{q^2}{8\pi r^4}. \tag{44}$$

Thus the geometric surface gravity,

$$\kappa = \frac{C}{2r^2} - \frac{q^2}{r^3}. \tag{45}$$

The equality of dynamical surface gravity and geometric surface gravity presents

$$f = k - \frac{C}{r} + \frac{q^2}{r^2}, \tag{46}$$

where we have imposed the proper Newtonian limit.

4. Modified Gravities

We always have two perspectives on the cosmological constant about its physical property. The first one is that it is matter, whose stress energy satisfies that energy density equals negative pressure. This property is exactly satisfies the requirement of vacuum in quantum field theory. In this sense we call it vacuum energy in a lot of cases. The second one is that the cosmological constant is a geometric property of the spacetime, which belongs to the gravity sector in the field equation. In the deduction of of the last section, we take the first perspective. This is because in the definition of Misner-Sharp mass (8), we have used the Einstein equation without cosmological constant $G_{ab} = 8\pi T_{ab}$. Hence the cosmological constant must be treated as form of matter at the right hand side of the equation. While in the second perspective, the field equation becomes,

$$G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}. \tag{47}$$

We still define the Misner-Sharp mass as in (9).

For a more general discussion, we work in an n-dimensional spacetime $\mathcal{M}=\mathcal{I} \times \mathcal{K}$,

$$ds^2 = -f(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2\gamma_{ij}dz^i dz^j, \tag{48}$$

where $(\mathcal{K},\ \gamma)$ is an $(n-2)$-dimensional maximally symmetric submanifold embedded in $(\mathcal{M}, g)$.

Then we obtain the Misner-Sharp mass for Einstein gravity with a cosmological constant for the above metric within radius r by a direct integration of (9),

$$\begin{aligned} &M_{ms} \\ &= \frac{V^k_{n-2} r^{n-3}}{8\pi G}\left[\frac{n-2}{2}(k - I^{ab}\partial_a r \partial_b r) - \frac{r^2\Lambda}{n-1}\right], \end{aligned} \tag{49}$$

where V^k_{n-2} denotes the volume of a unit subspace $\mathcal{K}$ with different sectional curvatures k, I represents the induced metric on $\mathcal{I}$, and the indexes $a,\ b = 0,\ 1$. For $n = 4$, it reduces to

$$M_{ms} = \frac{V^k_2 r}{8\pi G}\left[(k - I^{ab}\partial_a r\partial_b r) - \frac{\Lambda r^2}{3}\right]. \tag{50}$$

We call a system Misner-Sharp system, if its total energy is defined as its Misner-Sharp mass. Supposing the Misner-Sharp system in radius r is in adiabatic state in four dimensional spacetime, and making a variation with respective to r in (50), we arrive at,

$$k - h - \frac{\Lambda}{3}r^2 = r\left(h' + \frac{2}{3}\Lambda r\right). \tag{51}$$

The solution reads,

$$h = k - \frac{C}{r} - \frac{\Lambda}{3}r^2. \tag{52}$$

Substituting (52) into (50), we obtain the Misner-Sharp mass for this system,

$$M'_{ms} = \frac{C}{2}, \tag{53}$$

which is different from the result in the first perspective, in which $M_{ms} = \frac{C}{2} + \frac{\Lambda}{6}r^3$, as shown in (36). This difference is not difficult to explain: Λ belongs to the gravitational back ground and has been subtracted in (50) in the second perspective, while in the first perspective Λ is treated as a matter field, which has been considered in the Misner-Sharp mass.

Following the discussions in the above sections, we should then compare the dynamical surface gravity with the geometric surface gravity to determine f. However, unfortunately a proper definition of surface gravity in modified gravity as an extension of (20) is still absent. Thus we have to switch to the first perspective, i.e., to treat the cosmological constant as a vacuum matter with density and pressure,

$$\rho = \frac{\Lambda}{8\pi G}, \tag{54}$$

$$p = -\frac{\Lambda}{8\pi G}. \tag{55}$$

The following procedure exactly mimics what we done in Sec. 3. The resulted f is

$$f = k - \frac{C}{r} - \frac{\Lambda}{3}r^2. \tag{56}$$

It is not worthy of switching between the two perspectives if we only want to derive dS/AdS solution. This method will play important role in the discussions of more complicated modified gravities. In cosmology, the two perspectives of modified gravities have been studied long. In the first perspective, all the terms other than the Einstein tensor is casted to the right hand side of the field equation and treated as effective "matters". This is also called Einstein interpretation of the field equation. In the second perspective, all the geometric sectors are treated as gravity. The matter energy stress only contains realistic matters. This is the natural perspective of modified gravity. DS/AdS is a simple but instructive example, which is easy to understand for the switching between the two perspectives. In fact the switching between two perspectives is critical in search for solution in modified gravities via thermodynamic considerations.

Next, we start to discuss an important modified gravity, the Guass-Bonnet gravity, by thermodynamical method. In higher dimensional spacetime, one can find a proper combination of R^2-type or higher derivative terms in the Lagrangian which does not yield higher than two order derivatives with respect to metric in the field euqation [6]. In five dimensional spacetimes, this combination is Gauss-Bonnet term R_{GB}

$$R_{GB}=R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta}-4R_{\mu\nu}R^{\mu\nu}+R^2, \quad (57)$$

where $R_{\mu\nu\gamma\delta}$, $R_{\mu\nu}$, and R denote Reimann tensor, Ricci tensor, and Ricci scalar, respectively. More general Lovelock gravity permits combinations of more higher order power of R, such as R^4-type terms, which can also evade higher than two order derivatives with respect to the metric in more higher dimensional spacetimes [6]. A spherically symmetric static solution in Gauss-Bonnet gravity is derived in [7]. This result has been extended with introducing a cosmological constant [8]. We derive this Boulware-Deser-Cai solution via thermodynamic method.

We will work in the metric form (48) on $\mathcal{M}$. The Misner-Sharp mass for this metric is given by an direct integration (9) [9], see also [10],

$$\begin{aligned} &M_{ms}\\ =&\frac{V^k_{n-2}r^{n-3}}{8\pi G}\left[\frac{n-2}{2}(k-I^{ab}\partial_a r\partial_b r)\right.\\ &\left.-\frac{\Lambda}{n-1}r^2+\frac{n-2}{2}\tilde{\alpha}r^{-2}(k-I^{ab}\partial_a r\partial_b r)^2\right], \end{aligned} \quad (58)$$

which corresponds to the action of an Einstein-Hilbert term with an Gauss-Bonnet correction,

$$\begin{aligned} &S\\ =&\frac{1}{16\pi G}\int_{\mathcal{M}}d^n x\sqrt{-\det(g)}\left(R-2\Lambda+\alpha R_{GB}\right), \end{aligned} \quad (59)$$

in which

$$\tilde{\alpha}=\alpha(n-3)(n-4). \quad (60)$$

There are some constraints on α when we treat the Gauss-Bonnet gravity as a reduction from other theories, for example string/M theory. Phenomenologically, it is treated as a free parameter in this work. Consider an adiabatic Misner-Sharp system,

$$\delta M_{ms}=0. \quad (61)$$

Using (58), we have,

$$\begin{aligned} &\frac{n-3}{r}\left(k-\frac{2r^2\Lambda}{2-3n+n^2}+\frac{\tilde{\alpha}(k-h)^2}{r^2}-h\right)\\ =&\frac{4r\Lambda}{2-3n+n^2}+\frac{2\tilde{\alpha}(k-h)^2}{r^3}+h'\\ &+\frac{2\tilde{\alpha}(k-h)h'}{r^2}. \end{aligned} \quad (62)$$

By direct integration, we obtain

$$\begin{aligned} &h\\ =&k+\frac{r^2}{2\tilde{\alpha}}\left(1\mp\sqrt{1+\frac{8\tilde{\alpha}\Lambda}{(n-2)(n-1)}+\frac{4\tilde{\alpha}^2C}{r^{d-1}}}\right), \end{aligned} \quad (63)$$

where C is an integration constant. Comparing with the Schwarzschild case, we guess that C is a mass parameter. But it is not equal to mass here. Substituting (63) into (58), we find that C is proportional to mass,

$$M_{ms}=\frac{\tilde{\alpha}C(n-2)V^k_{n-2}}{16\pi G}. \quad (64)$$

We can confirm that M_{ms} reduces to the mass in the Newtonian sense at the limits $\Lambda\to 0$ and $\alpha\to 0$. Therefore, in this sense the Misner-Sharp mass seems a reasonable extension of the Newtonian mass in the Guass-Bonnet gravity.

Then we need to deal with f. One encounter a similar problem as that in the case of cosmological constant: we have no accurate definition of surface gravity in modified gravities. We also have two perspectives in modified gravity theories. The first perspective (Einstein interpretation) has been extensively studied in the researches of the cosmic acceleration [11]. As we have seen, the Misner-Sharp mass in the second perspective (modified gravity perspective) is critical to obtain h. Now we switch to the first perspective (Einstein interpretation). The effective stress energy $T^{(e)}$ in the first perspective is presented in [8],

$$\begin{aligned} & T^{(e)}_{\mu\nu} \\ = & -\frac{1}{8\pi}\Big[2\alpha(RR_{\mu\nu} - 2R_{\mu\alpha}R_{\nu}{}^{\alpha} - 2R^{\alpha\beta}R_{\mu\alpha\nu\beta} \\ & +R_{\mu}{}^{\alpha\beta\gamma}R_{\nu\alpha\beta\gamma}) - \frac{\alpha}{2}g_{\mu\nu}R_{GB} + \Lambda g_{\mu\nu}\Big], \quad (65) \end{aligned}$$

where the curvature tensors are calculated by the metric (48). The Misner-Sharp mass takes its original form in this perspective,

$$\overline{M}_{ms} = \frac{V^k_{n-2}r^{n-3}}{8\pi G}\frac{n-2}{2}(k - I^{ab}\partial_a r\partial_b r), \quad (66)$$

which is no more constant, because the "matter field" (65) is included. Mimicking the case of real matter fields, we define the work term

$$w = -\frac{1}{2}I^{ab}T^e_{ab}. \quad (67)$$

With these preparations, the surface gravity can be defined in n-dimensional ($n \geq 3$) spacetime

$$\kappa \equiv \frac{8\pi(n-3)}{(n-2)V^k_{n-2}}\frac{\overline{M}_{ms}}{r^{n-2}} - \frac{8\pi}{n-2}rw. \quad (68)$$

It is easy to confirm that when $n = 4$, this general form degenerates to (20). In this section we suppose $n \geq 5$ without special notations because we are discussing the Gauss-Bonnet gravity. Here, we would like to note that the above equation is valid for all the cases of $n \geq 3$. The three dimensional case is of special significance. κ vanishes for three dimensional pure Einstein gravity, which confirms the essential result in : Black holes with nontrivial geometries do not exist in three-dimensional Einstein gravity.

The geometric surface can be derived by substituting (66) and (67) into the above equation,

$$\kappa = \frac{1}{2G}(n-3)\frac{1-h}{r} + \frac{4\pi r}{n-2}I^{ab}T^e_{ab}\,. \quad (69)$$

The equality of dynamical surface gravity and geometric surface gravity presents the equation for f,

$$\left(1 - 2\sqrt{\frac{f}{h}}\right)hf' + fh' = 0. \quad (70)$$

h has been derived in (63). We thus obtain f,

$$f = \frac{1}{2}\left(h - D \pm \sqrt{h^2 - 2Dh}\right), \quad (71)$$

where D is the integration constant. D can be determined by studying some limit of f. It is clear that only $D = 0$ in the "+" branch reduces to correct Schwarzschild limit. The "−" branch should be treated to be an extraneous solution emerged in the thermodynamic method. Thus we write f,

$$\begin{aligned} f &= h \\ &= k + \frac{r^2}{2\tilde{\alpha}}\left(1 \mp \sqrt{1 + \frac{8\tilde{\alpha}\Lambda}{(n-2)(n-1)} + \frac{4\tilde{\alpha}^2 C}{r^{d-1}}}\right). \end{aligned} \quad (72)$$

So the Boulware-Deser-Cai solution in Gauss-Bonnet gravity is derived by thermodynamic considerations.

Divergence is a famous difficult problem in gravity theory. It has been found that the divergences are significantly alleviated if higher order derivatives are included [12]. The higher order terms are not something "put by hand". Such terms always come when quantum effects is considered [13] or some unified theory, for example string/M theory is taken into account [14]. $F(R)$ gravity is one of the most extensively explored theory in the higher order derivatives theories. $F(R)$ gravity has some distinctive properties. First of all, it is the unique one which successfully extricates from the catactrophic Ostrogradski instability amongst all higher derivative gravity theories [15]. Second, it is simple enough to handle, at the same time complicated enough to encode the principle framework of higher derivative theories.

All the above only reproduced the existing results. To display the power of this new method, we apply it to $F(R)$ gravity to get a class of new solution. The Misner-Sharp mass for $F(R)$ gravity in four-dimensional spherically symmetric spacetime is presented in [16]. This result is extended to the case of $F(R)$ gravity n-dimensional spacetime with three types of $(n-2)$-dimensional maximally symmetric submanifold [17]. Our starting point is still a Misner-Sharp system. We work on this n-dimensional manifold $(\mathcal{M}, g)$ of (48) with an $(n-2)$-dimensional maximally symmetric submanifold $(\mathcal{K},\ \gamma)$, on which the Misner-Sharp mass can be written as,

$$\begin{aligned} M_{ms} \ =\ & \frac{V_{n-2}^{k} r^{n-3}}{8\pi G}\Big[\frac{n-2}{2}(k-I^{ab}\partial_a r\partial_b r)F_R+\frac{1}{2(n-1)}r^2(F-F_R R)-rI^{ab}\partial_a F_R\partial_b r\Big] \\ & +\frac{V_{n-2}^{k}}{16\pi G}\int dr\Big[r^{n-2}h'+(n-2)r^{n-3}(h-k)+\frac{r^{n-1}}{n-1}R\Big]F_{R,r}, \end{aligned} \tag{73}$$

which corresponds to the action,

$$S=\frac{1}{16\pi G}\int_{\mathcal{M}} d^n x\sqrt{-\det(g)}\ F(R)+S_m, \tag{74}$$

where S_m is the action for the matter fields, and $F_R=\partial F(R)/\partial R$. Considering the vacuum case $S_m=0$, in which the Misner-Sharp system is adiabatic, we have

$$\delta M_{ms}=0. \tag{75}$$

Then, from (73), we obtain,

$$\begin{aligned} r^2 dR^3 \ &+\ (1+d)(n-2)R^2\left[3-n+(n-3)h+rh'\right]+2d(d^2-1)r^2hR'^2 \\ &+\ d(1+d)rR\left[rh'R'+2h\left((n-2)R'+rR''\right)\right]=0, \end{aligned} \tag{76}$$

where we take $F(R)=R^{d+1}$, and the Ricci scalar is given by

$$R=\frac{(n-3)(n-2)}{r^2}(k-h)-\frac{n-2}{r}\left(h'+\frac{hf'}{f}\right)+\frac{1}{2f}\left(\frac{hf'^2}{f}-h'f'-2hf''\right). \tag{77}$$

After carefully observing (77), we present an ansatz,

$$R=-\frac{kL}{r^2}, \tag{78}$$

where L is a constant, $k=0,\ \pm 1$. With this ansatz, (76) is tractable. The solution is

$$h=\frac{6-5n+n^2+d\left(6+L-5n+n^2\right)}{(1+d)\left(6+8d^2-4d(n-3)-5n+n^2\right)}\left(k+Cr^{3+2d+\frac{2d(1+2d)}{2+2d-n}-n}\right). \tag{79}$$

Similar to the case of Gauss-Bonnet gravity, we switch to the first perspective (Einstein interpretation). The effective stress energy is presented in [18],

$$T^e_{\mu\nu}=\frac{1}{8\pi G F_R}\Big[\frac{1}{2}\ g_{\mu\nu}(F-RF_R)+\nabla_\mu\nabla_\nu F_R-g_{\mu\nu}\Box F_R\Big]. \tag{80}$$

The geometric surface gravity κ in (69) becomes really involved. From (77) and (78), we arrive at,

$$\frac{(n-3)(n-2)}{r^2}(k-h)-\frac{n-2}{r}\left(h'+\frac{hf'}{f}\right)+\frac{1}{2f}\left(\frac{hf'^2}{f}-h'f'-2hf''\right)=-\frac{kL}{r^2}. \tag{81}$$

To obtain f, we then substitute h in (79),

$$f=r^{-\frac{2d(1+2d)}{1+d-n/2}}\left(k+Cr^{\frac{6+8d^2-4d(n-3)-5n+n^2}{2+2d-n}}\right), \tag{82}$$

and rewrite h

$$h=\frac{(3-n)(n-2-2d)^2}{(2+4d+4d^2-n)\left(6+8d^2-4d(n-3)-5n+n^2\right)}\left(k+Cr^{\frac{6+8d^2-4d(n-3)-5n+n^2}{2+2d-n}}\right). \tag{83}$$

Here all the integration constants and L have been calibrated by the Clifton-Barrow solution [19],

$$L=-\frac{(n-3)(n-2)d(d+1)}{1/2-n/4+d(d+1)}. \tag{84}$$

It is easy to check that this solution reduces to the Clifton-Barrow solution when $n=4$. We can confirm that the above solution satisfies the vacuum field equation of $F(R)$ gravity by direct calculation,

$$F_R R_{\mu\nu}-\frac{1}{2}Fg_{\mu\nu}-\nabla_\mu\nabla_\nu F_R+g_{\mu\nu}\Box F_R=0. \tag{85}$$

The physics of higher dimensional case of this solution certainly need to further study. Now we first present some property of the three-dimensional case of this solution. One may think that this solution is trivial because both h and R vanish. However, a special case with

$$d=\frac{1}{2}\left(-1\pm\sqrt{2}\right), \tag{86}$$

is none trivial. Under this condition, f and h become,

$$f=kr^2+Cr^{\sqrt{2}}, \tag{87}$$

$$h=k+Cr^{-2+\sqrt{2}}. \tag{88}$$

This represents a black hole with true singularity, which is completely deviate from the case of three-dimensional Einstein gravity, where the geometries are always trivial. A more general three-dimensional black hole solution in $F(R)$ theory has been obtained in our recent work [20]. The above solution is a special case of this more general solution.

5. Conclusion

In recent works we present a new view on gravi-thermodynamics [21]. We find that the spherically symmetric (more generally, an $n-$dimensional spacetime with an $(n-2)$-dimensional maximally symmetric submanifold) solutions can be derived in an adiabatic Misner-Sharp system in addition with an equality of geometric surface gravity and dynamical surface gravity. In this work we make a simple review of this approach. We first discuss the essential properties of Misner-Sharp mass, both in Einstein gravity and in modified gravity. In the discussions we explain our motivation to treat the Misner-Sharp mass within a radius r as an adiabatic system. In the vacuum case, it is a direct result of the definition of the Misner-Sharp mass. While when we extend this method to a system with matter fields, we must regard it as a principle (hypothesis). We first derive the most important solution in general relativity and astrophysics, the Schwarzschild solution. Then we derive some solutions with matter fields, including dS/AdS and RN solutions. We also make some explorations to this method for modified gravity theories. With the same definitions of Misner-Sharp mass by stress-energy, we can obtain h directly in Gauss-Bonnet gravity. To obtain f, we need a proper extension of the definition of geometric surface gravity to the higher dimensional case. We find such a definition, and then apply it to the Gauss-Bonnet gravity. We successfully find f through this thermodynamical manner. Furthermore, we find a new class solution via this method in $F(R)$ gravity. This new solution reduces to Clifton-Barrow solution in four dimensional $F(R)$ gravity, and to a special case of a solution in three dimensional $F(R)$ gravity in our recent work.

Principally the new thermodynamic method opens a new window to gravi-thermodynamics, which shows that the quasilocal mass form, especially the Misner-Sharp mass, may encode rich information of the gravity field. Practicably, it launch a new method to solve the gravity field equations. As seen in this article, the differential equations emerged in this thermodynamic demonstration are usually the first order equation. It is easier to solve than to solve the field equation directly, which is always at least the second order equation.

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References

[1] C. W. Misner and D. H. Sharp, Phys. Rev. **136**, B571 (1964).

[2] M. Cahill and G. McVittie, J. Math. Phys., 11, 1382 (1970).

[3] H. Kodama, Prog. Theor. Phys. **63**, 1217 (1980).

[4] S. A. Hayward, Class. Quant. Grav. **15**, 3147 (1998) [gr-qc/9710089].

[5] R. G. Cai and S. P. Kim, JHEP **0502**, 050 (2005) [hep-th/0501055].

[6] D. Lovelock, J. Math. Phys. **12** (1971) 498; C. Lanczos, Ann. Math. **39**, 842 (1938).

[7] D. G. Boulware and S. Deser, Phys. Rev. Lett. **55**, 2656 (1985).

[8] R. G. Cai, Phys. Rev. **D65**, 084014 (2002) [hep-th/0109133].

[9] H. Maeda and M. Nozawa, Phys. Rev. **D77**, 064031 (2008) [arXiv:0709.1199 [hep-th]].

[10] R. G. Cai, L. M. Cao, Y. P. Hu and N. Ohta, Phys. Rev. **D80**, 104016 (2009) [arXiv:0910.2387 [hep-th]].

[11] H. Zhang and Z. H. Zhu, Phys. Rev. **D75**, 023510 (2007) [astro-ph/0611834]; H. Zhang, H. Noh, Z. H. Zhu and H. Yu, Nucl. Phys. **B824**, 19 (2010) [arXiv:0908.1001 [astro-ph.CO]]; H. Zhang, arXiv:0909.3013 [astro-ph.CO]; Y. Du, H. Zhang and X. -Z. Li, Eur. Phys. J. **C71**, 1660 (2011) [arXiv:1008.4421 [astro-ph.CO]]; R. G. Cai, H. Zhang and A. Wang, Commun. Theor. Phys. **44**, 948 (2005) [hep-th/0505186].

[12] K. S. Stelle, Phys. Rev. **D16**, 953(1977).

[13] N. Birrell and P. Davies, *Quantum Fields in Curved Space*, Cambrige University Press (1984).

[14] J. Polchinski, *String Theory*, Cambridge University Press (2005).

[15] R. P. Woodward, Lect. Notes Phys. **720**, 403-433 (2007).

[16] R. G. Cai, L. M. Cao, Y. P. Hu, and N. Ohta, Phys. Rev. **D80**, 104016 (2009) [arXiv:0910.2387 [hep-th]].

[17] H. Zhang, Y. Hu, and X. -Z. Li, arXiv:1406.0577 [gr-qc].

[18] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. **82**, 451(2010) [arXiv:0805.1726 [gr-qc]]; S. Nojiri and S. D. Odintsov, Phys. Rept. **505** 59 (2011).

[19] T. Clifton and J. D. Barrow, Phys. Rev. **D72**, 103005 (2005) [gr-qc/0509059]. We note that there is an error in the published version in PRD. For the correct form of this solution, see the arXiv version.

[20] H. Zhang, D. Liu, and X. Li, arXiv:1405.7530.

[21] H. Zhang, S. Hayward, X. Zhai, and X. Li, Phys. Rev. **D89** 064052 (2014); H. Zhang and X. Z. Li, Phys. Lett. **B737** 395 (2014) [arXiv:1406.1553 [gr-qc]].