Aspects of Supergravity Compactifications and SCFT correlators

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Abstract

We begin by discussing aspects of supergravity compactifications and argue that the problem of finding lower-dimensional de Sitter solutions to the classical field equations of higher-dimensional supergravity necessarily requires understanding the back-reaction of whatever localized objects source the bulk fields. However, we also find that most of the details of the back-reacted solutions are not important for determining the lower-dimensional curvature. We find, in particular, a classically exact expression that, for a broad class of geometries, directly relates the curvature of the lower-dimensional geometry to asymptotic properties of various bulk fields near the sources. The near-source profile of the bulk fields thus suffices to determine the classical cosmological constant. We find that, due to the existence of a classical scaling symmetry, the on-shell supergravity action for IIA, IIB and 11d supergravity theories is a boundary term whose explicit form we also determine. Specializing to codimension-two sources, we find that the contribution involving the asymptotic behaviour of the warp factor is precisely canceled by the contribution of the sources themselves. As an application we show that *all* classical compactifications of Type IIB supergravity (and F-theory) to 8 dimensions are 8D-flat if they involve only the metric and the axio-dilaton sourced by codimension-two sources, extending earlier results to include warped solutions and more general source properties. We then proceed to study $3d$ SCFTs in the superspace formalism and discuss superfields and on-shell higher spin current multiplets in free $3d$ SCFTs. For $\mathcal{N} = 1$ $3d$ SCFTs we determine the superconformal invariants in superspace needed for constructing 3-point functions of higher spin operators, find the non-linear relations between the invariants and consequently write down all the independent invariant structures, both parity even and odd, for various 3-point functions of higher spin operators. We consider the additional constraints of higher spin current conservation on the structure of 3-point functions and show that the 3-point function of higher spin conserved currents is the sum of two terms- a parity even part generated by free SCFTs and a parity odd part.

Declaration

The research described in this dissertation was carried out in the Department of Applied Mathematics and Theoretical Physics at the University of Cambridge between May 2010 and December 2013. Chapters 3 and 4 include work which was done, in part, at the Tata Institute of Fundamental Research, Mumbai, India during a research visit (August 2012 - March 2013). Except where reference is made to the work of others, all the results are original and based on the following of my works:

- *Superspace formulation and correlation functions of 3d superconformal field theories*, Amin A. Nizami, Tarun Sharma and V. Umesh, arXiv:1308.4778 [hep-th].
- *On Brane Back-Reaction and de Sitter Solutions in Higher-Dimensional Supergravity*, C.P. Burgess, A. Maharana, L. Van Nierop, Amin A. Nizami, F. Quevedo, JHEP 1204 (2012) 018, arXiv:1109.0532 [hep-th].

None of the work contained in this dissertation has been submitted by me for any other degree, diploma or similar qualification.

Signed ____________________________________

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Date ____________________________________

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Contents

Chapter 1

Introduction

The AdS/CFT correspondence [20, 21, 22] provides a duality map between large N Superconformal Field Theories (SCFTs) and supergravity theories in higher dimensions. Certain large N SCFTs, for example, 4d $\mathcal{N} = 4$ Super Yang-Mills and 3d $\mathcal{N} = 6$ ABJ theory have a holographic dual description in terms of supergravity compactification geometries - IIB supergravity on the background geometry $AdS_5 \times S^5$ [20] or IIA supergravity on $AdS_4 \times \mathbb{CP}^3$ [28], respectively. In this thesis we will first study certain aspects of supergravity compactifications mainly pertaining to the maximally symmetric spacetime obtained on warped compactification. We will investigate, in particular, the feasibility of generating de Sitter solutions. We will also discuss a (classical) scaling symmetry possessed by the IIA, IIB and 11d supergravity theories and the on-shell action of these theories, consequently, being a boundary term. We will study the effects of codimension 2 brane sources on the lower dimensional curvature. Next we turn to the study of $3d$ SCFTs. We first discuss the superspace formalism for studying these theories, and in particular the construction of conserved higher spin currents in free $3d$ SCFTs. We then investigate the structure of 3-point functions of higher spin operators and the constraints of current conservation, extending earlier work of [35].

In this introductory chapter we will briefly review some underlying basic notions which should be useful for the later chapters.

1.1 Supergravity and de Sitter spacetime

Supergravity theories are supersymmetric theories where the global supersymmetry group is gauged. In this case the supersymmetry transformations depend on parameters which are (locally) space-time dependent. Such theories are theories of gravity where the graviton, described by the metric $g_{\mu\nu}$ has a supersymmetric counterpart- the gravitino ($\psi_{\mu\alpha}$). The actions of such theories, in varying number of dimensions, were constructed in the 1970's and provide a supersymmetric extension of the Einstein-Hilbert action by including terms corresponding to various bosonic/fermionic fields in the supergravity multiplet. We will be interested in the following basic supergravity theories from which most other supergravity theories (in lower dimensions) are naturally obtained by dimensional reduction.

IIA supergravity

This 10 dimensional supergravity theory has a spectrum comprising of the graviton (g_{ab}), dilaton (ϕ), the form potentials: A_a, B_{ab}, A_{abc} and two 16 component Majorana-Weyl spinors (of opposite chirality) in the fermionic part of the spectrum.

The action (bosonic part, in the Einstein frame) takes the form

$$S = -\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}\left(R + \frac{1}{2}(\partial\phi)^2 + \frac{e^{-\phi}}{2.3!}H_3^2 + \frac{e^{3\phi/2}}{2.2!}F_2^2 + \frac{e^{\phi/2}}{2.4!}\tilde{F}_4^2\right) - \frac{1}{4\kappa_{10}^2}\int B_2\wedge F_4\wedge F_4 \tag{1.1}$$

IIB supergravity

This 10 dimensional supergravity theory has a spectrum comprising of the graviton (g_{ab}), axio-dilaton (τ), the form potentials: B_{ab}, A_{ab}, A_{abcd} (the four form potential has a self-dual field strength) and two 16 component Majorana-Weyl spinors (of same chirality) in the fermionic part of the spectrum. Since the fermions are of same chirality, this theory is chiral.

The action (bosonic part, in the Einstein frame) takes the form

$$S_{IIB} = -\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}\left(R + \frac{\partial_A\bar{\tau}\partial^A\tau}{2(Im\tau)^2} + \frac{\bar{G}_3.G_3}{12.Im\tau} + \frac{\tilde{F}_5^2}{4.5!}\right) + \frac{1}{8i\kappa_{10}^2}\int C_4\wedge G_3\wedge\bar{G}_3 \tag{1.2}$$

This theory is self-dual under the action of the S-duality group $SL(2,\mathbb{R})$ (In IIB string theory, the duality group is a discrete subgroup of this group: $SL(2,\mathbb{Z})$)

The IIA and IIB supergravity arise (respectively) as the low energy limit ($\alpha' \to 0$) of the 10d IIA, IIB string theories. In this limit all massive stringy modes decouple (recall that the mass of the nth level $\sim n/\alpha'$) and one is left with the massless modes described by supergravity. The IIA and IIB string theories are also T-dual to each other.

11 dimensional supergravity

This is the unique supergravity theory in 11 dimensions. The spectrum comprises of the graviton g_{ab}, gravitino $\psi_{a\alpha}$ and a 3-form potential C_{abc} (a, b, c etc. are $SO(10,1)$ Lorentz indices while α is a 32 component spinor index). The bosonic part of the action of 11-D supergravity is

$$S = -\frac{1}{2\kappa_{11}^2}\int d^{11}x\sqrt{-g}\left(R + \frac{1}{2.4!}G_4^2\right) - \frac{1}{12\kappa_{11}^2}\int G_4\wedge G_4\wedge C_3 \tag{1.3}$$

On dimensional reduction on a circle one gets IIA supergravity from this theory. This theory is also the low energy limit of M-theory which includes in its degrees of freedom $M2$ and $M5$ branes. The 3-form is sourced by the $M2$ brane and the $M5$ brane is its magnetic dual.

de Sitter spacetime and no-go theorems

de Sitter spacetime is a maximally symmetric solution of Einstein's equations: $-R_{ab} = \Lambda g_{ab}$ with the cosmological constant $\Lambda > 0$.

Astronomical observations show that the Universe is currently in a period of accelerated expansion. If this is due to a cosmological constant the universe in the late time period would be in a de Sitter (dS) phase. Likewise several aspects of primordial cosmology are best explained by postulating an early "inflationary" phase of rapid accelerated expansion of the universe so that its early time behaviour was also dS to a fair degree of accuracy. This gives an added significance to understanding physics in de Sitter backgrounds. There are several aspects of de Sitter spacetime which are ill-understood. It possesses a cosmological event horizon and an associated temperature and entropy [1] which are hard to understand from a microscopic perspective. The Hilbert space of quantum gravity in de Sitter (dS) has been argued to be of finite dimension [2,3,4], a claim which is seemingly at variance with a proposed dS/CFT correspondence [5,6]. At a more basic level it is of interest to determine whether, in higher dimensional theories like supergravity and string theory, compactifications to de Sitter spacetime can be naturally obtained.

Kaluza-Klein compactifications of supergravity theories were extensively studied in the 70's and 80's with phenomenological applications in mind, and the particle spectrum and resultant possible compactified geometries investigated. There are no-go theorems, which we discuss and review next, which show that, under certain assumptions, such compactifications can not be realised as solutions of higher dimensional supergravity theories. The no-go result is that time independent compactifications of supergravity theories on compact manifolds with no singularities can't result in dS. Thus cosmological models of early (inflationary) and late (cosmological constant dominated) universe can not be based on such theories. As is usual with no-go theorems, it may be the case that altering the assumptions which go into the proof can potentially alter the conclusion. In particular, we aim to explore the effect of singular sources such as backreacting branes on the lower dimensional scalar curvature. Though we are not able to generate new dS solutions, we do extend the no-go theorems.

No-go Theorems on de Sitter compactifications in Supergravity

We consider warped compactifications in supergravity theories where the D dimensional spacetime is a warped product: $\mathcal{M}_D = X_d \times_w Y_{D-d}$ of a maximally symmetric d dimensional spacetime (X_d) and a compact $D-d$ dimensional space

(Y_{D-d}). The most general D dimensional line element that is consistent with d dimensional Poincaré invariance is

$$\begin{aligned} d\hat{s}^2 = \hat{g}_{MN}(x)dx^M dx^N &= \hat{g}_{\mu\nu}(x,y)dx^\mu dx^\nu + \tilde{g}_{mn}(y)dy^m dy^n \\ &= e^{2W(y)}g_{\mu\nu}(x)dx^\mu dx^\nu + \tilde{g}_{mn}(y)dy^m dy^n \end{aligned} \tag{1.4}$$

($W(y)$ is the warp factor) [1]. We have the relations

$$\hat{R}_{\mu\nu} = R_{\mu\nu} + \frac{e^{(2-d)W}g_{\mu\nu}}{d}\tilde{\nabla}^2 e^{dW} \tag{1.5}$$

$$\hat{R}_{mn} = \tilde{R}_{mn} + d[\tilde{\nabla}_m\tilde{\nabla}_n W + (\tilde{\nabla}_m W)(\tilde{\nabla}_n W)] \tag{1.6}$$

A simple no-go theorem [7] now follows if we take the compact manifold Y to have no boundaries (in particular no singular brane sources). It is known that the bosonic energy momentum tensor of all conventional supergravity theories obeys the Strong Energy Condition:

$$-\hat{R}_{MN}v^M v^N = \left(T_{MN} - \frac{\hat{g}_{MN}}{D-2}T\right)v^M v^N \geq 0 \tag{1.7}$$

for all timelike or null vectors v^M. In particular for v timelike and $\sim (1,0,0,....)$ we get

$$-\hat{R}_{00} = -R_{00} + \frac{e^{(2-d)W}}{d}\tilde{\nabla}^2 e^{dW} \geq 0 \tag{1.8}$$

Now X_d is maximally symmetric so $R_{\mu\nu} = \frac{R_d}{d}g_{\mu\nu}$. Multiplying the above equation by $e^{(d-2)W}\sqrt{\tilde{g}}$ and integrating over Y gives:

$$\frac{R_d}{d}V_W + \frac{1}{d}\int_Y d^{D-d}y\,\sqrt{\tilde{g}}\tilde{\nabla}^2 e^{dW} \geq 0 \tag{1.9}$$

[1] notation and convention: indices $M, N = 0, 1, \ldots, D-1$ run over all dimensions and give the coordinates on $\mathcal{M}_D$; greek indices $\mu, \nu = 0, 1, \ldots, d-1$ denote lower-dimensional coordinates(on X_d); and indices $m, n = 1, \ldots, n = D-d$ denote compactified coordinates (on Y_{D-d}). We use $\hat{\mathcal{R}}_{MN}$ to denote the D-dimensional Ricci curvature of the full D-dimensional metric, $\hat{g}_{MN}$; and $\hat{R}_{\mu\nu}$ to denote the d-dimensional Ricci curvature computed from the d-dimensional metric, $\hat{g}_{\mu\nu} = e^{2W}g_{\mu\nu}$. Also, $\hat{g}_D = \det\hat{g}_{MN}$ while $\hat{g}_d = \det\hat{g}_{\mu\nu}$ *etc.*. Also we use a 'mostly plus' metric and Weinberg's curvature conventions [55], which differ from those of MTW [56] only in the overall sign of the definition of the Riemann tensor. This means that it is the scalar curvature $-R$ that would be positive for dS and negative for AdS.

Here $V_W = \int_Y d^{D-d}y\sqrt{\tilde{g}}e^{(d-2)W}$ is the warped volume (it is the ratio of the D dimensional and d dimensional Newton's constants). The integrand above is a total derivative and so, for Y compact without boundary, does not contribute. We thus get

$$-R_d \leq 0 \tag{1.10}$$

so that X_d is necessarily Anti-de Sitter or Minkowski.

This shows that with our assumptions - of time independent non-singular compactification without boundary - the higher dimensional theory has to violate the strong energy condition to obtain dS on compactification. In fact, even an accelerating cosmological model more general than dS such as one given by an FRW metric: $ds^2 = -dt^2 + a^2(t)ds^2_{(3)}$ (this is a time dependent compactification) can not be obtained without violating the strong energy condition. Since $R_{00} = \partial_0^2 a$ here and acceleration implies $-R_{00} < 0$ this means that the strong energy condition has to be violated. The strong energy condition basically demands that gravity be locally attractive, so it is reasonable that using matter fields (like supergravity p-form fields) which obey it one can not get accelerating (deSitter) spacetimes.

Maldacena and Nunez [8] considered, along similar lines, a general higher dimensional supergravity lagrangian (with a potential for the scalars) with the following assumptions:

1) There are no higher derivative (for eg. stringy) corrections - the gravitational part of the action is the usual Einstein Hilbert form. This means we work in the supergravity (zero slope $\alpha' \to 0$) limit of string theory.

2) The kinetic terms of the p-form fields are positive.

3) The scalar potential is non-positive

4) Only the bosonic sector of the supergravity theories is considered.

5) The manifold Y_{D-d} is compact without boundary (Actually singularities which are such that the warp factor goes to zero on approaching them are allowed. These are singularities which may have a dual field theory interpretation [8].)

As [8] show these conditions imply that de Sitter spacetime can not be obtained through compactification.

Of course if any of the above assumptions are evaded then we may potentially realise positive curvature solutions. Although one may like to have a de Sitter realisation within a fully non-perturbative (finite α' and g_s) string/M theoretic framework it is typically quite hard to go much beyond assumption 1 and we will here not attempt to work beyond the supergravity regime. Assumption 2 seems quite reasonable though it is violated in Hull's II*A,B theories (obtained by T-dualising IIB,A on a timelike circle) where de Sitter compactifications (for eg. $dS_5 \times H_5$ in II*B, H being the hyperbolic space) are possible [9], see also [10]. However these theories seem to be ill-defined because of the negative sign kinetic term for the $R - R$ fields.

Assumption 3 would be violated, for example, if we start with a supergravity theory in higher dimensions with a positive cosmological constant. In $6d$ gauged supergravity with a positive (exponential) potential explicit $4d$ dS solutions have been constructed [14]. Typically we would expect a potential only to be generated through compactification and we will take the higher dimensional theory to be without an arbitrary potential.

It is to be noted that the strong energy condition is a quite strong one and violated by physically realistic systems (unlike, for example, the null energy condition). By incorporating fermions and thus coupling the theory to matter this condition can be violated. We will however take assumption 4 to hold.

The manifold Y_{D-d} being compact implies the lower dimensional Newton's constant (G_d) is finite. If one consider Y to be non-compact, in particular hyperbolic, then it is possible to get de Sitter solutions. In such a case, however, it is not clear how to obtain a discrete d dimensional spectrum. We willl take Y to be compact but allow it to have boundaries (thus partially evading assumption 5). In particular, we willl consider the boundaries to be singularities (of a type more general than allowed in 5) in the compact manifold Y due to the presence of brane sources. It may be noted that in the above no-go theorems the p-form field potentials which contribute to T_{MN} are included but the $(p-1)$-branes which source these fields are considered to be probe branes with negligible effect on the ambient geometry. We may, however, wish to include the effects of brane backreaction.

We also keep ourselves to considering only time-independent compactifications. Note that the above theorems need not hold if we consider Y to be Lorentzian and X a maximally symmetric space- such an accelerating cosmology would be a time-dependent compactification[10,11]. From the viewpoint the lower (d) dimensional observer this would give rise to time-dependent scalar (moduli) fields. In this case one considers more general time dependent compactifications along the lines of [15,16,17]. These authors discuss the constraints on realising accelerating cosmologies from higher dimensional compactifications. Considering two derivative higher dimensional theories compactified on a manifold without boundary which is flat (in the sense of having zero Ricci curvature scalar) or is conformally flat, they show that obtaining accelerating cosmologies requires violations of the null energy condition. More precisely, for an FRW cosmology with equation of state parameter w the null energy condition requires that there exists a threshold value w_{th} (which depends on the number of compact dimensions) such that $-1 \leq w \leq w_{th}$ and for which the number of e-foldings is bounded from the above (also this number goes to zero as $w \to -1$ and thus dS can not be realised). Thus, only transient acceleration can be obtained, as also shown earlier in [10], and in particular we can not have a dark energy due to a cosmological constant only. The maximum number of e-foldings possible is also too small to get a realistic description inflation. We note however that if we consider the compact manifold to have singularities or if it is not Ricci flat or conformally Ricci flat then realizing cosmic acceleration may not require violations of the null energy condition.

1.2 Higher spin operators in CFTs

Conformal Field Theories (CFTs) are of prime importance in theoretical physics for several reasons. They are important in the study of phase transitions as various statistical mechanical systems at criticality are described by CFTs. This historically was the principal reason for their introduction and motivation for their study. They describe fixed points of renormalization group flows and general QFTs can be defined and studied through deformations of CFTs by marginal operators. Through the AdS/CFT duality they map holographically to higher dimensional quantum theories of gravity and thus provide a non-perturbative construction of such theories. We give below, a brief overview of some well known basic CFT concepts.

The symmetry group of a CFT (the *Conformal Group*) is $SO(D,2)$ in D space-time dimensions ($D \geq 3$). All fields transform in representations of $SO(D,2)$. Representations are labelled by Cartans of the maximal compact subgroup $SO(D) \times SO(2)$: $\mathcal{R}$ and dimension Δ. In particular, for the 3 dimensional case we will be dealing with, the conformal group is $SO(3,2)$ (isomorphic to $OSp(2,\mathbb{R})$) with representations being labelled by Δ and the $SO(3)$ spin s. For 3d SCFTs with $\mathcal{N}$ extended supersymmetry, the supergroup of superconformal symmetries is $OSp(2,\mathbb{R}|\mathcal{N})$ which has the maximal compact bosonic subgroup $SO(2) \times SO(3) \times SO(\mathcal{N})$ with the associated Cartan charges labelling the representations: (Δ, s, h_i), h_i being the $SO(\mathcal{N})$ Cartan charges ($SO(\mathcal{N})$ is the R-symmetry group).

CFT Definition (usual): One considers local fields transforming in a representation $\mathcal{R}$ and the Action (more generally, Path Integral) invariant under this transformation on the field variables. This is a perturbative definition- the usual way QFTs are defined, about weakly coupled saddle points of the path integral.

For CFTs it is possible to give a *non-perturbative* definition by giving the spectrum of all local primary operators together with the Wilson coefficients $[O_\Delta, \mathcal{R}, c_{ijk}]$. Indeed many CFTs do not have any lagrangian description. This includes the (2,0) SCFT which is central to M-theory and describes $M5$ brane dynamics, and many other $\mathcal{N}=2$ 4d SCFTs (of the so-called S class) which can be obtained from the compactification of the (2,0) theory on a Riemann surface with punctures.

The CFT spectrum comprises of local primary operators O_Δ ($[K_\mu, O_\Delta] = 0$) with scaling dimension Δ; and representation $\mathcal{R}$ of $SO(D)$ in which O_Δ transforms (and the R-charges for an SCFT). All the local operators are in a one to one correspondence with states in the radial quantization scheme via the state-operator map.

The dynamical content of a CFT is encoded in the Wilson coefficients via the Operator Product Expansion:

$$O_i(x)O_j(0) = \sum_k c_{ijk} F(x, \partial_y) O_k(y) \mid_{y=0} \tag{1.11}$$

The OPE is an exact operator relation (with a finite radius of convergence) in any

CFT, unlike the usual case in QFTs where it is an asymptotic expansion.

Unitarity imposes additional constraints on the spectrum in terms of lower bounds on the dimensions of primaries: $\Delta \geq \Delta_{min}(\mathcal{R})$

Conformal symmetry is quite constraining. It fixes the form of the 2 and 3 point functions of scalar conformal primary operators. The form of the 2-point function is:

$$\langle \phi_\Delta(x_1)\phi_\Delta(x_2)\rangle = \frac{k}{x_{12}^{2\Delta}} \tag{1.12}$$

and we may normalise to set $k = 1$.

With the 2-point function normalised, the 3-point function is also completely fixed upto an overall constant

$$\langle \phi_{\Delta_1}(x_1)\phi_{\Delta_2}(x_2)\phi_{\Delta_3}(x_3)\rangle = \frac{c_{123}}{x_{12}^{2\alpha_{123}}x_{23}^{2\alpha_{231}}x_{31}^{2\alpha_{312}}} \tag{1.13}$$

with $\alpha_{ijk} = \frac{\Delta_i+\Delta_j-\Delta_k}{2}$

The overall constant c_{123} - a three point coupling, is not arbitrary but encodes dynamical information about the theory.

The spectrum together with the Wilson coefficients comprise the *CFT data* and its knowledge completely specifies the CFT. This is because the OPE can in principle be used recursively to reduce an n-point function of local primary operators to a sum of products of 2-point functions with various derivative operations. The Wilson coefficients being known, this expression is completely determined. Furthermore, since any descendent is determined by the action of some number of derivatives on a primary, it follows that the the n-point functions of all local operators are completely known.

However the operator dimensions and Wilson coefficients are not arbitrary real numbers. Apart from the constraints of unitarity they are constrained by OPE associativity (also called crossing symmetry) seen at the level of 4-point functions. 4-point functions are not fixed by conformal symmetry on kinematic grounds but their functional form is quite constrained.

$$\langle \phi_\Delta(x_1)\phi_\Delta(x_2)\phi_\Delta(x_3)\phi_\Delta(x_4)\rangle = \frac{1}{x_{12}^{2\Delta}x_{34}^{2\Delta}} f(u,v) \tag{1.14}$$

where u, v are the conformal cross-ratios:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \tag{1.15}$$

The function f can be expanded in terms of *conformal blocks*

$$f(u,v) = \sum_O c_O g_O(u,v) \tag{1.16}$$

The sum is over all the primaries in the spectrum and the conformal block - $g_O(u,v)$ encodes the contribution of the exchange of O within the 4-point function and all its conformal descendents.

Crossing symmetry (OPE associativity) states that one can do OPE contraction of different operators within the correlation function- and different ways should give same results. This leads to further constraints on f in the form of the *bootstrap equation*:

$$v^\Delta f(u,v) = u^\Delta f(v,u) \tag{1.17}$$

The basic idea of the bootstrap approach to QFTs is to use general principles like Symmetries, Unitarity, Analyticity, to determine physical observables of interest which may be S matrices. In CFTs one uses unitarity and crossing symmetry to constrain the correlators. Note that since u, v can take arbitrary real values, and the function f can be expanded using the OPE in terms of products of OPE coefficients (the conformal block expansion of the 4-point function), the above bootstrap equation in effect gives an infinite number of equations in infinitely many variables (OPE coefficients and operator dimensions). In general there is no way known to solve them but in special cases, for example 2d CFTs where the finite dimensional $SO(2,2)$ is in fact extended to the infinite dimensional Virasoro group one can find an explicit solution- these are the well known Minimal Model solutions of 2d CFTs with central charge $c < 1$ [23]

For CFTs with higher spin operators, the 2-point function is again completely fixed by conformal symmetry

$$\langle O_{s,\Delta}(1) O_{s,\Delta}(2) \rangle = \frac{unique\ tensor\ structure}{x_{12}^{2\Delta}} \tag{1.18}$$

The 3-point function is determined as a sum of a finite number of tensor structures with undetermined constant coefficients

$$\langle O_{s_1,\Delta_1}(1) O_{s_2,\Delta_2}(2) O_{s_3,\Delta_3}(3) \rangle = \frac{finitely\ many\ tensor\ structures}{x_{12}^{2\alpha_{123}} x_{23}^{2\alpha_{231}} x_{31}^{2\alpha_{312}}} \tag{1.19}$$

The 4-point functions of higher spin primary operators have not been extensively investigated (other than some work on spin 1 and spin 2 four-point functions).

In this thesis we will be dealing with superconformal field theories (SCFTs). These are special CFTs which additionally also have supersymmetry. Apart from the generators of the conformal group, the symmetry generators in this case include the supersymmetry generators (Q^a_α) and the generators of special superconformal transformations (S^a_α). The differential form of the action of all the symmetry generators in superspace (for $3d$ SCFTs) is given by eq. (3.1) in Chapter 3 and the full superconformal algebra is given by eq. (3.56) in an appendix to that chapter.

Superconformal symmetry provides additional constraints on the field theory. Superconformal representations are classified by superconformal primaries - these are lowest weight states annhilated by S^a_α (besides K_μ). The raising operator here is Q^a_α (like P_μ in the conformal case). Due to the nilpotent nature of the action of Q's, the superconformal multiplets are necessarily finite-dimensional and a single representation of the superconformal algebra headed by a superconformal primary contains within it many conformal primaries (its Q descendants), and hence many conformal representations. We will discuss in greater detail in Chapters 3 and 4, SCFTs in three dimensions and particularly their superspace formulation and correlators of higher spin operators.

Significance of higher spin operators and the Maldacena-Zhiboedov theorem

It is expected that CFTs which have any additional higher spin symmtery, and corresponding conserved higher spin currents, would be free. This is analogous to the Coleman-Mandula theorem for Poincaré symmetry. In 3 dimensions it was proven recently by Maldacena and Zhiboedov [31]. Under the assumptions of a unique spin 2 conserved current and finitely many degrees of freedom (finite N) for the CFT they showed, using lightcone OPE methods, that the existence of a single higher spin ($s > 2$) current suffices to demonstrate the existence of an infinite tower of higher spin currents. Furthermore n-point correlators of such conserved higher spin currents factorise into products of 2-point functions which signals that the theory is free.

In chapter 3 we formulate superspace methods for studying free $3d$ SCFTs and construct explicitly the higher spin currents that these theories have in terms of free superfields.

In subsequent work [32] QFTs with exact conformal symmtry but *weakly broken* higher spin symmtery ($1/N$ corrections being the source of symmetry breaking) were considered. Such theories are interacting- indeed a plethora of examples is known starting from the basic $O(N)$ vector model and including various superconformal Chern-Simons theories like ABJ theory. At large N, there is a weakly broken higher spin symmetry with an anomalous "conservation" law

$$\partial \cdot J_{(s)} = \frac{1}{N} J_{(s_1)} J_{(s_2)} + higher\ trace\ terms\ if\ possible \tag{1.20}$$

Here $s > 2$ (the energy-momentum tensor is always exactly conserved). This controlled breaking of higher spin symmetry in large N vector models can be used to further constrain correlators of these interacting CFTs as demonstrated in [32]

The Virasoro Algebra provides an infinite dimensional extension of the Conformal Algebra in two dimensions and enables the implementation of the conformal bootstrap - the $2d$ Minimal Model exact solutions [23]. In higher dimensions, Virasoro symmetry is lacking. However, it appears from recent work [31], [32] that higher spin symmetry might play an analogous role. The difference is that while $2d$

CFTs with exact Virasoro symmetry can be non-trivial, in $d > 2$ CFTs with exact higher spin symmetry are free, as shown in [31]. However, as mentioned above, CFTs can have a parametrically small weakly-broken higher symmetry, and this provides further constraints. This was seen at the level of 3-point functions in [32] but the same analysis is expected to work for higher correlators. It may thus be feasible that judicious use of (weakly broken) higher spin symmetry can be used for the conformal bootstrap (at least for large N) of higher dimensional CFTs.

1.3 Holographic interpretations

As is well known, the AdS/CFT correspondence [20] states that conformal quantum field theories can be holographically dual to certain quantum gravity theories in AdS backgrounds in higher dimensions. In particular, 4d $\mathcal{N} = 4$ super Yang-Mills theory is holographically dual to IIB string theory on $AdS_5 \times S^5$ and the 3d $\mathcal{N} = 6$ ABJ superconformal Chern-Simons theory is dual to IIA string theory on $AdS_4 \times \mathbb{CP}^3$. This is a strong-weak coupling duality, and in general the tractable domain is where the bulk/boundary theory is weakly coupled. In particular the strongly coupled large N, large λ limit of a CFT is well described by an AdS bulk geometry where the effective gravitational dynamics is that of Einstein gravity. In this limit, the equivalence $Z_{CFT} = Z_{QG}$ between partition functions becomes $Z_{CFT}[J] = exp(-S_{os} + \int J.\phi)$, which is the well-known GKPW prescription [22, 21] for computing correlators of strongly coupled CFTs (S_{os} is the on-shell action). This is the most extensively studied corner of the AdS/CFT duality.

It is of interest to determine what kinds of CFTs admit holographic duals with a geometric description. In other words, under what conditions is the dynamics of the CFT encoded in a metric based semi-classical description of a gravitational theory. This has been investigated [33, 34] and it is known that such a bulk geometric interpretation exists whenever there is a large parameter in the CFT such that the dimensions of a few (low spin) primary operators (the single-trace primary operators) do *not* become parametrically large as $N \to \infty$. This 'gap' in the spectrum, i.e, the existence of a level of low dimension primary operators ensures a dual geometric description in terms of an effective semi-classical gravitational description. $1/N$ corrections in the field theory amount to quantum corrections in the bulk theory.

In CFTs where there are infinitely many higher spin single-trace primary operators of minimal twist ($\tau = \Delta - s$) we do not expect the holographic duals to be classical bulk geometries described by the Einstein-Hilbert action, since Einstein gravity contains a unique spin two massless graviton and no higher spin massless particles.

Higher spin bulk theories and CFTs with higher spin operators

The holographic duals to CFTs with a tower of higher spin operators are theories of interacting higher spin massless fields in AdS. Although such theories do not exist in flat space-time (as demonstrated by no-go theorems proved by Weinberg [46]), the presence of a cosmological constant (dS/AdS spacetime) allows interacting massless higher spin theories to exist. The existence of such theories can also be inferred from string theory. In the usual infinite tension ($\alpha' \to 0\,,\, T \sim 1/\alpha' \to \infty$) supergravity limit of string theory, all massive stringy modes decouple (recall that the mass of the nth level $\sim n/\alpha'$) and one is left with the massless modes whose dynamics is described by supergravity. The opposite limit, the tensionless limit, is when the AdS curvature scale is much smaller than the string length ($R/l_s \ll 1$, which is the same as $\alpha' \to \infty$). In this limit all the massive levels become massless and one expects a complicated interacting theory of infintely many massless modes which captures the dynamics of string theory in the extreme stringy regime. Vasiliev has constructed a non-linear theory of interacting massless higher spin fields in AdS [39] and this construction is expected to be the tensionless limit of classical string theory (though this has not been demonstrated yet).

It was conjectured by Klebanov and Polyakov [24] that the bosonic $O(N)$ vector model (a $3d$ CFT) is dual to Vasiliev (type A) theory. When the singlet scalar in the theory has mimimal dimensionality $\Delta = 1$ we have a free bosonic CFT whereas for $\Delta = 2$ the theory is the critical $O(N)$ model (obtained by RG flow, from the free CFT to the Wilson-Fisher fixed point, triggered by the relevant deformation $(\phi.\phi)^2$). Similarily the type B Vasiliev theory is dual to the fermionic $O(N)$ vector model [25] - for $\Delta = 2$ we have the free fermion CFT whereas for $\Delta = 1$ the critical theory - the Gross-Neveu model. Thus these $3d$ CFTs are dual to higher spin theories (with even integer spin fields) where the boundary conditions (on the boundary of AdS) preserve the higher spin symmetry. It is also possible to choose boundary conditions which (weakly) break the higher spin symmetry (at $O(1/N)$ by multi-trace terms) and such theories have as boundary duals interacting $3d$ CFTs which are Chern-Simons gauge theories with bosons/fermions transforming in the fundamental (vector) representation of the gauge group. Examples include the $U(N)$ Chern-Simons theories studied in [26], [27]. Although supersymmetry is not an essential ingredient of the vector model/ higher spin duality one can indeed consider supersymmetric versions of Vasiliev's theory which would have superconformal field theories as duals [30, 29]. Although we will not explicitly discuss these theories in great detail in this thesis, the material presented in Chapters 3 and 4 - regarding the superspace formalism, higher spin operators and correlation functions - is of much revelance to their study.

Chapter 2

Supergravity compactifications and dS no-go theorems

de Sitter space, or slow-roll geometries close to de Sitter space, appear to play an important role in cosmology. This has motivated searching for explicit solutions to the higher-dimensional field equations for which the large four dimensions we see are de Sitter or de Sitter-like. Although a few such solutions are known [47, 48], more and more general no-go results [49, 50, 51, 52] show that such solutions are difficult to find[1] It is interesting to enquire about the reasons for this.

In this chapter we argue that part of the problem is that we are not yet using all of the ingredients that de Sitter solutions may require. In particular, contributions have been neglected that are the same size as some of the contributions that are usually kept when searching for (or ruling out) de Sitter-like solutions.

The neglected contributions come from the actions of any localized sources that may be present in the extra-dimensional configurations of interest. In particular, we argue here that for codimension-two sources these actions contribute to the curvature an amount that is competitive with the contribution of the bulk fields, including their back-reaction. In particular, the source action acts to systematically cancel the contribution from the warping of the noncompact geometry

[1] Four-dimensional effective field theories of string theory including non-perturbative effects and anti branes or D-terms [53] can give rise to de Sitter solutions. But at the moment there is no full understanding from the microscopic higher-dimensional theory. For other recent attempts for de Sitter solutions see [54].

across the extra dimensions. This is important because the sign of the warping contribution is usually definite, and because it is opposite to what is required for a de Sitter noncompact geometry it plays a role in the various extant de Sitter no-go results.

We study the effects of brane backreaction, source properties and bulk singularities on obtaining de Sitter compactifications in higher dimensional supergravity theories. We show how the lower dimensional scalar curvature (the cosmological constant) is determined by the on-shell bulk action, warping effects, source action and space-filling fluxes and is, in certain quite general cases, a sum over boundary terms and thus determined by the asymptotic form of the bulk fields in the near-brane limit. As an application we show that all codimension 2 brane solutions (warped or unwarped) in axio-dilaton-metric theories are flat.

This chapter is organised as follows. We first discuss the no-go theorems on de Sitter compactifications proved in the introduction. We then show how, in warped compactifications, the curvature of the compact manifold constrains the curvarure of the non-compact maximally symmetric part. In section 2.3 we establish our main result: a general expression that relates the lower dimensional scalar curvature to the on-shell bulk action of a theory and also includes effects due to warping, source action and any space-filling fluxes which might be present. In order to be able to put this relationship to use we show, in section 2.4, how the on-shell action of a theory with a classical scaling symmetry is just a boundary contribution. We show that the actions of 11-D supergravity, IIA, IIB supergravity (respectively) have this scaling behaviour and we explicitly evaluate the on-shell action as a sum over boundary contributions.

As an application we consider on-brane geometries for codimension 2 brane sources. Explicit analytical expressions for unwarped D7 brane solutions in IIB supergravity (axio-metric-dilaton sector) are known and are 8 dimensional flat. We show that even after incorporporating the effects of warping and source effects the solutions are still flat, thus generalising the result.

2.1 No-go results and the 6D loophole

Our interest is in D-dimensional metrics of the form

$$\mathrm{d}s^2 = \hat{g}_{MN}\,\mathrm{d}x^M \mathrm{d}x^N = e^{2W(y)}\,g_{\mu\nu}(x)\,\mathrm{d}x^\mu \mathrm{d}x^\nu + \tilde{g}_{mn}(y)\,\mathrm{d}y^m \mathrm{d}y^n\,, \tag{2.1}$$

where $D = d + n$; the d-dimensional metric, $g_{\mu\nu}$, is maximally symmetric (*i.e.* flat, de Sitter or anti-de Sitter); and the warp factor, W, can depend on position in the n compact directions (whose metric, $\tilde{g}_{mn}$, is so far arbitrary).

In particular, for cosmological applications there is much interest in identifying solutions to higher-dimensional field equations for which $g_{\mu\nu}$ is a de Sitter metric (which in our curvature conventions [2] satisfies $R = g^{\mu\nu}R_{\mu\nu} < 0$). The search for such solutions has been fairly barren, and this is partly explained by refs. [49], [50], [51] and [52], who identify increasingly general obstacles to finding this type of de Sitter solution to sensible, higher-dimensional, second-derivative field equations.

On the other hand, a handful of explicit solutions of this type *do* exist, including 4D de Sitter solutions [47] for six-dimensional Maxwell-Einstein systems,

$$S_{ME} = -\int \mathrm{d}^6x\sqrt{-\hat{g}}\;\left\{\frac{1}{2\kappa^2}\,\hat{g}^{MN}\hat{\mathcal{R}}_{MN} + \frac{1}{4}\,\mathcal{F}_{MN}\mathcal{F}^{MN} + \Lambda\right\}\,, \tag{2.2}$$

with positive 6D cosmological constant, Λ. Similar solutions [48] also exist for six-dimensional gauged, chiral supergravity [57], whose relevant bosonic action is

$$S_{\rm bulk} = -\int \mathrm{d}^6x\sqrt{-\hat{g}}\;\left\{\frac{1}{2\kappa^2}\,\hat{g}^{MN}\Big(\hat{\mathcal{R}}_{MN} + \partial_M\phi\,\partial_N\phi\Big) + \frac{1}{4}\,e^{-\phi}\mathcal{F}_{MN}\mathcal{F}^{MN} + \frac{2\,g_R^2}{\kappa^4}\,e^{\phi}\right\}\,. \tag{2.3}$$

For both of these actions $\hat{\mathcal{R}}_{MN}$ denotes the Ricci tensor for the 6D metric, $\hat{g}_{MN}$, and $\mathcal{F} = \mathrm{d}\mathcal{A}$ is the field strength for a 6D gauge potential, $\mathcal{A}_M$. The quantity $\kappa^2 = 8\pi G_6$ denotes the 6D gravitational coupling, while for the supersymmetric case g_R denotes the gauge coupling of a specific $U_R(1)$ gauge group that does not commute with 6D supersymmetry.

These examples do not contradict the various no-go theorems because they arise in systems which do not satisfy one of the assumptions of each. For instance,

[2] We use a 'mostly plus' metric and Weinberg's curvature conventions [55], which differ from those of MTW [56] only in the overall sign of the definition of the Riemann tensor.

the no-go result of [50] assumes that any extra-dimensional scalar potential must be negative (as it tends to be for higher-dimensional supergravities, but is not so for eqs. (2.2) and (2.3)). They evade the less restrictive assumptions of [51] and [52], some of which exclude [52] having only two extra dimensions, $n = 2$. More importantly they do not satisfy the average 'boundedness' assumptions [51] that exclude solutions that are too singular.

The potential relevance of back-reaction

There are two ways to view the possibility that singular behaviour can suffice to evade the no-go results. One view is to regard solutions with such singularities as unacceptable, and so draws the conclusion that de Sitter solutions may be impossible to find. And for some types of singularity (like negative-mass black holes) this is probably right, since the alternative requires admitting energies that are unbounded from below.

But some (apparent) singularities are known to be perfectly sensible, such as those seen in Coulomb's law at the position of a source charge. In the case of Coulomb's law, the singularity doesn't preclude taking the solution seriously because we don't intend to trust the solution in any case right down to zero size. The existence of apparent singularities might similarly be expected to arise in the gravitational theories relevant to cosmology, provided these are regarded as effective descriptions of some more-microscopic degrees of freedom. One can hope to get a handle on deciding whether a singularity might be reasonable for an effective description, by seeing what kinds of apparent singularities actually can emerge from localized sources governed by physically reasonable actions.

These considerations suggest that understanding the back-reaction of localized sources could be a crucial part of obtaining de Sitter solutions, or ruling them out. In particular the asymptotics, and apparent divergence, of bulk fields near a source is likely to be important, and is ultimately controlled by the action that describes the dynamics of that source. Notice for these purposes 'source' need not mean a fundamental object, like a D-brane. Rather, it could describe something more complicated, like a soliton or a higher-dimensional brane wrapping internal dimensions or a localized but strongly warped region. All we need know is that the sources are much smaller than the extra dimensions within which they sit.

How the properties of a source affect the properties of bulk fields is best understood at present for codimension-one and codimension-two sources. For codimension-one sources, the back-reaction is described by the Israel junction conditions [58], as is familiar from Randall-Sundrum models [59]. But bulk fields with codimension-one sources also tend not to diverge at the source positions, and so shed little light on how such singularities influence the low-energy curvature. It is only for higher-codimension sources that it is generic that bulk fields diverge at the source positions, and so where the relation between bulk singularity and source properties can be explored.

Of course, these bulk singularities make matching bulk solutions to source properties more complicated, usually requiring a renormalization of the source [60]. The tools for detailed bulk-source matching and renormalization are most explicitly known for codimension-two objects [61, 62, 63, 64, 65]. In particular, these tools have recently been used to identify [66] explicit objects that can source the de Sitter solutions [48] of the 6D supergravity action, eq. (2.3). Since the required source properties seem physically reasonable,[3] they show that the singularities in the corresponding bulk solutions need not be regarded as grounds for their rejection.

2.2 Constraints on scalar curvature of X due to that of Y

We discuss here how the scalar curvature of X (the non-compact maximally symmetric d-dimensional spacetime) is constrained by that of Y (the compact $D-d$ dimensional manifold) if we require X to be dS. In [12] it was noted that the scalar curvature ($-R_d$) of X gets a positive contribution from a negative scalar curvature ($-R_{D-d}$) of Y. We'll derive here a simple relationship between R_d, R_{D-d} ($\equiv \tilde{g}^{mn} R_{mn}$) and T_d ($\equiv g^{\mu\nu} T_{\mu\nu}$ - the d dimensional trace of the energy momentum

[3] As discussed in more detail below, their worst feature appears to be a requirement that the dilaton, ϕ, grows as one asymptotically approaches the sources, and so care must be taken to avoid leaving the weak-coupling regime before reaching the source.

tensor) for a manifold $\mathcal{M}_D = X_d \times_w Y_{D-d}$ with total energy-momentum tensor T_{MN}

We will use equations (1.5) and (1.6). The D dimensional Einstein equation is [4]

$$-\hat{R}_{MN} = T_{MN} - \frac{\hat{g}_{MN}}{D-2} T_D \tag{2.4}$$

Consider first the d dimensional $(\mu\nu)$ components of this equation. Since $\hat{g}_{\mu\nu} \equiv \bar{g}_{\mu\nu} = e^{2W} g_{\mu\nu}$ we have

$$-\hat{R}_{\mu\nu} = T_{\mu\nu} - \frac{e^{2W} g_{\mu\nu}}{D-2} T_D \tag{2.5}$$

Now using eq. (1.5) and that

$$T_D \equiv \hat{g}^{MN} T_{MN} = \hat{g}^{\mu\nu} T_{\mu\nu} + \hat{g}^{mn} T_{mn} = e^{-2W} T_d + T_{D-d} \tag{2.6}$$

(Here $\hat{g}_{mn} = \tilde{g}_{mn}$ and $T_{D-d} \equiv \tilde{g}^{mn} T_{mn}$) we get

$$-R_{\mu\nu} - \frac{e^{(2-d)W} g_{\mu\nu}}{d} \tilde{\nabla}^2 e^{dW} = T_{\mu\nu} - \frac{e^{2W} g_{\mu\nu}}{D-2} (e^{-2W} T_d + T_{D-d})$$

Contracting the above equation with $g^{\mu\nu}$ gives

$$-R_d = \frac{D-d-2}{D-2} T_d - \frac{d e^{2W}}{D-2} T_{D-d} + e^{(2-d)W} \tilde{\nabla}^2 e^{dW} \tag{2.7}$$

Likewise, we consider the $D-d$ dimensional (mn) components of eq. (2.4)

$$-\hat{R}_{mn} = T_{mn} - \frac{\tilde{g}_{mn}}{D-2} T_D \tag{2.8}$$

Now contracting equation (1.6) with $\hat{g}^{mn}$ and simplifying gives

$$\hat{g}^{mn} \hat{R}_{mn} = R_{D-d} + d \frac{\tilde{\nabla}^2 e^W}{e^W} \tag{2.9}$$

so eq. (2.8) upon contraction and using the above equation leads to

$$-R_{D-d} = \frac{d-2}{D-2} T_{D-d} - \frac{D-d}{D-2} e^{-2W} T_d + d \frac{\tilde{\nabla}^2 e^W}{e^W} \tag{2.10}$$

Eliminating T_{D-d} between eqs. (2.7) and (2.10) gives us

$$-R_d = -\frac{2}{d-2} T_d + \frac{d e^{2W}}{d-2} R_{D-d} + e^{(2-d)W} \tilde{\nabla}^2 e^{dW} + \frac{d^2}{d-2} e^W \tilde{\nabla}^2 e^W \tag{2.11}$$

[4] Here T_D is the D dimensional trace of the energy-momentum tensor: $T_D \equiv \hat{g}^{MN} T_{MN}$.

Now multiplying the above equation by $\sqrt{\tilde{g}}e^{d-2W}$ and integrating over Y_{D-d} gives us the desired relation:

$$-R_d V_W = -\frac{2}{d-2}\int d^{D-d}y\sqrt{\tilde{g}}e^{(d-2)W}T_d + \frac{d}{d-2}\int d^{D-d}y\sqrt{\tilde{g}}e^{dW}R_{D-d} + \int d^{D-d}y\sqrt{\tilde{g}}\tilde{\nabla}^2 e^{dW} + \frac{d^2}{d-2}\int d^{D-d}y\sqrt{\tilde{g}}e^{(d-1)W}\,\tilde{\nabla}^2 e^W \tag{2.12}$$

We note that for $d > 2$ the contribution of a negative T_d and negative $-R_{D-d}$ to $-R_d$ is positive. In the case of no warping ($W = 0$), $d = 4$ and $D = 10$, so that we are considering 4 dimensional compactifications of 10 dimensional supergravity theories the above relation reduces to eq. (1.1) of [13]:

$$-R_4 = -T_4 + 2R_6 \tag{2.13}$$

Thus even if some kind of energy condition enforces the positivity of T_d, it may be possible to compensate for this by having compact manifolds with scalar curvature everywhere negative thus leading to positive curvature for the d-dimensional space-time. Many compactications are known with X a Minkowski or AdS space-time and Y a manifold with non-negative scalar curvature. For example, $AdS_5 \times S^5$ in IIB string theory, $AdS_4 \times S^7$ in 11D supergravity (here Y has positive scalar curvature) and $\mathcal{M}_4 \times CY_3$ in heterotic string theory (here Y- the Calabi-Yau manifold- is Ricci flat and so the scalar curvature is zero). However, it seems quite difficult to realise compactifications with Y having negative scalar curvature. Finding any such solutions would help in realising de Sitter compactifications.

Summary of results

We examine how source back-reaction constrains the existence of de Sitter solutions in more general higher-dimensional theories than the six-dimensional ones already explored.

In particular, we explore some of these issues in eleven-dimensional supergravity, and in ten-dimensional Type IIB and Type IIA supergravity. Because our best-developed tools apply to codimension-two objects, it is these we largely explore in detail. If only D-branes were allowed as sources, this would restrict us to

$D7$-branes in Type IIB systems. But we also explore the other supergravities for two reasons: because some of our results apply equally well to higher-codimension sources; and because our sources might not be D-branes — or (p,q) branes for that matter — but instead be more complicated localized codimension-two quantities (like very small warped throats).

We find the following results:

- First, for geometries of the form of eq. (2.1), we find a very general classical relationship that gives the curvature in the non-compact dimensions parallel to the sources as the sum of four terms: $R \propto I + II + III + IV$, where IV vanishes for maximally symmetric geometries in the absence of space-filling fluxes.

- Second, we show that contribution I — which is proportional to the bulk action evaluated at the classical back-reacted solution — is very generally given as the integral of a total derivative, and so is controlled by the boundary values of a particular combination of bulk fields. This property relies only on the existence of a classical scale invariance that is shared by most higher-dimensional supergravities (and holds in particular for 11D and 10D Type IIA and IIB supergravity).

- Third, we show that for codimension-two sources the contributions II and III cancel one another. Here contribution II is an integral over a total derivative of the warp factor, W, whose definite sign plays an important role in the derivation of the general no-go results. Contribution III comes from the action of the localized source, which is left out of most no-go analyses.

- Finally, we explicitly identify the total derivative that appears in I for several examples of interest, including commonly used supergravities in 6, 10 and 11 dimensions. This identifies the combination of fields whose near-brane asymptotics is relevant to the low-energy curvature. As a simple application we show that the noncompact dimensions are always flat for all F-theory compactifications that involve only the metric and axio-dilaton with codimension-two sources.

These results carry two important messages. First, that back-reaction cannot be neglected when determining the curvature of the noncompact dimensions since the direct contributions from the source action cancel important contributions in the no-go theorems. But, because the nonzero contributions are total derivatives, the good news is that most of the details of the back-reacted solutions are *not* important. All that counts is the near-source asymptotics of a specific combination of back-reacted bulk fields.

Our explanation of these results is organized as follows. Section 2.3, develops general expressions for how the curvature of non-compact, maximally symmetric directions depends on the properties of the extra-dimensional bulk fields. Much of this section is similar in spirit to the arguments made when deriving no-go results [49, 50, 51, 52], and our main new contribution is to cleanly identify how the curvature is controlled by asymptotic forms near the sources, and to see how assumptions about source dynamics modifies this asymptotics. Section 2.4 explicitly identifies for 11D and 10D supergravity the precise combination of bulk fields whose asymptotic forms are relevant to the low-energy curvature. We then apply these general arguments to the special case of metric/axio-dilaton configurations in 10D Type IIB supergravity with codimension-two sources, showing in this case how all solutions are flat in the noncompact directions in the absence of bulk fluxes.

2.3 A general expression for the classical cosmological constant

The purpose of this section is to derive a general expression for the curvature of the noncompact directions. We make the connection between on-source curvatures and near-source asymptotics in three steps. First we show — at the classical level for maximally symmetric source geometries — that the integral of the low-energy curvature can be computed as the sum of four terms: $I + II + III + IV$. Of these, I is the higher-dimensional bulk action, evaluated at the compactified solution. II is the integral over a total derivative, which Gauss' theorem directly relates to the boundary values of the warp factor, at infinity and near any potential singularities. III is a direct contribution from the action of any sources, and IV is a term which vanishes in the absence of any space-filling fluxes.

Next we show that for all of the supergravities of interest the higher-dimensional bulk lagrangian density is itself also always a total derivative when evaluated at an arbitrary classical solution. Combining this with step one then shows that, in the absence of space-filling fluxes, the integrated low-energy curvature is completely controlled by source and boundary effects.

Finally, §2.5 demonstrates step three. By treating carefully the singular behaviour near any codimension-two sources, it is shown that contributions II and III precisely cancel one another. Taken together, these three steps show that only contribution I plays any role in a broad class of theories.

We first focus on step one: we use the higher dimensional equations of motion to derive a relationship between the lower dimensional curvature and the on-shell higher-dimensional action. For definiteness, we consider solutions to the field equations of a D-dimensional (super)gravity theory, with action[5]

$$S = \frac{1}{2\kappa_D^2} \int \mathrm{d}^D z \sqrt{-\hat{g}_D} \Big(- \hat{\mathcal{R}} + \mathcal{L}^D_{\text{matter}} \Big) + S_{\text{source}} \,, \tag{2.14}$$

where $\mathcal{L}_{\text{matter}}$ depends on a generic set of other D-dimensional fields (but not on the derivatives of the metric), denoted collectively by ψ. S_{source} denotes the action of any sources, which differs from the term explicitly written by only involving an integration over d dimensions, rather than D.

Now imagine we have a solution to the field equations for this action describing a compactification down to $0 < d = D - n$ dimensions, of the form of eq. (2.1). We wish to derive a general expression for $R = g^{\mu\nu} R_{\mu\nu}$ in terms of properties of the warp-factor, W, the compact metric, $\tilde{g}_{mn}$, and the bulk- and source-matter actions.

[5] An aside on notation: indices $M, N = 0, 1, \ldots, D-1$ run over all dimenesion; greek indices denote lower-dimensional coordinates $\mu, \nu = 0, 1, \ldots, d-1$; and indices $m, n = 1, \ldots, n = D - d$ denote compactified coordinates. We use $\hat{\mathcal{R}}_{MN}$ to denote the D-dimensional Ricci curvature of the full D-dimensional metric, $\hat{g}_{MN}$; and $\hat{R}_{\mu\nu}$ to denote the d-dimensional Ricci curvature computed from the d-dimensional metric, $\hat{g}_{\mu\nu} = e^{2W} g_{\mu\nu}$. Finally, $\hat{g}_D = \det \hat{g}_{MN}$ while $\hat{g}_d = \det \hat{g}_{\mu\nu}$ *etc.*

To this end consider the $\mu\nu$ component of Einstein's equation,

$$\sqrt{-\hat{g}_{D}}\left[\hat{\mathcal{R}}^{\mu\nu}+\frac{1}{2}\,\hat{g}^{\mu\nu}\Big(-\hat{\mathcal{R}}+\mathcal{L}^{D}_{\rm matter}\Big)+\frac{\partial\mathcal{L}^{D}_{\rm matter}}{\partial\hat{g}_{\mu\nu}}\right]+2\kappa_{D}^{2}\left(\frac{\delta S_{\rm source}}{\delta\hat{g}_{\mu\nu}}\right)=0\,, \quad (2.15)$$

which we contract with $\hat{g}^{\mu\nu}$, making use of

$$\begin{aligned}\hat{g}^{\mu\nu}\hat{\mathcal{R}}_{\mu\nu} &= e^{-2W}R+d\,\tilde{\nabla}^{2}W+d^{2}\,\tilde{g}^{mn}\partial_{m}W\partial_{n}W\\ &= e^{-2W}R+e^{-dW}\tilde{\nabla}^{2}e^{dW}\,,\end{aligned} \quad (2.16)$$

where $\tilde{\nabla}^{2}=\tilde{g}^{mn}\tilde{\nabla}_{m}\tilde{\nabla}_{n}$. Dividing the result by $2\kappa_{D}^{2}$, using $\sqrt{-\hat{g}_{D}}=e^{dW}\sqrt{-g_{d}}\,\sqrt{\tilde{g}_{n}}$, and integrating over all D dimensions then gives

$$\begin{aligned}-\frac{1}{2\kappa_{d}^{2}}\int \mathrm{d}^{d}x\sqrt{-g_{d}}\;R &= \frac{d}{2}\,S_{\rm on-shell}+\frac{1}{2\kappa_{D}^{2}}\int \mathrm{d}^{d}x\sqrt{-g_{d}}\int \mathrm{d}^{n}y\sqrt{\tilde{g}_{n}}\;\tilde{\nabla}^{2}e^{dW} \quad (2.17)\\ &\quad+\int \mathrm{d}^{d}x\;\hat{g}_{\mu\nu}\left(\frac{\delta S_{\rm source}}{\delta\hat{g}_{\mu\nu}}\right)+\frac{1}{2\kappa_{D}^{2}}\int \mathrm{d}^{D}z\sqrt{-\hat{g}_{D}}\;\hat{g}_{\mu\nu}\frac{\partial\mathcal{L}^{D}_{\rm matter}}{\partial\hat{g}_{\mu\nu}}\\ &:= I+II+III+IV\,,\end{aligned}$$

where $S_{\rm on-shell}$ means the bulk part of the action appearing in eq. (2.14), evaluated at a solution to the field equations, and the second last term uses that the source terms are localized within the extra dimensions. κ_{d}^{2} denotes the d-dimensional gravitational coupling given by $\kappa_{d}^{2}=\kappa_{D}^{2}/\mathcal{V}_{W}$, with the warped volume defined by

$$\mathcal{V}_{W}:=\int \mathrm{d}^{n}y\sqrt{\tilde{g}_{n}}\;e^{(d-2)W}\,. \quad (2.18)$$

Maximal symmetry and space-filling fluxes

Eq. (2.17) is the key equation, and so far it has been derived on very general grounds. We now specialize to the situation where the solution does not break the maximal symmetry of the d-dimensional metric $g_{\mu\nu}$.

Maximal symmetry is a very constraining condition. First, it implies R is a constant, so the left-hand-side of eq. (2.17) is proportional to the (divergent) volume of the noncompact dimensions. Furthermore, the left-hand-side vanishes only for flat d-dimensional space, and its sign is controlled by the sign of R.

Second, maximal symmetry strongly restricts the form of $\partial\mathcal{L}^{D}_{\rm matter}/\partial\hat{g}_{\mu\nu}$ for the field content usually found in higher-dimensional supergravity. In particular, the only fields that can be nonzero (classically) for maximally symmetric solutions are: the metric, $g_{\mu\nu}$; space-filling fluxes of the form

$$F^{(p)}_{\mu_1..\mu_d m_1..m_{p-d}} = \epsilon_{\mu_1..\mu_d} G_{m_1...m_{p-d}}\,; \tag{2.19}$$

and any number of d-dimensional scalar fields (like components of $\tilde{g}_{mn}$, *etc.*).

Because $\mathcal{L}^D$ is defined with an overall factor of $\sqrt{-\hat{g}_D}$ factored out, and because the Einstein term is also treated separately, in the absence of higher-derivative interactions $\partial\mathcal{L}^D_{\rm matter}/\partial\hat{g}_{\mu\nu} = 0$ if only scalar fields and the metric are present. For the supergravities of interest here the only nonvanishing contributions to $\partial\mathcal{L}^D_{\rm matter}/\partial\hat{g}_{\mu\nu}$ arise from p-form fields (with $p \geq d$), having nonzero space filling components.

For instance, for a p-form field with kinetic term

$$\mathcal{L}^D_{\rm p-form} = -\frac{1}{2\,p!}\,F^2_{(p)}\,, \tag{2.20}$$

and non-vanishing space filling components we have

$$\hat{g}_{\mu\nu}\frac{\partial\mathcal{L}^D_{\rm matter}}{\partial\hat{g}_{\mu\nu}} = -\frac{d}{2(p-d)!}\,G_{m_1..m_{p-d}}G_{n_1..n_{p-d}}\tilde{g}^{m_1n_1}\tilde{g}^{m_2n_2}\cdots\hat{g}^{m_{p-d}n_{p-d}} = -\frac{d\,G^2}{2(p-d)!}\,, \tag{2.21}$$

which contributes to the right-hand-side of eq. (2.17) the amount

$$-\frac{d}{2\kappa_D^2(p-d)!}\int {\rm d}^dx\sqrt{-g_d}\int d^ny\sqrt{\tilde{g}_n}\;e^{dW}\,G^2\,. \tag{2.22}$$

We note that this is negative definite, which (in our conventions) contributes to R with an anti-de Sitter-like sign.

Of course, space-filling fluxes need not contribute to eq. (2.17) only through their kinetic term. The quantity $\partial\mathcal{L}^D_{\rm matter}/\partial\hat{g}_{\mu\nu}$ can also receive contributions from Chern-Simons terms. In this case, because $\mathcal{L}^D_{\rm CS\,matter} = \mathcal{L}_{\rm CS}/\sqrt{-\hat{g}_D}$, the contribution is simply proportional to the Chern-Simons term itself:

$$\hat{g}_{\mu\nu}\frac{\partial\mathcal{L}^D_{\rm CS\,matter}}{\partial\hat{g}_{\mu\nu}} = -\frac{d}{2}\int\mathcal{L}_{\rm CS}\,. \tag{2.23}$$

Unlike for the kinetic term, this contribution can have indefinite sign.

We see that in the absence of space-filling flux, the last term in equation (2.17) vanishes. When this is so, eq. (2.17) relates the d-dimensional curvature, R, to a total derivative, a derivative of the source action, and the bulk action evaluated on shell (which we show below is often also a total derivative).

The restriction to no space-filling fluxes is also not very restrictive, because one can usually (Hodge) dualize a flux to get rid of any space filling components. But there can be some situations where this cannot be done, such as when the flux in question is the self-dual five form of Type IIB supergravity. In this case the self-duality condition relates the flux components in the internal and space-time directions. In an appendix to this chapter we use some well-known examples to illustrate how eq. (2.17) works in practice (in the absence of source terms), with and without space-filling flux.

We now make some comments about the implications of eq. (2.17). IV , as we showed above, gives an AdS contribution to R_d. This is a bulk contribution and is present only in the presence of space-filling fluxes. However if we have a space-filling flux we can equivalently use its Hodge dual which in general would not be space-filling (an exception is the five form self dual flux in the $AdS_5 \times S^5$ solution in IIB supergravity). Using the dual solution would then give no contribution from IV. Term III is manifestly a boundary term. The warping term (II) involves an integral over a total derivative and so is also a boundary term. It turns out (as we show in the next section) that for the supergravity theories arising as low energy limits of string/M theory there exists a classsical scaling symmetry which makes I into a boundary term as well. Thus, under certain quite general assumptions, the lower dimensional scalar curvature (equivalently the cosmological constant) is entirely determined by boundary data. Crucially, we need not know the full bulk profile of the solution, but only its asymptotic form in the near boundary limit, to determine R_d.

2.4 Scaling in Supergravity

In this section we first show that the on-shell action of a theory with a classical scaling symmetry is a boundary term. We then find explicitly the form of this boundary term for 11-D, IIA and IIB supergravity theories which possess such scaling behaviour (at least in the bosonic sector). This section now proves that

$S_{\rm on-shell}$ can generally also be expressed as the integral of a total derivative for the bulk supergravities of general interest.

This is actually a special case of a more general result [67] that states that any scale-invariant system has this property, as we review here. It is generic to higher-dimensional supergravities because these typically all have a classical scale invariance [68].

2.4.1 Scaling and the on-shell action

Consider a theory with a lagrangian density $\mathcal{L}(\phi_i, \partial\phi_i)$ with the following scaling property under the field scalings $\phi_i \rightarrow s^{a_i}\phi_i$:

$$\mathcal{L}(s^{a_i}\phi_i, s^{a_i}\partial\phi_i) = s^a\mathcal{L}(\phi_i, \partial\phi_i) \tag{2.24}$$

the field ϕ_i having scaling dimension a_i. With such scaling behaviour we can show that the onshell action is a total derivative. Differentiating the above equation with respect to s and then setting $s = 1$ gives

$$\sum_i a_i \left[\left(\frac{\partial \mathcal{L}}{\partial \left[\partial_\mu \varphi_i\right]} \right) \partial_\mu \varphi_i + \left(\frac{\partial \mathcal{L}}{\partial \varphi_i} \right) \varphi_i \right] = a\mathcal{L} \tag{2.25}$$

Now to put the fields on-shell we use the Euler-Lagrange field equations

$$\left(\frac{\partial \mathcal{L}}{\partial \varphi_i} \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left[\partial_\mu \varphi_i\right]} \right) = 0 \tag{2.26}$$

and this gives

$$\mathcal{L}_{on-shell} = \sum_i \frac{a_i}{a}\, \partial_\mu \left[\left(\frac{\partial \mathcal{L}}{\partial \left[\partial_\mu \varphi_i\right]} \right) \varphi_i \right] \tag{2.27}$$

Consequently the on-shell action of a theory with such scaling behaviour would be a boundary term (being the integral of a total derivative). In the rest of this section we determine the form of this boundary term for three supergravity theories- 11-D, IIA, IIB- in which, as we note below, the bosonic part of the action shows such scaling behaviour.

2.4.2 11 dimensional Supergravity

The bosonic part of the action of 11-D supergravity is

$$S = -\frac{1}{2\kappa_{11}^2}\int d^{11}x\sqrt{-g}\left(R + \frac{1}{2.4!}G_4^2\right) - \frac{1}{12\kappa_{11}^2}\int G_4 \wedge G_4 \wedge C_3 \tag{2.28}$$

For $g_{MN} \to s g_{MN} \quad C_{MNP} \to s^{3/2} C_{MNP}$ we have $S \to s^{9/2} S$. The existence of this scaling behaviour implies that the onshell action should be a boundary term. To find this boundary term we first take the trace of Einstein's equation:

$$R = -\frac{G_4^2}{(12)^2} \tag{2.29}$$

The equation of motion for the 3-form potential is:

$$d(*G_4) = -\frac{1}{2}G_4 \wedge G_4 \tag{2.30}$$

Using these two equations gives the following expression for the on-shell 11-D supergravity action:

$$S_{11-D}^{on-shell} = -\frac{1}{6\kappa_{11}^2}\int d[C_3 \wedge *G_4] \tag{2.31}$$

2.4.3 IIA Supergravity

The (bosonic part of) IIA supergravity action (in the string frame) is

$$S = S_{NS} + S_{RR} + S_{CS} \tag{2.32}$$

where,

$$\begin{aligned} S_{NS} &= -\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-\hat{g}}e^{-2\phi}\left(\hat{R} - 4\partial_\mu\phi\partial^\mu\phi + \frac{1}{2.3!}H_3^2\right) \\ S_{RR} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-\hat{g}}\left(-\frac{1}{2.2!}F_2^2 - \frac{1}{2.4!}\tilde{F}_4^2\right) \\ S_{CS} &= -\frac{1}{4\kappa_{10}^2}\int B_2 \wedge F_4 \wedge F_4 \end{aligned} \tag{2.33}$$

with,

$$\tilde{F}_4 = F_4 + C_1 \wedge H_3 \qquad H_3 = dB_2 \qquad F_2 = dC_1 \qquad F_4 = dC_3 \qquad F_p^2 = F_{a_1\ldots a_p}F^{a_1\ldots a_p} \tag{2.34}$$

The Einstein frame action is (with $\hat{g}_{MN} = e^{\phi/2} g_{MN}$):

$$S = -\frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left(R + \frac{1}{2}(\partial\phi)^2 + \frac{e^{-\phi}}{2.3!} H_3^2 + \frac{e^{3\phi/2}}{2.2!} F_2^2 + \frac{e^{\phi/2}}{2.4!} \tilde{F}_4^2 \right) + S_{CS} \tag{2.35}$$

This action scales as: $S \to s^2 S$ under the field rescalings :

$$e^{-\phi} \to s e^{-\phi} \ , \quad g_{MN} \to \sqrt{s} g_{MN} \ , \quad B_2 \to B_2 \ , \quad C_1 \to s C_1 \ , \quad C_3 \to s C_3 \tag{2.36}$$

so we expect the on-shell action to be a boundary term. To find the boundary contribution we write the Einstein frame action in terms of differential forms:

$$\begin{aligned} S &= -\frac{1}{2\kappa_{10}^2} \int \Big(* R - \frac{1}{2} d\phi \wedge * d\phi - \frac{e^{-\phi}}{2} H_3 \wedge * H_3 + \frac{e^{3\phi/2}}{2} F_2 \wedge * F_2 \\ &\quad + \frac{e^{\phi/2}}{2} \tilde{F}_4 \wedge * \tilde{F}_4 + \frac{1}{2} B_2 \wedge F_4 \wedge F_4 \Big) \end{aligned} \tag{2.37}$$

The equations of motion for the form fields following from the action are:

$$\begin{aligned} d\Big(e^{-\phi} * H_3 + e^{\phi/2} C_1 \wedge * \tilde{F}_4 \Big) &= -\frac{1}{2} F_4 \wedge F_4 \\ d(e^{3\phi/2} * F_2) &= - e^{\phi/2} H_3 \wedge * \tilde{F}_4 \\ d\Big(e^{\phi/2} * \tilde{F}_4 + F_4 \wedge B_2 \Big) &= 0 \end{aligned} \tag{2.38}$$

while the trace of Einstein's equation gives,

$$-R = \frac{1}{2}(\partial\phi)^2 + \frac{e^{-\phi}}{4.3!} H_3^2 + \frac{3e^{3\phi/2}}{8.2!} F_2^2 + \frac{e^{\phi/2}}{8.4!} \tilde{F}_4^2 \tag{2.39}$$

Substituting this in the action gives

$$S = -\frac{1}{4\kappa_{10}^2} \int \left(-\frac{e^{-\phi}}{2} H_3 \wedge * H_3 + \frac{e^{3\phi/2}}{4} F_2 \wedge * F_2 + \frac{3e^{\phi/2}}{4} \tilde{F}_4 \wedge * \tilde{F}_4 + B_2 \wedge F_4 \wedge F_4 \right) \tag{2.40}$$

Now using the equations of motion of the form fields, as before, we can put the action in the required form- the integral of a total derivative:

$$\begin{aligned} S_{IIA}^{on-shell} &= -\frac{1}{8\kappa_{10}^2} \int d\Big(- e^{-\phi} B_2 \wedge * H_3 + \frac{e^{3\phi/2}}{2} C_1 \wedge * F_2 + \frac{3e^{\phi/2}}{2} C_3 \wedge * \tilde{F}_4 \\ &\quad - e^{\phi/2} B_2 \wedge C_1 \wedge * \tilde{F}_4 + \frac{3}{2} C_3 \wedge F_4 \wedge B_2 \Big) \end{aligned} \tag{2.41}$$

2.4.4 IIB Supergravity

The (bosonic part of) IIB supergravity action (in string frame) is again given by eq.(2.33) above with the same S_{NS} but the R-R and Chern-Simons part of the action are now given by

$$S_{RR} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-\hat{g}}\Big(-\frac{1}{2!}F_1^2 - \frac{1}{2.3!}\tilde{F}_3^2 - \frac{1}{4.5!}\tilde{F}_5^2\Big) \tag{2.42}$$

$$S_{CS} = -\frac{1}{4\kappa_{10}^2}\int C_4\wedge H_3\wedge F_3 \tag{2.43}$$

with $F_k = dC_{k-1} \quad H_3 = dB_2$ and

$$\widetilde{F}_3 := F_3 - C_0 H_3 \quad \text{and} \quad \widetilde{F}_5 = *\widetilde{F}_5 := F_5 - \frac{1}{2}\,C_2\wedge H_3 + \frac{1}{2}\,B_2\wedge F_3\,. \tag{2.44}$$

We can go over to the Einstein frame with $\hat{g}_{MN} = e^{\phi/2}g_{MN}$ and combining fields into complex quantities

$$\tau = C_0 + ie^{-\phi} \qquad G_3 = F_3 - \tau H_3 \tag{2.45}$$

which transform simply under the $SL(2,\mathbb{R})$ duality group, to get the Einstein frame action:

$$S_{IIB} = -\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}\left(R + \frac{\partial_A\bar{\tau}\partial^A\tau}{2(Im\tau)^2} + \frac{\bar{G}_3.G_3}{12.Im\tau} + \frac{\tilde{F}_5^2}{4.5!}\right) + \frac{1}{8\kappa_{10}^2}\int C_4\wedge G_3\wedge\bar{G}_3 \tag{2.46}$$

This action also scales as: $S\to s^2 S$ under similar field rescalings ($C_k\to sC_k$and the rest, as before).

The Einstein frame action can be written in differential form notation:

$$\begin{aligned} S_{IIB} \;=\; & -\frac{1}{2\kappa_{10}^2}\int\Big(*\,R - \frac{1}{2}d\phi\wedge *d\phi - \frac{1}{2}e^{2\phi}dC_0\wedge *dC_0 - \frac{e^{\phi}}{2}\tilde{F}_3\wedge *\tilde{F}_3 \\ & - \frac{e^{-\phi}}{2}H_3\wedge *H_3 - \frac{1}{4}\tilde{F}_5\wedge *\tilde{F}_5 + \frac{1}{2}C_4\wedge H_3\wedge F_3\Big) \end{aligned} \tag{2.47}$$

The form field equations are

$$d(*e^{-\phi}H_3 - *e^{\phi}C_0\tilde{F}_3) = F_3\wedge\tilde{F}_5 \tag{2.48}$$

$$d(*e^{\phi}\tilde{F}_3) = \tilde{F}_5\wedge H_3 \tag{2.49}$$

$$d\tilde{F}_5 = H_3\wedge F_3 \tag{2.50}$$

whereas, the trace of Einstein's equations gives

$$-R = \frac{\partial_M \bar{\tau} \partial^M \tau}{2(Im\tau)^2} + \frac{\bar{G}_3.G_3}{24.Im\tau} \tag{2.51}$$

Using these the action can again be expressed as the integral of a total derivative

$$S_{IIB}^{on-shell} = \frac{1}{8\kappa_{10}^2} \int d\Big(C_4 \wedge C_2 \wedge H_3 - C_4 \wedge F_3 \wedge B_2 + B_2 \wedge e^{\phi} C_0 * \tilde{F}_3 - B_2 \wedge e^{-\phi} * H_3 - C_2 \wedge e^{\phi} * \tilde{F}_3 \Big) \tag{2.52}$$

Why should we care when the bulk contribution on the right-hand-side of eq. (2.17) is a total derivative? We care precisely because the bulk fields are generically singular at the specific points in the n compact dimensions where the sources are located. To deal with this singularity, as well as any singularities coming from S_{source}, we imagine surrounding these objects in the transverse dimensions by a 'Gaussian pillbox' at a small proper distance from the source. This removes the singularity at the source at the expense of introducing a new boundary on the Gaussian pillbox.

When the bulk contribution to the right-hand-side of eq. (2.17) is a total derivative, its integral depends only on the near-source limit of the back-reacted bulk fields at the pillbox. And these boundary conditions, in turn, are related to the physical properties of the source at y_c^m allowing them to be combined with the S_{source} terms in a general way, as the next section discusses in more detail.

The conclusion is that although explicitly finding the back-reacted bulk solution for a given source is very difficult, when the curvature depends only on a total derivative most of the details of these solutions are not important. It is only their near-brane boundary conditions that play any role in fixing the on-source curvature, R.

Note: The on-shell supergravity action figures prominently in the AdS/CFT correspondence via the GKPW prescription. It determines the generating functional for large N, large λ CFT correlators. Since we have shown that on-shell IIA, IIB and 11d supergravity actions are boundary terms (with their explicit form also determined above) it seems feasible that this should provide an efficient way to calculate correlators in CFTs whose duals are solutions with fluxes in such supergravity theories.

note on 6D supergravity

As a point of reference, we restate here the on-shell action as computed [67] for chiral, gauged supergravity [57] in six dimensions. The relevant bosonic action, S^6, is given in eq. (2.3) and scales as $S^6 \to s^2 \, S^6$ when $\hat{g}_{MN} \to s\hat{g}_{MN}$ and $e^{-\phi} \to s \, e^{-\phi}$. The on-shell lagrangian is therefore a total derivative, and is seen by explicit evaluation to be

$$S^6_{\rm on-shell} = \frac{1}{2\kappa_6^2} \int \mathrm{d}^6 x \, \sqrt{-\hat{g}_6} \, \Box\phi \, . \tag{2.53}$$

In our conventions, when used in eq. (2.17), this shows that an AdS sign corresponds to ϕ decreasing near the source, while a de Sitter sign arises when ϕ increases towards the source (a property that may also be directly verified of the explicit de Sitter solutions [48, 66]). Since $e^{2\phi}$ counts loops in this system, consistency of the classical approximation requires that one encounters the physics that regulates the source before leaving the weak-coupling regime $e^\phi \ll 1$. Although this sounds worrisome, similar considerations apply to the gravitational field of a macroscopic source like the Earth. The large curvatures encountered if this field were extrapolated to zero size would also eventually invalidate a semiclassical approximation; but are not a problem in practice due to the prior intervention of the Earth's surface.

2.5 Sources and singularities

The final step is to relate more precisely the boundary contributions to the bulk integrals encountered above to the properties of the source action, $S_{\rm source}$. As we now see, this allows contribution II to be related to contribution III in eq. (2.17), with the result that they cancel for codimension-two sources.

The trick when doing so is to deal properly with the singularity of the bulk configurations near the sources. We follow a strategy familiar from experience with the Coulomb singularity of electrostatics: we surround the sources with small 'Gaussian pillboxes,' and replace the singular extrapolation into the pillbox interior with an appropriate set of boundary conditions on the surface of the box. In this way the singular physics of a point charge is finessed into a finite flux through an arbitrary, but small, surface enclosing the charge.

Of course, this is only a useful construction if the size of the charge distribution is much smaller than the distances of interest for predicting the resulting electric field. If the box is too small compared with the charge distribution inside, the real charge distribution inside cannot be approximated by a point source with the same total charge. A similar problem arises if the box is too large compared with the scales over which the electric fields are to be computed. The construction is useful if a sufficiently large hierarchy exists between the size of the source and the distances of interest for the resulting electric fields.

The same is possible for gravitating systems, provided the physical size of the source is much smaller than the distance over which the gravitational field extends (like the size of any extra dimensions). To accomplish this in the present context [61, 63], we excise a small D-dimensional spacetime volume from around each source, and instead specify the boundary conditions on boundary to this small volume.

In the spirit of replacing a real charge distribution by an equivalent point charge, the boundary conditions are specified by doing so for a simple source distribution that shares the same energy. This is most simply done by imagining the source energy density to be distributed on the boundary of the pillbox itself, with the pillbox interior filled in with a smooth field configuration. Such a simple-minded procedure suffices to capture the long-distance physics of a generic real distribution if the pillbox is sufficiently small, with the size of the actual source of interest being much smaller still.

Formally this is done by specifying a $(D-1)$-dimensional codimension-one boundary action, $\widetilde{S}_{\rm bdy}$, on the pillbox surface, together with a smooth solution describing the pillbox interior. This construction allows boundary conditions to be inferred using standard methods involving the Israel junction conditions [58], which relate $\widetilde{S}_{\rm bdy}$ to the jump in bulk-field derivatives between inside and outside of the pillbox.

Once these junction conditions are found, a new point of view is possible for which the pillbox is regarded as a proper boundary of the bulk geometry, without reference to the pillbox interior. In this case one defines a new boundary action for the pillbox, $S_{\rm bdy}$, which is defined by the condition that its derivatives determine the near-source radial derivatives of the fields exterior to the pillbox. In general $S_{\rm bdy}$ differs from $\widetilde{S}_{\rm bdy}$ because it must now also include any effects that used to be generated by the now non-existent interior geometry. $S_{\rm bdy}$ also includes the Gibbons-Hawking action [71] for gravity on the boundary, both of the interior and exterior regions:

$$S_{\rm bdy} := \widetilde{S}_{\rm bdy} + S_{GH+} + S_{GH-} + S_{\rm int}\,, \tag{2.54}$$

with

$$S_{GH} = \frac{1}{\kappa_D^2}\int {\rm d}^{D-1}x\sqrt{-\gamma}\;K\,, \tag{2.55}$$

and $K = g^{ij}K_{ij}$, where K_{ij} is the extrinsic curvature of the boundary and γ_{ij} the induced metric. The subscript $\pm$ for $S_{GH\pm}$ indicates whether the extrinsic curvature is to be computed just inside or just outside of the codimension-one pillbox boundary. The Gibbons-Hawking action is required in the presence of boundaries to make the variation of the Einstein action well-posed. Finally, $S_{\rm int}$ describes the 'bulk' action describing the interior geometry, whose details are not important in what follows when the pillbox is sufficiently small.

In the limit of a vanishingly small pillbox, these codimension-one actions can be compactified into corresponding higher-codimension actions. We define $\widetilde{S}_{\rm source}$ to be the result obtained from $\widetilde{S}_{\rm bdy}$ in this way, but it is the dimensional reduction of $S_{\rm bdy}$ that compactifies to the d-dimensional source action, $S_{\rm source}$, used in previous sections.

This procedure has been worked through in detail for scalar-tensor-Maxwell theories with codimension-two sources in $D = d + 2$ dimensions [61], to which we now specialize. The resulting boundary conditions were then checked for $D7$-brane sources in Type IIB supergravity in 10 dimensions, for which the bulk and source actions are explicitly known, as are a broad class of solutions to the bulk field equations [69]. In all cases the solutions and actions satisfy the boundary conditions inferred using this construction [63].

For the present purposes it turns out that we need only the boundary conditions for the metric. Using the Israel junction conditions to relate an assumed smooth interior geometry for the pillbox to the geometry outside, one finds the following junction conditions, expressed in terms of the codimension-one action,

$\widetilde{S}_{\rm bdy}$, of the codimension-one source:[6]

$$\frac{1}{2\kappa_D^2}\sqrt{-\hat{g}_D}\,\left(K^{ij} - K g^{ij}\right) - (\mathrm{int})^{ij} = \frac{\delta \widetilde{S}_{\rm bdy}}{\delta \hat{g}_{ij}}\,. \tag{2.56}$$

This expression adopts coordinates near the pillbox for which ρ denotes radial proper distance away from the source, which is located at $\rho = 0$. The pillbox boundary lies on a surface of fixed, small ρ, for which K_{ij} is the extrinsic curvature of the fixed-ρ surface, for which the local coordinates are $\{x^i\} = \{x^\mu, \theta\}$, with $i = 0, 1, \cdots, d$ where $d = D-2$ and θ is an angular coordinate that runs from 0 to 2π as one encircles the source. Finally, '$(\mathrm{int})^{ij}$' denotes the same result evaluated for the smooth interior geometry, for which $\rho = 0$ is nonsingular.

As mentioned earlier, there are two equivalent ways to read eq. (2.56). The first is the way it was initially derived: where $\widetilde{S}_{\rm bdy}$ represents only the action of the boundary, and the interior region of the brane is matched onto the exterior one through eq. (2.56). The other viewpoint is that the pillbox is considered the actual boundary of spacetime, and the 'interior' of the branes is excised entirely. In this point of view, the properties of the interior solutions are encoded in the boundary action, $S_{\rm bdy}$:

$$\frac{1}{2\kappa_D^2}\sqrt{-\hat{g}_D}\,\left(K^{ij} - K g^{ij}\right) = \frac{\delta \widetilde{S}_{\rm bdy}}{\delta \hat{g}_{ij}} + (\mathrm{int})^{ij} = \frac{\delta S_{\rm bdy}}{\delta \hat{g}_{ij}}\,. \tag{2.57}$$

In the limit of a very small pillbox, these conditions dimensionally reduce to conditions that only refer to the codimension-two action.

$$\lim_{\rho\to 0}\oint_{x_b} \mathrm{d}\theta\,\left[\frac{1}{2\kappa_D^2}\sqrt{-\hat{g}}\,\left(K^{ij} - K\hat{g}^{ij}\right) - (\mathrm{int})^{ij}\right] = \frac{\delta \widetilde{S}_{\rm source}}{\delta \hat{g}_{ij}}\,, \tag{2.58}$$

where the integration is about a small circle of proper radius ρ encircling the brane position at $\rho = 0$, and N_M is the unit normal pointing towards the brane ($N_M \mathrm{d}x^M = -\mathrm{d}\rho$).

Thus we see that source-bulk matching relates the asymptotic, near-source radial derivatives of the bulk fields to the properties of the source action. In what

[6] The difference in signs compared to [63] arises from the choice of unit normal. Here, K is defined with respected to the outward pointing normal, to agree with the convention for the Gibbons-Hawking term.

follows, an important role is played by the function, $U_{\rm source}$, that controls the codimension-two boundary condition for the warp factor, W,

$$\begin{aligned} \frac{d}{\kappa_D^2} \lim_{\rho\to 0} \oint {\rm d}\theta \sqrt{-\hat g_D}\; N^M \partial_M W &= 2 \frac{\partial}{\partial g_{\theta\theta}} \Big[\sqrt{-g_d}\; \widetilde{\mathcal{L}}_{\rm source} \Big] \\ &:= d \sqrt{-g_d}\; U_{\rm source}\,, \end{aligned} \tag{2.59}$$

where the last equality defines $U_{\rm source}$, and $\widetilde{\mathcal{L}}_{\rm source}$ is the codimension-two lagrange density

$$\widetilde S_{\rm source} = \int {\rm d}^d x \sqrt{-\hat g_d}\; \widetilde{\mathcal{L}}_{\rm source}\,. \tag{2.60}$$

The function $U_{\rm source}$ is important[7] for other reasons, besides its above role in controlling the asymptotic behaviour of the warp factor. As we show below, for codimension-two sources $U_{\rm source}$ turns out also to be the Lagrange density of the full action, $S_{\rm source}$ [61, 63]. It turns out that $U_{\rm source}$ is generically non-negative, and this is related to the general property (described below) that the bulk field equations dictate that W does not increase as one approaches a codimension-two source.

Implications for the on-source curvature

We now show how the above matching conditions imply a dramatic cancelation in our key formula, eq. (2.17). In particular, after using Gauss' law to rewrite total derivatives in terms of surface terms at the position of the Gaussian pillboxes surrounding the sources, followed by eq. (2.59), one of the terms on the right-hand-side of eq. (2.17) can be written:

$$\begin{aligned} \frac{1}{2\kappa_D^2} \int {\rm d}^d x \sqrt{-g_d} \int {\rm d}^2 y \sqrt{\tilde g_2}\; \tilde\nabla^2 e^{dW} &= \frac{d}{2\kappa_D^2} \int {\rm d}^d x \sqrt{-g_d} \oint {\rm d}\theta \sqrt{\tilde g_2}\; (N\cdot \tilde\nabla W) e^{dW} \\ &= \frac{d}{2} \int {\rm d}^d x \sqrt{-g_d}\; U_{\rm source}\,. \end{aligned} \tag{2.61}$$

We wish to compare this with another term on the right-hand-side of eq. (2.17),

$$\int {\rm d}^d x\; \hat g_{\mu\nu} \left(\frac{\delta S_{\rm source}}{\delta \hat g_{\mu\nu}} \right) = \lim_{\rho\to 0} \int {\rm d}^{d+1} x\; \hat g_{\mu\nu} \left(\frac{\delta S_{\rm bdy}}{\delta \hat g_{\mu\nu}} \right)\,. \tag{2.62}$$

[7] Although determination of $U_{\rm source}$ appears to require knowing how $S_{\rm source}$ depends on $g_{\theta\theta}$, this is actually not necessary because the it is related [63] by an identity — the 'Hamiltonian' constraint for evolution in the ρ direction, since this relates the first derivatives of bulk fields with respect to ρ — to the easily computed derivatives $\delta S_{\rm source}/\delta\phi^a$ and $\delta S_{\rm source}/\delta g_{\mu\nu}$.

To evaluate this we use the matching condition, eq. (2.56), which implies

$$\int \mathrm{d}^{d+1}x \; \hat{g}_{ij} \frac{\delta \widetilde{S}_{\rm bdy}}{\delta \hat{g}_{ij}} = -\frac{d}{2\kappa_D^2} \int \mathrm{d}^{d+1}x \sqrt{-\hat{g}_D} \; \Big[K - ({\rm int}) \Big] = -\frac{d}{2} \Big(S_{GH+} + S_{GH-} \Big) \,, \tag{2.63}$$

to rewrite $S_{\rm bdy}$ as follows:

$$\begin{aligned} S_{\rm bdy} &= \widetilde{S}_{\rm bdy} + S_{GH+} + S_{GH-} \\ &= \widetilde{S}_{\rm bdy} - \frac{2}{d} \int \mathrm{d}^{d+1}x \; \hat{g}_{ij} \, \frac{\delta \widetilde{S}_{\rm bdy}}{\delta \hat{g}_{ij}} \\ &= \widetilde{S}_{\rm bdy} - \frac{2}{d} \int \mathrm{d}^{d+1}x \; \left(\hat{g}_{\mu\nu} \, \frac{\delta \widetilde{S}_{\rm bdy}}{\delta \hat{g}_{\mu\nu}} + \hat{g}_{\theta\theta} \, \frac{\delta \widetilde{S}_{\rm bdy}}{\delta \hat{g}_{\theta\theta}} \right) \,, \end{aligned} \tag{2.64}$$

Now, our interest is in maximally symmetric configurations with no space-filling fluxes, for which

$$\widetilde{S}_{\rm bdy} = \int \mathrm{d}^{d+1}x \sqrt{-\hat{g}_D} \; \widetilde{\mathcal{L}}_{\rm bdy} \,, \tag{2.65}$$

and $\widetilde{\mathcal{L}}_{\rm bdy}$ does not depend on curvatures. In this case $\delta \widetilde{S}_{\rm bdy}/\delta \hat{g}_{\mu\nu} = \frac{1}{2} \sqrt{-\hat{g}_D} \; \widetilde{\mathcal{L}}_{\rm bdy} \, \hat{g}^{\mu\nu}$. Using this in eq. (2.64) gives

$$S_{\rm source} = \lim_{\rho \to 0} S_{\rm bdy} = -\frac{2}{d} \lim_{\rho \to 0} \int \mathrm{d}^{d+1}x \; \hat{g}_{\theta\theta} \, \frac{\delta \widetilde{S}_{\rm bdy}}{\delta \hat{g}_{\theta\theta}} = - \int \mathrm{d}^d x \sqrt{-g_d} \; U_{\rm source} \,, \tag{2.66}$$

where the last equality uses eq. (2.59). This leads finally to our desired expression:

$$\int \mathrm{d}^d x \; \hat{g}_{\mu\nu} \left(\frac{\delta S_{\rm source}}{\delta \hat{g}_{\mu\nu}} \right) = -\frac{d}{2} \int \mathrm{d}^d x \sqrt{-g_d} \; U_{\rm source} \,. \tag{2.67}$$

As claimed, from eqs. (2.61) and (2.67) we see that the codimension-two matching conditions ensure the cancelation of two of the terms on the right-hand-side of eq. (2.17),

$$\frac{1}{2\kappa_D^2} \int \mathrm{d}^d x \sqrt{-g_d} \; \int \mathrm{d}^2 y \sqrt{\tilde{g}_2} \; \tilde{\nabla}^2 e^{dW} + \int \mathrm{d}^d x \; \hat{g}_{\mu\nu} \left(\frac{\delta S_{\rm source}}{\delta \hat{g}_{\mu\nu}} \right) = 0 \,, \tag{2.68}$$

leaving

$$\begin{aligned} -\frac{1}{2\kappa_d^2} \int \mathrm{d}^d x \sqrt{-g_d} \; R &= \frac{d}{2} \, S_{\rm on-shell} + \frac{1}{2\kappa_D^2} \int \mathrm{d}^D x \sqrt{-\hat{g}_D} \; \hat{g}_{\mu\nu} \frac{\partial \mathcal{L}^D_{\rm matter}}{\partial \hat{g}_{\mu\nu}} \\ &= \frac{d}{2} \, S_{\rm on-shell} \,, \end{aligned} \tag{2.69}$$

with the second line following because we already assumed there to be no space-filling fluxes. This, together with the earlier expressions that give $S_{\rm on-shell}$ as a total derivative, are our main results.

2.6 Example: the axio-dilaton IIB supergravity

In this section we first review the known [19] D7 brane solutions in axiodilaton-metric IIB supergravity first for the case in which there is no warping and the solutions are 8-dimensional flat. We'll then incorporate warping and source terms with the hope that we may be able to realise curved (in particular dS) on-brane geometries. It turns out however that even in this more general case the solutions are still 8d flat. In this general case we are unable to solve the coupled non-linear system of PDE's that results but we show the flatness of the brane solutions using the general expression for R_d established in the previous section.

Our goal in this section is to illustrate the generality of the result, eq. (2.69), obtained at the end of the last section. We use eq. (2.69) to show that the on-source curvature vanishes for F-theory axio-dilaton compactifications of 10D Type IIB supergravity with arbitrary codimension-two sources, generalizing a known result when the sources are supersymmetric [70]. Although this example corresponds to the choices $d = 8$ and $n = 2$, — with only the metric, g_{MN}, and the axio-dilaton, $\tau = C + i\, e^{\phi}$, (and no other fluxes) in play, in what follows we work instead with general d.

This choice is made for three reasons. First, because it includes a broad class of explicitly known solutions [69] with explicit sources: $D7$- and $O7$-planes, as well as various kinds of (p, q)-branes. Second, because the absence of bulk fluxes ensures that the right-hand-side of eq. (2.17) is particularly simple (and is a total derivative). Third, the d-dimensional sources in this case have codimension two, which is one of the few situations for which matching conditions relating near-source asymptotics to physical properties of the source are explicitly worked out [61]. In particular, they have been tested explicitly [63] for the solutions of ref. [69] with $D7$-brane sources — and implicitly, using $SL(2, R)$ invariance, for (p, q)-brane sources as well.

2.6.1 Bulk equations of motion

The Einstein frame action for the Einstein-axio-dilaton system in 10D Type IIB supergravity is $S = S_B + S_{\rm source}$, where

$$S_B = -\frac{1}{2\kappa^2} \int {\rm d}^{10}x \sqrt{-\hat g}\; \hat g^{MN} \left[\hat{\mathcal{R}}_{MN} + \frac{\partial_M \overline{\tau}\, \partial_N \tau}{2\,({\rm Im}\,\tau)^2} \right] . \tag{2.70}$$

This is invariant under PSL(2,R) transformations

$$\tau \to \frac{a\tau + b}{c\tau + d}\,, \tag{2.71}$$

with the real parameters a through d satisfying $a\,d - b\,c = 1$. The scaling symmetry boils down in this case to $\tau \to s\,\tau$ and $\hat g_{MN} \to \sqrt{s}\; \hat g_{MN}$, under which $S_B \to s^2\, S_B$.

The Einstein field equations for this action are

$$\hat{\mathcal{R}}_{MN} + \frac{1}{4({\rm Im}\,\tau)^2} \left(\partial_M \overline{\tau} \partial_N \tau + \partial_N \overline{\tau} \partial_M \tau \right) = (\hbox{source terms})\,, \tag{2.72}$$

whose trace with $\hat g^{MN}$ ensures that $S_{\rm on-shell} = 0$ (for all D). The axio-dilaton equation is, similarly

$$-i \hat\nabla^2 \tau + \frac{\partial^M \tau \partial_M \tau}{{\rm Im}\,\tau} = (\hbox{source terms})\,. \tag{2.73}$$

As ever, the solutions of interest have geometry

$${\rm d}\hat s^2 = \hat g_{MN}\, {\rm d}x^M {\rm d}x^N = e^{2W}\, g_{\mu\nu}\, {\rm d}x^\mu {\rm d}x^\nu + \tilde g_{mn}\, {\rm d}y^m {\rm d}y^n\,, \tag{2.74}$$

where $g_{\mu\nu}(x)$ is a d-dimensional maximally symmetric Minkowski-signature metric, and $W(y)$, $\tau(y)$ and $\tilde g_{mn}(y)$ depend only on the other n compact directions. We temporarily keep the variables d and n general, although at the end we specialize to our real interest in this section: $n = 2$ (and $D = 10$ and $d = 8$, though this is less crucial).

For general d and n the Ricci tensors satisfy (see eqs. 1.5 and 1.6)

$$\begin{aligned}
\hat{\mathcal{R}}_{\mu\nu} &= R_{\mu\nu} + \Bigl(\tilde\nabla^2 W + d\, \tilde g^{mn} \partial_m W\, \partial_n W \Bigr) e^{2W} g_{\mu\nu} \\
&= R_{\mu\nu} + \frac{1}{d}\, e^{(2-d)W} \left(\tilde\nabla^2 e^{dW} \right) g_{\mu\nu} \\
\hbox{and} \quad \hat g^{mn} \hat{\mathcal{R}}_{mn} &= \tilde R + d \Bigl(\tilde\nabla^2 W + \tilde g^{mn} \partial_m W \partial_n W \Bigr) = \tilde R + d\, e^{-W}\, \tilde\nabla^2 e^{W}\,,
\end{aligned} \tag{2.75}$$

and so the $(\mu\nu)$ Einstein equations, $\hat{\mathcal{R}}_{\mu\nu} = 0$,

$$R\, e^{-2W} + e^{-dW} \tilde{\nabla}^2 e^{dW} = (\text{source terms})\,, \tag{2.76}$$

while the n-dimensional trace of the remaining Einstein equations becomes

$$\tilde{R} + d\, e^{-W} \tilde{\nabla}^2 e^{W} + \frac{\tilde{g}^{mn} \partial_m \tau \partial_n \bar{\tau}}{2(\mathrm{Im}\, \tau)^2} = (\text{source terms})\,. \tag{2.77}$$

We next briefly review a situation where solutions are known fairly explicitly to the equations governing the metric and axio-dilaton in Type IIB supergravity. These are the unwarped, flat solutions of [69].

2.6.2 Flat unwarped solutions

When $n = 2$ a very broad class of explicit solutions to the Einstein equations are known [69] in the limiting case where the two transverse dimensions are not warped: $\partial_m W = 0$. In this case the $(\mu\nu)$ Einstein equation implies $R = 0$ and so the solutions are given by $\tau = \tau(z)$ and

$$\mathrm{d}s^2 = \eta_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu + e^{2C(z,\bar{z})}\, \mathrm{d}\bar{z}\, \mathrm{d}z\,. \tag{2.78}$$

The transverse components of Einstein equations simplify to

$$2\, \partial\bar{\partial} C - \frac{(\partial\tau\, \bar{\partial}\bar{\tau} + \partial\bar{\tau}\, \bar{\partial}\tau)}{(\tau - \bar{\tau})^2} \quad = \quad 0\,, \tag{2.79}$$

while the axio-dilaton equation of motion is:

$$\partial\bar{\partial}\, \tau + \frac{2\, \partial\tau \bar{\partial}\tau}{\bar{\tau} - \tau} = 0\,. \tag{2.80}$$

A broad class of solutions to eq. (2.80) are immediate when $\partial_m W = 0$ [69]: it is satisfied by any holomorphic function, $\tau = \tau(z)$, for which $\bar{\partial}\tau = 0$. The transformation properties of the axio-dilaton under the $PSL(2, Z)$ subgroup of the $PSL(2, R)$ symmetry are most easily tracked if $\tau(z)$ is written

$$j(\tau(z)) = P(z)\,, \tag{2.81}$$

where $j(\tau)$, is the standard bijection from the $PSL(2, Z)$ fundamental domain, $\mathcal{F}$, to the complex sphere, given in terms of Eisenstein modular forms, $E_k(\tau)$, [73].

$P(z)$ is a holomorphic function whose singularities are chosen by the properties of the source branes.

The singularities of the metric turn out to be just conical at positions, $z = z_i$, where $P(z)$ has isolated poles. The metric turns out to be compact when $P(z)$ is a ratio of polynomials of equal degree whose numerator has 24 zeroes, such as for the choice

$$P(z) = \frac{4(24f)^3}{27g^2 + 4f^3}\,, \tag{2.82}$$

with $f(z)$ a polynomial of degree 8 and $g(z)$ a polynomial of degree 12. This gives a compactification of Type IIB supergravity on CP^1, corresponding to an F-theory reduction on $K3$ [70].

The metric function $C(z,\overline{z})$ is found by solving Einstein's equations, giving

$$e^{2C(z,\overline{z})} = (\mathrm{Im}\,\tau)\left|\eta^2(\tau)\prod_{i=1}^{N}(z - z_i)^{-1/12}\right|^2\,, \tag{2.83}$$

where $\eta(\tau) = q^{1/24}\prod_k(1-q^k)$, for $q = e^{2\pi i\tau}$, denotes the Dedekind η-function [73], and the product runs over the singularities of $P(z)$. Having explicit expressions for $\tau(z)$ and $C(z)$ the full solution is thus determined.

Finally, the asymptotic form of $\tau(z)$ near the singularities may be found using the known properties of $j(\tau)$. In particular, for large $\mathrm{Im}\,\tau$, $j(\tau) \simeq e^{-2\pi i\tau} + \cdots$ and so where $P(z) \simeq c_i/(z - z_i)$ the above solution implies

$$\begin{aligned} \tau(z) &\simeq \frac{1}{2\pi i}\,\ln(z - z_i) + \cdots \\ \text{and}\quad e^{2C(z,\overline{z})} &\simeq k\,\mathrm{Im}\,\tau \simeq -\frac{k}{2\pi}\,\ln|z - z_i| + \cdots\,, \end{aligned} \tag{2.84}$$

as $z \to z_i$, for k a positive constant.

2.6.3 Warped solutions

Because source-bulk matching is best understood for codimension-two, we specialize now to the case $n = 2$, in which case several things simplify.

First, the trace leading to the last equation carries no loss of information, and so the full set of Einstein equations become completely equivalent to eqs. (2.76) and (2.77). Second, it becomes convenient to use complex coordinates, $z := x^8 + ix^9 = y^1 + iy^2$, and write the compact metric in conformally flat form

$$\tilde{g}_{mn}\,\mathrm{d}x^m\mathrm{d}x^n = e^{2C}\,\mathrm{d}z\,\mathrm{d}\bar{z} = \mathrm{d}\rho^2 + e^{2B}\,\mathrm{d}\theta^2\,. \tag{2.85}$$

With these choices $\tilde{\nabla}^2 f = e^{-2C}\,\delta^{mn}\partial_m\partial_n f = 4\,e^{-2C}\,\partial\bar{\partial} f$, for any scalar field f, and the scalar curvature becomes $\tilde{R} = 2\,\tilde{\nabla}^2 C$.

The Einstein equations simplify to

$$\begin{aligned}
\frac{1}{4}\,R\,e^{2C} + e^{-dW}\partial\bar{\partial}e^{dW} &= (\textit{source terms}) \\
2\,\partial\bar{\partial}C + d\,e^{-W}\partial\bar{\partial}e^{W} - \frac{(\partial\tau\,\bar{\partial}\bar{\tau} + \partial\bar{\tau}\,\bar{\partial}\tau)}{(\tau-\bar{\tau})^2} &= (\textit{source terms})\,,
\end{aligned} \tag{2.86}$$

while the axio-dilaton equation of motion becomes independent of C:

$$\partial\bar{\partial}\,\tau + \frac{d}{2}\,(\partial W\,\bar{\partial}\tau + \bar{\partial}W\,\partial\tau) + \frac{2\,\partial\tau\bar{\partial}\tau}{\bar{\tau}-\tau} = (\textit{source terms})\,. \tag{2.87}$$

These coupled non-linear second order partial differential equations together with the appropriate boundary conditions would determine the full solution (we have $d = 8$ here). When there was no warping though the equations were non-linear we were lucky to have explicit solutions. It's unlikely that this will be the case now.

From these system of equations, which we are unable to analytically solve, we need to infer the curvature of our brane solutions. To this end we'll use the expression eq.(2.17). We will identify the contributions on the right-hand-side of eq. (2.17) for this example. Contributions II and III can be related to each other using the bulk brane matching conditions resulting from the Israel junction conditions [18]. These relate the asymptotic near brane properties of the bulk fields to properties of the source action and as shown in section 2.5 II and III cancel each other for our codimension-2 example.

We also note that since we only have a 0-form potential there is no space-filling 8-form flux living on the 7 brane world-volume. So term IV in eq. (2.17) gives no contribution. Secondly, as there are no higher form fields, eq. (2.52) tells us immediately that the on-shell bulk action (term I) also vanishes [8].

[8]This is more generally true. For an action of the form $S = \int (R - f_{ab}(\phi)\partial_\mu\phi^a\partial_\nu\phi^b g^{\mu\nu} +$

$$R = \frac{d}{2}\, S_{\rm on-shell} = 0\,. \tag{2.88}$$

Thus we have inferred, without solving the complicated coupled non-linear PDE's above, that even in the more general case, making allowance for warping, the solutions are still flat.

Notice that if we had not included the source term, our conventions are such that the warping term contributes an AdS sign if $N \cdot \partial W < 0$; *i.e.* W *decreases* towards the boundary. As we show below, the explicit asymptotic form for the bulk solution near the sources can be found in general, and for a codimension-two source situated at $\rho = 0$ (where ρ denotes proper radius) has the form $e^W \propto \rho^\omega$ with $\omega \geq 0$, in agreement with the AdS sign found in the no-go results [49, 50, 51, 52].

2.6.4 Near-source Kasner solutions

To find asymptotic solutions in the vicinity of a source it is convenient to use an orthogonal coordinate system including proper distance ρ. We therefore take the following *ansatz* for the metric and dilaton

$$\begin{aligned} \hat{\mathrm{d}s}^2 &= \mathrm{d}\rho^2 + \mathcal{A}\rho^{2\alpha}\mathrm{d}\theta^2 + \mathcal{B}\rho^{2\omega}\, g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \\ \tau &= k\theta + i\mathcal{F}\rho^{-q}\,, \end{aligned} \tag{2.89}$$

where $\mathcal{A} = a_0 + a_1 \ln\rho$, $\mathcal{B} = b_0 + b_1 \ln\rho$ and $\mathcal{F} = f_0 + f_1 \ln\rho$. This form captures, in particular, the asymptotic form of the known unwarped solutions described in section 2.6.2. Since the quantity b_1 first arises in the field equations at subdominant order as $\rho \to 0$, we initially neglect it here.

Given this choice, and keeping only the most singular part as $\rho \to 0$, the

other), with an Einstein Hilbert term for the metric and an arbitrary target space metric for the scalars, the R part always cancels the scalar kinetic part when the metric is put on-shell. Thus only the "other" part of the action (involving, for example, higher p-form fields) can contribute to $S_{on-shell}$

dilaton equation becomes

$$\rho^{-q-2}\left[(\alpha+d\omega-1)(f_1-qf_0-qf_1\ln\rho)-\frac{f_1^2}{f_0+f_1\ln\rho}\right]$$
$$+\rho^{-q-2}\left[\frac{a_1}{2}\frac{f_1-qf_0-qf_1\ln\rho}{a_0+a_1\ln\rho}+\frac{k^2\rho^{2q+2-2\alpha}}{(a_0+a_1\ln\rho)(f_0+f_1\ln\rho)}\right]=0\,. \tag{2.90}$$

We keep the variable d general here, although our Type IIB application is to $d=8$. The $(\rho\rho)$ Einstein equation similarly is

$$\begin{aligned} 0 &= \frac{1}{\rho^2}\left[\alpha(\alpha-1)+d\omega(\omega-1)+\frac{1}{2}q^2\right]+\frac{1}{\rho^2}\left[\frac{a_1(2\alpha-1)}{2(a_0+a_1\ln\rho)}-\frac{qf_1}{f_0+f_1\ln\rho}\right] \\ &\quad +\frac{1}{\rho^2}\left[\frac{f_1^2}{2(f_o+f_1\ln\rho)^2}-\frac{a_1^2}{4(a_0+a_1\ln\rho)^2}\right]\,, \end{aligned} \tag{2.91}$$

while the $(\theta\theta)$ equation gives

$$\frac{g_{\theta\theta}}{\rho^2}\left[\alpha(\alpha+d\omega-1)+\frac{a_1(2\alpha+d\omega-1)}{2(a_0+a_1\ln\rho)}-\frac{1}{4}\frac{a_1^2}{(a_0+a_1\ln\rho)^2}+\frac{k^2\rho^{2q+2-2\alpha}}{4(a_0+a_1\ln\rho)(f_0+f_1\ln\rho)^2}\right]=0\,. \tag{2.92}$$

To leading approximation the most singular part of these equations as $\rho\to 0$ is solved — upto terms of relative order $1/\ln\rho$ or more — if the powers satisfy the two 'Kasner' conditions,

$$\begin{aligned} \alpha+d\omega-1 &= 0 \\ \alpha(\alpha-1)+d\,\omega(\omega-1)+\frac{q^2}{2} &= 0\,. \end{aligned} \tag{2.93}$$

Using the first of these to simplify the latter allows it to be written

$$\alpha^2+d\,\omega^2+\frac{q^2}{2}=1\,. \tag{2.94}$$

This result holds if terms that depend on k are suppressed, which is true if the condition $q+1>\alpha$ is satisfied. In the case of interest, with $d=8$, α can be eliminated from the Kasner conditions to give

$$72\,\omega^2-16\,\omega+\frac{q^2}{2}=0\,, \tag{2.95}$$

with solutions

$$\omega=\frac{1}{9}\left(1\pm\sqrt{1-\frac{9q^2}{16}}\right)\,. \tag{2.96}$$

This shows that the only real solutions have $\omega \geq 0$, and consequently $\alpha \leq 1$. The limiting case with $q = \omega = 0$ and $\alpha = 1$ corresponds to a conical singularity at the brane position. Hence positive q is sufficient to have the Kasner condition satisfy the leading terms in the field equations near $\rho = 0$, with additional contributions of order $1/\ln\rho$ and smaller.

Notice in particular that because $\omega \geq 0$, the warp factor always either goes to zero or to a finite value when approaching a source. This ensures that the warping contribution to eq. (2.17) is never of the de Sitter sign.

We can now consider what happens if we do not neglect the logarithm, $b_1 \ln\rho$, in the warping. In this case

$$\hat{g}_{\mu\nu} = \rho^{2\omega}(W_0 + W_1 \ln\rho) g_{\mu\nu} \, . \tag{2.97}$$

In the dilaton equation, we get the additional (suppressed) terms

$$... + \rho^{-q-2} \left[\frac{W_1}{2} \frac{f_1 - q f_0 - q f_1 \ln\rho}{W_0 + W_1 \ln\rho} \right] . \tag{2.98}$$

In the $(\rho\rho)$ Einstein equation this gives

$$... + \frac{1}{\rho^2} \left[\frac{\omega}{W_0 + W_1 \ln\rho} - \frac{1}{2} \frac{W_1}{W_0 + W_1 \ln\rho} - \frac{1}{4} \frac{W_1^2}{(W_0 + W_1 \ln\rho)^2} \right] , \tag{2.99}$$

and finally for $(\theta\theta)$

$$... - \frac{g_{\theta\theta}}{\rho^2} \left[\frac{d}{2} \frac{\alpha W_1}{W_0 + W_1 \ln\rho} - \frac{d}{4} \frac{a_1 W_1}{(a_0 + a_1 \ln\rho)(W_0 + W_1 \ln\rho)} \right] . \tag{2.100}$$

From this we see that a log-term in W only modifies the field equations at a suppressed $1/\ln\rho$ level.

Appendix

Curvature and fluxes for simple Freund-Rubin examples

In this appendix we review several familiar Freund-Rubin $AdS_d \times S_p$ solutions to higher-dimensional supergravity, where $d + p = D$. We do so in order to explore how space-filling fluxes show up in eq. (2.17) of the main text.

Freund-Rubin solutions

Consider solutions to the field equations for the action

$$S = -\frac{1}{2\kappa_D^2} \int \mathrm{d}^D x \sqrt{-g_D} \, \left(\mathcal{R} + \frac{1}{2\,p!} F^2 \right) . \tag{2.101}$$

For the p-form threading a p-sphere, $F_{m_1.....m_p} = k \, \epsilon_{m_1.....m_p}$, Einstein's equations

$$\mathcal{R}_{MN} - \frac{1}{2} g_{MN} \mathcal{R} + \frac{1}{2(p-1)!} \left(F_{MABC..} F_N{}^{ABC..} - \frac{1}{2p} g_{MN} \, F^2 \right) = 0 \,, \tag{2.102}$$

yield solutions that are product spaces,

$$\mathrm{d}s^2 = g_{MN} \mathrm{d}x^M \mathrm{d}x^N = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + \tilde{g}_{mn} \, \mathrm{d}x^m \mathrm{d}x^n \,, \tag{2.103}$$

with curvatures

$$\tilde{R} = -\frac{k^2 p(D-p-1)}{2(D-2)} \quad \text{and} \quad R = \frac{k^2(2p-D)}{2(D-2)} \,. \tag{2.104}$$

Here $\tilde{R}$ is the Ricci scalar associated with the p-sphere metric (which is negative in our conventions), $\tilde{g}_{mn}$, R is the (positive) Ricci scalar of a d-dimensional anti-de Sitter metric, $g_{\mu\nu}$. $\mathcal{R}_{MN}$ is the Ricci tensor for the full D-dimensional metric g_{MN}. (In the absence of warping we need not distinguish $\hat{g}_{\mu\nu}$ from $g_{\mu\nu}$.)

Example: 11D supergravity

In this section we consider several examples from 11D supergravity that illustrate the equality (2.17) with and without space-filling fluxes.

Since the Chern-Simons term does not contribute, Freund-Rubin solutions for 11-D supergravity can be obtained using the 4-form field strength, G_{MNPQ}, and the following action

$$S_{11} = -\frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-g_{11}} \left[\mathcal{R} + \frac{1}{2(4!)} G_4^2 \right] . \tag{2.105}$$

There are two natural choices, depending on whether the 4-form flux threads the anti-de Sitter or spherical dimensions.

$AdS_7 \times S_4$

First consider solutions of the form $AdS_7 \times S_4$, for which the only nonzero components of G_4 are along the 4-sphere directions:

$$G_{mnpq} = 3n \, \epsilon_{mnpq} \quad \text{and so} \quad G_4^2 = (9n^2)4! \, . \tag{2.106}$$

Einstein's equations are

$$\mathcal{R}_{MN} - \frac{1}{2} g_{MN} \mathcal{R} + \frac{1}{12} \left(G_{MABC} G_N{}^{ABC} - \frac{1}{8} g_{MN} G_4^2 \right) = 0 \, , \tag{2.107}$$

and so taking the 11-, 7- and 4-dimensional traces of eq. (2.107) one finds

$$\mathcal{R} = -\frac{3n^2}{2} \, , \qquad R = g^{\mu\nu} \mathcal{R}_{\mu\nu} = \frac{21n^2}{2} \quad \text{and} \quad \tilde{R} = \tilde{g}^{mn} \tilde{R}_{mn} = -12n^2 \, , \tag{2.108}$$

corresponding to $AdS_7 \times S_4$.

One can use these to check eq. (2.17):

$$\begin{aligned} -\frac{1}{2\kappa_7^2} \int d^7x \sqrt{-g_7} \, R &= -\frac{21\, n^2}{4\, \kappa_7^2} \int d^7x \\ \text{and} \quad S_{\text{on-shell}} &= -\frac{1}{2\kappa_{11}^2} \int d^{11} \sqrt{-g_{11}} \left[-\frac{3n^2}{2} + \frac{(9n^2)4!}{2(4!)} \right] \\ &= -\frac{3n^2}{2\, \kappa_{11}^2} \int d^{11} \sqrt{-g_{11}} \, , \end{aligned}$$

and so

$$-\frac{1}{2\kappa_7^2} \int d^7x \sqrt{-g_7} \, R = \frac{7}{2} \, S_{\text{on-shell}} \, , \tag{2.109}$$

as required by (2.17) for a unwarped solution of maximal symmetry without space filling flux.

$AdS_4 \times S_7$

Now consider the solution $AdS_4 \times S_7$, which involves a space-filling flux: $G_{\mu\nu\rho\sigma} = 3m\,\epsilon_{\mu\nu\rho\sigma}$. From Einstein's equations one finds

$$\mathcal{R} = \frac{3m^2}{2}\,, \quad \tilde{R} = \tilde{g}^{mn}\tilde{R}_{mn} = -\frac{21m^2}{2} \quad \text{and} \quad R = g^{\mu\nu}R_{\mu\nu} = 12m^2\,. \tag{2.110}$$

In this case one finds a mismatch between

$$-\frac{1}{2\kappa_4^2}\int \mathrm{d}^4x\sqrt{-g_4}\;R \quad \text{and} \quad \frac{4}{2}\,S_{\text{on}-\text{shell}}\,. \tag{2.111}$$

This difference is accounted for by including the flux contribution to $g^{\mu\nu}\partial\mathcal{L}^{11}/\partial g^{\mu\nu}$, which gives a term of the form of eq. (2.22), as required by eq. (2.17).

Alternatively, one can work with a dual Lagrangian containing a kinetic term for the 7-form, H, that is dual to G:

$$S_{\text{dualized}} = -\frac{1}{2\kappa_{11}^2}\int \mathrm{d}^{11}x\sqrt{-g_{11}}\;\left[\mathcal{R} + \frac{1}{2(7!)}\,H_7^2\right]\,. \tag{2.112}$$

In this description the seven form threads only internal directions and has no space-filling components, and the dualized action evaluates to

$$-\frac{1}{2\kappa_4^2}\int \mathrm{d}^4x\sqrt{-g_4}\,R = \frac{4}{2}\,S_{\,\text{on-shell (dualized)}}\,. \tag{2.113}$$

Recall for these purposes that although dualization is a symmetry of the equations of motion, it is not a symmetry of the action.

Chapter 3

Superspace formulation of SCFTs with higher spin operators

We now turn to 3d SCFTs. This is a vast arena of current research. These theories exist aplenty as fixed lines of renormalisation group flows in three dimensions in the form of superconformal Chern-Simons theories. A classic example, much studied recently, is ABJ theory [28]. This is an $\mathcal{N} = 6$ superconformal Chern-Simons theory with the gauge group $U_k(N) \times U_{-k}(M)$. We'll not explicitly study superconformal Chern-Simons theories in this thesis but our general results on 3d SCFTs apply to these theories in particular. The techniques and formalism developed in this chapter and the next could probably be used with advantage in studying higher spin operators/currents and their correlation functions in ABJ theory and its bulk holographic dual.

In section 3.1 we consider 3d superspace, and the differential form of various operators which act in it. The construction of superconformally covariant structures in superspace is reviewed. In section 3.2 on-shell supercurrent multiplets for higher spin currents in the free theory are constructed out of the superfields. In section 3.3 we make a few remarks about the structure of anomalous conservation equations for 3d CFTs and SCFTs with weakly broken higher spin symmetry. In an appendix we list our conventions and some useful identities.

3.1 Superspace

We begin by reviewing superspace in three dimensions and the covariant structures that it admits, following the paper of Park [36]. Our conventions are summarized in an appendix to this chapter.

In order to study $\mathcal{N} = m$ superconformal field theories in 3 dimensions we employ a superspace whose coordinates are the 3 spacetime coordinates x^μ together with the $2m$ fermionic coordinates θ^a_α. Here $\alpha = 1, 2$ is a spacetime spinor index while $a = 1 \ldots m$ is the R-symmetry index, where the θs (and the supercharges Q^a_αs) are Majorana spinors that lie in the vector representation of the R-symmetry group $SO(\mathcal{N})$. The superconformal algebra, listed in (3.56) in the appendix, is implemented in superspace by the construction

$$
\begin{aligned}
P_\mu &= -i\partial_\mu, \\
M_{\mu\nu} &= -i\left(x_\mu\partial_\nu - x_\nu\partial_\mu - \frac{1}{2}\epsilon_{\mu\nu\rho}(\gamma^\rho)_\alpha{}^\beta\theta^a_\beta\frac{\partial}{\partial\theta^a_\alpha}\right) + \mathcal{M}_{\mu\nu}, \\
D &= -i\left(x^\nu\partial_\nu + \frac{1}{2}\theta^{\alpha a}\frac{\partial}{\partial\theta^\alpha_a}\right) + \Delta, \\
K_\mu &= -i\left(\left(x^2 + \frac{(\theta^a\theta^a)^2}{16}\right)\partial_\mu - 2x_\mu\left(x\cdot\partial + \theta^{\alpha a}\frac{\partial}{\partial\theta^\alpha_a}\right) + (\theta^a X_+\gamma_\mu)^\beta\frac{\partial}{\partial\theta^\beta_a}\right) \\
&= x^\nu M_{\nu\mu} - x_\mu D + \frac{i}{2}(\theta^a\gamma_\mu X)^\alpha\frac{\partial}{\partial\theta^\alpha_a} - \frac{i}{16}(\theta^a\theta^a)^2\partial_\mu + \frac{(\theta^a\theta^a)}{4}(\theta^b\gamma_\mu)^\alpha\frac{\partial}{\partial\theta^\alpha_b}, \\
Q^a_\alpha &= \frac{\partial}{\partial\theta^\alpha_a} - \frac{i}{2}\theta^{\beta a}(\gamma^\mu)_{\beta\alpha}\partial_\mu, \\
S^a_\alpha &= -(X_+)_\alpha{}^\beta Q^a_\beta - i\theta^a\theta^b\frac{\partial}{\partial\theta^\alpha_b} - i\theta^a_\alpha\theta^{b\beta}\frac{\partial}{\partial\theta^\beta_b} + \frac{i}{2}(\theta^b\theta^b)\frac{\partial}{\partial\theta^\alpha_a} \\
&= -(X_-)_\alpha{}^\beta\frac{\partial}{\partial\theta^\beta_a} + \frac{\theta^a_\alpha}{2}D + \frac{1}{4}\epsilon_{\mu\nu\rho}(\gamma^\rho\theta^a)_\alpha M^{\mu\nu} - \frac{(\theta^b\theta^b)}{8}\theta^{a\beta}\partial_{\beta\alpha} - \frac{3i}{4}\left(\theta^a_\alpha\theta\frac{\partial}{\partial\theta} + \theta^a\theta^b\frac{\partial}{\partial\theta^\alpha_b}\right), \\
I^{ab} &= -i\left(\theta^a\frac{\partial}{\partial\theta_b} - \theta^b\frac{\partial}{\partial\theta_a}\right) + \mathcal{I}^{ab}\ .
\end{aligned}
\tag{3.1}
$$

Here the derivative expressions act on superspace coordinates while the operators $\mathcal{M}$, Δ and $\mathcal{I}^{ab}$ act on the operators (states) which carry tensor structure, non-zero scaling dimensions and transform non-trivially under R-symmetry. All indices are contracted in matrix notation (the spinors are contracted from north-west to south-east, see the appendix to this chapter) and the definitions of X_+, X_- are given in (3.9). Note that $x^2 + \frac{(\theta^a\theta^a)^2}{16} = \frac{1}{2}(X_+X_-)_\alpha^{\ \alpha}$ (this combination appears in the expression for K_μ above). The supercovariant derivative operator D^a_α is defined by

$$D^a_\alpha = \frac{\partial}{\partial\theta^\alpha_a} + \frac{i}{2}\theta^{a\beta}\partial_{\beta\alpha}, \tag{3.2}$$

The operator D^i_α has the property that it anticommutes will all supersymmetry generators

$$\{D^a_\alpha, Q^b_\beta\} = 0 \tag{3.3}$$

Note also that

$$\{D^a_\alpha, D^b_\beta\} = -P_{\alpha\beta}\delta^{ab} \tag{3.4}$$

In what follows we will sometimes need to construct functions built out of coordinates in superspace that are invariant under superconformal transformations. Given two points in superspace, (x_1, θ_1) and (x_2, θ_2), it is obvious that $\theta_{12} = \theta_1 - \theta_2$ is annihilated by the supersymmetry generators. It is also easy to verify that the supersymmetrized coordinate difference

$$\tilde{x}^\mu_{12} = x^\mu_{12} + \frac{i}{2}\theta_1^{a\alpha}(\gamma^\mu)_\alpha^{\ \beta}\theta^a_{2\beta} \tag{3.5}$$

is also annihilated by all Q_α.

Any vector of $SO(2,1)$ may equally be regarded as a symmetrized bispinor. So x^μ may be represented in terms of bispinors by the 2×2 matrix $X = x\cdot\gamma$. In this notation (3.5) may be rewritten as

$$(\tilde{X}_{12})_\alpha^{\ \beta} = (X_{12})_\alpha^{\ \beta} + i\theta^a_{1\alpha}\theta_2^{a\beta} + \frac{i}{2}(\theta^a_1\theta^a_2)\delta_\alpha^{\ \beta} \tag{3.6}$$

While an arbitrary function of θ_{12} and $\tilde{X}_{12}$ is annihilated by the supersymmetry operator, it is not, in general, annihilated by the generator of superconformal transformations. In order to build superconformally invariant expressions it is useful to note that

$$S^a_\alpha = IQ^a_\alpha I \tag{3.7}$$

where I is the superinversion operator, whose action on the coordinates of superspace is given by

$$I(x^\mu) = \frac{x^\mu}{x^2 + \frac{(\theta^a\theta^a)^2}{16}} \tag{3.8}$$

To define the superinversion properties of spinors, it is useful to define the objects

$$X_{\pm} = X \pm \frac{i}{4}(\theta^a \theta^a)\mathbb{1}. \tag{3.9}$$

It follows from (3.8) that this object transforms homogeneously under inversions

$$\begin{aligned} I(X_{\pm}) &= X_{\pm}^{-1} \\ I(\theta^a_{\alpha}) &= (X_{+}^{-1}\theta^a)_{\alpha} \\ I(\theta^{a\beta}) &= -(\theta^a X_{-}^{-1})^{\beta} \end{aligned} \tag{3.10}$$

(Here X is the 2×2 matrix corresponding to a particular superspace point, not a coordinate difference).

Using these rules it follows that the following objects (see [36, 37, 38]) transform homogeneously under inversions:

$$(X_{ij+})_{\alpha}^{\ \beta} = (X_{i+})_{\alpha}^{\ \beta} - (X_{j-})_{\alpha}^{\ \beta} + i\theta^a_{i\alpha}\theta^{\beta a}_j \tag{3.11}$$

$$(X_{ij-})_{\alpha}^{\ \beta} = (X_{i-})_{\alpha}^{\ \beta} - (X_{j+})_{\alpha}^{\ \beta} - i\theta^a_{j\alpha}\theta^{\beta a}_i \tag{3.12}$$

For example,

$$I\left(X_{ij+}\right)_{\alpha}^{\ \beta} = I\left((X_{i+})_{\alpha}^{\ \beta} - (X_{j-})_{\alpha}^{\ \beta} + i\theta^a_{i\alpha}\theta^{a\beta}_j\right) = -(X_{i+}^{-1})_{\alpha}^{\ \gamma}(X_{ij+})_{\gamma}^{\ \delta}(X_{j-}^{-1})_{\delta}^{\ \beta} \tag{3.13}$$

Thus $X_{ij\pm}$ transform homogeneously under inversions and are also annihilated by the generators of supersymmetry. Moreover it may be demonstrated [36, 37, 38] that

$$X_{ij\pm} = \tilde{X}_{ij} \pm \frac{i}{4}\theta^2_{ij}\mathbb{1} \qquad X_{ij\pm} = -X_{ji\mp} \tag{3.14}$$

The second relation above implies that once all the X_{ij+} are known, all the X_{ij-} are determined (and vice-versa) by this relation. In performing various manipulations it is useful to note that

$$X_{+}X_{-} = \left(x^2 + \frac{1}{16}(\theta^a\theta^a)^2\right)\mathbb{1} \tag{3.15}$$

$$X_{ij+}X_{ij-} = \left(\tilde{x}^2_{ij} + \frac{1}{16}(\theta^a_{ij}\theta^a_{ij})^2\right)\mathbb{1} \tag{3.16}$$

so that

$$
\begin{aligned}
(X_{\pm})^{-1} &= \frac{X_{\mp}}{x^2 + \frac{1}{16}(\theta^a\theta^a)^2} \\
(X_{ij\pm})^{-1} &= \frac{X_{ij\mp}}{\tilde{x}_{ij}^2 + \frac{1}{16}(\theta^a_{ij}\theta^a_{ij})^2}
\end{aligned}
\tag{3.17}
$$

(note that the the R-symmetry index a is summed over but that, throughout, i, j $(= 1, 2, 3)$ label points in superspace and are not summed over).

There also exist fermionic covariant structures (which are identically zero in the non-supersymmetric case) which are constructed out of the superspace coordinates as follows [36, 37, 38]:

$$
\Theta^a_{1\alpha} = \left((X^{-1}_{21+}\theta^a_{21})_\alpha - (X^{-1}_{31+}\theta^a_{31})_\alpha\right) \tag{3.18}
$$

Θ_2, Θ_3 are defined similarly. Its transformation properties under superinversion are

$$
\Theta^a_{i\alpha} \to -(X_{i-})_\alpha^{\ \beta}\Theta^b_{i\beta}I_b^{\ a} \qquad \Theta^{\alpha\, a}_i \to I^{T\,a}_{\ \ b}\,\Theta^{\beta\, b}_i(X_{i+})_\beta^{\ \alpha} \tag{3.19}
$$

The basic covariant structures $X_{ij\pm}$, $\Theta^a_{i\alpha}$ are annihilated by the generators of supersymmetry. For this reason they form the basic building blocks for the construction of superconformal invariants, as we will explain in a later section.

Polarization spinors: Since we will be dealing extensively with higher spin operators and their correlators, it will be useful to adopt a formalism, developed in [35], in which the information about the tensor structure is encoded in *polarization spinors*: λ_α. These auxiliary objects are book-keeping devices to keep track of the tensorial nature of correlators in an efficient manner. They are defined to be real, bosonic, two-component objects transforming as spinors of the $3d$ Lorentz group (see [35]). Being spinors in 2+1 dimensions fixes their transformation law under superinversions:

$$
\lambda_\alpha \to (X_+^{-1}\lambda)_\alpha \quad , \quad \lambda^\beta \to -(\lambda X_-^{-1})^\beta \tag{3.20}
$$

(This is the same as the transformation law of the θ's).

A higher spin primary operator $J_{\mu_1\mu_2.....\mu_s}$ with spin s can be represented in spinor components by $J_{\alpha_1\alpha_2.....\alpha_{2s}} \equiv (\sigma^{\mu_1})_{\alpha_1\alpha_2}(\sigma^{\mu_2})_{\alpha_3\alpha_4}....(\sigma^{\mu_s})_{\alpha_{2s-1}\alpha_{2s}}J_{\mu_1\mu_2.....\mu_s}$. We note that this represents an operator supermultiplet in contradistinction to [35] where the non-supersymmetric conformal case was considered (also, J need not necessarily be a conserved current). We then define $J_s \equiv \lambda^{\alpha_1}\lambda^{\alpha_2}...\lambda^{\alpha_{2s}}J_{\alpha_1\alpha_2.....\alpha_{2s}}$.

The 3-point function $\langle J_{s_1}(x_1,\theta_1,\lambda_1)J_{s_2}(x_2,\theta_2,\lambda_2)J_{s_3}(x_3,\theta_3,\lambda_3)\rangle$ is then a superconformal invariant constructed out of three points in (augmented) superspace with co-ordinates labelled by (x_i,θ_i,λ_i). The tensor structure of the correlator, instead of being represented by indices, is encoded by the polynomial in λ's (the 3-point function being a multinomial with degree $\lambda_1^{2s_1}\lambda_2^{2s_2}\lambda_3^{2s_3}$ for each term).

3.2 Free SCFTs in superspace and conserved higher spin currents

In this section we study free superconformal theories, with $\mathcal{N} = 1, 2, 3, 4$ and 6 supersymmetry in superspace and describe the construction of conserved higher spin currents which these theories possess.

These currents constitute the full local gauge-invariant operator spectrum of the theories considered. In the non-supersymmetric case the bosonic conserved currents and the violation, due to interactions, of their conservation by $\frac{1}{N}$ effects play a central role in the solution of three point functions in these theories [31, 32]. The currents we consider in this section are the supersymmetric extension of the bosonic currents considered in [35, 31, 32]. We construct the supercurrents, using the superspace formalism described in sections 3.1, in terms of onshell superfields and supercovariant derivatives.

3.2.1 General structure of the current superfield

Let us start by first describing the structure of the $\mathcal{N} = 1$ supercurrents. A general spin s supercurrent multiplet can be written as a superfield carrying $2s$ spacetime spinor indicies and can be expanded in components as follows

$$\Phi^{\alpha_1\alpha_2...\alpha_{2s}} = \phi^{\alpha_1\alpha_2...\alpha_{2s}} + \theta_\alpha\psi^{\alpha\alpha_1\alpha_2...\alpha_{2s}} + \theta^{\{\alpha_1}\chi^{\alpha_2...\alpha_{2s}\}} + \theta^\alpha\theta_\alpha B^{\alpha_1\alpha_2...\alpha_{2s}} \quad (3.21)$$

where all the indices α_1,$\alpha_2, \ldots \alpha_{2s}$ are symmetrized. The conservation (shortening) condition for the supercurrent is

$$D_{\alpha_1}\Phi^{\alpha_1\alpha_2\ldots\alpha_{2s}} = 0 \tag{3.22}$$

where D_α is the supercovariant derivative given by

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{i}{2}\theta^\beta\partial_{\beta\alpha} \tag{3.23}$$

Using eqs.(3.23) and (3.21) we obtain

$$\begin{aligned}\delta_{\alpha_1}^{\{\alpha_1}\chi^{\alpha_2\ldots\alpha_{2s}\}} \;&+\; \theta_{\alpha_1}(2B^{\alpha_1\alpha_2\ldots\alpha_{2s}} - \frac{i}{2}\partial_\beta{}^{\alpha_1}\phi^{\beta\alpha_2\ldots\alpha_{2s}}) \\ &-\; \frac{i}{2}\theta^2\partial_{\alpha\alpha_1}\psi^{\;\alpha\alpha_1\alpha_2\ldots\alpha_{2s}} + \frac{i}{2}\theta^\beta\partial_{\beta\alpha_1}\theta^{\{\alpha_1}\chi^{\alpha_2\ldots\alpha_{2s}\}} = 0\end{aligned} \tag{3.24}$$

This implies

$$\chi^{\alpha_2\ldots\alpha_{2s}} = 0 \tag{3.25}$$

while the symmetric part of the θ component gives

$$B^{\alpha_1\alpha_2\ldots\alpha_{2s}} = \frac{i}{4}\partial_\beta{}^{\{\alpha_1}\phi^{|\beta|\alpha_2\ldots\alpha_{2s}\}} \tag{3.26}$$

whereas the antisymmetric part gives

$$\epsilon_{\alpha_1\alpha_2}\partial^{\alpha_1}_\beta\phi^{\beta\alpha_2\ldots\alpha_{2s}} = 0 \Rightarrow \partial_{\alpha_1\alpha_2}\phi^{\alpha_1\alpha_2\ldots\alpha_{2s}} = 0 \tag{3.27}$$

which is the current conservation equation for the current ϕ. Since $\chi = 0$, the $\theta\theta$ component gives the current conservation equation for ψ

$$\partial_{\alpha\alpha_1}\psi^{\alpha\alpha_1\ldots\alpha_{2s}} = 0 \tag{3.28}$$

Thus the form of the supercurrent multiplet for a spin s conserved current is

$$\Phi^{\alpha_1\alpha_2\ldots\alpha_{2s}} = \phi^{\alpha_1\alpha_2\ldots\alpha_{2s}} + \theta_\alpha\psi^{\alpha\alpha_1\alpha_2\ldots\alpha_{2s}} + \frac{i}{4}\theta^\alpha\theta_\alpha\partial^{\{\alpha_1}_\beta\phi^{|\beta|\alpha_2\ldots\alpha_{2s}\}} \tag{3.29}$$

The general structure of the current superfield described above goes through for higher supersymmetries as well. For higher supersymmetries the conservation equation reads

$$D^a{}_{\alpha_1}\Phi^{\alpha_1\alpha_2\ldots\alpha_{2s}} = 0 \tag{3.30}$$

where $a = 1, 2 \ldots \mathcal{N}$ is the R-symmetry index[1]. In the case of an $\mathcal{N} = m$ spin-s current multiplet, the currents $\phi^{\alpha_1\alpha_2\ldots\alpha_{2s}}$ and $\psi^{\alpha\alpha_1\alpha_2\ldots\alpha_{2s}}$ are themselves $\mathcal{N} = m-1$ spin s and spin $s+\frac{1}{2}$ conserved current superfields (depending on the grassmann coordinates θ^a_α: $a = 1, \ldots m-1$) while the θ_α in (3.29) is the left over grassmann coordinate θ^m_α. Thus we see the general structure of the supercurrent multiplets: An $\mathcal{N} = m$ spin s supercurrent multiplet breaks up into two $\mathcal{N} = m-1$ supercurrents with spins s and $s+\frac{1}{2}$ respectively.

This structure can be used to express higher supercurrents superfields in term of components. For instance, the $\mathcal{N} = 2$ spin s currents superfield can be expanded in components as follows

$$\begin{aligned}\Phi^{\alpha_1\alpha_2\ldots\alpha_{2s}} = \varphi^{\alpha_1\alpha_2\ldots\alpha_{2s}} + \theta^a_\alpha(\psi^a)^{\alpha\alpha_1\alpha_2\ldots\alpha_{2s}} + \frac{1}{2}\epsilon_{ab}\theta^a_\alpha\theta^b_\beta\mathcal{A}^{\alpha\beta\alpha_1\alpha_2\ldots\alpha_{2s}}\\ + \text{term involving derivatives of } \varphi, \psi^a \text{ and } \mathcal{A}\end{aligned} \tag{3.31}$$

where a, b are R-symmetry indicies and take values in $\{1, 2\}$. The conformal state content so obtained, namely $(\varphi, \psi^1, \psi^2, \mathcal{A})$ above, match exactly with the decomposition of spin s supercurrent multiplet into conformal multiplets.

3.2.2 Free field construction of currents

In this section we give an explicit construction of the conserved higher spin supercurrents in term of superfields for $\mathcal{N} = 1, 2, 3, 4, 6$ free SCFTs. [2]

$\mathcal{N} = 1$

In a free field theory the exactly conserved higher spin currents are bilinear in the free fields with symmetrised derivatives. For example, in the free bosonic $O(N)$ vector models the currents have the schematic form: $J_{(s)} \sim \phi_i\partial^s\phi^i + \ldots$ where any number of the s derivatives can act left or right and the whole expression is symmetrised and traceless. The coefficients in this linear combination can be

[1] Note that for $\mathcal{N} > 1$, (3.30) is true only for R-symmetry singlet currents. For currents carrying non-trivial R-symmetry representation the shortening condition is different. In this paper we will only need the shortening condition (3.30).

[2] This part was worked out by V. Umesh and T. Sharma in collaboration with Shiraz Minwalla. It is included here for the sake of completeness.

determined by current conservation ($\partial.J_{(s)} = 0$), where the free field equations of motion ($\partial^2\phi = 0$) are also used.

In a similar manner, the exactly conserved higher spin currents in a free SCFT are constructed from scalar superfield bilinears together with supercovariant derivatives. The scalar superfield satisfies $D_\alpha D^\alpha \Phi = 0$, and using the current conservation equation:

$$\frac{\partial}{\partial \lambda^\alpha} D^\alpha J^{(s)} = 0. \tag{3.32}$$

the spin s supercurrent for the $\mathcal{N} = 1$ free SCFT can be expressed in term of the $\mathcal{N} = 1$ free superfield Φ as follows

$$J^{(s)} = \sum_{r=0}^{2s} (-1)^{\frac{r(r+1)}{2}} \binom{2s}{r} D^r \bar{\Phi} D^{2s-r} \Phi \tag{3.33}$$

where $J^{(s)} = \lambda^{\alpha_1}\lambda^{\alpha_2}\cdots\lambda^{\alpha_{2s}} J_{\alpha_1\alpha_2\cdots\alpha_{2s}}$ and $D = \lambda^\alpha D_\alpha$, and λ_α's are polarization spinors and $s = 0, \frac{1}{2}, 1 \ldots$. The currents are of both integral and half-integral spins. It can be verified that the above is the unique expression for the conserved spin-s current in $\mathcal{N} = 1$ free field theory. We note here that the stress tensor lies in the spin $\frac{3}{2}$ current supermultiplet (which also contains the supersymmetry current), and thus is conserved exactly even in interacting theory.

$\mathcal{N} = 2$

We give the expression of the conserved current in terms of the free $\mathcal{N} = 2$ superfield Φ and its complex conjugate $\bar{\Phi}$.

$$J^{(s)} = \sum_{r=0}^{s} \left\{ (-1)^{r(2r+1)} \binom{2s}{2r} \partial^r \bar{\Phi} \partial^{s-r} \Phi + (-1)^{(r+1)(2r+1)} \binom{2s}{2r+1} \partial^r \bar{D}\bar{\Phi} \partial^{s-r-1} D\Phi \right\} \tag{3.34}$$

where $\partial = i\lambda^\alpha \gamma^\mu_{\alpha\beta} \lambda^\beta \partial_\mu$, $D = \lambda^\alpha D_\alpha$ and $s = 0, 1, 2 \ldots$. The spin 1 supercurrent multiplet contains the stress tensor, supersymmetry current and R-current, and its conservation holds even in the interacting superconformal theory.

As described above in subsection 3.2.1 these $\mathcal{N} = 2$ currents can be decomposed into $\mathcal{N} = 1$ currents. It is straightforward to check that the currents 3.34 when expanded in θ^2_α as in (3.29) correctly reproduce the $\mathcal{N} = 1$ currents (3.33). This give a consistency check of these $\mathcal{N} = 2$ currents.

$\mathcal{N}=3$

The $\mathcal{N}=3$ chirality constraint on the matter superfield Φ^k is

$$\begin{aligned} D^{\{ij}\Phi^{k\}} &= \frac{1}{3!}(D^{ij}\Phi^k + D^{ik}\Phi^j + D^{jk}\Phi^i) = 0 \\ \text{or equivalently} \qquad D^{ij}\Phi^k &= -\frac{1}{3}\big(D^{il}\Phi_l\epsilon^{jk} + D^{jl}\Phi_l\epsilon^{ik}\big) \end{aligned} \tag{3.35}$$

where $D^{ij}_\alpha = (\sigma^a)^{ij} D^a_\alpha$.

From this chirality constraint the following identities, which would be useful in proving current conservation, can be derived[3]

$$D^{ij}_\alpha D^{mn}_\beta \Phi^k = \frac{1}{2}\big(i\partial_{\alpha\beta}\Phi^i\epsilon^{jm}\epsilon^{nk} + i\partial_{\alpha\beta}\Phi^i\epsilon^{jn}\epsilon^{mk} + i\partial_{\alpha\beta}\Phi^j\epsilon^{im}\epsilon^{nk} + i\partial_{\alpha\beta}\Phi^j\epsilon^{in}\epsilon^{mk}\big) \tag{3.36}$$

Contracting various indicies, the following relations can be obtained from (3.36) as corollaries

$$\begin{aligned} D^{\alpha ij} D^{mn}_\alpha \Phi^k &= 0 \\ D^{ij}_\alpha D^{mk}_\beta \Phi_k &= -\frac{3}{2}\big(i\partial_{\alpha\beta}\Phi^i\epsilon^{jm} + i\partial_{\alpha\beta}\Phi^j\epsilon^{im}\big) \\ D^{ij}_\alpha D_{ij\beta}\Phi^k &= -3i\partial_{\alpha\beta}\Phi^k = \frac{2}{3}D^k_j D^{ji}\Phi_i \end{aligned} \tag{3.37}$$

We give here the expression for the conserved currents in terms of the $\mathcal{N}=3$ superfield Φ^i.

$$\begin{aligned} J^{(s)} &= \sum_{r=0}^{s}(-1)^r\binom{2s}{2r}\partial^r\bar{\Phi}^i\partial^{s-r}\Phi_i + \frac{2}{9}\sum_{r=0}^{s-1}(-1)^{r+1}\binom{2s}{2r+1}\partial^r D_i{}^j\bar{\Phi}^i\partial^{s-r-1}D_j{}^k\Phi_k \\ J^{(s+\frac{1}{2})} &= \sum_{r=0}^{s}\left\{(-1)^r\binom{2s+1}{2r}\partial^r\bar{\Phi}^i\partial^{s-r}D_i{}^j\Phi_j + (-1)^{r+1}\binom{2s+1}{2r+1}\partial^r D_i{}^j\bar{\Phi}^i\partial^{s-r}\Phi_j\right\} \end{aligned} \tag{3.38}$$

where $\partial = i\lambda^\alpha\gamma^\mu_{\alpha\beta}\lambda^\beta\partial_\mu$, $D = \lambda^\alpha D_\alpha$ and $s = 0, 1, 2\ldots$. The stress energy tensor in this case lies the spin $\frac{1}{2}$ supercurrent multiplet along with the R-current and supersymmetry currents. The conservation of this supercurrent holds exactly even in the interacting superconformal theory.

[3]see appendix for $SO(3)$ conventions

$\mathcal{N} = 4$

The R-symmetry in this case is $SO(4)$ (equivalently $SU(2)_l \times SU(2)_r)$[4]. The supercharges $Q^{\tilde{i}i}_\alpha$ transform in the 4 of $SO(4)$(equivalently $(2,2)$ of $SU(2)_l \times SU(2)_r$). The two matter superfields transform in the $(2,0)$ representation which implies that the scalar transforms in the $(2,0)$ while the fermions transform in $(0,2)$. The matter multiplet again satisfies a 'chirality' constraint

$$\begin{aligned} D^{\tilde{i}\{i}\Phi^{j\}} &= \frac{1}{2}(D^{\tilde{i}i}\Phi^j + D^{\tilde{i}j}\Phi^i) = 0, \\ \text{or equivalently} \qquad D^{\tilde{i}i}_\alpha \Phi^j &= -\frac{1}{2}\epsilon^{ij} D^{\tilde{i}k}_\alpha \Phi_k. \end{aligned} \tag{3.39}$$

where $D^{\tilde{i}j}_\alpha = (\bar{\sigma}^a)^{\tilde{i}j} D^a_\alpha$.

From this chirality constraint the following identities, useful in proving current conservation, can be derived[1]

$$D^{\tilde{i}i}_\alpha D^{\tilde{j}j}_\beta \Phi^k = 2i\partial_{\alpha\beta}\Phi^i \epsilon^{\tilde{i}\tilde{j}}\epsilon^{jk} \tag{3.40}$$

Contracting various indices, the following equations can be obtained from (3.40) as corollaries

$$\begin{aligned} D^{\alpha\tilde{i}i} D^{\tilde{j}j}_\alpha \Phi^k &= 0 \\ D^{\tilde{i}i}_\alpha D^{\tilde{j}j}_\beta \Phi_j &= -4i\partial_{\alpha\beta}\Phi^i\epsilon^{\tilde{i}\tilde{j}} \\ D^{\tilde{i}j}_\alpha D_{\beta\tilde{i}k}\Phi^k &= 2D^{\tilde{i}i}_\alpha D_{\beta\tilde{i}i}\Phi^j = 8i\partial_{\alpha\beta}\Phi^j. \end{aligned} \tag{3.41}$$

Using these equations it is straightforward to show that the following currents are conserved.

$$J^{(s)} = \sum_{r=0}^{s}(-1)^r \binom{2s}{2r}\ \partial^r\bar{\Phi}^i\ \partial^{s-r}\Phi_i + \frac{1}{8}\sum_{r=0}^{s-1}(-1)^r\binom{2s}{2r+1}\ \partial^r D^{\tilde{i}i}\bar{\Phi}_i\ \partial^{s-r-1}D_{\tilde{i}j}\Phi^j. \tag{3.42}$$

where $\partial = i\lambda^\alpha\gamma^\mu_{\alpha\beta}\lambda^\beta\partial_\mu$, $D = \lambda^\alpha D_\alpha$ and $s = 0, 1, 2 \ldots$. In this theory the stress energy tensor lies in the R-symmetry singlet spin zero supercurrent multiplet $(1, 0, \{0, 0\})$.

[4]The indices $a, b..$ take values $1, 2, 3, 4$ and represent the vector indices of $SO(4)$ while the fundamental indices of the $SU(2)_l$ and $SU(2)_r$ are denoted by $i, j...$ and $\tilde{i}, \tilde{j}...$

$\mathcal{N} = 6$

The field content of this theory is double of the field content of the $\mathcal{N} = 4$ theory. In $\mathcal{N} = 2$ language the field content is 2 chiral and 2 antichiral multiplets in fundamental of the gauge group. The R-symmetry in this theory is $SO(6)$ ($\equiv SU(4)$) under which the supercharges transform in vector representation (6 of $SO(6)$) while the 2+2 chiral and antichiral multiplets transform in chiral spinor representation (4 of $SU(4)$).

The $\mathcal{N} = 6$ shortening (chirality) condition on the matter multiplet is[5]

$$\begin{aligned} D^{ij}_\alpha \Phi^k &= D^{jk}_\alpha \Phi^i = D^{ki}_\alpha \Phi^j \\ \text{or equivalently} \quad D^a_\alpha \Phi^k &= -\frac{1}{10} D^b_\alpha \Phi^l (\bar{\gamma}^{ab})_l{}^k \end{aligned} \tag{3.43}$$

From this chirality constraint the following identities, which are useful in proving current conservation, can be derived[2]

$$\begin{aligned} D^a_\alpha D^b_\beta \Phi^k &= \frac{i}{2} \partial_{\alpha\beta} \Phi^k \delta^{ab} + \frac{i}{4} \partial_{\alpha\beta} \Phi^l (\bar{\gamma}^{ab})_l{}^k, \\ \text{or equivalently} \quad D^{ij}_\alpha D^{mn}_\beta \Phi^k &= -i \partial_{\alpha\beta} \left(\epsilon^{ijmn} \Phi^k + \epsilon^{kjmn} \Phi^i + \epsilon^{ikmn} \Phi^j - \epsilon^{ijkn} \Phi^m - \epsilon^{ijmk} \Phi^n \right) \end{aligned} \tag{3.44}$$

Taking the complex conjugate of equations (3.43) and (3.44), and using the property that γ^{ab} and $\bar{\gamma}^{ab}$ are antihermitian, we get

$$\begin{aligned} D^{ij}_\alpha \bar{\Phi}_k &= \frac{1}{3} \left(D^{il}_\alpha \bar{\Phi}_l \delta^j_k - D^{jl}_\alpha \bar{\Phi}_l \delta^i_k \right) \\ \text{or equivalently} \quad D^a_\alpha \bar{\Phi}_k &= \frac{1}{10} D^b_\alpha (\bar{\gamma}^{ab})_k{}^l \bar{\Phi}_l \end{aligned} \tag{3.45}$$

and

$$\begin{aligned} D^a_\alpha D^b_\beta \bar{\Phi}_k &= \frac{i}{2} \partial_{\alpha\beta} \bar{\Phi}_k \delta^{ab} - \frac{i}{4} \partial_{\alpha\beta} (\bar{\gamma}^{ab})_k{}^l \bar{\Phi}_l, \\ \text{or equivalently} \quad D^{ij}_\alpha D^{mn}_\beta \bar{\Phi}_k &= -i \partial_{\alpha\beta} \left(\epsilon^{ijmn} \Phi^k - \epsilon^{ljmn} \bar{\Phi}_l \delta^i_k - \epsilon^{ilmn} \bar{\Phi}_l \delta^j_k + \epsilon^{ijln} \bar{\Phi}_l \delta^m_k + \epsilon^{ijml} \bar{\Phi}_l \delta^n_k \right) \end{aligned} \tag{3.46}$$

[5]Here we revert back to lower case letters for the $SU(4)$ indices i, j (taking values $1, \ldots 4$) as there is no confusion with other R indices.

Using the above relation a straightforward computation shows that the following R-symmetry singlet integer spin currents are conserved

$$J^{(s)} = \sum_{r=0}^{s} (-1)^r \binom{2s}{2r} \partial^r \bar{\Phi}_p \, \partial^{s-r} \Phi^p - \frac{1}{24} \sum_{r=0}^{s-1} (-1)^{r+1} \binom{2s}{2r+1} \epsilon_{ijkl} \, \partial^r D^{ij} \bar{\Phi}_p \, \partial^{s-r-1} D^{kl} \Phi^p. \tag{3.47}$$

where $\partial = i\lambda^\alpha \gamma^\mu_{\alpha\beta} \lambda^\beta \partial_\mu$, $D = \lambda^\alpha D_\alpha$ and $s = 0, 1, 2 \ldots$. The stress-energy tensor of this theory lies, as in the $\mathcal{N} = 4$ theory, in the R-symmetry singlet spin zero multiplet $(1, 0, \{0, 0, 0\})$.

3.3 Weakly broken conservation

The free superconformal theories discussed above have an exact higher spin symmetry algebra generated by the charges corresponding to the infinite number of conserved currents that these theories possess. These free theories can be deformed into interacting theories by turning on $U(N)$ Chern-Simons (CS) gauge interactions, in a supersymmetric fashion and preserving the conformal invariance of free CFTs, under which the matter fields transform in fundamental representations. The CS gauge interactions do not introduce any new local degrees of freedom so the spectrum of local operators in the theory remains unchanged. Turning on the interactions breaks the higher spin symmetry of the free theory but in a controlled way which we discuss below. These interacting CS vector models are interesting in there own right as non trivial interacting quantum field theories. Exploring the phase structure of these theories at finite temperature and chemical potential, provides a platform for studying a lot of interesting physics, at least in the large N limit, using the techniques developed in [29].

From a more string theoretic point of view, a very interesting example of this class of theories is the $U(N) \times U(M)$ ABJ theory in the vector model limit $\frac{M}{N} \to 0$. ABJ theory in this vector model limit has recently been argued to be holographically dual a non-abelian supersymmetric generalization of the non-minimal Vasiliev theory in AdS_4 [29]. The ABJ theory thus connects, as its holographic duals, Vasiliev theory at one end to a string theory at another end. Increasing $\frac{M}{N}$ from 0 corresponds to increasing the coupling of $U(M)$ gauge interactions in the bulk Vasiliev theory. Thus, understanding the ABJ theory away from the vector model limit in an expansions in $\frac{M}{N}$ would be a first step towards understanding of how string theory emerges from 'quantum' Vasiliev theory.

In [31, 32] theories with exact conformal symmetry but weakly broken higher spin symmetry were studied. It was first observed in [31], and later used with great efficiency in [32], that the anomalous "conservation" equations are of the schematic form

$$\partial \cdot J_{(s)} = \frac{a}{N} J_{(s_1)} J_{(s_2)} + \frac{b}{N^2} J_{(s_1')} J_{(s_2')} J_{(s_3')} \tag{3.48}$$

plus derivatives sprinkled appropriately. The structure of this equation is constrained on symmetry grounds - the twist $(\Delta_i - s_i)$ of the L.H.S. is 3. If each J_s has conformal dimension $\Delta = s + 1 + O(1/N)$, and thus twist $\tau = 1 + O(1/N)$, the two terms on the R.H.S. are the only ones possible by twist matching. Thus we can have only double or triple trace deformations in the case of weakly broken conservation and terms with four or higher number of currents are not possible.

In the superconformal case that we are dealing with, since D has dimension $1/2$, $D \cdot J_{(s)}$ is a twist 2 operator. Thus in this case the triple trace deformation is forbidden and the only possible structure is more constrained:

$$D \cdot J_{(s)} = \frac{a}{N} J_{(s_1)} J_{(s_2)} \tag{3.49}$$

In view of this, it is feasible that in large N supersymmetric Chern-Simons theories the structure of correlation functions is much more constrained (compared to the non-supersymmetric case).

Appendix

Conventions

Spacetime spinors

The Lorentz group in $D = 3$ is $SL(2,\mathbb{R})$ and we can impose the Majorana condition on spinors, i.e., the fundamental representation is a real two component spinor $\psi_\alpha = \psi_\alpha^*$ ($\alpha = 1, 2$). The metric signature is mostly plus. $D = 3$ superconformal theories with $\mathcal{N}$ extended supersymmetry posses an $SO(\mathcal{N})$ R-symmetry which is part of the superconformal algebra, whose generators are real antisymmetric matrices I^{ab}, where a, b are the vector indices of $SO(\mathcal{N})$. The supercharges carry a vector R-symmetry index, Q^a_α, as do the superconformal generators S^a_α.

In $D = 3$ we can choose a real basis for the γ matrices

$$(\gamma_\mu)_\alpha{}^\beta \equiv (i\sigma^2, \sigma^1, \sigma^3) = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \tag{3.50}$$

Gamma matrices with both indices up (or down) are symmetric

$$(\gamma_\mu)_{\alpha\beta} \equiv (\mathbb{1}, \sigma^3, -\sigma^1) \qquad (\gamma_\mu)^{\alpha\beta} \equiv (\mathbb{1}, -\sigma^3, \sigma^1) \tag{3.51}$$

The antisymmetric ϵ symbol is $\epsilon^{12} = -1 = \epsilon_{21}$. It satisfies

$$\begin{aligned} \epsilon\gamma^\mu\epsilon^{-1} &= -(\gamma^\mu)^T \\ \epsilon\Sigma^{\mu\nu}\epsilon^{-1} &= -(\Sigma^{\mu\nu})^T \end{aligned} \tag{3.52}$$

where $\Sigma^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]$ are the Lorentz generators. The charge conjugation matrix C can be chosen to be the identity, which we take to be

$$-\epsilon\gamma^0 = C^{-1} \qquad \gamma^0\epsilon^{-1} = C \tag{3.53}$$

$C^{\alpha\beta}$ denotes the inverse of $C_{\alpha\beta}$. Spinors transform as follows

$$\psi'_\alpha \to -(\Sigma_{\mu\nu})_\alpha{}^\beta\psi_\beta.$$

Spinors are naturally taken to have index structure down, i.e., ψ_α.

The raising and lowering conventions are

$$\begin{aligned}\psi^\beta &= \epsilon^{\beta\alpha}\psi_\alpha \\ \psi_\alpha &= \epsilon_{\alpha\beta}\psi^\beta\end{aligned} \tag{3.54}$$

There is now only one way to suppress contracted spinor indices,

$$\psi\chi = \psi^\alpha\chi_\alpha,$$

and this leads to a sign when performing Hermitian conjugation

$$(\psi\chi)^* = -\chi^*\psi^*.$$

The γ matrices satisfy

$$(\gamma_\mu\gamma_\nu)_\alpha^{\ \beta} = \eta_{\mu\nu}\delta_\alpha^{\ \beta} + \epsilon_{\mu\nu\rho}(\gamma^\rho)_\alpha^{\ \beta} \tag{3.55}$$

where $\epsilon_{\mu\nu\rho}$ is the Levi-Civita symbol, and we set $\epsilon_{012} = 1$ ($\epsilon^{012} = -1$). The superconformal algebra is given below:

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\lambda}] &= i\left(\eta_{\mu\rho}M_{\nu\lambda} - \eta_{\nu\rho}M_{\mu\lambda} - \eta_{\mu\lambda}M_{\nu\rho} + \eta_{\nu\lambda}M_{\mu\rho}\right), \\
[M_{\mu\nu}, P_\lambda] &= i(\eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu), \\
[M_{\mu\nu}, K_\lambda] &= i(\eta_{\mu\lambda}K_\nu - \eta_{\nu\lambda}K_\mu), \\
[D, P_\mu] &= iP_\mu\ , \quad [D, K_\mu] = -iK_\mu, \\
[P_\mu, K_\nu] &= 2i(\eta_{\mu\nu}D - M_{\mu\nu}), \\
[I_{ab}, I_{cd}] &= i\left(\delta_{ac}I_{bd} - \delta_{bc}I_{ad} - \delta_{ad}I_{bc} + \delta_{bd}I_{ac}\right), \\
\{Q^a_\alpha, Q^b_\beta\} &= (\gamma^\mu)_{\alpha\beta}P_\mu\delta^{ab}, \\
[I_{ab}, Q^\alpha_c] &= i(\delta_{ac}Q^\alpha_b - \delta_{bc}Q^\alpha_a), \\
\{S^a_\alpha, S^b_\beta\} &= (\gamma^\mu)_{\alpha\beta}K_\mu\delta^{ab}, \\
[I_{ab}, S^\alpha_c] &= i(\delta_{ac}S^\alpha_b - \delta_{bc}S^\alpha_a), \\
[K_\mu, Q^a_\alpha] &= i(\gamma_\mu)_\alpha^{\ \beta}S^a_\beta, \\
[P_\mu, S^a_\alpha] &= i(\gamma_\mu)_\alpha^{\ \beta}Q^a_\beta, \\
[D, Q^a_\alpha] &= \frac{i}{2}Q^a_\alpha\ , \quad [D, S^a_\alpha] = -\frac{i}{2}S^a_\alpha, \\
[M_{\mu\nu}, Q^a_\alpha] &= -(\Sigma_{\mu\nu})_\alpha^{\ \beta}Q^a_\beta, \\
[M_{\mu\nu}, S^a_\alpha] &= -(\Sigma_{\mu\nu})_\alpha^{\ \beta}S^a_\beta, \\
\{Q^a_\alpha, S^b_\beta\} &= \left(\epsilon_{\beta\alpha}D - \frac{1}{2}\epsilon_{\mu\nu\rho}(\gamma^\rho)_{\alpha\beta}M^{\mu\nu}\right)\delta^{ab} + \epsilon_{\beta\alpha}I^{ab}.
\end{aligned} \tag{3.56}$$

All other (anti)-commutators vanish.

R-symmetry

$SO(3)$

Gamma matrices are chosen to be the sigma matrices

$$(\sigma^a)_i{}^j = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right). \tag{3.57}$$

Indicies are raised and lowered by $\epsilon^{12} = -1 = -\epsilon_{12}$. Note that σ matrices with both lower or both upper indicies are symmetric.

The following identities are useful

$$\begin{aligned}
&\epsilon^{ij}\phi^k + \epsilon^{jk}\phi^i + \epsilon^{ki}\phi^j = 0, \\
&\epsilon^{ij}\epsilon_{kl} = \delta^i_l \delta^j_k - \delta^i_k \delta^j_l, \\
&\epsilon_{ij}\epsilon_{kl} = \epsilon_{ik}\epsilon_{jl} - \epsilon_{il}\epsilon_{jk}, \quad \text{(same for upper indices)} \\
&(\sigma^a)_i{}^j (\sigma^a)_k{}^l = 2\delta^l_i \delta^j_k - \delta^j_i \delta^l_k \\
&(\sigma^a)_{ij}(\sigma^a)_{kl} = -(2\epsilon_{il}\epsilon_{jk} + \epsilon_{ij}\epsilon_{kl}) = -(\epsilon_{ik}\epsilon_{jl} + \epsilon_{il}\epsilon_{jk})
\end{aligned} \tag{3.58}$$

$SO(4)$

Gamma matrices are chosen to be

$$\begin{aligned}
\Gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \bar{\sigma}^a & 0 \end{pmatrix} \text{ for } a = 1,2\ldots 4 \\
\text{where } (\sigma^a)_i{}^{\tilde{i}} = (\sigma^1, \sigma^2, \sigma^3, i\mathbb{1}_2), \quad (\bar{\sigma}^a)_{\tilde{i}}{}^i = (\sigma^1, \sigma^2, \sigma^3, -i\mathbb{1}_2).
\end{aligned} \tag{3.59}$$

Indicies are raised and lowered by $\epsilon^{12} = -\epsilon_{12} = -1 = \tilde{\epsilon}^{12} = -\tilde{\epsilon}_{12}$. With these definitions, the following identities would be useful.

$$\begin{aligned}
(\bar{\sigma}^a)^{\tilde{i}i} = (\bar{\sigma}^{aT})^{\tilde{i}i} \quad \left((\bar{\sigma}^a)^T = -\epsilon\sigma^a\tilde{\epsilon}^{-1}\right), \\
(\sigma^a)_i{}^{\tilde{i}} (\bar{\sigma}^a)_{\tilde{j}}{}^j = 2\delta^{\tilde{i}}_{\tilde{j}} \delta^j_i, \\
(\sigma^a)^{i\tilde{i}} (\bar{\sigma}^a)^{\tilde{j}j} = -2\epsilon^{ij}\epsilon^{\tilde{i}\tilde{j}}, \quad (\sigma^a)_{i\tilde{i}} (\bar{\sigma}^a)_{\tilde{j}j} = -2\epsilon_{ij}\epsilon_{\tilde{i}\tilde{j}}.
\end{aligned} \tag{3.60}$$

$SO(6)$

We choose the gamma matrices to be

$$\Gamma^a = \begin{pmatrix} 0 & \gamma^a \\ \bar{\gamma}^a & 0 \end{pmatrix} \text{ for } a = 1, 2 \ldots 6$$

where $\gamma^a = (\gamma^1, \gamma^2, \gamma^3, \gamma^4, \gamma^5, i\mathbb{1}_4)$, $\bar{\gamma}^a = (\gamma^1, \gamma^2, \gamma^3, \gamma^4, \gamma^5, -i\mathbb{1}_4)$, $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$.

$$\text{and } \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \bar{\sigma}^i & 0 \end{pmatrix} \text{with } \sigma^i = (\sigma^1, \sigma^2, \sigma^3, i\mathbb{1}_2), \bar{\sigma}^i = (\sigma^1, \sigma^2, \sigma^3, -i\mathbb{1}_2) \text{ for } i = 1 \ldots 4 \tag{3.61}$$

In these basis we the 'chirality' projection matrix is diagonal and is given by

$$\Gamma^7 = -i\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5 = \begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix} \tag{3.62}$$

The charge conjugation matrix is

$$C = \Gamma^0\Gamma^2\Gamma^4 = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix} \text{with } c = i\gamma^2\gamma^4 \tag{3.63}$$

which satisfies

$$\begin{aligned} C^* = C^{-1} = -C, &\quad (\Gamma^a)^* = C^{-1}\Gamma^a C \\ \Rightarrow \quad c = -c^* = c^{-1}, &\quad (\bar{\gamma}^a)^* = -c^{-1}\gamma^a c \end{aligned} \tag{3.64}$$

$$\text{In index notation:} \quad (\bar{\gamma}^{a\ j}_{\ i})^* = (\bar{\gamma}^{a*})^i_{\ j} = -c^{ik}(\gamma^a)_k^{\ l} c_{lj} = c^{ik} c_{jl} (\gamma^a)_k^{\ l}$$

Indicies are raised and lowered with using the charge conjugation matrix C for Γ^a and c for γ^a. With both indicies up or down the γ matrices are antisymmetric[6]. The last equation in (3.64) implies the following useful properties for the generators Let us define

$$\gamma^{ab} = \gamma^a\bar{\gamma}^b - \gamma^b\bar{\gamma}^a, \quad \bar{\gamma}^{ab} = \bar{\gamma}^a\gamma^b - \bar{\gamma}^b\gamma^a,$$

then we have following useful relations

$$\begin{aligned} \gamma^{ab\dagger} = -\gamma^{ab}, &\quad \bar{\gamma}^{ab\dagger} = -\bar{\gamma}^{ab}, \\ (\bar{\gamma}^{ab*})^i_{\ j} = (c^{-1}\gamma^{ab}c)^i_{\ j}, &\quad (\bar{\gamma}^{ab})_i^{\ j} = -(c^{-1}\gamma^{ab}c)^j_{\ i}. \end{aligned} \tag{3.65}$$

[6]This should be the case as the vector of $SO(6)$ is $(4 \times 4)_{\text{antisym}}$ of $SU(4)$.

The first line says that the generators of $SO(6)$ transformation are Hermitian[7] while the two equation in the second line follows from (3.64).

The following identities are useful[8]:

$$\begin{aligned}
&\bar{\gamma}^a_{ij} = \gamma^a_{ij} + 2\delta^{a0}c_{ij}, \quad (\bar{\gamma}^a)_i{}^j = (\gamma^a)_i{}^j - 2\delta^{a0}\delta^j_i, \\
&\gamma^a_{ij}\gamma^a_{kl} = -2\epsilon_{ijkl} = 2(c_{ik}c_{jl} - c_{il}c_{jk} - c_{ij}c_{kl}), \\
&\gamma^a_{ij}\bar{\gamma}^a{}_{kl} = -2\epsilon_{ijkl} + 2c_{ij}c_{kl} = 2(c_{ik}c_{jl} - c_{il}c_{jk}), \\
&(\gamma^a)^{ij}(\bar{\gamma}^a)_{kl} = 2\delta^i_k\delta^j_l - 2\delta^i_l\delta^j_k, \\
&(\gamma^{ab})^j_i(\gamma^{ab})^l_k = -32\delta^l_i\delta^j_k + 8\delta^j_i\delta^l_k,
\end{aligned} \tag{3.66}$$

Useful relations

Some useful relations and identities are given below

$$\epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^{a\beta}} = -\frac{\partial}{\partial\theta^a_\alpha} \tag{3.67}$$

$$(\gamma^\mu)_\alpha{}^\beta(\gamma_\mu)_\sigma{}^\rho = 2\delta_\alpha{}^\rho\delta_\sigma{}^\beta - \delta_\alpha{}^\beta\delta_\sigma{}^\rho \tag{3.68}$$

$$\theta_\alpha\theta_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\theta\theta, \quad \theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta \tag{3.69}$$

$$\theta_{1\alpha}\theta_2^\beta + \theta_{2\alpha}\theta_1^\beta + (\theta_1\theta_2)\delta^\beta_\alpha = 0 \tag{3.70}$$

$$X^2 \equiv X_\alpha{}^\beta X_\beta{}^\alpha = 2x_\mu x^\mu \equiv 2x^2\,, \qquad X_\alpha{}^\beta X_\beta{}^\gamma = x^2\delta^\gamma_\alpha = \frac{X^2}{2}\delta^\gamma_\alpha \tag{3.71}$$

$$D_{1\alpha}\tilde{X}_{12}^{-\Delta} = i\Delta(\tilde{X}_{12})^\beta_\alpha(\theta_{12})_\beta \tag{3.72}$$

$$D_{1\alpha}(\tilde{X}_{12})_\beta{}^\gamma = -i\delta^\gamma_\alpha(\theta_{12})_\beta + \frac{i}{2}\delta^\gamma_\beta(\theta_{12})_\alpha \tag{3.73}$$

$$D_{1\alpha}(X_{12-})_\beta{}^\gamma = -i\delta^\gamma_\alpha\theta_{12\beta}\,, \quad D_{1\alpha}(X_{12+})_\beta{}^\gamma = i\epsilon_{\alpha\beta}\theta^\gamma_{12} \tag{3.74}$$

[7]The generator of $SO(6)$ acting on chiral and antichiral transformation are respectively $-\frac{i}{4}\gamma^{ab}$ and $-\frac{i}{4}\bar{\gamma}^{ab}$

[8]Note that representation theory ($SU(4)$) wise C shouldn't be used to raise or lower indicies as it is not an invariant tensor of $SU(4)$. Only ϵ^{ijkl} and ϵ_{ijkl}(which are specific combinations of product of c's) can be used to raise or lower $SU(4)$ indices. we will explicitly see that all the $SU(4)$ tensor equations can be written using just ϵ tensors.

Chapter 4

3-point functions of higher spin operators

In this chapter we turn to correlation functions of $\mathcal{N} = 1$ 3d SCFTs. We discuss the form of the 2-point function of a spin s operator and give an elementary derivation, on the basis of symmetry and dimensional arguments, of the 2-point function of two spin half operators and explicitly compute a 2-point correlator in the free theory. Next we turn to 3-point correlation functions. The structure of correlation functions in SCFTs have been earlier studied by J-H Park [36, 38] and H. Osborn [37]. We build on their results on the supercovariant structures in superspace (reviewed in chapter 3) and use them together with the polarization spinor formalism of [35] to carry out our analysis.

We first construct parity even and odd superconformal invariants in superspace, determine the myriad non-linear relations between them and then use these results (in section 4.2.3) to determine the independent invariant structures which can arise in various 3-point functions of higher spin operators. This chapter is essentially an extension, to the superconformal case, of many of the results of [35]. We subsequently (in section 4.2.3) apply the constraints of current conservation and find evidence that the 3-point function of conserved higher spin currents is the sum of two parts- a parity even part generated by free SCFTs and a parity odd part.

4.1 Two-point functions

The two-point function of two spin-s operators in a 3d SCFT has a form completely determined (upto overall multiplicative constants) by superconformal invariance. Since, as we saw in chapter 2, $X_{12\pm}$ is the only superconformally covariant structure built out of two points in superspace, the only possible expression for the two point function which also has the right dimension and homogeneity in λ is:

$$\langle J_s(1)J_s(2)\rangle \propto \frac{P_3^{2s}}{\tilde{X}_{12}^2} \tag{4.1}$$

where P_3 is the superconformal invariant defined on two points, given in Table 4.1.

As an illustrative example, we consider the two-point function of two spin half supercurrents. On the basis of symmetry and dimension matching we can have the following possible structure for the 2-point function:

$$\langle J_{1/2}(x_1,\theta_1,\lambda_1)J_{1/2}(x_2,\theta_2,\lambda_2)\rangle = b\frac{\lambda_1\lambda_2}{\tilde{X}_{12}^{\Delta_1+\Delta_2}}\frac{\theta_{12}^2}{\tilde{X}_{12}} + \frac{\lambda_1\tilde{X}_{12}\lambda_2}{\tilde{X}_{12}^{\Delta_1+\Delta_2+1}}(c+d\frac{\theta_{12}^2}{\tilde{X}_{12}}) \tag{4.2}$$

where [1] $\tilde{X}_{12} \equiv \sqrt{(\tilde{X}_{12})_\alpha^{\ \beta}(\tilde{X}_{12})_\beta^{\ \alpha}}$. The shortening condition on the above 2-point function gives

$$d = 0 \qquad b = \frac{ic}{4}(\Delta_1+\Delta_2-2) \tag{4.3}$$

For $J_{1/2}$ a superconformal primary $\Delta_1 = \Delta_2 = 3/2$ so $b = ic/4$ and the two point function (upto some undetermined overall normalization) is given by

$$\langle J_{1/2}(x_1,\theta_1,\lambda_1)J_{1/2}(x_2,\theta_2,\lambda_2)\rangle \propto \frac{\lambda_1\tilde{X}_{12}\lambda_2}{\tilde{X}_{12}^4} + \frac{i}{4}\frac{\lambda_1\lambda_2\theta_{12}^2}{\tilde{X}_{12}^4} \tag{4.4}$$

A natural generalization, that reduces correctly to the above equation for $s = 1/2$, is

$$\langle J_s(1)J_s(2)\rangle \propto \frac{(\lambda_1\tilde{X}_{12}\lambda_2)^{2s-1}}{\tilde{X}_{12}^{4s+2}}(\lambda_1\tilde{X}_{12}\lambda_2 + \frac{is}{2}\lambda_1\lambda_2\theta_{12}^2) \tag{4.5}$$

[1] $\tilde{X}_{12}$, wherever it occurs uncontracted and without indices, is a scalar and stands for the supersymmetric distance between points 1 and 2

with $\langle J_0 J_0 \rangle = 1/\tilde{X}_{12}^2$ (since the superconformal shortening condition is different for spin zero). Note that the above can be written as

$$\langle J_s(1) J_s(2) \rangle \propto \frac{(\lambda_1 \tilde{X}_{12} \lambda_2 + \frac{i}{4} \lambda_1 \lambda_2 \theta_{12}^2)^{2s}}{\tilde{X}_{12}^{4s+2}} \tag{4.6}$$

which is the same as (4.1) . The shortening condition on this is satisfied, as may be explicitly checked.

As a check, we also work out, by elementary field theory methods, the two point function of the spin $\frac{1}{2}$ current constructed out of the free $\mathcal{N} = 1$ superfield which is defined as

$$\begin{aligned} \Phi &= \phi + i\theta\psi \\ \bar{\Phi} &= \bar{\phi} + i\theta\psi^* \end{aligned} \tag{4.7}$$

We find that the 2-point function computed explicitly in the free theory is in agreement with our result (4.1) above. The spin half supercurrent is

$$J_\alpha = \bar{\Phi} D_\alpha \Phi - (D_\alpha \bar{\Phi}) \Phi \tag{4.8}$$

Using the equation of motion for Φ this obeys the shortening condition $D^\alpha J_\alpha = 0$. The two point function of two such currents can be obtained after doing Wick contractions to write 4-point functions in terms of 2-point functions. We use the free field propagator $\langle \bar{\Phi} \Phi \rangle = \frac{1}{\tilde{X}_{12}}$, and also that,

$$D_{1\alpha} D_{2\beta} \frac{1}{\tilde{X}_{12}} = \frac{-i(\tilde{X}_{12})_{\alpha\beta}}{(\tilde{X}_{12})^3} \qquad , \quad D_{1\alpha} \frac{1}{\tilde{X}_{12}} D_{2\beta} \frac{1}{\tilde{X}_{12}} = \frac{\epsilon_{\alpha\beta} \theta_{12}^2}{4(\tilde{X}_{12})^4} \tag{4.9}$$

This gives (upto multiplicative factors which we neglect)

$$\langle J_\alpha(1) J_\beta(2) \rangle = \frac{((-\tilde{X}_{12})_{\alpha\beta} + \frac{i}{4} \theta_{12}^2 \epsilon_{\alpha\beta})}{\tilde{X}_{12}^4}. \tag{4.10}$$

Contracting with λ_1^α and λ_2^β we find, in free field theory,

$$\langle J_{\frac{1}{2}}(1) J_{\frac{1}{2}}(2) \rangle = \frac{P_3}{\tilde{X}_{12}^2} \tag{4.11}$$

which, indeed, is what was expected.

4.2 Three-point functions

In this section we undertake the task of determining all the possible structures that can occur in the three-point functions of higher spin operators $\langle J_{s_1} J_{s_2} J_{s_3} \rangle$. For the non-supersymmetric case this was done in [35]. We will use superconformal invariance to ascertain what structures can occur in three-point functions. We find that there exist new structures for both the parity even and odd part of $\langle J_{s_1} J_{s_2} J_{s_3} \rangle$ which were not present in the nonsupersymmetric case. The parity-odd superconformal invariants are of special interest as they arise in interacting 3d SCFTs. We will here restrict ourselves to the case of $\mathcal{N} = 1$ SCFTs (no R-symmetry). The results are summarized in the table given below:

	Parity even	Parity odd
Bosonic	$P_1 = \lambda_2 X_{23-}^{-1} \lambda_3$ $Q_1 = \lambda_1 X_{12-}^{-1} X_{23+} X_{31-}^{-1} \lambda_1$ and cyclic	$S_1 = \frac{\lambda_3 X_{31+} X_{12+} \lambda_2}{\tilde{X}_{12} \tilde{X}_{23} \tilde{X}_{31}}$ and cyclic
Fermionic	$R_1 = \lambda_1 \Theta_1$ and cyclic	$T = \tilde{X}_{31} \frac{\Theta_1 X_{12+} X_{23+} \Theta_3}{\tilde{X}_{12} \tilde{X}_{23}}$

Table 4.1: Invariant structures in $\mathcal{N} = 1$ superspace.

4.2.1 Superconformal invariants for 3-point functions of $\mathcal{N} = 1$ higher spin operators

We need to determine all the superconformal invariants that can be constructed out of the co-ordinates of (augmented) superspace : x_i, θ_i and the (bosonic) polarization spinors λ_i $(i = 1, 2, 3)$. Using the covariant objects of chapter 3, which transformed homogeneously under superinversions, we can begin to write down the superconformal invariants constructed out of $(x_i,\, \theta_i,\, \lambda_i)$.

We have

$$\lambda_i X_{ij-}^{-1} \lambda_j \to -(\lambda_i X_{i-}^{-1})(-X_{i-} X_{ij-}^{-1} X_{j-})(X_{j+}^{-1} \lambda_j) = \lambda_i X_{ij-}^{-1} \lambda_j \tag{4.12}$$

Thus we have the three superconformal invariants

$$P_1 = \lambda_2 X_{23-}^{-1} \lambda_3 \,, \quad P_2 = \lambda_3 X_{31-}^{-1} \lambda_1 \,, \quad P_3 = \lambda_1 X_{12-}^{-1} \lambda_2 \tag{4.13}$$

Also, under superinversion,

$$\mathfrak{X}_{1+} = X_{12-}^{-1} X_{23+} X_{31-}^{-1} \to -X_{1-} \mathfrak{X}_{1+} X_{1+} \tag{4.14}$$

and similarly for $\mathfrak{X}_{2+}, \mathfrak{X}_{3+}$, so we also have the following as superconformal invariants:

$$Q_1 = \lambda_1 \mathfrak{X}_{1+} \lambda_1 \,, \quad Q_2 = \lambda_2 \mathfrak{X}_{2+} \lambda_2 \,, \quad Q_3 = \lambda_3 \mathfrak{X}_{3+} \lambda_3 \tag{4.15}$$

Furthermore,

$$\lambda_3 X_{31+} X_{12+} \lambda_2 \to -\frac{1}{x_1^2 x_2^2 x_3^2} \lambda_3 X_{31+} X_{12+} \lambda_2 \,, \quad \tilde{X}_{ij}^2 \to \frac{\tilde{X}_{ij}^2}{x_i^2 x_j^2} \tag{4.16}$$

so there are the additional (parity odd) superconformal invariants

$$S_1 = \frac{\lambda_3 X_{31+} X_{12+} \lambda_2}{\tilde{X}_{12} \tilde{X}_{23} \tilde{X}_{31}} \,, \quad S_2 = \frac{\lambda_1 X_{12+} X_{23+} \lambda_3}{\tilde{X}_{12} \tilde{X}_{23} \tilde{X}_{31}} \,, \quad S_3 = \frac{\lambda_2 X_{23+} X_{31+} \lambda_1}{\tilde{X}_{12} \tilde{X}_{23} \tilde{X}_{31}} \tag{4.17}$$

which transform to minus themselves under inversion. Together these constitute the supersymmetric generalizations of the conformally invariant P, Q, S structures discussed in [35] [2]

Using the covariant Θ structures of chapter 3 it follows that we have the additional (parity even) fermionic invariants

$$R_1 = \lambda_1 \Theta_1 \,, \quad R_2 = \lambda_2 \Theta_2 \,, \quad R_3 = \lambda_3 \Theta_3 \tag{4.18}$$

It may be checked that

$$R_1^2 = R_2^2 = R_3^2 = R_1 R_2 R_3 = 0 \tag{4.19}$$

Construction of the parity odd fermionic invariant T

We can construct more superconformally covariant structures from the building blocks (X_{jk+}, $\mathfrak{X}_{i+}$, Θ_i, λ_i) - these are the fermionic analogues of P, S, Q. We define them below and also give there transformation under superinversion.

[2] Note that the S_k in [35] has an extra factor of iP_k compared to ours.

a) Fermionic analogues of P_i: Define

$$\pi_{ij} = \lambda_i X_{ij+} \Theta_j \tag{4.20}$$

Then under superinversion

$$\pi_{ij} \to -\lambda_i X_{i-}^{-1} X_{i+}^{-1} X_{ij+} X_{j-}^{-1} X_{j-} \Theta_j = -\frac{1}{x_i^2} \pi_{ij} \tag{4.21}$$

Similarly,

$$\Pi_{ij} = \Theta_i X_{ij+} \Theta_j \,, \quad \Pi_{ij} \to \Pi_{ij} \tag{4.22}$$

It turns out, however, that

$$\Pi_{ij} = 0 \tag{4.23}$$

This result was found using Mathematica.

b) Fermionic analogues of S_i:

$$\sigma_{13} = \frac{\lambda_1 X_{12+} X_{23+} \Theta_3}{\tilde{X}_{12} \tilde{X}_{23} \tilde{X}_{31}}, \quad \sigma_{13} \to x_3^2 \sigma_{13} \tag{4.24}$$

$$\Sigma_{13} = \frac{\Theta_1 X_{12+} X_{23+} \Theta_3}{\tilde{X}_{12} \tilde{X}_{23} \tilde{X}_{31}} \,, \quad \Sigma_{13} \to -x_1^2 x_3^2 \Sigma_{13} \tag{4.25}$$

$\sigma_{32}, \sigma_{21}, \Sigma_{32}, \Sigma_{21}$ are similarly defined through cyclic permutation of the indices. It follows that

$$\tilde{X}_{ij}^2 \Sigma_{ij} \to -\tilde{X}_{ij}^2 \Sigma_{ij} \tag{4.26}$$

c) Fermionic analogues of Q_i:

$$\omega_i = \lambda_i \mathfrak{X}_{i+} \Theta_i \,, \quad \omega_i \to -x_i^2 \omega_i \tag{4.27}$$

$$\Omega_i = \Theta_i \mathfrak{X}_{i+} \Theta_i \,, \quad \Omega_i \to x_i^4 \Omega_i \tag{4.28}$$

However, using Mathematica, we find

$$\Omega_i = 0 \tag{4.29}$$

The invariants constructed out of the product of two parity odd (or two parity even) covariant structures would be parity even, and since we have already listed all the parity even invariants, would be expressible in terms of P_i, Q_i, R_i. Thus, we find the following relations for the above covariant structures

$$\pi_{ij}^2 = \sigma_{ij}^2 = \omega_i^2 = 0 \tag{4.30}$$

$$\pi_{ij}\omega_i = 0 \tag{4.31}$$

$$\frac{1}{\tilde{X}_{12}^2}\pi_{12}\pi_{23} = -R_1R_2\,, \quad \frac{1}{\tilde{X}_{23}^2}\pi_{23}\pi_{31} = -R_2R_3\,, \quad \frac{1}{\tilde{X}_{31}^2}\pi_{31}\pi_{12} = -R_3R_1 \tag{4.32}$$

$$\frac{1}{\tilde{X}_{ij}^2}\pi_{ij}\pi_{ji} = R_iR_j = \tilde{X}_{ij}^2\sigma_{ij}\sigma_{ji} \tag{4.33}$$

$$\tilde{X}_{12}^2\sigma_{21}\sigma_{32} = R_2R_3\,, \quad \tilde{X}_{23}^2\sigma_{32}\sigma_{13} = R_3R_1\,, \quad \tilde{X}_{31}^2\sigma_{13}\sigma_{21} = R_1R_2 \tag{4.34}$$

$$\tilde{X}_{ij}^2\,\omega_i\omega_j = -R_iR_j \tag{4.35}$$

From the above covariant structures it is possible to build additional parity odd fermionic invariants by taking products of a parity even and a parity odd covariant structure.[3]Thus, we have

$$T_{ij} = \pi_{ij}\sigma_{ji} \tag{4.36}$$

and under superinversion

$$T_{ij} \to -T_{ij} \tag{4.37}$$

Note that $\pi_{ij} \neq \pi_{ji}$ so $\{\pi_{12},\, \pi_{23},\, \pi_{31}\}$ is a different set of parity odd covariant structures than $\{\pi_{21},\, \pi_{32},\, \pi_{13}\}$ (the same is true for the even structures σ_{ij}). However, because the following relation is true

$$T_{ij} = -T_{ji} \tag{4.38}$$

it follows that we have only three odd invariant structures:

$$T_1 \equiv T_{23} = \pi_{23}\sigma_{32}\,, \quad T_2 \equiv T_{31} = \pi_{31}\sigma_{13}\,, \quad T_3 \equiv T_{12} = \pi_{12}\sigma_{21} \tag{4.39}$$

[3]Note that structures like $x_i\omega_i$, π_{ij}/x_i would be parity odd invariants under inversion. However, these are not Poincaré invariant (since correlation functions should depend only on differences (x_{ij}) of the coordinates). We could also construct structures like $U = \tilde{X}_{12}\tilde{X}_{23}\tilde{X}_{31}\omega_1\omega_2\omega_3$ which would be an odd invariant ($U \to -U$) . However, it is identically zero because the product of three different Θ's vanishes.

We may also define

$$T'_{23} = \pi_{12}\sigma_{31}\,, \quad T'_{31} = \pi_{23}\sigma_{12}\,, \quad T'_{12} = \pi_{31}\sigma_{23} \quad T'_{ij} \to -T'_{ij} \tag{4.40}$$

with $T'_{32} = \pi_{13}\sigma_{21}\,, \quad T'_{13} = \pi_{21}\sigma_{32}\,, \quad T'_{21} = \pi_{32}\sigma_{13}$ again being related to the above by

$$P_3 T'_{21} = -P_2 T'_{31}\,, \quad P_1 T'_{32} = -P_3 T'_{12}\,, \quad P_2 T'_{13} = -P_1 T'_{23} \tag{4.41}$$

Also

$$\bar{T}_{ij} = \tilde{X}^2_{ij}\sigma_{ji}\omega_j\,, \quad \bar{T}_{ij} \to -\bar{T}_{ij} \tag{4.42}$$

Again, we have the relation

$$\bar{T}_{ij} Q_i = \bar{T}_{ji} Q_j \tag{4.43}$$

thus we have only three $\bar{T}_{ij}$'s.

Likewise, we have

$$\hat{T}_{12} = \tilde{X}^2_{12}\sigma_{31}\omega_2\,, \hat{T}_{23} = \tilde{X}^2_{23}\sigma_{12}\omega_3\,\hat{T}_{31} = \tilde{X}^2_{31}\sigma_{23}\omega_1 \quad \hat{T}_{ij} \to -\hat{T}_{ij} \tag{4.44}$$

with $\hat{T}_{21}\,, \hat{T}_{32}\,, \hat{T}_{13}$ being related to the above by

$$P_j \hat{T}_{ij} = P_i \hat{T}_{ji} \tag{4.45}$$

We also have the following relations involving Σ_{ij}

$$\Sigma_{ij} = \Sigma_{ji}\,, \quad \tilde{X}^2_{12}\Sigma_{12} = \tilde{X}^2_{23}\Sigma_{32} = \tilde{X}^2_{31}\Sigma_{31} \tag{4.46}$$

Therefore, here we get just one parity odd invariant

$$T \equiv \tilde{X}^2_{ij}\Sigma_{ij} \tag{4.47}$$

It turns out that T'_{ij}, $\bar{T}_{ij}$, $\hat{T}_{ij}\,, \tilde{X}^2_{ij}\Sigma_{ij}$ can be expressed in terms of T_i by means of the following relations

$$\begin{gathered}
P_1 T'_{31} = P_3 T_1\,, \quad P_2 T'_{12} = P_1 T_2\,, \quad P_3 T'_{23} = P_2 T_3 \\
P_3 \bar{T}_{12} = -Q_2 T_3\,, \quad P_1 \bar{T}_{23} = -Q_3 T_1\,, \quad P_2 \bar{T}_{31} = -Q_1 T_2 \\
\frac{1}{2} P_2 \tilde{X}^2_{13}\Sigma_{13} = T_2\,, \quad \frac{1}{2} P_3 \tilde{X}^2_{21}\Sigma_{21} = T_3\,, \quad \frac{1}{2} P_1 \tilde{X}^2_{32}\Sigma_{32} = T_1 \\
P_1 \hat{T}_{23} = -P_2 T_1\,, \quad P_2 \hat{T}_{31} = -P_3 T_2\,, \quad P_3 \hat{T}_{12} = -P_1 T_3
\end{gathered} \tag{4.48}$$

Making use of the above equation and eq.(4.47) we can express all parity odd fermionic structures in terms of T

$$T_2 = \frac{1}{2}P_2 T\,, \quad T_3 = \frac{1}{2}P_3 T\,, \quad T_1 = \frac{1}{2}P_1 T \tag{4.49}$$

$$\bar{T}_{12} = -\frac{1}{2}Q_2 T\,, \quad \bar{T}_{23} = -\frac{1}{2}Q_3 T\,, \quad \bar{T}_{31} = -\frac{1}{2}Q_1 T \tag{4.50}$$

$$T'_{31} = \frac{1}{2}P_3 T\,, \quad T'_{12} = \frac{1}{2}P_1 T\,, \quad T'_{23} = \frac{1}{2}P_2 T \tag{4.51}$$

$$\hat{T}_{12} = -\frac{1}{2}P_1 T\,, \quad \hat{T}_{23} = -\frac{1}{2}P_2 T\,, \quad \hat{T}_{31} = -\frac{1}{2}P_3 T \tag{4.52}$$

To summarize, from our fermionic covariant structures we could construct five parity odd invariants T_i, T'_{ij}, $\bar{T}_{ij}$, $\hat{T}_{ij}\,, T$. However, only T suffices as the other four are related to it through the above simple relations.

We have thus obtained the superconformal invariants P_i, Q_i, R_i, S_i, T (listed in tabular form at the beginning of this section) out of which the invariant structures for particular 3-point functions can be constructed as monomials in these variables. Before we do this, however, we need to determine all the relations between these variables using which we can get a linearly independent basis of monomial structures for 3-point functions.

4.2.2 Relations between the invariant structures

Following [35] we can do a counting of the number of independent (parity even) invariant structures for 3-point functions in 3d SCFTs with $\mathcal{N} = 1$ supersymmetry: the superconformal group in this case has 14 generators (10 bosonic, 4 fermionic) and out of $(x_i,\, \theta_i,\, \lambda_i)$ ($i = 1, 2, 3$, so 7×3 real variables) we can construct $7 \times 3 - 14 = 7$ superconformal invariants. Thus among the nine parity even structures $(P_i\,, Q_i\,, R_i)$ we must have two relations. One of them is the supersymmetrized version of the non-linear relation (2.14) in [35]

$$P_1^2 Q_1 + P_2^2 Q_2 + P_3^2 Q_3 - 2P_1P_2P_3 - Q_1Q_2Q_3 - \frac{i}{2}(R_1R_2P_3Q_3 + R_2R_3P_1Q_1 + R_3R_1P_2Q_2) = 0 \tag{4.53}$$

This cuts down the number of independent invariants by one. We also have the following triplet of relations which vanishes identically when the Grassmann variables are set to zero (fermionic relations) and reduces the number of invariants to seven :

$$\begin{aligned} P_2R_1R_2 + Q_1R_2R_3 + P_3R_3R_1 &= 0 \\ P_3R_2R_3 + Q_2R_3R_1 + P_1R_1R_2 &= 0 \\ P_1R_3R_1 + Q_3R_1R_2 + P_2R_2R_3 &= 0 \end{aligned} \tag{4.54}$$

There are further non-linear relations involving the S's. Since the squares or products of S's are parity even, we expect them to be determined in terms of the parity even structures. Indeed, we find

$$\begin{gathered} S_1^2 = P_1^2 - Q_2Q_3 - iP_1R_2R_3\,, \quad S_2^2 = P_2^2 - Q_3Q_1 - iP_2R_3R_1\,, \quad S_3^2 = P_3^2 - Q_1Q_2 - iP_3R_1R_2 \\ S_1S_2 = P_3Q_3 - P_1P_2\,, \quad S_2S_3 = P_1Q_1 - P_2P_3\,, \quad S_3S_1 = P_2Q_2 - P_3P_1 \end{gathered} \tag{4.55}$$

They imply that the most general odd structures that can occur in any three point function are linear in S_i. It turns out there exist further *linear* relations between the parity odd structures. We find the following basic linear relationships between the various parity odd invariant structures:

At $O(\lambda_1\lambda_2\lambda_3)$:

$$R_1S_1 + R_2S_2 + R_3S_3 = 0 \tag{4.56}$$

At $O(\lambda_1^2\lambda_2\lambda_3,\ \lambda_1\lambda_2^2\lambda_3,\ \lambda_1\lambda_2\lambda_3^2)$:

$$\begin{aligned} Q_1S_1 + P_2S_3 + P_3S_2 - \frac{i}{2}P_2P_3T &= 0 \\ Q_2S_2 + P_3S_1 + P_1S_3 - \frac{i}{2}P_1P_3T &= 0 \\ Q_3S_3 + P_1S_2 + P_2S_1 - \frac{i}{2}P_1P_2T &= 0 \end{aligned} \tag{4.57}$$

and

$$\begin{aligned} S_2R_1R_2 + S_3R_3R_1 + T(Q_1P_1 - P_2P_3) &= 0 \\ S_3R_2R_3 + S_1R_1R_2 + T(Q_2P_2 - P_3P_1) &= 0 \\ S_1R_3R_1 + S_2R_2R_3 + T(Q_3P_3 - P_1P_2) &= 0 \end{aligned} \tag{4.58}$$

From eq. (4.56) follows:

$$S_2R_1R_2 - S_3R_3R_1 = 0$$

$$S_3R_2R_3 - S_1R_1R_2 = 0 \tag{4.59}$$

$$S_1R_3R_1 - S_2R_2R_3 = 0$$

From these follow other linear relations at higher orders in $\lambda_1\,,\lambda_2\,,\lambda_3$:

$$\begin{aligned}
&Q_1P_1S_1 + Q_2P_2S_2 - Q_3P_3S_3 + 2P_1P_2S_3 - \frac{i}{2}TP_1P_2P_3 = 0\\
&Q_2P_2S_2 + Q_3P_3S_3 - Q_1P_1S_1 + 2P_2P_3S_1 - \frac{i}{2}TP_1P_2P_3 = 0 \qquad (4.60)\\
&Q_3P_3S_3 + Q_1P_1S_1 - Q_2P_2S_2 + 2P_3P_1S_2 - \frac{i}{2}TP_1P_2P_3 = 0
\end{aligned}$$

Adding the above equations gives

$$Q_1P_1S_1 + Q_2P_2S_2 + Q_3P_3S_3 - \frac{3i}{2}TP_1P_2P_3 + 2(P_1P_2S_3 + P_2P_3S_1 + P_3P_1S_2) = 0 \tag{4.61}$$

Also, we get

$$R_1R_2(S_1P_2 + \frac{1}{2}Q_3S_3) + R_2R_3(S_2P_3 + \frac{1}{2}Q_1S_1) + R_3R_1(S_3P_1 + \frac{1}{2}Q_2S_2) = 0 \tag{4.62}$$

$$\begin{aligned}
&(P_1^2Q_1 - P_2^2Q_2)P_3S_3 + (P_3^2 - Q_1Q_2 - iP_3R_1R_2)(Q_1P_1S_1 - Q_2P_2S_2) = 0\\
&(P_2^2Q_2 - P_3^2Q_3)P_1S_1 + (P_1^2 - Q_2Q_3 - iP_1R_2R_3)(Q_2P_2S_2 - Q_3P_3S_3) = 0 \qquad (4.63)\\
&(P_3^2Q_3 - P_1^2Q_1)P_2S_2 + (P_2^2 - Q_3Q_1 - iP_2R_3R_1)(Q_3P_3S_3 - Q_1P_1S_1) = 0
\end{aligned}$$

and so on. All these relations can be put to use in eliminating linearly dependent structures in 3-point functions. The above relations between the invariant structures extend the corresponding non-supersymmetric ones in [35].

We also have the following relations

$$T^2 = 0\,, \quad FT = 0\,, \quad S_iT = -\epsilon_{ijk}R_jR_k \quad sum\, over\, j,k \tag{4.64}$$

where F stands for any of the fermionic covariant/invariant structures. This implies that for any 3-point function it suffices to consider parity odd structures linear in T, S_i. Thus S_i, T comprise all the parity odd invariants we need in writing down possible odd structures in the 3-point functions of higher spin operators and we need only terms linear in these invariants.

4.2.3 Simple examples of three point functions

Independent invariant structures for three point functions

Below we write down the possible superconformal invariant structures that can occur in specific three point functions $\langle J_{s_1}(1)J_{s_2}(2)J_{s_3}(3)\rangle$. We consider the case of abelian currents so that, when some spins are equal, the correlator is (anti-) symmetric under pairwise exchanges of identical currents. We use only superconformal invariance to constrain the correlators, so the results of this section apply even if the higher spin symmetry is broken (that is, if J_s is not conserved for $s > 2$). All that is required is that J_s are higher spin operators transforming suitably under superconformal transformations[4].

Under the pairwise exchange $2 \leftrightarrow 3$ we have

$$A_1 \to -A_1\,, \quad A_2 \to -A_3\,, \quad A_3 \to -A_2\,, \quad T \to T \tag{4.65}$$

where A stands for any of P, Q, R, S.

$\langle J_{\frac{1}{2}}J_{\frac{1}{2}}J_0\rangle$: Here J_0 is a scalar operator with $\Delta = 1$. It is clear that any term that can occur is of order $\lambda_1\lambda_2$. Thus the possible structures that can occur in this correlator are:

$$P_3\,,\; R_1R_2\,,\; S_3\,,\; P_3T \tag{4.66}$$

We also computed this correlator explicitly in the free field theory (like the $\langle J_{\frac{1}{2}}J_{\frac{1}{2}}\rangle$ correlator in the previous section) and the result is (with $\Delta_1 = \Delta_2 = \frac{3}{2}\,,\; \Delta_3 = \frac{1}{2}$):

$$\frac{1}{\tilde{X}_{12}\tilde{X}_{23}\tilde{X}_{31}}(P_3 - \frac{i}{2}R_1R_2) \tag{4.67}$$

The odd piece can not occur in the free field case.

$\langle J_{\frac{1}{2}}J_{\frac{1}{2}}J_{\frac{1}{2}}\rangle$: Note that this has to be antisymmetric under exchange of any two currents. However the only two possible structures $\sum R_iP_i\,,\; \sum R_iS_i$ are symmetric under this exchange. Thus $\langle J_{\frac{1}{2}}J_{\frac{1}{2}}J_{\frac{1}{2}}\rangle$ vanishes.

$\langle J_sJ_0J_0\rangle$: For s an even integer, the correlator is

[4] We take $J_{\alpha_1\alpha_2.....\alpha_{s_i}}$ to be a primary with arbitrary conformal dimension Δ_i so that $J_{s_i} \equiv \lambda^{\alpha_1}\lambda^{\alpha_2}...\lambda^{\alpha_{s_i}}J_{\alpha_1\alpha_2.....\alpha_{s_i}}$ has dimension $\Delta_i - s_i$. In general J_{s_i} need not be conserved. However, if the unitarity bound is attained - $\Delta_i = s_i + 1$ for $s_i \geq \frac{1}{2}$; $\Delta_i = \frac{1}{2}$ for $s_i = 0$- then J_{s_i}, being a short primary, is necessarily conserved: $D_{(i)\alpha}\frac{\partial}{\partial\lambda_{(i)\alpha}}J_{s_i} = 0$

$$\langle J_s J_0 J_0 \rangle = \frac{1}{\tilde{X}_{12}\tilde{X}_{23}\tilde{X}_{31}} Q_1^s \tag{4.68}$$

In this case no other structure can occur. For s odd or half-integral, the correlator is zero.

$\langle J_s J_{\frac{1}{2}} J_{\frac{1}{2}} \rangle$: For s an even integer, the possible structures are

$$Q_1^s P_1\,,\ Q_1^{s-1} P_2 P_3\,,\ R_2 R_3 Q_1^s\,,$$
$$Q_1^{s-1}(P_2 S_3 + P_3 S_2)\,,\ Q_1^s P_1 T\,,\ Q_1^{s-1} P_2 P_3 T$$

The structure $R_1 Q_1^{s-1}(R_2 P_2 - R_3 P_3)$ is also possible but using eq.(4.54) equals $-R_2 R_3 Q_1^s$ and hence can be eliminated while writing down independent superconformal invariant structures. Similarly, the structure $Q_1^s S_1$ can be written in terms of others listed above by using eq. (4.57) and $R_1 Q_1^{s-1}(R_2 S_2 - R_3 S_3)$ in terms of the last two structures above by using eq. (4.58)

For s odd, antisymmetry under the exchange $2 \leftrightarrow 3$ allows only the following possible structures

$$R_1 Q_1^{s-1}(R_2 P_2 + R_3 P_3)\,,\ Q_1^{s-1}(P_2 S_3 - P_3 S_2)$$

The structure $R_1 Q_1^{s-1}(R_2 S_2 + R_3 S_3)$ vanishes on using eq. (4.56).

$\langle J_1 J_1 J_0 \rangle$: The possible structures are

$$Q_1 Q_2\,,\ P_3^2\,,\ R_1 R_2 P_3\,,\ R_1 R_2 S_3\,,\ P_3 S_3\,,\ Q_1 Q_2 T\,,\ P_3^2 T$$

$\langle J_1 J_1 J_1 \rangle$: Note that all the parity even structures that can occur in $\langle J_1 J_1 J_1 \rangle$ are those that are present in the non-linear relation eq.(4.53) but all these structures are antisymmetric under the exchange of any two currents whereas this correlator is symmetric under the same exchange. Hence the parity even part of $\langle J_1 J_1 J_1 \rangle$ vanishes. For the same reason no possible parity odd structures can occur either. Thus $\langle J_1 J_1 J_1 \rangle$ vanishes in general.

$\langle J_{\frac{3}{2}} J_{\frac{1}{2}} J_0 \rangle$: Here the possible structures are

$$Q_1 P_3\,,\ R_1 R_2 Q_1\,,\ Q_1 S_3\,,\ Q_1 P_3 T$$

$\langle J_{\frac{3}{2}} J_{\frac{1}{2}} J_{\frac{1}{2}} \rangle$: The linearly independent structures are

$$R_1 Q_1 P_1 \, , \, R_1 P_2 P_3 \, , \, Q_1 (R_2 P_2 + R_3 P_3) \, , \, R_1 Q_1 S_1$$

Two other possible fermionic parity odd structures can be eliminated using eqs. (4.56,4.57)

$\langle J_{\frac{3}{2}} J_{\frac{1}{2}} J_1 \rangle$: After eliminating some structures using the relations in sec. (7.2) we get the following linearly independent structures:

$$Q_1 Q_2 P_2 \, , \, Q_1 P_1 P_3 \, , \, P_3^2 P_2 \, , \, R_1 R_2 Q_1 P_1 \, , \, R_1 R_2 P_2 P_3 \, , \, R_3 R_1 Q_1 Q_2 ,$$
$$Q_1 P_1 S_3 \, , \, Q_1 P_3 S_1 \, , \, P_2 P_3 S_3 \, , \, R_1 R_2 P_2 S_3 \, , \, Q_1 Q_2 P_2 T \, , \, Q_1 P_1 P_3 T \, , \, P_3^2 P_2 T$$

$\langle J_{\frac{3}{2}} J_{\frac{3}{2}} J_{\frac{3}{2}} \rangle$:

$$Q_1 Q_2 Q_3 \sum_i R_i P_i \, , \quad \sum_{cyclic} R_1 Q_2 Q_3 P_2 P_3 \, , \quad \sum_i R_i Q_i P_i^3 \, , \quad P_1 P_2 P_3 \sum_i R_i P_i \, ,$$
$$\sum_i R_i Q_i P_i^2 S_i$$

The structure $\sum_{cyc} R_1 P_1 (P_2^2 Q_2 + P_3^2 Q_3)$ can, by using the non-linear identity eq.(4.53), be expressed in terms of the above structures and hence need not be included. The structure $\sum_{cyclic} R_1 Q_2 Q_3 (P_2 S_3 + P_3 S_2)$ vanishes on using eqs. (4.57,4.56)

$\langle J_2 J_1 J_1 \rangle$: The possible linearly independent structures are

$$Q_1^2 Q_2 Q_3 \, , \, Q_1^2 P_1^2 \, , \, Q_1 P_1 P_2 P_3 \, , \, P_2^2 P_3^2 \, ,$$
$$R_2 R_3 P_1 Q_1^2 \, , \, R_2 R_3 P_2 P_3 Q_1 \, ,$$
$$Q_1 Q_2 P_2 S_2 + Q_1 Q_3 P_3 S_3 \, , \, P_2^2 P_3 S_3 + P_3^2 P_2 S_2 \, ,$$
$$R_1 R_2 P_2^2 S_3 + R_3 R_1 P_3^2 S_2 \, ,$$
$$Q_1^2 Q_2 Q_3 T \, , \, Q_1^2 P_1^2 T \, , \, Q_1 P_1 P_2 P_3 T \, , \, P_2^2 P_3^2 T$$

Other structures are possible, but can be written in terms of the other structures listed above by using the relations in section 4.2.2.

$\langle J_3 J_1 J_1 \rangle$: As before, after eliminating some structures which are antisymmetric under the exchange $2 \leftrightarrow 3$ we are left with the following linearly independent basis for $\langle J_3 J_1 J_1 \rangle$:

$$Q_1^2(P_2^2 Q_2 - P_3^2 Q_3)\,,$$
$$Q_1^2(R_1 R_2 P_1 P_2 - R_3 R_1 P_3 P_1)\,,\; Q_1(R_1 R_2 P_2^2 P_3 - R_3 R_1 P_3^2 P_2)\,,$$
$$Q_1^2(P_2 Q_2 S_2 - P_3 Q_3 S_3)\,,\; Q_1(P_3^2 P_2 S_2 - P_2^2 P_3 S_3)\,,$$
$$Q_1(R_1 R_2 P_2^2 S_3 - R_3 R_1 P_3^2 S_2)\,,\; Q_1^2(P_2^2 Q_2 - P_3^2 Q_3) T$$

Again, linearly dependent structures have been eliminated using the relations of section 4.2.2.

$\langle J_4 J_1 J_1 \rangle$: The structures that occur here are the same as Q_1^2 times the structures in $\langle J_2 J_1 J_1 \rangle$.

$\langle J_s J_1 J_1 \rangle$: For s even this again equals $Q_1^{s-2}\langle J_2 J_1 J_1 \rangle$ (this was noted, for the non-supersymmetric case, in ref. [35]- it continues to hold in our case). For s odd and greater than three this correlator equals $Q_1^{s-2}\langle J_3 J_1 J_1 \rangle$. Thus the number of possible tensor structures in $\langle J_s J_1 J_1 \rangle$ does not increase with s.

$\langle J_2 J_2 J_2 \rangle$: The following are the possible independent invariant structures

$$Q_1^2 Q_2^2 Q_3^2\,, \quad P_1^2 P_2^2 P_3^2\,, \quad Q_1 Q_2 Q_3 P_1 P_2 P_3\,, \quad \sum_i Q_i^2 P_i^4\,,$$
$$Q_1 Q_2 Q_3 \sum_{cyclic} Q_3 P_3 R_1 R_2\,, \quad P_1 P_2 P_3 \sum_{cyclic} Q_3 P_3 R_1 R_2\,,$$
$$P_1 P_2 P_3 \sum_{cyclic} P_1 P_2 S_3\,, \quad \sum_i Q_i^2 P_i^3 S_i\,,$$
$$Q_1 Q_2 Q_3 \sum_{cyclic} Q_3 S_3 R_1 R_2\,, \quad P_1 P_2 P_3 \sum_{cyclic} Q_3 S_3 R_1 R_2\,,$$
$$Q_1^2 Q_2^2 Q_3^2 T\,, \quad P_1^2 P_2^2 P_3^2 T\,, \quad Q_1 Q_2 Q_3 P_1 P_2 P_3 T\,, \quad \sum_i Q_i^2 P_i^4 T$$

Many other linearly dependent structures have been eliminated using the relations in sec. (7.2).

As is evident, the number of invariant structures needed to construct the 3-point correlator increases rapidly as the spins of the operators increase and we will not consider more examples.

It is clear from the above examples that the general structure of the 3-point function is the following:

$$\langle J_{s_1} J_{s_2} J_{s_3} \rangle = \frac{1}{\tilde{X}_{12}^{m_{123}} \tilde{X}_{23}^{m_{231}} \tilde{X}_{31}^{m_{312}}} \sum_n \mathcal{F}_n(P_i,\, Q_i,\, R_i,\, S_i,\, T) \tag{4.69}$$

where $m_{ijk} \equiv (\Delta_i - s_i) + (\Delta_j - s_j) - (\Delta_k - s_k)$ and the sum is over all the independent invariant structures $\mathcal{F}_n$, each of homogeneity $\lambda_1^{2s_1}\lambda_2^{2s_2}\lambda_3^{2s_3}$. Since the 3-point function is linear in the parity odd invariants and linear or bilinear in the R's (either R_i or R_jR_k , $j \neq k$), we have the following structure for $\mathcal{F}_n$:

$$\begin{aligned}\mathcal{F}_n = F_n^{(1)}(P_i,Q_i) + a_n^{(1)}F_n^{(1)}(P_i,Q_i)T + a_n^{(2)}F_n^{(2)}(P_i,Q_i)S_i + a_n^{(3)}F_n^{(3)}(P_i,Q_i)R_i \\ + a_n^{(4)}F_n^{(4)}(P_i,Q_i)R_iS_j + a_n^{(5)}F_n^{(5)}(P_i,Q_i)R_jR_k + a_n^{(6)}F_n^{(6)}(P_i,Q_i)R_jR_kS_l\end{aligned}$$

Here each $F_n^{(a)}(P_i,Q_i)$ is a monomial in P's and Q's such that each term on the r.h.s above has homogeneity $\lambda_1^{2s_1}\lambda_2^{2s_2}\lambda_3^{2s_3}$.[5]

Three point functions of conserved currents

We have so far considered the constraints on the structure of the three-point functions of higher spin operators arising due to superconformal invariance alone. We will now see how the structure is further constrained by current conservation, i.e, when the operators are actually conserved higher spin currents. In this section we present evidence for the claim that the three point function of the conserved higher spin currents in $\mathcal{N}=1$ superconformal field theory consists of two linearly independent parts, i.e.,

$$\langle J_{s_1}J_{s_2}J_{s_3}\rangle = \frac{1}{\tilde{X}_{12}\tilde{X}_{23}\tilde{X}_{31}}\left(a\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\text{even}} + b\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\text{odd}}\right) \tag{4.70}$$

where a and b are independent constants, and the 'even' structure arises from free field theory.

The procedure, quite similar to that used by [35], is as follows. For any particular three point function we first consider the linearly independent basis of monomial structures (listed in section 4.2.3) and take an arbitrary linear combination of these structures.

$$\langle J_{s_1}J_{s_2}J_{s_3}\rangle = \frac{1}{\tilde{X}_{12}\tilde{X}_{23}\tilde{X}_{31}}\sum_n a_n\mathcal{F}_n \tag{4.71}$$

[5] The six $F_n^{(a)}(P_i,Q_i)$ are not independent functions. $F_n^{(2)}$, $F_n^{(4)}$, $F_n^{(6)}$ can be obtained from $F_n^{(1)}$, $F_n^{(3)}$, $F_n^{(5)}$, respectively, by replacing a P_i^p in the latter by $P_i^{p-1}S_i$ (suitably (anti-) symmetrized if some spins are equal in the 3-point function)

Current conservation $D_{\alpha_1} J^{\alpha_1\alpha_2.....\alpha_{2s}} = 0$ is tantamount to the following equation on the contracted current $J_s(x, \lambda)$:

$$D_\alpha \frac{\partial}{\partial \lambda_\alpha} J_s = 0 \tag{4.72}$$

Thus the equation

$$D_i \frac{\partial}{\partial \lambda_i} \langle J_{s_1} J_{s_2} J_{s_3} \rangle = 0 \tag{4.73}$$

for each $i = 1, 2, 3$ gives additional constraints in the form of linear equations in the a_n's- some of these constants can thus be determined. The algebraic manipulations get quite unwieldy- we used superconformal invariance to set some co-ordinates to particular values and took recourse to Mathematica[6]. The results obtained are given below (the known $\tilde{X}_{ij}$ dependent factors in the denominator are not listed below):

Three-pt function	Even	Odd
$\langle J_{\frac{1}{2}} J_{\frac{1}{2}} J_0 \rangle$	$P_3 - \frac{i}{2} R_1 R_2$	$S_3 - \frac{i}{2} P_3 T$
$\langle J_1 J_{\frac{1}{2}} J_0 \rangle$	$P_3 R_1 + \frac{1}{2} Q_1 R_2$	0
$\langle J_1 J_1 J_0 \rangle$	$\frac{1}{2} Q_1 Q_2 + P_3^2 - i R_1 R_2 P_3$	$S_3 P_3 + \frac{i}{2}(S_3 R_1 R_2 - Q_1 Q_2 T)$
$\langle J_{\frac{3}{2}} J_{\frac{1}{2}} J_0 \rangle$	$P_3 Q_1 - \frac{i}{2} Q_1 R_1 R_2$	$Q_1 S_3 - i Q_1 P_3 T$
$\langle J_{\frac{3}{2}} J_{\frac{1}{2}} J_{\frac{1}{2}} \rangle$	$Q_1 R_1 P_1 + Q_1(R_2 P_2 + R_3 P_3) + 2 R_1 P_2 P_3$	0
$\langle J_2 J_{\frac{1}{2}} J_{\frac{1}{2}} \rangle$	$Q_1^2 P_1 - 4 Q_1 P_2 P_3 - \frac{5i}{2} R_2 R_3 Q_1^2$	$Q_1(P_2 S_3 + P_3 S_2)$ $+\frac{i}{2}(Q_1^2 P_1 - 3 Q_1 P_2 P_3) T$

Table 4.2: Explicit examples of conserved three-point functions.

Using expression (3.33) for the currents in the $\mathcal{N} = 1$ free theory, some 3-point functions were explicitly evaluated (again using Mathematica, as the computations get quite cumbersome beyond a few lower spin examples). It must be emphasized that the (tabulated) even structures obtained above match with the expressions obtained from free field theory (upto overall constants). We thus have some evidence for the claim that the three-point function of conserved currents has a parity even part (generated by a free field theory) and a parity odd piece.

[6]We would like to acknowledge the use of Matthew Headrick's *grassmann* Mathematica package for doing computations with fermionic variables.

Chapter 5

Summary and Outlook

In this chapter we summarise the results presented in chapters 2, 3 and 4 and give a brief overview of some possible directions for future work. To begin with, in Chapter 2, we first examined solutions to higher-dimensional field equations for geometries of the form of eq. (2.1), with maximal symmetry in the noncompact dimensions. We asked what features of a solution control the curvature in the maximally symmetric, noncompact dimensions. We discussed no-go theorems on obtaining dS compactifications in supergravity theories and considered the effects of bulk singularities, backreacting brane sources and space-filling fluxes on the lower dimensional curvature of the solution.

Our main result is given by eq. (2.17), which gives the noncompact curvature scalar as a sum of four terms: $R \propto I + II + III + IV$. Here I corresponds to the bulk action evaluated at the appropriate back-reacted solution; II denotes an integral over a total derivative involving the warp factor (whose sign is usually definite, and not de Sitter-like); III denotes the direct contribution of the actions of any localized sources; and IV denotes a term which vanishes for solutions that are maximally symmetric in the non-compact dimensions, in the absence of space-filling fluxes.

We showed that the (classical) cosmological constant of the d dimensional spacetime is related, under quite general assumptions, to the asymptotic form of the bulk fields in the near boundary/source limit- we need **not** know the full bulk profile the solution. The boundary data includes contributions from the on-shell action, warping effects and the source action. Thus understanding the backreaction of the localised objects that source the bulk fields is crucial in looking for classical de Sitter solutions.

Eq. (2.17) relates the curvature to the on-shell bulk action. Remarkably, it is very often true that this on-shell action is also a total derivative. A sufficient condition for this turns out to be the existence of a scale invariance of the classical equations of motion [67], which in particular is present for most higher-dimensional supergravity theories of general interest. When $S_{\rm on-shell}$ is the integral of a total derivative, the curvature of the noncompact dimensions is completely determined by the asymptotic form of a particular combination of bulk fields near any sources that are distributed around the extra dimensions.

These arguments have two main implications. First, they show (at least for codimension-two sources) that source back-reaction and the source actions cannot be neglected when seeking de Sitter solutions. But they also show that all of the details of the complete back-reacted solution are not required; it often suffices to know the asymptotic behaviour of the bulk fields in the near-source limit. We explicitly derive which bulk fields play this role for 11D supergravity and 10D Type IIA and Type IIB supergravity.

We also demonstrated that all solutions (warped or unwarped) of the metric-axio-dilaton theory with only codimension two sources are d-dimensional flat. This generalises the known F theory result for unwarped solutions. This result is true because the boundary conditions that must be satisfied near the sources relate the near-source asymptotics of the bulk fields in such a way that the contributions II and III precisely cancel. We can look at more general solutions with higher form fluxes and higher codimension branes in the hope of obtaining dS.

Subsequent work [74] based on the results presented in [40] (and discussed in chapter 2) deals in particular with the application of these methods to type II supergravity flux compactifications. In such solutions it was investigated further how the classical cosmological constant is fully determined by the boundary conditions of the fields in the near-source region. The implications for meta-stable de Sitter solutions in IIB theory, obtained by placing anti-D3 branes at the tip of a warped Klebanov-Strassler geometry [72], were considered. A topological argument was presented which demonstrates the presence of a singularity in the flux energy density of the backreacted solution.

Also, classical de Sitter solutions in higher ($d > 4$) dimensions were considered by [75]. It was shown here that orientifold compactifications of type II supergravity theories to d dimensions can not give rise to meta-stable de Sitter solutions for $d > 6$. There are only a few possibilities in $d = 5, 6$ and they all require negative tension orientifold planes and negative curvature compact manifolds, as discussed in section 2.2

Besides potentially addressing issues of dark energy and inflation in cosmology a natural de Sitter embedding in supergravity/string theory would also help in gaining a more complete understanding of holography for dS gravity.

In chapter 3 we embarked on the study of $3d$ superconformal theories in an on-shell superspace formalism. We summarize the main results obtained in this regard, and presented in chapters 3 and 4, below:

- Classification of superconformal invariants formed out of 3 polarization spinors and 3 superspace points (following [36]) and using it to constrain 3 point functions of higher spin operators in 3d superconformal field theories in section 4.2.1.

- A conjecture - and evidence - that there are exactly two structure allowed in the 3 point functions of the conserved higher spin currents for $\mathcal{N} = 1$ in section 4.2.3.

- An explicit construction of higher spin conserved current supermultiplets in terms of on shell elementary superfields in free superconformal field theories in section 3.2.2.

- The superspace structure of higher spin symmetry breaking on adding interactions to large N gauge theories in section 3.3.

The 3-point functions of higher spin operator supermultiplets were constructed in terms of superconformal invariants built out of superspace co-ordinates and bosonic polarization spinors. For this purpose, following earlier work of Hugh

Osborn and J.H. Park , we constructed the various parity even and odd superconformal invariant structures out of the covariant structures which superspace admits.

The myriad non-linear relations between these invariants were discovered (this required extensive manipulations of grasmannian variables on Mathematica). This enabled the determination of a basis of linearly independent monomial structures (built out of the invariants) for the possible structures which can occur in 3-point functions of higher spin operators. Thus the form of the 3-point functions was constrained on the basis of superconformal invariance alone.

Thereafter, the constraints arising from imposition of conservation of currents of higher spin were considered. It was conjectured, and evidence through examples was provided, that the 3-point function of conserved higher spin currents is the sum of a parity even and a parity odd part with the parity even piece arising from free SCFTs. This 3-point function analysis extended the earlier work of Giombi, Prakash and Yin [35].

It should be possible to build on this line of work in some different, but related, directions:

- Extending the 3-point function analysis to SCFTs with R-symmetry (i.e. with extended supersymmetry). Several interesting theories which are currently being studied intensively are of this class.

- The analysis can also be extended to higher dimensions - for four and six dimensional SCFTs. The superconformal covariant structures in four and six dimensions are known [37, 38]. We may also note here that for $d > 3$ CFTs it is known that the invariants are all independent and so there are no complicated non-linear relationships between them which need to determined prior to writing down the structures for 3-point functions. Also there is only one parity odd invariant in $d = 4$ and none in $d = 6$, so the analysis may be considerably simpler.

- Extending the various recent results of Maldacena and Zhiboedov on CFTs [32] with weakly broken higher spin symmetry. In SCFTs with weakly broken higher spin symmetry (eg. Chern-Simons SCFTs) the superspace structure of the anomalous ”conservation” equation is more constrained (see section 3.3) and this presumably can be used profitably to constrain correlators to a greater degree.

- 3d SCFTs with higher spin operators include a range of theories describing a variety of Renormalization group fixed points. In particular they include Large N supersymmetric Chern-Simons theories with vector matter (i.e. with matter fields, bosonic/fermionic, transforming in the fundamental representation of the gauge group) or bi-fundamental matter (such as ABJ theory) which are currently an active avenue of research. This includes determining the single-trace operator spectrum, the exact large N solution of various such theories on different 3-manifolds by computing the partition function and their phase structure at finite temperature.

- 3d SCFTs are expected to be holographically dual to 4-dimensional theories of gravity (via the AdS/CFT duality map). In particular, field theories with higher spin operators are expected to be dual to Vasiliev type theories of massless higher spins in AdS spacetime. Thus, insight into the complicated non-linear dynamics of these theories (which themselves describe string theory in the highly stringy tensionless limit) can be obtained by studying such CFTs.

- Extending the analysis of 3-point functions to 4 and higher point functions in view of implementing the (super) conformal bootstrap for higher spin operators. This has been implemented mainly for scalar operators, but it may be possible to take forward some of this work using the formalism developed in chapter 4 by using polarisation spinor and superspace techniques, perhaps together with the embedding formalism for CFTs ([44, 45], see also [42, 43]).

It would be interesting to work on this range of topics in the future as they would shed light on the dynamics of a variety of CFTs. One may hope for the exact solution of vector model CFTs (at least in the large N limit) and the application of bootstrap methods to learn more about CFTs in general, in various dimensions.

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