Cosmological models, nonideal fluids and viscous forces in general relativity

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Daniele Gregoris is an Erasmus Mundus Joint Doctorate IRAP PhD student and is supported by the Erasmus Mundus Joint Doctorate Program by Grant Number 2011-1640 from the EACEA of the European Commission.

Doctoral Thesis in Theoretical Physics at Stockholm University, Sweden 2014

Abstract

This thesis addresses the open questions of providing a cosmological model describing an accelerated expanding Universe without violating the energy conditions or a model that contributes to the physical interpretation of the dark energy. The former case is analyzed considering a closed model based on a regular lattice of black holes using the Einstein equation in vacuum. In the latter case I will connect the dark energy to the Shan-Chen equation of state. A comparison between these two proposals is then discussed.

As a complementary topic I will discuss the motion of test particles in a general relativistic spacetime undergoing friction effects. This is modeled following the formalism of Poynting-Robertson whose link with the Stokes' formula is presented. The cases of geodesic and non-geodesic motion are compared and contrasted for Schwarzschild, Tolman, Pant-Sah and Friedmann metrics respectively.

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Public defense day: 7th November 2014, room FP41 AlbaNova University Center

ISBN 978-91-7447-982-9

Typeset in LATEX

Printed in Sweden by Universitetsservice US AB, Stockholm 2014

Distributor: Department of Physics, Stockholm University

List of Papers

The following papers, referred to in the text by their Roman numerals, are included in this thesis.

PAPER I: **Radiation pressure vs. friction effects in the description of the Poynting-Robertson scattering process**
D. Bini, D. Gregoris, S. Succi, EPL (Europhysics Letters) 97, 40007 (2012)
DOI:10.1209/0295-5075/97/40007

PAPER II: **Particle motion in a photon gas: friction matters**
D. Bini, D. Gregoris, K. Rosquist, S. Succi, General Relativity and Gravitation: Volume 44, Issue 10 (2012),
Page 2669-2680
doi:10.1007/s10714-012-1425-5

PAPER III: **Effects of friction forces on the motion of objects in smoothly matched interior/exterior spacetimes**
D. Bini, D. Gregoris, K. Rosquist and S. Succi, Class. Quantum Grav. **30** (2013) 025009 (17pp)
doi:10.1088/0264-9381/30/2/025009

PAPER IV: **Friction forces in cosmological models**
D. Bini, A. Geralico, D. Gregoris, S. Succi, Eur.Phys.J. C73 (2013) 2334
doi:10.1140/epjc/s10052-013-2334-9

PAPER V: **Dark energy from cosmological fluids obeying a Shan-Chen nonideal equation of state**
D. Bini, A. Geralico, D. Gregoris, S. Succi, Physical Review D **88**, 063007 (2013)
doi:10.1103/PhysRevD.88.063007

PAPER VI: **Exact Evolution of Discrete Relativistic Cosmological**

Models
T. Clifton, D. Gregoris, K. Rosquist, R. Tavakol, *JCAP* vol. 11, Article 010, ArXiv 1309.2876
doi:10.1088/1475-7516/2013/11/010

PAPER VII: **Scalar field inflation and Shan-Chen fluid models**
D. Bini, A. Geralico, D. Gregoris, S. Succi, Physical Review D **90**, 044021 (2014)
doi:10.1103/PhysRevD.90.044021

PAPER VIII: **Piecewise silence in discrete cosmological models**
T. Clifton, D. Gregoris, K. Rosquist, Class. Quantum Grav. **31** (2014) 105012 (17pp), arXiv:1402.3201
doi:10.1088/0264-9381/31/10/105012

Proceedings (not included in this thesis):

Kinetic theory in curved space-times: applications to black holes
D. Bini, D. Gregoris, proceeding for the 12th Italian-Korean Symposium on relativistic astrophysics,
Il nuovo Cimento Vol. 36 C, N. 1 Suppl. 1
doi:10.1393/ncc/i2013-11484-7

Friction forces in general relativity
D. Bini, D. Gregoris, K. Rosquist, submitted as proceeding of the 13th Marcel Grossmann meeting, 2011

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Contents

1. Introduction

The modern theory of gravitation formulated as General Relativity is a backbone of the current approach to theoretical physics. In fact it exhibits many physical applications for example related to astrophysical phenomena like the study of black holes, of massive stars, cosmology, cosmogony, gravitational waves and many others. In this thesis I will deal in particular with cosmological applications of Einstein's theory discussing two original cosmological models; at the end I will also speak about the motion of particles in general relativistic spacetimes undergoing friction effects comparing the modified orbits with the geodesic motion. The plan of the present thesis is the following:

- In this introductory chapter I will briefly review the most famous cosmological models discussed so far in literature to explain how our original ones fit inside the current research and the context in which they have been formulated. Particular attention will be devoted to the role played by the symmetries of the geometry and the role of the equation of state in the cosmological modeling. Then I will compare and contrast the methods employed in the construction of the two above mentioned models enlightening their merits and suggesting possible future directions of research for refining them. The last part of the chapter introduces the problem of friction forces in general relativity discussing its connection with the analogous topic in newtonian mechanics.

- In chapter (2) I will introduce a relativistic inhomogeneous discrete cosmological model in vacuum explaining on which assumptions it is based and summarizing how we improved its understanding. I will focus my attention on the construction of the initial data for the Cauchy formulation of General Relativity discussing their symmetries which are inherited by the the complete solution of the Einstein equations. I will show how the local rotational symmetry and the reflection symmetry constrain the physical variables of the configuration. An application to the propagation of the gravitational waves in the configuration can then be derived.

- In chapter (3) I will formulate an original cosmological model in which the matter content is modeled by a nonideal fluid whose equation of state exhibits asymptotic freedom. I will mention how this equation of state has been derived in statistical mechanics by Shan and Chen, its most important features and the observational relations we used to test the theory with the astronomical observations provided by the current space missions.

- Chapter (4) includes instead a digression about the general-relativistic version of the Poynting-Robertson effect with reference to the Tolman, Schwarzschild, Pant-Sah and Friedmann metrics respectively. This force affects the motion of a massive test particle acting as a dissipative term. The results will be discussed with applications to astrophysics and cosmology. For example the Pant-Sah metric can describe a gas cloud which behaves like a gravitational lensing distorting the motion of a body crossing it: our formalism enables us to evaluate the deflection angle for a particle moving inside it. Moreover we can use our results to estimate the peculiar velocity of an astrophysical object when a cosmological metric is assumed. I would like to mention also that friction effects are always present in motion of particles, and usually the computations are only approximations neglecting this kind of effects. Thus the formalism presented in this chapter admits wider applications also unrelated to cosmology or astrophysics. Our most important original result in this context is the proof that the Poynting-Robertson formula is the correct general relativistic extension of the famous Stokes' law.

- At the end of the thesis the reader can find attached the papers I co-authored. Hence in the next pages I will focus mainly in the methodology adopted, in the initial hypotheses of my work and the final conclusions leaving all the details of the full derivation of the original results to the referred articles.

The computations and the plots presented in this thesis and in the attached papers have been created with the help of the software for algebraic and symbolic manipulations Maple$^{\text{TM}}$ and Mathematica$^{\text{TM}}$.

1.0.1 Notation

In this paragraph I will introduce the notation used throughout this thesis:

- I will adopt geometric units: $G = c = \hbar = 1$

- Einstein convention of sum over repeated indices is understood
- the metric has signature [-,+,+,+]
- ; and ∇ denote covariant derivative
- round parentheses denote symmetrization: $T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba})$
- square parentheses denote antisymmetrization: $T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba})$
- Volume element for the rest space:

$$\eta_{\alpha\beta\gamma} = \eta_{\alpha\beta\gamma\delta}u^{\delta} = \eta_{[\alpha\beta\gamma]}\,, \tag{1.1}$$

  where $\eta_{\alpha\beta\gamma\delta}$ is the 4-dimensional Levi-Civita symbol.

- Fully orthogonally projected covariant derivative:

$$\tilde{\nabla}_{\alpha}T^{\beta}{}_{\gamma} = h^{\beta}{}_{\delta}h^{\rho}{}_{\gamma}h^{\varepsilon}{}_{\alpha}\nabla_{\varepsilon}T^{\delta}{}_{\rho} \tag{1.2}$$

- Covariant time derivative along the fundamental worldlines:

$$\dot{T}^{\alpha}{}_{\beta} = u^{\gamma}\nabla_{\gamma}T^{\alpha}{}_{\beta} \tag{1.3}$$

- Angle brackets:

$$v^{\langle\alpha\rangle} = h^{\alpha}{}_{\beta}v^{\beta}\,, \quad T^{\langle\alpha\beta\rangle} = \left(h^{(\alpha}{}_{\gamma}h^{\beta)}{}_{\delta} - \frac{1}{3}h^{\alpha\beta}h_{\gamma\delta}\right)T^{\gamma\delta}\,. \tag{1.4}$$

1.1 The current state of modern cosmology

The search for the correct model describing our Universe is a subject of constant debate since the first days of general relativity. The current concordance model of cosmology we are familiar with from many textbooks is named Λ-Cold Dark Matter (ΛCDM). It assumes spatial homogeneity and isotropy on large enough scales as working hypotheses and consequently is based on the Friedmann metric written in comoving Robertson-Walker coordinates. The matter source is a mixture of perfect fluids like a photon gas, a baryonic matter density, a dark matter and dark energy components. These fluids are assumed to be non-interacting and hence they are separately conserved. The experimental observations, like the ones coming from the study of the type Ia supernovae, the analysis of the cosmic microwave background and of the baryonic acoustic oscillations, are then fitted to derive the amount of the different matter contents of the Universe. The orthogonal plots which can be created by superimposing the above mentioned observations, after fixing a Friedmann metric as a background, are currently interpreted in terms of an almost spatially flat Universe dominated by a dark energy fluid, compatible with the cosmological constant term entering the Einstein equations, which in particular gives raise to an accelerated expansion of the Universe [1; 2].

The symmetries of the theory of general relativity does not forbid the presence of this latter term, but its physical nature and consequently the one of the dark energy has not been established up to now. The current literature contains many theoretical proposals which address the question of the physical interpretation of the dark energy. For example the dark energy has been modeled as a quintessence fluid [3; 4], or the existence of modified and exotic equations of state, the most popular of them being nowadays the Chaplygin gas [5–7], have been considered. This latter case has also been compared to the *k*-essence model whose lagrangian can be formulated in terms only of its kinetic part like in the Born-Infeld theory [8–10]. Braneworld models, scalartensor theories of gravity and modified/massive gravity models play some role in the current research [11; 12]. Other authors describe the large scale accelerated expansion of the Universe invoking the bulk viscosity of scalar theories [13], while others analyzed the effects of small local inhomogeneities and of the formation of structures on the global dynamics of the Universe. In this latter case both perturbative [14–16] and non perturbative approaches [17–19] have been adopted to evaluate the so-called backreaction. This term follows from the non-commutativity of the averaging operation over small scale structures and of the Einstein field equations. The two complementary methods for the analysis of spatial inhomogeneities are referred to as top-down and bottom-up approaches [20–22]. In the former we begin with a completely homogeneous background to which we then add some perturbations which should mimic the observed astronomical structures. In the latter the point of view is the opposite: we build up a cosmological model starting from the truly observed structures in the physical Universe, i.e. galaxies and clusters of galaxies, and deriving then the consequent large scale dynamics which in this case can be considered as an emergent property of the system resulting from the interaction between the different constituents. The most important aim of this specific proposal is to solve the structure formation problem and the accelerated expansion of the Universe together without violating the energy conditions. For the state of the art in the analysis of inhomogeneities in the cosmological modeling see [23–29] and references therein. I would like to warn the reader that I just named few of the most important lines of research to give an idea of how much the work about the dark energy problem is important, and not to review all the possible analysis published so far. The reader can find the references to these approaches in the bibliographies of the attached papers.

To summarize, the current research in theoretical cosmology falls mainly into two different lines: one can act on the l.h.s or on the r.h.s of the Einstein equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}\,, \tag{1.5}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{1.6}$$

is the Einstein tensor, $R_{\mu\nu}$ and R being the Ricci tensor and scalar respectively function of the metric $g_{\mu\nu}$ and its derivatives, Λ is the above mentioned cosmological constant and $T_{\mu\nu}$ the stress-energy tensor.

In the former case we act on the geometry of the Universe usually relaxing the initial assumptions of perfect homogeneity and isotropy (i.e. reducing the symmetry group of the geometry), while in the latter we usually maintain the assumptions that we can base geometrically our model on the Friedmann metric and we modify the matter content involved in the stress-energy tensor for example considering not only simple polytropic equations of state for the fluids entering the stress-energy tensor. Again in the former case the aim is to account for the observational data without the need of any dark side of the Universe or any cosmological constant or any exotic fluids looking for new and original exact or approximate solutions of the Einstein field equations. In the latter, overturning the point of view, we postulate the existence of some exotic fluid permeating the Universe which behaves like dark energy to explain its physical nature: the starting point is a lagrangian formulation motivated from microscopic arguments (usually from the theory of elementary particles), which is connected to an effective equation of state through the canonical equations whose free parameters are then constrained by the experimental observations.

1.1.1 **Original contribution of this thesis**

During my Ph.D., performed inside the project "Erasmus Mundus joint IRAP Ph.D. program"for relativistic astrophysics, I have analyzed both these approaches.

In one case, under the supervision of prof. Kjell Rosquist and in collaboration with Dr. Timothy Clifton and Prof. Reza Tavakol, we have considered a spatially closed Universe in vacuum made by a regular lattice of an increasing number of non-rotating and uncharged Schwarzschild black holes. They are supposed to be a rough schematization of the observed astronomical structures like galaxies and clusters of galaxies: their positions have been fixed by tiling a 3-sphere with regular polychora (four-dimensional analogue of the platonic solids). This model is part of the research in the bottom-up approach to inhomogeneous cosmology: this is a genuinely inhomogeneous model on small scale and instead approaches a homogeneous one on large scale after coarse-graining. Moreover this is a fully general relativistic exact non-perturbative model which does not possess any global continuous symmetry. Its dynamics can thus be completely solved analytically during all the evolution of the system only along special lines admitting local rotational symmetry, examples of them being the edges of the cells and the lines connecting two masses. The face of the cell, whose center is occupied by the black hole, admits instead invariance under reflection that can be exploited to derive some conclusions about the propagation of gravitational waves in these models. The kinematical quantities like the expansion rate, the shear tensor, the spatial gravito-electric and gravito-magnetic Weyl tensors have been discussed on these lines and surfaces. In particular we can follow the time evolution of the length of some special lines and we can introduce a Hubble function and a deceleration parameter based on this length, formally in the same way as in the Friedmann model where they are instead defined in terms of the scale factor. It turns out that different regions of the space-time admit completely different behaviors. The results have also been compared with the ones of the simpler Friedmann model. We can also show that the discrete symmetries of the face induce these models to be piecewise silent, and consequently they are more realistic than previously considered silent models.

On the other hand, under the supervision of Drs. Donato Bini, Sauro Succi and Andrea Geralico, I have considered a Friedmann model without cosmological constant whose matter content is given by the Shan-Chen non-ideal equation of state with asymptotic freedom with the purpose of giving a physical interpretation of nature of dark energy. This is a modified equation of state introduced in the framework of kinetic lattice theory describing a fluid which behaves like an ideal gas (pressure and density change in linear proportion to each other) at both low and high density regimes (for this reason we speak of asymptotic freedom), with a liquid-gas coexistence loop in between. This equation of state has also been compared to the bag model of hadronic matter and thus a cosmological application is well motivated because the Shan-Chen equation of state is not just a numerical trick. We showed that when we plug this equation of state in the Einstein equation we can evolve from an initially radiation dominated universe, as required by the hot big bang model, to a dark energy dominated one. This means that we have a phase transition in which the pressure switches its sign at a certain instant in the past and stays negative for a long time interval including the present day. After adding a pressure-less matter content to our picture of the Universe, we proved that our model can fit the supernova data where the distance modulus is plotted with respect to the redshift for an appropriate choice of the free parameters entering the equation of state without any need of vacuum energy. We also showed that for this specific choice of the parameters inside the equation of state, our model is stable under small initial perturbations and so is self-consistent. In this way we can provide a microscopic interpretation of the dark energy when we take into account the form of the potential modeling the interaction on which this equation of state is based. We also applied the model to the description of the inflationary era of the Universe evaluating the experimental quantities like the ratio of tensor to scalar perturbations, the scalar spectral index and its running. A graceful exit mechanism from the inflationary era is also provided by this model.

1.1.2 The author's contribution to the accompanying published papers

I will explain in this section my personal contribution to the attached papers that I co-authored.

In papers **I**-**IV** my work consisted mainly in deriving the equations of motion and in solving them generating the plots for the particle orbit. Before the beginning of this analysis I was trained on the use of the software Maple™ in similar but simplified situations than the ones consider in this series of papers. I also proposed the extension of the Poynting-Robertson formalism to the case of motion inside a massive fluid subsequently re-interpreted in terms of the analogy with the Stokes' law and the introduction of direct and indirect interaction between test particle and fluid.

I re-derived independently all the results discussed in paper **V** and **VII**, where in the latter I played a major role in clarifying the role of the measurable quantities in the physical situation under exam and their connection with respect to our specific cosmological model and in proposing an application of the Shan-Chen equation of state to the inflationary epoch.

In paper **VI** I focused my attention mainly in the 5 and 8 mass configurations, in their lattice construction, in the initial data construction and in the derivation of the dynamical equations in these specific cases; in paper **VIII** I calculated the explicit simplifications of the gravito-electric, gravito-magnetic and shear tensors on the reflection symmetry surfaces we were dealing with.

I took active part in all the discussions underlying all the papers I am submitting for my final dissertation not only providing handwritten computations and the Maple$^{\text{TM}}$ and Mathematica$^{\text{TM}}$ codes for the quantitative derivation of the results presented but also in suggesting their connection to the broader scientific context inside which I have been working and in writing preliminary versions of the manuscripts (note the alphabetic ordering of the names of the authors in the attached papers).

1.2 The geometric approach: a symmetry group point of view

In the next chapter I will demonstrate that the Einstein equations (1.5) constitute a system of non-linear partial differential equations. Thus their solution is not known in general. To find manageable solutions usually some symmetries are imposed as assumptions at the very beginning: this has been done also in cosmology. Then the derived solution must be compared with the astrophysical observations to see if the mathematical result represents also a physical meaningful one. In this section I will review the role played by the space-time symmetry group in the development of some remarkable cosmological models.

As already stated the Friedmann-Lemaître-Robertson-Walker (FLRW) models, derived from the hypotheses of spatial homogeneity and isotropy to account for the cosmological principle, agree very well with the astronomical data sets in the description of our Universe only by introducing a dark and undetected side of the Universe. The Friedmann metrics are solutions of the Einstein equations (1.5) where the matter content is described by the stress-energy tensor

$$T_{\alpha\beta} = (p+\rho)u_\alpha u_\beta + p g_{\alpha\beta} \tag{1.7}$$

for a perfect fluid assumed to be at rest with respect to the coordinates with p denoting the pressure, ρ the energy density and $u^\alpha = (1,0,0,0)$ the four-velocity of the geodesic congruence. To take into account also the observational constraint of an expanding Universe (formulated by the Hubble law) the metric has been chosen in the form

$$ds^2 = -dt^2 + a^2\left[dr^2 + \Sigma_k^2(d\theta^2 + \sin^2\theta d\phi^2)\right] , \tag{1.8}$$

where $a = a(t)$ is the scale factor, while the function $\Sigma_k(r) = [\sin r, r, \sinh r]$ assumes different expressions reflecting the curvature of a spherical, flat and hyperbolic universe respectively, and $k = -\Sigma''/\Sigma$, where a prime denotes derivative with respect to r. An equation of state $p = p(\rho)$ relating the pressure and the energy density of the fluid must be chosen to have a closed system of equations.

The simplest and most symmetric model belonging to this family is the Einstein static model which is based on the line element (1.8) with $k = +1$ and $a = const$. This model is homogeneous both in space and time and isotropic in space. To obtain a static solution it is necessary to impose a relation of the type $\Lambda = \frac{1}{2}(\rho + p)$ between the cosmological constant, the energy density and the pressure of the fluid permeating the Universe [30]. However the discovery of the expansion of the Universe mathematically expressed by the Hubble law clearly showed that this mathematical solution was not the one chosen by nature [31].

Thus to improve our description of the physical Universe, the static assumption has been removed moving to the class of the Friedmann expanding models [32]. In this context the assumptions of spatial homogeneity and isotropy are maintained, and thus the energy density, the pressure and the scale factor are function of the time only. The Einstein equations (1.5) in this case are reduced to the Friedmann equation expressing the evolution in time of the scale factor

$$\dot{a}^2 = -k + \frac{8}{3}\pi\rho a^2 \,, \tag{1.9}$$

where a dot corresponds to derivative with respect to the time, and the energy conservation equation

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p) \,, \tag{1.10}$$

where a fluid with $p = -\rho$ can mimic the cosmological constant Λ. With further assumptions on the spatial curvature and on the type of fluid, a set of geometrical cosmological solutions can be derived, the most famous ones being the de Sitter [33], anti de Sitter, Milne [34] and Eintein-de Sitter ones [35]. As I mentioned above the Friedmann class of solutions are the basis of the concordance model of cosmology.

However in this picture the existence of astronomical structures is completely neglected. Two approaches can now be followed: add some perturbations on a fixed Friedmann background and follow their evolutions or look for other exact solutions. We will consider in this thesis the latter case. For example if one wants to eliminate the hypothesis of isotropy it is possible to substitute the scale factor $a(t)$ with a family of two or three scale factors $a_1(t), a_2(t), a_3(t)$, which are again only function of the time because we are still considering a homogeneous Universe. In this case we refer to the scale factor of the Universe as the quantity $a = (a_1 a_2 a_3)^{\frac{1}{3}}$ [36]. On the other hand if we want to consider an inhomogeneous Universe the scale factor, the energy density and the pressure must be function also of the spatial coordinates and not only of the time. For example the first step is to consider an inhomogeneous Universe admitting spherical symmetry, mathematically meaning that $a(t) \to a(t,r)$ (then we can of course consider different scale factors along the different spatial directions to obtain an even more general solution) [37]. As a next step the scale factor must be allowed to be function also of the angular coordinates. In these cases analytical solutions are not available, but the dynamics of the Universe can be treated at least qualitatively exploiting a dynamical system formalism. In the next chapter I will apply it for particular points of the configurations considered in our original model. For a complete classification of the most famous cosmological models in terms of their symmetries see [38].

The real Universe is of course less symmetric than the solutions available in literature. Moreover we can hypothesize that the postulated existence of an exotic and undetected fluid like the dark energy in the Λ-CDM model can be a consequence of a naive choice of the metric. It is then worthy to relax even more the initial assumptions about the symmetries of the geometry of the Universe, as I will do in the second chapter of this thesis where I will deal with an inhomogeneous discrete model. I would like to stress now that the above mentioned symmetries are continuous and are thus connected to the presence of a Killing vector field of the metric. As a next step in relaxing these assumptions I will consider a model admitting only discrete symmetries. To realize this project I will consider a regular lattice of black holes playing the role of the astronomical structures. Appropriately arranging the Schwarzschild mass sources, there would be points and lines which admit local rotational symmetry and surfaces which exhibit instead symmetry under reflection. We will treat the symmetries as follow: after constructing the initial data we will see what kind of symmetries they exhibit; then it is possible to prove that their symmetries are inherited by the complete solution of the Einstein equations. When we will consider a discrete symmetry, we must define it not in terms of Killing vector fields, but in terms of a diffeomorphism between the spacetime manifold and itself which preserves the first and second fundamental forms. See [39; 40] for the proofs of such theorems about the preservations of the initial symmetries along all the time evolution of the system.

1.3 The role of the equation of state in cosmology

In this paragraph I will introduce a complementary approach to cosmology than the one discussed in the previous section. In fact I will maintain that the assumptions of spatial homogeneity and isotropy of the Friedmann class of cosmological models are well motivated at least on large scales. The interpretation of the space mission data in this framework implies the present epoch of the Universe to be dark energy-matter dominated; the goal is now to picture physically the characteristics of this exotic fluid. In this section I will outline the procedure usually adopted to reach this target.

First of all it is well known that in nature three fundamental forces exist: the electroweak, strong and gravitational interactions. The first two can be successfully described by the theory of relativistic quantum field and are the dominant ones on microscopic scales. On the other hand we assume that in the late time cosmology the only force playing a role is the gravitational one. However it has not been possible so far to provide a physical interpretation of the dark energy in the framework of these fundamental forces: up to now together with the mass density it is only a parameter of the fit of the standard model of cosmology after assuming a Friedmann metric as background, as already explained.

Once admitted that the dark energy can really be a physical component of our Universe, even if not yet detected, and not only an unwanted consequence of the chosen metric, the challenge is to set it in terms of a well posed field theory. The method usually adopted in this context is the following:

- We start from a lagrangian formulation in terms of a complex scalar field: the action of the cosmological model is usually provided by the theory of elementary particles; I stress that the action itself is not a measurable quantity and consequently can not be compared directly to experimental data and consequently observational relations must be derived.

- From the lagrangian, the Friedmann metric and the definition of the stress-energy tensor in terms of the pressure and energy density of the fluid, we can derive the canonical equations which connect the scalar field to the pressure and energy density of the fluid that it describes.

- We eliminate the scalar field from the canonical equations to obtain a "phenomenological"equation of state for the fluid stating the pressure in terms of the energy density; usually this equation of state contains one or two free parameters not fixed by the initial assumptions.

- We plug this equation of state inside the Einstein equations, which are now constituted by the Friedmann and energy conservation equations, and solve them obtaining the evolution of the energy density and of the scale factor as functions of the time.

- We use these solutions to fit the observed data, for example plotting the the distance modulus and/or the Hubble function with respect to the redshift, or reconstructing the cosmic microwave background and the baryonic acoustic spectra. If we can fit all the observations with the same interval for the free parameters of the equation of state, the theory can be considered compatible with the observations.

- As a final check it is necessary to show that the model is self-consistent: it must be stable under small perturbations.

In real life, it can happen that we start fitting the observed data with a parametric equation of state *ad hoc* and then the lagrangian is inferred integrating the canonical equations. A remarkable example in this direction is the modified Chaplygin gas formulated by a two-parameter equation of state of the form

$$p = -\frac{A}{\rho^{\alpha}} \,. \tag{1.11}$$

In fact in a recent paper [5] its relation to the Nambu-Goto string action plus soft-core corrections has been analytically derived, while in a series of other papers the free parameters have been constrained using the astronomical data set [6; 7].

Following the same procedure, we have considered the modified equation of state with asymptotic freedom of Shan-Chen [41]; it is possible to show that:

- This equation of state can be compared to the quark model for hadrons.
- Plugging this equation of state inside the Einstein equations (1.5) it is possible to evolve naturally from a radiation dominated Universe to a dark energy dominated one without the need of any vacuum energy: the pressure switches sign at a certain instant in the past remaining negative for a long time interval which includes the present era.
- It replaces the repulsive action of the cosmological constant with a purely attractive interaction with asymptotic freedom.
- With an appropriate choice of the free parameters we can fit the type Ia supernova data, the scalar spectral index, its running and the ratio of tensor to scalar perturbations.
- This model is stable under small perturbations for the same choice of parameters.
- This equation of state admits an interpretation in terms of a chameleon scalar field.
- It provides a natural exit mechanism from the inflationary era of the Universe, which instead is not present in the previous modeling of the dark energy in terms of the cosmological constant.

I refer to chapter 3 of this thesis and to papers **V** and **VII** and references therein for more details about the derivation of the above mentioned equation of state by Shan and Chen in the framework of the lattice kinetic methods employed in statistical mechanics, and the observational tests we applied. I would like to stress here instead the relationship between a physical interaction and an equation of state [42]. In fact if a microscopic potential $U = U(\mathbf{r})$ describes the interaction between the particles constituting the fluid, the statistical mechanics methods allow us to derive a virial expansion of the equation of state where the pressure is expressed through a power series of the density N/V:

$$\frac{p}{T} = \frac{N}{V} + B_2(T)\left(\frac{N}{V}\right)^2 + B_3(T)\left(\frac{N}{V}\right)^3 + \ldots, \tag{1.12}$$

as a function of the potential, where T denotes the temperature. The second coefficient of this expansion is for example given by

$$B_2(T) = -\frac{1}{2}\int d\mathbf{r}(e^{-\beta U} - 1), \tag{1.13}$$

where β is the inverse temperature. This well-known example shows that also in cosmology we can reconstruct the properties of the fluid permeating the Universe by fitting the data sets, and then connect this fluid to a fundamental interaction using the methods of statistical mechanics.

1.4 Geometry versus matter content of the Universe

In the second and third chapter of this thesis I will introduce two different and original models for relativistic cosmology. As explained there and in the attached papers they belong to two completely different and opposite schools of thinking appearing nowadays in the scientific debate. The initial assumptions are consequently not the same and according to me they deserve to be discussed in some details once again even more than the conclusions they imply. Any theory or model with the ambitions of describing the nature and the physical world we are living in must clearly separate and distinguish between what *we assume*, what *we derive*, what *we observe* and what *we interpret*; cosmologists must always remember and apply these steps.

In the specific case of this thesis I have discussed two models both exhibiting an accelerated expansion Universe; in the concordance model of cosmology such dynamical behavior is explained in terms of the presence of a dark energy in the matter content of the Universe. In this thesis in one case the dark energy is considered as an *interpretative consequence* of the model, in the other as an *observational phenomenon*. From the operative point of view this means that, once general relativity is considered the correct arena for the modeling of this effect, we can act on the right hand side or on the left hand side of the Einstein equations (i.e. on the geometry or on the matter content of the Universe). The physical meaning must be always remembered and is the following:

- Assuming the dark energy to be an interpretative effect means that we expect that it is not a real fluid: it does not exist in the physical world. Its presence is regarded as a consequence of a not completely correct choice of the underlying geometry for the spacetime: the available data sets point out its existence only because they are analyzed with this bias. In the ΛCDM model the most important of these "prejudices"is the copernican principle stating that we do not live in a particular point of the Universe that combined with the Cosmic Microwave Background data allows us to say that the Universe is homogeneous and isotropic on large scales. However a test of this cosmological principle would require the acquisition of the same information about the Cosmic Microwave Background with a family of satellites and astronomical observatories covering the full space each of them set in a different spatial point. Only if they all provide the same data are we permitted to say that the copernican principle, on which the current concordance model is based on, holds [43]. Such a confirmation does not exist at the present time of this research. Another assumption in the current concordance model of cosmology is that there is no interaction energy between the matter sources.

- Assuming the dark energy to be an (indirect) observational evidence means that we assume the validity of homogeneity and isotropy of the Universe and we think that this fluid really exists. In this case we study its physical properties and characteristics.

For the sake of completeness I must aware the reader that the research on the inhomogeneous cosmological model is not only physically motivated, but also mathematically motivated because of the lack of a well-defined definition of the process of averaging for tensorial quantities. However constraining ourselves to the physical aspects, the two approaches are both valid until they are incompatible with at least one observation (the scientific method is hypothetical-deductive-negative-asymmetric [44]). For example if one day there will be a direct detection of a particle of the dark energy fluid we will be led to admit that the small-scale inhomogeneities have less importance on the large-scale dynamics than the one derived in this thesis and that we must reconsider the vacuum hypotheses for our model.

On the other hand the method helps us in outlining some general lines for future short-term researches trying to eliminate unrealistic models:

- After analyzing the dynamical backreaction we would like to study its observational counterpart through the Sachs optical equations: are the lattice models compatible with supernova data?
- The Shan-Chan equation of state coupled to the Friedmann metric is compatible with the above mentioned supernova plot. What can we say about the agreement with cosmic microwave background data? And with the baryonic acoustic oscillations data?

1.5 Friction forces in general relativity: formulation of the problem

In this section I will explain how the motion of a test particle (a particle whose mass-energy m can not perturb the fixed background) is described in general relativity; I will begin with the geodesic motion to add next a friction force term. Here I will only derive the formal equations governing these phenomena in their general form, while their applications to physical interesting situations will be discussed in chapter (4) of this thesis. To really understand this formulation of the problem it is important to consider how it has been treated in newtonian mechanics and then generalize to a curved space-time all the classical relations.

I start reminding the reader that in an Euclidean space the motion of a massive particle is governed by the second law of dynamics:

$$m\mathbf{a} = m\frac{d\mathbf{V}}{dt} = \mathbf{F}\,, \tag{1.14}$$

where $\mathbf{a}$ is the vectorial expression of the acceleration (which can then be expressed as the time derivative of the velocity), while $\mathbf{F}$ is the resultant of all the forces driving the motion. In the case that the force field is conservative it is possible to express it through the gradient of a scalar field, named potential energy U, implying that the set of three equations of motion (1.14) can be re-formulated as

$$m\ddot{\mathbf{x}} = -\nabla U\,, \tag{1.15}$$

where I have eliminated the velocity of the particle in terms of the time derivative of its position. Assuming then a set of initial conditions for the motion, one can fully determine the orbit of the particle (at least numerically). In the particular case $\mathbf{F} = 0$ the motion reduces from a dynamical to a pure geometrical effect. If we are interested in the description of a friction effect, in the r.h.s of the equation of motion we can consider a dissipative force proportional to the velocity of the body moving inside the viscous fluid

$$m\frac{dV}{dt} = -\beta V\,, \tag{1.16}$$

where I considered here only a scalar equation, while β denotes just a numerical factor. If we assume also that the object is moving inside a gravitational field quantified by the acceleration of gravity g the equation to be solved is reduced to

$$\frac{dV}{dt} = g - \frac{\beta}{m}V\,, \tag{1.17}$$

whose solution is given by

$$V = \frac{mg}{\beta}\left(1 - e^{-\frac{\beta t}{m}}\right), \tag{1.18}$$

where I assumed without loss of generality that the particle was initially at rest [45]. When we will move to a general relativistic context, the first part of equation (1.17) will be given by the geodesic term $U^{\alpha}\nabla_{\alpha}U^{\beta}$, U being the four velocity of the particle, and thus our first goal is to understand how to generalize the friction force employed here deriving an expression valid also in a curved background.

For this purpose we must look more in detail how the proportional factor β is written in classical mechanics. The Stokes'law is explicitly given by

$$f_{(\text{Stokes})} = -6\pi R\nu\rho V \tag{1.19}$$

where R is the radius of the body (assumed spherical) whose motion we are interested in, ν is the so-called kinematic viscosity of the medium while ρ denotes its density: the Stokes' force is given by the product of a geometric factor, the viscosity and the group ρV. Starting from this expression we proved that the same force in general relativity is given by the Poynting-Robertson effect [46; 47]:

$$f_{(\text{fric})}(U)^{\alpha} = -\sigma P(U)^{\alpha}{}_{\beta}T^{\beta\mu}U_{\mu}\,, \tag{1.20}$$

where σ can be regarded as the cross section of the interaction, $P(U) = g + U \otimes U$ projects orthogonally to the velocity (here g denoting the background metric) and $T^{\beta\mu}$ is the stress-energy tensor inside which we must consider an appropriate equation of state describing the physical properties of the fluid inside which the body is moving. I want to stress that the Poynting-Robertson formula is not our original result, but it is its connection with the Stokes' law, as proved in paper **III**. In fact considering a stress-energy tensor for a perfect fluid

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}\,, \tag{1.21}$$

in terms of its four-velocity u_μ, and parameterizing the four-velocity of the body as

$$U = \gamma(U,u)[u + v(U,u)]\,, \tag{1.22}$$

$\gamma(U,u)$ denoting here a Lorentz factor and $v(U,u)$ the spatial part of the velocity, the Poynting-Robrtson formula (1.20) is reduced to

$$f_{(\text{fric})} = -\sigma c \gamma^2 \left(\rho + \frac{p}{c^2}\right)\frac{v}{c} \sim -\sigma\gamma^2\left(\rho + p\right)v\,, \tag{1.23}$$

after some algebraic manipulations. In the first step of the equation above I explicitly restored the speed of light to show that σ should be interpreted as an area because of its dimension. Equation (1.23) is exactly the Stokes' law (1.19) with all the modifications that one could expect in a relativistic regime:

- The Lorentz factor γ which reduces to unity in the non-relativistic regime characterized by a small value for the velocity;
- The density ρ now must take into account also the pressure term, as expected from the analogy with the equations of the relativistic hydrostatic equilibrium;
- We can then identify $6\pi R\nu \quad \leftrightarrow \quad \sigma c$.

The constant σ is consequently given by the cross-section of the body whose motion we are interested in:

$$\sigma = \frac{wR\nu}{c} \sim L_{(\text{body})} L_{(\text{visc})}\,, \tag{1.24}$$

where wR can be regarded as the form factor of the body; from this expression we can give a numerical estimate for its value. We also have

$$L_{(\text{visc})} = \lambda c_s / c\,, \tag{1.25}$$

where c_s denotes the speed of sound, while λ is the mean free path, which is indirectly proportional to the density of the medium inside which the motion takes place. Rewriting the cross section of the scattering process between test particle and medium as

$$\sigma = L^2_{(\text{body})} \frac{\lambda}{L_{(\text{body})}} \frac{c_s}{c}\,, \tag{1.26}$$

shows that the dissipative effects we are talking about are expected to drastically modify the motion of macroscopic bodies crossing relativistic fluids: this represents the astrophysical case we are interested in. This is true because in this context the factor σ can become comparable to the geometrical cross section of the body; in fact when the density is sufficiently low, the mean free path is of the same order of magnitude of the dimension of large bodies, $\lambda \sim L_{(\mathrm{body})}$, and on the other hand the speed of sound of a relativistic medium can approach the speed of light. Moreover I would like to observe that this derivation holds for both the cases of a particle moving inside a massive or massless fluid, just changing the equation of state entering the stress-energy tensor which is one of the term of the Poynting-Robertson formula. Thus the analysis of the modified geodesic motion by a friction term is in order, as we will discuss in the chapter dedicated to the physical applications of the Poynting-Robertson effect in general relativity.

2. Inhomogeneous discrete relativistic cosmology

In this chapter I will introduce an inhomogeneous discrete relativistic cosmological model based on a black hole lattice. After providing a physical motivation for the adoption of this class of models, I will review the basic underlying equations and how they have been constructed in literature. Then I will explain how we characterized these models both from the static and dynamic point of views on some special spatial surfaces, lines and points which exhibit particular symmetries, like the reflection and the locally rotational symmetries. Moreover the static characterization can then be used as initial data for the dynamical evolution of the system. This latter original part can be considered as a warming up exercise before moving to the attached papers **VI**[1] and **VIII**.

2.1 Motivation of this study

As already stated in the previous chapter the Friedmann-Lemaître-Robertson-Walker (FLRW) models, relying on the assumptions of spatial homogeneity and isotropy, are in very good accordance with the astronomical data sets in the description of our Universe only at the expenses of introducing a dark and undetected side of the matter content with no physical explanations up to now. On the other hand the astronomers observe that the Universe is filled with galaxies and galaxy clusters and in particular that at the present moment the volume fraction of matter in our Universe is of the order $\sim 10^{-30}$ as explained in paper **VIII**. This suggests the

[1]In the printed version of paper **VI** at page 12 we consider also the 2-fold symmetry case. There is stated that rank-2 tensors have no fixed eigen-direction, implying that eigenvectors must be degenerate and a condition between the components of such a tensor was presented from which we derived the locally rotational symmetry property. This reasoning is not correct. This is because the eigenvector is only defined up to a multiplicative factor, meaning that its direction is undetermined even if we have fixed its length (for example multiplying it by -1). Thus we can not invoke the equation used in that paper. Anyway we never used this argument throughout the paper and it does not affect its conclusions.

needs both to relax the assumption of homogeneity at some level of our description and to consider a in-vacuum model. In particular we are interested in investigating if and in which amount the observed local small scale inhomogeneities affect the large scale evolution of the whole Universe. Moreover our Universe can be considered homogeneous only in the limit of large enough scales and not locally. Unfortunately the Einstein equations (1.5) are not linear and so they do not commute with the averaging operation (this means that a metric describing the Universe in average is not the solution of the average of the Einstein equations). This forces us to find other ways to prove that a homogeneous Universe can arise at some level starting from a discrete modeling of the matter content. Moreover we think that a discrete description will at least enrich our knowledge because more motivated by the observations, for example it contains a gravitational interaction energy term between the matter sources not present in the current dust homogeneous models.

In order to realize a universe which is genuinely inhomogeneous on small scales approaching homogeneity on large ones we can follow the methods pointed out by by Lindquist and Wheeler [20]. They are considered the pioneers of the Wigner-Seitz cell approach to cosmology: considering a closed topology we can tile the 3-sphere with regular polyhedra, put a central mass in each cell and approximate the true space-time geometry with the Schwarzschild geometry of the closest mass. However this approach exhibits discontinuities in the metric and in its derivatives at the boundaries of the cells. We must also remember that the Wigner-Seitz approximation used in condensed matter theories works very well in electrodynamics, but in the theory of gravity it requires some more attention. This is because the initial data are not completely arbitrary but follow from the solution of the constraint Einstein equations (see next section for their derivation). In this chapter I will follow an exact analogue using the Misner static lattice [48]. The possible regular lattices are [49]:

- Tetrahedra 5-cell
- Cubes 8-cell
- Tetrahedra 16-cell
- Octahedra 24-cell
- Dodecahedra 120-cell
- Tetrahedra 600-cell.

Then we will introduce the physical 3-metric related to an auxiliary 3-metric (S^3 metric) by the relation $\gamma_{ij} = \psi^4 \hat{\gamma}_{ij}$ where ψ is the conformal factor solution of the Helmholtz equation $\hat{\triangle}\psi = \frac{1}{8}\hat{R}\psi$, $\hat{\triangle}$ and $\hat{R}$ being respectively the laplacian operator and the scalar curvature for the 3-sphere metric. I want to stress that this is an in-vacuum model because we have sources with an approximately Schwarzschild structure and we are interested in the physics happening outside the horizons of these sources. One possible tool to characterize in a static way these models is to study the curvature at the moment on maximum expansion in some highly symmetric points to move next to the analysis of the same quantity along some lines and surfaces. Moreover these results are the initial data for the full evolution of the dynamics using the Cauchy formulation of general relativity introduced in the section below.

2.2 The Cauchy formulation of General Relativity

This paragraph reviews some results about the formulation of General Relativity as an initial value problem discussed in [38; 50; 51]. In this chapter I will follow the same index convention as in the paper that I co-authored **VI**:

- μ, ν, ρ,..., run between 0 and 3 and denote spacetime coordinates;
- i, j, k,..., run between 1 and 3 and denote spatial coordinates;
- a, b, c,..., run between 0 and 3 and denote orthonormal frame spacetime coordinates;
- α, β, γ,..., run between 1 and 3 and denote spatial orthonormal frame coordinates.

Thanks to the standard 1+3 decomposition we can foliate the 4-dimensional manifold in terms of a family of constant-time hypersurfaces denoted by $\{\Sigma\}$ with the properties of being spacelike and 3-dimensional. The future-pointing timelike unit normal to the slice is

$$n^\mu = -N\nabla^\mu t, \tag{2.1}$$

where N is called *lapse function*. Accounting the general definition of the time vector

$$t^\mu = Nn^\mu + N^\mu \tag{2.2}$$

we can introduce the *shift vector* N^μ with the property $N^\mu n_\mu = 0$. Then the four-dimensional coordinate system adapted to the foliation introduced above is given by x^i (the spatial coordinates in the slice), t is parameterizing the slices and finally γ_{ij} is the 3-dimensional metric of the hypersurfaces. In terms of these quantities the invariant interval can be decomposed as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\,. \tag{2.3}$$

In the Cauchy formulation of General Relativity γ_{ij} is called “first fundamental form”and its value must be chosen in agreement with the constraint equations for a well-posed initial value problem. As second fundamental form (the Einstein equations are second order) we introduce the extrinsic curvature

$$K_{ij} = -\frac{1}{2}\pounds_n\gamma_{ij} = u_{(i;j)}\,, \tag{2.4}$$

$\pounds_n$ denoting the Lie derivative along the n^μ direction. The first and the second fundamental forms together constitute the set for the initial data. Moreover the values of the induced metric depend on the embedding of the hypersurface Σ in the 4-dimensional space and so we must consider also the Gauss-Codazzi constraints:

$$\bar{R} = -K^2 + K_i{}^j K_j{}^i + 2G_{ij}u^i u^j \tag{2.5}$$
$$K^i{}_{j;i} - K_{;j} = \bar{R}_{ik}u^i\gamma^k{}_j\,, \tag{2.6}$$

where $K = K^i{}_i$ is the trace of the extrinsic curvature, $\bar{R}_{ik}$ the Ricci tensor of the hypersurface Σ and $\bar{R}$ its trace. Introducing also the following decomposition of the stress-energy tensor

$$T_{\mu\nu} = \rho u_\mu u_\nu + q_\mu u_\nu + u_\mu q_\nu + p(g_{\mu\nu} + u_\mu u_\nu) + \pi_{\mu\nu} \tag{2.7}$$
$$q_\mu u^\mu = 0, \quad \pi^\mu{}_\mu = 0, \quad \pi_{\mu\nu} = \pi_{(\mu\nu)}, \quad \pi_{\mu\nu}u^\mu = 0\,, \tag{2.8}$$

where q_μ is the momentum density and $\pi_{\mu\nu}$ is the anisotropic pressure, the complete evolution equations can be cast in the following form:

$$\begin{aligned}\partial_t K_{ij} &= N[\bar{R}_{ij} - 2K_{ik}K^k{}_j + KK_{ij} - 8\pi\pi_{ij} + 4\pi\gamma_{ij}(\pi^i{}_i - \rho)] \\ &- \bar{\nabla}_i\bar{\nabla}_j N + N^k\bar{\nabla}_k K_{ij} + K_{ik}\bar{\nabla}_j N^k + K_{jk}\bar{\nabla}_i N^k \end{aligned}\tag{2.9}$$
$$\partial_t\gamma_{ij} = -2NK_{ij} + \bar{\nabla}_i N_j + \bar{\nabla}_j N_i\,, \tag{2.10}$$

where the quantities ith a bar are referred to the three-dimensional hypersurface, while the Hamiltonian and the momentum constraints (just a reformulation of the Gauss-Codazzi ones) are respectively:

$$\bar{R} + K^2 - K_{ij}K^{ij} = 16\pi\rho \tag{2.11}$$
$$\bar{\nabla}_j(K^{ij} - \gamma^{ij}K) = 8\pi q^i\,. \tag{2.12}$$

I underline that only the initial data satisfying the constraints (2.11)-(2.12) can be accepted for the Cauchy formulation of General Relativity and the conservation law $G^{\mu}{}_{\nu;\mu} = 0$ then guarantees that the evolution equations will preserve the constraints on the other slices.

Comment. Introducing an auxiliary 3-metric $\hat{\gamma}_{ij}$ related to the physical 3-metric via the relation

$$\gamma_{ij} = \psi^4 \hat{\gamma}_{ij}, \tag{2.13}$$

where ψ is the so-called conformal factor, the Hamiltonian constraint (2.11) can be written as [50]

$$\hat{\triangle}\psi - \frac{1}{8}\psi\hat{R} - \frac{1}{8}\psi^5 K^2 + \frac{1}{8}\psi^5 K_{ij}K^{ij} = -2\pi\psi^5\rho, \tag{2.14}$$

where $\hat{\triangle}$ is the Laplacian associated with the auxiliary 3-metric and $\hat{R}$ is its scalar curvature. If we are in vacuum and we consider a time-symmetric situation (happening for example in closed cosmological models with a moment of maximum expansion like ours) which implies $K_{ij} = 0$, we obtain the Helmholtz equation

$$\hat{\triangle}\psi = \frac{1}{8}\psi\hat{R} \tag{2.15}$$

which is the equation used for the construction of the initial data of the model we are working on in [17]. If we have also $\hat{R} = 0$ Eq. (2.15) reduces to the well-known Laplace equation $\hat{\triangle}\psi = 0$.

2.2.1 Orthonormal frame approach

In this paragraph I will present the basic equations for the study of a cosmological model like ours. I will follow an orthonormal frame approach [38; 52; 53], which completes the covariant one used in the previous paragraph. The first frame vector corresponds to the unitary time-like (free falling) observer velocity vector u^{μ}; then we introduce the projection tensors

$$h^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + u^{\mu}u_{\nu}, \qquad h^{\mu}{}_{\nu}u^{\nu} = 0, \qquad U^{\mu}{}_{\nu} = u^{\mu}u_{\nu}. \tag{2.16}$$

In terms of the two projectors (2.16) it is possible to irreducibly decompose the covariant derivative of the four-velocity as

$$\nabla_{\mu}u_{\nu} = -u_{\mu}\dot{u}_{\nu} + \theta_{\mu\nu} = -u_{\mu}\dot{u}_{\nu} + \sigma_{\mu\nu} + \frac{1}{3}\theta h_{\mu\nu} - \omega_{\mu\nu}, \tag{2.17}$$

where the rate of expansion $\theta = \tilde{\nabla}_{\mu}u^{\mu}$ is the trace part, the shear $\sigma_{\mu\nu} = \tilde{\nabla}_{\langle\mu}u_{\nu\rangle}$ with $\sigma^2 = \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}$ is the symmetric trace-free part, while the vorticity $\omega_{\mu\nu} = \tilde{\nabla}_{[\mu}u_{\nu]}$ is the antisymmetric one. These are also called the kinematical variables and in a cosmological context the rate of expansion is related to the Hubble function via $\theta = 3H$.

Moreover we can introduce the gravito-electric-magnetic variables. We begin decomposing the Riemann tensor into trace and trace-free parts as:

$$R^{\mu\nu}{}_{\rho\sigma} = C^{\mu\nu}{}_{\rho\sigma} + 2\delta^{[\mu}{}_{[\rho}R^{\nu]}{}_{\sigma]} - \frac{1}{3}R\delta^{\mu}{}_{[\rho}\delta^{\nu}{}_{\sigma]}\,, \tag{2.18}$$

where the Ricci tensor and the Ricci scalar can be eliminated in terms of the matter content of the Universe via the Einstein equations (1.5), while the trace-free Weyl curvature tensor can be decomposed into its electric and magnetic parts relative to u^{μ}:

$$E_{\mu\nu} = C_{\mu\rho\nu\sigma}u^{\rho}u^{\sigma} \tag{2.19}$$

$$H_{\mu\nu} = \frac{1}{2}\eta_{\tau\xi}{}^{\psi\mu}C_{\psi\mu\chi\omega}u^{\xi}u^{\omega} = {}^{\star}C_{\mu\rho\nu\sigma}u^{\rho}u^{\sigma}\,. \tag{2.20}$$

To complete the orthonormal basis we introduce then three more space-like unit vectors, with the property of being mutually orthogonal and each of which is orthogonal to u^{μ}. One can denotes such a set of vectors with $\{e^{\mu}_{\alpha}\}$. They satisfy commutation relations of the form:

$$[e_a\,, e_b] = \gamma^{c}{}_{ab}\,e_c\,, \tag{2.21}$$

where the spatial commutation functions can be further decomposed as

$$\gamma^{\alpha}{}_{\beta\gamma} = 2a_{[\beta}\delta^{\alpha}{}_{\gamma]} + \varepsilon_{\beta\gamma\delta}n^{\delta\alpha}\,. \tag{2.22}$$

The equations we need for our cosmological application are a simplified version of the following:

- Ricci identities: $2\nabla_{[a}\nabla_{b]}u^{c} = R_{ab}{}^{c}{}_{d}u^{d}$
- Twice-contracted Bianchi identities $T^{ab}{}_{;a} = 0$
- Bianchi identities: $\nabla_{[a}R_{bc]de} = 0$
- Jacobi identities: $[[e_a, e_b], e_c] + [[e_b, e_c], e_a] + [[e_c, e_a], e_b] = 0$

As a next step we separate out the orthogonally projected part into trace, symmetric trace-free and skew-symmetric parts and the parallel part similarly obtaining a set of propagation and constraint equations (Cauchy formulation of general relativity). Some of them are referred to in literature with historical names; from the Ricci identities the propagation equations are:

- Raychaudhuri equation (trace part)
- Vorticity propagation equation (anti-symmetric part)
- Shear propagation equation (trace-free symmetric part)

while the constraint equations are

- (0α)-equation
- Vorticity divergence identity
- H_{ab}-equation.

From the Bianchi identities the propagation equations are:

- $\dot{E}$-equation
- $\dot{H}$-equation

while the constraint equations are

- divE-equation
- divH-equation.

I refer to the papers [38; 53] for their general and complete expressions. Finally to express the geometric quantities through the physical ones we write the extrinsic curvature as

$$-K_{ij} = \sigma_{ij} + \frac{1}{3}\gamma_{ij}\theta\,. \tag{2.23}$$

The above mentioned set of equations constitutes a system of non-linear partial differential equations, and its solution is not known in general. One way of dealing with them is to assume some kind of symmetry and/or other simplifications imposing the vanishing of some terms for hypotheses and derive a simpler set of equations. A fundamental task in this procedure of simplification is to check that the simplified version of the constraint equations are really correctly evolved by the simplified version of propagation equations, otherwise the model is inconsistent. Maartens [54] showed that under the assumptions I am considering in this chapter, the complete covariant set of equations to be studied in a generic point of the spacetime reduces to the following. The evolution equations are

$$\dot{\theta} = -\frac{1}{3}\theta^2 - 2\sigma^2 \tag{2.24}$$

$$\dot{\sigma}_{ab} = -\frac{2}{3}\theta\sigma_{ab} - \sigma_{c\langle a}\sigma_{b\rangle}{}^{c} - E_{ab} \tag{2.25}$$

$$\dot{E}_{ab} = -\theta E_{ab} + 3\sigma_{c\langle a}E_{b\rangle}{}^{c} + \mathrm{curl}H_{ab} \tag{2.26}$$

$$\dot{H}_{ab} = -\theta H_{ab} + 3\sigma_{c\langle a}H_{b\rangle}{}^{c} - \mathrm{curl}E_{ab}\,, \tag{2.27}$$

while the constraint equations are

$$\mathrm{div}\sigma_a = \frac{2}{3}D_a\theta \tag{2.28}$$

$$H_{ab} = \mathrm{curl}\sigma_{ab} \tag{2.29}$$

$$\mathrm{div}E_a = \varepsilon_{abc}\sigma^b{}_d H^{cd} \tag{2.30}$$

$$\mathrm{div}H_a = -\varepsilon_{abc}\sigma^b{}_d E^{cd}\,, \tag{2.31}$$

where $\mathrm{div}A_a = D^b A_{ab}$ and $D_a A^b{}_c = h^d{}_a h^b{}_e h^f{}_c \nabla_d A^e{}_f$.

In this chapter I will further simplify these equations to derive the dynamics of our model in special highly symmetric points, while in the attached papers **VI** and **VIII** a similar analysis is extended to particular curves exhibiting local rotational symmetry. I will also use the well-known algebraic decomposition of a spatial symmetric trace-free tensor [53]:

$$\begin{aligned} &A_+ := -\frac{3}{2}A_{11} = \frac{3}{2}(A_{22}+A_{33})\,, \quad A_- := \frac{\sqrt{3}}{2}(A_{22}-A_{33}) \\ &A_1 := \sqrt{3}A_{23}\,, \quad A_2 := \sqrt{3}A_{31}\,, \quad A_3 := \sqrt{3}A_{12}\,. \end{aligned} \tag{2.32}$$

2.3 Curvature of a discrete mass distribution

I am considering an inhomogeneous model of a closed Universe filled with a discrete distribution of masses. To begin with I want to compare them to the FLRW models at the moment of maximum expansion to provide a static characterization and to generate the initial data for the dynamical evolution which will follow. The lattice of black holes on the 3-sphere is described by the metric [17]

$$ds^2 = \psi^4\left(d\chi^2 + \sin^2\chi d\theta^2 + \sin^2\chi\sin^2\theta d\phi^2\right), \quad \psi = \sum_k \frac{\sqrt{m_k}}{2f_k(\chi,\theta,\phi)}\,, \tag{2.33}$$

where

$$ds^2 = d\chi^2 + \sin^2\chi d\theta^2 + \sin^2\chi \sin^2\theta d\phi^2, \tag{2.34}$$

is the metric of the 3-sphere. The index k runs over the masses involved in the configuration, m_k is the numeric value of each mass (that in the following I will put $m_k = 1$ without loss of generality since all the physical quantities should be rescaled with respect to the values of the total mass of the model); the functions f_k are solutions of the Helmholtz equation (2.15) where now $\hat{R} = 6$ is the scalar curvature of the 3-sphere metric (2.34). Being the Helmholtz equation a linear equation, it allows the use of the superposition principle to construct lattice type solutions [55].

As I explained in the section above, the Riemann tensor, which quantifies the curvature of a manifold, is irreducibly decomposed into a trace part given by the Ricci tensor and scalar and a tracefree part, the Weyl tensor. Moreover the latter can be decomposed into a spatial electric and a spatial magnetic observer-dependent fields. I will refer to these quantities, and in particular to the electric field, as the curvature of the configuration.

To compare the results we will obtain for the different configurations between themselves and with the analogue Friedmann solution it is necessary to take into account the presence of the binding energy between the black holes constituting the lattices which instead does not enter the standard cosmological model which is based on a (continuous) fluid description.

This evaluation can be done following [17; 56]. Observing that all our configurations have a mass at the north pole $\chi = 0$ (at least up to a rotation), and observing that a series expansion around this point gives

$$\psi(\chi, \theta, \phi) = A + \frac{B}{\chi} + O(\chi), \tag{2.35}$$

and comparing with the Schwarzschild metric which approximates the spacetime in that region, we can introduce the “proper”mass defined as:

$$\hat{m} = 2AB. \tag{2.36}$$

The appearance of an interaction term between the constituents is a basic property of the discrete models and will play a fundamental role in the computations which follow. For the values of the binding energy in the models considered below see [17].

The Friedmann metric (1.8) can be showed to be conformally flat, meaning that it can be expressed as the Minkowski metric multiplied by a conformal factor. A straightforward computation thus shows that both the gravito-electric

and gravito-magnetic tensors are identically zero: $E_{\mu\nu} = H_{\mu\nu} = 0$ along all the evolution. The Ricci tensor and the Ricci scalar, the remaining parts of the curvature, are instead functions of the particular fluid permeating the Universe via the Einstein equations (1.5) and they are functions only of the time and not of the spatial coordinates for the homogeneity assumption.
The picture of our discrete models is instead completely different from the Friedmann one: the trace parts of the Riemann tensor are identically zero along all the evolution because we are dealing with a vacuum model (in particular the Riemann tensor reduces to the Weyl one), while the electric and magnetic fields are in general non-zero and depend both on the time and on the spatial coordinates. On the time-symmetric hypersurface the electric field is reduced to the three-dimensional Ricci tensor of the hypersurface which can be evaluated directly from the line element (2.33) once the functions f_k solutions of the constraint equations are known. They depend on the positions of the mass sources n_0:

$$f_k = \sqrt{2(1 - n \cdot n_0)}\,, \qquad (2.37)$$

where $n = (\cos\chi, \sin\chi\cos\theta, \sin\chi\sin\theta\cos\phi, \sin\chi\sin\theta\sin\phi)$. We focused our attention on the lines admitting local rotational symmetry, the edges of the cells being one example. We showed that along the edges the magnetic field is zero at the beginning for symmetry under time reversal and its evolution equation imposes it to remain zero along all the time evolution. The situation displayed by the electric field is instead richer: only one independent component is non zero and its numerical value depends on the position along the edge. Particular points are the vertices: they are spherically symmetric and we expect that the electric field is identically zero at the beginning and remains zero also during the evolution: its initial numerical value directly derived form the line element (2.33) confirmed this intuition, while its evolution is discussed in the next section. These points are referred to as locally minkowskian because the complete Riemann tensor is zero. The spatial evolution along the edge of the non-zero electric field component at the initial moment exhibits two different behaviors depending if the configuration admits contiguous or non-contiguous edges since this affects the value of its first derivative. Moreover the electric field has always the same sign in all the points constituting the edge and admits a local extremum in the middle point between two vertices. Moreover the flat region around the vertices is dominant when the number of masses is increased. See figure 4 and 5 of paper **VI** for a graphical representation.

To summarize, we started showing that the discrete configurations we are dealing with in our cosmological models admit a wider behavior than the previously considered Friedmann models. In particular it is already evident from this static analysis that different spatial regions of the same configuration behave completely differently from each other and from their Friedmannian analogue. This suggests to improve the comparison between these different models and with the Friedmann one also from a dynamical point of view.

2.4 Equations of the dynamics for the discrete models

In this section I will move from the static characterization of the discrete models discussed above to their dynamical one. The equations presented here are, of course, simplified relations coming from the general ones presented at the beginning of this chapter. Some attention will be devoted to their qualitative analysis which allows us to determine the relationships between our lattice model with other exact cosmological solutions already studied in literature. As in the previous section, the computations mentioned here must be intended as a simplified paradigm of the one presented in the published articles on which this chapter is based.
The initial values for the integration of the evolution equations are:

- $\theta = 0$ from time symmetry
- $\sigma^{ij} = 0$ from the Hamiltonian constraint taking into account that we are in vacuum ($\bar{R} = 0$)
- $E^{ij} = \bar{R}^{ij}$ from (2.9) with the substitution $-K_{ij} = \sigma_{ij} + \frac{1}{3}\gamma_{ij}\theta$ and taking into account the shear propagation equation and the preceding initial conditions.

In particular at the center of the cell faces and at the vertices we have $\tilde{\nabla}_i E^{ij} = 0$ and $\tilde{\nabla}_i \sigma^{ij} = 0$ because for example the center of the face corresponds to a middle point between two masses whose position is preserved during the evolution. Thus in this peculiar point both E^{ij} and σ^{ij} have only one independent component each. In fact:

- E^{ij} has 9 components (it is a 3x3 tensor)
- From the symmetry $E^{ij} = E^{(ij)}$ the independent components are 6
- From the constraint $E^{i}{}_{i} = 0$ the independent components reduce to 5
- From the 3 equations $\tilde{\nabla}_i E^{ij} = 0$ the independent components reduce to 2
- The directions inside one face are equivalent and so in the center of the cell face E^{ij} must have only one independent component.

Therefore in the center of the face in an orthonormal frame we can parametrize the electric field and the shear tensor as $E_{ij} = \mathrm{diag}[-2E, E, E]$ and $\sigma_{ij} = \mathrm{diag}[-2\tilde{\sigma}, \tilde{\sigma}, \tilde{\sigma}]$.
The equations driving the evolution of the components of the electric field and the shear tensor are thus given by:

$$\begin{aligned} \dot{E} &= -\theta E - 3\tilde{\sigma}E && (2.38) \\ \dot{\tilde{\sigma}} &= -\frac{2}{3}\theta\tilde{\sigma} + \tilde{\sigma}^2 - E && (2.39) \\ \dot{\theta} &= -\frac{1}{3}\theta^2 - 6\tilde{\sigma}^2\,, && (2.40) \end{aligned}$$

where a dot denotes derivative with respect to the time. Since the evolution equations change the values of the components of the electric tensor and of the shear, but not their structure [39; 40](for example the tensor E_{ab} will continue to be of the form presented above only with a different numerical value for the quantity E), we can understand from the $\dot{H}$-equation that the initial condition $H_{ij} = 0$ will hold also during the evolution. The general class of models characterized by zero pressure and vanishing magnetic component of the Weyl tensor are named in literature with the name of *silent universe* because there is no exchange of information between different fluid elements, either by sound waves (since $p = 0$) or by gravitational waves (since $H_{ab} = 0$) [57–60]. I want to stress that also in the case of dust, where the density energy ρ is non zero, one can impose the condition $H_{ab} = 0$ and assume that the fluid is irrotational to have a silent universe [59; 61]. Moreover it is well-known that these simplifications allow us to decouple the evolution equations for E and for $\tilde{\sigma}$ with a suitable change of variables from the evolution equation for θ and write them as an autonomous system of ordinary differential equations [62]. This manipulation allows us to study them qualitatively as a dynamical system looking for the possible equilibrium points.

Initial values for the first derivatives. Inserting the initial data $\theta = 0$ and $\tilde{\sigma} = 0$ inside the evolution equations (2.38), (2.39) and (2.40) we can obtain information about the first derivatives of the physical quantities which describe our system. In fact we have:

$$\begin{aligned} \dot{E}_{\mathrm{in}} &= 0 && (2.41) \\ \dot{\tilde{\sigma}}_{\mathrm{in}} &= -E_{\mathrm{in}} && (2.42) \\ \dot{\theta}_{\mathrm{in}} &= 0\,, && (2.43) \end{aligned}$$

which allow us to make some observations:

- At the beginning the variation of the eigenvalue of the electric component of the Weyl tensor is small and this quantity exhibits an extremum in correspondence of the time-symmetric hypersurface;

- At the moment of maximum expansion it is reasonable to have $\dot{\sigma} = 0$ but there are not hypotheses about the time derivative of the eigenvalue $\tilde{\sigma}$;

- The same discussion for the initial value of the derivative of E is valid also for θ.

A remarkable property of the evolution system we obtained is to admit the Milne solution and the Bianchi flat type III as invariant subsets as proved in the chapter 13 of the book [62] exploiting the spatial variable method. These solutions are characterized by

- Milne universe:

$$\Omega_m = 0\,, \quad \Omega_\Lambda = 0\,, \quad \Sigma_+ = 0\,, \quad E_+ = 0\,, \quad M_1 = M_2 = M_3 = \frac{1}{3}$$

- Flat Bianchi III:

$$\Omega_m = 0\,, \quad \Omega_\Lambda = 0\,, \quad \Sigma_+ = \frac{1}{2}\,, \quad E_+ = 0\,, \quad M_1 = \frac{3}{4}\,, \quad M_2 = M_3 = 0$$

where

$$M_1 = \frac{1}{3}(K - 2S_+)\,, \qquad M_{2,3} = \frac{1}{3}(K + S_+ \pm \sqrt{3} S_-) \tag{2.44}$$

$$K = -\frac{\bar{R}}{6H^2}\,, \qquad S_\pm = \frac{{}^3S_\pm}{H^2}\,, \qquad \Sigma_+ = \frac{\sigma_+}{H}\,, \tag{2.45}$$

with H the Hubble function and ${}^3S_\pm$ the spatial curvature. Of course, this is a subset of the analysis of a generic silent universe; although the results obtained in [62] are valid in all the spacetime points while now only in some due to their symmetries. In paper **VI** instead we showed how to decouple this system of equations.

2.5 From generic points to particular surfaces, lines and points

This section is devoted to the analytic derivation of the equations for the lines connecting two masses or two vertices in the 5 masses configuration. These lines play an important role since they are locally rotationally symmetric. I start therefore moving from a generic point of the configuration to some special surfaces.

2.5.1 Reflection symmetry hypersurfaces

The symmetries of the configuration help us not only in characterizing the locally rotational lines, but also the hypersurfaces invariant under reflection. Assuming that on these hypersurfaces the reflection is implemented by the transformation $x_1 \to -x_1$, the metric exhibits the important property

$$g_{\mu\nu}(x_0, x_1, x_2, x_3) = g_{\mu\nu}(x_0, -x_1, x_2, x_3)\,; \tag{2.46}$$

reflection symmetry in particular means that all the odd functions of the coordinate x_1 must vanish on the surface we are dealing with. Consequently it is important to construct an orthonormal frame in terms of only even either odd functions of the coordinate under which we have the reflection symmetry: this is the starting point of paper **VIII**. After some algebraic manipulations an appropriate construction of the frame shows explicitly that all the commutation functions (2.21) exhibiting an odd number of indices equal to 1 must vanish. This result can then be restated in terms of the Ricci rotation coefficients which are combination of the commutation functions

$$\Gamma_{abc} = \frac{1}{2}\left(\gamma_{acb} + \gamma_{bac} - \gamma_{cba}\right). \tag{2.47}$$

In terms of the Ricci rotation coefficients the Riemann tensor, which in our case corresponds to the Weyl tensor being in vacuum, can be written as [52]

$$C^a{}_{bcd} = R^a{}_{bcd} = e_c(\Gamma^a{}_{bd}) - e_d(\Gamma^a{}_{bc}) + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc} - \Gamma^a{}_{be}\gamma^e{}_{cd}\,. \tag{2.48}$$

A direct inspection of the electric and magnetic tensors under the reflection symmetry just introduced provides the following restrictions:

$$E_{12} \equiv 0\,, \qquad E_{13} \equiv 0\,, \qquad H_{11} \equiv 0\,, \tag{2.49}$$

$$H_{22} \equiv 0\,, \qquad H_{23} \equiv 0\,, \qquad H_{33} \equiv 0\,. \tag{2.50}$$

These relations state that the component with an odd number of indices equal to 1 of the electric tensor are zero, while for the magnetic tensor the same condition holds for an even number. In particular these simplifications combined together impose the relation

$$E_{ab}H^{ab} = 0 \tag{2.51}$$

on the reflection symmetry surfaces: on the cell faces, for example, we then have a weaker condition than the one defining a silent universe. Moreover this condition, contrary to the electric and magnetic tensors themselves, is observer-independent and consequently is a true property of the space-time under exam. What discussed so far is valid for any reflection symmetry hypersurface. In the specific case of our discrete inhomogeneous cosmological model we start observing that the initial data exhibit this kind of discrete symmetry and when we apply the theorems for the conservation of a symmetry in general relativity proved in [39; 40] to derive a result which is valid all along the time evolution of the system.

2.5.2 Lines connecting two masses

I consider explicitly the masses 4 and 5 in table II of [17], whose positions in the coordinates of the embedding space are:

$$w=-\frac{1}{4}, \qquad x=-\frac{1}{4}\sqrt{\frac{5}{3}}, \qquad y=-\frac{1}{2}\sqrt{\frac{5}{6}}, \qquad z=\lambda\,. \tag{2.52}$$

Normalizing the previous expression we obtain:

$$\begin{aligned} w&=-\frac{1}{\sqrt{6+16\lambda^2}}, \qquad x=-\sqrt{\frac{5}{3(6+16\lambda^2)}}, \\ y&=-\sqrt{\frac{10}{3(6+16\lambda^2)}}, \qquad z=\frac{4\lambda}{\sqrt{6+16\lambda^2}}, \end{aligned} \tag{2.53}$$

which gives the parametric equation of the line connecting two masses if we consider $-\frac{1}{2}\sqrt{\frac{5}{2}} \leq \lambda \leq \frac{1}{2}\sqrt{\frac{5}{2}}$. Using the coordinate transformation

$$\begin{aligned} w &= \cos\chi \\ x &= \sin\chi\cos\theta \\ y &= \sin\chi\sin\theta\cos\phi \\ z &= \sin\chi\sin\theta\sin\phi\ , \end{aligned} \tag{2.54}$$

the equation of the line connecting two masses can be derived in the coordinates intrinsic to the 3-sphere. It reads as:

$$\begin{aligned} \chi &= \arccos\left(-\frac{1}{\sqrt{6+16\lambda^2}}\right) \\ \theta &= \arccos\left(-\sqrt{\frac{5}{3(5+16\lambda^2)}}\right) \\ \phi &= \arctan\left(-2\sqrt{\frac{6}{5}}\lambda\right)\,. \end{aligned} \tag{2.55}$$

2.5.3 **Lines connecting two vertices**

I consider explicitly the vertices 4 and 5 in table (2.1), whose positions in the coordinates of the embedding space are:

$$w = \frac{1}{4}, \qquad x = \frac{1}{4}\sqrt{\frac{5}{3}}, \qquad y = \frac{1}{2}\sqrt{\frac{5}{6}}, \qquad z = \lambda \,. \tag{2.56}$$

Normalizing the previous expression we obtain:

$$\begin{aligned} w &= \frac{1}{\sqrt{6+16\lambda^2}}, \qquad x = \sqrt{\frac{5}{3(6+16\lambda^2)}}, \\ y &= \sqrt{\frac{10}{3(6+16\lambda^2)}}, \qquad z = \frac{4\lambda}{\sqrt{6+16\lambda^2}}, \end{aligned} \tag{2.57}$$

which gives the parametric equation of the line connecting two vertices if we consider $-\frac{1}{2}\sqrt{\frac{5}{2}} \leq \lambda \leq \frac{1}{2}\sqrt{\frac{5}{2}}$. Using the coordinate transformation (2.54) the equation of the line connecting two vertices can be derived in the coordinates intrinsic to the 3-sphere. It reads as:

$$\begin{aligned} \chi &= \arccos\left(\frac{1}{\sqrt{6+16\lambda^2}}\right) \\ \theta &= \arccos\left(\sqrt{\frac{5}{3(5+16\lambda^2)}}\right) \\ \phi &= \arctan\left(2\sqrt{\frac{6}{5}}\lambda\right) \,. \end{aligned} \tag{2.58}$$

The computation of this section gives an analytic picture of what instead is presented only graphically in paper **VI**. Thus in the study of the cosmological model introduced in this chapter I started from the analysis of particularly symmetric surfaces to move next to the case of locally rotationally symmetric lines to move then to the analysis of specific points; in this latter case the reflection symmetry substitutes the local rotational one and allows us to restrict the non-zero components of the curvature tensor.

2.5.4 **Application of the 1+3 orthonormal frame formalism**

Our dynamical equations (2.38), (2.39) and (2.40) have been derived following a 1+3 covariant approach. However it is possible to determine the same system of equations following a more complete 1+3 orthonormal formalism. Moreover this method let us check the conservation of the

Point	w, x, y, z	χ,θ,ϕ
1	-1, 0, 0, 0	$\pi, -, -$
2	$\frac{1}{4}$, $-\frac{\sqrt{15}}{4}$, 0, 0	$\arccos\frac{1}{4}$, π, $\frac{3\pi}{2}$
3	$\frac{1}{4}, \frac{1}{4}\sqrt{\frac{5}{3}}, -\sqrt{\frac{5}{6}}, 0$	$\arccos\frac{1}{4}$, $\arccos\frac{1}{3}$, π
4	$\frac{1}{4}, \frac{1}{4}\sqrt{\frac{5}{3}}, \frac{1}{2}\sqrt{\frac{5}{6}}, -\frac{1}{2}\sqrt{\frac{5}{2}}$	$\arccos\frac{1}{4}$, $\arccos\frac{1}{3}$, $\frac{5\pi}{3}$
5	$\frac{1}{4}, \frac{1}{4}\sqrt{\frac{5}{3}}, \frac{1}{2}\sqrt{\frac{5}{6}}, \frac{1}{2}\sqrt{\frac{5}{2}}$	$\arccos\frac{1}{4}$, $\arccos\frac{1}{3}$, $\frac{\pi}{3}$

Table 2.1: Positions of the vertices in the coordinates of the embedding space and in the hyperspherical polar coordinates intrinsic to the 3-sphere. We can note that the vertices are at the antipode respect to the masses and we remind that the coordinate transformation for going to the antipode is $w \to -w$, $x \to -x$, $y \to -y$, $z \to -z$ or equivalently $\chi \to \pi - \chi$, $\theta \to \pi - \theta$, $\phi \to \phi + \pi$.

constraints $E_{22} = E_{33}$ and $\sigma_{22} = \sigma_{33}$ along the evolution. Here I want to demonstrate this statement starting from the results obtained in [52; 53]. In the 1+3 orthonormal formalism the coupled system of ordinary differential evolution equations is given by:

$$\mathbf{e}_0(\theta) = -\frac{1}{3}\theta^2 - 2\sigma^2 \tag{2.59}$$

$$\mathbf{e}_0(\sigma_+) = -\frac{1}{3}(2\theta - \sigma_+)\sigma_+ - \frac{1}{3}(\sigma_-)^2 - E_+ \tag{2.60}$$

$$\mathbf{e}_0(\sigma_-) = -\frac{2}{3}(\theta + \sigma_+)\sigma_- - E_- \tag{2.61}$$

$$\mathbf{e}_0(E_+) = -(\theta + \sigma_+)E_+ + \sigma_- E_- \tag{2.62}$$

$$\mathbf{e}_0(E_-) = -(\theta - \sigma_+)E_- + \sigma_- E_+ \,. \tag{2.63}$$

Starting from our initial data we obtain:

$$E_+ = 3E\,, \qquad E_- = 0\,, \qquad \sigma_+ = 3\tilde{\sigma}\,, \qquad \sigma_- = 0\,. \tag{2.64}$$

In this way the equations (2.59), (2.60) and (2.62) reduce respectively to our evolution equations written in a covariant way (2.38), (2.39) and (2.40), while equations (2.61) and (2.63) are identically satisfied proving our assertion.

2.5.5 Commutators and symmetries in the orthonormal frame approach

Now I would like to discuss the dynamics following the $1+3$ orthonormal frame approach focusing my attention on the center of the face and in the vertex of our discrete configurations starting from the papers [52; 53] and inserting our assumptions regarding the matter content of the Universe. The commutators of the orthonormal frame vectors are

$$
\begin{aligned}
[e_0, e_\alpha] &= -\left[\frac{1}{3}\theta\delta^\beta{}_\alpha + \sigma^\beta{}_\alpha\right] e_\beta && (2.65)\\
\left[e_\alpha, e_\beta\right] &= \left[2a_{[\alpha}\delta^\gamma{}_{\beta]} + \varepsilon_{\alpha\beta\delta}n^{\delta\gamma}\right] e_\gamma . && (2.66)
\end{aligned}
$$

Moreover we can invert relation (2.22) to obtain the quantities a and n:

$$
\begin{aligned}
a_\beta &= \frac{1}{2}\gamma^\alpha{}_{\beta\alpha} && (2.67)\\
n^{\alpha\beta} &= \frac{1}{2}\gamma^{(\alpha}{}_{\gamma\delta}\varepsilon^{\beta)\gamma\delta} . && (2.68)
\end{aligned}
$$

We can further simplify the commutators focusing our attention on some specific points of our configuration characterized by certain discrete symmetries, the most important ones being symmetry under reflection which preserves a given cell face and locally rotational symmetry. In particular the 1+3 orthonormal frame approach helps us in showing explicitly the conservation of such symmetries during the evolution and the connection between them and the gravito-magnetic tensor.

For example the center of a face is a locally rotational symmetric point (in the plane) and in this particular case the commutators become

$$
\begin{aligned}
[e_0, e_1] &= -\frac{1}{3}(\theta - 2\sigma_+)e_1 && (2.69)\\
\left[e_0, e_2\right] &= -\frac{1}{3}(\theta + \sigma_+)e_2\\
\left[e_0, e_3\right] &= -\frac{1}{3}(\theta + \sigma_+)e_3\\
\left[e_1, e_2\right] &= ae_2\\
\left[e_2, e_3\right] &= n_{11}e_1 + 2n_{31}e_3\\
\left[e_3, e_1\right] &= -ae_3
\end{aligned}
$$

leading to the following evolution equations

$$\begin{aligned}
e_0(\theta) &= -\frac{1}{3}\theta^2 - \frac{2}{3}(\sigma_+)^2 && (2.70)\\
e_0(\sigma_+) &= -\frac{1}{3}(2\theta - \sigma_+)\sigma_+ - E_+ && (2.71)\\
e_0(a) &= -\frac{1}{3}(\theta + \sigma_+)a && (2.72)\\
e_0(n_{11}) &= -\frac{1}{3}(\theta + 4\sigma_+)n_{11} && (2.73)\\
e_0(n_{31}) &= -\frac{1}{3}(\theta + \sigma_+)n_{31} && (2.74)\\
e_0(E_+) &= -\theta E_+ - \sigma_+ E_+ - \frac{3}{2}n_{11}H_+ && (2.75)\\
e_0(H_+) &= -(\theta + \sigma_+)H_+ + \frac{3}{2}n_{11}E_+\,, && (2.76)
\end{aligned}$$

to the following constraint equations

$$\begin{aligned}
0 &= -\frac{2}{3}e_1(\theta) - \frac{2}{3}(e_1 - 3a)(\sigma_+) && (2.77)\\
0 &= (e_1 - a)(a) + \frac{1}{9}\theta^2 - \frac{1}{9}(\theta + 2\sigma_+)\sigma_+ + \frac{1}{4}(n_{11})^2 + \frac{1}{3}E_+ && (2.78)\\
0 &= (e_1 - 2a)(n_{11}) && (2.79)\\
0 &= (e_1 - a)(n_{31}) && (2.80)\\
0 &= H_+ + \frac{3}{2}n_{11}\sigma_+ && (2.81)\\
0 &= (e_1 - 3a)(E_+) && (2.82)\\
0 &= (e_1 - 3a)(H_+)\,, && (2.83)
\end{aligned}$$

and to two consistency conditions:

$$\begin{aligned}
0 &= a n_{11} && (2.84)\\
0 &= n_{11}\left[E_+ + \frac{1}{3}(\theta - \sigma_+)\theta - \frac{2}{3}(\sigma_+)^2 + \frac{3}{4}(n_{11})^2\right]\,,
\end{aligned}$$

where no assumptions were made about the silent properties of the Universe. When we will move to the vertices of the cells these equations simplify further because they are spherically rotational symmetric points in three dimensions and not only in the plane.

Now I will specialize to the center of the face in the eight mass case to specify the preceding equations. To be specific I consider the face defined by the four vertices whose Euclidean coordinates are $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$,

$\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, where I used the fact that the positions of the vertices in the eight masses configuration correspond to the positions of the masses in the sixteen one. Thus the center of the face is the point

$$c = \left(0, 0, \frac{1}{2}, \frac{1}{2}\right) \tag{2.85}$$

and an arbitrary point in the face can be expressed as

$$f = c + \lambda_1 s_1 + \lambda_2 s_2 = \left(\lambda_1, \lambda_2, \frac{1}{2}, \frac{1}{2}\right), \tag{2.86}$$

where the two vectors $s_1 = (1,0,0,0)$ and $s_2 = (0,1,0,0)$, which are constructed by the difference of the coordinates of the vertices, are the ones which span the space. To obtain the equation of the face in spherical coordinates I must introduce the normalization factor

$$N = ||f|| = \sqrt{\lambda_1^2 + \lambda_2^2 + \frac{1}{2}} \tag{2.87}$$

so that

$$f_N = \left(\frac{\lambda_1}{N}, \frac{\lambda_2}{N}, \frac{1}{2N}, \frac{1}{2N}\right), \tag{2.88}$$

which shows that the equation of the face is $\phi = \frac{\pi}{4}$. Thus the face normal is

$$n = \sqrt{g_{\phi\phi}}\,\phi_{,i} = \psi^2 \sin\chi \sin\theta\, \delta_i^\phi \tag{2.89}$$

and thanks to the diagonal property of the metric the orthonormal basis vectors at the initial moment can be chosen as:

$$\begin{aligned} e_2 &= e_\chi = \frac{1}{\psi^2}\partial_\chi & (2.90)\\ e_3 &= e_\theta = \frac{1}{\psi^2 \sin\chi}\partial_\theta & (2.91)\\ e_1 &= e_\phi = \frac{1}{\psi^2 \sin\chi \sin\theta}\partial_\phi\,. & (2.92) \end{aligned}$$

The commutators at the initial moment are given by:

$$\begin{aligned} [e_\chi, e_\theta] &= \frac{2\partial_\theta\psi}{\psi^3 \sin\chi} e_\chi - \frac{2\sin\chi\,\partial_\chi\psi + \cos\chi\,\psi}{\psi^3 \sin\chi} e_\theta & (2.93)\\ [e_\theta, e_\phi] &= \frac{2\partial_\phi\psi}{\psi^3 \sin\chi \sin\theta} e_\theta - \frac{2\sin\theta\,\partial_\theta\psi + \cos\theta\,\psi}{\psi^3 \sin\chi \sin\theta} e_\phi & (2.94)\\ [e_\phi, e_\chi] &= -\frac{2\partial_\phi\psi}{\psi^3 \sin\chi \sin\theta} e_\chi + \frac{2\sin\chi\,\partial_\chi\psi + \cos\chi\,\psi}{\psi^3 \sin\chi} e_\phi\,, & (2.95) \end{aligned}$$

or equivalently the non-zero structure functions are reduced to:

$$\begin{aligned} \gamma^{\chi}{}_{\chi\theta} &= \frac{2\partial_\theta \psi}{\psi^3 \sin\chi} \\ \gamma^{\theta}{}_{\chi\theta} &= -\frac{1}{\psi^3}\left[2\partial_\chi \psi + \cot\chi\,\psi\right] = \gamma^{\phi}{}_{\chi\phi} \\ \gamma^{\theta}{}_{\theta\phi} &= \frac{2\partial_\phi \psi}{\psi^3 \sin\chi \sin\theta} = \gamma^{\chi}{}_{\chi\phi} \\ \gamma^{\phi}{}_{\theta\phi} &= -\frac{1}{\psi^3 \sin\chi}\left[2\partial_\theta \psi + \cot\theta\,\psi\right]. \end{aligned} \tag{2.96}$$

I stress that up to now this result for the structure functions is valid in all the points of the face. When specializing to a locally rotational symmetric point we can identify:

$$a = -\frac{2}{\psi^3 \sin\chi \sin\theta}\partial_\phi \psi \tag{2.97}$$

$$n_{11} \equiv 0 \tag{2.98}$$

$$n_{31} = -\frac{1}{2\psi^3 \sin\chi}\left[2\sin\chi\,\partial_\chi \psi + \cos\chi\,\psi\right]. \tag{2.99}$$

One can note that in the eight mass configuration in all the face defined by the equation $\phi = \frac{\pi}{4}$ we have $\partial_\phi \psi \equiv 0$ so that the structure functions reduce to:

$$\begin{aligned} \gamma^{\chi}{}_{\chi\theta} &= \frac{2\partial_\theta \psi}{\psi^3 \sin\chi} \\ \gamma^{\theta}{}_{\chi\theta} &= -\frac{1}{\psi^3}\left[2\partial_\chi \psi + \cot\chi\,\psi\right] = \gamma^{\phi}{}_{\chi\phi} \\ \gamma^{\phi}{}_{\theta\phi} &= -\frac{1}{\psi^3 \sin\chi}\left[2\partial_\theta \psi + \cot\theta\,\psi\right]. \end{aligned} \tag{2.100}$$

Center of the face

Substituting then ψ in terms of the functions f_k and inserting the coordinate of the center of the face $\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$ one obtains that the basis vectors commute at the initial moment, i.e.:

$$[e_\chi, e_\theta] = [e_\theta, e_\eta] = [e_\eta, e_\chi] = 0, \tag{2.101}$$

where I have introduced $\eta = \phi + \frac{\pi}{4}$ giving $a = 0$ and $n^{ij} = 0$ at the initial instant. In this way we obtain the constraint equations at the initial moment:

$$e_1(\theta) = -e_1(\sigma_+) \tag{2.102}$$
$$0 = 3e_1(a) + \frac{1}{3}\theta^2 - \frac{1}{3}(\theta + 2\sigma_+)\sigma_+ + E_+ \tag{2.103}$$
$$0 = e_1(n_{11}) \tag{2.104}$$
$$0 = e_1(n_{31}) \tag{2.105}$$
$$0 = H_+ \tag{2.106}$$
$$0 = e_1(E_+) \tag{2.107}$$
$$0 = e_1(H_+) \tag{2.108}$$

while the consistency equations are automatically satisfied. In particular it is important to observe that when applying the evolution equations (2.72), (2.73) and (2.74) with the above derived initial conditions we obtain that during all the time evolution:

$$a \equiv n_{11} \equiv n_{31} \equiv 0\,. \tag{2.109}$$

Returning back to the constraint equation (2.81) we see in particular that we stay in a “piecewise”silent model along all the evolution:

$$H_+ \equiv 0\,. \tag{2.110}$$

Vertex

In this paragraph I will discuss the analytic solutions of the evolution equation in the vertex of the configuration. As already noticed the vertex is a locally spherically symmetric point, so that the result $n_{11} \equiv 0$ and $H_+ \equiv 0$ of the preceding paragraph are still valid and we now also have that $E_+ = 0$ on the time symmetric hyper-surface considered as initial condition (remember the connection between the gravito-electric Weyl tensor and the curvature). Moreover thanks to the evolution equation (2.75) we obtain:

$$E_+ \equiv 0\,, \tag{2.111}$$

showing that in this point the full Riemann tensor vanishes. Therefore the evolution equation for the shear becomes

$$e_0(\sigma_+) = -\frac{1}{3}(2\theta - \sigma_+)\sigma_+ \tag{2.112}$$

and accounting the initial condition $\sigma_+ = 0$ we obtain:

$$\sigma_+ \equiv 0\,. \tag{2.113}$$

Thus in this point the only dynamical quantity is the rate of expansion θ whose evolution is given by the Raychaudhuri equation that now reads as

$$\dot{\theta} = -\frac{1}{3}\theta^2 , \tag{2.114}$$

which can be integrated giving:

$$\theta(t) = \frac{3}{t} , \tag{2.115}$$

or

$$\theta \equiv 0 , \tag{2.116}$$

where only the latter is physically meaningful since it satisfies the initial condition $\theta = 0$. Introducing the relation between the rate of expansion and the scale factor $\theta = 3\frac{\dot{a}}{a}$ we obtain the time dependence for the latter:

$$a(t) = const. \tag{2.117}$$

meaning that we are in a locally minkowski space.

2.6 Summary

Applying the method outlined in this chapter we have obtained an interesting physical picture of our cosmological configuration:

- Starting from the parametric solution of the dynamical equations in a locally rotational symmetric curve we have introduced the length of an edge $l(t)$; and from this quantity we have considered a Hubble function and a deceleration parameter defined in the same formal way as in the previously considered Friedmann metric just substituting the scale factor with the length of the edge:

  $$H_l = \frac{\dot{l}}{l} , \qquad q_l = -\frac{\ddot{l}l}{\dot{l}^2} . \tag{2.118}$$

  Following their time dependence we have shown that under the assumptions of this class of models an accelerated expansion of the Universe in vacuum is possible (the deceleration parameter can be negative). This suggests that the violation of the energy condition is not necessarily required by the astrophysical observations.

- We have proved that the reflection symmetries acting on the face of the cells impose the observer-independent condition $E_{ab}H^{ab} = 0$ on them.

- Exploiting the reduction of the Weyl tensor on a reflection symmetry surface, we have also shown that there is no gravitational radiation crossing the cell boundaries according to Bel's first criterion [63]. In fact the perpendicular component of the super-Poynting vector $P_a = \varepsilon_{abc} E^b{}_d H^{cd}$ vanishes implying that the system has standing gravitational waves with nodes placed on these surfaces.

The role of the symmetries and of their time conservation in deriving all the these results should now be clear: they allow us to eliminate many degrees of freedom of the system. In fact this procedure of breaking the homogeneity and isotropy symmetries of the Friedmann model in a controlled way is a standard procedure in dealing with new cosmological solutions of the Einstein equations. In particular we have considered here discrete symmetries since our space-time does not admit any Killing vector field, while instead the standard literature is based on continuous ones and before our study it was possible to treat these models only numerically. The real physical Universe is of course less symmetric than the ones used to model it; anyway it is fundamental to note that in our model the local inhomogeneities imply an accelerated expansion of the whole configuration suggesting that the dark energy matter content should be considered as an interpretative and not observational aspect in the modeling of the Universe which follows from a naive choice of the underlying metric which completely neglects the presence of the astronomical structures. In a step forward in this program analogue stronger results should be re-derived considering even weaker configurations without any symmetry at all (for example considering different values for the masses of the configurations).

3. On the role of the Shan-Chen equation of state in the cosmological modeling

I will now adopt a completely different approach to cosmology than the one presented in the previous chapter: I will focus my attention on the matter content of the Universe (r.h.s. of the Einstein field equations) and not on the geometric side (l.h.s. of the Einstein equations). Particularly in this original class of cosmological models I will introduce here there are two fundamental hypotheses: the geometric description of the Universe is provided by the Friedmann metric (1.8) meaning that, contrary to the discrete inhomogeneous cosmological model, we assume that the small scale structures play a negligible role in the large scale expansion of the Universe; on the other hand the matter content is accounted by the Shan-Chen equation of state. More broadly speaking, in this direction of research the stress energy-tensor (2.7) can contain also anisotropic pressure terms or heat fluxes that instead are neglected in the simpler Friedmann models. Another possibility is to maintain the hypothesis that the cosmic fluid can be fully characterized by its energy and pressure and to propose a non-polytropic modified equation of state. Our attempt falls inside this line of research. One famous example discussed in literature is the Chaplygin and anti-Chaplygin gas whose equations of state are respectively

$$p = \pm \frac{A}{\rho^{\alpha}}, \tag{3.1}$$

with A and α numerical constants constrained using the observational relations (cosmic microwave background peaks, type Ia distant supernovae, baryonic acoustic oscillations,...). This expression allows analytical results many times, while in our case we will be forced to employ numerical methods.

In other words I will postulate that the accelerated expansion of the Universe is not a geometric effect or a consequence of the presence of a cosmological constant term inside the Einstein equations, or of a vacuum energy but that can instead be accounted for in terms of a nonideal fluid with asymptotic freedom. In this way I can provide a possible microscopic interpretation of the dark energy. To realize this goal it is consequently fundamental to understand under which assumptions the Shan-Chen equation of state has been derived and that when we plug this equation of state inside the Einstein equations it is possible to satisfy the observational tests currently available in literature both in the contexts of late time cosmology and inflationary epoch, as explained in the two next sections. As for the previous chapter the reader can find all the technicalities of the full derivation in the two attached papers **V** and **VII** and references therein.

3.1 The Shan-Chen equation of state: a review

The lattice Boltzmann equation method can be systematically employed to evolve fluids which exhibit phase transition where analytical methods fail [64–66]. The basic starting point is that the gaussian function e^{-x^2} is the functional generator of the Hermite polynomials. This mathematical trick can then be applied to decompose the classical Maxwell-Boltzmann velocity distribution in terms of a series of Hermite polynomials. In a step forward in the research in statistical mechanics, the lattice Boltzmann algorithm has been extended to the description of relativistic systems replacing the maxwellian function with the Maxwell-Jüttner distribution moving from the Euclidean to the Minkowskian space. Recent literature contains attempts to extend this treatment also to general relativity. Particular attention has been devoted to the case of a fluid placed in a flat but expanding background, whose hydrodynamic evolution we want to follow.

An interesting physical application of the manifestly relativistic lattice Boltzmann algorithm has been discussed with respect to the quark-gluon plasma which can be produced for example in the collisions of heavy ions. For the modeling of this phenomenon the chosen background metric is the Milne Universe because it describes an expanding spacetime [67].

Another remarkable application in this context is the derivation of the Shan-Chen equation of state [41], which exhibits a phase transition, that I will briefly review in this section and that I will use for the construction of an original family of cosmological models in the next.

It is in fact well known that in the Boltzmann equation the details of microscopic interactions between the fluid constituents are set inside a collision integral [68]. The role of statistical mechanics, which is a branch of theoretical physics, is in fact to provide methods for the full derivation of all the macroscopic properties of a thermodynamic system (pressure, temperature, volume and relations between them) starting from the laws of the molecular dynamics (velocity of the molecules inside the fluids). A typical example is the connection between the temperature of a gas and the average speed of the particles constituting it. Therefore if we derive an equation of state, relating the energy of the fluid to its pressure, employing the methods of statistical mechanics we know that the equation of state is a macroscopic expression of the underlying microscopic physics: it must necessarily reflect the nature of the interactions between the particles of the fluid. Therefore we can expect that statistical mechanics can also enlighten the peculiar form of the equation of state of the dark energy.

For the derivation of the Shan-Chen equation of state we start from the pair-potential

$$V(\mathbf{x},\mathbf{x}') = \psi(\mathbf{x})G(\mathbf{x}-\mathbf{x}')\psi(\mathbf{x}'), \tag{3.2}$$

where $\mathbf{x}$ and $\mathbf{x}'$ denote the position of two points along the lattice, $G(\mathbf{x}-\mathbf{x}')$ is the Green function which quantifies the strength of the interaction and $\psi(\mathbf{x})$ will become later on a functional of the density energy ρ. Consequently when I will move to a cosmological application of the Shan-Chen equation of state we can interpret the quantity ψ as a chameleon scalar field, i.e. a particle which exhibits a non-linear self-interaction and whose mass is consequently a function of the environment inside which the particle lives and thus depends on the presence or absence of other particles and fields. The existence of a field with such peculiar characteristics has been postulated to provide a possible candidate for the dark energy and dark matter contents of our Universe whose direct detection is still a subject of investigation [69].

If we are interested in the description of a system in which only nearest-neighbor interactions appear, we can approximate the Green function with a number setting $G(\mathbf{x}-\mathbf{x}') = 0$ if the particles are at distance bigger than the lattice scale length and $G(\mathbf{x}-\mathbf{x}') = G$ otherwise. In this latter case we can derive the force between two particles from the underlying potential (3.2) employing a Taylor expansion:

$$F(\mathbf{x}) \simeq -G\psi(\mathbf{x})\nabla\psi(\mathbf{x}). \tag{3.3}$$

From this expression of the inter-particle interaction it is possible to observe that the equation of state we are looking for is given by a linear part typical of an ideal gas plus an excess pressure term:

$$p = w\left(\rho + \frac{G}{2}\psi^2\right), \qquad \psi = 1 - e^{-\alpha\rho}, \tag{3.4}$$

w, $G < 0$ and α, the latter one having the dimension of the inverse of an energy density in this formulation, being free parameters of the model. Imposing the first and second derivatives of the pressure with respect to the energy equal to zero

$$\frac{\partial p}{\partial \rho} = w(1+G\alpha\psi(1-\psi)) = 0 \tag{3.5}$$
$$\frac{\partial^2 p}{\partial \rho^2} = wG\alpha^2(\psi-1)(2\psi-1) = 0\,,$$

Shan-Chen observed that the system they were describing admits a phase transition with critical energy

$$\rho_{\mathrm{crit}} = \frac{\ln 2}{\alpha}\,, \tag{3.6}$$

and $G\alpha = -4$. Considering this equation of state as an isotherm of a pressure-volume-temperature thermodynamic system and applying the Maxwell equal-area construction, they interpreted the phase transition as between liquid and vapor. It is important to note that the above specified expression for the field ψ (3.4) is fundamental to have such phase transition. Our original intuition consists in coupling the equation of state derived by Shan and Chen to the Friedmann and the energy conservation equations (1.9)-(1.10) to show that in the cosmological context we can naturally evolve from an initially radiation dominated Universe to a dark energy dominated one exploiting only one fluid during the time evolution carefully selecting the free parameters of the model; this has been explicitly proved in paper **V**: this was not possible previously when the dark energy has been modeled as a cosmological constant.

Our original cosmological interpretation of the Shan-Chen equation of state is not its only possible physical application. In fact rescaling the energy ρ with respect to a reference energy ρ_* the equation of state (3.4) reduces to

$$p = w\rho_*\mathcal{P}\,, \qquad \mathcal{P} = \xi + \frac{G}{2}(1-e^{-\alpha\xi})^2 \tag{3.7}$$
$$\xi = \frac{\rho}{\rho_*}\,, \tag{3.8}$$

where in particular the constant α has been rescaled with respect to this reference energy to a dimensionless quantity. The sound speed squared is given by

$$c_s^2 = \frac{\partial p}{\partial \rho} = w[1+G\alpha\psi(1-\psi)]\,, \tag{3.9}$$

which is shown in figure (3.1) as a function of ξ for the values $w = \frac{1}{3}$, $G = -5$ and a family of values for α. In particular at high densities the sound speed approaches the constant value $c_s^2 \to w$.

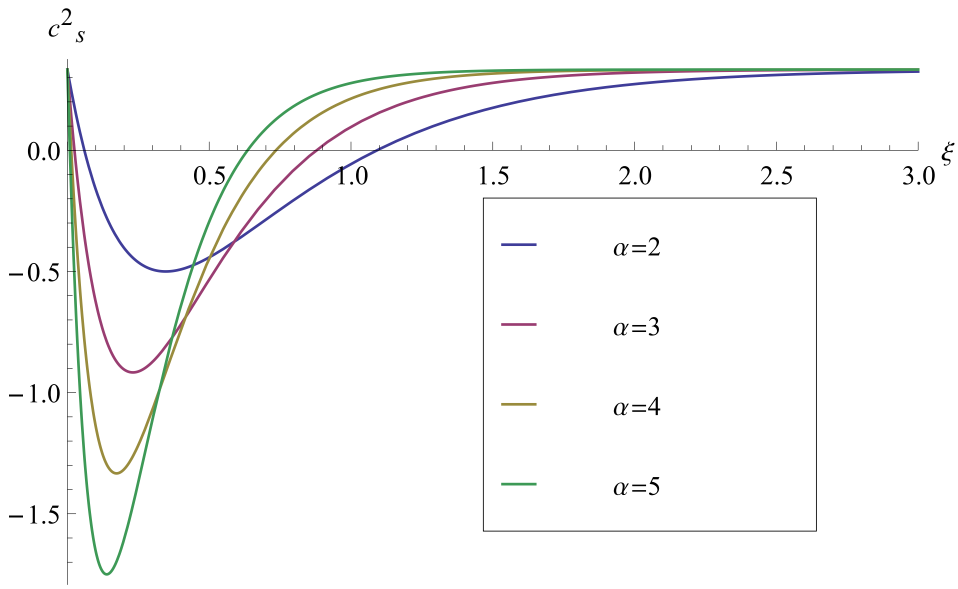

Figure 3.1: Sound speed of a Shan-Chen fluid. The figure shows the speed of sound squared for a Shan-Chen fluid for the values $w = \frac{1}{3}$, $G = -5$ and $\alpha = [2, 3, 4, 5]$ as a function of the dimensionless energy density. The sound speed approaches asymptotically the constant value $c_s^2 \sim 0.33$.

At low and high energy density regimes the Shan-Chen equation of state is approximated by

$$\mathcal{P} \simeq \xi \qquad \text{for} \qquad \xi << 1 \tag{3.10}$$
$$\mathcal{P} \simeq \xi + \frac{G}{2} \qquad \text{for} \qquad \xi >> 1\,,$$

which show that we are considering an asymptotic-free equation of state because it reduces to an ideal gas behavior both at low and high densities. Figure (3.2), which shows the rescaled pressure $\mathcal{P}$ as a function of ξ for a fixed value of the free parameter G and a set of possible values for α, confirms this behavior.

This is its most important physical feature: it replaces hard-core repulsive interaction with a purely attractive one which becomes negligible above a given density threshold; this phenomenon is referred to as "asymptotic freedom". This property of the fluid considered was necessary in the lattice Boltzmann equation theory for numerical reasons, but it can help us also in dealing with the repulsive nature of the cosmological constant, meaning that according to our study the Shan-Chen equation of state should not be interpreted just as a numerical trick but can have also a physically meaningful foundation. Moreover the expansion (3.10) shows that this equation of state can be applied also to the hadronic model of nuclear matter: in this case the

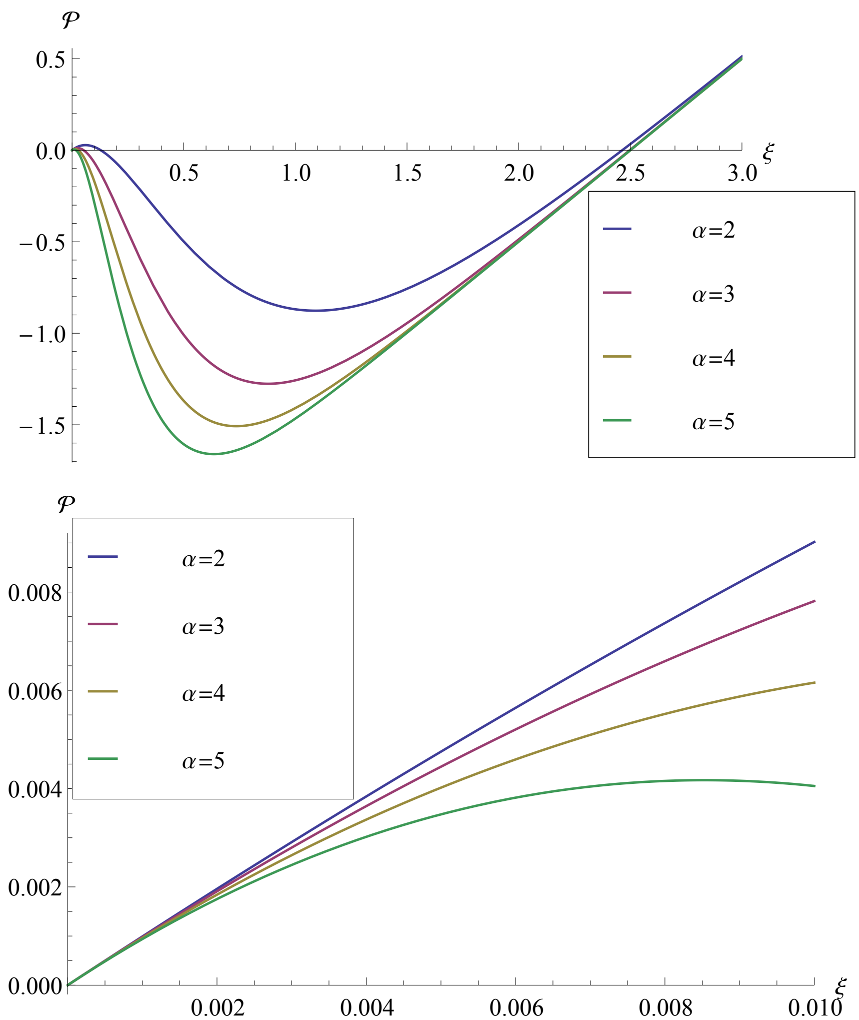

Figure 3.2: Shan-Chen equation of state. This figure shows the rescaled pressure with respect to the dimensionless energy density of the Shan-Chen equation of state for the value $G = -5$ and $\alpha = [2, 3, 4, 5]$ confirming the asymptotic freedom (linear proportionality between pressure and energy density) at both high and low regimes.

constant G should be regarded as the bag constant which accounts for the difference between the energy density of the true vacuum state and the perturbative one. In particular in the parton model of hadrons the authors referred to “bag”as the region of space where the strongly interacting fields/particles are confined. Also in this case when the quarks are close to each other the confining force becomes weaker reaching zero for close confinement and consequently the quarks are free to move (another non-cosmological application of the idea of asymptotic freedom underlying the derivation of the Shan-Chen equation of state) [70].

3.2 Observational tests

In the previous section of the present chapter I have reviewed the derivation of the nonideal equation of state of Shan-Chen with asymptotic freedom focusing in particular on the form of the underlying potential and on the fact that it has been proved that it can be used to simulate a liquid-vapor phase transition, phase transition that we expect to appear also when we couple the equation of state (3.4) to the Einstein equations (1.5). In this section I will describe how to prove this statement rigorously showing how we can connect the observational data collected by the space missions to the quantities entering our theoretical formulation. The demonstration can be split up into two parts having the aim of describing two completely different stages of the evolution of our Universe: we can apply the Shan-Chen equation of state to late time cosmology and/or to the inflationary era of the Universe. In both cases the entry and exit mechanisms are crucial for the improvement of the current models.

3.2.1 Late time cosmology

When we apply the Shan-Chen equation of state to the cosmological modeling we start solving the Friedmann and the energy conservation equations once we have plugged in this specific equation of state. For the sake of completeness they are explicitly given by

$$\frac{d\xi}{d\tau} = -\frac{3}{x}\frac{dx}{d\tau}\left[(1+w)\xi + \frac{w}{2}G\left(1-e^{-\alpha\xi}\right)^2\right], \tag{3.11}$$

$$\frac{dx}{d\tau} = \pm\sqrt{\Omega_{k,0}+x^2\xi}, \qquad x = \frac{a(t)}{a_0} \tag{3.12}$$

where $\tau = H_0 t$, H_0 being the Hubble constant, is a dimensionless time and x is given by the ratio of the scale factor at an arbitrary time $a(t)$ and at today a_0. To find a numerical solution we must then implement the initial conditions $\xi(\tau_0) = \Omega_0$, where Ω_0 is the amount of dark energy today, and $x(\tau_0) = 1$.

We can then plot both the pressure, the energy density and the "effective"parameter of the equation of state $w_{\text{eff}} = \frac{\mathcal{P}}{\xi}$ versus the time. The first important feature of the model is that the pressure as a function of the energy changes its sign during the time evolution and stays negative for a long time interval which includes the present day: the physical meaning is that the fluid permeating the Universe naturally and smoothly evolves from an ordinary matter content to an energy component whose equation of state is compatible with dark energy. To have a more realistic picture of the Universe we can consider two non-interacting fluids permeating the Universe which are separately conserved, namely we add a dust component to the matter content in terms of a pressure-less fluid. The first basic property of the model, i.e. the evolution from ordinary to dark energy of the Shan-Chen fluid, remains valid also in this extension for another numerical choice of the free parameters entering the equation of state.

From a stricter observational point of view we can compare our model to the supernova type Ia data, after proving that the Shan-Chen fluid can mimic dark energy. This data set is the first test that a cosmological model must pass and it is based on the fact that the supernovae are standard candles since they exhibit fixed and known luminosity [1; 2]. The test is usually done by plotting the distance modulus with respect to the redshift; the expression for the distance modulus is [71]

$$\mu = 5\log\frac{d_L}{\text{Mpc}} + 25\,, \tag{3.13}$$

where I have inserted the luminosity distance introducing the redshift z

$$d_L = (1+z)\frac{d_H}{\sqrt{|\Omega_{k,0}|}}\Sigma_k\left(\sqrt{|\Omega_{k,0}|}\frac{d_c}{d_H}\right)\,. \tag{3.14}$$

$d_H = 1/H_0$ is called "Hubble distance", while d_c is the comoving distance and $\Omega_{k,0}$ is the curvature parameter, where a subscript here and in what follow states that the quantity is considered at the present time; moreover $\Sigma_k(f) = [\sin(f), f, \sinh(f)]$ for a closed, flat and hyperbolic universe respectively. To obtain the optical properties of the Universe we must integrate the radial null geodesic equation in the Friedmann metric $dr/dt = -1/a$ which gives the path of a light signal moving from a galaxy to us. From this solution we evaluate then the comoving distance $d_c = a_0 r$.

In the case of the previously considered ΛCDM model that we want to compare with, the expression for the comoving distance can be found in [71]:

$$\begin{aligned} d_c &= d_H\int_0^z \frac{dz'}{E(z')}\,, \\ E &\equiv \frac{H}{H_0} = \sqrt{\Omega_{\Lambda,0} + \Omega_{k,0}(1+z)^2 + \Omega_{m,0}(1+z)^3}\,. \end{aligned} \tag{3.15}$$

In the formulas above H denotes the Hubble function, $\Omega_{\Lambda,0}$ and $\Omega_{m,0}$ the mass parameters today, where the first accounts for the presence of dark energy while the latter quantifies the dust; the presence of photons is instead assumed to be negligible. In our original cosmological model, instead, we must integrate simultaneously the Friedmann and the energy conservation equations (3.11)-(3.12) and

$$\frac{d}{d\tau}\left(\frac{d_c}{d_H}\right) = -\frac{1}{x}\,. \tag{3.16}$$

The initial condition for the radial coordinate is $r(\tau_0) = 0$ which means $d_c(\tau_0) = 0$ for the comoving distance. Appropriately choosing the numerical values for w, α and G, we have obtained that our curve for the distance modulus can be exactly superimposed to the ΛCDM one.

As a second test we can plot the Hubble function versus the redshift for the same choice of parameters as in the previous case: also in this case our model is in agreement with the ΛCDM curve and consequently with the observational data. This data set has been experimentally obtained exploiting the aging of passively evolving galaxies (in which there is a negligible star formation activity [72]) [73] and baryon acoustic oscillations [74]. In particular the latter ones are density fluctuations of the baryonic matter and are used as a standard ruler in cosmology; they also give information about the large scale structure formation in the Universe.

Finally we have shown the dependence of the deceleration parameter on the redshift. As I mentioned in the second chapter of this thesis when I discussed the inhomogeneous discrete cosmological model, the deceleration parameter is defined through the scale factor and its first and second time derivatives by

$$q = -\frac{\ddot{a}a}{\dot{a}^2}\,, \tag{3.17}$$

where the time dependence of the scale factor is obtained via the Einstein equations (1.5); its evolution consequently depends on the equation of state we are dealing with. Our formalism allows us also to evaluate the present day value of the deceleration parameter and the age of the Universe which are compatible with the observed ones.

Role of instabilities in the hydrodynamic fluid evolution

Even if the theoretical model is compatible with the observational data, as I discussed so far in this section, we are required to do a further check. In fact it is important to show explicitly that the model is stable under small initial perturbations, otherwise the mathematical formulation can not be accepted as a description of the physical world. This is a self-consistency exam for the model itself rather than an experimental test. This is because the hydrodynamic approach we followed in the modeling of the matter content of the Universe would not hold anymore if it contains growing inhomogeneities, especially under the effect of gravity. Moreover accounting for what I explained in the first section of this chapter about the derivation and the basic features of the Shan-Chen equation of state, we know that the hard-core repulsive effects, which prevent this kind of instabilities in liquids, have now been replaced with soft-core attractive ones with asymptotic freedom: the attractive interactions can consequently facilitate the growth of the instabilities causing a density blowup and a destruction of the model. In the Shan-Chen model the interparticle interaction reaches a saturation value above which the interaction strength vanishes; in this specific case the attractive force goes to zero at short distance, or in other words at high densities. For these reasons a stability analysis of our cosmological model was necessary. The study shows that the density contrast defined as the ratio between the perturbation of the energy density and the energy density itself is bounded for a long time interval for the same choice of the free parameters in the equation of state needed for the data fit of the modulus distance and the Hubble function versus the redshift, while it explodes only approaching the initial singularity. In this latter case it is possible to assume without loss of generality that our model is too naive for a complete and correct description of the big bang since we are here interested only in a physical application to late time cosmology. We also noticed that the presence of a pressure-less fluid component plays a stabilizing role, which was not guaranteed at priori (for example the Einstein static Universe in which the dark energy is modeled as a cosmological constant term rather than a Shan-Chen fluid is unstable [38]).

3.2.2 Inflationary era

So far I have discussed the experimental tests to compare the Shan-Chen cosmological model to observations in late time cosmology, however we can expect that the phase transition exhibited by the fluid during the time evolution in that context can be employed also for the description of the inflationary era of the Universe, in particular providing a natural exit mechanism from this stage of the evolution of the Universe. This is because in the standard model of cosmology the dust density dilutes as the volume $a^3(t)$, the photons density as $a^4(t)$, while the cosmological constant does not dilute at all: when this latter term will become dominant it is impossible to escape from that stage of the evolution. I recall that in the inflationary epoch the Universe undergoes a phase of exponential expansion which is needed to solve the

- flatness paradox: at the present time we observe an almost flat Universe, which requires the Universe to be flat also at the initial moment, if the Universe is matter or radiation dominated.

- horizon paradox: why do we observe an almost constant temperature even in spatial points of the Universe which could not have been in causal contact with each other?

and which has also been proposed in reference to the

- formation of the large scale structure

- origin of the anisotropies in the cosmic microwave background radiation

in the framework of the standard Big Bang model [71; 75].

The simplest possible theoretical model for this phenomenon assumes that the inflationary epoch is dominated by a quintessence fluid which can be described in terms of a real scalar field $\phi = \phi(t)$, which depends only on the time for the assumptions of homogeneity and isotropy. The lagrangian is given by

$$\mathcal{L} = \frac{\dot{\phi}^2}{2} - V\,, \tag{3.18}$$

$V = V(\phi)$ being the potential of the field causing the expansion of the Universe. The canonical equations give us the relations between the energy and pressure of the fluid and the field:

$$\rho = \frac{\dot{\phi}^2}{2} + V\,, \qquad p = \frac{\dot{\phi}^2}{2} - V\,, \tag{3.19}$$

or equivalently

$$V = \frac{1}{2}(\rho - p)\,, \qquad \dot{\phi}^2 = \rho + p\,, \tag{3.20}$$

where in our case we will postulate that the pressure is given by the energy by the equation of state of Shan-Chen (3.4). We can use this formalism also to connect the Hubble function to the field obtaining

$$H^2 = \frac{1}{3}\left(\frac{\dot{\phi}^2}{2} + V\right)\,. \tag{3.21}$$

This couple of equations shows that the potential driving the inflationary era is fixed when an equation of state is assumed and when the background metric is decided (an extension of the quintessence formalism to a background with small inhomogeneities has been considered in the context of the k-essence [8]).

During the inflationary era it is assumed that the kinetic part of the lagrangian (3.18) is negligible respect to its potential part in order to have an exponential growth of the Universe. To define mathematically this condition we start introducing the so called slow-roll parameters [71; 75]

$$\varepsilon = \frac{1}{16\pi}\left(\frac{V'}{V}\right)^2 \tag{3.22}$$

$$\eta = \frac{1}{8\pi}\frac{V''}{V} \tag{3.23}$$

$$\Xi = \frac{1}{64\pi^2}\frac{V'''V'}{V^2} \tag{3.24}$$

where here a prime denotes derivative with respect to the scalar field ϕ. The inflationary era takes place when the above defined slow-roll parameters are small:

$$\varepsilon \ll 1\,, \qquad |\eta| \ll 1\,, \qquad |\Xi| \ll 1\,, \tag{3.25}$$

which in particular implies that during this stage the equation of state describing the fluid permeating the Universe can be approximated by $p \sim -\rho$. In the attached paper **VII**, after deriving the relations between the slow-roll parameters and the energy density of the model we plot the slow-roll parameters as a function of the scalar field ϕ showing that they respect the conditions required to describe an inflationary scenario for an appropriate choice of the free parameters of the model. After this initial check we deepen the connection between the Shan-Chen class of cosmological models and the inflationary paradigm comparing our formalism with the experimental data characterizing this era of the Universe. In particular one can consider the ratio of scalar to tensor perturbations, the scalar spectral index and its running which can be expressed in terms of the above defined slow roll parameters following standard literature [75]:

$$r = 16\varepsilon \tag{3.26}$$

$$n_s = 1-6\varepsilon+2\eta \tag{3.27}$$

$$\alpha_s = \frac{dn_s}{d\ln k} = 16\varepsilon\eta - 24\varepsilon^2 - 2\Xi\,, \tag{3.28}$$

and consequently as functions of the energy density of the Shan-Chen fluid.

The data provided by the Planck mission analyzing the structure of the cosmic microwave background can be combined with the ones of the large angle polarization of the Wilkinson Microwave Anisotropy Probe (WMAP) to constrain observationally these parameters [76; 77]:

$$r < 0.11\,, \qquad n_s = 0.9603 \pm 0.0073\,, \qquad \alpha_s = -0.0134 \pm 0.0090\,, \tag{3.29}$$

where in particular the last one suggests that there are no indications for a running of the spectral index. The spectral index instead comes from the assumption that the power spectrum can be written as a power function whose exponent is $n_s - 1$; its numerical value is fundamental to discriminate between different proposals of the inflationary mechanism and it is important to observe that its current experimental value is not compatible with 1 which characterizes gaussian (or adiabatic) initial perturbations [75]. According to our model instead this is not a problem at all since we can reproduce it within our model.

3.2.3 A word of warning: the BICEP2 experiment

In the previous section I have chosen to use the Planck data, in particular for the ratio of scalar to tensorial perturbations, for a test of our original cosmological model. However a recent result by the BICEP2 collaboration suggests a different observational value for this quantity [78] claiming the detection of the B-mode. Following the analogy between gravity and electromagnetism discussed in the second chapter of this thesis, the two polarizations which could be present in the cosmic microwave background are called E-mode and B-mode, which are respectively the curl-free and grad-free components. The latter can not be produced only by the physics of the plasma (Thomson scattering) and are interpreted as a signal coming from the cosmic inflation quantifying the presence of possible primordial gravitational waves. The perturbations in the CMB can then be classified in scalar, vectorial and tensorial with different consequences on the temperature anisotropies. Looking at the temperature pattern BICEP2 obtains the following quantitative result

$$r = 0.20^{+0.07}_{-0.05}\,, \tag{3.30}$$

clearly incompatible with Planck's one

$$r < 0.11\,. \tag{3.31}$$

At the time of the first submission of paper **VII** in January 2014 only Planck results were available justifying my choice of the experimental value to compare to our Shan-Chen cosmological model.

3.3 Summary

To summarize, in this chapter I have proposed a possible physical interpretation of the nature of the dark energy in terms of the nonideal equation of state with asymptotic freedom of Shan-Chen, once I have assumed that instead the geometric side of the Universe can be well enough described by the Friedmann metric. In this original approach the presence of the cosmological constant and of the vacuum energy is not needed for accounting for many observational data both in the late time cosmology and in the inflationary era. In particular the improvements respect to the previous approaches are:

- We have understood both the numerical formulation of the equation of state describing the fluid permeating the Universe and its theoretical meaning; for example this same equation of state can also be applied in nuclear physics to the extended model for hadrons. Considering the form of the potential underlying the derivation of this equation of state we can provide its microscopic interpretation. Thus this is not an *ad hoc* formula just derived from the fit of astronomical observations.

- Our model of the Universe exhibits a radiation dominated phase at the beginning in accordance with the hot big bang model and naturally evolves to a dark energy dominated one and after we can escape from this phase; when the dark energy is modeled as a cosmological constant instead there are no mechanisms available for such phase transition invoking only one fluid.

- We can fit the distant type Ia supernova data with an appropriate choice of the free parameters appearing in the model without the need of the vacuum energy.

- Our model is also stable under small initial perturbations for the same choice of parameters that allows us to satisfy the point above; not all the cosmological models are stable, the Einstein static Universe formulated on the same Friedmann line element like ours and involving a cosmological constant, a energy density and a pressure is not; our model is thus a step forward also in this direction.

- We have shown that the repulsive action of the cosmological constant can be handled with an equation of state describing an attractive effect with asymptotic freedom.

- We can apply our formalism also to describe the inflationary phase of the Universe using the quintessence model: we have an exit mechanism.

- In this latter case we can also show the slow-roll conditions to be satisfied and that we can correctly reproduce the numerical values for the scalar to tensor perturbations, the spectral index and its running.

Connecting the first point above with other ones in the list we can say that we have suggested a possible microscopic interpretation of the dark energy, interpretation that can also be formulated in terms of a chameleon field. I refer to the attached papers **V** and **VII** for the details of the derivation of all the statements claimed here.

4. Applications of the Poynting-Robertson effect in general relativity

4.1 Characterizing a spacetime through the motion of particles

In the previous chapters of this thesis I have tried to characterize a given spacetime solving the Einstein equations for the metric after fixing an appropriate matter content of the Universe. A complementary point of view in the search for the correct modeling of the spacetime consists in the analysis of the orbit of a test particle moving inside it. The first step in this direction is the analysis of the geodesic motion, a purely geometric effect which depends on the metric and its first derivatives (the Christoffel symbols): setting the acceleration equal to zero the mass of the body drops out and we distinguish only between the cases of massive or massless test particles. The gravitational effects on a particle motion are therefore encoded inside the line element which is coupled to the matter distribution via the Einstein equations. Thus according to general relativity the test particle interacts with the matter through this process; reversing this point of view we can reconstruct some properties of the metric and consequently of the matter distribution generating it following the motion of a particle. We can refer to this phenomenon as a kind of indirect interaction between test particle and matter.

To obtain a more accurate picture of the matter content in the region crossed by our test particle we should also consider its direct interaction with the fluid. In fact we can suppose that during its motion the body collides with the fluid elements exchanging energy with them. Assuming that the energy is transferred from the object to the matter content, the fluid acts as a viscous medium leading to the presence of a dissipative force term inside the equations of motion. In classical mechanics the expression for such a friction effect is modeled by the Stokes' law, which can be extended to general relativity using the Poynting-Robertson formula as I proved in section (1.5) of this thesis. Moreover the general relativistic formalism that I will follow in this chapter can describe both the case of motion inside a massive or massless (photons) fluids that instead was not possible in the classical treatment.

We can also distinguish between two more approaches to this problem. In the first one a background is fixed and described by the metric $g_{\mu\nu}$ and we superpose to it a test fluid whose stress-energy tensor is a function of this background:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}\,, \tag{4.1}$$

ρ and p being its pressure and density energy connected to each other by a specific equation of state, while u_μ denotes the fluid four-velocity. This expression shows that the formalism is enough general to describe both the cases of massive or massless media. The quantity (4.1) is the stress-energy tensor which enters the Poynting-Robertson formula that I recall here for the sake of completeness

$$f_{(\mathrm{fric})}(U)^\alpha = -\sigma P(U)^\alpha{}_\beta T^{\beta\mu} U_\mu\,. \tag{4.2}$$

In the formula above the sign minus accounts for the dissipative nature of the force, the cross section of the process σ contains the details of the interactions between test particle and fluid, U_μ is the body four-velocity and $P(U)^\alpha{}_\beta$ projects orthogonally to this four-velocity. In a first attempt we can consider the weak field approximation assuming that the fluid field does not perturb the spacetime. The derivation of the expression of the stress-energy tensor for a fluid placed in a curved background can also be seen as an interesting application of the kinetic theory and of the Boltzmann equation in general relativity [79]. We have followed this procedure in particular in paper **I** studying the motion of a massive object around a Schwarzschild black hole to which we superpose a test photon field comparing and contrasting the geodesic and scattered motions. Our original analysis was necessary for the clarification of the role of friction and of radiative pressure in the Poynting-Robertson formula [80; 81]. The literature is rich of examples of this kind of applications: from the motion of meteors to the modeling of an accretion disk around a star due to this phenomenon [82–84].

The first question which arises at this point is: how good is the approximation of a test field for the photon gas? Can the motion of the test particle deviate also qualitatively and not only quantitatively when we leave this regime? To answer this question we must solve the Einstein equation (1.5) to describe a system of a black hole surrounded by a photon gas. This analytical solution is not known yet. Thus we have decided to “switch off”the black hole and to consider a spacetime generated by a photon gas in equilibrium. The Tolman metric is the solution of the field equation in this case [85]. Inserting the same stress-energy tensor in both the Einstein equations and the Poynting-Robertson formula (4.2) we can compare and contrast the geodesic and the modified motions in the Tolman spacetime. The geodesic case is what we have called the “indirect”interaction between test particle and fluid, while the scattered motion accounts also for the “direct”interactions. The analysis allows us to separate the role of the curvature of the spacetime to the true friction effects inside this phenomenon. This is the subject of paper **II**. The analysis can also be compared to a similar one in the Vaydia metric which instead mimics a photon field not in equilibrium like a radiating star [86].

In paper **III** we have extended the Poynting-Robertson formula, which was initially proposed for the motion inside a massless fluid, to the motion inside a massive fluid. In fact there can be gas clouds in the Universe which distort the orbits of particles crossing them acting as a gravitational lensing. The goal of this specific analysis is the evaluation of the deflection angle which quantifies the difference between the case of a particle moving in vacuum and a particle traveling this region. Also in this case the geodesic and non-geodesic cases can be compared and we have proved that when we add a viscous term to the equations of motion the body can not escape the gas cloud once it has entered it because it dissipates all its initial energy and it is consequently condemned to fall at the center of the configuration. The friction effects are thus much more important than the space curvature ones. To be specific, for this derivation we have considered the Pant-Sah metric as a background.

In the cosmological modeling our formalism can be applied to the quantification of the peculiar velocities, which is the motion of a particle with respect to an observer in a rest frame. In paper **IV** we have considered a Friedmann expanding Universe whose matter source driving its evolution influences also the friction force (4.2). In this case when we add the Poynting-Robertson formula to the geodesic equation of motion we can therefore speak of direct interaction between the test particle and the fluid. In the case of a closed Universe the asymptotic behavior of a particle moving geodesically and geodesically plus friction is completely different: in the former the object accelerates to the speed of light, while in the latter it dissipates all its initial energy approaching zero velocity. As for the previous analysis of the gravitational lensing, the friction term plays a fundamental role in driving the long-term result and should not be neglected in a realistic modeling of the motion.

In this chapter I will review all these astrophysical applications of the Poynting-Robertson formula leaving the details of the complete derivation to the attached papers at the end of the thesis and [46; 47; 80–84] and references therein. I would like to stress that in this analysis we have assumed that all the noise effects are negligible even if they should be accounted by the relaxation-dissipation theorem: this opens a line for future possible investigations which will provide a more complete picture of this phenomenon. Our formalism is also enough general to be applied to non astrophysical systems in which friction effects anyway occur.

4.2 Motion inside a photon gas in the Schwarzschild metric

In this section I am interested in describing the motion of a massive test particle around a Schwarzschild black hole to which I have superimposed a test photon gas (the strength of the field is sufficiently weak not to perturb the fixed background metric). The first task is consequently to derive the stress-energy tensor describing such a gas in a curved spacetime and then insert it inside the Poynting-Robertson formula (4.2). This can be seen as an application of the relativistic kinetic theory in a generic curved spacetime.

We can follow for example [87]. I start from a generic line element

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta\,. \tag{4.3}$$

Assuming then that all the particles of the gas are indistinguishable with the same mass m, which can be set equal to zero at the end of the computations without problems, we have the following constraint in the momentum space:

$$p_\mu p^\mu = -m^2\,, \tag{4.4}$$

where p_μ is the four-momentum of the particles. Introducing the phase space, the gas is characterized by the distribution function $f(x^\mu, p^\mu)$, x^μ denoting the coordinates in the real space, whose evolution can be followed along a line parameterized by τ :

$$\frac{d}{d\tau} f(x^\mu(\tau), p^\mu(\tau)) = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\tau} + \frac{\partial f}{\partial p^\mu}\frac{dp^\mu}{d\tau}. \tag{4.5}$$

This relation reduces to the homogeneous relativistic Boltzmann equation

$$L(f) = \left(p^\mu \frac{\partial}{\partial x^\mu} - \Gamma^\mu{}_{\alpha\beta}\, p^\alpha p^\beta \frac{\partial}{\partial p^\mu} \right) f = 0, \tag{4.6}$$

where L is referred to in the literature as the Liouville operator, in the case that the gas particles interact only gravitationally moving along geodesics

$$\frac{dp^\mu}{d\tau} = -\Gamma^\mu{}_{\alpha\beta}\, p^\alpha p^\beta\,. \tag{4.7}$$

Generalizing the previous result accounting for the collisions between the gas molecules, assumed to be elastic (the total four-momentum is conserved) and binary, we obtain

$$L(f)(x^\mu, p^\mu) = \int \mathcal{C}\, d^4\tilde{p} d^4\tilde{q} d^4 q, \tag{4.8}$$

where I have introduced

$$\mathcal{C} \equiv \mathcal{C}(f; x^\mu, \tilde{p}^\mu, \tilde{q}^\mu, p^\mu, q^\mu) = \omega[f(x^\mu, \tilde{p}^\mu) f(x^\mu, \tilde{q}^\mu) - f(x^\mu, p^\mu) f(x^\mu, q^\mu)], \tag{4.9}$$

where $\omega = \omega(x^\mu; \tilde{p}^\mu, \tilde{q}^\mu, p^\mu, q^\mu)$ represents the phenomenological cross section of the process; here x^μ denotes the point in the physical space where the collision takes place and p^μ, q^μ, $\tilde{p}^\mu$, $\tilde{q}^\mu$ the four-momenta of the colliding particles before and after the collision. Summarizing, the collision integral describes the variation rate of the number of particles in a given region of the phase space [68].

The physical properties of the gas considered are contained in the momenta of the distribution function: the current of particles density, the stress-energy tensor and the current of entropy density:

$$\begin{aligned} N^\mu &= \int f\, 2\,\delta^+(p^2+m^2)\, p^\mu \frac{\sqrt{-g} d^4 p}{(2\pi)^3} = \int f \frac{p^\mu}{|p_t|} \frac{\sqrt{-g} d^3 p}{(2\pi)^3} \qquad (4.10) \\ T^{\mu\nu} &= \int f\, 2\,\delta^+(p^2+m^2)\, p^\mu p^\nu \frac{\sqrt{-g} d^4 p}{(2\pi)^3} = \int f \frac{p^\mu p^\nu}{|p_t|} \frac{\sqrt{-g} d^3 p}{(2\pi)^3} \\ S^\mu &= -k_B \int (f \ln f)\, \frac{p^\mu}{|p_t|} \frac{\sqrt{-g} d^3 p}{(2\pi)^3}, \end{aligned}$$

where g is the determinant of the background metric and k_B the Boltzmann constant. It can be proved by direct inspection that in a generic spacetime the solution of the collisional relativistic Boltzmann equation is given by $f = \alpha e^{\beta \xi_\mu p^\mu}$, ξ_μ being a timelike Killing vector of the background metric, α a normalization constant and $\beta = 1/k_B T$ the inverse temperature. In the specific case of the Schwarzschild spacetime such Killing vector is $\xi_\mu = (\partial_t)_\mu$. For the explicit evaluation of the statistical momenta, the functional generator technique

$$N^\mu = \frac{1}{\beta} \frac{\partial I}{\partial \xi_\mu} \tag{4.11}$$

$$T^{\mu\nu} = \frac{1}{\beta^2} \frac{\partial^2 I}{\partial \xi_\mu \partial \xi_\nu} = \frac{1}{\beta} \frac{\partial N^\mu}{\partial \xi_\nu} \tag{4.12}$$

$$S^\mu = -k_B \int (f \ln f)\, \frac{p^\mu}{|p_t|} \frac{d^3 p}{(2\pi)^3} = -k_B \left[\ln(\alpha) N^\mu + \frac{\partial N^\mu}{\partial \xi_\nu} \xi_\nu \right], \tag{4.13}$$

where I have introduced

$$I = \int f(\xi_\mu p^\mu)\, 2\delta^+(p^2+m^2)\, \sqrt{-g} d^4 p\,, \tag{4.14}$$

is of great help. After some algebraic manipulations [79], the stress-energy tensor for a massless gas in a curved spacetime is reduced to

$$T_{\mu\nu} \,=\, (\rho+p)u_\mu u_\nu + p g_{\mu\nu} \,=\, \frac{C}{3(\beta\xi)^4}[g^{\mu\nu}+4u^\mu u^\nu]\,, \tag{4.15}$$

where $u_\mu = \frac{\xi_\mu}{\xi}$ is a unit vector where $\xi = \sqrt{-\xi_\mu \xi^\mu}$. In equation (4.15) we have reobtained in a more formal way both the expected equation of state for a photon system $p = \frac{1}{3}\rho$, and the Tolman law $\beta \to \beta\xi$ for the temperature. Finally the constant $C = \frac{\pi^2}{15}$ has been determined requiring to be in agreement with the black body theory in a flat spacetime. Direct inspection of the expression (4.15) shows that the conservation law $T^{\mu\nu}{}_{;\mu} = 0$ and the tracefree condition of an radiation field $T^\mu{}_\mu = 0$ hold.

Considering a Schwarzschild spacetime

$$ds^2 \,=\, -N^2 dt^2 + \frac{1}{N^2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\,, \qquad N = N(r) = \sqrt{1-\frac{2M}{r}}\,, \tag{4.16}$$

M denoting the mass of the central black hole, then the explicit expression for the Poynting-Robertson formula (4.2) is reduced to

$$f(U)^\alpha{}_{\rm rad} \,=\, -\frac{m\tilde{A}}{N^4}\gamma^3[\nu^2 u^\alpha + \nu^\alpha]\,, \tag{4.17}$$

after inserting the stress-energy tensor (4.15). In this expression m is the mass of the test particle moving inside the photon gas around the black hole, $\tilde{A}$ a constant of proportionality containing information about the temperature of the photons, the cross section of the process, and other numerical constants, γ is the Lorentz factor and ν^α the spatial part of the test particle four-velocity (1.22).

I recall that the equation of motion are given by the superposition of the geodesic part and of the friction term:

$$mU^\beta \nabla_\beta U^\alpha \,=\, f(U)^\alpha{}_{\rm rad}\,. \tag{4.18}$$

Numerically integrating this set of equations we observed that:

- The motion is still planar as in the pure geodesic case.

- Considering as initial condition the circular geodesic orbit for a pure geodesic case with $r = 4M$ and $\nu_K = \frac{1}{\sqrt{2}}$, here ν_K being the keplerian velocity, the test particle falls inside the black hole crossing its horizon. Thus the circular motion is modified to a spiral one. This behavior can be understood in terms of the physical nature of the Poynting-Robertson formula: it is a friction effect and consequently the test particle is condemned to dissipate its initial energy.

- Our analysis can be compared to a previous one in which the stress-energy tensor for a null dust around a Schwarzschild black hole was written as
$$T^{\mu\nu} = \Phi^2 k^\mu k^\nu \,, \qquad k^\mu k_\mu = 0 \,, \tag{4.19}$$
where k^μ is a vector tangent to a null geodesic, while Φ is a function to be determined imposing the conservation law $T^{\mu\nu}{}_{;\mu} = 0$ in a specific spacetime. In this previous analysis the null dust acts as a wind and the test particle, whose motion is based on the same initial condition as in the point above, reaches an equilibrium point (outside the horizon) where the gravitational attraction of the black hole equates the radiative pressure of the field.

- Our original result based on a statistical modeling of the photon gas was fundamental in separating the two parts, the radiation pressure and the friction term inside the Poynting-Robertson effect.

- According to our analysis the Poynting-Robertson effect can play some role in the formation of an accretion disk in the vicinity of a star or a black hole because the test particle stops its motion closer to the horizon than the innermost stable geodesic circular orbit (classical description of the photons moving along a preferred direction), or crosses it (original approach).

I would like to mention that I started studying the topic presented in this section with my master thesis [88], but it has been completed only during my Ph.d. In the future this same analysis should be extended to a rotating Kerr black hole.

4.3 Metric curvature versus friction effects

In the previous section the photons were considered as a test field which does not perturb the Schwarzschild black hole. However also the strong limit case is physically interesting. To study a configuration in which the presence of the photons models the spacetime, we have eliminated the black hole and on the other hand we have solved the Einstein equation (1.5) for a photon system in gravitational equilibrium as a matter source. Since the gas is assumed to be in equilibrium the metric must be spherically symmetric and we can start from the ansatz

$$ds^2 = -e^{\nu(r)}dt^2 + \left(1 - \frac{2M(r)}{dr}\right)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \,. \qquad (4.20)$$

Its explicit expression can be derived using the Tolman-Oppenheimer-Volkoff formalism once a specific matter content is assumed. The TOV set of equations are given by [89; 90]:

$$\begin{aligned}
\frac{dp(r)}{dr} &= -\frac{1}{r^2}(\rho(r)+p(r))(M(r)+4\pi r^3 p(r))\left(1-\frac{2M(r)}{r}\right)^{-1} \qquad (4.21)\\
\frac{dM(r)}{dr} &= 4\pi\rho(r)r^2\\
\frac{d\nu(r)}{dr} &= -\frac{2}{\rho(r)+p(r)}\frac{dp(r)}{dr}\,.
\end{aligned}$$

In this set of equations $p(r)$ and $\rho(r)$ are the pressure and the energy density of the fluid function only of the radial coordinate and related to each other by an equation of state which reflects the fluid we are considering, while $M(r)$ can be regarded as the mass of the configuration inside a sphere of radius r. In the case under examination we choose $p(r) = \frac{1}{3}\rho(r)$ and the Tolman metric gives the solution we are looking for [85]:

$$ds^2 = -ardt^2 + \frac{1}{a}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \qquad a = \frac{4}{7}\,, \qquad (4.22)$$

and the energy density

$$\rho = \frac{3}{56\pi r^2}\,, \qquad (4.23)$$

is in agreement with the Tolman law for static spacetime [91]. Three features of the line element (4.22) are evident:

- The energy density approaches zero as a function of the radial coordinate only asymptotically: we can not have a bounded photons configuration in equilibrium from which photons can not escape otherwise we will have a black hole.

- The metric exhibits a naked singularity in the point $r = 0$ as it comes from the evaluation of the Kretschmann scalar

$$K = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \sim \frac{1}{r^4}\,. \qquad (4.24)$$

- The metric describes a photon gas configuration in equilibrium and not a radiating star (it is static) and consequently its form is completely different from the Vaidya metric [86]

$$ds^2 = -\left(1-\frac{2M(v)}{r}\right)dv^2+2dvdr+r^2(d\theta^2+\sin^2\theta d\phi^2). \quad (4.25)$$

To quantify the relative importance between spacetime curvature and friction term in the modeling of the Poynting-Robertson effect three different situations for the motion of a massive test particle must be compared:

- We consider a test photon field superimposed to a fixed Minkowski background: only friction effect, no curvature (being the Riemann tensor identically zero).

- We consider a geodesic motion inside the Tolman metric (4.22): only curvature, no friction; we can refer to this case as an indirect interaction between test particle and photon gas through the Einstein field equations.

- We consider a Poynting-Robertson effect in the Tolman metric: both friction and curvature of the space are accounted for; in this case there is a direct interaction between the test particle and the fluid generating the space since there are collisions between them (which microscopically can be regarded as a Thomson scattering).

I would like to mention that the analysis of the motion of a particle in such a metric should not be considered only as a theoretical exercise because the Tolman metric can also describe a static Universe radiation dominated. The characteristics of the motion are the following.

- Only friction, no curvature: the massive test particle moves along a straight line, as in the case of Minkowski metric without the Poynting-Robertson term, but decelerating and reaching a motion endpoint with zero speed. As in the classical mechanics counterpart in which the friction force is given by the Stokes' law, the velocity is exponentially decaying. On the other hand we have observed that in a relativistic context, contrary to the newtonian one, the object crosses a maximum distance independently of its initial speed:

$$d_{\max} = \frac{\pi}{2A}, \qquad A = \frac{4\rho_0\sigma}{3m}, \quad (4.26)$$

  where ρ_0 is the constant energy density of the photon gas, σ the cross section of the process appearing in the Poynting-Robertson formula

(4.2), while m is the mass of the test particle. The quantity A can therefore be interpreted as an effective coupling constant between test particle and field which quantifies the energy absorption and re-emission from the photons. The upper limit stated above opens the question if a similar condition holds also in a curved spacetime in which the energy of the photon gas is not anymore constant but goes to zero asymptotically with r (4.23).

- Only curvature, no friction: considering a geodesic motion in the Tolman metric, the test particle orbit is bounded between two specific values of the radial coordinate whose numerical values depend on the initial condition of the motion. The radial coordinate plotted as a function of the proper time exhibits an oscillating behavior.

- Curvature plus friction: in this case the massive test particle reaches the center of the configuration following a spiral with vanishing speed for all the numerical tests analyzed. The speed of the test particle shown in terms of the proper time exhibits a damped oscillating behavior.

To summarize we have showed that in this specific spacetime the friction dominates over the curvature because asymptotically the massive object stops its motion, as in the case in which the curvature non-linearities were sent to zero. The assumption of test field for the photons consequently does not influence the long-time modeling of the motion. A stronger analytical proof of this claim opens a direction for future possible research.

4.4 Extending the Poynting-Robertson formalism to the case of a massive fluid

In this section I will report about the results of the second part of the attached paper **III**; the first part has already been discussed in section (1.5) where the extension of the Poynting-Robertson formula from the motion inside a massless gas to the motion inside a massive fluid has been derived. This proves that their formalism is the correct general relativistic counterpart of the Stokes' law. I will focus here instead on the physical application of this interpretation studying the motion of a body inside a massive gas cloud formally described by the exact Pant-Sah solution of the Einstein equations. Thus I start reviewing how the solutions of the field equations have been constructed in general relativity to describe a system of self-gravitating particles in equilibrium [89; 90; 92; 93; 95–97]; in particular I will follow closely [94] since this set of solutions received poor attention in literature so far. In this latter paper the authors start from the general form of the line element in isotropic coordinates for a spherically symmetric system:

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\omega(r)}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2)\,, \tag{4.27}$$

and the well-known expression for the stress-energy tensor as function of the energy density and pressure of the fluid:

$$T^t{}_t = -\rho\,, \qquad T^r{}_r = T^\theta{}_\theta = T^\phi{}_\phi = p\,. \tag{4.28}$$

Then Pant and Sah have proved that the Einstein equations for such a system are reduced to:

$$8\pi p = e^{-\omega}\left[\frac{\omega'^2}{4} + \frac{\omega'}{r} + \frac{\omega'\nu'}{2} + \frac{\nu'}{r}\right] \tag{4.29}$$

$$8\pi p = e^{-\omega}\left[\frac{\omega''}{2} + \frac{\nu''}{2} + \frac{\nu'^2}{4} + \frac{\omega'}{2r} + \frac{\nu'}{2r}\right] \tag{4.30}$$

$$8\pi\rho = -e^{-\omega}\left[\omega'' + \frac{\omega'^2}{4} + 2\frac{\omega'}{r}\right]\,, \tag{4.31}$$

where a prime denotes, as usual, derivative with respect to r. The first two equations of the system above can be combined together to give:

$$\nu'' + \omega'' + \frac{\nu'^2}{2} - \frac{\omega'^2}{2} - \nu'\omega' - \frac{1}{r}(\nu' + \omega') = 0\,, \tag{4.32}$$

which admits the solution

$$e^{\frac{\nu}{2}} = A\frac{1-k\delta}{1+k\delta}\,, \qquad e^{\frac{\omega}{2}} = \frac{(1+k\delta)^2}{1+\frac{r^2}{a^2}}\,, \tag{4.33}$$

where

$$\delta = \delta(r) = \frac{\left(1+\frac{r^2}{a^2}\right)^{\frac{1}{2}}}{\left(1+b\frac{r^2}{a^2}\right)^{\frac{1}{2}}}\,, \tag{4.34}$$

with A, a, b and k numerical constants. This is exactly the procedure followed by Pant and Sah for deriving their namesake solution. The pressure and energy density can then be expressed in terms of this set of quantities as

$$8\pi p = \frac{4(bk^2\delta^6 - 1)}{a^2(1+k\delta)^5(1-k\delta)} \tag{4.35}$$

$$8\pi\rho = \frac{12(1+bk\delta^5)}{a^2(1+k\delta)^5}\,, \tag{4.36}$$

which are consequently connected by the equation of state

$$p = p(r) = \frac{\rho^{\frac{6}{5}}}{1+2q_c(1-\rho^{\frac{1}{5}})}, \qquad a = \frac{2q_c}{1+q_c}, \tag{4.37}$$

which should be regarded as a generalized polytropic equation of state of index $n = 5$. The sound speed for such a system is:

$$c_s = \sqrt{\frac{\partial p}{\partial \rho}} = \frac{p}{\rho}\sqrt{\frac{2}{5\rho^{\frac{1}{5}}}(3+6q_c-5q_c\rho^{\frac{1}{5}})}. \tag{4.38}$$

This last expression motivates why the Pant-Sah solutions have been looked for: they are more realistic than the interior Schwarzschild solution (which can be derived from the Tolman-Oppheneimer-Volkoff equations (4.21) for a sphere of constant density everywhere):

$$ds^2 = -N_1^2 dt^2 + \frac{dr^2}{N_2^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.39}$$

$$N_1 \equiv N_1(r) = \frac{3}{2}\sqrt{1-\frac{2M}{R}} - \frac{1}{2}\sqrt{1-\frac{2Mr^2}{R^3}}, \quad N_2 \equiv N_2(r) = \sqrt{1-\frac{2Mr^2}{R^3}},$$

M denoting the total mass of the configuration, to have a system in which the sound speed assumes physical values. I would like also to mention that the particular case for which $b = 0$, implying that

$$\delta_{\text{Buchdahl}} = \sqrt{1+\frac{r^2}{a^2}}, \tag{4.40}$$

is referred to in the literature as the Buchdahl solution.

Figures (4.1)-(4.2)-(4.3) picture what I have explained so far in this section. The first two plots show the equation of state p as a function of ρ and the sound speed versus the energy density for the set of values $a = [0.1, 1.0, 1.4, 1.5]$. The third instead gives an estimate of the radius of the configuration for the specific values $k = 4$, $a = 1.5$ and $b = [0.010, 0.050, 0.000, -0.002]$, where the first two numerical values of b correspond to bounded Pant-Sah solution, the third to the unbounded Buchdahl solution and the fourth to the unbounded Pant-Sah solution.

Moreover it is important to stress that the case $b > 0$ describes a finite boundary fluid distribution, while $b \leq 0$ (being the Buchdahl one part of this second class) models unbounded distributions. Thus if we would like to interpret the Pant-Sah solution as a modeling of an astrophysical gravitational lensing we must restrict ourselves to the former case. Literature also explains

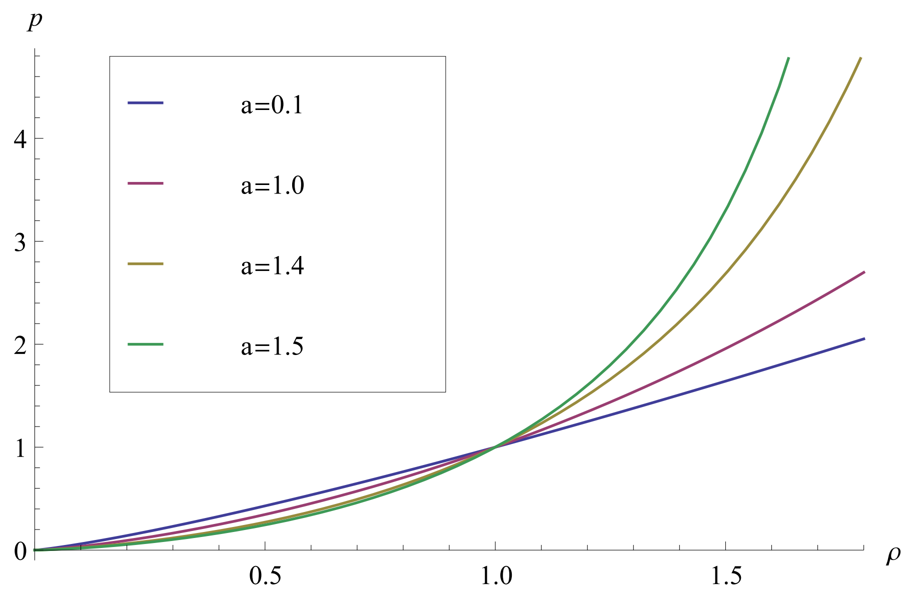

Figure 4.1: Equation of state for a Pant-Sah star. The figure shows the generalized polytropic equation of state of index $n = 5$ (4.37): the pressure is plotted versus the energy density for a Pant-Sah star for the set of values $a = [0.1, 1.0, 1.4, 1.5]$.

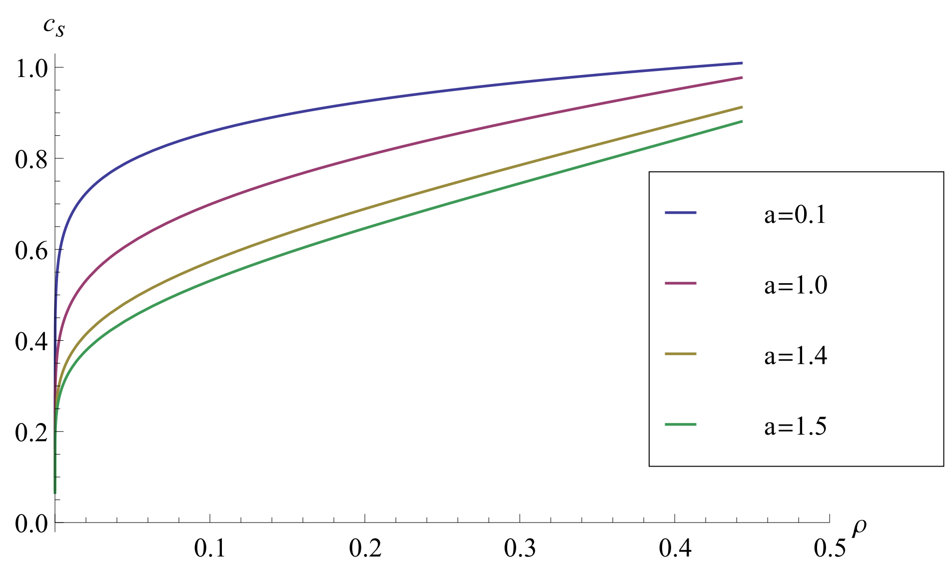

Figure 4.2: Sound speed inside a Pant-Sah star. The figure shows the sound speed inside a Pant-Sah star (4.38) for the same numerical values of the free parameter a as in the plot of the equation of state.

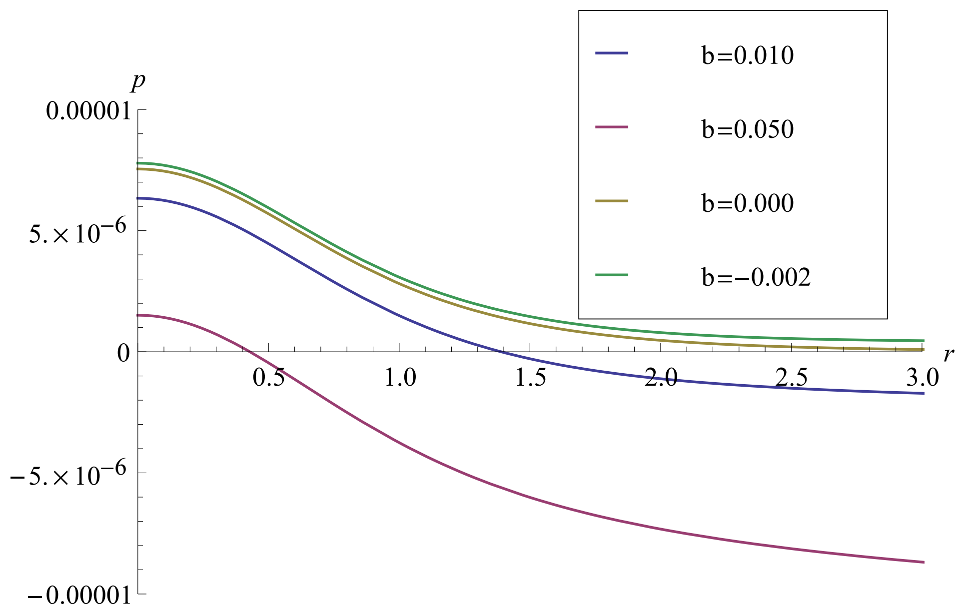

Figure 4.3: Radius for a Pant-Sah star. The figure shows the pressure in terms of the radial coordinate (4.35) for a Pant-Sah star for the values $k = 4$, $a = 1.5$ and a set of values for b. The case $b = 0.000$ corresponds to the Buchdahl solution in which the pressure approaches zero only asymptotically. For $b = 0.010$ and $b = 0.050$ the radius of the configuration is given by the value at which the pressure switches its sign. Moreover it can be noted that p is a monotonically decreasing function of the distance from the center of the star, as expected.

in details how the interior Pant-Sah and exterior vacuum Schwarzschild solutions are matched on the surface for the bounded case and also analyzes the ranges of validity for the numerical values of the free parameters to have a physical meaningful solution.

Moving ahead to our original analysis, we have compared and contrasted the geodesic and geodesic plus Poynting-Robertson effect (where we have plugged the same expression of the stress-energy tensor both inside the Einstein equation (1.5) and the friction force (4.2)) for the Schwarzschild interior solution, the Buchdahl solution and the Pant-Sah one. These cases can be considered as examples of indirect and direct interactions between the test particle and the background respectively in the meaning introduced in the previous section.

- We have considered a massive test particle moving in a vacuum space, then entering a Schwarzschild interior region. If the orbit is geodesic the body deflects its motion and can escape from the massive configuration; when also the friction term is accounted for this behavior is not possible.

- Similar qualitative result has been obtained when the fluid has been modeled by the Pant-Sah line element.

- In the presence of a direct interaction between test particle and fluid through scattering, the test particle entering the gas cloud reaches the center of the configuration both for the Schwarzschild interior and Pant-Sah metrics following a spiral orbit (consequently the radial coordinate of the motion exhibits a damped oscillating behavior as a function of the proper time). The shape of the spiral is different reflecting the different matter distribution of the two cases, being constant in the former and smoothly approaching the vacuum region in the latter.

- For a test particle moving inside the Buchdahl and unbounded Pant-Sah solutions we have re-obtained qualitatively the same motion as in the previous section of this chapter where I have considered the Tolman metric for a photon gas. The geodesic orbit is confined between two specific values of the radial coordinate, while the friction does not allow the body to escape to infinity and condemns it to fall to the center of the configuration.

- In all the cases considered the motion is planar meaning that the friction term does not change this symmetry of the system.

- We have also analytically proved that the equation of motion admits an equilibrium (constant) solution for the angular coordinate ϕ: the presence of a gravitational lensing along the path of an object does not necessarily distort its motion.

I would like to conclude this section recalling that evaluating the angular distortion for the orbit of the test particle in terms of the radial distance from the center of a cluster of galaxies, we can reconstruct the mass of this self-gravitating system [98; 99]. Moreover the evaluation of the observational distortion can also provide an indirect estimate for the coupling constant between the test particle and the background fluid.

4.5 Peculiar velocities in astrophysics

The formalism used in the previous sections of this chapter for the description of the motion of an object inside a gas cloud can be extended to the case of an expanding or contracting spacetime, like the Friedmann one (1.8) about which I have spoken in the third chapter of this thesis in the framework of the construction of an original class of cosmological models. Thus this section is meant to deepen the understanding of the concordance model of cosmology considering the motion of a galaxy inside a homogeneous and isotropic Universe when its direct interaction with the intergalactic medium is accounted for. The motivation for this study is that the search for the correct model for our Universe is a hot matter of debate in the context of general relativity and the motion of a galaxy strongly depends on the description of the relativistic spacetime. A complementary analysis about peculiar velocities and matter horizons has recently been discussed in [100]

The explicit form of the Poynting-Robertson formula (4.2) is in this case given by

$$f(U)_{\rm rad} = -\sigma(1+w)\gamma^2\rho\nu\bar{U}\,, \tag{4.41}$$

where σ is as usual the cross section of process, $w = \frac{p}{\rho}$ describes a one-parameter polytropic equation of state for the fluid driving the dynamics of the Universe (we have assumed the presence of only one matter component), p and ρ being its pressure and energy density respectively; γ represents the dilation Lorentz factor, ν the magnitude of the spatial component of the test particle four-velocity and finally $\bar{U}$ is a spacelike vector orthogonal to U in the plane of motion. It is interesting to note that in the exotic case of dark energy the motion reduces to a purely geodesic one because of the vanishing of the r.h.s of equation (4.41).

The system of equation to be integrated can be shown to reduce to

$$\dot{\nu} = -\frac{\nu}{\gamma}\left[A(1+w)\rho + \frac{1}{\gamma}\frac{\dot{a}}{a}\right],$$
$$\dot{\alpha} = \frac{\nu\Sigma'\cos\alpha}{a\Sigma}, \quad \dot{r} = \frac{\nu\sin\alpha}{a}, \quad \dot{\phi} = \frac{\nu\cos\alpha}{a\Sigma},$$
$$\dot{a} = \sqrt{-k + \frac{8}{3}\pi\rho a^2}, \tag{4.42}$$

where α is the inclination of the motion, $A = \frac{8\pi\sigma}{m}$ the effective coupling constant between test particle and field, while k and Σ have been defined in section (1.2) of this thesis. The energy density depends on the scale factor of the Universe as in the concordance model of cosmology:

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)}, \tag{4.43}$$

where ρ_0 and a_0 are the initial values for the energy density and the scale factor respectively. I have presented explicitly this system of equation to explain what I mean by direct interaction between test particle and field: the spacetime is evolving (equation for the scale factor) because of the presence of a time-dependent energy density; this same energy density appears explicitly also in the equation for the evolution of the magnitude of the spatial velocity of the particle (it enters only indirectly the equation for the inclination). The equations governing the time evolution of the scale factor and the velocity must consequently be integrated simultaneously.

The first equation of the system (4.42) enlightens also the two separate contributions in the spatial acceleration, the friction and gravitational ones:

$$\dot{\nu} = a_{(\text{fric})} + a_{(\text{grav})}, \tag{4.44}$$

with

$$a_{(\text{fric})} = -A(1+w)\rho\frac{\nu}{\gamma}, \qquad a_{(\text{grav})} = -\frac{\nu}{\gamma^2}\frac{\dot{a}}{a}, \tag{4.45}$$

whose ratio can be considered as a function of time to evaluate which one is dominant in the different epochs of the Universe. In particular the friction term will always decelerate the test particle, while the gravitational one can also accelerates it during the re-collapsing phase of a closed Friedmann Universe.

According to our analysis, the motion exhibits the following characteristics:

- The motion is radial for all the cases of hyperbolic, closed and flat Universe.

- Hyperbolic Universe: the test particle velocity is approaching zero both in the pure geodesic ($A = 0$ in the system of equations (4.42)) and geodesic plus Poynting-Robertson effect cases. The friction effects thus play a more minor role than in the case of a closed Universe.

- Flat Universe: qualitatively the same behavior as for the hyperbolic Universe.

- Closed Universe: the motion undergoing friction effects radically deviates from the geodesic one; in the latter case the test particle returns to the ultrarelativistic equilibrium limit for the velocity $\nu \to 1$, as expected from the big crunch model, while in the former goes to zero. This is true because in the recollapsing phase of the Universe the energy density increases causing a dominant effect of the friction deceleration versus the gravitational (positive) acceleration.

- Our formalism does not allow us to discriminate between different equations of state for the fluid permeating the Universe because of the great ignorance inside the modeling of the effective coupling constant between test particle and field (namely inside the cross section of the process σ whose value must be sharply refined in a possible future extension of this work).

The analysis of this section could be applied to deepen our knowledge of the standard model of cosmology as follows. I begin introducing the time-dependent proper distance between an observer and a galaxy $D = ar$ where r indicates the matter comoving radial coordinate and $a = a(t)$ the scale factor at a specific instant. The so-called coordinate velocity between galaxy and observer is then derived from the Hubble law $v = \dot{D} = \dot{a}r = HD$, where I have introduced the Hubble function $H = \frac{\dot{a}}{a}$ [31]. In our case instead also r depends on the time t with the matter non comoving, implying that the Hubble law should be modified as $\mathrm{v} = HD + \nu$, where I have explained in this section how to determine quantitatively the contribution of ν. The observed astronomical structures consequently deviate from the Hubble motion and are expected to approach it only as a limiting case. It is expected that future space missions will quantify experimentally the deviation theoretically derived here [101; 102] increasing our knowledge of the line element of the Universe and its matter content.

To summarize, in this chapter I have dealt with the scattered motion of a massive body in a given spacetime undergoing geodesic plus friction effects.

This latter term has been modeled *à* la Poynting-Robertson. Both the cases of motion in static and non-static spacetimes have been treated, as well as the motion inside a viscous massless (for which the formalism we followed was initially proposed in the standard literature) or massive medium. Physical applications have been discussed to the modeling of the formation of an accretion disk around a star, motion distorted by a gravitational lensing and astrophysical peculiar velocities. A possible future analysis requires the extension of such formalism to other contexts in which friction has been neglected so far: we want to move from the indirect interaction between test particle and fluid contained in the geodesic equation to the direct interaction (collisions between test particle and fluid generating the spacetime). Also its more realistic modeling accounting for the noise effects will be addressed in future. The technicalities of the derivation of all the results briefly presented in this chapter can be found in the attached papers **I**, **II**, **III** and **IV** at the end of this thesis.

Sammanfattning

Denna avhandling behandlar de öppna frågorna att ge en kosmologisk modell som beskriver ett accelererat expanderande universum utan att bryta mot energivillkoren eller en modell som bidrar till den fysikaliska tolkningen av den mörka energin. Det första fallet analyseras med hjälpav en sluten modell baserad på ett regelbundet gitter av svarta hål med användning av Einsteins ekvation i vakuum. I det senare fallet kommer jag att förbinda den mörka energin med Shan-Chens tillståndsekvation. En jämförelse mellan dessa två förslag diskuteras därefter. Som ett kompletterande ämne diskuterar jag rörelse av testpartiklar som i en allmänrelativistisk rumtid genomgår friktionseffekter. Detta är modellerat efter Poynting-Robertsons formalism, vars koppling till Stokes formel presenteras. Fallen av geodetisk och icke-geodetisk rörelse jämförs och kontrasteras för metrikerna Schwarzschild, Tolman, Pant-Sah och Friedmann respektive.

Acknowledgements

In this page I would like to thank the people without whose support I would have never achieved this goal.

Innanzitutto la mia famiglia: un sentito grazie ai miei genitori Patrizia e Dario e a mio fratello Paolo per la loro continua presenza in questi anni di dottorato e ai miei nonni, in particolare Bepi, per avermi sempre incoraggiato nell'intraprendere tali studi.

It would have never been possible to me to write my thesis without the help and the advice of my supervisor Kjell: I thank you for accepting me as a student inside your working group, for correcting my numerous mistakes with endless patience and for visiting me in Potsdam in November 2012 and in February 2014.

I would like to thank the founder and director of the International Center for Relativistic Astrophysics Network (ICRANet), professor Remo Ruffini, for giving me the opportunity of being a member of the international relativistic astrophysics doctorate program and for his renewed trust after my graduation in Rome inside his research group.

A special thank you to my mentor Donato first of all for keeping considering me as one of his students even during this Ph.D., for all the time he dedicated me, for visiting me in Stockholm in July 2012, for our many discussions on Skype and for teaching me how to think in a scientific rigorous way.

I would like to thank the director of the Institute "M. Picone"of the Italian National Council of Research, professor Sauro Succi, for many helpful discussions about possible applications of my work and for visiting me in Stockholm in May 2012.

Many thanks to Andrea for sharing with me his precious experience with Maple^{TM}, to Timothy for sharing his knowledge about Mathematica^{TM} with me and to Reza for our stimulating chats during his visit in Stockholm in June 2012.

I also thank the students, the scientists and the staff of the CoPS group in Stockholm, of the Albert Einstein Institute in Potsdam and of ICRANet for helpful and wise advice.

I acknowledge the University of Nice for support during our Erasmus Mundus schools, the University of Stockholm and the Max Planck Institute for gravitational physics (Albert Einstein Institute) for hospitality during my research.

Daniele Gregoris is an Erasmus Mundus Joint Doctorate IRAP PhD student and is supported by the Erasmus Mundus Joint Doctorate Program by Grant Number 2011-1640 from the EACEA of the European Commission

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