

# THEORETICAL PHYSICS

ITED AT A SEMINAR, TRIESTE, 16 JULY - 25 AUGUST 1962

ohys



INTERNATIONAL ATOMIC ENERGY AGENCY, VIENNA 1963

The following States are Members of the International Atomic Energy Agency:

AFGHANISTAN ALBANIA ARGENTINA AUSTRALIA AUSTRIA BELGIUM BRAZIL BULGARIA BURMA BYELORUSSIAN SOVIET SOCIALIST REPUBLIC CAMBODIA CANADA CEYLON CHILE CHINA COLOMBIA CONGO (LÉOPOLDVILLE) CUBA CZECHOSLOVAK SOCIALIST REPUBLIC DENMARK DOMINICAN REPUBLIC ECUADOR EL SALVADOR ETHIOPIA FINLAND FRANCE FEDERAL REPUBLIC OF GERMANY GHANA GREECE GUATEMALA HAITI HOLY SEE HONDURAS HUNGARY ICELAND INDIA INDONESIA IRAN IRAO ISRAEL

ITALY JAPAN REPUBLIC OF KOREA LEBANON LIBERIA LUXEMBOURG MALI MEXICO MONACO MOROCCO NETHERLANDS NEW ZEALAND NICARAGUA NORWAY PAKISTAN PARAGUAY PERU PHILIPPINES POLAND PORTUGAL ROMANIA SAUDI ARABIA SENEGAL 1 SOUTH AFRICA SPAIN SUDAN SWEDEN SWITZERLAND THAILAND TUNISIA TURKEY UKRAINIAN SOVIET SOCIALIST REPUBLIC UNION OF SOVIET SOCIALIST REPUBLICS UNITED ARAB REPUBLIC UNITED KINGDOM OF GREAT BRITAIN AND NORTHERN IRELAND UNITED STATES OF AMERICA VENEZUELA VIET-NAM YUGOSLAVIA

The Agency's Statute was approved on 26 October 1956 at an international conference held at United Nations headquarters, New York, and the Agency came into being when the Statute entered into force on 29 July 1957. The first session of the General Conference was held in Vienna, Austria, the permanent seat of the Agency, in October, 1957.

The main objective of the Agency is "to accelerate and enlarge the contribution of atomic energy to peace, health and prosperity throughout the world".

Printed by the IAEA in Austria January 1963

# THEORETICAL PHYSICS

LECTURES PRESENTED AT THE SEMINAR ON THEORETICAL PHYSICS ORGANIZED BY THE INTERNATIONAL ATOMIC ENERGY AGENCY HELD AT THE PALAZZINO MIRAMARE, TRIESTE, FROM 16 JULY TO 25 AUGUST \_962

#### Contributions by:

R. CAPPS, A. P. CONTOGOURIS, K. E. ERIKSSON,
S. FRAUTSCHI, M. FROISSART, S. FUBINI, G. FURLAN,
A. GALINDO, R. R. GATTO, H. GOLDBERG, S. HAYAKAWA,
K. JOHNSON, B.W. LEE, S. MANDELSTAM, S.W. Mc DOWELL,
A. MORALES, A. PAIS, T. REGGE, H. RUEGG,
J. J. SAKURAI, A. SALAM, R. SAWYER, S. SCHLIEDER,
J. SCHWINGER, R. STROFFOLINI, W. THIRRING,
J. TIOMNO, J. WERLE, A. S. WIGHTMAN, E. P. WIGNER.

DIRECTOR: A. SALAM

INTERNATIONAL ATOMIC ENERGY AGENCY VIENNA 1963

THEORETICAL PHYSICS, IAEA, VIENNA, 1963 STI/PUB/61

## FOREWORD

Theoretical physics is at present in a stage of rapid development. This makes it a very exciting field of research which attracts some of the most talented scientists all over the world. It is of vital interest to theoretical physicists to report on their latest work, to discuss the results of recent research, and to exchange ideas on the many advanced theories which are constantly being put forward. Work in theoretical physics can be done away from the gigantic research centres which are becoming ever more essential in other fields of science, but which only few countries can afford. Therefore, theoretical physics provides an opportunity for scientists in developing countries to achieve important results and to make their mark without suffering the handicaps with which experimentalists in these countries have to contend. They depend, however, on ways of communication and on a continuous flow of information which will keep them abreast of research which is performed in other countries.

For these reasons the International Atomic Energy Agency organized the Seminar on Theoretical Physics which was held at Miramare, near Trieste, in July and August 1962. The lectures given there are now published in full. These proceedings, together with those of the Summer School on Selected Topics in Nuclear Theory, held in Czechoslovakia in September 1962, will, it is believed, make a substantial contribution to the dissemination of knowledge in this field.

January 1963

SIGVARD EKLUND Director General

# EDITORIAL NOTE

The Seminar on Theoretical Physics was organized so that physicists could discuss mutual problems as they would do in a research centre. The preprints were typed, corrected and distributed on the spot. In several cases, for instance in those of Professors R. Capps, S. Mandelstam, J.J. Sakurai, J. Schwinger and W. Thirring, lecture notes were prepared by participants and later checked by the lecturers. In other cases, papers were prepared by the authors, (Professors M. Froissart, S. Fubini, T. Regge J. Tiomno, A. Pais, S. Hayakawa, A. Salam, E. Wigner).

Theoretical physics is a rapidly developing subject and it is therefore necessary to publish the proceedings with the minimum delay. For this reason, no proofs could be submitted to the authors; however, the texts were carefully proofread by the Agency's staff. The Agency apologizes for any errors which may nevertheless have occurred.

# CONTENTS

| INTRODUCTION  | 1          |
|---|------------|
| BOOK I: QUANTUM FIELD THEORY IN AXIOMATIC AND LAGRANGIAN FORMULATION  | S j        |
| PART I: LORENTZ GROUP AND AXIOMATIC FIELD THEORY         Recent achievments of axiomatic field theory         A.S. Wightman | . 9<br>11  |
| Invariant quantum mechanical equations of motion  | 59<br>83   |
| S. Schlieder  |            |
| PART II: GAUGE FIELDS<br>Gauge theories of vector particles<br>J. Schwinger   | . 87<br>89 |
| Strong coupling in a gauge invariant field theory   | 135        |
| BOOK II: SYMMETRY PROPERTIES OF ELEMENTARY PARTICLES  |            |
| PART I: EXPERIMENTAL REVIEW OF KN AND $\overline{K}$ N INTERACTIONS<br>KN and KN interactions<br><i>R. Capps</i>            | 145<br>147 |
| PART II: SYMMETRIES OF INTERACTIONS<br>The formalism of Lie groups  | 171<br>173 |
| Consequences of higher symmetry for electromagnetic and for weak transitions  | 197        |
| Experimental tests of $SU_3$ and $G_2$ symmetries in strong interactions  | 211        |
| The decomposition of direct products of irreducible representations of SU(3)<br>H. Goldberg                                 | 219        |
| J. J. Sakurai   | 227        |
| Octonions and super-global symmetry   | 251        |
| Pomeranchuk conjecture and symmetry schemes   | 265        |
| BOOK III: COMPLEX ANGULAR MOMENTA AND MANDELS TAM REPRESENTATION  | ×.         |
| Mathematical theory of potential scattering   | 275        |
| Determinal method for complex angular momenta in potential scattering<br>B. W. Lee  | 331        |
| Regge poles in field theory   | 339        |
| Asymptotic limits in potential scattering and in field theory   | 347        |
| On the equation for the Regge trajectories  | 365        |
| High-energy properties of the Mandelstam representation   | 379        |
| Regge poles and strip approximation   | 401        |

| The one-pion exchange model and its relation to Regge pole exchange  | 421        |
|--|------------|
| Regge poles and Drell's production of high-energy particles  | 429        |
| BOOK IV: COMPOUND SYSTEMS  |            |
| PART I: DYNAMICAL INVESTIGATIONS OF $\pi$ N AND NN SYSTEMS<br>Unitarity and production amplitudes (Application to $\pi$ N higher resonances)<br>B. W. Lee                                | 435<br>437 |
| PART II: COMPOUND PARTICLE MODELS  | 449<br>451 |
| BOOK V: VERY HIGH ENERGIES   |            |
| PART I: STRONG INTERACTIONS  | 483<br>485 |
| A model for high-energy nucleon-nucleus collisions   | 521        |
| Strong damping of large momentum transfers, and its consequences for<br>high-energy inelastic processes  | 529        |
| PART II:       ELECTROMAGNETIC INTERACTIONS         Methods for very accurate calculations in high-energy quantum electrodynamics          K. E. Ericsson                                | 541<br>543 |
| Soft photon contribution to electrodynamical cross-sections at very high energies  | 555        |
| Infrared divergence phenomena, high-energy processes, and Regge poles  | 563        |
| Experiments with electron colliding beams  | 569        |
| PART III: WEAK INTERACTIONS  | 591        |
| Weak interactions at high energies   | 593        |
| The interactions in $K_{\mu3}$ and $K_{e3}$ decay $\ldots \ldots \ldots$ | 619        |
| STAFF OF THE SEMINAR   | 630        |
| LIST OF PARTICIPANTS   | 632        |

## INTRODUCTION

In planning the scientific programme for the IAEA seminar on theoretical aspects of high-energy and elementary-particle physics, one overriding consideration was kept in view. More than at any other seminar, we expected to welcome a fairly large proportion of physicists from countries rather far afield from active centres of research. Instead of specializing in a limited field it was felt that the plan of the seminar should provide for a synoptic review of the whole span of the subject at a fairly high level. In this decision we were reinforced by the feeling that apart from the Annual High Energy Physics Conferences at Rochester, CERN and Dubna no such review has recently been attempted. The record of the proceedings reflects this decision.

In mid-1962, one can distinguish three major areas of ferment in theory; firstly, there is the intensive development of quantum field theory; secondly, the search for the symmetry properties of strong and weak interactions; and thirdly, the exploration of the analyticity of the scattering matrix in energy and angular momentum variables. The first four books of this volume survey these developments.

Book I is concerned with quantum field theory in its axiomatic as well as lagrangian formulations. Part 1 is a survey by Wightman of the recent achievements of the axiomatic approach followed by an account from Wigner of some of the less known representations of the Lorentz Group (continuous spin and imaginary mass representations) which may possibly acquire relevance in connection with theories of Regge poles. Part two of Book I consists of Schwinger's lectures on the structure of Gauge Theories of Vector Particles and an account of his recent ideas about gauge invariance and its connection with mass.

Book II is devoted to the symmetry properties of elementary particles with an experimental review by Capps and a survey of the formalism of Lie groups by Salam. A number of contributions by Gatto, Sakurai and others specialize to particular Lie groups, exploring the possibility of testing which, if any, of the higher symmetries are in fact realized in nature.

Book III is concerned with complex angular momenta and Mandelstam representation, with major lecture courses from Regge, Fubini, Mandelstam and Froissart. A shorter Book IV surveys some recent dynamical investigations of  $\pi$ N and NN Systems as well as compound models of elementary particles (Thirring).

The concluding part of this volume (Book V) is different in spirit from the rest. Its concern is with the emerging topic of very high energies, with a survey of strong interactions from Hayakawa, of electromagnetic interactions from Ericsson and others and of weak interactions at very high energies from Pais.

Besides these major courses, at least twenty-five additional seminar lectures were delivered. Some of these are printed but the majority could not be reproduced. This has inevitably meant that some of the continuity from one topic to another achieved in the seminar is lost in its record. This, however, is the price one has to pay in order to keep the volume to a reasonable size. It is hoped that in spite of this, the excitement of the new ideas which we all lived through during the six weeks of the seminar may come out in the published record of its proceedings.

Finally I wish to express on behalf of the participants our appreciation to the International Atomic Energy Agency for making the seminar possible.

Abdus Salam Scientific Director

# INTRODUCTION

Lors de l'établissement du programme scientifique des semaines d'études organisées par l'AIEA sur les aspects théoriques de la physique des hautes énergies et des particules élémentaires, une considération l'a emporté sur toutes les autres. Cette réunion semblait, plus que toute autre, devoir compter une assez forte proportion de physiciens originaires de pays relativement éloignés des centres qui se consacrent activement à la recherche. Aussi a-t-on estimé qu'au lieu de cantonner le programme dans un domaine particulier, il convenait de prévoir une étude générale de l'ensemble de la question d'un niveau assez élevé. Cette façon de voir se trouvait d'ailleurs corroborée par le fait qu'en dehors des conférences annuelles sur la physique des hautes énergies, qui se tiennent à Rochester, au CERN et à Doubna, aucun effort semble n'avoir été déployé récemment dans ce sens.

Au milieu de 1962, on pouvait distinguer trois principaux domaines où les études théoriques étaient particulièrement poussées : la théorie quantique des champs, qui a considérablement progressé; la recherche des propriétés de symétrie dans les interactions fortes et faibles; enfin, l'étude de l'analyticité de la matrice de diffusion en fonction des variables d'énergie et de moment angulaire. Les quatre premiers livres du présent ouvrage rendent compte des progrès accomplis dans ces différents domaines.

Le Livre premier porte sur la théorie quantique des champs, sous sa forme axiomatique ainsi que sous la forme lagrangienne. Il comprend deux parties : la première contient une étude de Wightman sur les récents progrès obtenus par la méthode axiomatique, à laquelle fait suite un exposé de Wigner sur certaines des représentations les moins connues du groupe de Lorentz - représentations en spin continu et en masse imaginaire - qui pourront peut-être se rattacher aux théories des pôles de Regge. La seconde partie contient des exposés de Schwinger sur la structure des théories des particules vectorielles à invariance de jauge ainsi qu'un aperçu de ses récentes théories sur l'invariance de jauge en relation avec la masse.

Le Livre II est consacré aux propriétés de symétrie des particules élémentaires; on y trouve une étude expérimentale de Capps et un exposé de Salam sur le formalisme des groupes de Lie. Différentes communications présentées par Gatto, Sakurai et d'autres physiciens portent sur des groupes de Lie particuliers; ces auteurs étudient la possibilité de vérifier si certaines des symétries d'ordre supérieur se trouvent réalisées dans la nature.

Le Livre III porte sur les moments angulaires complexes et la représentation de Mandelstam, et comprend le texte d'importants exposés de Regge, Fubini, Mandelstam et Froissart. Le Livre IV, de dimensions plus restreintes, rend compte, dans une première partie, de certaines recherches récentes sur la dynamique des systèmes  $\pi N$  et NN; sa deuxième partie est consacrée à un exposé de Thirring sur les modèles composés des particules élémentaires.

Le cinquième et dernier Livre n'est pas rédigé dans le même esprit que les autres. Il a trait au domaine nouveau des très hautes énergies et contient une étude de Hayakawa sur les interactions fortes, une autre d'Ericsson et de plusieurs autres physiciens, sur les interactions électromagnétiques et une troisième de Pais, sur les interactions faibles dans la gamme des très hautes énergies. Outre ces travaux essentiels, vingt-cinq autres exposés au moins ont été présentés au cours des semaines d'études. Certains d'entre eux sont reproduits dans le présent ouvrage, mais la plupart n'ont pu y trouverplace. Aussi le compte rendu des réunions ne reflète-t-il pas entièrement la cohérence qui a marqué l'ensemble des travaux. Ce n'est pourtant qu'à ce prix qu'il a été possible de maintenir les dimensions de l'ouvrage dans les limites raisonnables. J'espère que, malgré cela, il n'en donnera pas moins une idée de l'atmosphère passionnante qui a régné durant ces six semaines d'études.

En conclusion, je tiens à exprimer, au nom des participants, ma reconnaissance à l'Agence internationale de l'énergie atomique grâce à laquelle les semaines d'études ont pu avoir lieu.

> Abdus Salam Directeur scientifique

## ВВЕДЕНИЕ

При составлении научной программы семинара МАГАТЭ по теоретическим аспектам физики высоких энергий и физики элементарных частиц учитывалось одно, ранее не принимавшееся во внимание соображение. На этом семинаре, более чем на каком-либо другом, мы предлагали принять, безусловно, больше физиков от стран, расположенных вдалеке от действующих исследовательских центров. Кроме того, было сочтено уместным составить план семинара таким образом, чтобы в нем предусматривался конспективный обзор предмета во всей его полноте на достаточно высоком уровне. Принятию нами такого решения способствовало то обстоятельство, что, не считая ежегодных конференций по физике высоких энергий, проводимых в Рочестере, ЦЕРНЕ и Дубне, за последнее время не было попыток составить подобный обзор.

В середине 1962 года определились три главных направления в теории: во-первых, интенсивное развитие теории квантового поля; во-вторых, поиски симметричных свойств сильных и слабых взаимодействий; и, в-третьих, исследование аналитичности рассеивающей матрицы в энергии и исследование переменных углового момента. В первых четырех книгах этого тома дается обзор развития этих направлений.

В книге I затрагивается вопрос о теории квантового поля как в аксиоматической формулировке, так и в формулировке Лагранжиана. Часть 1 этой книги представляет собой подготовленный Витманом обзор недавних достижений в области аксиоматического приближения, сопровождаемого расчетом Вигнера некоторых малоизвестных способов представления групп Лоренца (непрерывный спин и представление мнимой массы), которые, возможно, необходимы здесь в связи с теориями полюсов Регге. Часть вторую книги I составляют лекции Швингера относительно структуры теорий измерения вектора частиц, и в ней содержится упоминание о его недавних идеях относительно инвариантности измерения и его связи с массой.

Книга II посвящена симметричным свойствам элементарных частиц. В ней дается также экспериментальный обзор, сделанный Каппсом, и обзор формализма групп Ли, сделанный Саламом. Ряд статей Гатто, Сакураи и других затрагивает, в частности, группы Ли, исследуя возможность их проверки, если таковая вообще имеется, хотя их наибольшая симметричность существует в природе.

В книге III разбирается комплекс угловых моментов и моделирование Мандельштама и Фруассара. В небольшой по объему книге IV дается обзор некоторых недавних динамических исследований систем тN и NN, а также составных моделей элементарных частиц (Тирринг).

Заключительная часть этого тома (книга V) отличается по своему духу от остальных. Она касается темы очень высоких энергий. В ней дается обзор сильных взаимодействий на основе работ Хаякавы, электромагнитных взаимодействий по Эрриксону и других авторов, а также слабых взаимодействий при очень высоких энергиях по Пайсу. Кроме этих общирных курсов представлено, по крайней мере, 25 дополнительных семинарских лекций. Некоторые из этих лекций напечатаны,но большая часть их не воспроизводится. Такое положение неминуемо означает то, что последовательность изучения тем, достигнутая на семинаре, не соблюдена в издании. Однако это является своебразной жертвой во имя того, чтобы сохранить приемлемый размер тома. Выражается надежда, что, несмотря на такое сокращение, новые идеи, которыми все мы жили в течение шести недель семинара, появятся в опубликованных трудах семинара.

И, наконец, от имени участников семинара я хочу выразить нашу признательность Международному агентству по атомной энергии за то, что оно провело такой семинар.

> Абдас Салам, Ученый директор

# INTRODUCCIÓN

Hemos elaborado el programa científico del Seminario del OIEA sobre los aspectos teóricos de la física de las altas energías y de las partículas elementales pensando sobre todo que en el Seminario iba a participar un número sin precedentes de físicos de países que no cuentan con centros de investigación. Por eso resolvimos que el plan de la reunión, en lugar de circunscribirse a un campo muy concreto, ofreciera una sinopsis suficientemente especializada de todo el tema objeto del Seminario. Excepción hecha de las Conferencias anuales sobre física nuclear de las altas energías celebradas en Rochester, el CERN y Dubno, en los últimos tiempos no se ha intentado llevar a cabo un examen general de este género; esto nos impulsó a adoptar nuestra decisión, de la que dan constancia las actas del Seminario.

A mediados de 1962 cabía distinguir tres sectores principales en los que abundaban las investigaciones teóricas: en primer lugar, la teoría de los campos cuánticos, que pasaba por una fase de intenso estudio; en segundo lugar, las investigaciones sobre las propiedades simétricas de las interacciones fuertes y débiles y, en tercer lugar, el estudio de la posibilidad de analizar la matriz de dispersión en las variables de momento angular y de energía. De los cinco libros en que se divide el presente volumen, los cuatro primeros estudian esta cuestiones.

El libro I trata de la teoría de los campos cuánticos en sus formulaciones axiomática y lagrangiana. Su primera parte consiste en un estudio de Wightman sobre los más recientes progresos conseguidos con el método axiomático, seguido de una reseña de Wigner que trata de algunas de las representaciones menos conocidas del grupo de Lorentz (representaciones de espín continuo y de masa imaginaria), que posiblemente lleguen a tener importancia en relación con las teorías de los polos de Regge. La segunda parte del libro I contiene las conferencias de Schwinger sobre la estructura de las teorías del "gauge" de las partículas vectoriales, y una reseña de sus ideas más recientes sobre la invariancia del "gauge" y su relación con la masa.

El libro II está dedicado a las propiedades de simetría de las partículas elementales, y contiene un estudio experimental de Capps y otro trabajo de Salam acerca del formalismo de los grupos de Lie. Una serie de memorias de Gatto, Sakurai y otros investigadores estudian determinados grupos de Lie y exploran las posibilidades de comprobar cuáles de las simetrías superiores se dan realmente en la naturaleza.

El libro III trata de los momentos angulares complejos y de la representación de Mandelstam e incluye una serie de conferencias importantes dadas por Regge, Fubini, Mandelstam y Froissart. En el libro IV, más breve, se exponen algunas investigaciones recientes sobre la dinámica de los sistemas  $\pi$ N y NN, y se describen algunos modelos compuestos de partículas elementales (Thirring).

La parte final del volumen (libro V) difiere en espíritu de las restantes. Trata del novísimo tema de las energías ultraelevadas y contiene un estudio de Hayakawa sobre las interacciones fuertes, otro de Ericsson y sus colaboradores sobre las interacciones electromagnéticas, y otro de Pais sobre las interacciones débiles con energías ultraelevadas. Aparte de los cursillos principales se pronunciaron en el Seminario otras veinticinco conferencias por lo menos. Algunas de ellas se han impreso, pero en la mayoría de los casos no fue posible hacerlo. Esto quiere decir que, en el presente volumen, al pasar de un tema a otro, se acusan forzosamente soluciones de continuidad que no existieron en el Seminario. Pero no cabía obrar de otra manera si se quería evitar que la presente publicación resultara excesivamente voluminosa. Esperamos que los trabajos publicados reflejen el ambiente de interés por las nuevas ideas en que todos vivimos durante las seis semanas que duró el Seminario.

Permítaseme, por último, manifestar en nombre de todos los participantes nuestro agradecimiento al Organismo Internacional de Energía Atómica por haber hecho posible que se celebrase el Seminario.

> Abdus Salam Director Científico

# BOOK I

# QUANTUM FIELD THEORY IN AXIOMATIC AND LAGRANGIAN FORMULATIONS

# PART I

LORENTZ GROUP AND AXIOMATIC FIELD THEORY

# RECENT ACHIEVEMENTS OF AXIOMATIC , FIELD THEORY

# A. WIGHTMAN PRINCETON UNIVERSITY, PRINCETON, N. J. UNITED STATES OF AMERICA

#### INTRODUCTION

These lecture notes are an attempt to describe something of what has been achieved in so-called axiomatic field theory in the last couple of years with the emphasis on those results which are particularly neat.

Two significant projects currently under way which probably are very deep and certainly are very difficult will not be mentioned: Symanzik's structure analysis and the pursuit of the so-called "linear programme" by Källén and others. Fortunately, these are excellently summarized in [43].

The paper is divided into two parts. The results presented in the first half are characterized by the fact that, once one has had the proper insight, they can be proved with a few simple manipulations. In the second part there is a steep rise in the difficulty of the analysis.

No attempt will be made to rationalize the rather mathematical preoccupations of these lectures; for one reason, the author has tried it before [1]. The root-mean-square deviation from the mean of opinion on what is a sensible thing to try to do in elementary particle theory seems to be one of those unrenormalizable infinities one hears about.

Of all the work reported, the most significant seems Borcher's discovery of equivalence classes of local fields and Ruelle's rigorization of Haag's collision theory. The first was totally unsuspected and represents the kind of insight which is indispensable if one is ever going to be able to get back to calculating cross-sections in relativistic quantum field theory. The second shows that in relativistic quantum field theory the collision theory (or asymptotic particle description) is already uniquely determined by the fields, a result which accords with one's physical intuition and supplies strong evidence that axiomatic field theory is on the right track.

#### PART ONE

This first part will describe a number of results which have simplicity and generality in common. All mathematical technicalities will be deferred to Part 2.

## 1.1. RECOLLECTION OF THE PCT THEOREM

The PCT theorem will be used again and again in the course of this paper so it will be presented here briefly in the form given by JOST [1].

#### A. WIGHTMAN

If A(x) is a charged scalar field, its transform under the PCT operation is  $A(-x)^*$ . The anti-unitary operator  $\Theta$  on the states which generates this transformation of the fields therefore satisfies

$$\Theta A(\mathbf{x}) \Theta^{-1} = A(-\mathbf{x})^* . \tag{1}$$

(A charged rather than a neutral scalar field will be considered temporarily to bring out the role of the Hermitian adjoint in the definition of PCT.) In any theory of a field (or a denumerable set of fields) that has the vacuum  $\Psi_0$  as cyclic vector (i.e. for which polynomials in the smeared fields  $\mathscr{P}(A(g)...)$  applied to the vacuum  $\Psi_0$  yield a dense set in  $\mathscr{H}$  the Hilbert space of states), (1) is equivalent to an identity of the vacuum expectation values:

$$(\Psi_0, A_1(x_1), \dots, A_n(x_n)\Psi_0) = [(\Psi_0, A_1(-x)^{\mp}, \dots, A_n(-x_n)^{*}\Psi_0)]^{*}$$
(2)

or equivalently:

$$(\Psi_0, A_1(x_1), \dots, A_n(x_n)\Psi_0) = (\Psi_0, A_n(-x_n), \dots, A_1(-x_1)\Psi_0).$$
 (3)

This reduces the problem of determining whether a theory has PCT symmetry to an examination of its vacuum expectation values. If (3) or equivalently (2) holds for all  $x_1 \ldots x_n$ , we say the n-fold vacuum expectation value has PCT symmetry. On the other hand, from the Lorentz invariance of the field

$$U(a, \Lambda)A(x) U(a, \Lambda)^{-1} = A(\Lambda x + a), \qquad (4)$$

the vacuum expectation values satisfy

$$(\Psi_0, A_1(\Lambda x_1 + a), \dots, A_n(\Lambda x_n + a)\Psi_0) = (\Psi_0, A_1(x_1), \dots, A_n(x_n)\Psi_0).$$
 (5)

(Only invariance under restricted Lorentz transformations det  $\Lambda = 1$ , sgn  $\Lambda_0^0 = 1$ , is assumed.)

From this and the spectral condition it follows that

$$(\Psi_0, A_1(x_1), \dots, A_n(x_n)\Psi_0)$$

$$= \int \left[ \exp -i \sum_{j=1}^{n-1} p_j (x_j - x_{j+1}) \right] G^{A_1 \dots A_n} (p_1, \dots, p_{n-1}) dp_1 \dots dp_{n-1}, \quad (6)$$

where  $G^{A_1...,A_n}$  vanishes for  $p_1..., p_{n-1}$  outside the physical spectrum which must be in the future light cone. From this in turn it follows that there is an analytic function  $F^{A_1...,A_n}$  of n-1 complex from vector variables,

$$z_j = (x_j - x_{j+1}) - i \eta_j$$
 (where  $j = 1, 2, ..., n-1$ ),  
(7)

$$\mathbf{F}^{A_{1}...A_{n}}(z_{1},...,z_{n-1}) = \int \left[ \exp(-i\sum_{j=1}^{n-1} p_{j} z_{j}) \right] \mathbf{G}^{A_{1}...A_{n}}(p_{1},...,p_{n-1}) dp_{1}...dp_{n-1},$$

analytic in the <u>tube</u>,  $\Im_{n-1}$ , which is the set of  $z_1 \dots z_{n-1}$  for which  $\eta_j \in V_+$ , the future light cone for  $j=1,\dots,n-1$  and such that

$$(\Psi_{0}, A_{1}(x_{1}), \dots, A_{n}(x_{n})\Psi_{0}) = \lim_{\substack{\eta_{1}, \dots, \eta_{n-1} \to 0 \\ \text{in } V_{+}}} F^{A_{1}, \dots, A_{n}}(z_{1}, \dots, z_{n-1}).$$
(8)

 $\mathbf{F}^{\mathbf{A_1}...,\mathbf{A_n}}$  is also Lorentz invariant

$$\mathbf{F}^{\mathbf{A}_1 \cdots \mathbf{A}_n}(\mathbf{z}_1, \ldots, \mathbf{z}_{n-1}) = \mathbf{F}^{\mathbf{A}_1 \cdots \mathbf{A}_n} (\Lambda \mathbf{z}_1, \ldots, \Lambda \mathbf{z}_{n-1}), \qquad (9)$$

which implies that  $F^{A_1...,A_n}$  possesses a single-valued continuation to the <u>extended tube</u>  $\mathcal{J}'_{n-1}$ , which consists of all points of the form  $\Lambda z_1, \ldots, \Lambda z_{n-1}$  with  $\Lambda$  a complex Lorentz transformation of determinant one and

$$z_1, \ldots, z_{n-1} \in \mathcal{J}_{n-1}$$
.

In particular,

$$F^{A_1...A_n}(z_1,...,z_{n-1}) = F^{A_1...A_n}(-z_1,...-z_{n-1})$$
 (10)

at each point of  $\mathfrak{I}_{n-1}'$ . Finally, it should be remembered that the extended tube contains real points, the so-called <u>Jost points</u>;  $\xi_1, \ldots, \xi_{n-1}$  is a Jost point if it is real and  $\sum_{j=1}^{n-1} \lambda_j \xi_j$  is space-like for all  $\lambda_j$  (j=1,...,n-1) such that

$$\lambda \ge 0$$
 and  $\sum_{j=1}^{n-1} \lambda_j > 0.$  (11)

PCT Theorem

If W(eak) L(ocal) C(ommutativity)

$$(\Psi_{0}, A_{1}(x_{1}), \dots, A(x_{n})\Psi_{0}) = (\Psi_{0}, A_{n}(x_{n}), \dots, A_{1}(x_{1})\Psi_{0})$$
(12)

holds for  $x_1, \ldots, x_n$  such that  $x_1 - x_2, \ldots, x_{n-1} - x_n$  fill a real neighbourhood of a Jost point, then (3) holds for all  $x_1, \ldots, x_n$  and the n-fold vacuum expectation value has PCT symmetry.

Conversely, if the n-fold vacuum expectation value has PCT symmetry, then WLC holds in the neighbourhood of every Jost point.

Proof

If WLC holds in the neighbourhood of the Jost point  $x_1 - x_2, \ldots, x_{n-1} - x_n$ , then

$$\mathbf{F}^{A_1...A_n}(z_1,...,z_{n-1}) = \mathbf{F}^{A_1...A_1}(-z_{n-1},...,-z_1)$$
(13)

in an open set of real space. Therefore, the analytic functions on the left-hand side and right-hand side coincide throughout  $\mathfrak{J}'_{n-1}$ , using the fact that

#### A. WIGHTMAN

two functions analytic in an open set of complex space and coinciding on a real subset which is open in the real subspace coincide everywhere.

Using (10), this says:

$$\mathbf{F}^{A_{1},\ldots,A_{n}}(z_{1},\ldots,z_{n-1}) = \mathbf{F}^{A_{n},\ldots,A_{1}}(z_{n-1},\ldots,z_{1})$$
(14)

throughout  $\mathcal{J}_{n-1}$ . (Note that if  $z_1, \ldots, z_{n-1}$  is a Jost point so is  $-z_{n-1}, \ldots, -z_n$  and if  $z_1, \ldots, z_{n-1} \in \mathcal{J}_{n-1}$  then  $z_{n-1}, \ldots, z_1 \in \mathcal{J}_{n-1}$ .) Passing to the boundary values with  $\eta_i \in V_+$ , one gets

$$(\Psi_0, A_1(x_1), \dots, A_n(x_n)\Psi_0) = (\Psi_0, A_n(-x_n), \dots, A_1(-x_1)\Psi_0)$$
(15)

for all  $x_1, \ldots, x_n$ , which is PCT symmetry.

Conversely, suppose (15) holds for all  $x_1 \dots x_n$ , then it holds for a real neighbourhood of a Jost point. Then (14) and (13) follow at every real point of analyticity, and that is exactly WLC at every Jost point.

Of course, WLC is implied by LC:

$$[A_{i}(x), A_{j}(y)] = [A_{i}(x), A_{j}(y)^{*}] = 0$$
(16)

It is important in applications that the PCT operator of an irreducible set of fields is essentially uniquely determined [3].

If 
$$\Theta_1 A_i(x) \Theta_1^{-1} = A_i(-x)^*$$

and 
$$\Theta_2 A_i(x) \Theta_2^{-1} = A_i(-x)^*$$

then 
$$\Theta_2 \Theta_1 A_i(\mathbf{x}) \Theta_1^{-1} \Theta_2^{-1} = A_i(\mathbf{x})$$
 (17)

so by the irreducibility of  $A_i(x)$ ,

$$\Theta_2 \Theta_1 = \lambda \, 1 \, . \tag{18}$$

Now because  $(PCT)^2 = 1$ ,  $\Theta^2 = \mu \mathbf{1}$  with  $|\mu| = 1$ . (A priori  $\Theta_j^2$  need only be constant in each coherent subspace of states, i.e. states not separated by super selection rules. But (17) implies  $[\Theta_1^2, A_j(\mathbf{x})] = 0$ , so, by the irreducibility of  $A_j, \Theta_1^2 = \mu \mathbf{1}$ . If one had a more complicated transformation law, say that for appropriate two-component scalar field, it could be arranged to have  $[\Theta^2, A_j(\mathbf{x})]_+ = 0$ , then one would have  $\Theta^2 = +1$  on states obtained from the vacuum by applying an even number of  $A_{j\alpha}$  and -1 on those obtained by applying an odd number to the vacuum. In that case  $\Theta^2$  generates a super selection rule. While these applications have an interest of their own they will not be pursued here.) For anti-unitary operators  $\Theta_j^2 = \mu \mathbf{1}$  with  $|\mu| = 1$  implies  $\mu = \pm 1, (\Theta(\Theta\Theta) = \Theta(\mu 1) = (\Theta\Theta)\Theta = \mu\Theta$  so  $\mu$  is real and therefore  $= \pm 1$ ); thus (18) implies

$$\Theta_1^2 = |\lambda|^2 \Theta_2^{-2}$$

so  $|\lambda|^2 = 1$  and  $\Theta_1$  and  $\Theta_2$  differ only by a phase factor. It is customary to fix this phase factor so that

$$\Theta \Psi_0 = \Psi_0 \quad . \tag{19}$$

Then  $\Theta$  is unique. That the left-hand side of (19) must be proportional to the right follows from a comparison of the transformation law (4) of A<sub>i</sub> under U(a,  $\Lambda$ ) and (1) under  $\Theta$ . One immediately deduces that  $\Theta^{-1}$ U(a,  $\Lambda$ )<sup>-1</sup> $\Theta$ U(-a,  $\Lambda$ ) commutes with  $A_i$  so

$$\omega(\mathbf{a}, \Lambda) \mathbf{U}(\mathbf{a}, \Lambda) = \Theta \mathbf{U}(-\mathbf{a}, \Lambda) \Theta^{-1}(\text{where } |\omega| = 1),$$

and since the inhomogeneous Lorentz group possesses no one-dimensional representations,  $\omega = 1$ .

$$U(a, \Lambda) = \Theta U(-a, \Lambda) \Theta^{-1} .$$
 (20)

Thus the energy momentum operator satisfies

$$\mathbf{P}^{\mu} = \Theta \mathbf{P}^{\mu} \Theta^{-1} \cdot \tag{21}$$

The anti-unitary character of  $\Theta$  is essential here; if  $\Theta$  were unitary, (21) would have a minus sign and negative energy states would exist. Finally, (21) and the convention (19) imply that  $\Theta \Psi_0 = \Psi_0$ . The essential point is that the algebraic structure of the set of field operators, as displayed in the symmetries of their vacuum expectation values, uniquely determines a  $\Theta$  and a transformation law of the fields under  $\Theta$ .

The relation of  $\Theta$  to scattering theory is very simple:

$$\Theta A^{\text{in}}(\mathbf{x}) \Theta^{-1} = A^{\text{out}}(-\mathbf{x})^* .$$
(22)

This is easy to see if one has a theory in which the simple form of the asymptotic condition is valid.

$$A^{in}(\mathbf{x}) = A(\mathbf{x}) - \int \Delta_R (\mathbf{x} - \mathbf{y}) \mathbf{j}(\mathbf{y}) d\mathbf{y} ,$$
  

$$\Theta A^{in}(\mathbf{x}) \Theta^{-1} = A(-\mathbf{x})^* - \int \Delta_R (\mathbf{x} - \mathbf{y}) \mathbf{j}(-\mathbf{y})^* d\mathbf{y}$$
  

$$= A(-\mathbf{x})^* - \int \Delta_A (\mathbf{x} - \mathbf{y}) \mathbf{j}(\mathbf{y})^* d\mathbf{y}$$
  

$$= A^{out} (-\mathbf{x})^*$$
  
because  

$$\Delta_A (-\mathbf{x}) = \Delta_R (\mathbf{x}) .$$

(22) is still true in the most general scattering theory we know where the correspondence between  $A_i$  and  $A_i^{in}$  need not be one to one. This will be discussed below.

It is clear from (22) that  $\Theta$  is not the PCT operator for  $A^{out}$  (x); by the PCT theorem there must be another anti-unitary operator U satisfying

#### A. WIGHTMAN

$$UA_{j}^{out}$$
 (x)  $U^{-1} = A_{j}^{out} (-x)^{*}$  (23)

because  $A^{in}$  is local and the  $A_j^{in}$  are irreducible, which we assume for the collision states to be complete. Now we know a <u>unitary</u> operator, the S operator, which satisfies

$$A^{out}(x) = S^{-1}A^{in}(x) S$$
 (24)

By comparing (24), (23) and (22) and using the familiar argument above, we get:  $\Theta^{-1} U = S$ . The collision operator is the <u>relative</u> PCT transformation of the basic fields A<sub>j</sub> and the out fields A<sub>j</sub><sup>out</sup>. It is clear from this that one can define a "relative S operator" of two fields even when they do not satisfy the asymptotic condition S<sub>AB</sub> =  $\Theta_A \Theta_B *$ .

## 1.2. THE TRANSITIVITY OF WLC AND LC; EQUIVALENCE CLASSES OF LOCAL FIELDS [3]

One of the most striking recent discoveries in quantum field theory was made by Borchers. Roughly, it says (a) that if A is an irreducible field which is LC and B is LC relative to A, i.e.

$$[A(x), B(y)] = 0 \text{ for } (x^2 - y^2) < 0 , \qquad (25)$$

then B is LC; and (b) if A is irreducible and is LC and B and C are LC relative to A then B is LC relative to C. This shows that local fields fall into equivalence classes (also called Borchers classes), two being equivalent if they are relatively local. Similar statements hold for WLC. Finally, Borchers showed [3] that if two fields lie in the same equivalence class and satisfy the LSZ asymptotic condition they have the same S operator. He also shows that if the fields are A and B,  $A^{in} = \pm B^{in}$ . This shows that in order to get theories with a non-trivial S operator one must use fields outside the equivalence class of any free field. It should be emphasized that each member of equivalence classes of fields acts in the same Hilbert space and has the same representation of the inhomogeneous Lorentz group. Two free fields of different mass are not comparable in this classification. It remains an open question whether there are Borchers classes, other than those of free fields, which have the same representation as free fields. Of course, there is nothing now known to prevent different Borchers classes from having the same S operator. In fact, this happens for free fields of the same mass which are not local relative to one another. Incidentally, it should also be emphasized that one can prove the required properties of the equivalence classes only by assuming that there is at least one irreducible field in the class. Thus B and C local relative to A need not imply B and C relatively local unless A is irreducible.

<sup>\*</sup> The simple but interesting remark that the S operator is a relative PCT transformation was made by SYMANZIK [4]. The relative S operator is definable even for models with a space-time containing a finite number of points [4a].

Before the proof of Borchers' result, an example of a Borchers class and an application of his theorems to prove the non-existence of solutions of certain theories will be given.

Example: the equivalence class of an irreducible free neutral scalar field

Denote the field A. Then  $D^{\alpha}A$  is again a field (no longer scalar!) and LC with respect to A. Here

$$D^{\alpha}A(x) = \partial^{|\alpha|} A(x) / (\partial x^0)^{\alpha_0} \partial (x^1)^{\alpha_1} \dots \partial (x^3)^{\alpha_3}, \qquad (26)$$

where  $|\alpha| = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$ . Furthermore, the Wick ordered product :  $D^{\alpha}A(x)D^{\beta}A(x)$ : is again a field and LC with respect to A. It is defined by

$$\lim_{\substack{x_{1},...,x_{\ell} \to x}} : D^{\alpha^{(1)}}A(x_{1}) D^{\alpha^{(2)}}A(x_{2}),..., D^{\alpha^{(\ell)}}A(x_{\ell}) :$$

$$= \sum_{r=0}^{\lfloor \ell/2 \rfloor} (-1)^{r} \sum_{c_{r}} [D^{\alpha^{(j_{1})}}A(x_{j_{1}}),...,D^{\alpha^{(j_{2}r)}}A(x_{j_{2}r})]$$

$$: D^{\alpha^{(k_{1})}}A(x_{k_{1}}),...,D^{\alpha^{(k_{\ell}-2r)}}A(x_{k_{\ell-2r}}), \quad (27)$$

where  $\lfloor \ell/2 \rfloor$  is the largest integer less than  $\ell/2$ . The sum  $\Sigma_{c_r}$  is over all partitions of the integers  $\ell$  into two subsets  $j_1 \ldots j_{2r}$  and  $k_1 \ldots k_{\ell-2r}$  satisfying  $j_1 < j_2 \ldots < j_{2r}$  and  $k_1 < k_2 \ldots k_{\ell-2r}$ . The Hafnian  $[\ldots ]$  is defined by:

$$\begin{bmatrix} D^{\alpha^{(j_1)}} A(x_{j_1}) \dots D^{\alpha^{(j_{2r})}} A(x_{j_{2r}}) \end{bmatrix} = \sum_{\substack{k_1 \dots k_r \\ k'_1 \dots k'_r}} \prod_{s=1}^{2r} (\psi_0, D^{\alpha^{(k_s)}} A(x_{k_s}) + D^{\alpha^{(k_s)}} A(x_{k_s}))$$

where, here, the summation is over all partitions  $(k_1, k'_1) \dots (k_r, k'_r)$  of  $j_r \dots j_{2r}$  in disjoint subsets so that  $k_s < k'_s$  (s=1,...r). Thus the equivalence class of the free field must include all invariant Wick polynomials of the form:

$$\sum_{n,\alpha,\beta} c_{n\alpha\beta} : D^{\alpha} A(x) D^{\beta} A(x):$$

where the indices on the derivatives are summed to give invariant combinations. For example,

$$\sum_{j=1}^{n} \alpha_{j} : A(x)^{j} : , : \frac{\partial}{\partial x^{\mu}} A(x) \frac{\partial}{\partial x_{\mu}} A(x) :$$

SCHROER [5] has shown recently that the invariant Wick polynomials exhaust the equivalence class of an irreducible neutral scalar free field of mass m. An obvious possibility,

$$\sum_{j=1}^{\infty} \alpha_j : A(x)^j : \qquad (28)$$

#### A. WIGHTMAN

with  $\alpha_j$  decreasing very fast with j, is excluded because it describes a theory with an infinite number of subtractions<sup>\*</sup>. This will be discussed in detail later.

The fact that the invariant Wick polynomial in a free field and its derivatives were then the only known examples of local fields suggested to the author some years ago [7] that one should try to use them as currents, i.e. to look for local solutions of,

$$A(x) = A^{in}(x) + \int \Delta_R(x-y) j(y) dy,$$
 (29)

where j is an invariant Wick polynomial in a given free field  $A^{(0)}(x)$ .  $A^{in}$  is also a free field but <u>a priori</u> not in any way related to  $A^{(0)}(x)$ . EPSTEIN and the author have shown that there are no LC solutions in the special case j(x) = g:  $A^{(0)}(x)^2$  [7]. ARAKI, HAAG and SCHROER [8] pointed out that when j is irreducible, Borchers' result enables one to give a very much more general and certainly neater discussion.

Theorem 1

If

$$(\Box + m^2) A(x) = j(x),$$
 (30)

one of A and j is irreducible and A is LC, then A and j lie in the same equivalence class.

If j is an invariant Wick polynomial in a free field  $A^{(0)}$  and its derivatives and is irreducible, S = 1. Furthermore, (29) has no non-trivial solutions unless j = 0 and A =  $A^{in} = A^{(0)}$ .

#### Proof

The first statement is an immediate consequence of Borchers' result. To obtain the second, note that first-degree Wick polynomials in  $A^{(0)}$  are inadmissible in (29) because their retarded potentials do not exist. Because of the assumed irreducibility of j and the assumption that it is an invariant polynomial in  $A^{(0)}$  and its derivatives A,  $A^{(0)}$  and j lie in the same equivalence class. (A and  $A^{(0)}$  are LC relative to an irreducible j. Therefore, A is LC relative to  $A^{(0)}$ .) Therefore, the "in" fields associated with A and  $A^{(0)}$  coincide up to a sign  $A^{in}(x) = \pm A^{(0)}(x)$ , and the Borchers theorem implies S = 1. If A were local, this would imply

$$A^{0}(x) + \int \Delta_{R}(x-y) j(y) dy$$
 (31)

is local which is impossible as a direct calculation shows. Sticklers for completeness can support this last step by the somewhat more general statement [9].

<sup>\*</sup> The fact that (28) does not satisfy the ordinary axioms of quantum field theory if an infinite number of  $\alpha$ 's  $\neq$  0 was pointed out by GLASER [6]. It is a freak that this statement is true in three- and four-dimensional but not in two-dimensional space-time where such expressions occur in the Thirring model. That operator guage transformations give rise to such "unrenormalizable fields" was emphasized by KÄLLÉN [6].

Theorem 2

If A is a neutral scalar field of the form

$$A(x) = \sum_{n=1}^{N} \int dx_{1} \dots dx_{n} f_{n}(x, x_{1} \dots x_{n}) : A^{0}(x_{1}) \dots A^{(0)}(x_{n}) : , \quad (32)$$

where  $A^{(0)}$  is a free field and A is LC then A is an invariant Wick polynomial in  $A^{(0)}$  and its derivatives.

Now let us consider the precise statement of the Borchers theorem and its proof. It comes in four parts, the first two relating to WLC and the second two to LC.

#### Theorem 3

Let A and B be neutral scalar fields but not necessarily LC. Suppose A irreducible and that A satisfies WLC. Then B satisfies WLC if the identities,

$$(\Psi_0, A(x_1), \dots, A(x_j)B(y)A(x_{j+1}), \dots, A(x_n) \Psi_0)$$
  
=  $(\Psi_0, A(x_n), \dots, A(x_{j+1})B(y)A(x_j), \dots, A(x_1) \Psi_0),$  (33)

hold for  $x_1-x_2, \ldots, x_{j-1}-x_j, x_j-y, y-x_{j+1}, \ldots, x_{n-1}-x_n$  in a neighbourhood of some Jost point for each  $n = 0, 1, \ldots$  and each  $j, 1 \leq j \leq n$ . Furthermore, the PCT operator of B coincides with that of A so A and B together satisfy WLC.

#### Proof

Assume (33) holds. Because  $\Theta$  is anti-unitary, one has:

$$(\Theta \Phi, \Theta B(z) \Theta^{-1} \Theta \Psi) = [(\Phi, B(z) \Psi)]^{*} = (\Psi, B(z) \Phi).$$
(34)

In particular, (34) holds for vectors of the form

$$\Phi = \sum_{k} \int \dots \int d\mathbf{x}_{1} \dots d\mathbf{x}_{k} \mathbf{f}_{k} (\mathbf{x}_{1}, \dots, \mathbf{x}_{k}) \mathbf{A}(\mathbf{x}_{1}) \dots \mathbf{A}(\mathbf{x}_{k}) \Psi_{0}$$

$$\Psi = \sum_{\ell} \int \dots \int d\mathbf{x}_{1} \dots d\mathbf{x}_{\ell} \mathbf{g}_{\ell} (\mathbf{x}_{1}, \dots, \mathbf{x}_{\ell}) \mathbf{A}(\mathbf{x}_{1}) \dots \mathbf{A}(\mathbf{x}_{\ell}) \Psi_{0} \quad (35)$$

for which

ΘΦ

$$= \sum_{k} \int \dots \int d\mathbf{x}_{1} \dots d\mathbf{x}_{k} (\mathbf{f}_{k} (\mathbf{x}_{1}, \dots, \mathbf{x}_{k}))^{*} A(-\mathbf{x}_{1}) \dots A(-\mathbf{x}_{k}) \Psi_{0}$$

$$\Theta \Psi = \sum_{\boldsymbol{\ell}} \int \dots \int dx_1 \dots dx_{\boldsymbol{\ell}} (g_{\boldsymbol{\ell}} (x_1, \dots, x_{\boldsymbol{\ell}}))^* A(-x_1) \dots A(-x_{\boldsymbol{\ell}}) \Psi_0.$$

The identities (33) in the neighbourhood of a Jost point imply the identities A. WIGHTMAN

$$(\Psi_0, A(x_1), \dots, A(x_j) B(y) A(x_{j+1}), \dots, A(x_n) \Psi_0)$$
  
=  $(\Psi_0, A(-x_n), \dots, A(-x_{j+1}) B(-y) A(-x_j), \dots, A(-x_1) \Psi_0)$  (36)

for all  $x_1, \ldots x_n$  and y. (The argument is that used in the proof of the PCT theorem.) Thus,

$$(\Psi, B(z)\Phi) = \sum_{k,\ell} \int \dots \int dy_1 \dots dy_\ell dx_1 \dots dx_k \cdot [g_\ell(y_1, \dots, y_\ell)]^*$$
  

$$\cdot f_k(x_1, \dots, x_k) (\Psi_0, A(y_0), \dots, A(y_1) B(z) A(x_1), \dots, A(x_k) \Psi_0)$$
  

$$= \sum_{k,\ell} \int \dots \int dx_1, \dots dx_k dy_1 \dots dy_\ell f_k(x_1, \dots, x_k)$$
  

$$\cdot [g_\ell(y_1, \dots, y_\ell)]^* (\Psi_0, A(-x_k), \dots, A(-x_1) B(-z) A(-y_1), \dots, A(-y_\ell) \Psi_0)$$
  

$$= (\Theta \Phi, B(-z) \Theta \Psi). \qquad (37)$$

Since by assumption states of the form (35) are dense in  $\mathcal{H}$ , (37) implies

$$\Theta \mathbf{B}(\mathbf{z}) \Theta^{-1} = \mathbf{B}(-\mathbf{z}); \tag{38}$$

i.e. B has a PCT operator which is the same as that of A. This implies the statements of the theorem.

It is worth noting that the last statement of the theorem is equivalent to the non-trivial result that the identities (33) linear in B imply the analogous identities with an arbitrary number of B's. When the identities (33) hold, we say, B is weakly local relative to A; or is WLC relative to A.

#### Theorem 4

Suppose A, B and C are WLC and A is irreducible. Let B be WLC relative to A and C be WLC relative to A, then B is WLC relative to C.

#### Proof

By theorem (3), A, B and C all have the same PCT operator, say  $\Theta$ , which implies immediately B is WLC relative to C. In fact, it implies that A, B and C altogether are WLC.

Theorems 3 and 4 together establish a kind of weakened transitivity for WLC. Recall that a relation r is transitive if arb and brc implies arc.

Theorem 5

If A is LC and irreducible and B is LC relative to A, i.e.

[A(x), B(y)] = 0

# for space-like x-y, then B is LC.

This theorem is a special case of the following (take B = C).

Theorem 6

If A is LC and irreducible and B and C are each LC relative to A,then B is LC relative to C; i.e.

$$[B(x), C(y)] = 0$$

for space-like x-y.

Proof

By Theorem 4, A, B and C are together WLC. From this and the assumptions of the theorem one gets for any  $x_1...x_n, y_1, y_2$  such that the set of successive difference vectors  $(x_1-x_2, x_{j-1}-x_j, x_j-y_1, y_1-y_2, y_2-x_{j+1},..., x_{n-1}-x_n)$  is a Jost point

$$(\Psi_{0}, A(x_{1})...A(x_{j}) B(y_{1}) C(y_{2}) A(x_{j+1})...A(x_{n}) \Psi_{0})$$
  
=  $(\Psi_{0}, A(x_{n})...A(x_{j+1}) C(y_{2}) B(y_{1}) A(x_{j})...A(x_{1}) \Psi_{0})$   
=  $(\Psi_{0}, A(x_{1})...A(x_{j}) C(y_{2}) B(y_{1}) A(x_{j+1})...A(x_{n}) \Psi_{0})$  (39)

(the first step by WLC; the second by assumption).

Now the first and third expressions in (39) are boundary values of analytic functions, being

 $\lim_{\substack{\eta_1,\ldots,\eta_{i-1},\eta',\eta''\to 0}} F^{(1)}([x_1-x_2-i\eta_1],\ldots,[x_{j-1}-x_j-i\eta_{j-1}],[x_j-y_1-i\eta_j],$ 

 $[y_1-y_2-i\eta'], [y_2-x_{j+1}-i\eta'], [x_{j+1}-x_{j+2}-i\eta_{j+1}] \dots$ 

 $[\mathbf{x}_{n-1} - \mathbf{x}_n - i\eta_{n-1}])$ 

and

$$\lim_{\substack{\eta_1,\ldots,\eta_{n-1}, \eta', \eta'' \to 0 \\ \text{in } V^+}} F^{(2)}([x_1 - x_2 - i\eta_1], \ldots, [(x_j - y_1 - i\eta_j) + (y_1 - y_2 - i\eta')],$$

$$- [y_1 - y_2 + i\eta''], [(y_1 - y_2) - i\eta' + (y_2 - x_{j+1} - i\eta')].... [x_{n-1} - x_n - i\eta_{n-1}]),$$

respectively. For the next step in the argument we use not the functions  $F^{(1)}$  and  $F^{(2)}$  but two functions derived from them by setting  $\eta' = 0$  and smearing in  $(y_1-y_2)$  with a test function  $\varphi$  where support consists entirely of space-like vectors  $f^{(j)} = \int \varphi(y_1-y_2) d(y_1-y_2) F^j(\ldots, y_1-y_2, \ldots)$ . The  $f^{(j)}$  are then analytic in  $\mathcal{J}_n$  in the variables  $x_1-x_2-i\eta_1, \ldots, [x_j-y_1-i\eta], \ldots, [y_2-x_{j+1}-i\eta''], \ldots, x_{n-1}-x_n-i\eta_{n-1}$ , and therefore the same is true of  $f = f^{(1)} - f^{(2)}$ . Furthermore, the boundary value of f vanishes in an open set of real vectors, at least if

the support of  $\varphi$  is sufficiently small. (This statement is obtained by smea ing (39) with  $\varphi$  in the variable  $y_1$ - $y_2$ .)

This vanishing of f's boundary values implies that  $f_1-f_2$  vanishes identically, so the first and third expressions in (39) are equal for all  $x_1 \dots x_n$  when  $y_1-y_2$  is space-like; thus

$$[B(y_1), C(y_2)] = 0$$
 for  $(y_1-y_2)^2 < 0$ .

The fact that f's boundary values vanishing in an open set implies f = 0is a generalization of a large class of theorems in one complex variable of which the <u>Theorem of the Brothers Riesz</u> is typical: let f(z) be analytic in the unit disc |z| = 1 and continuous on |z| = 1. If f(z) = 0 for |z| = 1 and arg z in an open interval, then f = 0 throughout the closed unit disc [44]. If one takes the "Edge of the wedge" theorem [10] for granted, one has an easy proof.  $f(\varphi, z_1 \dots z_n)$  is analytic in  $\mathcal{J}_n$ ,  $[f(\varphi, \bar{z}_1 \dots \bar{z}_n]^*$  in  $-\mathcal{J}_n$ , their boundary values coincide in an open set S of real space (and are zero!) and therefore f(z) is analytic there. Since the value in S is zero, f = 0. This implies tha the identity given by equating the first and third expressions in (39) is valid for all space-like  $y_1-y_2$  and all  $x_1 \dots x_n$ . Since A is irreducible, this mean that B is LC relative to C.

Now let us examine the question of the equality of the S operator for different fields. Borchers gives us the simple criterion.

#### Theorem 7

Let A be LC and irreducible and the same for B. Suppose

$$A^{in} = B^{in} \tag{40}$$

and the in fields are irreducible. Then the S-operator of the two theories is the same if A and B are together WLC.

#### Remarks

The theorem has been stated as though there were a single "in" field in each theory. This is by no means necessarily so, as will be seen from the proof. What is assumed is that the set of "in" fields for the two theories coincide and are determined by A and B in such a way that (41) and (42) below hold.

#### Proof

Suppose A and B are together WLC; then by the PCT theorem both have the same PCT operator  $\Theta$ . Then

$$\Theta A^{\text{in}}(\mathbf{x}) \Theta^{-1} = A^{\text{out}}(-\mathbf{x})$$
(41)

$$\Theta \mathbf{B}^{\mathrm{in}}(\mathbf{x}) \Theta^{-1} = \mathbf{B}^{\mathrm{out}}(-\mathbf{x}) \tag{42}$$

so  $A^{in} = B^{in}$  implies  $A^{out} = B^{out}$  and therefore  $S_A S_B^{-1}$  commutes with  $A^{in}$ , which implies  $S_A = S_B$  (since we normalize  $S_A \Psi_0 = S_B \Psi_0 = \Psi_0$ ).

Conversely, suppose

$$B^{out} = S^{-1}B^{in}S = S^{-1}A^{in}S = A^{out}$$
 (43)

Since A and B are LC, they have PCT operators  $\Theta_A$  and  $\Theta_B$ , respectively. Now  $\Theta_A$  and  $\Theta_B$  are uniquely determined by  $A^{in}$  and  $A^{out}$  and  $B^{in}$  and  $B^{out}$  via the relations

$$\Theta_A A^{in} (x) \Theta_A^{-1} = A^{out} (-x)$$
$$\Theta_B B^{in} (x) \Theta_B^{-1} = B^{out} (-x)$$

(the argument is always the same: assume two  $\Theta_A$  and  $\Theta'_A$ , say; then prove  $\Theta_A \Theta'_A$  commutes with  $A^{in}$ ). Therefore, by (43),  $\Theta_A = \Theta_B$ , and A and B are together WLC.

One can, of course, make this theorem "covariant". Assume instead of (40) that

$$A^{\text{in}} = R B^{\text{in}} R^{-1}, R \Psi_0 = \Psi_0 \text{ (deducible as usual);}$$
(44)

then in order that the theory of A and B should predict the same results for collision one wants

$$S_A = R S_B R^{-1} , \qquad (45)$$

because then

$$A^{out} = S_A^{-1} A^{in} S_A \text{ and } B^{out} = S_B^{-1} B^{in} S_B$$
(46)

are consistent with

$$A^{out} = R B^{out} R^{-1}$$
(47)

and the S matrix elements are the same in the two theories:

$$(A^{in}(x_1), \dots, A^{in}(x_j)\Psi_0, S_A A^{in}(x_{j+1}), \dots, A^{in}(x_n)\Psi_0)$$
  
=  $(B^{in}(x_1), \dots, B^{in}(x_j)\Psi_0, S_B B^{in}(x_{j+1}), \dots, B^{in}(x_n)\Psi_0),$  (48)

which is what is meant by predicting the same results for collisions.

Under assumption (44) one has merely to replace B by  $R^{-1}BR$  in Theorem 7 to get the appropriate criterion. The covariant form of Theorem 7 therefore reads: (45) follows if A and  $R_{in}^{-1}BR_{in}$  have the same PCT operator where  $A^{in} = R_{in} B^{in} R_{in}^{-1}$ . This is not the situation in practice which may be described as follows: Let

$$\Theta \mathbf{A}^{\text{in}}(\mathbf{x}) \Theta^{-1} = \mathbf{A}^{\text{out}}(-\mathbf{x}); \Theta \mathbf{B}^{\text{in}}(\mathbf{x}) \Theta^{-1} = \mathbf{B}^{\text{out}}(-\mathbf{x}).$$
(48a)

$$S_{A}^{-1} A^{in}(x) S_{A} = A^{out}(x); S_{B}^{-1} B^{in}(x) S_{B} = B^{out}(x)$$
 (48b)

$$R_{in}^{-1} A^{in}(x) R_{in} = B^{in}(x); R_{out}^{-1} A^{out}(x) R_{out} = B^{out}(x).$$
 (48c)

From (48a) and (48b)

$$S_A \Theta A^{in}(x) \Theta^{-1} S_A^{-1} = A^{in}(-x),$$

and therefore

$$[(S_A\Theta)^2, A^{in}(x)] = 0;$$

so, by the usual argument,

$$\Theta S_{\mathbf{A}} \Theta^{-1} = S_{\mathbf{A}}^{-1} \tag{48d}$$

and similarly

$$\Theta S_{\mathbf{R}} \Theta^{-1} = S_{\mathbf{R}}^{-1} \,. \tag{48e}$$

(This is the PCT symmetry of the S operator.) From (48a) and (48c),

$$\Theta \operatorname{R_{in}^{-1} A^{in}(x) R_{in}} \Theta^{-1} = \operatorname{R_{out}^{-1} } \Theta \operatorname{A^{in}(x) } \Theta^{-1} \operatorname{R_{out}^{-1}},$$

and so

$$\mathbf{R}_{\text{out}} = \Theta \, \mathbf{R}_{\text{in}} \, \Theta^{-1} \, . \tag{48f}$$

From (48b) and (48c)

$$S_{B}^{-1} R_{in}^{-1} A^{in}(x) R_{in} S_{B} = R_{out}^{-1} S_{A}^{-1} A^{in}(x) S_{A} R_{out};$$

so

$$R_{out} = S_A^{-1} R_{in} S_B . \qquad (48g)$$

Thus

$$S_{B} = R_{in}^{-1} S_{A} (\Theta R_{in} \Theta^{-1}).$$

$$(48h)$$

The results (48d) to (48h) follow from (48a), (48b) and (48c). Conversely, if  $\Theta A^{in}(x) \Theta^{-1} = A^{out}(-x)$  and  $S_A$  satisfies (48d), one can <u>define</u>  $R_{out}$  by (48f) and  $S_B$  by (48h); and then (48a), (48b) and (48c) will be satisfied for any unitary  $R_{in}$  that commutes with U(a, A). This shows that to get  $[\Theta, R_{in}] = 0$  and therefore the physical equivalence (48) of the operators  $S_A$  and  $S_B$ , one must use more details of the relationship between A, B,  $A_{out}^{in}$ ,  $B_{out}^{in}$  and  $\Theta$ . How this works out for the Haag-Ruelle collision theory will be discussed later.

The remaining step in Borchers' theory is as follows:

Theorem 8

Let A and B be LC and A be irreducible. Suppose B is LC relative to A. Then if A and B have asymptotic fields of the same mass,  $B^{in} = \pm A^{in}$ .

24

The proof as it stands in his paper uses the LSZ asymptotic condition and will not be reproduced here.

## 1.3. GENERALIZED FREE FIELDS AND THE SUPPORT PROPERTIES OF FIELDS IN MOMENTUM SPACE

In an effort to get out of the Borchers class of the free field, GREEN-BERG introduced the notion of <u>generalized free field</u> as any field A for which the commutator is a c-number [11]. The standard spectral representation then gives

$$[A(x), A(y)] = \int d\mu(a) (1/i) \Delta_a(x-y).$$
 (49)

It turns out that all the vacuum expectation values of a generalized free field are obtained from those of a free field of mass m by replacing the free propagator  $\frac{1}{2} \Delta_m^{+1}(\mathbf{x})$  by  $(1/i) \int d\mu(\mathbf{a}) \Delta_a(\mathbf{x})$ . Although generalized free fields are physically rather uninteresting, they illustrate a number of points of principle. For example, a generalized free field may be irreducible and its "in" and "out" fields exist according to LSZ prescriptions, but the "in" and "out" fields need not be irreducible. This makes evident a complication already mentioned before. The Borchers classes are not strictly equivalence classes unless one restricts one's attention to irreducible fields. Compare the result of Schroer alluded to just before equation (28) with that of Greenberg just quoted. One says that all elements of the equivalence classes of an irreducible free field of mass m are of the form (27); the other says that a reducible free field can have a generalized free field in its equivalence class and that generalized free field need not be of the form (27). When a generalized free field has an "in" field, it is LC relative to it so one does not get a new Borchers class except in pathological cases where no "in" fields exist.

A principal reason for discussing generalized free fields is that a number of elegant criteria have been given which guarantee that a field is a generalized free field. This gives some idea of what to avoid in trying to make a non-trivial theory.

Theorem 9 [12, 13, 14]

If A is LC and is irreducible and

$$[A(x), A(y)] = B(x-y)$$

(B may be an operator but must depend on x-y and not x+y), then A is a generalized free field.

 $Proof^*$ 

Consider

$$[B(x-y), A(z)] = \left[ [A(x+\xi), A(y+\xi)], A(z) \right],$$

which holds for all  $\xi$ .

By the Jacobi identity it is

$$-\left[\left[A(y+\xi), A(z)\right], A(x+\xi)\right] - \left[\left[A(z), A(x+\xi)\right], A(y+\xi)\right].$$

For sufficiently large space-like  $\xi$  this vanishes, so

$$[B(x-y), A(z)] = 0$$

and by the irreducibility of A, B must be a constant multiple of the identity operator so A is a generalized free field.

The second kind of criterion for a field to be a generalized free field relates to the support of the field in momentum space, i.e. the points of the spectrum of  $\widetilde{A}(p) = \int e^{ip \cdot x} A(x) dx$ . (This should not be confused with the spectrum of physical states, with which it is only indirectly connected.)

• There is another proof of Theorem 9 by J. Katzin [13a] which is about as neat as that by Licht and Toll. It goes as follows. Because the commutator is by assumption translation invariant

$$U(a)[A(x), A(y)] U(a)^{-1} = [A(x), A(y)]_{\bullet}$$

Then

#### $U(a)[A(x), A(y)] \Psi_0 = [A(x), A(y)] \Psi_0$

and therefore by the uniqueness of the vacuum

$$[A(x), A(y)] \Psi_0 = b(x-y) \Psi_0$$

where b is a c number. But then

$$(\Psi_0, A(x_1)...A(x_j)([A(x), A(y)] - b(x - y)]A(x_{j+1})...A(x_n) \Psi_0 = 0$$

for all lost points in the successive differences  $f(x_1...x_j xy x_{j+1}...x_n)$  and so by analytic continuation for all  $x_1...x_n$ . Therefore

$$[A(x), A(y)] = b(x - y)$$

Theorem 10 [12, 15, 16\*]

Let A be LC and have the vacuum as cyclic vector. If the spectrum of  $\widetilde{A}$  omits an open set of space-like p, then A is a generalized free field. Two local fields whose Fourier transforms agree on such a set differ only by a generalized free field in their Borchers class.

The results of Robinson and Greenberg have been quoted. Other cases are considered by Greenberg and Dell'Antonio. For example, it is shown that, if the spectral weight of the 2-fold vacuum expectation value vanishes above some mass, then the field is a generalized free field. The proofs involve a systematic use either of the Dyson representation or holomorphy envelope calculations. Since these techniques will not be explained here, the proofs will also not be given.

It is worth noting that, unlike the case in Theorem 2, smeared polynomials in generalized free field operators can be LC [11].

#### 1.4. THE CLUSTER DECOMPOSITION PROPERTY

Given a vacuum expectation value,

$$(A(x_1)...A(x_j)A(x_{j+1}+a)...A(x_n+a))_0$$
,

one would expect that, if  $a \rightarrow \infty$  in a space-like direction, it should approach

$$\langle A(x_1), \ldots, A(x_j) \rangle_0 \langle A(x_{j+1}), \ldots, A(x_n) \rangle_0$$

This can in fact be proved under appropriate assumptions and is an example of a cluster decomposition property. More refined statements can be obtained in which the  $x_1...x_n$  are divided into k clusters which are then allowed to separate.

The significance of cluster decomposition properties for the theory of collisions was first emphasized by HAAG [17], and one of the most significant developments mentioned here is the work by RUELLE [18], which puts Haag's arguments on a rigorous mathematical foundation. Ruelle's results are based on a proof that a very refined form of the cluster decomposition property can holds in any theory of local fields in which the vacuum is cyclic. Before going into detail, I shall give two neat results which show the power of the method. Of course, the required cluster decomposition properties will be assumed here.

Theorem 11 [19]

Let A and B be two fields which satisfy

\* Borchers has obtained a number of the same results independently.
but not necessarily LC. (They could be components of general spinor fields.) Suppose

$$[A(x), B(y)]_{+} = 0 = [A(x), B^{*}(y)]_{+}$$
(51)

hold for all space-like x-y.

Then either  $A(\phi)\Psi_0 = 0 = A(\phi)^* \Psi_0$  or  $B(\phi)\Psi_0 = 0 = B(\phi)^* \Psi_0$  for all test functions  $\phi$ . If A and B together have  $\Psi_0$  as a cyclic vector and belong to some sets of operators which transform under homogeneous Lorentz transformation like spinors, then either A = 0 or B = 0.

# Proof

Let  $\varphi$  and  $\psi$  be any two test functions of compact support whose supports are space-like with respect to one another. Taking

 $A(\phi) = \int dx \phi(x) A(x), B(\psi) = \int dy \psi(y) B(y),$ 

$$|| B (\psi) A(\phi)^* \Psi_0 ||^2 = (\Psi_0, A(\phi) B(\psi)^* B(\psi) A(\phi)^* \Psi_0)$$
$$= - (\Psi_0, B(\psi)^* B(\psi) A(\phi) A(\phi)^* \Psi_0).$$
(52)

If we let the support of  $\phi$  run off in a space-like direction, the last expression converges to

٠

- 
$$(\Psi_0, B(\psi)^* B(\psi) \Psi_0) (\Psi_0, A(\phi) A(\phi)^* \Psi_0)$$

by the cluster decomposition property. (This proves incidentally that the left-hand side also converges.) But  $(\Psi_0, B(\psi)B(\psi)^* \Psi_0)$  and  $(\Psi_0, A(\phi)A(\phi)^*\Psi_0)$  are non-negative, so either

$$A(\varphi)^* \Psi_0 = 0 \text{ or } B(\psi) \Psi_0 = 0$$

A precisely similar argument starting from  $||B(\psi)^*A(\varphi)^* \Psi_0||^2$  yields

$$A(\phi)^* \Psi_0 = 0 \text{ or } B(\psi)^* \Psi_0 = 0$$

Finally, starting from the adjoint of the relations (51), one has the same statements with  $A(\phi)^*$  replaced by  $A(\phi)$ . Thus either

$$A(\phi)^* \Psi_0 = 0 = A(\phi) \Psi_0 \text{ or } B(\psi)^* \Psi_0 = 0 = B(\psi) \Psi_0$$
. (53)

The last statement of the theorem is based on an argument which is, by now, standard. Look at an arbitrary vacuum expectation value:

$$(\Psi_0,\ldots,A(x),\ldots,B(y),\ldots,\Psi_0)$$
(54)

If all arguments are taken as space-like and the first of the alternatives (53) holds, take the farthest A or A\* to the right and move it through B's and B\*'s until it hits  $\Psi_0$ ; conclude that (54) vanishes for such space-like sepa-

28

rators. But the hypothesis on the transformation law of the A's and B's guarantees that the vacuum expectation values are analytic at Jost points, so the preceding argument shows all vacuum expectation values containing an A or an  $A^*$  are zero. Therefore A = 0.

This argument of Dell' Antonio actually first occurs in a slightly different connection in a paper by ARAKI [20]\* in which he discusses the possible commutation relations of different fields and shows that a theory with anomalous commutation relations, distinct integer spin fields anti-commuting or half-odd-integer spin fields commuting or integer spin fields anti-commuting with half-odd-integer spin fields, is always physically equivalent to one with normal commutation relations (all integer spin fields commute with each other and all half-odd-integer spin fields, all half-odd-integer spin fields anti-commute). These two papers together with the original BURGOYNE [21], LÜDERS-ZUMINO [22] proof bring the theorem of the connection of spin with statistics to a dazzling polish.

As a second application of the <u>cluster decomposition property</u>, an example of SUDARSHAN and BARDACKI [23] in which it is violated will be discussed.

Consider two theories of a neutral scalar field labelled respectively by 1 and 2: Hilbert spaces  $\mathcal{H}_i$ , vacua  $\Psi_{0i}$ , representations of the Lorentz group  $U_j(a, \Lambda)$ , fields  $A_j(x)$ . Form a new theory with Hilbert space  $\mathcal{H}_1 \oplus \mathcal{H}_2$ representations of the Lorentz group  $U_1 \oplus U_2$  and field  $A = A_1 \oplus A_2$ . In this theory, the state vectors are pairs  $\{\Psi_1, \Psi_2\}$  with the scalar product,

$$(\{\Psi_1, \Psi_2\}, \{\Phi_1, \Phi_2\}) = (\Psi_1, \Phi_1) + (\Psi_2, \Phi_2).$$

Clearly, there is a two-dimensional subspace of the Hilbert space  $\mathcal{H}_1 \oplus \mathcal{H}_2$ , each of whose vectors is left invariant by the representation of the Lorentz group:

 $(U_1(a,\Lambda) \oplus U_2(a,\Lambda)) (\alpha \{\Psi_{01},0\} + \beta \{0,\Psi_{02}\})$ =  $\alpha \{\Psi_{01},0\} + \beta \{0,\Psi_{02}\},$ 

which shows a grave defect of this theory; the vacuum ought to be unique. How does one recognize this defect in the vacuum expectation values? Pick a particular vacuum, say  $\Psi^{(\alpha)} = \sqrt{\alpha} \{\Psi_{01}, 0\} + \sqrt{1-\alpha} \{0, \Psi_{02}\}, \ 0 \le \alpha \le 1$ , and compute

$$(\Psi^{(\alpha)}, A(x_1), \dots, A(x_n) \Psi^{(\alpha)}) = \alpha \ (\Psi_{01}, A_1(x_1), \dots, A_1(x_n) \Psi_0) + (1-\alpha) \ (\Psi_{02}, A_2(x_1), \dots, A_2(x_n) \Psi_{02}) , \qquad (55)$$

This just gives the proposal of Sudarshan and Bardacki: one takes two theories and forms a new one whose vacuum expectation values are convex linear

<sup>\*</sup> Note that Araki does not show that the normal case is physically equivalent to the abnormal case, but rather that the abnormal case is necessarily very restricted. By virtue of its abnormal commutation relation it must have selection rules which in turn yield the result that it is physically equivalent to a normal case with the same selection rules.

combinations of the vacuum expectation values of the two theories. But (55) does not have the cluster decomposition property even if the theories of  $A_1$  and  $A_2$  do because

$$(\Psi^{(\alpha)}, A(x_1), \dots, A(x_j) A(x_{j+1}+a), \dots, A(x_n+a)\Psi^{(\alpha)})$$

$$= \alpha(\Psi_{01}, A_1(x_1), \dots, A_1(x_j) A_1(x_{j+1}+a), \dots, A_1(x_n+a) \Psi_{01})$$

$$+ (1-\alpha)(\Psi_{02}, A_2(x_1), \dots, A_2(x_j) A_2(x_{j+1}+a), \dots, A_2(x_n+a)\Psi_{02})$$

$$\rightarrow \alpha(\Psi_{01}, A_1(x_1), \dots, A_1(x_j) \Psi_{01}) (\Psi_{01}, A_1(x_{j+1}), \dots, A_1(x_n) \Psi_{01})$$

$$+ (1-\alpha) (\Psi_{02}, A_2(x_1), \dots, A_2(x_j) \Psi_{02}) (\Psi_{02}, A_2(x_{j+1}), \dots, A_2(x_n) \Psi_{02}),$$

whereas it ought to approach

$$(\Psi^{(\alpha)}, A(x_1), \dots, A(x_j) \Psi^{(\alpha)}) (\Psi^{(\alpha)}, A(x_{j+1}), \dots, A(x_n) \Psi^{(\alpha)})$$
  
=  $[\alpha (\Psi_{01}, A_1(x_1), \dots, A_1(x_j) \Psi_{01}) + (1 - \alpha) (\Psi_{02}, A_2(x_1), \dots, A_2(x_j) \Psi_{02})]$   
 $\cdot [\alpha (\Psi_{01}, A_1(x_{j+1}), \dots, A_1(x_n) \Psi_{01}) + (1 - \alpha) (\Psi_{02}, A_2(x_{j+1}), \dots, A_2(x_n) \Psi_{02})]$ 

Equating these two and assuming that some at least of the vacuum expectation values are non-zero, one finds  $\alpha = 0$  or 1; i.e. the only theories of this kind with cluster decomposition property are the original constituents. Of course, there are other things wrong with these models but the fundamental trouble is the non-uniqueness of the vacuum as was first shown by HEPP, JOST, RUELLE and STEINMANN [24]. Actually, BORCHERS [25] has shown that the cluster decomposition property is not only necessary but sufficient for the uniqueness of the vacuum, if there is at least one cyclic vacuum. This point will be discussed further in the next section.

A third application of the cluster decomposition property comes about as follows. The author considers that finding non-trivial examples of internally consistent field theories is one of the most important problems of the subject at the present moment. One approach to this problem which might be attempted is to simplify it mathematically without losing its essential nature. For example, suppose one assumes that U contains only the vacuum and one irreducible representation. Can one find local fields which transform according to (4)? The answer is no, if  $\Psi_0$  is cyclic:

Theorem 12 [26]

In a theory of a neutral scalar field with cyclic vacuum, the physical spectrum must be additive.

# Remark

A point p lies in the physical spectrum if for each open set W of four momenta containing p there is a non-zero vector whose energy momentum spectrum lies in W. That the spectrum is additive means  $p_1$  in the spectrum, and  $p_2$  in the spectrum implies  $p_1 + p_2$  in the spectrum.

# Proof

Let  $S_1$  be an open neighbourhood of  $p_1$  and  $S_2$  an open neighbourhood of  $p_2$ . The first step in the proof consists in choosing field operators  $B_1(x)$  and  $B_2(x)$  satisfying

$$U(a, 1) B_i(x) U^{-1}(a, 1) = B_i(x + a)$$
 (56)

and test functions  $\phi_j$  which have Fourier transforms with supports in  $S_1$  and  $S_2\,$  such that

$$B_1(\varphi_1) \Psi_0 \neq 0 \text{ and } B_2(\varphi_2) \Psi_0 \neq 0.$$
 (57)

It follows from (56) that the energy momentum spectra of these vectors are in  $S_1$  and  $S_2$ , respectively. (Note that  $U(a, 1)B_j(\phi_j)\Psi_0 = B_j(\{a, 1\}\phi_j)\Psi_0$  where  $(\{a, 1\}\phi_j)(x) = \phi_j(x-a)$ , so a momentum analysis of the vector is equivalent to a momentum analysis of  $\phi_i$ .)

To get the required B's, choose open neighbourhoods  $T_1$  and  $T_2$  of  $p_1$ and  $p_2$ , respectively, such that the closures  $\overline{T}_1$  and  $\overline{T}_2$  satisfy  $\overline{T}_1 \subset S_1, \overline{T}_2 \subset S_2$ . Let  $\mathcal{H}_{\overline{T}_j}$  be the closed subspace of  $\mathcal{H}$  consisting of all vectors whose spectrum lies in  $\overline{T}_i$ . Then because  $\Psi_0$  is cyclic there exist vectors of the form

$$\sum_{n=1}^{N} \int \dots \int f_{nj}(x_1, \dots, x_n) A(x_1) \dots A(x_n) dx_1 \dots dx_n \Psi_0$$
 (58)

which are respectively not orthogonal to  $\mathcal{H}_{\overline{T}_j}$ . Define

$$B_{j}(x) = \sum_{n=1}^{N} \int \dots \int f_{nj}(x - x_{1}, \dots, x - x_{n}) A(x_{1}) \dots A(x_{n}) dx_{1} \dots dx_{n}.$$
 (59)

Then clearly (56) holds. (Quantities of the form (59) are called almost local fields by Haag.) Let  $\tilde{\phi_j}$  have a support in  $S_j$  that includes  $\overline{T_j}$ . Then  $B_j(\phi_j)\Psi_0 \neq 0$  for some such  $\phi_j$ ; otherwise (58) would be orthogonal to  $\mathcal{H}_i$   $\overline{T_j}$ . Thus the required  $B_j(\phi_i)\Psi_0 \neq 0$  can be constructed.

Now consider the vectors

$$B_1(\phi_1) U(a, 1) B_2(\phi_2) \Psi_0$$
.

Their support must lie in  $S_1 + S_2$  by the same argument as before. Can they vanish for all a? To prove not, assume the contrary:

$$0 = ||B_{1}(\varphi_{1}) U(a, 1) B_{2}(\varphi_{2}) \Psi_{0}||^{2}$$
  
=  $\langle B_{2}(\varphi_{2})^{*} U(a, 1)^{*} B_{1}(\varphi_{1})^{*} B_{1}(\varphi_{1}) U(a, 1) B_{2}(\varphi_{2}) \rangle_{0}$  (60)

Now apply the cluster decomposition property in a stronger form than be-

fore. It is asserted and will be discussed in detail later that, as a  $\rightarrow \infty$  in a space-like direction, (60) converges to

$$\langle B_2(\varphi_2)^* B_2(\varphi_2) \rangle_0 \langle B_1(\varphi_1)^* B_1(\varphi_1) \rangle_0$$
,

so either  $B_2(\phi_2) \Psi_0 = 0$  or  $B_1(\phi_1) \Psi_0 = 0$  is a contradiction. Therefore,  $p_1 + p_2$  lies in the spectrum.

To get a neat statement of the required cluster decomposition property it is advisible to introduce the notion of the <u>truncated part</u> of a vacuum expectation value [27]. This is defined by induction:

$$\langle A(\mathbf{x}) \rangle_0 = \langle A(\mathbf{x}) \rangle_{0T} ,$$

$$\langle A(\mathbf{x}_1) A(\mathbf{x}_2) \rangle_0 = \langle A(\mathbf{x}_1) A(\mathbf{x}_2) \rangle_{0T} + \langle A(\mathbf{x}_1) \rangle_{0T} \langle A(\mathbf{x}_2) \rangle_{0T} ,$$

$$\langle A(\mathbf{x}_1) A(\mathbf{x}_2) A(\mathbf{x}_3) \rangle_0 = \langle A(\mathbf{x}_1) A(\mathbf{x}_2) A(\mathbf{x}_3) \rangle_{0T} + \langle A(\mathbf{x}_1) A(\mathbf{x}_2) \rangle_{0T} \langle A(\mathbf{x}_3) \rangle_{0T}$$

$$+ \langle A(\mathbf{x}_1) A(\mathbf{x}_3) \rangle_{0T} \langle A(\mathbf{x}_2) \rangle_{0T} + \langle A(\mathbf{x}_2) A(\mathbf{x}_3) \rangle_{0T} \langle A(\mathbf{x}_1) \rangle_{0T}$$

$$+ \langle A(\mathbf{x}_1) \rangle_{0T} \langle A(\mathbf{x}_2) \rangle_{0T} \langle A(\mathbf{x}_3) \rangle_{0T} ,$$

$$(61)$$

or generally

$$\langle A(x_1), \ldots, A(x_n) \rangle_0 = \Sigma \prod \langle A(x_j) \rangle_{0T}$$
, (62)

where the sum is overall partitions of  $1 \ldots n$  into non-empty subsets and the product is over the truncated vacuum expectation values of the subsets, all x's occurring in the subsets in the order they occur in  $1 \ldots n$ . The definition works both for the almost local fields defined by (59) and for the field A.

The truncated part calculated in perturbation theory is just the sum of all connected diagrams. The various cluster decomposition properties can be stated thus: the truncated parts go to zero as their arguments separate (under various conditions).

The actual calculation for (60) is the following:

$$\langle B_{2} (\varphi_{2})^{*} B_{1} (\{-a,1\}\varphi_{1})^{*} B_{1} (\{-a,1\}\varphi_{1}) B_{2} (\varphi_{2}) \rangle_{0}$$

$$= \langle B_{2} (\varphi_{2})^{*} B_{1} (\{-a,1\}\varphi_{1})^{*} B_{1} (\{-a,1\}\varphi_{1}) B_{2} (\varphi_{2}) \rangle_{0T}$$

$$+ \langle B_{2} (\varphi_{2})^{*} \rangle_{0T} \langle B_{1} (\{-a,1\}\varphi_{1})^{*} B_{1} (\{-a,1\}\varphi_{1}) B_{2} (\varphi_{2}) \rangle_{0T}$$

$$+ \langle B_{1} (\{-a,1\}\varphi_{1})^{*} \rangle_{0T} \langle B_{2} (\varphi_{2})^{*} B_{1} (\{-a,1\}\varphi_{1}) B_{2} (\varphi_{2}) \rangle_{0T}$$

$$+ \langle B_{1} (\{-a,1\}\varphi_{1}) \rangle_{0T} \langle B_{2} (\varphi_{2})^{*} B_{1} (\{-a,1\}\varphi_{1})^{*} B_{2} (\varphi_{2}) \rangle_{0T}$$

$$+ \langle B_{2} (\varphi_{2}) \rangle_{0T} \langle B_{2} (\varphi_{2})^{*} B_{1} (\{-a,1\}\varphi_{1})^{*} B_{1} (\{-a,1\}\varphi_{1}) \rangle_{0T}$$

$$+ \langle B_{2} (\varphi_{2})^{*} B_{1} (\{-a,1\}\varphi_{1})^{*} \rangle_{0T} \langle B_{1} (\{-a,1\}\varphi_{1}) B_{2} (\varphi_{2}) \rangle_{0T}$$

$$+ \langle B_{2}(\varphi_{2})^{*} B_{1}(\{-a,1\}\varphi_{1})\rangle_{0T} \langle B_{1}(\{-a,1\}\varphi_{1})^{*} B_{2}(\varphi_{2})\rangle_{0T}$$

$$+ \langle B_{2}(\varphi_{2})^{*} B_{2}(\varphi_{2})\rangle_{0T} \langle B_{1}(\{-a,1\}\varphi_{1})^{*} B_{1}(\{-a,1\}\varphi_{1})\rangle_{0T}$$
(i)
$$+ \langle B_{2}(\varphi_{2})^{*}\rangle_{0T} \langle B_{2}(\varphi_{2})\rangle_{0T} \langle B_{1}(\{-a,1\}\varphi_{1})^{*} B_{1}(\{-a,1\}\varphi_{1})\rangle_{0T}$$
(ii)
$$+ \langle B_{2}(\varphi_{2})^{*}\rangle_{0T} \langle B_{1}(\{-a,1\}\varphi_{1})\rangle_{0T} \langle B_{1}(\{-a,1\}\varphi_{1})^{*} B_{2}(\varphi_{2})\rangle_{0T}$$

$$+ \langle B_{2}(\varphi_{2})^{*}\rangle_{0T} \langle B_{1}(\{-a,1\}\varphi_{1})^{*}\rangle_{0T} \langle B_{1}(\{-a,1\}\varphi_{1}) B_{2}(\varphi_{2})\rangle_{0T}$$

$$+ \langle B_{1}(\{-a,1\}\varphi_{1})^{*}\rangle_{0T} \langle B_{1}(\{-a,1\}\varphi_{1})\rangle_{0T} \langle B_{2}(\varphi_{2})^{*} B_{2}(\varphi_{2})\rangle_{0T}$$
(iii)
$$+ \langle B_{1}(\{-a,1\}\varphi_{1})^{*}\rangle_{0T} \langle B_{2}(\varphi_{2})\rangle_{0T} \langle B_{2}(\varphi_{2})^{*} B_{1}(\{-a,1\}\varphi_{1})\rangle_{0T}$$

$$+ \langle B_{1}(\{-a,1\}\varphi_{1})\rangle_{0T} \langle B_{2}(\varphi_{2})\rangle_{0T} \langle B_{2}(\varphi_{2})^{*} B_{1}(\{-a,1\}\varphi_{1})\rangle_{0T}$$

$$+ \langle B_{1}(\{-a,1\}\varphi_{1})\rangle_{0T} \langle B_{1}(\{-a,1\}\varphi_{1})\rangle_{0T} \langle B_{2}(\varphi_{2})^{*} \delta_{0T} \langle B_{2}(\varphi_{2})\rangle_{0T}$$
(iiii)

Of all these terms only the numbered ones (i), (ii), (iii), (iiii) are constant in a; the rest go to zero as  $a \rightarrow \infty$  in a space-like direction, because the expressions separate into two clusters.

Clearly, here one needs the cluster decomposition property for almost local fields rather than the local fields of which the almost local fields are constructed. This will be developed later.

A much stronger result than Theorem 12 can be derived from the work of Ruelle described below. It can be shown that U necessarily contains as sub-representation the representation belonging to the theory of free fields, one for each irreducible representation contained in U. This shows that there are no non-trivial mathematical idealizations of local field theory which simplify U. U must be as complicated as physics tells us it is in a theory of particles.

# PART TWO

This part will be quite precise mathematically and will begin with axioms for a theory of scalar fields.

# 2.1. AXIOMS AND THE RECONSTRUCTION THEOREM

Such a theory has a continuous unitary representation of the restricted inhomogeneous Lorentz group  $\{a, \Lambda\} \rightarrow U(a, \Lambda)$  and a unique vacuum,  $\Psi_0$ , in a separable Hilbert space  $\mathcal{H}$ . A field is a linear function A with domain  $\mathcal{H}$ , and values linear operators in  $\mathcal{H}$ . It is assumed:

I. As  $\phi$  runs over  $\mathscr{D}_{\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!}$   $A(\phi)$  and  $A(\phi)^*$  possess a common linear dense domain D such that

$$\begin{array}{ll} A(\phi) D \subset D & A(\phi)^* D \subset D \\ \Psi \in D & U(a, \Lambda) D \subset D \end{array}$$
(63)

A is an operator valued distribution in the sense that for each  $\Phi, \Psi \in D$ ,  $(\Phi, A(\phi) \Psi)$  is a distribution in  $\mathcal{D}$ , i.e. a continuous linear functional on  $\mathcal{D}$ . II. On D

$$U(a, \Lambda) A(\varphi) U(a, \Lambda)^{-1} = A(\{a, \Lambda\}\varphi)$$
(64)

III. On D

$$[A(\phi), A(\psi)] = 0 = [A(\phi), A(\psi)^*]$$
(65)

for  $\varphi, \psi \in \mathcal{D}$  such that

$$\varphi(x)\psi(y) = 0 \text{ for } (x-y)^2 \ge 0$$
 (66)

If

$$A(\varphi)^* = A(\bar{\varphi}) \text{ on } D, \qquad (67)$$

A is called neutral or Hermitian.

It follows directly from I that the vacuum expectation values

$$(\Psi_0, A_{j_1}(\phi_1), \dots, A_{j_n}(\phi_n) \Psi_0)$$

are multilinear functionals in  $\varphi_1 \dots \varphi_n$  separately continuous in their arguments. The Schwartz Nuclear Theorem asserts that these functionals can be uniquely extended by continuity to be distributions in the n variables [28].

$$\int d\mathbf{x}_1 \dots d\mathbf{x}_n \phi(\mathbf{x}_1 \dots \mathbf{x}_n) (\Psi_0, \mathbf{A}_{j_1}, (\mathbf{x}_1) \dots \mathbf{A}_{j_n}, (\mathbf{x}_n) \Psi_0).$$

Conversely, as was shown some time ago, one can take a set of distributions satisfying certain conditions and construct a theory having just those for vacuum expectation values [29]. The only reason for talking about this now is that these have significant recent improvements in the sharpness of this reconstruction theorem.

Let us briefly recapitulate the conditions for a single neutral scalar field. Then the vacuum expectation values may be labelled

$$F^{(n)}(x_1-x_2,\ldots,x_{n-1}-x_n) = (\Psi_0, A(x_1),\ldots,A(x_n),\Psi_0)$$

where n = 0, 1... From (67) and hermiticity:

$$(\Psi_0, A(\phi_1), \dots, A(\phi_n) \Psi_0) = [(\Psi_0, A(\phi_n)^*, \dots, A(\phi_1)^* \Psi_0)]^*;$$
  
and  $\mathbf{F}^{(n)}(\xi_1, \dots, \xi_{n-1}) = [\mathbf{F}^{(n)}(\xi_{n-1}, \dots, \xi_1)]^*.$  (68)

# The hermiticity conditions

From Schwartz's inequality

$$\begin{aligned} \left| \Psi_{0}, \int \varphi_{1}(\mathbf{x}_{1}, \ldots, \mathbf{x}_{j}) \, \mathrm{d}\mathbf{x}_{1}, \ldots, \mathrm{d}\mathbf{x}_{j} \, \mathrm{A}(\mathbf{x}_{1}), \ldots, \mathrm{A}(\mathbf{x}_{j}) \, \mathrm{U}(\mathbf{a}, \mathbf{I}) \right. \\ \left. \cdot \int \varphi_{2}\left(\mathbf{x}_{j+1}, \ldots, \mathbf{x}_{n}\right) \, \mathrm{A}(\mathbf{x}_{j+1}), \ldots, \mathrm{A}(\mathbf{x}_{n}) \, \Psi_{0}\right) \right| \\ \leq \left| \left| \int [\varphi_{1}(\mathbf{x}_{1}, \ldots, \mathbf{x}_{j})]^{*} \, \mathrm{d}\mathbf{x}_{1}, \ldots, \mathrm{d}\mathbf{x}_{j} \, \mathrm{A}(\mathbf{x}_{j})^{*}, \ldots, \mathrm{A}(\mathbf{x}_{1})^{*} \, || \right. \\ \left. \cdot \left| \left| \int \varphi_{2}\left(\mathbf{x}_{j+1}, \ldots, \mathbf{x}_{n}\right) \, \mathrm{d}\mathbf{x}_{j+1}, \ldots, \mathrm{d}\mathbf{x}_{n} \, \mathrm{A}(\mathbf{x}_{j+1}), \ldots, \mathrm{A}(\mathbf{x}_{n}) \, \Psi_{0} \, || \right. \right| \right. \end{aligned}$$

$$(69)$$

this shows that

$$(\Psi_0, \int \varphi_1(\mathbf{x}_1, \ldots, \mathbf{x}_j) \, d\mathbf{x}_1, \ldots, d\mathbf{x}_j \, \mathbf{A}(\mathbf{x}_1), \ldots, \mathbf{A}(\mathbf{x}_j) \, \mathbf{U}(\mathbf{a}, 1)$$

$$\int \varphi_2(\mathbf{x}_{j+1}, \ldots, \mathbf{x}_n) \, d\mathbf{x}_{j+1}, \ldots, d\mathbf{x}_n \, \mathbf{A}(\mathbf{x}_{j+1}), \ldots, \mathbf{A}(\mathbf{x}_n) \Psi_0)$$
(70)

is bounded in a. Since it is also infinitely differentiable in a (moving U(a,1) to the right, it can be expressed as a translation of  $\varphi_2$  which is infinitely differentiable) we can Fourier transform it and find that the Fourier transform is zero except for p in the physical spectrum [30]. (These are the spectral conditions.) The boundedness of (70) in a also has the consequence that  $F^{(n)}(\varphi_1,\ldots,\varphi_{n-1})$  can be extended to a continuous linear functional on A, the space of infinitely differentiable functions which, together with their derivatives, vanish at infinity faster than any power of the distance. (Continuity is then defined in the standard manner of SCHWARTZ [31].) Finally,

$$\lim_{a\to\infty} (70) = (\Psi_0, \int \varphi_1(\mathbf{x}_1, \ldots, \mathbf{x}_j) \, d\mathbf{x}_1, \ldots \, d\mathbf{x}_j \, \mathbf{A}(\mathbf{x}_1), \ldots, \mathbf{A}(\mathbf{x}_j) \, \Psi_0)$$

$$\cdot (\Psi_0, \int \varphi_2(\mathbf{x}_{j+1}, \ldots, \mathbf{x}_n) \, d\mathbf{x}_{j+1}, \ldots \, d\mathbf{x}_n \, \mathbf{A}(\mathbf{x}_{j+1}), \ldots, \mathbf{A}(\mathbf{x}_n) \, \Psi_0),$$

which expressed as a property of  $F^n$  is

$$\lim_{a \to \infty} F^{(n)}(\phi_1, \dots, \{a, 1\} \phi_j, \dots, \phi_{n-1}) = F^{(j)}(\phi_1, \dots, \phi_{j-1})$$

$$\cdot F^{(n-j)}(\phi_j, \dots, \phi_{n-1}), \quad (71)$$

as  $a \rightarrow \infty$  in a space-like direction. This is the <u>cluster decomposition property</u> [27, 30, 33].

Lastly, because

$$\left|\left|\Sigma \alpha_{\mathbf{k}} A(\varphi_{\mathbf{k}1}) \dots A(\varphi_{\mathbf{k}\mathbf{k}})\Psi_{\mathbf{0}}\right|\right|^{2} \ge 0$$
,

for any finite set of complex numbers  $\alpha_{\rm b}$ ,

$$\sum_{\mathbf{k}, \boldsymbol{\ell}} \bar{\alpha}_{\mathbf{k}} \alpha_{\boldsymbol{\ell}} \int \dots \int [\phi_{\mathbf{k}\mathbf{k}} (\mathbf{x}_{\mathbf{k}}), \dots \phi_{\mathbf{k}1} (\mathbf{x}_{1})]^{*} [\phi_{\boldsymbol{\ell}1} (\mathbf{y}_{1}), \dots \phi_{\boldsymbol{\ell}\boldsymbol{\ell}} (\mathbf{y}_{\boldsymbol{\ell}})]$$

$$\cdot \mathbf{F}^{(\mathbf{k}+\boldsymbol{\ell})} (\mathbf{x}_{\mathbf{k}} - \mathbf{x}_{\mathbf{k}+1}, \dots, \mathbf{x}_{2} - \mathbf{x}_{1}, \mathbf{x}_{1} - \mathbf{y}_{1}, \mathbf{y}_{1} - \mathbf{y}_{2}, \dots, (\mathbf{y}_{\boldsymbol{\ell}-1} - \mathbf{y}_{\boldsymbol{\ell}})$$

$$\cdot d\mathbf{x}_{\mathbf{k}} \dots d\mathbf{x}_{1} d\mathbf{y}_{1}, \dots d\mathbf{y}_{\boldsymbol{\ell}} \ge 0$$
(72)

these are usually referred to as the positive definiteness conditions.

Now the reconstruction theorem can be stated precisely.

Theorem 13

For each n = 0, 1, 2,.... let  $F^{(n)}$  be a distribution in  $\mathcal{B}'$  depending on (n-1) four-vector variables and invariant under the transformations

$$\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{n-1} \rightarrow \Lambda \boldsymbol{\xi}_1, \ldots, \Lambda \boldsymbol{\xi}_{n-1}$$

Suppose the  $F^{(n)}$  are extendable to  $\mathscr{J}$  in each of their arguments, the others being held fixed. If the  $F^{(n)}$  satisfy the hermiticity conditions, the spectral conditions, the positive definiteness conditions and the cluster decomposition property, then there exists a Hilbert space  $\mathcal{H}$ , a continuous unitary representation of the Lorentz group  $\{a, \Lambda\} \rightarrow U(a, \Lambda)$  with energy-momentum spectrum in or on the future light cone and unique vacuum  $\Psi_0$ , and a Hermitian scalar field  $A(\varphi)$  satisfying Axioms I and II with  $D = D_0$  and such that

$$(\Psi_0, A(x_1), \ldots, A(x_n)\Psi_0) = F^{(n)}(x_1 - x_2, \ldots, x_{n-1} - x_n).$$

This realization is unique up to unitary equivalence.

Axiom III is also satisfied if in addition the  $F^{(n)}$  satisfy the local commutativity conditions.

The proof will not be given here; it is the same as in [29] or [25], except for the uniqueness of the vacuum which is obtained from [25].

# 2.2. $\mathcal{D}$ VERSUS $\mathcal{S}$ AS DEFINITION DOMAIN FOR A( $\varphi$ ); DISCUSSION OF D; SELF ADJOINTNESS FOR HERMITIAN FIELDS

Those things which could be proved by assuming test functions in  $\mathscr{D}$ and those which also required assuming the fields defined for test functions in  $\mathscr{S}$  were not very carefully distinguished in Part One. Clearly some of the constructions required the latter, for example, that in the proof of Theorem 12. Physically, it is very natural to assume fields defined for test functions in  $\mathscr{D}$ : then  $A(\varphi)$ ,  $\varphi$  real would describe a field measurement in a bounded region of space time. It would be very satisfactory if one could prove from this that  $A(\varphi)$  could be extended to  $\mathscr{S}$ . Fields defined for test functions in  $\mathscr{S}$  are desirable for a very practical reason. They permit one to use Fourier transforms freely and to derive dispersion relations for scattering amplitudes. It should be borne in mind that what one is excluding in such a proof that fields can be extended to  $\mathscr{S}$  is worse than polynomial growth in x-space. The argument in connection with the spectral conditions (just before Theorem 13) shows that the vacuum expectation values are bounded in any one difference variable with the others held fixed. So the worse-than-exponential growth to be excluded appears only when two or more difference vectors go to infinity simultaneously. Such a growth is wildly implausible behaviour for a quantity which measures correlations between field measurements in the vacuum.

On the other hand, field quantities do behave in a way which would lead one to use test functions with compact support in p-space rather than x-space.

One finds in the perturbation theory of unrenormalized field theories evidence that one must expect momentum space vacuum expectation values which would grow faster than any power of the momentum. To make sense of these one needs test functions of compact support in p-space and therefore entire functions of exponential growth in x-space. The idea that one should adapt the axioms to such possibilities has been urged particularly by GÜTTINGER [34]. It provides a natural way of making the distinction between renormalizable and unrenormalizable theories independent of any detailed classification of Lagrangians.

Let us now discuss the domain D, again a subject which was glossed over in Part One. The first natural question is: Why not simplify the problem by assuming the field operators are everywhere defined, i.e.  $D = \mathcal{H}_c$ ? The answer is that for  $\varphi$  real (and therefore A( $\varphi$ ) Hermitian) this would imply that A( $\varphi$ ) is a bounded and therefore continuous operator, i.e.

 $\sup_{||\Phi||=1} ||A\Phi|| < \infty.$  This happens to be false for the free field, and there is every reason to believe that interesting theories should be worse rather than better than the free field. Thus D must not be all of  $\mathcal{H}$ . The best we can hope for is that the Hermitian unbounded  $A(\varphi)$  are self-adjoint,  $A(\varphi)^* = A(\varphi)$ . But it is known that such operators are everywhere discontinuous on their domain of definition, so it appears that one must face up to unbounded discontinuous operators.

Recall that the <u>adjoint</u> of an operator T with dense <u>domain</u>  $D(T) \subset \mathcal{H}$ , and <u>range</u>  $R(T) \subset \mathcal{H}_2$  and <u>graph</u>  $\Gamma_T$ , consisting of all pairs  $\{\Phi, T\Phi\}$  with  $\Phi \in D(T)$  is the uniquely defined linear operator T\* from  $\mathcal{H}_2$  to  $\mathcal{H}_1$  whose graph  $\Gamma_T*$  is  $\{-\Psi^*, \Psi\}$  where  $\{\Psi^*, \Psi\}$  runs over the orthogonal complement of  $\Gamma_T$  in  $\mathcal{H}_1 \oplus \mathcal{H}_2$ . That means that  $\Psi$  lies in  $D(T^*)$  and  $T^*\Psi = \Psi^*$ , if for all  $\Phi \in D(T)$ 

$$(\Psi^*, \Phi) = (\Psi, T\Phi).$$

An operator T is <u>Hermitian</u> if  $T \in T^*$ , i.e. if  $D(T) \in D(T^*)$  and  $T = T^*$  on D(T). An operator T is <u>self-adjoint</u> if  $T = T^*$ . It is <u>essentially self-adjoint</u> if  $T^{**} = T^*$ . A self-adjoint operator cannot be extended to any other vector without losing the property  $T = T^*$ . A useful criterion for the essential self-adjointness of an Hermitian operator is that there are no solutions of the equations:

In general, when T is Hermitian the number of linearly independent solutions of these two equations are respectively the <u>defect indices</u> of T. If the defect indices of T are equal, then T possesses at least one self-adjoint extension. Evidently, in the first half of these notes the precise distinctions made in this paragraph were not noted, but they will be from here on [35].

The very best we can presume for the operators  $A(\varphi)$ ,  $\varphi$  real, is that they are essentially self-adjoint on the domain  $D_0$ , whose vectors are of the form  $P(A(\psi), \ldots) \Psi_0$  where P is a polynomial in the smeared operators for  $\psi \in \mathscr{D}$ . Clearly,  $D_0 \in D$ , so I write  $A(\varphi)|_{D_0}$  for the restriction of  $A(\varphi)$  to  $D_0$ . Written out, the required essential self-adjointness is

$$[A(\varphi)|_{D_{\alpha}}]^{**} = [A(\varphi)|_{D_{\alpha}}]^{*}.$$

It is possible to prove this for the free field.

Theorem 14

If A is a free field and  $\varphi$  is real and  $\in \mathscr{S}$  then  $A(\varphi)|_{D_0}$  is essentially selfadjoint.

The proof is not long but makes very explicit use of a configuration space realization of the free field [36].

For a general field satisfying I, II or I, II and III, there is no such result proved at present. However, one can prove that the defect indices of  $A(\phi)|_{D_0}$  are equal. In outline, the proof is as follows: From the discussion just before Theorem 13, it follows that  $F^{(n)}$  is the boundary value of an analytic function in each of its variables, the others being held fixed and smeared with test functions in  $\mathscr{S}$ . The analyticity in question is in the tube  $\mathcal{J}$ . It then follows from a theorem of ZERNER [37]\* that there exists a unique function analytic in  $\mathcal{J}_{n-1}$  which reduces to  $F^{(n)}$ . This function is invariant under the homogeneous Lorentz group so that one can use the theorem of Hall to prove the PCT theorem as at the beginning of Part One Thus the PCT theorem is valid for an irreducible field satisfying I, II and III. The PCT operator  $\Theta$  leaves  $D_0$  invariant.

Now suppose  $\varphi$  is not only real but even under  $x \rightarrow -x$ . Then  $\Theta$  satisfies

$$\Theta A(\varphi) |_{D_{\varphi}} \Theta^{-1} = A(\varphi) |_{D_{\varphi}}$$

But then if  $\Phi$  satisfies

$$(A(\varphi)|_{D_{\varphi}})^{*} \Phi = \pm i\Phi,$$
  
$$(A(\varphi)|_{D_{\varphi}})^{*} \Theta \Phi = \mp i\Theta \Phi.$$

 $\Theta \Phi$  will satisfy

(If  $\Theta$  commutes with  $A(\varphi)$  and leaves  $D_0$  invariant, it maps  $D_0$  one to one onto itself and commutes with  $A(\varphi)|_{D_0}^*$  as can easily be verified directly from

<sup>\*</sup> In the simplest case of two complex variables Zemer's result is as follows: if  $f(x_1, z_2)$  is analytic for  $z_2 > 0$  for each real value of  $x_1$  and  $f(z_1, x_2)$  is analytic in  $z_1 > 0$  for each real  $x_2$  and  $f(x_1, x_2)$  is continuous, then there exists a unique function f analytic for  $z_1 > 0$  and  $z_2 > 0$  which reduces to the given data on  $z_1 = 0$ ,  $z_2 \ge$  and  $z_2 = 0$ ,  $z_1 \ge 0$ .

the definitions.) Thus, there are as many solutions with the plus sign as with the minus sign and the defect indices of  $A(\phi)$  are equal when  $\phi$  is real and even. The general case of  $\phi$  real is easily reduced to this.

There does not appear to be any evidence against the conjecture that  $A(\phi)|_{D_0}$  is essentially self-adjoint in the general case. At the moment, however, the best we have is the following:

#### Theorem 15

If  $\varphi$  is real and  $\varphi \in \mathscr{D}$  and A is an irreducible field satisfying I, II and III, then  $A(\varphi)|_{D_{\varphi}}$  has equal defect indices and therefore possesses at least one self-adjoint extension.

The importance of self-adjointness is that it makes available one of the most powerful tools for the study of operators in Hilbert space, the spectral theorem. If  $\widehat{A}(\phi)$  is a self-adjoint extension of  $A(\phi)|_{D_{\alpha}}$ , then

$$\widehat{A}(\varphi) = \int_{-\infty}^{\infty} \lambda \, dE(\lambda, \varphi),$$

where  $E(\lambda, \varphi)$  is a spectral resolution.

There may be physical requirements which single out a particular selfadjoint extension (for example, LC for the extended operators). If it turns out that even after these additional requirements have been applied the  $A(\phi)|_{D_0}$  do not possess unique self-adjoint extensions, one will have to say that the theory is not completely given by its vacuum expectation values. This would not be a catastrophe.

There is one additional simple remark about domains: The extension of the vacuum expectation values from multilinear functionals  $(\Psi_0, A(\phi_1), \ldots, A(\phi_n)\Psi_0)$  to distributions in all the variables,

$$\int dx_1 \dots dx_n \phi(x_1, \dots, x_n) (\Psi_0, A(x_1), \dots, A(x_n)\Psi_0),$$

permits an analogous extension for vectors:

$$A(\varphi_1).\ldots A(\varphi_n)\Psi_0 \rightarrow \int dx_1\ldots dx_n \varphi(x_1,\ldots,x_n)A(x_1)\ldots A(x_n)\Psi_0.$$
(73)

This last expression is then a vector valued distribution where continuity for the vectors is in the norm topology of Hilbert space [18, 30]. This permits an extension of the operators  $A(\varphi)$  to the domain D of all vectors such as (73).

# 2.3. VON NEUMANN ALGEBRAS ASSOCIATED WITH A DOMAIN OF SPACE-TIME AND A FIELD

It is natural to try to associate an algebra of bounded operators with the field. (This is the reverse situation from that customary in mathematics where one is given an algebra of bounded operators and associates un-

bounded operators with it.) HAAG has particularly emphasized the significance of associating an algebra of bounded operators  $R(\theta)$  with the set of field operators  $A(\phi)$  where the supports of the  $\phi$  lie in a fixed domain  $\theta$  of space-time [38].

There would be a straightforward way to define  $\mathcal{R}_{o}(\theta)$  if we knew that the  $A(\varphi)|_{D_{o}}$  were essentially self-adjoint: take the von Neumann algebra generated by the spectral projections of the self-adjoint operators  $(A(\varphi)|_{D_{o}})^{*}$ . (Recall that a von Neumann algebra is a set  $\mathcal{R}_{o}$  of bounded operators with the properties:  $\mathbf{l} \in \mathcal{R}_{o}$ ; if  $A \in \mathcal{R}_{o}$ , then  $A^{*} \in \mathcal{R}_{o}$ ; if A and  $B \in \mathcal{R}_{o}$ , then AB and a  $A+b B \in \mathcal{R}_{o}$ ; if  $A_{n}$  (where n = 1, 2, ...) is a weakly convergent sequence of operators  $\in \mathcal{R}_{o}$ , then  $\lim A_{n} \in \mathcal{R}_{o}$ .) This definition would still work with our present knowledge but might give different  $\mathcal{R}_{o}(\theta)$  depending on which self-adjoint extension of  $A(\varphi)|_{D_{o}}$  is used. Alternatively one can proceed as follows [18]. Define: C, a bounded operator, commutes with  $A(\varphi)$  if

$$(A(\varphi)^* \Phi, C\Psi) = (\Phi, CA(\varphi) \Psi)$$
(74)

for all  $\Phi, \Psi$  in D. Then define  $X \in \mathcal{P}_{\Phi}(\theta)$  if X commutes with all C that satisfy (14) for every  $A(\phi)$  and  $A(\phi)$  with support of  $\phi$  in  $\theta$ . The relations among the various possible definitions are well worth exploring. The first steps in this direction are in [39]. One particular result is so simple and important that it must be given here [40].

Theorem 16

Let A be a neutral field satisfying I and II, but with test functions in  $\checkmark$  (including as usual the requirement that the vacuum be unique). Suppose  $\Psi_0$  is cyclic. Then A is irreducible in the sense that any operator C satisfying

$$(A(\phi)^* \Phi, C\Psi) = (\Phi, CA(\phi) \Psi)$$
(75)

for all  $\varphi \in \mathscr{D}$  and all  $\Phi, \Psi \in D_0$  is a constant multiple of the identity.

Proof

If (75) holds for the  $A(\phi)$ , it also holds with  $A(\phi)$  replaced by

$$\sum_{n}\int\ldots\int dx_{1}\ldots dx_{n}\phi_{n}(x_{1}\ldots x_{n}) A(x_{1})\ldots A(x_{n}),$$

a fact that will be used in a moment.

Now it may be assumed that  $C\Psi_0 \neq 0$  because, if  $C\Psi_0 = 0$ ,  $C\Psi = 0$  for any  $\Psi \in \mathcal{D}_0$  and therefore C = 0.

Write  $||C\Psi_0|| = \rho > 0$ ,  $(\Psi_0, C\Psi_0) = \alpha$ . Schwartz's inequality then implies  $|\alpha| \le \rho$ . To prove the required result it suffices to show  $|\alpha| = \rho$ , because then  $C\Psi_0 = \alpha\Psi_0$  and this implies  $C\Phi = \alpha\Phi$  for all  $\Phi \in D_0$ , because C commutes with the  $A(\phi)$  according to (75).

Because  $\Psi_0$  is cyclic a polynomial exists in the smeared fields, say  $\rho$ , such that  $||(C-\rho)\Psi_0|| < \epsilon$ . Then

$$|(\Psi_0, C^* C \Psi_0) - (\Psi_0, \mathcal{O}^* C \Psi_0)| = |((C - \mathcal{O}) \Psi_0, C \Psi_0)| < \rho \epsilon .$$
(76)

So far the commutation relation (75) has not been used.

Now analyse the form of  $\mathscr{O} \Psi_0$  in momentum space.  $\mathscr{O}$  may have a pspace support which runs over all of p-space; but when it is applied to  $\Psi_0$ , all of the contribution save that from the physical spectrum is annihilated. By multiplying the Fourier transform of the test function occurring in  $\mathscr{O}$  by a function which is 1 on the physical spectrum and zero for points which are in the negative of the continuous spectrum, one can get a new operator,  $\widetilde{\mathscr{O}}$ , of the same form as  $\mathscr{O}$ , which satisfies

$$\widehat{\mathcal{O}} \Psi_0 = \widehat{\mathcal{O}} \Psi, \quad \widehat{\mathcal{O}}^* \Psi_0 = (\widehat{\mathcal{O}} \Psi_0, \Psi_0) \Psi_0. \quad (77)$$

(Crudely, what is being done is this: Replace

$$\langle \mathbf{p} | \boldsymbol{\rho} | \mathbf{q} \rangle \operatorname{by} \langle \mathbf{p} | \boldsymbol{\hat{\rho}} | \mathbf{q} \rangle = \theta(\mathbf{p}^{0} - \mathbf{q}^{0})\theta((\mathbf{p} - \mathbf{q})^{2}) \langle \mathbf{p} | \boldsymbol{\rho} | \mathbf{q} \rangle;$$

$$\langle \mathbf{p} | \boldsymbol{\hat{\rho}}^{*} | \mathbf{q} \rangle = \theta(\mathbf{q}^{0} - \mathbf{p}^{0}) \theta((\mathbf{q} - \mathbf{p})^{2}) \langle \overline{\mathbf{q}}^{<} | \boldsymbol{\rho} | \mathbf{p} \rangle,$$

then

$$\langle \mathrm{p}| \mathscr{P} \left| 0 \right\rangle \ = \langle \mathrm{p}| \widehat{\mathscr{P}} \left| 0 \right\rangle \mathrm{but} \langle \mathrm{p}| \widehat{\mathscr{P}}^* \left| 0 \right\rangle = \left. heta(-\mathrm{p}^0) \, heta \left( \mathrm{p}^2 \right) \left\langle \overline{0| \, \mathscr{P}| \mathrm{p}} \right\rangle,$$

which can only be different from zero when p = 0 because  $\langle 0 | \mathcal{P} | p \rangle = 0$  unless p is in the physical spectrum. Actually,  $\theta$  has to be replaced by an infinitely differentiable function, so we need the hypothesis that p = 0 is an isolated point of the spectrum in order to get enough room for the smoothed  $\theta$  to fall to zero from the value 1 it has at 0.) Then, using (75),

$$\rho \epsilon > \left|\rho^{2} - \left(\widehat{\mathcal{O}} \Psi_{0}, C\Psi_{0}\right)\right| = \left|\rho^{2} - \left(\widehat{\mathcal{O}} \Psi_{0}, C\Psi_{0}\right)\right| = \left|\rho^{2} - \left(\Psi_{0}, C\widehat{\mathcal{O}}^{*} \Psi_{0}\right)\right|$$
$$= \left|\rho^{2} - \alpha \left(\widehat{\mathcal{O}} \Psi_{0}, \Psi_{0}\right)\right|.$$
(78)

But  $\epsilon$  can be chosen arbitrarily small; and when it is,  $(\mathscr{P} \Psi_0, \Psi_0)$  is arbitrarily close to  $\overline{\alpha}$ . Therefore  $|\alpha| = \rho$ .

A second remarkable result of this type has been produced by REEH and SCHLIEDER [39].

# Theorem 17

Suppose A is a field satisfying I and II with  $D_0$  dense in  $\mathcal{H}$  (test functions in  $\mathcal{D}$ ).  $D_0(\theta)$  is also dense for any open set of space-time  $\theta$ .  $D_0(\theta)$  is the set of all vectors of the form  $\mathcal{P}(A(\varphi), \ldots) \Psi_0$  where P is a polynomial in the fields smeared with test functions whose supports lie in  $\theta$ .

# Proof

A matrix element of the form

$$(\chi, A(x_1), \ldots, A(x_n) \Psi_0)$$

is the (distribution!) boundary value of an analytic function G of the vari-

41

ables  $-x_1 - i\eta_0$ ,  $x_1 - x_2 - i\eta_1$ , ...,  $x_{n-1} - x_n - i\eta_{n-1}$  defined in  $\mathcal{J}_n$ . This follows immediately from the arguments described above in connection with the proof of the PCT theorem under the weakened hypothesis that test functions are in  $\mathcal{O}$ . But then the hypothesis of the theorem implies that the boundary value of G is zero in an open set of real space. Thus by the argument given in the proof to Theorem 6, G vanishes everywhere in  $\mathcal{J}_n$  and therefore so do its boundary values ( $\chi$ , A( $x_1$ )..., A( $x_n$ ) $\Psi_0$ ). Since D<sub>0</sub> has been assumed dense, we see that  $\chi$  orthogonal to D<sub>0</sub>( $\theta$ ) implies  $\chi = 0$ , so the theorem is proved.

One might think that, by combining the arguments of the preceding theorem with the present one, one could prove the irreducibility of the set of operators  $\mathscr{P}(A(\varphi)...)$  with  $\varphi$  restricted to have support in any fixed open set of space-time. However, this is not and cannot be so because the result is false. As was first shown by HAAG and SCHROER [41], there are generalized free fields such that the set of  $\mathscr{P}(A(\varphi))$  is irreducible when  $\varphi$ ranges over all  $\mathscr{B}$  but the set of  $\mathscr{P}(A(\varphi)...)$  is not irreducible when the supports of the  $\varphi$  are restricted to lie in any time slice  $-\infty \le a \le x^0 \le b \le \infty$ . The reason the proof does not go through is that the construction of the  $\widehat{\mathscr{P}}$ used in (78) requires test functions  $\varphi$  which cannot be of compact support in x space.

# 2.4. HAAG-RUELLE COLLISION THEORY; GENERAL ACCOUNT

The first step in Haag's theory is the construction of what he calls almost local fields. These are quantities of the form

$$B(x) = \sum_{n} \int \dots \int f_n (x - x_1, \dots, x - x_n) A(x_1) \dots A(x_n) dx_1 \dots dx_n$$
(79)

which satisfy

$$U(a,\Lambda) B(x) U(a,\Lambda)^{-1} = B(\Lambda x + a)$$

 $(\Psi_0, B(x) \Psi_0) = 0.$ 

where  $f_n \in A$ . We assume finite sums in (79). At one time or another Haag has considered using some kind of limit of finite sums but that does not appear to be necessary and has not been possible till now. Furthermore, it is desirable that for each irreducible representation contained in U, say of mass m<sub>i</sub>, there exists an almost local field such that  $B_i(x)\Psi_0$  lies in the subspace of that irreducible representation. (This actually implies  $(\Psi_0, B(x)\Psi_0) = 0.)$  Haag refers to the construction of almost local operators satisfying these requirements as the "solution of the one-body problem". It would seem that neither Haag nor Ruelle tells one in print how to "solve the one-body problem". It is clear that under some circumstances it can always be done. Suppose, for example, that the discrete mass state in question is isolated in the mass spectrum. Then the construction used in the proof of Theorem 12 will yield the required  $B_i$ . The same holds true even if the discrete mass value is not isolated, provided that conserved quantum numbers exist which label the fields and the mass value is isolated in the subspace of states with definite values of the quantum numbers. The sort of

#### AXIOMATIC FIELD THEORY

thing meant here is, say, the case of the deuteron which lies in the middle of the mass continuum if all states are considered, but which is isolated if one confines one's attention to states of baryon number 2. It should always be possible to "solve the one-body problem" with sufficient accuracy, so that the following calculations would work, but the author has not carried out the details. (The idea is that although  $B_i(x)\Psi_0$  is not a pure one-particle state the left-over piece can be made sufficiently small not to matter.) For the purpose of the present exposition it is assumed that one can "solve the one-body problem" exactly.

Now define

$$B_{i}^{f}(\mathbf{x}_{i}^{0}) = i \int d\mathbf{x}_{i} \left[ f_{i}(\mathbf{x})^{*} \frac{\partial}{\partial \mathbf{x}_{i}^{0}} \quad B_{i}(\mathbf{x}_{i}) - \frac{\partial}{\partial \mathbf{x}_{i}^{0}} f_{i}(\mathbf{x}_{i})^{*} B_{i}(\mathbf{x}_{i}) \right], \quad (80)$$

where the Fourier transform of  $f_i$  is of the form

$$heta(\mathrm{p}^0)\,\delta\,(\mathrm{p}^2\,\text{-}\,\mathrm{m}_\mathrm{i}^2\,)\,\widehat{\mathrm{f}}\,(\overline{\mathrm{p}})\,\mathrm{with}\,\widehat{\mathrm{f}}oldsymbol{\in}\,oldsymbol{\mathscr{D}}$$
 .

Then Haag's assertion is as follows:

Theorem 18

Let  $B_i$  be an almost local field such that  $B_i(x_i)$  lies in the subspace of  $\mathcal{H}$  belonging to the irreducible representation  $[m_i, s_i]$  of mass  $m_i$  and spin  $s_i$ . Form the states,

$$\Phi(t) = \prod B_i^f(t) \Psi_0;$$

then  $\lim_{t\to\pm\infty} \Phi(t)$  exists in norm.

Proof

Note first that  $\frac{d\Phi}{dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} [\Phi(t + \Delta t) - \Phi(t)]$  exists where the limit is to be understood in the norm. This is an immediate consequence of the continuity properties discussed earlier in connection with the domain D. Furthermore, in order to verify the strong convergence of  $\Phi(t)$  it is sufficient to prove that  $|t|^{3/2} ||d\Phi/dt|| \to 0$  as  $t \to \pm \infty$ , because then

$$\left|\left|\Phi\left(t'\right)-\Phi\left(t''\right)\right|\right|=\left|\left|\int_{t'}^{t''} d\tau \frac{d\Phi(\tau)}{d\tau}\right|\right| \leqslant \left|\int_{t'}^{t''} d\tau \left|\left|\frac{d\Phi(\tau)}{d\tau}\right|\right|\right| \leqslant C \left|\int_{t'}^{t'''} \frac{d\tau}{\tau^{3/2}}\right|,$$

and this can be made arbitrarily small for sufficiently large t' and t''. Thus, to prove the theorem it is sufficient to prove

$$\left|t\right|^{3/2} \left|\left|d\Phi / dt\right|\right| \rightarrow 0$$
.

Now  $\left| d\Phi / dt \right|$  can be written out as a sum of terms of the form

$$\int d\vec{x_1} d\vec{x_2} \dots d\vec{x_k} f_1(\vec{x_1}, t) f_2(\vec{x_2}, t) \dots f_k(\vec{x_k}, t) F(\vec{x_1} - \vec{x_2}, \dots, \vec{x_{k-1}} - \vec{x_k}), (81)$$

where two of the  $f_j$  are actually time derivatives of the f's appearing in the theorem and F is the vacuum expectation value of the  $B_i$  fields. Note that F is time independent because  $x_i^0 = x_j^0 = t$ . F can now be expanded in terms of truncated vacuum expectation values. Then (81) appears as a sum of products of integrals which are again of the form (81); however, now F stands for a truncated vacuum expectation value.

There are now two steps in the proof. First, one must establish that  $\sup_{x \to 0} |f_j(\vec{x}, t)| \leq C/|t|^{3/2}$  for large (t) and

$$\int d\vec{x} \left| f(\vec{x}, t) \right| \leq C_1 \left| t \right|^{3/2}.$$

Secondly, it must be shown that the (truncated) F's fall off faster than any power of

$$\sum_{j=1}^{k-1} |x_j - x_{j+1}|^2 \text{ for } k \ge 2.$$

If both these things have been established, then (81) will decrease as  $|t|^{(-s/2)(k-2)}$ . It remains to show that no terms with k = 2 contribute. This is a result of the hypothesis that the B's "solve the one-body problem". The two steps in the proof will be returned to in the two following sections.

Some remarks about the relativistic invariance of the procedure are necessary here. What has to be shown at this point is that the same limiting state is arrived at if one carries out the same procedures along another time-like direction. For this it suffices to show that  $(1+i\epsilon \vec{n}, \vec{N}) \Phi(t)$  yields the same result as  $\Phi(t)$ , where  $\vec{n}, \vec{N}\epsilon$  is an infinitesimal pure Lorentz transformation along the direction  $\vec{n}$ . The term  $\vec{n}, \vec{N}\Phi(t)$  will give rise to no contribution in the limit because it will involve one extra derivative of the term which approached a constant in the preceding calculation.

The next step is to define "in" and "out" operators on the "in" and "out" states which have just been defined. One writes

$$B_{in}^{f} \Phi_{in} = \lim_{t \to \pm \infty} B^{f}(t) \Phi(t) ,$$

$$\left(B_{in}^{f}\right)^{*} \Phi_{in} = \lim_{t \to \pm \infty} (B^{f}(t))^{*} \Phi(t) .$$
(82)

To be sure that these equations actually define linear operators one has only to check the single valuedness; i.e. suppose  $\Psi(t) = \sum_{j=0}^{f} \Phi_j(t)$  and  $\Psi_{in} = 0$ or  $\Psi_{out} = 0$ , then one must have  $\lim_{t \to \pm \infty} B^f(t) \Psi(t) = 0$  for the appropriate case. But the families of vectors  $\Psi(t)$  and  $(B^f(t))^* B^f(t) \Psi(t)$  both have a strong limit, that of the first family being zero. Therefore  $\lim_{t \to \pm \infty} (\Psi(t), (B^{i}(t))^{\star} B^{f}(t)\Psi(t)) = 0$ , so  $B_{in}^{f} \Phi_{in} = 0$  or  $B_{out}^{f} \Phi_{out} = 0$ , whichever is appropriate. The  $B_{in}^{f}$  and  $B_{out}^{f}$  and their adjoints are respectively defined on the "in"

The  $B_{in}^{f}$  and  $B_{out}^{o}$  and their adjoints are respectively defined on the "in" and "out" states which span two subspaces of Hilbert space  $\mathcal{H}_{in}$  and  $\mathcal{H}_{out}$  respectively.

We have no assurance that  $\mathcal{H}_{in} = \mathcal{H}_{out}$  nor that  $\mathcal{H}_{in} = \mathcal{H} = \mathcal{H}_{out}$  at the present stage, and in fact examples show that the asymptotic states need not be complete. (There are generalized free fields such that  $\mathcal{H}_{in} \neq \mathcal{H}$  and  $\mathcal{I}_{i} \neq \mathcal{H}_{out}$ .) That is Axiom IV (Ruelle):

IV.  $\mathcal{H}_{in} = \mathcal{H} = \mathcal{H}_{out}$ 

Notice that  $\Theta \Phi_{in}$  is an "out" state; thus if  $\chi$  is orthogonal to  $\mathcal{H}_{in}$ , then  $\Theta \chi$  is orthogonal to  $\mathcal{H}_{out}$ . Thus it suffices to assume  $\mathcal{H}_{in} = \mathcal{H}_{out}$  to get  $\mathcal{H}_{out} = \mathcal{H}_{out}$ .

The  $B_{in}$  and  $B_{out}$  which have been defined are associated with the correct discrete masses m but do not have any simple transformation law under Lorentz transformation. Ruelle's next step is to extract from the B-free spinor fields with the appropriate transformation law under Lorentz transformations to describe particles of spin  $s_i$ . The construction will not be described here, but the author believes that this is the first place where the collision theory of particles of arbitrary spin has been treated systematically in so-called axiomatic field theory.

There is one subject not explored in Ruelle's paper where further investigation would seem very valuable. That is the relation between the domains of the operators  $B_{in}$ ,  $B_{out}$  and the domain of the original operators A. A typical problem here would be whether one can show that all these operators can be extended to the subspace of  $\mathcal{H}$  consisting of all states whose energy is less than  $E < \infty$ .

# 2.5. ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF THE KLEIN GORDON EQUATION [18]

An important role was played in Haag's original argument for the asymptotic condition by an estimate of the asymptotic behaviour for large times of the solutions of the Klein Gordon equation:

$$\frac{1}{(2\pi)^{3/2}} \int e^{-ik.x} \tilde{f}(k) d\Omega(k)$$

$$\sim \sqrt{m} i^{-3/2} (1 - \vec{v}^2)^{-3/4} \exp\left[-imt(1 - \vec{v}^2)^{1/2}\right] t^{-3/2} \tilde{f}(m\vec{v}(1 - \vec{v}^2)^{-1/2}),$$
where  $\vec{v} = \vec{x}/t$ . (83)

This was one of the weak points of Haag's argument, because the class of functions for which it is valid was not determined. Ruelle replaces this by the following:

Lemma

Let f be the solution of the Klein Gordon equation  $(\Box + m^2)f(x) = 0$  given by

$$f(x) = (2\pi)^{-2} \int dp \theta(p^0) \delta(p^2 - m^2) \widetilde{f}(\vec{p}) e^{-ip \cdot x}$$
(84)

where  $\tilde{f}(\vec{p})$  is infinitely differentiable and of compact support. Then f is infinitely differentiable and  $f(\lambda u)$  goes to zero as  $\lambda \to +\infty$  in two different ways depending on whether the vectors  $\lambda u$  (where  $0 < \lambda < \infty$ ) intersect the support of  $\delta(p^2 - m^2)\tilde{f}(\vec{p})$  or not; such vectors determine a cone C.

- (a) If  $u \in C$ ,  $|f(\lambda u)| \leq A(u) \lambda^{-\frac{3}{2}} \qquad 0 \leq \lambda \leq \infty$  (85) where A(u) is continuous;
- (b) If  $u \notin C$ ,  $\lim_{\lambda \to +\infty} \lambda^n |f(\lambda u)| = 0 \qquad \text{for all } n = 0, 1, 2, \dots \qquad (86)$

and uniformly for u in compact subsets of  $(u^0)^2 + \vec{u}^2 = 1$ .

## Remark

It is helpful to recall the Riemann Lebesgue Lemma and one of its proofs in order to see why the cone C appears. Consider

$$f(x) = \int e^{ikx} dk \widetilde{f}(k)$$

and suppose  $\widetilde{f}$  is integrable and has an integrable derivative. Then

$$\begin{aligned} \mathbf{f}(\mathbf{x}) &= \int \widetilde{\mathbf{f}}(\mathbf{k}) \, \mathrm{d}\mathbf{k} \, (1/\mathrm{i}\mathbf{x}) \frac{\mathrm{d}}{\mathrm{d}\mathbf{k}} \, (\mathrm{e}^{\mathrm{i}\,\mathbf{k}\,\mathbf{x}}) = \frac{\mathrm{i}}{\mathrm{x}} \, \int \frac{\mathrm{d}\,\widetilde{\mathbf{f}}(\mathbf{k})}{\mathrm{d}\mathbf{k}} \mathrm{d}\mathbf{k} \, \, \mathrm{e}^{\mathrm{i}\,\mathbf{k}\,\mathbf{x}} \\ &\text{so } \left| \mathbf{f}(\mathbf{x}) \right| \leq \left( \int \left| \mathrm{d}\,\widehat{\mathbf{f}}(\mathbf{k})/\mathrm{d}\mathbf{k} \right| \mathrm{d}\mathbf{k} \right) / \left| \mathbf{x} \right| \, . \end{aligned}$$

This procedure can be repeated if  $\tilde{f}$  has more integrable derivatives each yields one more power of |x| in the denominator.

For an integral of the form

$$\int e^{i\sqrt{k^2 + m^2} x} \widetilde{f}(k) dk$$

the situation is different because

$$\frac{1}{ix} \left[ \frac{\sqrt{k^2 + m^2}}{k} \right] \frac{d}{dk} \left( e^{i\sqrt{k^2 + m^2}x} \right) = e^{i\sqrt{k^2 + m^2}x}$$

and the square bracket is singular at 0. Thus the previous argument cannot be repeated indefinitely.

Proof

(84) can be written

$$f(x) = [2(2\pi)^2]^{-1} \int d\Omega_m(p) e^{-ipx} \tilde{f}(\vec{p}) , \qquad (87)$$

where the integral runs over  $p^2 = m^2$ ,  $p^0 > 0$  and  $d\Omega_m(p) = dp^3 / \sqrt{\vec{p}^2 + m^2}$ . Because the integral runs over a compact subset of  $\vec{p}$  space, one can differen-





tiate with respect to  $x^{\mu}$  under the integral sign and always get convergent integrals. Therefore f(x) is infinitely differentiable.

To study the asymptotic behaviour in  $\lambda$  when x =  $\lambda u$ , rewrite (87) as

$$f(\lambda u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-is\lambda} \widetilde{f}_{u}(s) ds , \qquad (88)$$

$$\widetilde{f}_{u}(s) = \frac{1}{2(2\pi)^{\sqrt{2}}} \int d\Omega(p) \,\delta(s - p \cdot u) \,\widetilde{f}(\vec{p}) \,. \tag{89}$$

Now  $s = p \cdot u$  is a 3 plane with normal u. It intersects the hyperboloid in a two-dimensional surface, which is the Lorentz transform of a sphere if u is plus time-like and s is sufficiently large (Fig. 1). They do not intersect

where

for sufficiently small s and in the transition case the plane is tangent to the hyperboloid. For light-like u the plane intersects in a two-dimensional surface which runs to infinity; the same is true for space-like u. When the  $\delta$  function is eliminated, there appears in the remaining integral over the curve a Jacobian which is analytic in s as long as s does not take the value for which the plane becomes tangent. If the support of  $\tilde{f}$  does not contain the  $\vec{p}$  of the point of tangency,  $f_u(s)$  is infinitely differentiable. Since whatever u is,  $\tilde{f}_u(s)$  is of compact support because the integrand will get too singular at k = 0. If the support of  $\tilde{f}$  does not include zero, however, the preceding argument is valid. The analogue of  $\vec{K}=0$  in the integral is  $p\alpha \lambda u$ , which shows that one expects different behaviour for  $u \in C$  and for  $u \notin C$ . (88) shows that  $f(\lambda u)$  vanishes faster than any power of the distance. Furthermore, it will be uniformly continuous in u as long as u stays away from C. This establishes (b).

To prove (a) note that under the assumption  $u \in C$ , u is plus time-like, so by a Lorentz transformation it can be brought into the time axis. Then choosing for convenience u = (1, 0, 0, 0), we get for (89)

$$\widetilde{f}_{u}(s) = \frac{1}{2(2\pi)^{\frac{3}{2}}} \int \frac{d^{3}p}{\sqrt{p^{2} + m^{2}}} \delta(s - \sqrt{\vec{p}^{2} + m^{2}}) \widetilde{f}(\vec{p})$$

$$= \frac{1}{4(2\pi)^{\frac{3}{2}}} \sqrt{s^{2} - m^{2}} \theta(s - m) \int d\omega(\vec{p}) \widetilde{f}(\vec{p})$$

$$= \sqrt{s - m} \widetilde{g}(s - m), \qquad (90)$$

where  $\widetilde{g}$  (s-m) is infinitely differentiable and of compact support on the closed half axis  $0 \le s \le \infty$ . Then

$$f(\lambda u) = (2\pi)^{-1/2} \int_m^\infty ds \, e^{-is\lambda} \sqrt{s-m} \, \widetilde{g}(s-m)$$
$$= (2\pi)^{-1/2} \, e^{-im\lambda} \int_0^\infty ds \, e^{-is\lambda} \sqrt{s} \, \widetilde{g}(s) \, .$$

Write

$$\sqrt{s} \widetilde{g}(s) = \sqrt{s} \widetilde{g}(0) e^{-s} + \sqrt{s} (\widetilde{g}(s) - \widetilde{g}(0) e^{-s}).$$

The contribution from the first can be done exactly because

$$\int_{0}^{\infty} e^{-ist} e^{-s} \sqrt{s} \, ds = \sqrt{\pi} i \left[1 + it\right]^{-s/2} , \qquad (91)$$

while the second has two integrable derivatives, so that its Fourier transform is bounded in absolute value by  $a(u) |\lambda|^{-2}$ . Thus

$$\left|f(\lambda u)\right| \leq A(u) \left|\lambda\right|^{-\sqrt{2}}$$
(92)

Here A(u) can be taken to be continuous because the integral varies continuously under Lorentz transformations.

This Lemma has as an immediate consequence the following:

Lemma

If f satisfies the hypotheses of the preceding Lemma, then

$$\begin{split} \sup_{\vec{x}} & \left| f(x^0, \vec{x}) \right| \text{ decreases as } \left| x^0 \right|^{-3/2} \text{ when } x^0 \to +\infty \\ & \int d \, \vec{x} \, \left| f(x^0, \vec{x}) \right| \text{ does not increase faster than } (x^0)^{3/2} \\ & \text{ when } x^0 \to +\infty \,. \end{split}$$

Proof

and

Because of the uniformity of the estimates in u one has that  $\sup_{x \in C} |f(x^0, \vec{x})|$ decreases as  $|x^0|^{-3/2}$ .

The intersection of the plane  $x^0 = \text{const.}$  with C is a compact set  $C_1$  of three-space which lie inside a sphere of radius  $\langle x^0 \rangle$ . The integral  $\int d\vec{x} |f(x^0, \vec{x})|$  can be split into an integral over  $C_1$  and over the rest of space. The contribution from the rest goes to zero faster than any power of  $x^0$ , while  $|\int_{C_1} |\leq \text{const.} |x_0|^{-3/2} |x^0|^3$ .

#### 2.6. THE REFINED CLUSTER DECOMPOSITION THEOREM [18]

First, a notation must be introduced to describe the n + 1 clusters:

$$A_{i}(x_{i}) = A(x_{i_{0}}) A(x_{i_{1}}) \dots A(x_{i_{r}(i)})$$
 (93)

(The second index labels the points within a cluster;  $\underline{x}_i$  stands for the set of vector variables  $x_{i_0} \dots x_{i_{f(i)}}$ ,  $i = 0, \dots, n$ .) Define also

$$A_i(x_i + a_i) = U(a_i, 1) A_i(x_i) U(a_i, 1)^{-1}$$
. (94)

(If we had to deal with a set of fields A, an analogous definition could be made by putting an extra index on  $\underline{A}_i$  to indicate what the constituent fields are in the i<sup>th</sup> cluster.  $\underline{A}_i$  would then be called a <u>Bose</u> or <u>Fermi field</u> respectively if the product contained an even or odd number of anti-commuting fields.) (93) will be referred to as a <u>cluster product</u> and (94) as a <u>trans-</u> lated cluster product.

The translated cluster products will appear in vacuum expectation values in different orders, and the next bit of notation labels these vacuum expectation values. Let  $\pi$  be the element (permutation) of the symmetric group on n + 1 objects such that  $\pi(0, 1, \ldots, n) = (i_0, \ldots, i_n)$  (and  $\alpha_{\pi} = \pm 1$  according to whether, when acting on  $\underline{A}_1, \ldots, \underline{A}_n$  the permutation of the Fermi fields is even or odd). Then define

$$T^{\pi} (\underline{\mathbf{x}} + \mathbf{a}) = T^{\pi} (\underline{\mathbf{x}}_{0} + \mathbf{a}_{0}, \underline{\mathbf{x}}_{1} + \mathbf{a}_{1}, \dots, \underline{\mathbf{x}}_{n} + \mathbf{a}_{n})$$
$$= \sigma_{\pi} < \underline{A}_{i_{0}} (\underline{\mathbf{x}}_{i_{0}} + \mathbf{a}_{i_{0}}) \underline{A} (\underline{\mathbf{x}}_{i_{1}} + \mathbf{a}_{i_{1}}) \dots \underline{A}_{i_{n}} (\underline{\mathbf{x}}_{i_{n}} + \mathbf{a}_{i_{n}}) >_{0}, \quad (95)$$

$$\mathbf{F}_{\underline{\varphi}}^{\pi}(\mathbf{a}) = \int d\underline{\mathbf{x}} \ \underline{\varphi}(\underline{\mathbf{x}}) \ \mathbf{T}^{\pi}(\underline{\mathbf{x}} + \mathbf{a})$$
(96)

where  $\underline{\phi} \in \mathcal{J}$  in the  $\sum_{k=0}^{n} [r(i_k) + 1]$  vector variables,

$$x_{00}x_{01}\ldots x_{or(o)}\ldots x_{no}\ldots x_{nr(n)}$$

Note that in (95) and (96)  $\underline{x}$  stands for the set  $\underline{x}_i$ ,  $i = 0, \ldots n$ , and a for the set  $a_i$ ,  $i = 0, \ldots n$ .

The  $a_i$  that will be under discussion here are purely space-like, so  $a = (0, \vec{a_i})$ . The diameter  $\lambda$  of the set  $\vec{a_0} \dots \vec{a_n}$  is given by  $\lambda^2 = \sup_{i, \vec{f}} (\vec{a_i} - \vec{a_{i'}})^2$ . Let this maximum be obtained for i = j and i' = j'. Then  $\lambda^2 = (\vec{a_j} - \vec{a_{j'}})^2$ . Now consider the family of all partitions of  $\{0, 1, \dots, n\}$  into two subsets X and X' such that  $j \in X$  and  $j' \in X'$ . The maximum of the distance of the set  $\{\vec{a_i}; i \in X\}$  from the set  $\{\vec{a_i'}; i' \in X'\}$  as X varies over the family is given by

$$\mu^2 = \sup_{\mathbf{X}} \left[ \inf_{i \in \mathbf{X}, i' \in \mathbf{X}'} (\overrightarrow{\mathbf{a}_i} - \overrightarrow{\mathbf{a}_{i'}})^2 \right].$$

In the following discussion it will be assumed that this maximum is obtained for the partition X = Y and X' = Y' and that  $\mu^2 = (\vec{a}_{\ell} - \vec{a}_{\ell'})^2$ ,  $\ell \in Y$  and  $\ell' \in Y'$ .

There is an elementary but basic inequality connecting  $\mu$  with the diameter  $\lambda$ :

$$n\mu \geqslant \lambda . \tag{97}$$

Proof

We divide the points  $\vec{a_i}$ , into two classes: those which can be joined to  $\vec{a_j}$  by a chain of points such that 1) no point repeats, 2) the distance between successive points is  $\leq \mu$ , and those which cannot. We claim  $\vec{a_j}$  lies in the former class, because every point of the latter class lies a distance  $\geq \mu$  from every point of the former and if  $\vec{a_j}$  belonged to it we would have a partition violating the definition of  $\mu$ . Therefore, there is a chain of points  $\vec{a_j}$ ,  $\vec{a_p}, \ldots, \vec{a_j}$ , such that

$$\lambda = \left| \overrightarrow{a_j} - \overrightarrow{a_j} \right| \leqslant \left| \overrightarrow{a_j} - \overrightarrow{a_p} \right| + \left| \overrightarrow{a_p} - \dots \right| + \dots + \dots - a_j \left| \leqslant n\mu \right|.$$

Notice that  $n\mu = \lambda$  when the  $\vec{a_i}$  are equally spaced along a line.

A final bit of notation: the truncated vacuum expectation values corresponding to (95) will be denoted  $T_T^{\pi}$  and

$$F_{\underline{T}\underline{\varphi}}^{\pi}(\mathbf{a}) = \int d\underline{x} \ \underline{\phi}(\mathbf{x}) \ T_{\underline{T}}^{\pi}(\underline{x} + \mathbf{a})$$
$$Y = \{i_0, i_1, \dots, i_k\}, \ Y' \{i'_0, i'_1, \dots, i'_{k'}\} \text{ with } \mathbf{k} + \mathbf{k}' = \mathbf{n} - 1,$$

where the elements  $i_{\tau}$  within each of the subsets are written in their natural order as integers. Define permutations I and J by

$$I(0, 1, ..., n) = (0, 1, ..., n); J(0, 1, ..., n) = (i_0, i_1, ..., i_k, i'_0, i'_1, ..., i'_{k'}).$$

I is the identity permutation.

Now we are ready for the second step in the proof. Let A be a field satisfying I, II and III but with test functions in  $\mathcal{S}$  rather than  $\mathcal{S}$ .

#### Theorem

Let  $\lambda$  be the diameter of the set  $\{\vec{a_0}, \ldots, \vec{a_n}\}$ . Then, for any positive integer N,

$$\lim_{\lambda \to \infty} \lambda^{N} \left[ F_{T_{\underline{\varphi}}}^{I} \left( \vec{a} \right) - F_{T_{\underline{\varphi}}}^{J} \left( \vec{a} \right) \right] = 0$$
(97.a)

provided that the configuration of the  $\vec{a}$ 's remains such that the above defined j, j', Y, Y', and  $\ell$ ' stay the same.

# Remarks

(1) This theorem already has been stated by HAAG [17]. He gave a plausible but somewhat hand-waving-type proof.

(2) It is this theorem which enables the commutation relations for the "in" and "out" fields to be proved.

#### Proof

Note first that  $T_T^I(\underline{x}) - T_T^J(\underline{x})$  vanishes when all  $x_{i\alpha}$ ,  $(i \in Y)$ , are spacelike to all  $x_{i'\alpha'}$ ,  $(i' \in Y')$ , because of III (LC). Therefore,  $\underline{\phi}(\underline{x})$  does not contribute to the integral:

$$F_{T_{\underline{\varphi}}}^{I}(\vec{a}) - F_{T_{\underline{\varphi}}}^{J}(\vec{a}) = \int d\underline{x} \ \underline{\varphi}(\underline{x}) \left[ T_{T}^{I}(\underline{x} + \vec{a}) - T_{T}^{J}(\underline{x} + \vec{a}) \right] ,$$

$$(x_{i\alpha}^{0} - x_{i'\alpha'}^{0})^{2} \leq \left[ (\vec{x}_{i\alpha} - \vec{x}_{i'\alpha'}) + (\vec{a}_{i} - \vec{a}_{i'}) \right]^{2} ,$$

$$||x_{i\alpha} - x_{i'\alpha'}||^{2} \leq (\vec{x}_{i\alpha} - \vec{x}_{i'\alpha'})^{2} + \left[ (\vec{x}_{i\alpha} - \vec{x}_{i'\alpha'}) + (\vec{a}_{i} - \vec{a}_{i'}) \right]. \qquad (98)$$

Now the square bracket is always greater than  $||\vec{x}_{i\alpha} - \vec{x}_{i'\alpha'}| - |\vec{a}_i - \vec{a}_{i'}||$ ; and if, when  $[(x_{i\alpha} - x_{i'\alpha'}) + (a_i - a_{i'})]^2 < 0$  for all  $\alpha = 0, \ldots, r(i)$ ,

If

 $\alpha' = 0, \ldots, r(i')$  and all  $i \in Y$ ,  $i' \in Y'$ .

Introducing the Euclidean distance,

$$||x_{i\alpha} - x_{i'\alpha'}||^2 = (x_{i\alpha}^0 - x_{i'\alpha'}^0)^2 + (\vec{x}_{i\alpha} - \vec{x}_{i'\alpha'})^2, \qquad (99)$$

one can get a sufficient condition for (99) to be satisfied as follows: Note (99) can be rewritten as

$$\left|\left|\mathbf{x}_{\mathbf{i}\alpha}-\mathbf{x}_{\mathbf{i}'\alpha'}\right|\right|^{2} \leq (\vec{\mathbf{x}}_{\mathbf{i}\alpha}-\vec{\mathbf{x}}_{\mathbf{i}'\alpha'})^{2} + [(\vec{\mathbf{x}}_{\mathbf{i}\alpha}-\vec{\mathbf{x}}_{\mathbf{i}'\alpha'})+(\vec{\mathbf{a}}_{\mathbf{i}}-\vec{\mathbf{a}}_{\mathbf{i}'})]^{2}.$$
 (100)

The second term on the right-hand side is always bigger than

$$[\left|\vec{x}_{i\alpha} - \vec{x}_{i'\alpha'}\right| - \left|\vec{a}_{i} - \vec{a}_{i'}\right|]^2,$$

so the right-hand side is bigger than

$$2\left|\vec{\mathbf{x}}_{i\alpha} - \vec{\mathbf{x}}_{i'\alpha'}\right|^2 + \left|\vec{\mathbf{a}}_i - \vec{\mathbf{a}}_{i'}\right|^2 - 2\left|\vec{\mathbf{x}}_{i\alpha} - \vec{\mathbf{x}}_{i'\alpha'}\right| \left|\vec{\mathbf{a}}_i - \vec{\mathbf{a}}_{i'}\right|.$$

This takes its minimum as  $|\vec{x}_{i\alpha} - \vec{x}_{i'\alpha'}|$  varies when

$$\left| \vec{\mathbf{x}}_{i\alpha} - \vec{\mathbf{x}}_{i'\alpha'} \right| = (1/2) \left| \vec{\mathbf{a}}_{i} - \vec{\mathbf{a}}_{i'} \right|;$$

then it is  $(1/2) |\vec{a_i} - \vec{a_i'}|^2$ ; thus (99) is guaranteed if

 $||x_{i_{\alpha}} - x_{i'_{\alpha'}}||^2 \le \mu^2/2;$ 

or, using (97);

$$\left|\left|\mathbf{x}_{i_{\alpha}}-\mathbf{x}_{i_{\alpha}'}\right|\right|^{2} \leq \lambda^{2}/2n^{2}$$
.

Because  $||x_{i_{\alpha}} - x_{i'_{\alpha'}}||^2 \leq (||x_{i_{\alpha}}|| + ||x_{i'_{\alpha'}}||)^2$ , if one makes

$$\left|\left|\mathbf{x}\right|\right|^2 \equiv \sum_{i=0}^{r(i)} \left|\left|\mathbf{x}_{i\alpha}\right|\right|^2 \le \lambda^2 / 8n^2$$
,

one has each  $||x_{i\alpha}|| \le \lambda/2 \sqrt{2}n$  so  $||x_{i\alpha} - x_{i'\alpha'}||^2 \le (\lambda/\sqrt{2}n)^2 = \lambda^2/2n^2$ . Thus there is a sphere in <u>x</u> space whose radius is  $\lambda\sqrt{2}n$  such that  $\underline{\phi}(\underline{x})$  does not contribute to the integral (98) for x in the sphere.

Next note that the transformation  $x \to x+a$ , where all a are identical, leaves  $T_T^{\pi}$  invariant, so one can assume without loss of generality that the cluster labelled zero has its first  $\vec{a}$  at the origin. Then

$$\begin{aligned} \left|\left|\underline{a}\right|\right|^{2} &= \sum_{i=0}^{n} \sum_{\alpha=0}^{r(i)} \left|\left|\underline{a}_{i}\right|\right|^{2} < \sum_{i=1}^{n} (r(i)+1) \lambda^{2} = L\lambda^{2}, \text{ where } L = n + \sum_{i=1}^{n} r(i); \end{aligned}$$
i.e.
$$\left|\left|\underline{a}\right|\right| \leq \lambda \sqrt{L}. \tag{101}$$

To complete the proof, Ruelle introduces an important technical device: a partition of unity adapted to the problem. Partitions of unity are a standard device of distribution theory [42], but the one used here has some special features.

What is wanted is a family of non-negative functions  $f_{\nu}(\underline{x}) \in A$ ;  $\nu = 1, 2.$ . such that

- (1)  $\sup_{x} f_{\nu}(\underline{x})$  is bounded in  $\nu$  and the same holds true for each derivative of  $f_{\nu}$ ;
- (2)  $f_{\nu}(\underline{x}) \equiv f_{\nu}(||x||) = 0$  both if  $||x|| > \nu + 1$  and  $||x|| < \nu 1$ ;
- (3)  $\sum_{v} f_{v}(x) = 1$ .

Recall that for an arbitrary open covering of space-time  $\{O_i : i \in I\}$ (where, according to the definition of open covering, I is some index set,  $O_i$  is open for all i and every x lies in some  $O_i$ ) a partition of the identity is a family of  $\varphi_i$ ;  $i \in I$  of infinitely differentiable non-negative functions with support of  $\varphi_i \subset O_i$  and such that if C is any compact set of space-time, C intersects the support of almost a finite number of  $\varphi_i$ . In the present case, the sets may be taken as  $O_i$ , the interiors of spherical shells of thickness  $(2 + \epsilon)$  and integer radius, and one has to look into the details of the proof, for example that of SCHWARTZ [42], to see that the property 1, which is usually not required for a partition of unity, can be secured. It is true but will not be proved here.

Taking the  $f_{y}$  for granted, then one gets

$$\mathbf{F}_{\mathbf{T}_{\underline{\varphi}}}^{\mathbf{I}}\left(\overrightarrow{\mathbf{a}}\right) - \mathbf{F}_{\mathbf{T}_{\underline{\varphi}}}^{\mathbf{J}}\left(\overrightarrow{\mathbf{a}}\right) = \sum_{\nu > \lambda/2n\sqrt{2}-1} \left[\mathbf{F}_{\mathbf{T}_{\underline{\varphi}}}^{\mathbf{I}}\left(\overrightarrow{\mathbf{a}}\right) - \mathbf{F}_{\mathbf{T}_{\underline{\varphi}}\nu}^{\mathbf{J}}\left(\overrightarrow{\mathbf{a}}\right)\right],$$

where  $\underline{\phi}_{\nu} = f_{\nu}(\underline{x}) \phi(\underline{x})$ . (The series  $\sum_{r=1}^{\infty} \phi_{\nu}$  converges to  $\phi$  in  $\measuredangle$ . There is no contribution from the terms with  $\nu + 1 < \lambda/2n\sqrt{2}$  because support of  $\phi_2$  is then entirely in the sphere  $||\mathbf{x}|| < \lambda/2n\sqrt{2}$ ). Since  $T_T^T - T_T^T$  is a temperate distribution, it may be written as  $T_T^I - T_T^J$ 

Since  $T_T^{T} - T_T^{T}$  is a temperate distribution, it may be written as  $T_T^{T} - T_T^{T}$ =  $D^{\alpha}g$ , where g is a continuous function of x of at most polynomial growth.  $D^{\alpha}$  is the differentiation operator defined in Eq. (26). Thus

$$F_{T\underline{\varphi}_{\nu}}^{I}(\vec{a}) - F_{T\underline{\varphi}_{\nu}}^{J}(\vec{a}) = \int d\underline{x} \underline{\varphi}_{\nu}(\underline{x}) D^{\alpha}g(\underline{x} + a)$$
$$= \int d\underline{x} \left[ D^{\alpha}\underline{\varphi}_{\nu}(\underline{x}) \right] g(\underline{x} + a).$$
(102)

Now the numbers  $\sup_{\underline{x}} |D^{\alpha} \underline{\phi}_{\nu}(\underline{x})|$  decrease with  $\nu$  faster than any power of  $\nu^{-1}$ . (The reason for this is that  $\varphi \in \mathcal{A}$  so  $\sup_{\underline{x}} |x^{\beta}D^{\gamma}\varphi(x)| < \infty$ . But the derivatives of  $\varphi_{\nu}$  are uniformly bounded in  $\nu$ . This supplies  $\sup_{\underline{x}} |x^{\beta}\varphi_{\nu}(x)| < C$  independent of  $\nu$  so

$$\sup_{\underline{\mathbf{x}}} \left| \mathbf{D}^{\alpha} \boldsymbol{\varphi}_{\nu} (\mathbf{x}) \right| \leq \mathbf{C} (\boldsymbol{\alpha}, \boldsymbol{\beta}) / \nu^{\boldsymbol{\beta}}$$

for all integer  $\nu$  and each  $\beta$ .)

Thus, since

$$\begin{aligned} \left| g(\mathbf{x}) \right| &\leqslant C \left( 1 + \left| \left| \mathbf{x} \right| \right|^2 \right)^{k/2} \\ \left| F_{T_{\underline{\sigma}}}^{\mathbf{I}} \left( \overrightarrow{\mathbf{a}} \right) - F_{T_{\underline{\sigma}}}^{\mathbf{J}} \left( \overrightarrow{\mathbf{a}} \right) \right| &\leqslant S(\nu+1) \sup_{\mathbf{x}} \left| D^{\alpha} \underline{\phi}_{\nu}(\mathbf{x}) \right| C , \end{aligned}$$
$$\sup_{\left| \left| \mathbf{x} \right| \right|^2} \left( 1 + \left| \left| \mathbf{x} \right| \right|^2 \right)^{k/2} &\leqslant S(\nu+1) \sup_{\mathbf{x}} \left| D^{\alpha} \underline{\phi}_{\nu}(\mathbf{x}) \right| C \left( 1 + 2(\nu+1)^2 \right)^{k/2} \left( 1 + 2\lambda^2 L \right)^{k/2}, \end{aligned}$$

where  $S(\nu+1)$  is the volume of the sphere in x space of radius  $\nu+1$  and the inequality  $1+||x+a||^2 < (1+2||x||^2) \times (1+2||a||^2)$  has been used. Now the numbers  $C_{\nu} = \max_{x} |D^{\alpha} \underline{\phi}_{\nu}(\underline{x})| [C S(\nu+1) \times (1+2(\nu+1)^2)^{k/2}]$  de-

Now the numbers  $C_{\nu} = \max_{\underline{x}} |D^{\alpha} \underline{\phi}_{\nu}(\underline{x})| |C S(\nu+1) \times (1+2(\nu+1)^{\nu})^{\nu}|$  decrease faster than any power of  $\nu^{-1}$ ; therefore, in the inequality

$$\left| \mathbf{F}_{\mathbf{T}_{\underline{\varphi}}}^{\mathbf{I}}(\vec{\mathbf{a}}) - \mathbf{F}_{\mathbf{T}_{\underline{\varphi}}}^{\mathbf{J}}(\vec{\mathbf{a}}) \right| < \left( \sum_{\nu > \lambda/2} C_{\nu} \right) \left( 1 + 2 \mathbf{L} \lambda^{2} \right)^{k/2}$$

the first factor decreases faster than any power of  $\lambda^{-1}$ .

 $(\Sigma C_{\nu} \text{ decreases as } N^{-(\ell-1)} \text{ for } \ell \ge 2; \text{ as proof of this compare with an integral which can be integrated explicitly.) Therefore,$ 

$$\lim_{\boldsymbol{\ell} \to \infty} \lambda^{N} \left[ \mathbf{F}_{T\underline{\varphi}}^{I} \left( \overrightarrow{\mathbf{a}} \right) - \mathbf{F}_{T\underline{\varphi}}^{J} \left( \overrightarrow{\mathbf{a}} \right) \right] = 0$$

for all N, as was to be proved.

It is well to look over the proof to see why it works. Evidently, it uses the sphere in  $\underline{x}$ , within which there is no contribution to the integral. Furthermore, it uses the assumption that the  $T_T^{\pi}$  are temperate in order to conclude that they can be written in terms of a derivative of a continuous polynomial bounded g.

The next theorem is the one which gives the title to this section.

# Theorem

With the same hypotheses as in the previous theorem but, in addition, the requirement that p = 0 be an isolated point of the physical momentum spectrum,  $F_{T,\varphi}^{T}(\vec{a})$  as well as  $D_0 F_{T,\varphi}^{T}(\vec{a})$  where  $D_0$  is any derivative with respect to the a a refunctions in  $\boldsymbol{\delta}$ .

# Proof '

Introduce now in x-space the new variables,

54

That is, single out one point with index in Y,  $x_{i_00}$  and one with index in Y',  $x_{i'_00}$ . Introduce the first as x and their difference as  $\xi$ . Then introduce the differences of the first points of the clusters in Y relative to  $x_{i_00}$  and call them  $\xi_i$ ; introduce the differences of the first points of the clusters in Y' relative to  $x_{i'_00}$  and call them  $\xi_{i'}$ . Finally, introduce the differences between the  $x_{i\alpha}$  and the first point of their clusters  $\xi_{i\alpha} = x_{i\alpha} - x_{i0}$ , and the corresponding differences between the  $x_{i'\alpha'}$  and the first point of their clusters  $\xi_{i'\alpha'} = x_{i'\alpha'} - x_{i'0'}$ .

Denote by  $\underline{\xi}$  the family of all  $\xi_i \xi_{i'} \xi_{i\alpha} \xi_{i'\alpha'}$ . Then  $T_T^{\pi}$  is a function of  $\xi$  and  $\xi'$ , and  $\varphi$  a function of x,  $\xi$ ,  $\xi$ . Define Fourier transforms by

$$(\mathcal{F} \mathbf{T}_{\mathbf{T}}^{\pi})(\mathbf{P}, \underline{\mathbf{P}}) = (2\pi)^{-2L} \int \dots \int d\boldsymbol{\xi} d\boldsymbol{\xi} e^{-i(\mathbf{p}\,\boldsymbol{\xi} + \underline{\mathbf{P}}\,\underline{\xi})} \mathbf{T}_{\mathbf{T}}^{\pi}(\boldsymbol{\xi}, \underline{\boldsymbol{\xi}}) ,$$
$$(\mathcal{F} \underline{\phi})(\mathbf{p}, \mathbf{P}, \underline{\mathbf{P}}) = (2\pi)^{-2(L+1)} \int \dots \int d\mathbf{x} d\boldsymbol{\xi} d\underline{\boldsymbol{\xi}} e^{+i(\mathbf{p}\,\mathbf{x} + \underline{\mathbf{P}}\,\underline{\boldsymbol{\xi}} + \underline{\mathbf{P}}\,\underline{\boldsymbol{\xi}})} \underline{\phi}(\mathbf{x}, \boldsymbol{\xi}, \underline{\boldsymbol{\xi}}).$$

Here the P's are labelled in the same way as the  $\underline{\xi}$ 's. Incidentally, this formula displays what was already clear from first principles:  $F_{T\varphi}^{\pi}$  is an infinitely differentiable function of at most polynomial growth. Then

$$F_{T\underline{\varphi}}^{\pi}(\underline{a}) = (2\pi)^{2} \int dP d\underline{P} (\mathcal{F} \underline{\phi}) (0, P, \underline{P}) (\mathcal{F} T_{T}^{\pi}) (\mathcal{F} T_{T}^{\pi}) (P, \underline{P})$$

$$\times \exp + i \left[ P(a_{i_{0}} - a_{i_{0}}) + \sum_{i=i_{1}}^{i_{k}} P_{i}(a_{i} - a_{i_{0}}) + \sum_{i'=i_{1}}^{i'_{k'}} P_{i'}(a_{i'} - a_{i'_{0}}) \right].$$

Up to this point in the proof there is essentially nothing but notation for Fourier-transforms. Now comes the idea. Notice that  $(\mathcal{F} T_1^J)(P, P) = 0$ unless  $P \in V_+^M$  (where  $V_+^M$  stands for all vectors P with  $Q^2 > M^2$ ,  $\varphi^0 > 0$  and the bar denotes closure). This is true because P is conjugate to the difference  $\boldsymbol{\xi} = x_{i_0'0} - x_{i_0^0}$ . (Insert U(a, 1) just after  $A(x_{i_0^0})$  in the vacuum expectation value, multiply by  $e^{-iQ.\alpha}$  and integrate. The result has to be zero except when Q is in the physical spectrum but has the effect  $\boldsymbol{\xi} \rightarrow \boldsymbol{\xi} + \mathbf{a}$  so that P must be in the physical spectrum.) M is the assumed lower limit on the mass of the system. The vacuum does not appear as an intermediate state because the vacuum expectation values have been truncated. A full formal proof of this last intuitively obvious statement is contained in [27]. Furthermore if K is the permutation  $K(0, 1, \ldots, n) \rightarrow (i'_0, \ldots, i'_k, i_0, \ldots, i_k)$ , K changes  $\boldsymbol{\xi}$  into  $\boldsymbol{\xi}$  without changing  $\boldsymbol{\xi}$  so  $(\mathcal{F} T_T^K)(P, P) = 0$  unless  $P \in V_-^M$ . Now define  $(\mathcal{F} \Psi)(p, P, P) = h(P) \widetilde{\Phi}(p, P, P) \in \boldsymbol{\mathcal{S}}$  where h is infinitely differentiable on  $\overline{V}_+^M$ and vanishes outside of V<sub>+</sub>. Then, clearly

$$\mathbf{F}_{T\underline{\psi}}^{\mathbf{J}}(\mathbf{a}) = \mathbf{F}_{T\underline{\psi}}^{\mathbf{J}}(\mathbf{a}), \quad \mathbf{F}_{T\underline{\psi}}^{\mathbf{J}}(\mathbf{a}) = 0 \quad . \tag{103}$$

Now the argument of the preceding theorem was made for two permutations, I and J, but it would differ only in notation if carried out for J and K. Thus

$$\lim_{\lambda \to \infty} \lambda^{N} \operatorname{F}_{\mathbf{T}_{\underline{\sigma}}}^{J}(\vec{\mathbf{a}}) = 0$$
(104)

under the same conditions described in the preceding theorem. Those conditions involve the points j, j',  $\ell$ ,  $\ell'$  and the sets Y and Y'. But if the  $\vec{a}$  have a configuration such that j, j', etc. are different, the conclusion is the same and there are only a finite number of possible choices for the j, j'... Thus, whatever the configuration of the  $\vec{a}$ , (104) holds. Applying D<sub>0</sub> to F is equivalent to changing  $\varphi$ , so the theorem is proved.

For the application to Haag collision theory one needs the preceding conclusion but for almost local fields. Actually, this case is covered by the preceding argument if a change in notation is made. Write

$$B_i(x_i) = U(x_i, 1) A_i(\phi_i) U(x_i, 1)^{-1}$$

and call  $x_i$  the former variables  $a_i$ , and replace a by x. Then

$$F(x) = (\Psi_0, B_0(x_0) B_1(x_1), \dots, B_n(x_n) \Psi_0)$$

is a special case of the (untruncated) F's considered before with  $\underline{\phi} = \underline{\phi}_0 \otimes \underline{\phi}_1 \otimes \ldots \otimes \underline{\phi}_n$ . The truncated vacuum expectation values are defined with respect to the B's as in Part One, not as above with respect to the A's, but one sees immediately from the above proof that the vacuum will be eliminated equally well in the intermediate states by this procedure.

# Corollary

The preceding theorem is also true for truncated vacuum expectation values of almost local fields built out of local fields (test functions again in  $\mathcal{S}$ ) provided the vacuum is an isolated point of the spectrum.

# 2.7. FINAL REMARKS ON THE HAAG-RUELLE COLLISION THEORY

The preceding sections have explained how one can construct collision states of all the elementary systems associated with irreducible representations of the Poincaré group contained in U. A natural question is then: Are the collision states unique? The answer is yes. Suppose that by choosing two different sets of B's, say B and  $\hat{B}$ , and carrying out the preceding constructions, one was led to two states  $\Phi(t)$  and  $\hat{\Phi}(t)$ . The argument which follows Eq. (81) shows that they actually converge to the same "in" or "out" state. The argument goes just as before, except that instead of the terms with two operators not contributing because their time derivatives are zero, here it is because the contributions of  $\Phi$  and  $\hat{\Phi}$  cancel. Both cases are covered by the statement that there is no contribution because the one-body problem has been solved, assuming the-one particle states  $B_f \Psi_0$  and  $\hat{B}_f \Psi_0$ are normalized in the same way. Thus, the Haag-Ruelle Collision Theory will give a unique set of "in" and "out" fields and consequently a unique collision matrix.

These statements hold even if Axiom IV does not hold. Then, however, the S operator is a unitary mapping of  $\mathcal{H}_{out}$  onto  $\mathcal{H}_{in}$  which is undefined on those vectors of  $\mathcal{H}$  which are not in  $\mathcal{H}_{out}$ . There might be some point in

investigating (in the spirit of Heisenberg's elementary particle theory) theories for which Axiom IV does not hold.

# REFERENCES

- [1] JOST, R., Eine Bemerkung zum CTP Theorem, Helv. Phys. Acta 30 (1957) 409.
- [2] HALL, D. and WIGHTMAN, A.S., A theorem on invariant analytic functions with applications to relativistic quantum field theory, Dan. Mat. Fys. Medd. 31 (1957) 5.
- BORCHERS, H.J., Über die Mannigfaltigkeit der interpolierenden Felder zu einer kausalen S Matrix, Nuovo Cim. 15 (1960) 784.
- [4] SYMANZIK, K., Grundlagen und gegenwärtiger Stand der feldgleichungsfreien Feldtheorie, Werner Hersenberg und die Physik unserer Zeit, Vieweg, Braunschweig (1961).
- [4a] JOOS, H., Group theoretical models of local field theories, Math. Phys. (in press).
- [5] SCHROER, B., private communication.
- [6] GLASER, V., private communication; KÄLLEN, G., private communication.
- [7] WIGHTMAN, A.S., Quelques problèmes mathématiques de la théorie quantique relativiste, Les Problèmes Mathématiques de la Théorie Quantique des Champs, CNRS, Paris (1959) 1-38.
- [8] ARAKI, H., HAAG, R. and SCHROER, B., The determination of local or almost local field from given current, Nuovo Cim. 19 (1961) 40.
- [9] BARDACKI, K. and SUDARSHAN, E.C.G., Local fields with terminating expansions, Nuovo Cim. <u>21</u> (1961) 722.
- [10] EPSTEIN, H., Generalization of the "edge of the wedge" theorem, J. Math. Phys. 1 (1960) 524.
- [11] GREENBERG, O.W., Generalized free fields and models of local field theory, Ann. Phys. 16 (1961)158.
- [12] DELL'ANTONIO, G.F., Support of a field in p-space, J. Math. Phys. 2 (1961) 759.
- [13] LICHT, A.M. and TOLL, J., Two point function and generalized free fields, Nuovo Cim. <u>21</u> (1961) 346.
- [13a] GREENBERG, O.W., private communication.
- [14] ACHARYA, R., Some Borchers' type theorems in quantum field theory, Nuovo Cim. 23 (1962) 580.
- [15] ROBINSON, D.W., Support of a free field in momentum space (in press).
- [16] GREENBERG, O.W., Heisenberg fields which vanish on domains of momentum space (in press).
- [17] HAAG, R., Quantum field theories with composite particles and asymptotic conditions, Phys. Rev. <u>112</u> (1958) 669-673.
- [18] RUELLE, D., On the asymptotic condition in quantum field theory, Helv. Phys. Acta 35 (1962) 1,7.
- [19] DELL'ANTONIO, G.F., On the connection between spin and statistics, Ann. Phys. 16 (1961) 153.
- [20] ARAKI, H., On the connection of spin and the commutation relations between different fields, J.Math. Phys. 2 (1961) 267.
- [21] BURGOYNE, N., On the connection of spin with statistics, Nuovo Cim. 8 (1958) 607.
- [22] LÜDERS, G. and ZUMINO, B., Connection between spin and statistics, Phys. Rev. 110 (1958) 1450.
- [23] SUDARSHAN, E.C.G. and BARDACKI, K., The nature of the axioms of quantum field theory.
- [24] HEPP, K., JOST, R., RUELLE, D. and STEINMANN, O., Necessary condition on Wightman functions, Helv. Phys. Acta <u>34</u> (1961) 542.
- [25] BORCHERS, H.J., On the structure of the algebra of field observables, Nuovo Cim. 24 (1962) 214.
- [26] WIGHTMAN, A.S., Proc. Int. Congress of Mathematicians (14-23 Aug. 1962).
- [27] ARAKI, H., On the asymptotic behaviour of vacuum expectation values at large space-like separations, Ann. Phys. 11 (1960) 260.
- [28] GELFAND, I. and VILENKIN, N. Ya., Generalized functions 4, 32.
- [29] WIGHTMAN, A.S., Quantum field theory in terms of vacuum expectation values, Phys. Rev. <u>101</u> (1956) 860.
- [30] JOST, R. and HEPP, K., Über die Matrixelemente des Translations-operators, Helv. Phys. Acta <u>35</u> (1962) 34; UHLMANN, A., Spectral integral for the representation of the space-time translation group in relativistic quantum field theory, Ann. Phys. <u>13</u> (1961) 453-462.
- [31] SCHWARTZ, L., "Théorie des distributions", Paris 2 (1957) 95.
- [32] Ibid. 2 90.

[33] ARAKI, H., HEPP, K. and RUELLE, D., On the asymptotic behaviour of Wightman functions in spacelike directions, Helv. Phys. Acta 35 (1962) 164.

- [35] STONE, M.H., Linear transformations in Hilbert space (to be published).
- [36] WIGHTMAN, A.S. (in preparation).
- [37] ZERNER, M., Seminaire de Physique Mathématique de Marseille.
- [38] HAAG, R., Discussion des axiomes, Problèmes mathématiques de la théorie quantique des champs, CNRS, Paris (1959) 151-162.
- [39] REEH, H. and SCHLIEDER, S., Bemerkungen zur Unitäräquivalenz von Lorentz invarianten Feldern, Nuovo Cim. <u>22</u> (1961) 1051; Über den Zerfall der Feldoperatoralgebra im Falle einer Vacuumentartung (in press).
- [40] RUELLE, D., Helv. Phys. Acta 35 (1962) 162-163.
- [41] HAAG, R. and SCHROER, B., The postulates of quantum field theiry (in press).
- [42] SCHWARTZ, L., "Théorie des distributions", Paris 1 (1957) 22-23.
- [43] SYMANZYK, K., Green's functions and the quantum theory, Lectures in theoretical physics, BUTTEN, W., Ed., New York <u>3</u> (1961): On the many-particle structure of Green's functions in quantum field theory, J. Math, Phys. <u>1</u> (1960) 249; Green's function method and the renormalization of renormalizable quantum field theories, Lectures in theoretical physics, JAKŠIČ, B., ed., Hercegnovi (1961); KÄLLÉN, G., Properties of vacuum expectation, Values of field operators in dispersion relations and elementary particles, Wiley, New York (1960).
- [44] BIEBERBACH, "Lehrbuch der Functionen Theorie", Bielefeld (1952) 156.

<sup>[34]</sup> GUTTINGER, W.

# INVARIANT QUANTUM MECHANICAL EQUATIONS OF MOTION

E. P. WIGNER PRINCETON UNIVERSITY, PRINCETON, N.J. UNITED STATES OF AMERICA

#### I. INTRODUCTORY REMARKS

One of the last few years' most important developments in theoretical physics is the recognition that it is useful to extend to complex numbers the definition domain of intrinsically real variables, such as energy or angular momentum. This leads one to review many subjects which were considered to be closed. It should not have surprised me, therefore, when Dr. Salam asked me to report, at this seminar, on equations for elementary particles which are not believed to exist in nature. such as particles with imaginary mass. Even though the equations which describe such particles will play no role in the theory as long as the variables such as energy or angular momentum have physically meaningful values, that is, as long as they are real, they may play a significant role when the definition domain of these variables is extended.

I was, at one time, greatly interested in establishing all linear equations which are invariant under the inhomogeneous Lorentz group and much of what I will talk about originates from this interest. The inhomogeneous Lorentz group contains displacements in space and time in addition to Lorentz transformations; it will be called Poincaré group after the mathematician who first became convinced of the basic significance of this group for physics. It turns out that the representations of a group essentially determine all linear equations which are invariant under the group in question and one is thus led naturally to the theory of the representations of the Poincaré group. The term "representation" will mean, throughout this article, a group of linear operators which is homomorphic to the group to be represented; the space of the vectors on which these operators act is a complex Hilbert space, usually infinite dimensional, which will be called representation space.

Only some of the representations of the Poincaré group will be discussed: those which are irreducible and unitary. The first restriction means, in the domain of real masses and spins, that only equations for elementary particles will be considered, and these only on the Schrödinger, that is not second quantized, level. In the extended domain of the variables it should mean that the Regge poles to be considered are primitive but at the present time this point has not been fully elucidated. Naturally, it would be desirable to consider also the second quantized form of the equations, but I am not able to do this. My excuse for considering only unitary representations is similar: the non-unitary ones present complications which have not yet been surmounted, even though Dr. Froissart has made significant progress in their investigation.

59

There is one other respect in which my discussion will be limited: by the very fact that the Poincaré group will be the basic group throughout. It would be desirable to consider equations which are invariant under one of the generalizations of the Poincaré group, in particular equations invariant under the usual de Sitter group. However, the doctoral thesis of T. Philips shows that even the interpretation of the real mass-real spin representations of the usual de Sitter group encounters serious difficulties and I want to avoid these. Hence, the discussion will be concerned solely with the Poincaré group and almost solely with the unitary irreducible representations of this, or the linear equations which correspond to these.

The relation between representations and equations of motion justifies a few remarks. In one sense, the representation gives much more information than the equations of motion: whereas the equations of motion, as ordinarily conceived, give only the change of the state vector (or whatever characterizes the instantaneous state of the system) with the passage of time, and this directly only for an infinitesimal increment of time, the representation gives the change of the state vector for arbitrary Poincaré transformations, and for finite ones as well as for infinitesimal ones. The time displacement. the effect of which is given by the equations of motion, is only one special type of Poincaré transformations. Hence, the representation is more informative than the equation of motion in two regards: because it gives the effect of finite, rather than only of infinitesimal, transformations, and because it gives the effect of all Poincaré transformations, not only of timedisplacements. It may even happen that it is, on the basis of the equation of motion alone, not possible to determine without further assumptions how the state vector changes under a proper Lorentz transformation. Thus, to mention a rather trivial example, the Dirac equation in empty space is invariant under Lorentz transformations not only if the four components are considered to be spinors, but also if they are considered to be scalars.

In another less mathematical but much more suggestive sense, the equation of motion is much more informative than the representation from which it arises. The reason is that it invites the application of the methods of second quantization and hence the replacement of the particle by a quantum field. Once this is accomplished, one may be led by analogies to assumptions concerning interactions. Without any knowledge of its interactions, the picture of a particle is rather empty. All these remarks apply, for the present, only to representations or equations which describe particles which exist in some sense in nature. It does not apply to characteristics of Regge poles or anything similar; for these the relation between representations and equations of motion (if such exist) is much less clear.

It should be mentioned, finally, that the relation of representations to equations of motion is not one-to-one. We shall see several examples for this; one of the principal objectives of these lectures being the establishment of a general method to obtain one equation of motion for every representation. This equation of motion will, in some cases, not be the common and well-known one. However, one example for the lack of uniqueness of the correspondence between representation and equation of motion is already known to all of us: the electromagnetic field can be described either by the scalar and vector potentials, or by the electric and magnetic fields. The representation is, however, the same for both:  $0_1$  in the usual notation. In fact, the representation is always uniquely determined by the properties of an elementary particle because it merely expresses the relation between the deccriptions of the particle by observers using different but equivalent frames of reference. In particular, the two frames of reference whose relation gives the equations of motion are at rest with respect to each other, but their time scales have different starting points.

# II. THE UNITARY REPRESENTATIONS OF THE POINCARÉ GROUP

One further general observation will be useful for the understanding of the connections which will form the subject of these lectures. This observation relates to the greater effectiveness of invariance considerations in quantum than in classical theory. The reason for this greater effectiveness was spelled out already by C. N. Yang: the states in quantum theory constitute a linear manifold whereas there is no similar structure of the states in classical mechanics. However, it will be useful to pursue somewhat more in detail the way this difference manifests itself. We shall choose for this a very simple and elementary example in which only rotational symmetry is present.

Hamel, Klein and Noether have shown how the conservation laws for angular momentum, for instance, can be derived in classical mechanics directly from the invariance of the equations with respect to rotations. However, the considerations leading from the invariance to the conservation laws are rather subtle, being based on the principle of least action. If one just considers a possible classical trajectory, such as a planetary orbit. an unsophisticated application of the invariance principle only leads to the conclusion that there are other similar orbits, obtained from the given orbit by a rotation. This is not a very fruitful conclusion. In quantum theory, given one orbit, one can also obtain other orbits by rotation. However, all the orbits obtained in this way form a linear manifold and one can select from this manifold a linearly independent set in terms of which all the "orbits" can be expressed linearly. If one then subjects the members of the selected set to a rotation, and expresses these rotated orbits linearly in terms of the originally selected set, one obtains at once a representation of the rotation group.

When carrying out the procedure just outlined, one of two situations may be encountered. If starting with one orbit, the orbits obtained by different rotations are all linearly independent, no significant conclusion results. The representation obtained in this case is the infinite dimensional socalled regular representation of the rotation group, but even with a detailed analysis it is clear that, in this case, no significant conclusion concerning the properties of the orbits can be arrived at. In fact, the situation is very much the same as in classical theory. The most interesting and significant conclusions concerning the properties of the "orbits" will result if there is only a finite number of linearly independent states in terms of which all states obtained by rotation can be expressed. In the well-known case when this number is 1, all states are spherically symmetric and the conclusions are usually only little less striking.

#### E.P. WIGNER

Let us now review briefly a way in which the representations of the Poincaré group can be determined. The procedure has actually been given by Frobenius long before the Poincaré group was known. Its application is based on the fact that the Poincaré group has an invariant subgroup, consisting of all displacements. Matters become particularly simple because this invariant subgroup is abelian (commutative). The mathematics which will be used is not rigorous because members of the continuous spectrum will be treated as if they were bona fide vectors in Hilbert space. However, the procedure can be justified rigorously, principally on the basis of the investigations of Mautner and von Neumann.

Let us consider states which belong to irreducible representations of the group of displacements. Since this group is abelian, the unitary irreducible representations are one-dimensional. Denoting the displacement vector by a, its operator by  $T_a$ , there will be "states"  $|p, \zeta \rangle$  for which

$$\mathbf{T}_{\mathbf{a}} | \mathbf{p}, \boldsymbol{\zeta} \rangle = e^{-i\mathbf{p} \cdot \mathbf{a}} | \mathbf{p}, \boldsymbol{\zeta} \rangle$$
(2.1)

where p.a is the Lorentz scalar product of the two vectors p and a:

$$p.a = p_t a_t - p_x a_x - p_y a_y - p_z a_z.$$
 (2.1a)

The reason for the apparently arbitrary sign convention adopted in (2.1) will become evident soon. It also follows from the unitary nature of the representation that the components of p must be real. Otherwise, T<sub>a</sub> would not be unitary. However, the existence of vectors for which (2.1) holds would be rigorously assured only if the components of p were discrete variables. As we shall see at once, this is not the case and it follows that the "vectors"  $|p, \zeta\rangle$  are not normalizable. This is the point where the derivation is not rigorous. The variable  $\zeta$  was introduced because it is possible that there are several vectors which transform, under the operations of the displacement group, according to the representation (e<sup>-ip. a</sup>); the index t distinguishes these vectors. It can be assumed to be a discrete variable but if there are infinitely many vectors which belong to the  $(e^{-ip.a})$  representation, it will assume infinitely many values. Naturally, we do not yet know for which four-vectors p there are Hilbert vectors  $|p, \zeta \rangle$ , i.e., which representations (e<sup>-ip. a</sup>) of the displacement subgroup occur in the Poincaré group's representation which is being analyzed. As a matter of fact, this representation is not yet specified.

It will be shown now that if a representation of the Poincaré group contains the representation ( $e^{-ip.a}$ ) of the displacement subgroup, it also contains all representations ( $e^{-ip'.a}$ ) of this subgroup if p' = Lp can be obtained from p by a proper Lorentz transformation L. The representation ( $e^{-ip.a}$ ) is contained in a representation of the Poincaré group if there is a vector  $|p, \varsigma \rangle$  in the Hilbert space of the latter for which (2.1) is valid. Similarly, ( $e^{-iLp.a}$ ) is contained in the same representation if there is a vector for which (2.1) with p replaced by p' = Lp is valid. Since the vector  $|p, \varsigma \rangle$  is expecte to describe a state with four-momentum p, one will expect that the operatio:  $O_L$ , which corresponds to the Lorentz transformation L, will transform this state into one with momentum p' = Lp. Hence, one will expect that

$$T_{a}(O_{L} | p, \boldsymbol{\xi} >) = e^{-iLp. a} O_{L} | p, \boldsymbol{\xi} >.$$

$$(2.2)$$

This is indeed the consequence of the equation

$$T_a O_L = O_L T_{L^{-1}a}$$
 (2.3)

This equation expresses the fact that a Lorentz transformation L followed by the displacement a is identical to a displacement by  $L^{-1}a$ , followed by the Lorentz transformation L. If (2.3) is applied to the vector  $|p, \zeta\rangle$ , the left side will be identical with the left side of (2.2). The right side becomes

$$O_L T_{L^{-1}a} | p, \varsigma \rangle = O_L e^{-ip. L^{-1}a} | p, \varsigma \rangle = e^{-ip. L^{-1}a} O_L | p, \varsigma \rangle.$$

The second member is a consequence of (2.1) as applied to the displacement L<sup>-1</sup>a, the last member follows because the exponential is a numerical factor and O<sub>L</sub> is linear. Furthermore, it follows from the properties of the Lorentz scalar product that

$$p.L^{-1}a = Lp.a$$
 (2.4)

so that indeed, (2.2) is established. This then proves that if the Hilbert space of a representation of the Poincaré group contains vectors  $|p, \zeta\rangle$  with four-momentum p, it also contains vectors with all the momentum Lp, where L is any Lorentz transformation. According to (2.2),  $O_L | p, \zeta \rangle$  is such a vector.

Since the  $|Lp, \eta\rangle$ , for all possible values of  $\eta$ , form a complete set of vectors which transform under the displacement group according to the representation (e<sup>iLp. a</sup>), one can conclude that

$$O_{L} | p, \zeta \rangle = \sum c_{\eta} | Lp, \eta \rangle.$$
(2.5)

The coefficients  $c_{\eta}$  can depend on p,  $\zeta$ , and L. We shall use only a special case of (2.5) to define what has come to be called the "little group".

# III. THE LITTLE GROUP

We have seen that the four-vectors p for which there are Hilbert vectors satisfying (2.1) form a set which is invariant under all proper Lorentz transformations. In an irreducible representation, all such vectors can be obtained from a single one by applying all possible Lorentz transformations to it. Hence, the Lorentz length p.p of the momenta is the same for all state vectors which are present in the representation space of an irreducible representation. Altogether, one has to distinguish six qualitatively different cases.

1. p.p =  $m^2 > 0$ ,  $p_t > 0$ . The corresponding representations describe the transformation properties of real particles with finite rest mass. 2. p.p = 0,  $p_t > 0$ . These representations refer to particles with zero rest mass. The equations which correspond to some of these representations are well-known, but we shall discuss all of them.

3.  $p.p = m^2 < 0$ , i.e., p is space-like, m imaginary. In this case  $p_t$  can

63
assume arbitrarily large negative (as well as positive) values. It is axiomatic that no particle can exist which corresponds to such a representation because, if it existed, it could transfer any amount of energy to a particle of class 1 by going over into a state with sufficiently large negative  $p_t$ . Neve theless, the representations of this class will be described and equations of motion given which correspond to these representations. Also, some of the properties will be given which particles corresponding to these representations would have, if they existed. This conforms to the program giver in the first section.

4. p.p = 0,  $p_t < 0$ . Again,  $p_t$  can assume arbitrarily large negative values. However, the representations of this class are simply conjugate complex to the representations of class 2 and will not be discussed further. 5. p.p =  $m^2 > 0$ ,  $p_t < 0$ . These representations are conjugate complex to the representations of class 1 and will not be discussed further either. Agai  $p_t$  can assume arbitrarily large negative values.

6.  $p_t = p_x = p_y = p_t = 0$ . All states would be displacement invariant. Again, i is axiomatic that no particles with these transformation properties can exist.

The preceding enumeration gives the possible momentum vectors p for which states  $|p, \zeta \rangle$  exist in the irreducible representation in question. The transformation properties of these states with respect to translations are given by (2.1); we shall now discuss their transformation properties with respect to (homogeneous) Lorentz transformations L. This discussion will be based on (2.5).

Let us select in every case, except the last one which will be disregarded, from all possible momentum vectors a definite one which will be called  $p^0$ . In the case of class 1,  $p^0$  is best chosen to be parallel to the time axis, in case 3, parallel to the z axis. In case 2, it will be the vector with components 1, 0, 0, 1. The choice of  $p^0$  is arbitrary, but it is useful to ma. it in order to fix the ideas.

We next define the "little group" as the group of all Lorentz transform: tions which leave  $p^0$  invariant.

$$Lpo = po. \tag{3.1}$$

The L which satisfy (3.1) evidently form a group and this group does not depend essentially on the arbitrary choice of  $p^{0}$ . If another momentum  $p^{1} = L_{1} p^{0}$  had been chosen, the transformations  $L_{1} \perp L_{1}^{-1}$  which leave it invariant would have formed a group which is isomorphic to the group of L which leave  $p^{0}$  invariant. However, with the preceding choice of  $p^{0}$ , it is clear that in case 1 the little group is the three-dimensional rotation group in case 3 the 2+1 dimensional Lorentz group, i.e. the group which leaves the form  $t^{2} - x^{2} - y^{2}$  invariant. In case 2, the group is not quite so obvious. It clearly contains the rotations in the xy plane and, as will be seen at once it also contains two sets of commuting operations  $T_{\xi} (\alpha)$  and  $T_{\eta} (\beta)$  which form, together with the rotations in the xy plane, a group isomorphic to the two-dimensional Euclidean group, i.e. the group of rotations and displace ments in the plane.  $T_{\xi} (\alpha)$  and  $T_{\eta} (\beta)$  are:

$$T_{\xi}(\alpha) = \begin{vmatrix} 1 + \frac{1}{2}\alpha^{2} & \alpha & 0 & -\frac{1}{2}\alpha^{2} \\ \alpha & 1 & 0 & -\alpha \\ 0 & 0 & 1 & 0 \\ \frac{1}{2}\alpha^{2} & \alpha & 0 & 1 - \frac{1}{2}\alpha^{2} \end{vmatrix}, \quad (3.2)$$
$$T_{\eta}(\beta) = \begin{vmatrix} 1 + \frac{1}{2}\beta^{2} & 0 & \beta & -\frac{1}{2}\beta^{2} \\ 0 & 1 & 0 & 0 \\ \beta & 0 & 1 & -\beta \\ \frac{1}{2}\beta^{2} & 0 & \beta & 1 - \frac{1}{2}\beta^{2} \end{vmatrix}, \quad (3.2a)$$

The first row (and column) refers to the t component, the second, third and last to the x, y and z components. No simple argument is known to this writer to show directly that the group of Lorentz transformations which leave a null vector invariant is isomorphic to the two-dimensional Euclidean group, desirable as it would be to have such an argument. Clearly, there is no plane in the four-space of momenta in which these transformations could be interpreted directly as displacements and rotations because all transformations considered are homogeneous. The simplest geometrical picture known to me uses two vectors  $p^{\xi}$  and  $p^{\eta}$ , of length -1 and orthogonal to each other as well as to  $p^{\circ}$ . These vectors could be unit vectors parallel to the x and y axes. The  $T_{\xi}(\alpha)$  then adds  $\alpha p^{0} to p^{\xi}$ , whereas  $T_{\eta}(\beta)$  adds  $\beta p^{\circ}$  to the  $p^{\eta}$ .

In summary, then, the little groups for the first three cases are: 1.  $p.p = m^2 > 0$ ,  $p_t > 0$ : the three-dimensional rotation group 2. p.p = 0,  $p_t > 0$ : the two-dimensional Euclidean group 3.  $p.p = m^2 < 0$ : the 2+1 dimensional Lorentz group.

The little groups for cases 4 and 5 are the same as for 2 and 1, but we shall not be concerned with these cases.

The significance of the operations of the little group becomes evident if (2.5) is specialized to  $p = p^{0}$  and L a member of the little group. One then has

$$O_{L} | p^{0}, \zeta \rangle = \sum_{\eta} D(L)_{\eta \zeta} | p^{0}, \eta \rangle.$$
(3.3)

The dependence of the  $c_{\eta}$  on the remaining variables,  $\zeta$  and L, is made explicit in (3.3). The L is, however, restricted to members of the little group. One now concludes in the usual way, by applying another operation  $O_M$  of. the little group to (3.3), that the coefficients  $D(L)_{\eta\zeta}$  form a representation of the little group. This representation will be unitary and irreducible if the representation of the Poincaré group which we are analyzing is unitary and irreducible. It can be shown, further, that all the coefficients  $c_{\eta}$  in (2.5) are essentially determined once the D(L) are given. Hence, the unitary irreducible representations of the Poincaré group are characterized by two entities: (a) the set of momentum vectors which can be obtained from a singlemomentum vector  $p^{\circ}$  by applying to it all proper Lorentz transformations and (b) an irreducible unitary representation of the little group, i.e. the

### E.P. WIGNER

group of proper Lorentz transformations which leave  $p^{0}$  invariant. We shall take up the three cases of the preceding section separately.

# IV. INFINITESIMAL AND CASIMIR OPERATORS

The infinitesimal operators of a unitary representation are skew hermitean; they become hermitean when multiplied by i and correspond to conserved quantities. Because of (2.1), the infinitesimal operators for a displacement parallel to the t, x, y, z axes are  $-ip_t$ ,  $ip_x$ ,  $ip_y$  and  $ip_z$ . Hence,  $p_t$ ,  $-p_x$ ,  $-p_y$ ,  $-p_z$  are conserved quantities; they are the covariant components of the momentum. The covariant components of the angular momentum tensor will be denoted by  $M_{k\ell} = -M_{\ell k}$ . The commutation relations are, then,

$$\left[\mathbf{p}_{k}, \mathbf{p}_{\ell}\right] = 0 \left[\mathbf{M}_{k\ell}, \mathbf{p}_{m}\right] = i(g_{\ell m}\mathbf{p}_{k} - g_{km}\mathbf{p}_{\ell})$$
(4.1)

and

$$\begin{bmatrix} M_{k\ell}, M_{mn} \end{bmatrix} = i(g_{\ell m} M_{kn} - g_{km} M_{\ell n} + g_{kn} M_{\ell m} - g_{\ell n} M_{km})$$
(4.1a)

where g is the metric tensor,  $g_{tt} = -g_{xx} = -g_{yy} = -g_{zz} = 1$ , all other components of g vanishing.

The significance of the infinitesimal operators in the present context derives from the fact that the equation of motion gives the change of the state vector for an infinitesimal displacement of time. Hence, the equation of motion will be an equation which permits the calculation of the infinitesimal operator for such a displacement.

Functions of the infinitesimal operators which commute with all infinitesimal operators — such functions are called Casimir operators commute with all operators of the representation. These are, after all, exponentials, and products of exponentials, of the infinitesimal operators. Each Casimir operator of an irreducible representation must be equivalent with multiplication by a number, at least if the Casimir operator in question is hermitean. In other words, all vectors in the representation space of an irreducible representation must be a characteristic vector of every hermitean Casimir operator and the corresponding characteristic value can depend only on the Casimir operator and the irreducible representation, not on the vector in the representation space. In fact, the vectors which belong to a given characteristic value of a Casimir operator form an invariant subspace and the only non-empty invariant subspace of an irreducible representation is the whole representation space.

It follows that the irreducible representations of any group can be characterized, at least partially, by the values of the Casimir operators for the representation in question, i.e. by the numbers with which the Casimir operators multiply the vectors in the representation space of the irreducible representation in question. The Poincaré group has two Casimir operators. One of these was implicitly determined before: it describes the manifold of momenta

$$P = m^2 = p.p \tag{4.2}$$

by the common length of the momentum vectors of the states of the representation. The second Casimir operator is Lubanski's invariant; this characterizes the representation of the little group. It is the square of the total angular momentum in the coordinate system in which the particle is at rest, multiplied with the square of the mass. Mathematically, Lubanski's invariant is the negative Lorentz square of a vector w

$$W = -W.W.$$
 (4.3)

The contravariant components of this vector are

$$w^{k} = \frac{1}{2} \epsilon^{k\ell m n} p_{\ell} M_{mn}, \qquad (4.3a)$$

 $\epsilon^{klmn}$  being the fully antisymmetric tensor and (3.3a) implying summation of the repeated indices.

We shall not use the Casimir operators to derive the various irreducible representations of the Poincaré group. However, having derived the irreducible representations, we shall calculate the Casimir operators and ascertain the extent to which they characterize the representation or can even replace them.

# V. CASE OF POSITIVE REST MASS

The results are, in this case, well known. The irreducible representations of the little group, which is the three-dimensional rotation group in this case, can be characterized by a quantity s which can assume the values  $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ ; it is called the spin. The dimension of the representation s is 2s+1 so that  $\zeta$  can assume 2s+1 values and there are 2s+1 states with the same four-momentum. The representation with  $p.p=m^2=P$  and the s representation of the little group can be denoted by  $P_s$ . Equations of motion for the particles which belong to the representation  $P_s$  have also been given; in fact, there are several forms for these equations. It should be noted, however, that the solutions of these equations do not all belong to the representation  $P_s$ . They all have negative energy solutions which belong to the conjugate complex of  $P_s$ , i.e. to the fifth class of the section III. These spurious solutions are then eliminated, or rather reinterpreted, when the transition to the field theory is undertaken.

The first Casimir invariant is  $m^2$ , the second one, W, can easily be calculated for one of the states  $|p^0, \zeta\rangle$ . For these,  $p_x = p_y = p_z = 0$ ,  $p_t = m$ , so that  $w_x = mM_{yz}$ ,  $w_y = mM_{zx}$ ,  $w_z = mM_{xy}$  and

$$W = m^2 s(s+1),$$
 (5.1)

so that indeed P and W suffice to characterize the representations with real rest mass, except that for the two conjugate complex representations — one of class 1 and the corresponding one of class 5 — the Casimir operators

have the same value. It was mentioned before that the equations of motion also permit the vectors of these two representation spaces.

The equations for positive rest mass have been discussed in the literature repeatedly and will not be given in detail.

# VI. CASE OF ZERO REST MASS

A. The representations of the little group.

The representations of the little group, that is the two-dimensional Euclidean group, are not as commonly known as those of the three-dimensional rotation group. They could be easily determined, however, by the method used for the Poincaré group. The operators of displacement,  $T_{\xi}(\alpha) T_{\eta}(\beta) = T_{\eta}(\beta) T_{\xi}(\alpha)$ , form an abelian invariant subgroup and one can choose "vectors" in the space of the Euclidean group's representation which belong to an irreducible representation of this invariant subgroup. If these are denoted by  $||\pi', \pi''\rangle$ , one has

$$T_{\xi}(\alpha) \mid |\pi', \pi''\rangle = e^{-i\pi'\alpha} \mid |\pi', \pi''\rangle, \qquad (6.1a)$$

$$\mathbf{T}_{n}(\boldsymbol{\beta}) \mid \mid \boldsymbol{\pi}', \ \boldsymbol{\pi}'' \rangle = \mathrm{e}^{-\mathrm{i}\,\boldsymbol{\pi}''\boldsymbol{\beta}} \mid \mid \boldsymbol{\pi}', \ \boldsymbol{\pi}'' \rangle. \tag{6.1b}$$

It is good to remember that the "displacements"  $T_{\xi}$  and  $T_{\eta}$  are not displacements in any physical space, their most visualizable interpretation in terms of physical quantities being given after equations (3.2). Similarly, the representation space is not a physical space but the space of the coordinate axes which were denoted before by  $\xi$  (see (3.3)). The argument proceeds from this point just as in the case of the Poincaré group but is simpler because the group is much more simple. The possible values of  $(\pi', \pi'')$  can all be obtained by an orthogonal transformation from one such two-dimensional vector, i.e., in an irreducible representation only such  $||\pi', \pi''\rangle$  occur for which  $\pi'^2 + \pi''^2 = \Xi^2$  has a fixed value.

We shall not follow this method but use the same one which will be used also to determine the representations of the little group in the case of imaginary rest mass. This method is based on the solution of the commutation relations of the infinitesimal operators. Since Gårding's construction of an everywhere dense set of vectors in representation space to which all infinitesimal operators can be applied, this is entirely legitimate. The only disadvantage of this method, as compared with the usual one, is,that it gives only the infinitesimal operators, not those for the actual group elements. However, the determination of the infinitesimal operators will suffice for our purposes.

The infinitesimal operators of the little group are  $M_{xy}$ , and, as can be seen from (3.2) by setting  $\alpha$  and  $\beta$  infinitely small,  $\pi' = M_{zx} - M_{tx}$  and  $\pi'' = M_{zy} - M_{ty}$ . The communication relations between these operators are

$$\left[\pi', \pi''\right] = 0$$
 ,  $\left[M_{xy}, \pi'\right] = i\pi''$  ,  $\left[M_{xy}, \pi''\right] = -i\pi'$ . (6.2)

The characteristic values of  $M_{xy}$  can be either integer, or half-integer. In either case, these are discrete numbers so that one can assume a form of the representation in which  $M_{xy}$  is diagonal. Let us denote the diagonal elements by  $m_a$ ; the  $\alpha\beta$  matrix elements of the second and third equations of (6.2) are then

$$(m_a - m_\beta)\pi'_{a\beta} = i\pi''_{a\beta}, \qquad (m_a - m_\beta)\pi''_{a\beta} = -i\pi'_{a\beta}. \qquad (6.3)$$

One easily concludes that  $\pi'_{a\beta} = \pi''_{a\beta} = 0$  unless  $|m_a - m_\beta| = 1$ , that is, if one arranges the diagonal elements of  $M_{xy}$  in increasing order, both  $\pi'$  and  $\pi''$  have non-vanishing matrix elements only between consecutive values of the diagonal elements of  $M_{xy}$ . One can then transform all infinitesimal elements by a unitary diagonal matrix in such a way that the matrix elements of  $\pi'$  which are above the diagonal become real. Since  $\pi'$  is hermitean, all its matrix elements will then be real whereas the matrix elements of  $\pi''$  will be all imaginary. The first of the equations (6.2) then shows that all the non-vanishing matrix elements of  $\pi'$  are equal. Except for this last point, the situation reminds one of the representations of the rotation group in the form in which  $M_{xy}$  is diagonal.

Two cases have to be distinguished now. These are the analogues of the six cases encountered in Section III for the Poincaré group. If all the matrix elements of  $\pi'$  are zero, the same holds for  $\pi''$ . In these representations the unit element corresponds to all the "displacements"  $T_{\xi}(\alpha) T_{\eta}(\beta)$  and the representation is faithful only for the factor group of this representation, i.e. the two-dimensional rotation group. The representation can be irreducible only if it is one-dimensional. It coordinates to a rotation by  $\theta$  in the xy plane the matrix ( $e^{is\theta}$ ) where s can be an integer or a half integer, positive, negative or zero. These representations are denoted by 0; they are the well-known representations associated with a null-mass Klein-Gordon particle (s=0), neutrino of positive or negative chirality (s =  $\pm \frac{1}{2}$ ), a right or left circular polarized light quantum (s =  $\pm 1$ ). The quantity |s|is called the spin of the particle. Both Casimir invariants P and W vanish so that they cannot be used to distinguish these representations. Since the representations of the little group are one-dimensional, there is only one state with any definite momentum; the doubling of the number of states for  $s \neq 0$  is a result of reflection symmetries. These representations have been adequately discussed in the literature.

The non-singular case, in which  $\pi'$  and  $\pi''$  do not vanish, is less well known. The non-vanishing elements of  $\pi'$  will be denoted by  $\frac{1}{2}\Xi$  so that  $\pi'$  and  $\pi''$  are given by

|      | <b>  ···</b> | •••              | •••              |                  | •••              | • •   |                     | $\ \cdot\cdot\cdot$ | • • •         | • • • •                        | • • •              | ••                | • • • |              |
|------|--------------|------------------|------------------|------------------|------------------|-------|---------------------|---------------------|---------------|--------------------------------|--------------------|-------------------|-------|--------------|
| π' = |              | 0                | $\frac{1}{2}\Xi$ | 0                | 0                | • • • | , π <sup>''</sup> = |                     | 0             | <sup>1</sup> / <sub>2</sub> iΞ | 0                  | 0                 | •••   | •  <br>•   • |
|      |              | $\frac{1}{2}\Xi$ | 0                | $\frac{1}{2}\Xi$ | 0                | ••••  |                     |                     | - <u>1</u> iΞ | 0                              | ¹i≘                | 0                 | •••   |              |
|      |              | 0                | $\frac{1}{2}\Xi$ | 0                | $\frac{1}{2}\Xi$ | •••   |                     |                     | 0             | - <u>1</u> i≘                  | 0                  | <sup>1</sup> /₂iΞ |       |              |
|      |              | 0                | 0                | $\frac{1}{2}\Xi$ | 0                |       |                     |                     | 0             | 0                              | $-\frac{1}{2}i\Xi$ | 0                 | •••   |              |
|      |              |                  |                  |                  |                  | • •   |                     | ∥                   |               |                                | . <b>.</b> .       |                   |       |              |
|      |              |                  |                  | ~                |                  |       |                     |                     |               |                                |                    |                   | (6.4  | )            |

 $\Xi$  can be assumed to be positive because its sign can be changed by transforming all infinitesimal elements with a diagonal matrix whose diagonal elements are, alternately, 1 and -1. Clearly, these representations of the little group are infinite dimensional; they can be characterized by the Casimir operator  $\pi^{12} + \pi^{2} = \Xi^2$  of the two-dimensional Euclidean group. However, this invariant does not characterize them completely; the diagonal elements of  $M_{xy}$  can be either the integers ... -2, -1, 0, 1, 2, ..., or the half integers ...  $-\frac{3}{2}$ ,  $-\frac{1}{2}$ ,  $\frac{1}{2}$ ,  $\frac{3}{2}$ , ... In the former case, the representation is single valued, in the latter case two valued. As far as representations of the two-dimensional Euclidean group are concerned, the characteristic values of  $M_{xy}$  could be any arithmetic series with difference 1. However, unless these arithmetic series consist either of the integers, or of the half integers, the representation of the Euclidean group will be more than two valued so that no one or two valued representation of the Poincaré group can be constructed from these representations of the little group. There is a theorem according to which all representations up to a factor of the Poincaré group can be made one or two valued by multiplying the operators of the representation by suitable factors. Hence, the many valued representations are of no interest.

Since the characteristic value of the operator  $M_{xy}$ , for a state in which the momentum vector is in the tz plane, can extend to infinity, these representations are also called "infinite spin" representations. The values of the Casimir operators P and W are 0 and  $\Xi^2$ . The single-valued representation,  $O(\Xi)$ , is not distinguished from the two-valued representation  $O'(\Xi)$  by the values of the Casimir operators.

Numerous arguments can be adduced to show that no real particles can exist which would transform according to the representation  $O(\Xi)$  or  $O'(\Xi)$ . The simplest of these arguments is that the heat capacity of vacuum due to the possibility of the formation of particles, or of pairs of particles, is proportional to the number of polarizations of the particle in question. This number is infinite for particles with one of the representations  $O(\Xi)$  or  $O'(\Xi)$  because the representation of the little group is infinite dimensional. Hence, the mere possibility of the existence of any of these particles would give an infinite heat capacity to vacuum.

### **B.** Equations

The equations for the well-known zero mass cases  $0_s$  are adequately discussed in the literature. Again, all known equations permit not only solutions which belong to the representation  $0_s$ , but also solutions which belong to the conjugate complex of  $0_s$ . These are the negative energy solutions which are then eliminated or reinterpreted in the second quantized form of the theory. However, equations for the  $0(\Xi)$  and  $0^{\circ}(\Xi)$  cases were obtained only after a general procedure for obtaining equations from representations was devised. This will be described next and illustrated also on one of the earlier, well established cases. It should be admitted, though, that the procedure to be described can be used only in conjunction with single valued representations. The reason for this will be evident at once. If one wants to derive similar equations for the two valued representations, one has to use a space appropriate for these representations: a two-dimensional complex space in which the two valued representations of the Lorentz group are isomorphic to unimodular matrices. However, this will not be spelled out in detail.

The term "equation for a representation" is not clearly defined and, in fact, we have seen that several equations may correspond to the same representation. The quantity to which the equation applies will be called wave function; it may have one or more components. The wave functions which satisfy the equation, or equations, should transform, under the operations of the Poincaré group, according to the representation in question, but this condition does not yet determine the equation, not even the variables on which the wave function depends. It is always possible, for instance, to introduce extraneous, that is unnecessary, variables and then neutralize these by equations as a consequence of which the wave function is either independent of these unnecessary variables, or depends on them only in a trivial fashion. We shall postulate, however, that the variables be of such a nature that they clearly indicate how the wave function transforms under the operations of the Poincaré group. This means that the variables are either the components of a four vector x, or of the difference of two vectors. A four vector x goes over, under a displacement by a, into x+a, under a Lorentz transformation L into Lx. The difference of two vectors is invariant under displacements and transforms like a four vector under Lorentz transformations. Since I do not know a better expression for vectors of this nature, I will call them difference vectors. The position vector is an example for the first case and one is indeed inclined to interpret the components of a vector which occurs in a wave equation as the position vector. This may or may not be justified.\* The momentum vector is a difference vector. It is because of this restriction of the variables which are admitted that the equations will always correspond to single valued representations.

We shall determine next the number of vectors and difference vectors which are needed as variables of the wave function. One will be sure to have introduced enough variables into the wave function only if every Poincaré transformation changes the set of values of the variables. If this is not the case, some Poincaré transformations will necessarily leave the wave function unchanged, whereas it may follow from the representation that the wave function is changed by the transformation in question. Hence, the variables should be able to describe completely a frame of reference. A frame of reference can be given by an ordinary vector which describes the origin of the coordinate system, and four difference vectors which give the direction of the four coordinate axes. These vectors have, together, twenty components - surely too many variables, but it will not be difficult to eliminate the unnecessary ones by restricting the variability domain of some and by pointing out that the wave function is independent of the others. Nevertheless, we do not want to go too far with such an elimination because the final variables should be quadruplets of vector components.

Let us consider first the difference vectors. One of these may be identified with the momentum vector and it is convenient, then, to give it the length of the momentum vector. This is purely a matter of convenience,

<sup>•</sup> In a recent article, (Dubna report P 939), M.I. Shirokov criticizes the replacement of the variables of the wave equation by other position operators, as proposed by T.D. Newton and the present writer. Unfortunately, his considerations contain a serious error.

giving the final equations a slightly simpler form; the frame of reference could be specified also if all the difference vectors were normalized in some other fashion. The other difference vectors can be assumed to be mutually orthogonal, orthogonal to the momentum vector, and of length 1 or -1, whichever is possible. Since these conditions completely specify the last difference vector in terms of the first three, the components of this are surely unnecessary variables and can be omitted. Even the third difference vector contributes only one independent variable — since it is normalized and is perpendicular to two other vectors. It turns out, although this is not evident at this point, that it is also "unnecessary", i.e., its omission does not automatically entail the invariance of a function of the remaining variables under a Poincaré transformation under which it should not be invariant. Hence, we are left with two difference vectors, one of them p. The other will be denoted by  $\xi$ ; its condition of normalization and perpendicularity to p give the wave equations

$$(p, p)\psi = m^2\psi$$
, (6.5)

$$(\boldsymbol{\xi},\boldsymbol{\xi})\psi = -\psi, \qquad (6.6)$$

$$(p.\xi)\psi = 0.$$
 (6.7)

At this point,  $\psi$  depends on the eight components of two difference vectors, p and  $\xi$ , and the four components of a normal vector which will be denoted by x and which will permit  $\psi$  to change under displacements. The equations (6.5), (6.6), (6.7) are common to the equations of all representations; the remaining equations with one exception will be characteristic of the representation according to which the solutions of the equation should transform. Our remaining task is, therefore, to express the equations of the representation in terms of the variation of the wave function. We shall carry this out in detail at this point for only two cases: the Klein-Gordon equation for a finite mass P<sub>0</sub>, and the case of present interest 0( $\Xi$ ).

The first representation equation, still common to all representations, is (2.1). According to this, an infinitesimal displacement by ha changes the state vector by a factor 1 - ihp.a. Hence, if we use a vector notation for the variables of  $\psi$ 

$$\psi(x + ha, p, \xi) = (1 - ihp.a) \psi(x, p, \xi)$$
 (6.8)

or, since this is valid for all a,

$$\frac{\partial \psi}{\partial \mathbf{x}_{k}} = -\mathrm{i}\mathbf{p}^{k}\psi. \tag{6.9}$$

It follows from (6.9) that the components of x are "unnecessary variables". If  $\psi$  is given as function of p and  $\xi$  for one vector x, say x=0, it is determined by (6.9) for all other x. One has

$$\psi(\mathbf{x}, \mathbf{p}, \xi) = e^{-i\mathbf{p} \cdot \mathbf{x}} \psi(0, \mathbf{p}, \xi).$$
 (6.10)

Hence,  $\psi$  can be considered to depend only on p and  $\xi$ .

Alternately, one can integrate (6.10) over p and obtain a function of x and  $\xi$  only:

$$\widehat{\psi}(\mathbf{x},\xi) = (2\pi)^{-2} \int d^4 p \,\psi(\mathbf{x}, \mathbf{p},\xi) = (2\pi)^{-2} \int \psi(\mathbf{0}, \mathbf{p},\xi) \, e^{-i\mathbf{p}\cdot\mathbf{x}} \, d^4 \mathbf{p}. \quad (6.11)$$

From  $\hat{\psi}(\mathbf{x}, \boldsymbol{\xi})$ , the original  $\psi(\mathbf{x}, \mathbf{p}, \boldsymbol{\xi})$  can be recovered by Fourier inversion,

$$\psi(\mathbf{x}, \mathbf{p}, \boldsymbol{\xi}) = \psi(0, \mathbf{p}, \boldsymbol{\xi}) e^{-i\mathbf{p}\cdot\mathbf{x}} = (2\pi)^{-2} \int d^4\mathbf{x}' \, \hat{\psi}(\mathbf{x}', \boldsymbol{\xi}) e^{i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})} \,.$$
 (6.11a)

The relation between  $\psi(0, p, \xi) = \psi(p, \xi)$  and  $\hat{\psi}(x, \xi)$  is, except for a proportionality factor, the usual one. It follows from (6.5) and (6.7) that  $\psi(x, p, \xi)$  contains the factors  $\delta$  (p.p - m<sup>2</sup>) and  $\delta(p, \xi)$  (as well as  $\delta(\xi, \xi + 1)$ ), but these do not interfere with the integrations in (6.11). It follows, however, from these equations for  $\hat{\psi}$  that

$$-\frac{\partial^2}{\partial x_k \partial x^k} \hat{\psi} = m^2 \hat{\psi}, \qquad (6.5a)$$

$$\xi_k \frac{\partial}{\partial x_k} \hat{\psi} = 0. \qquad (6.7a)$$

The relation between  $\psi(\mathbf{p}, \boldsymbol{\xi})$  and  $\hat{\psi}(\mathbf{x}, \boldsymbol{\xi})$  is so simple that it makes little difference which of these wave functions one uses. In the present note the momentum space representation,  $\psi(\mathbf{p}, \boldsymbol{\xi})$ , will be preferred.

Let us now consider a representation with finite rest mass. Equation (6.7) restricts  $\xi$  to a three-dimensional space-like manifold which is perpendicular to p. In particular, if  $p = p^0$ , i.e., is parallel to the time axis,  $\xi_{f} = 0$  and (6.6) further restricts the spatial part of  $\xi$  to the unit sphere,  $\xi_x^2 + \xi_y^2 + \xi_z^2 = 1$ ,  $\xi_t = 0$ . If we apply an element of the little group to the two vectors  $p^0$  and  $\xi$ , the former will remain unchanged, the latter point to another point of the unit sphere. However, if the representation is  $P_0$ , the representation of the little group is the identical representation,  $\psi$  has the same value for any two positions which can be transformed into each other by an element of the little group. Since the little group is the group of all three-dimensional rotations,  $\psi$  has the same value no matter to which point of the unit sphere  $\xi$  points. It follows that  $\psi$  is independent of  $\xi$  within the domain of this variable, as restricted by (6.6) and (6.7). Hence,  $\xi$  is an unnecessary variable in this case and can be dropped. Thus, for the representation  $P_0$ , the wave function depends only on p and obeys the single equation (6.5). This, or rather the Fourier transform of this, is the usual Klein-Gordon equation so that our procedure led, in this case, to the usual equation.

It would be quite interesting to derive the equations for the other representations  $P_s$ , and also for  $O_s$ . Instead, we proceed at once to  $O(\Xi)$ . In this case, (6.6) and (6.7) restrict the variables  $\xi$  to a cylinder-like structure the axis of which is p. At  $\xi_1 = 0$ , the spatial components of  $\xi$  are restricted to a unit circle in the plane which is perpendicular to the direction of the spatial part of p. If we denote by  $\xi'$  and  $\xi''$  two perpendicular purely spatial vectors (i.e., whose t component is 0) which are orthogonal to p, the general purely spatial  $\xi$  vector will be  $\xi' \cos \theta + \xi'' \sin \theta$ . The other  $\xi$ 

### E.P. WIGNER

vectors which are consistent with (6.6) and (6.7) can be obtained by adding to one of the purely spatial  $\boldsymbol{\xi}$  vectors an arbitrary multiple of p. This, then, forms the aforementioned cylinder-like structure. Since p is orthogonal to itself, the vectors just obtained are also orthogonal to p. It follows similarly that the length of all vectors  $\boldsymbol{\xi}' \cos \theta + \boldsymbol{\xi}'' \sin \theta + cp$  is -1.

We yet have to express the condition that the representation of the little group is given by (6.4) so that  $W = \Xi^2$ . In order to express this condition, we recall that the infinitesimal operators of the Lorentz transformations of the wave function are

$$\mathbf{M}_{mn} = \mathbf{i} \left( \mathbf{p}_m \frac{\partial}{\partial \mathbf{p}^n} - \mathbf{p}_n \frac{\partial}{\partial \mathbf{p}^m} + \boldsymbol{\xi}_m \frac{\partial}{\partial \boldsymbol{\xi}^n} - \boldsymbol{\xi}_n \frac{\partial}{\partial \boldsymbol{\xi}^m} \right). \tag{6.12}$$

Both p and  $\xi$  are vectors, hence both change upon a Lorentz transformation. On the other hand, (6.9) shows that the infinitesimal operator of displacement is simply multiplication by -ip. Hence, the w<sup>k</sup> of (4.3a) will have two types of terms: those arising from the first two terms of (6.12), involving only p, and those arising from the last two terms. However, because of the antisymmetry of the  $\epsilon$ , all the terms vanish which involve only the p. This is natural since all w vanish if there is no spin variable as in the case of the representation P<sub>0</sub>. Hence, we have

$$w^{k} = \frac{1}{2}i \epsilon^{k\ell mn} \left( \frac{\partial}{\partial \xi^{n}} \xi_{m} - \frac{\partial}{\partial \xi^{m}} \xi_{n} \right) p_{\ell}$$
$$= i \epsilon^{k\ell mn} \frac{\partial}{\partial \xi^{n}} \xi_{m} p_{\ell}. \qquad (6.13)$$

If one now calculates W

$$W = -w^{k} w_{k} = \epsilon^{k\ell m n} \epsilon_{k\ell' m' n'} \frac{\partial}{\partial \xi^{n}} \xi_{m} P_{\ell} \frac{\partial}{\partial \xi^{n'}} \xi^{m'} p^{\ell'}$$
(6.14)

one can make use of the identity

$$\epsilon^{k\ell m n} \epsilon_{k'\ell'm'n'} = -\delta_{\ell\ell'} (\delta_{mm'}\delta_{nn'} - \delta_{mn'}\delta_{nm'}) - \delta_{\ell'm} (\delta_{m'n}\delta_{n'\ell} - \delta_{m'\ell}\delta_{n'n})$$

$$- \delta_{\ell'n} \left( \delta_{m'\ell} \delta_{n'm} - \delta_{m'm} \delta_{n'\ell} \right). \tag{6.15}$$

If one inserts this into (6.14) and applies both sides to  $\psi$ , the first two terms give zero because both  $p_{\ell} p^{\ell} \psi$  and  $p_{\ell} \xi^{\ell} \psi$  vanish. Hence one can set in any scalar product:

$$W\psi = -\frac{\partial}{\partial\xi^{n}} \xi_{m} \frac{\partial}{\partial\xi_{\ell}} p_{\ell} (\xi^{m}p^{n} - p^{m}\xi^{n})\psi$$

One can now push the  $\xi_{g}$  across the  $\partial /\partial \xi_{m}$  and obtain using again (6.5), (6.7), and also (6.6)

$$W\psi = -\frac{\partial^2}{\partial \xi^n \partial \xi_\ell} p^n p_\ell \psi = V^2 \psi \qquad (6.16)$$

where

$$V = i p_{\ell} \frac{\partial}{\partial \xi_{\ell}} . \qquad (6.16a)$$

It now follows from  $W\psi = \Xi^2 \psi$  that the linear space of the  $\psi$  can be decomposed into two subspaces. In one of these subspaces

$$\nabla \psi = \sum i \mathbf{p}_{\boldsymbol{\ell}} \frac{\partial \psi}{\partial \xi_{\boldsymbol{\ell}}} = \Xi \psi , \qquad (6.17)$$

in the other subspace (6.17) holds with the opposite sign. Evidently, both subspaces are relativistically invariant — and also equivalent as the remark after (6.4) shows. Hence, we may adopt as well (6.17) as the last equation for  $\psi$  (p,  $\xi$ ). It determines the variation of  $\xi$  along the lines in  $\xi$  space which are parallel to p. We recall that  $\xi$  is confined to a cylinder-like structure the axis of which is parallel to p. It follows that  $\psi$  can be freely chosen only on a line around this cylinder, for instance on the line  $\xi_1 = 0$ . This is, as was also mentioned before, a unit circle in the plane perpendicular to the direction of the spatial part of p. The reason why  $\psi$  is defined not only on this circle — where it can be chosen arbitrarily — but all over the cylinder is, that the relativistic invariance is manifest only if the variables of  $\xi$  are restricted only in a relativistically invariant fashion. This is done by (6.6) and (6.7). On the contrary, (6.5) is more properly an equation of motion,

There are altogether four equations for  $\psi(p,\xi)$ : (6.5) with m=0, (6.6), (6.7) and (6.17). The common solutions of these equations actually give two invariant linear manifolds: the positive energy solutions belong to  $0(\Xi)$ , the negative energy solutions to the conjugate complex of  $0(\Xi)$ . These can be obtained from the positive energy solution by complex conjugation and replacement of  $\xi$  by  $-\xi$ . It is of interest to apply the compatibility criterion to the four equations for  $\psi$  which postulates that the commutator of the operators of any two of them shall vanish if applied to  $\psi$ , and that this shall be a consequence of the original equations. Evidently, (6.5), (6.6) and (6.7) commute so that these do not-lead to any condition. However, the commutator of (6.6) and (6.17) gives just (6.7) whereas the commutator of (6.7) and (6.17) gives (6.5) with m=0. Hence, the compatibility criterion is satisfied — but it would not be satisfied for similar equations with a non zero rest mass.

It is clearly possible to transform the equations from momentum space into coordinate space. The infinitesimal operators of the little group in (6.4) use the coordinate system in which the  $\xi$  dependence of  $\psi$ , on the unit circle described before, is expanded into harmonic functions  $e^{im\theta}$ ,  $\theta$  being the polar coordinate. Actually, had we determined the representations of the twodimensional Euclidean group by the method outlined at the beginning of this section, using equations (6.1), the crucial equation (6.17) would have appeared as a more direct translation of the little group's representation. However, the method here used is somewhat quicker.

# VII. CASE OF IMAGINARY REST MASS

A. The representations of the little group.

The case of imaginary rest mass will not be treated in as much detail as the case of zero rest mass. It is believed that the general principles are adequately illustrated in the preceding section and their detailed application should not be too difficult. The little group in this case is clearly the 2+1 dimensional Lorentz group, the group of three-dimensional linear transformations which leave the form  $p_t^2 - p_v^2 - p_v^2$  invariant.

The representations of the 2+1 dimensional Lorentz group can be determined in the same way in which the representations of the two-dimensional Euclidean group were determined in the preceding section. Actually, even the representations of the 3+1 dimensional Lorentz group were determined by L.H. Thomas, using this method. The representations of the 2+1 dimensional Lorentz group were investigated in most detail by V. Bargmann. At the time he carried out this investigation it was not clear that he obtained all representations because he used infinitesimal operators in his calculation. However, Gårding's construction subsequently fully justified Bargmann's work.

The 2+1 dimensional Lorentz group has three infinitesimal elements. Their commutation relations are

$$[M_{xy}, M_{xt}] = i M_{yt}, \qquad [M_{xy}, M_{yt}] = -i M_{xt}, \qquad (7.1)$$

$$[M_{xt}, M_{yt}] = -i M_{xy}.$$
 (7.1a)

They differ from the commutation relations of the rotation group only in the signs. Since the representations in which we are interested are either single or double-valued, the characteristic values of  $M_{xy}$  are either integers or half integers. At any rate, they are discrete numbers so that we can assume, as in the preceding section, that  $M_{xy}$  is diagonal. Since the equations (7.1) are the same as the last two of equations (6.2), with  $\pi'$  and  $\pi''$  replaced by  $M_{xt}$  and  $M_{yt}$ , we can infer again that  $M_{xt}$  and  $M_{yt}$  have non-vanishing matrix elements only just above and just below the main diagonal and that it is possible to transform  $M_{xt}$  into a real matrix.  $M_{yt}$  will then be purely imaginary and both will have the form illustrated in (6.4) except that the  $\frac{1}{2} \Xi$  will be replaced by numbers which are in general different from each other. We denote the diagonal elements of  $M_{xy}$  by m; the non-vanishing elements of  $M_{xt}$  and  $M_{yt}$  will be denoted, then

$$(M_{xt})_{m,m+1} = (M_{xt})_{m+1,m} = N_{m+1}$$
, (7.2a)

$$(M_{yt})_{m,m+1} = -(M_{yt})_{m+1,m} = i N_{m+\frac{1}{2}}.$$
 (7.2b)

The last commutation relation (7.1a) now gives

$$N_{m+\frac{1}{2}}^2 - N_{m+\frac{1}{2}}^2 = \frac{1}{2}m, \qquad (7.3)$$

INVARIANT QUANTUM MECHANICAL EQUATIONS

$$N_{\rm m} = \frac{1}{2} \sqrt{{\rm m}^2 + {\rm c}}.$$
 (7.4)

Since  $N_m$  must be real, only such m must occur as a diagonal element of  $M_{xy}$  for which  $m^2 + c \ge 0$ . This will be automatically satisfied if  $m^2 + c \ge 0$  for all m and this gives rise to the first type of representations of the 2+1 dimensional Lorentz group. From (7.2) and  $(M_{xy})_{mm'} = m \delta_{mm'}$  one can calculate the Casimir operator of this group:

$$Q = M_{xt}^{2} + M_{yt}^{2} - M_{xy}^{2} = c + \frac{1}{4}.$$
 (7.5)

In the present case, the characteristic values of  $M_{xy}$  are either all the integers, positive, negative, and zero, or all half integers, positive and negative. In the latter case, c > 0,  $Q > \frac{1}{4}$ . In the former case, c need not be positive but only larger than  $-\frac{1}{4}$ , the smallest possible value of  $-m^2$ . Hence, Q > 0 holds for single-valued representations of this class,  $Q > \frac{1}{4}$  holds for the two-valued representations of this class. The former are called  $C_Q$ , the latter  $C'_Q$ .

It might appear, first, that this exhausts all the representations. This is not so, however, because if an  $N_{s-\frac{1}{2}}$  vanishes, the matrices whose rows and columns are labeled by m=s, s+1, s+2, ... are disconnected from the rows and columns with lower m and provide in themselves a solution of the commutation relations. Hence, if

so that

$$Q = c + \frac{1}{4} = -s(s - 1)$$
(7.6)

we have a second class of solutions of the commutation relations. For these, the characteristic values of  $M_{xy}$  are s, s+1, s+2, ... so that s is the <u>lowest</u> characteristic value. Clearly,  $s \ge 0$  must hold, otherwise  $N_{3\frac{1}{2}}^2$  would become negative. Hence, s can assume the values  $\frac{1}{2}$ , 1,  $\frac{3}{2}$ , ... and the values of the Casimir operator are quantized in this case. The representations of this subclass are denoted by  $D_5^+$ . (The case s=0 will be treated separately.) Similarly, if m assumes only the values -s, -s-1, -s-2, ... and  $N_{-s+\frac{1}{2}} = 0$ , the matrices  $M_{xt}$ ,  $M_{yt}$  given by (7.2), (7.4), and the diagonal matrix  $M_{xy}$ , will satisfy the commutation relations. The representations of this subclass are conjugate complex to the representations of the previous subclass. The value of the Casimir operator Q, and of the parameter c, will be the same as for the representations just discussed. The representations of this subclass are denoted by  $D_{s}^{-}$ .

The case s=0 remains to be discussed. It follows that, in this case, not only  $N_{s-\frac{1}{2}} = N_{-\frac{1}{2}}$  but also  $N_{s+\frac{1}{2}} = N_{\frac{1}{2}}$  vanishes. The matrices with the single row and column m=0 therefore separate from the rest and we obtain the trivial solution  $M_{xy} = M_{xt} = M_{yt} = 0$  of the commutation relations and the trivial representation in which every group element is represented by the unit operator. This representation will be denoted by  $D_0$ .

The preceding discussion assumes, implicitly, that the 2+1 dimensional Lorentz group has only one and two-valued representations up to a factor.

$$c = -(s - \frac{1}{2})^2$$

E.P. WIGNER

This is a permissible assumption because only these representations of the little group can be used to form a representation of the Poincaré group.

Let us return briefly to the Poincaré group in order to calculate the Casimir invariant W, defined by (4.3). Since this invariant has the same value for all vectors which belong to an irreducible representation, we may as well calculate it for a state of momentum  $p^0$ . Hence, we set  $p_t = p_x = p_y = 0$ ,  $p_z = m$  in (4.3a) and obtain

 $w^t = m \ M_{xy}, \qquad w^x = m \ M_{yt}, \qquad w^y = -m \ M_{xt}, \qquad w^z = 0.$  It follows that

$$W = -w \cdot w = m^2 Q.$$
 (7.7)

B. Equations.

It would seem offhand that one should be able to obtain the equations for the imaginary mass case by replacing m by im in the equations for positive rest mass. Thus, one can replace the Klein-Gordon equation by

$$(p_t^2 - p_x^2 - p_y^2 - p_z^2)\psi = -m^2\hat{\psi}$$
(7.8)

or

$$-\frac{\partial^2}{\partial x_k \partial x^k} \hat{\psi} = -m^2 \hat{\psi}. \qquad (7.8a)$$

It is clear, on the other hand, that this procedure cannot work because it would yield equations with a finite number, 2s+1, of polarizations (linearly independent states of the same momentum). Since the representations of the little group are, with one exception, infinite dimensional, there will be, except for that single case, infinitely many "directions of polarization".

As a matter of fact, the replacement of m by im gives a self-adjoint expression for the infinitesimal operators of the Poincaré group only in case of the Klein-Gordon equation — and this corresponds to the representation of the Poincaré group for which the little group's representation is the identical one,  $D_0$ . Thus, if one replaces m by im in Dirac's equation, the expression for i  $\partial/\partial t$  will not be self-adjoint any more. This resolves the paradox of the preceding paragraph but shows, at the same time, how strongly the results derived in these notes depend on the assumption of the unitary nature of the representations.

It is of some interest to investigate the behavior of the solution of the equation (7.8a) and to contrast it with the solutions of the equations with positive rest mass. We can further simplify the situation by assuming that there is only one space-like dimension x. The positive rest mass Klein-Gordon equation is then

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2} = \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} - \mathrm{m}^2\psi. \tag{7.9}$$

J.

It is well known that if, say, at time t=0, the wave function and its time deri-

78

vative vanish outside the interval (a, b), they will vanish, at time t (or have vanished at time -t), outside the interval (a - |t|, b + |t|). This expresses the finite propagation velocity of disturbances and can be proven in a variety of ways. The proof which is perhaps simplest starts from a Dirac equation

$$\left[\begin{array}{c|c} i & 0 & 1 \\ 1 & 0 & \frac{\partial}{\partial t} + i \\ \end{array}\right] \left[\begin{array}{c|c} 0 & 1 \\ \hline \partial x \\ \hline \partial x \end{array}\right] \phi = m \phi$$

and identifies the first component of  $\varphi$ , that is  $\varphi_1$ , with the solution of (7.9) which vanishes, together with its time derivative, outside the interval (a, b) at time 0. The second component  $\varphi_2$  will then be equal to  $m^{-1}(i\partial/\partial t + i\partial/\partial x)\varphi_1$  and have the same property. Hence, the time component of the Dirac current  $|\varphi_1|^2 + |\varphi_2|^2$  vanishes at time 0 on the half line x > b. Applying now the divergence theorem to the shaded region in Fig. 1, one sees that the



Fig.1

integral of the current across the tilted line, which goes through the point b', t, also vanishes. If the tilted line is space-like, the current across it is positive definite. Hence, it vanishes at every point. From this, the vanishing of both components of  $\varphi$  follows, just as both components of  $\varphi$  vanish on a t = const line if the current across this line,  $|\varphi_1|^2 + |\varphi_2|^2$ , vanishes. This, then, proves the theorem on the finite propagation velocity.

Interchanging now x and t in (7.9), we note that if  $\psi$ , together with its x derivative, is zero at x=0 outside an interval (a, b) of t, it will be zero at x after b + |x|, and was zero before a - |x|. Hence,  $\psi$  will be zero this time in the shaded area of Fig. 2. Instead of the maximum velocity



of propagation, one has a minimum velocity of propagation. This is, of course, what was to be expected. Actually, one is more interested in the initial condition which underlies the first figure: that  $\psi$  and  $\partial \psi/\partial t$  are zero

### E.P. WIGNER

for a fixed t, say t=0, in an interval of x. In this case it is not true that  $\psi$  is zero in the shaded area of Fig. 3. However, as t increases,  $\psi$  goes to zero in this area faster than any power of t.



Fig.3

As was mentioned before, the equations for all other representations will give an infinite number of polarizations so that the introduction of a second vector  $\boldsymbol{\xi}$  as a variable into the wave function is quite appropriate. The equations (6.6) and (6.7) define in this case a hyperboloid as the domain of the variables  $\boldsymbol{\xi}_i$ . This is unnecessarily complicated as there are null vectors which are perpendicular to the space-like vector p. We therefore replace (6.6) by

$$(\xi,\xi) \psi = 0$$
 with  $\psi = 0$  for  $\xi_t < 0$ , (7.10)

retain (6.7)

$$(p.\xi)\psi = 0$$
 (7.11)

and in order to denote a real number by m, we set

$$(p,p)\psi = -m^2\psi$$
 (7.12)

instead of (6.5). The calculation of the Casimir operator W then becomes very similar to that in the preceding section, with the roles of  $\xi$  and p interchanged. The only difference is that the non-commuting nature of  $\xi_m$  and  $\partial/\partial \xi_m$  has to be taken into account. The resulting expression is

$$W\psi = -m^{2}\left(\sum_{l}\frac{\partial}{\partial \xi_{l}}\xi_{l}\right)\left(\sum_{l}\frac{\partial}{\partial \xi_{l}}\xi_{l}-1\right)\psi. \qquad (7.13)$$

The variability domain of  $\psi$  is now a light cone in the three-space of  $\xi$  which is perpendicular to p. We can transform (7.13) in such a way that it contains only the operator  $\sum \xi_l \partial/\partial \xi_l$  which is the derivative along the straight null-lines of the cone. Further, we can express by (7.7) the Casimir operator W of the Poincaré group in terms of the Casimir operator Q of the 2+1 dimensional Lorentz group. This gives

$$- V (V+1) \psi = - \left[ (V + \frac{1}{2})^2 - \frac{1}{4} \right] \psi = Q \psi, \qquad (7.14)$$

$$V = \sum \boldsymbol{\xi}_{\boldsymbol{\ell}} \ \partial / \partial \boldsymbol{\xi}_{\boldsymbol{\ell}} . \tag{7.14a}$$

This V is different from that defined by (6.16a) for the equations for the  $0(\Xi)$  representations. However, we can conclude again, just as we did when deriving the equations for the  $0(\Xi)$  representations, that the linear set of wave functions for which (7.10), (7.11), (7.12) hold and for which the second Casimir operator has the value W, decomposes into two invariant linear subsets. For the first of these

$$\nabla \psi = \sum \xi_{\ell} \frac{\partial}{\partial \xi_{\ell}} \psi = \left(-\frac{1}{2} + \sqrt{\frac{1}{4} - Q}\right) \psi \tag{7.15}$$

holds, for the second

$$\nabla \psi = \sum \xi_{\ell} \frac{\partial}{\partial \xi_{\ell}} \psi = \left(-\frac{1}{2} - \sqrt{\frac{1}{4} - Q}\right) \psi. \qquad (7.15a)$$

We recall that  $Q = W/m^2$  is a function of the two Casimir operators W and  $m^2$ . These equations are quite similar to (6.17) but whereas the latter gives the change of  $\psi$  for an increment of the vector  $\xi$  which is parallel to p, (7.15) gives the change of  $\psi$  for an increment of  $\xi$  which is parallel to  $\xi$  itself. Both increments are, however, along the straight lines of the developable surface which is the definition domain of  $\xi$ . The resultant set of equations, (7.10), (7.11), (7.12), (7.15), was not discussed in detail.

### VIII. PROBLEMS WHICH REMAIN

The preceding discussion of the equation for representations with imaginary mass is even more perfunctory than the discussion of the  $O(\Xi)$  equations. Furthermore, apparently, no more complete discussion is available in the literature. Whereas for the  $O(\Xi)$  equations several equivalent forms of the relativistically invariant scalar product are known, the preceding discussion gives no expression therefor. This should be supplemented.

A more serious omission is our failure to give equations for the twovalued representations, that is for the representations which describe particles with half integer spin. In order to do this, one should again introduce a space in which a relativistic transformation can be defined. Such a space is, in this case, a two-dimensional space with complex coordinates. In that space then the total manifold of functions must be limited by as many relativistically invariant linear equations as possible. A "relativistically invariant" equation in this case is invariant under complex unimodular transformations. The statement "as many as possible" means that the elimination of a single further function, by a new equation or otherwise, together with

81

the postulate of relativistic invariance, eliminates all functions from the linear manifold so that this becomes vacuous. The task of obtaining equations for the two-valued representations in this way has not been carried out.

# REDUCTION OF THE ALGEBRA OF FIELD OPERATORS FOR THE CASE OF SEVERAL VACUUM STATES

# S. SCHLIEDER

# MAX PLANCK INSTITUTE OF PHYSICS, MUNICH, GERMANY

With the word vacuum a state shall be meant which is invariant under the inhomogeneous restricted Lorentz-group.

The result of the following proof will be:

If there are several vacua, the algebra  $\mathcal{A}$  of field operators will be fully reducible. This means, speaking loosely, that you have as many invariant subspaces of the Hilbert space under  $\mathcal{A}$  as you have vacuum-states.

For simplicity, the following is written down only for a scalar field which may interact with itself. The proof is still correct if one has a manifold of fields, if one assumes that each of the fields commutes spacelike with itself and with each of the other fields of the manifold.

The assumptions we make are essentially those given in the axioms of Wightman:

- (1) Lorentz covariance of the field;
- (2) Existence of one or several vacua;
- (3) The spectrum-condition for the momentum-operator;
- (4) The vacuum expectation values of products of the field-operator are temperate distributions;
- (5) Locality: [A(x), A(y)] = 0 for (x y) spacelike.
- (6) Definite metric in the space of states.

The following proof uses a stronger spectrum condition: (3a)  $\Sigma p_{\mu}^2 \ge m^2$  with  $m^2 > 0$  for all states except the vacuum-states; this means  $p_{\mu}$  = 0 is an isolated point in the momentum spectrum. The collection of all vacuum-states defines a subspace  $\mathcal{S}_{\Omega}$  of the Hilbert space  $\mathcal{S}$ ; we assume  $\delta_{\Omega}$  to be separable and notate the projection-operator on to  $\delta_{\Omega}$ by  $P_{\Omega}$ :

$$P_{\Omega} \mathcal{S} = \mathcal{S}_{\Omega}$$

With the algebra of field operators  $\mathcal{O}$  we mean the collection of linear combinations of products of field operators:

$$\{\sum_{m} c_m A(f_1) A(f_2) \dots A(f_{k_m})\} \text{ with } A(f_1) = \int A(x) f_1(x) dx.$$

 $f_1(x)$  is a testfunction from D or S.

It is assumed that the elements of  $\mathcal{O}$  are defined everywhere in  $\mathscr{S}_{\alpha}$  , and that  $\delta_{\Omega}$  is a generating subspace for  $\delta$  if we apply  $\mathcal{X}$  to  $\delta_{\Omega}$ . This means that the linear space built from  $\{A_k\Omega_i\}$ ,  $A \in \mathcal{A}$ ,  $\Omega_i \in \mathcal{S}_{\Omega}$  is dense in  $\mathcal{S}$ . Written symbolically:

$$\alpha S_{\Omega} = S$$

## S. SCHLIEDER

After these preliminaries we can start with the proof for the reduction of  $\mathcal{G}$ .

. We use the property of cluster decomposition [1] which follows from the assumptions made above. We write it down for a special case and only in the weak form we need:

$$\lim_{\lambda \to \infty} \int f_{1}(x_{1}) \dots f_{n}(x_{n}) g_{1}(y_{1} - \lambda \overrightarrow{a}) \dots g_{m}(y_{n} - \lambda \overrightarrow{a}) [(\Omega, A(x_{1}) \dots A(x_{n})A(y_{1}) \dots A(y_{m})\Omega) - (\Omega A(x_{1}) \dots A(x_{n})\Omega)(\Omega A(y_{1}) \dots A(y_{m})\Omega)] dxdy = 0$$
(1)

with  $\lambda$  real and  $\overrightarrow{a}$  a space-like vector.

If one looks to the proofs of this cluster decomposition one sees easily the generalization of (1) for the case of several vacuum-states:

$$\lim_{\lambda \to \infty} \int f_1(x_1) \dots f_n(x_n) g_1(y_1 - \lambda \vec{a}) \dots g_m(y_m - \lambda \vec{a}) (\Omega_k A(x_1) \dots A(x_n)(1 - P_{\Omega}) A(y_1) \dots A(y_m) \Omega_{\ell}) dxdy = 0.$$
(2)

Let us take  $f_1, \ldots f_n, \ldots g_1, \ldots g_m$  with compact supports that means from D. With the property of the locality of the field (assumption (5)) we get the equation:

$$\lim_{\lambda \to \infty} \int f_1(x_1) \dots f_n(x_n) g_1(y_1 - \lambda \vec{a}) \dots g_m(y_m - \lambda \vec{a}) (\Omega_k A(x_1) \dots A(x_n) A(y_1) \dots A(y_m) \Omega_\ell) dxdy$$
$$= \lim_{\lambda \to \infty} \int f_1(x_1) \dots f_n(x_n) g_1(y_1 - \lambda \vec{a}) \dots g_m(y_m - \lambda \vec{a}) (\Omega_k A(y_1) \dots A(y_m) A(x_1) \dots A(x_n) \Omega_\ell) dxdy$$

and applying (2):

$$\begin{split} \lim_{\lambda \to \infty} \int f_1(x_1) \dots f_n(x_n) g_1(y_1 - \lambda \vec{a}) \dots g_m(y_m - \lambda \vec{a}) (\Omega_k A(x_1) \dots \\ \dots A(x_n) P_\Omega A(y_1) \dots A(y_m) \Omega_\ell) dxdy \\ &= \lim_{\lambda \to \infty} \int f_1(x_1) \dots f_n(x_n) g_1(y_1 - \lambda \vec{a}) \dots g_m(y_m - \lambda \vec{a}) (\Omega_k A(y_1) \dots \\ \dots A(y_m) P_\Omega A(x_1) \dots A(x_n) \Omega_\ell) dxdy \,. \end{split}$$

The obtained equation does not depend on  $\boldsymbol{\lambda},$  so we get:

$$(\Omega_{k}A(f_{1})...A(f_{n})P_{\Omega}A(g_{1})...A(g_{m})\Omega_{\ell}) = (\Omega_{k}A(y_{1})...A(g_{m})P_{\Omega}A(f_{1})...A(f_{n})\Omega_{\ell}) (3)$$
  
with  $A(f_{1}) = \int A(x_{1})f_{1}(x_{1})dx_{1}..., A(g_{1}) = \int A(y_{1})g_{1}(y_{1})dy_{1},...$ 

Let us write down (3) in a shorter notation and more impressively with

$$A(f_1) A(f_2) \dots A(f_n) = A$$

$$A(g_1) A(g_2) \dots A(g_m) = B$$

$$(\Omega_k P_0 A P_0 P_0 B P_0 \Omega_\ell) = (\Omega_k P_0 B P_0 P_0 A P_0 \Omega_\ell).$$
(4)

Equation (4) is correct for each pair of vacuum-states. Therefore the operators  $P_{\Omega} \land P_{\Omega}, P_{\Omega} \land P_{\Omega}, \dots$  now taken as operators in  $\&_{\Omega}$  are commutative. Or in short notation:

The algebra.

$$\tilde{\mathcal{O}}_{t}$$
 =  $P_{\Omega} \mathcal{O}_{t} P_{\Omega}$  is

(a) Commutative;

(b) It is involutive. This means with  $\widetilde{A}$  is also the adjoint  $\widetilde{A}^{*}$  in  $\widetilde{\mathcal{A}}_{*}$ ;

(c) It consists of bounded operators. For one can write:

$$\widetilde{A} = \frac{1}{2} (\widetilde{A} + \widetilde{A}^{*}) + \frac{1}{2i} (i\widetilde{A} - i\widetilde{A}^{*}).$$

The two operators on the right side are Hermitian operators defined everywhere in  $\& \Omega$  and are therefore bounded, which follows from a theorem of Hellinger and Toeplitz [2]. From (a), (b), (c) and the separability of  $\& \Omega$  one concludes [3], that all operators of  $\widetilde{\mathcal{A}}$  can be expressed as functions of the same bounded self-adjoint operator which we will call S (S can be taken with a simple spectrum):

$$\mathbf{S} = \int_{b}^{c} \lambda \, \mathrm{d} \, \mathbf{E}_{\lambda}, \qquad \widetilde{\mathbf{A}}_{k} = \int_{b}^{c} \widetilde{\mathbf{a}}_{k}(\lambda) \mathrm{d} \, \mathbf{E}_{\lambda} \ .$$

This means in the case that one has a n-dimensional  $\mathscr{S}_{\Omega}$ : choosing an appropriate base in  $\mathscr{S}_{\Omega}$ , the elements of  $\widetilde{\mathscr{A}}$  are represented by n-dimensional diagonal matrices.

If we have an infinite-dimensional  $\mathscr{S}_{\Omega}$ , the form of the decomposition depends on the spectrum of S. If it is a pure point spectrum, the operators of  $\mathscr{J}$  are represented by infinite-dimensional diagonal matrices if one chooses the appropriate base.

These two cases give the same structure of  $\mathcal{A}$  in  $\mathcal{S}$ . Let  $\Omega'_1, \Omega'_2, \ldots \Omega'_m$ . be the appropriate base (finite or infinite). Then  $A_k \Omega'_l$  is orthogonal to  $A_j \Omega'_m$  for  $m \neq l$ ; for

$$(A_k\Omega'_\ell, A_j \Omega'_m) = (\Omega'_\ell A_k^* A_j \Omega'_m) = (\Omega'_\ell A_r \Omega'_m) = (\Omega'_\ell P_\Omega A_r P_\Omega \Omega'_m) = 0$$

with  $A_k$ ,  $A_j$ ,  $A_r \in \mathcal{C}$ .

$$\mathcal{S} = \overline{\mathcal{U} \Omega_1'} \oplus \overline{\mathcal{U} \Omega_2'} \oplus \ldots \oplus \overline{\mathcal{U} \Omega_n'} \oplus \ldots$$

 $\theta_k = \mathcal{O}_{\Omega_k}$  is invariant under the algebra  $\mathcal{O}$  and dense in  $\mathcal{S}_k = \overline{\mathcal{O}_{\Omega_k}}$ . If S has also a continuous spectrum one gets correspondingly also a direct integral for  $\mathcal{S}$ . If for instance one has a continuous spectrum from  $\lambda_1$  to  $\lambda_2$ 

$$\lambda_1 \quad d_1 \quad d_2 \quad d_3 \quad d_4 \quad \lambda_2 \qquad \lambda$$

and takes as an example the two orthogonal projection-operators

$$\Delta_{(d_1,d_2)} \mathbf{E} = \int_{-d_1}^{d_2} d\mathbf{E}_{\lambda}, \quad \Delta_{(d_3,d_4)} \mathbf{E} = \int_{-d_3}^{d_4} d\mathbf{E}_{\lambda}$$
$$\Delta_{(d_1,d_2)} \mathbf{E} \cdot \Delta_{(d_3,d_4)} \mathbf{E} = 0$$

one gets at first the two orthogonal subspaces

$$\boldsymbol{\beta}_{\Omega}^{(1,2)} = \Delta_{(d_1,d_2)} \mathbf{E} \cdot \boldsymbol{\beta}_{\Omega} \text{ and } \boldsymbol{\beta}_{\Omega}^{(3,4)} = \Delta_{(d_3,d_4)} \mathbf{E} \cdot \boldsymbol{\beta}_{\Omega}$$

of  $\mathcal{B}_{\Omega}$  and then the orthogonal subspaces

$$\mathcal{S}^{(1,2)} = \mathcal{O} \mathcal{S}^{(1,2)}_{\Omega}, \ \mathcal{S}^{(3,4)} = \mathcal{O} \mathcal{S}^{(3,4)}_{\Omega}$$

of &. The form of the spectrum of S will depend on the manifold of fields, which operate in &.

Finally the following further remarks should be made:

(1) If there are also anticommutative fields in the manifold of fields, in the first step one can choose the greatest sub-algebra  $\mathcal{U}_c$  of  $\mathcal{O}$  with operators in  $\mathcal{O}_c$  which commute spacelike.  $\mathcal{O}_c$  is reduced in the way mentioned above. The remaining elements of  $\mathcal{O}$ , if applied to a vacuum-state, produce a state with half-integer spin. If C is such an element  $P_{\Omega} \ell P_{\Omega}$  must be zero. Therefore the proof is also correct in this more general case.

(2) The cluster decomposition property can be proved [1] without using locality. If one wishes to go from Eq. (2) to Eq. (3) one only needs the weaker condition

$$\lim_{\lambda \to \infty} [A(x + \lambda \vec{a}), A(y)] = 0$$

where  $\lambda$  real,  $\vec{a}$  spacelike.

To get the results of the proof, therefore, only this weaker condition (instead of locality) is needed.

(3) The reduction of  $\mathcal{A}$  can also be proved, if one does not assume that  $p_{\mu} = 0$  is isolated in the spectrum, but if one assumes CPT - invariance [4].

# REFERENCES

- [1] ARAKI, H., preprint (1961); JOST, R. and HEPP, preprint (1961).
- [2] RIESZ-NAGY, Vorlesungen über Funktionalanalysis," Berlin (1956) 114.
- [3] RIESZ-NAGY, "Vorlesungen über Funktionalanalysis", Berlin (1956) 130.
- [4] BORCHERS, H.J., Phys. Rev., preprint (1961); REEH, H. and SCHLIEDER, S., Nuovo Cim., preprint (1962).

# BOOK I

# QUANTUM FIELD THEORY IN AXIOMATIC AND LAGRANGIAN FORMULATIONS

# PART II

GAUGE FIELDS

# GAUGE THEORIES OF VECTOR PARTICLES\*

J. SCHWINGER HARVARD UNIVERSITY, CAMBRIDGE, MASS. UNITED STATES OF AMERICA

# 1. INTRODUCTION

The general topic of this paper is the vector theory of gauge fields, but I like to think that these lectures are really concerned with the future of the relativistic field theory as an effective force in the development of fundamental physics. Two basic positions are at present under investigation as the possible organizing forces for the rapidly growing empirical data on elementary particles. To put it as extremely as possible, we might call these two positions :

(i) The particle point of view,

(ii) The field point of view.

By the particle point of view,I mean those investigations in which the physical particles, as we see them, are the basic elements. This is the whole line of development associated with the S-matrix, with the idea that the only function of the theory is to compute and to correlate the results of scattering measurements. It also underlies those further attempts intended to give a physical content to this essentially empty framework, such as dispersion relations, Regge poles, etc. And, to adopt this point of view systematically, one must necessarily accept the Orwellian philosophy that no particle is more fundamental than any other. That is the strict particle point of view; the particles are unanalysable. To our mind it is an extremely conservative position.

Opposed to this is the field point of view which supports the idea that there is a deeper dynamical level, that the empirical information we have is very complicated and that the purpose of theory is to discover simplicity not necessarily in terms of the observed properties, but in terms of concepts, of properties which are at the moment not directly observable but which undoubtedly will become so in the course of future developments. This is the way that physics has always proceeded. The field point of view is thus the idea that there exists something more fundamental than the phenomenological particles. This is a very general statement and we should say that field theory as it now stands is based upon the tentative identification of these more fundamental entities with some localizable fields. We would almost try to make a distinction between the idea that there is something dynamically deeper than the particles, and the particular association of the deeper structure with localizable fields. Such fields may be what is required, but the important thing to our mind is the alternative between accepting the particles as they are and seeking for something more fundamental. At the

\* Based on notes taken by A.P. Balachandran, B. Jaksic, I. Saavedra, J. Kvasnica, P.W. Nielsen, D. Boulware, C. Bollini and P. Rastall.

#### J. SCHWINGER

moment, the latter is identified with the idea of fields which are operator functions of the space and time coordinates. But even within this framework, there are various possible viewpoints. There is the extreme viewpoint this is Heisenberg's attitude - that there is only one fundamental field. Everything we know must come out of this one field. This is rather hard to accept, and I myself will adopt here the intermediate position that there are several fundamental fields. As to which fundamental fields are necessary I would say that the clue must be found in the exact, or almost exact, conservation laws we know in nature. This is the line of thought that leads to the idea of vector gauge fields which I am going to explore.

It should be emphasized that in the field viewpoint the fundamental fields are not immediately correlated with observable things. This is the deeper dynamical level; out of the interplay of the dynamics that govern these entities emerges the world of particles as we know it. In other words, the important thing is to recognize that the fields we begin with in this viewpoint are not necessarily directly correlated in a simple way with the observed particles. It has become fasionable to describe field theory as "oldfashioned". I would insist upon the following: what is old-fashioned is the naive confusion of these two points of view, in which one speaks indiscriminatly of particles and fields and associates with every particle a field which is inserted in some Lagrangian for the purpose of applying perturbation theory. This is the old-fasioned, naive point of view but it is not the one I am advocating here. We must clearly understand that we are dealing with a much more sophisticated approach, in which the fundamental fields are not simply correlated with particles, although there may be an ardent relation in some individual cases. The basic physical problem, from this point of view, is to explore the possibilities of postulating various funda- • mental fields with their dynamics and by proving the existence of special states of definite or almost definite energy-momentum relations to identify these with physical particles.

Such are the two extreme viewpoints, and obviously it is the second one which is adopted here. I shall try to indicate some of the possibilities that are inherent within it. Now, I said that the clue to which fields are fundamental is given by the exact, or perhaps almost exact, conservation laws. And I point here, inevitably, to the example upon which the whole field theory has been built more or less by analogy, i.e. electrodynamics. The electromagnetic field has the very special feature of gauge variation; while it might be possible to advocate, as Heisenberg does, that there is no fundamental electromagnetic field, I regard this property of gauge variation to be so basic that it seems necessary to postulate a fundamental electromagnetic field.

It should be remembered that the electromagnetic field is one such that the vector potential must be allowed freedom of transformation by gradients of an arbitrary scalar function, at the moment a numerical scalar function, say:

$$A_{\mu}(\mathbf{x}) \to A_{\mu}(\mathbf{x}) + \partial_{\mu}\lambda(\mathbf{x}). \tag{1.1}$$

Now we know that as the theory has been constructed to be invariant under such a gauge transformation, it follows automatically that the current vector  $j\mu(x)$ , which is the source of this field, must be conserved:

$$\partial_{\mu} j^{\mu}(\mathbf{x}) = 0.$$
 (1.2)

This is the important aspect of gauge invariance: the concept of the absolute conservation of the electrical charge is not explained as the result of specific dynamical restrictions on every conceivable system, but is understood in terms of the structure of the Maxwell field itself. To put it in another way, we know that the field strength tensor  $F^{\mu\nu}$  is antisymmetrical and obeys the equation:

$$\partial_{\mu} F^{\mu\nu} = j^{\mu} \tag{1.3}$$

and in virtue of that antisymmetry, a structural property of the field, it follows automatically that the current obeys

$$\partial_{\mu} j^{\mu} \equiv 0 . \tag{1.4}$$

That is, the equation of local electrical charge conservation is an identity characteristic of the structure of the Maxwell equations, and, therefore, once the Maxwell field is introduced, non-conservation of the electrical charge is inconceivable. This is the perfect model of a dynamical explanation of an absolute conservation law.

One may attempt to build an explanation of another absolute conservation law along these lines. One may say that what has been explained here is, in a sense, the absolute stability of the electron. The electron, being the lightest object that carries an electrical charge, is a stable object in virtue of the conservation of the electrical charge, since there is nothing lighter for it to go into while maintaining its charge. There is an analogy between the stability of the electron and the conservation of electrical charge, on the one hand, and the stability of nuclear matter and the conservation of the nuclear charge, on the other. This nucleonic charge must be possessed by all the heavy baryons and is handed on from the cascade particle to the  $\Lambda_{,}$ the  $\Sigma$  and the nucleon in the process of all their disintegrations. But with a nucleon, or more precisely with a proton, as the lightest object carrying this nucleonic charge, the process of decay ceases because there is nowhere else to transmit the nucleonic charge. That is, in the absolute conservation of nucleonic charge we have a description of the stability of matter and one would like to have an understanding of this most fundamental of all conservation laws on some general dynamic grounds rather than merely as a statement, since it is a rule which has to be superimposed on every possible interaction. It is natural, then, to introduce a hypothetical vector field, a gauge field analogous to the electromagnetic field and to insist that its dynamics be governed by the requirement of gauge invariance from which would follow the existence of an absolute conservation law. This dynamical explanation involves a new field and the question now is what will be the dynamic consequences of that field. Here is where the idea appears to run into immediate difficulties. If the analogy with the electromagnetic field is complete, a physical particle with zero mass, analogous to the photon should exist, and we know of no such particle. One could assume that the coupling to the new field is arbitrarily weak, there are arguments that the field must

### J. SCHWINGER

be unobservable even on a cosmological scale. That is hardly the kind of fundamental field which should be introduced to explain the conservation of the strongly interacting particles. This was the great objection: gauge invariance should imply the existence of a zero mass particle. And this is the decisive point at which we want to introduce the ideas of the new field theory.

It may be helpful to give a simple form of the argument relating gauge invariance to a massless particle. Using the notation of electrodynamics, we have the charge density equation

$$\vec{\nabla} \cdot \vec{E} = \rho$$
 (1.5)

and an integration over a large volume gives :

$$\int (\mathbf{d}\mathbf{r})\rho = \mathbf{Q} = \int \mathbf{d} \, \vec{\mathbf{S}} \, \cdot \vec{\mathbf{E}}$$
(1.6)

where Q is the total charge of the system and a constant of the motion. Therefore, the electric field at large distances must fall off like

$$E \sim (Q/4\pi) (n/r^2)$$
 (1.7)

which is a long range field. This is a static field, but one could argue, not incorrectly, that if one finds a static field which is long range, there must be a zero-mass particle or the field would be of finite range. But it is implicitly assumed here that the total charge Q is different from zero. And it is precisely at this point that the argument fails. When a charge is inserted into the vacuum, the accompanying electric field polarizes the vacuum producing a partial compensation of the charge. That is the origin of charge renormalization. But it is conceivable that the compensation of charge is not partial, but complete is present. That is, if a charge is placed in the system, there may come into being in the course of time a vacuum polarization in which, loosely speaking, one part of the charge escapes to infinity and the compensating charge exactly balances the charge that was originally inserted. Under these conditions, the constant total charge that will be observable in any arbitrarily large volume will be zero. This is not intended as a convincing argument, but merely an indication that there is a loophole in the assertion that there must be a long-range field - or massless particlefor this depends upon the assumption that there is no complete compensation charge. The massless physical particle disappears when a non-zero total charge can no longer be maintained in the vacuum.

# 2. THE ONE DIMENSIONAL MODEL

Rather than indicate by general agreements that this is a very real possibility, a very simple physical model will be used to show that such a new situation can occur.

The model I want to discuss is completely physical, in the sense that no general principles of physics are violated. On the other hand, it is an unworldly one, since it is a special case of electrodynamics in one spatial dimension. Of course, all the arguments we have given until now about gauge invariance apply equally well to one spatial dimension, apart from specifically geometrical factors.

Let me write down the basic equations we shall be concerned with for electrodynamics, and then we shall specialize and solve exactly in one-dimensional space. We shall begin with the Lagrange function

$$\mathcal{L} = -(1/2)F^{\mu\nu}(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}) + (e^{2}/4)F^{\mu\nu}F_{\mu\nu} + (i/2)\Psi\alpha^{\mu}\partial_{\mu}\Psi + (i m_{0}/2)\Psi\beta\Psi + (1/2)A_{\mu}\Psi\alpha^{\mu}q\Psi$$
(2.1)

where we have introduced two fundamental fields, a gauge field characterized by a vector  $A_{\mu}$  and an antisymmetric tensor  $F_{\mu\nu}$ , and a Fermi field  $\Psi$ . The matrices  $\alpha$  and  $\beta$  are connected with the usual Dirac  $\gamma$ -matrices by  $i\alpha^{\mu} = \beta \gamma^{\mu}$ . The  $\alpha^{\mu}$  are all real and symmetrical, and  $\alpha^{0} = 1$ ;  $\beta$  is real and antisymmetrical; e is the coupling constant and  $m_{0}$  the mass constant associated with the field  $\Psi$ . The antisymmetric matrix

$$q = \begin{bmatrix} o & -i \\ i & o \end{bmatrix}$$

is specifically associated with the charge, and is introduced here in order to work with Hermitian  $\forall$ fields.Under the transformation  $A_{\mu} \rightarrow e A_{\mu}, F_{\mu\nu} \rightarrow (i/e)F_{\mu\nu}$ , the Lagrangian  $\mathscr{L}$  goes back to its more familiar form (i.e. the coupling constant e appears at its usual place in the coupling term  $e A^{\mu} j_{\mu}$ ). The Fermi field obeys the anticommutation relation

$$\left\{ \Psi_{\alpha}(\mathbf{x}), \Psi_{\beta}(\mathbf{x}') \right\} = \delta_{\alpha\beta} \delta(\overrightarrow{\mathbf{x}} - \overrightarrow{\mathbf{x}'}), \quad \mathbf{x}_{0} = \mathbf{x}_{0}',$$

where the indices  $\alpha$  and  $\beta$  refer to spin and charge.

The one essential point to be emphasized about the distinction between three dimensions and one dimension is the question of the dimensionality of the charge, i.e. of the coupling constant. The action operator

$$w = \int (dx) \mathcal{L}$$

- is dimensionless in the system of units where  $\hbar = 1$ . Then the dimension of the Lagrangian

$$[\mathcal{L}] = 1/(L^{n+1})$$

where L is length and n the number of spatial dimensions. From this, we obtain the dimensionality of  $e^2$ :

$$[e^2] = 1/(L^{3-n})$$

and hence for the particular case n = 3, e is dimensionless, while for n = 1,

$$[e^2] = 1/L^2$$
,

i.e. the coupling constant itself carries a length, carries a mass.

#### J. SCHWINGER

We shall now show that it is possible to find an exact solution of the one-dimensional problem in the special case in which the mass constant associated with the Fermi field vanishes,  $m_0 = 0$ . In this one-dimensional model there are only two  $\alpha$ -matrices,  $\alpha^0 = 1$  and  $\alpha^1 = \alpha_1$ , which can be represented by 2 X 2 matrices. The spatial metric adopted is positive and the time metric negative.

## 3. EXACT SOLUTION OF THE ONE-DIMENSIONAL PROBLEM

What we want to do now is to solve a preliminary problem : the polarization of the vacuum of a Fermi field  $\Psi$  by an externally imposed field  $A_{\mu}$ . We must then introduce a certain requirement of self-consistency : the charge brought in creates a field, this field polarizes the vacuum which creates a charge that polarizes the vacuum and so on. The problem can be solved exactly in our model, because of the assumption of one spatial dimension and the zero mass of the fermion field. This is familiar, for example, from the discussions that have gone on about the Thirring model which is also a one-dimensional model though not electrodynamic.

Our preliminary problem is then a Dirac field  $\Psi$  plus an external (electromagnetic) field  $A_{\mu}$ . In terms of its solution we shall have the exact solution to our problem. We begin with a simplified Lagrange function

$$\mathcal{L} = (i/2) \Psi \alpha^{\mu} (\partial_{\mu} - iq A_{\mu}) \Psi.$$
(3.1)

We want to find the current induced in the vacuum by the external field  $A_{\mu}$ . Let this be:

$$\langle \mathbf{j}_{\mu}(\mathbf{x}) \rangle = 1/2 \langle \Psi(\mathbf{x}) \alpha q \Psi(\mathbf{x}) \rangle^{A}$$
. (3.2)

Since  $j_{\mu}(x)$  is a bilinear combination of fields taken at the same point x, we construct its expectation value by first solving another problem, which is to find the expectation value of a bilinear combination of fields at arbitrary points of space and time. This is, in other words, the construction of the Green's function associated with the field. We define this Green's function as

$$G(\mathbf{x}, \mathbf{x}'; \mathbf{A}) = \langle (\Psi(\mathbf{x})\Psi(\mathbf{x}'))_+ \rangle \in (\mathbf{x} - \mathbf{x}')$$
(3.3)

which is the vacuum expectation value of the time ordered product of the fields.  $\epsilon(x-x')$  is a sign function. This is the basic physical quantity in terms of which we extract physical information about the states that are created in the vacuum of the field  $\Psi$ , and in terms of which, by a limiting process with  $x' \rightarrow x$ , we shall construct the current operator.

The Green's function obeys an inhomogeneous differential equation which incorporates the field equations and the anti-commutation relations

$$\alpha^{\mu}(\partial_{\mu} - iq A_{\mu}(\mathbf{x}))G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}'). \qquad (3.4)$$

Under a gauge transformation Eq. (1) this Green's function transforms according to

$$G(\mathbf{x}, \mathbf{x}') = \exp\left[iq\lambda(\mathbf{x})\right]G(\mathbf{x}, \mathbf{x}')\exp\left[-iq\lambda(\mathbf{x}')\right]. \tag{3.5}$$

The current operation  $j_{\mu}(x)$  is given by:

$$j_{\mu}(\mathbf{x}) = (1/2)\Psi(\mathbf{x})\alpha_{\mu} q \Psi(\mathbf{x}).$$
 (3.6)

This is a singular expression and therefore it must be defined by a suitable limiting process as stated above. We must let  $x' \rightarrow x$  along a space-like direction, since we do not want to bring dynamics into the definition of an operator. We must carry out this definition in such a way that gauge invariance is guaranteed and then we must check the covariance of the procedure.

We now rewrite the expectation value Eq.(3.2) in terms of a Green's function. We get:

$$\langle \mathbf{j}_{\mu}(\mathbf{x}) \rangle = -(1/2) \operatorname{Tr} \alpha_{\mu} \mathbf{G} (\mathbf{x}, \mathbf{x})$$
 (3.7)

where G(x, x) is defined by:

$$G(x, x) = \lim_{x' \to x} G(x, x') \exp[-iq \int_{\xi'}^{x} d\xi A_{\mu}(\xi)], \qquad (3.8)$$

the limit being taken from a spatial direction maintaining all symmetries, i.e. taking an average of the values of the limits attained from the left and from the right. The exponential factor is required in order to maintain gauge invariance for  $x \neq x$ .

The solution of Eq. (3.3) can be written as:

$$G(x, x') = G^{0}(x, x') \exp iq[\Phi(x) - \Phi(x')]$$
(3.9)

where  $\Phi(\mathbf{x})$  satisfies :

$$\alpha^{\mu}\partial_{\mu}\Phi(\mathbf{x}) = \alpha^{\mu}A_{\mu}(\mathbf{x}). \qquad (3.10)$$

and  $G^0$  is the Green's function for  $A_{ij} = 0$ 

$$\alpha^{\mu} \partial_{\mu} G^{0}(\mathbf{x}, \mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}'). \qquad (3.11)$$

The solution of this equation with the proper boundary conditions is:

$$G^{0}(\mathbf{x}, \mathbf{x}') = (1/2\pi) \int_{0}^{\infty} d\mathbf{p} \exp[i\mathbf{p} \,\alpha^{\mu}(\mathbf{x}_{\mu} - \mathbf{x}'_{\mu}), \mathbf{x}^{0} > \mathbf{x}^{0}'$$
  
=  $-(1/2\pi) \int_{0}^{\infty} d\mathbf{p} \exp[i\mathbf{p} \,\alpha^{\mu}(\mathbf{x}_{\mu} - \mathbf{x}'_{\mu}), \mathbf{x}^{0} < \mathbf{x}^{0}'.$  (3.12)

To perform the limiting process (along a space-like direction), let us consider the right hand side of Eq. (3.8) for  $x_0 = x'_0$ :

$$G(\mathbf{x},\mathbf{x}') \{ \exp iq[\Phi(\mathbf{x}) - \Phi(\mathbf{x}')] \} \left\{ \exp \left[ -iq \int_{\mathbf{x}'}^{\mathbf{x}_1} d\xi A_1 \right] \right\}$$

But, for equal times

95

#### J. SCHWINGER

$$G^0 = (1/2\pi)[\alpha_1/(x_1-x')]$$
.

Expanding the exponentials in a Taylor series for  $x'_1 \rightarrow x'_1$ , we get:

$$G \cong (i/2\pi) [\alpha_1/(x_1 - x_1')] [1 + iq(x_1 - x_1')(\partial_1 \Phi - A_1).$$
(3.13)

Taking now the symmetrical limit as explained above, we obtain:

$$G(\mathbf{x}, \mathbf{x}) = \cdot (1/2\pi) \alpha_1 q[\partial_1 \Phi(\mathbf{x}) - A_1(\mathbf{x})] = (1/2\pi) q(\partial_0 \Phi(\mathbf{x}) - A_0(\mathbf{x})). \quad (3.14)$$

Inserting this into Eq. (3.7), we obtain the covariant expression:

$$\langle j_{\mu}(\mathbf{x}) \rangle = -(1/\pi) [A_{\mu}(\mathbf{x}) - \partial_{\mu}(1/4) \operatorname{Tr} \Phi(\mathbf{x})].$$
 (3.15)

Writing explicitly Eq. (3.10)

$$(\partial_0 + \alpha^1 \partial_1) \Phi(\mathbf{x}) = A_0(\mathbf{x}) + \alpha^1 A_1(\mathbf{x})$$
 (3.16)

and multiplying it from the left by  $(\partial_0 - \alpha^1 \partial_1)$ , we obtain the second order differential equation:

$$-\partial^2 \Phi(\mathbf{x}) = -\partial_{\mu} A^{\mu}(\mathbf{x}) + \alpha^1 [\partial_0 A_1(\mathbf{x}) - \partial_1 A_0(\mathbf{x})] \cdot \qquad (3.17)$$

By taking the trace, this reduces to

$$-\partial^{2}(1/4) \operatorname{Tr} \Phi(\mathbf{x}) = -\partial_{\mu} A^{\mu}(\mathbf{x}) \qquad (3.18)$$

which we can solve for  $Tr \Phi$  by means for the corresponding Green's function D(x, x'):

$$(1/4) \operatorname{Tr} \Phi(\mathbf{x}) = - \int (d\mathbf{x}') D(\mathbf{x}, \mathbf{x}') \partial_{\nu} A^{\nu}(\mathbf{x}') \qquad (3.19)$$

or, symbolically:

$$(1/4) \operatorname{Tr} \Phi(\mathbf{x}) = - D \partial_{y} A^{y}(\mathbf{x}).$$
 (3.20)

Hence, our final result in this notation is:

$$\langle \mathbf{j}_{\mu}(\mathbf{x}) \rangle^{=} - (1/\pi) \left[ \mathbf{A}_{\mu}(\mathbf{x}) + \partial_{\mu} \mathbf{D} \partial_{\nu} \mathbf{A}^{\nu} \right].$$
 (3.21)

This is an obviously covariant expression, it is also conserved and it is gauge invariant. To show that it is conserved, let us take the divergence of Eq. (3.21). We get:

$$\partial^{\mu} j_{\mu} = -(1/\pi) \left[\partial^{\mu} A_{\mu} - \partial^{\mu} A_{\mu}\right] = 0.$$

Let us now indicate some of its physical implications by a simple but not wrong method. We will then justify it. Let us think of the idea of selfconsistency in the simplest possible way.  $A_{\mu}$  has been until now an external field, but suppose this field somehow is brought into existence propagating in accordance with Maxwell's equations. Then this field induces a current and this current in turn reacts back to change the nature of the field. What then is the condition of self-consistency ?

We go back to Maxwell equations

$$\partial_{\nu} \mathbf{F}^{\mu\nu} = \mathbf{j}_{\mu}, \quad .$$
$$\partial_{\mu} \mathbf{A}_{\nu} - \partial_{\nu} \mathbf{A}_{\mu} = \mathbf{e}^{2} \mathbf{F}_{\mu\nu}.$$

From here,  $e^2 j^{\mu} = -\partial^2 A^{\mu} + \partial^{\mu} \partial_{\lambda} A^{\lambda}$ . Adopting now the Lorentz gauge  $\partial_{\mu} A^{\mu} = 0$ . and using Eq. (3.21), we get the propagation equation for the vector potential:

$$(-\partial^2 + \mu^2)A_{\mu}(\mathbf{x}) = 0 \tag{3.22}$$

where

$$\mu^2 = (e^2/\pi). \tag{3.23}$$

This is an equation describing non-interacting particles of finite mass  $\mu = \sqrt{(e^2/\pi)}$  and shows that gauge invariance of a vector field does not necessarily require zero-mass particles.

The expression found for the vacuum expectation value of the current in the presence of an external vector potential A is

$$\langle j^{\mu}(x) \rangle^{A} = -(1/\pi) [A^{\mu}(x) + \int \partial^{\mu} D(x - x') \partial_{\nu}' A^{\nu}(x')],$$
 (3.24)

where  $j^{\mu} \, is$  the electrical current carried by the fermion field. This may be symbolically written

$$\langle j \rangle^{A} = -(1/\pi)(1 + \partial D \partial) A,$$
 (3.25)

where the projection operator  $(1 + \partial \Box \partial)$  guarantees the conservation of charge and gauge invariance. We also found, by a simple self-consistency argument, that the condition for the vector potential to maintain itself is that it satisfy the field equation

$$[\partial^2 - (e^2/\pi)]A = 0. \tag{3.26}$$

Here  $e^2/\pi = \mu^2$  plays the part of the square of a mass; so the result is - at least in a simple-minded way - that the propagation equation for A is the same as that for a particle of mass  $\mu$ . We shall give a precise derivation of this result here and also show how to calculate all other properties of the system.

But before we begin the precise derivation, let me come back to another general qualitative remark that I made. The equation  $\nabla \cdot \mathbf{E} = \rho$  implies that at great distances from the sources the electric field E satisfies  $\mathbf{E} \sim \mathbf{Q}$  in the one-dimensional case and  $\mathbf{E} \sim \mathbf{Q}/4\pi r^2$  in the three-dimensional case, where  $\mathbf{Q}$  is the total charge. It is argued, quite correctly, that a long-range electrical field can only be maintained and propagated by zero-mass particles. However, the course of the argument is that the total charge should be different from zero, and this is not the case under the conditions we are talking about because the vacuum polarization effect acts to annihilate any given charge.

Suppose that we insert a static external charge density  $J^0$  which total charge  $Q_0$  into the vacuum. A charge density  $j^0$  will then be induced, whose expectation value, in the Lorentz gauge, is given by our previous equation

$$\langle j^0 \rangle = -(1/\pi) A^0$$
. (3.27)

The potential A<sup>0</sup> has its source in the total charge density  $J^0 + \langle j^0 \rangle$ :

$$-\partial^2 A^0 = e(J^0 + \langle j^0 \rangle).$$
 (3.28)

Substituting for  $\langle j^0\rangle$  , and using the fact that the fields are time independent, we get

$$(d^2/dx'^2 - \mu^2) A^0 = -e^2 J^0.$$
 (3.29)

The solution of this is:

$$A^{0} = (e^{2}/2\mu) \int (dx'^{1}) \{ \exp[-\mu (x'-x'^{1})] \} J^{0}(x').$$
 (3.30)

The total charge induced in the vacuum is therefore

$$\int \langle j^{0}(\mathbf{x}') \rangle d\mathbf{x}' = [-e^{2}/(2\mu \pi) [\int d\mathbf{x}' d\mathbf{x}'^{1} \{ \exp [-\mu | \mathbf{x}' - \mathbf{x}'^{1} | ] \} J^{0}(\mathbf{x}') = -\mathbf{Q}_{0}$$
(3.31)

which exactly cancels the inserted charge  $Q_0$ . Thus there is no long-range field and no longer an argument for a zero-mass particle.

# 4. SOLUTIONS OF THE GENERAL EQUATIONS WITH EXTERNAL SOURCES AND THE GREEN'S FUNCTIONAL.

Now we must write down the general equations of this relativistic field system and solve them exactly. The method we shall use is that of external sources and the Green's function. This is the general technique for dealing with any field problem. It depends on the idea of introducing simple excitations into the system, in terms of which all possible states can be created.

To the Lagrangian written down previously we add the source terms

$$A^{\mu}(x) J_{\mu}(x) - i\psi(x) \eta(x).$$
 (4.1)

The total Lagrangian must still be gauge invariant, which implies that the

external current  $J_{\mu}$  is conserved (this external current can be considered simply as an idealization of other dynamic systems which act upon our system). As for the Fermi source term  $\eta$ , it is a fully anticommutative quantity (just as the boson source  $J^{\mu}$  is fully commutative). The source  $\eta(x)$  anticommutes with  $\eta(x')$  and  $\psi(x')$  for all x and x'. Such quantities can be perfectly well realized in terms of familiar algebraic structures. The change in  $\eta(x)$ under a gauge transformation must be just such as to compensate the change in  $\psi$ .

With the addition of the source terms, the field equations become inhomogeneous. That for the Fermi-field, written for the case of zero-mass constant - is

$$\alpha^{\mu}(\partial_{u} - i \in A_{u})\psi = \eta, \qquad (4.2)$$

while that for the electromagnetic field tensor is

$$\partial_{\mu} \mathbf{F}^{\mu\nu} = \mathbf{J}^{\mu} + \mathbf{j}^{\mu}_{\mathrm{C}} \cdot \tag{4.3}$$

Notice that this is  $j_c^{\mu}$ , and not  $j^{\mu}$ . The quantity  $j^{\mu} = \frac{1}{2}\psi \alpha^{\mu}q\psi$  is no longer conserved in the presence of sources and therefore it would be inconsistant to write  $J^{\mu} + j^{\mu}$  as the right hand side of our previous equation. A proper calculation, which takes account of the fact that there is a transfer of charge from outside the system we are considering, shows that one must extract from  $j^{\mu}$  its conserved part  $j^{\mu}$ .

The point of introducing these external source terms is that one can convert the Hilbert-space operator field equations by their aid into numerical functional differential operator equations. For our problem the latter equations turn out to be soluble. But how do we make the transition from one kind of equation to the other? Well, we consider that the system begins in the vacuum state and the sources are, so to speak, turned on. The system is then disturbed and by choosing the disturbance correctly one may generate any state into which the vacuum may be thrown by the action of the field operators. By watching how these states propagate in time we see their properties. Finally, we switch off the sources and return to the vacuum state. The mathematical quantity which contains all the information about this process is the transformation function which relates the vacuum state  $|0\rangle$  before the disturbance to the vacuum state  $|0\rangle$  after it. I shall call this transformation function the Green's functional: it is the generating functional of all the Green's functions, or propagation functions, which describe processes in our system. We write it:

$$G[\eta J] = \langle 0_+ | 0_- \rangle^{\eta J}. \qquad (4.4)$$

We must find how the Green's functional depends on the external sources  $\eta$  and J. The idea is to consider its response to infinitesimal changes  $\delta \eta$ ,  $\delta J_u$ . These produce a change in the Lagrangian:

$$\delta_{\eta I} \mathcal{L} = A^{\mu} \delta J_{\mu} - i \psi \, \delta \eta, \qquad (4.5)$$

and a change in the Green's functional
#### J. SCHWINGER

$$\delta_{\eta J} \mathbf{G} [\eta \mathbf{J}] = \delta_{\eta J} \langle \mathbf{0}_{+} | \mathbf{0}_{-} \rangle^{\eta J} = \mathbf{i} \rangle \mathbf{0}_{+} | \int (\mathrm{d}\mathbf{x}) \mathbf{A}^{\mu} \delta \mathbf{J}_{\mu} - \mathbf{i} \psi \, \delta \eta | \mathbf{0}_{-} \rangle.$$
(4.6)

(The matrix elements are always taken between vacuum states). If one imagines a disturbance  $\delta J_{\mu}$ ,  $\delta \eta$  localized around the point x and if one somehow knows  $\delta_{\eta J} G[\eta J]$ , then immediately one gets the matrix elements of the operators  $A^{\mu}(x)$  and  $\psi(x)$ . If this can be repeated at all points of space-time, one gets a general correspondance between matrix elements of the field operators and functional derivatives of  $G[\eta J]$ . This gives the general differential operator representation of the field operators, very much analogous to the representation of p's by differential operators with respect to q's.

Now we come to an important point. I have written down the variation  $\delta J^{\mu}$ . To find one differential operator representation we should like to make arbitrary infinitesimal variations of the  $J^{\mu}$ . But this we cannot do: the  $J^{\mu}$  are not independent. If we vary the m independently we shall violate charge conservation and therefore gauge invariance. The way to overcome this difficulty is to work in a specific gauge. By choosing a suitable gauge we shall be able to vary our  $J^{\mu}$  arbitrarily while conserving charge.

How is this to be accomplished? Take an arbitrary vector J and project it by means of a projection operator  $\Pi$  into a vector J<sub>c</sub> which is conserved

$$J_c = \pi J. \qquad (4.7)$$

I want to make this projection so that it does not upset the temporal development of the system, so we shall choose a projection operator which is local in time. Let us introduce, in addition to the usual space-time gradient  $\partial_{\mu}$ , the purely spatial gradient  $\nabla_{\mu}$ . The spatial components, or component since what we say applies to both one and three dimensions, of  $\nabla_{\mu}$ are the same as those of  $\partial_{\mu}$ , but the time component is zero. The projection equation will be taken to be

$$J_{c} = (1 + \nabla D \delta) J, \qquad (4.8)$$

where  $\mathcal D$  is the Green's function associated with the spatial gradient

$$\nabla^2 \mathcal{D}(\mathbf{x}, \mathbf{x}') = -\delta(\underline{\mathbf{x}} - \underline{\mathbf{x}}'). \tag{4.9}$$

The conservation equation  $\partial J_c = 0$  follows immediately.

The conservation of charge can now be ensured by replacing  $\delta J$  by  $\delta J_c$  in the J term of our variational integral

$$\int (\mathrm{dx}) A(1 + \nabla \mathcal{D} \partial) \delta J.$$
 (4.10)

We can now certainly perform arbitrary variations of J, but at the cost of some awkwardness. However, if we now choose the radiation gauge, in which  $\nabla$ . A = 0, and perform an integration by parts (this must be validated by appropriate restrictions, which we will not go into), then the extra term  $\nabla D \partial$  simply disappears. So we have exploited the gauge freedom of the theory in such a way that we can replace the variation of the conserved current

by the variation of an arbitrary vector. This is true only for our special choice of gauge. Other gauges are possible, but the projection operator will no longer be local in time. However, once we have presented the whole formalism in terms of functional differentials, we shall be quite free to change the gauge as we whish. In fact, I shall immediately switch over from the radiation gauge to the Lorentz gauge, which is much more symmetrical.

Treating J as arbitrary, which is justified with our particular choice of gauge, we can write down a correspondence between the variational derivative with respect to  $J^{\mu}$  and the vector potential  $A^{\mu}$ 

$$(1/i)[\delta/\delta J_{\mu}(\mathbf{x})] \subseteq A^{\mu}(\mathbf{x}). \tag{4.11}$$

(The coefficient of  $\delta J_{\mu}$  in the variational integral is just  $iA^{\mu}$ ). In the same way, considering the  $\psi \delta \eta$  term in the variational integral, we find the correspondence

$$-\delta_{\varrho}/\delta\eta(\mathbf{x}) - \psi(\mathbf{x}). \tag{4.12}$$

The  $\ell$  suffix indicates that this is a left derivative. In making the correspondence we must bring  $\delta\eta$  to the left of  $\psi$ , which accounts for the minus sign ( $\psi \delta\eta = -\delta\eta.\psi$ ).

These correspondences suggest that one can convert the field equations for  $\psi$  and  $A^{\mu}$  into functional differential equations for  $G[\eta J]$  by simply substituting  $\delta/\delta J_{\mu}$  for i  $A^{\mu}$  and  $\delta/\delta \eta$  for  $\psi$ . First, from the Dirac equation we get:

$$[\alpha^{\mu}(\partial_{\mu}-q\frac{\delta}{\delta J^{\mu}(x)})\frac{\delta}{\delta \eta(x)}+\eta(x)]G[\eta J]=0. \tag{4.13}$$

Secondly, there is the Maxwell set and at this point we shall change over to the Lorentz gauge. The radiation gauge was described first because it is most immediate, but now let us define the conserved current  $J_c$  by

$$J_{c} = (1 + \partial D \partial) J, \qquad (4.14)$$

where D is the Green's function associated with -  $\partial^2$ . This equation for  $J_c$  is not local in time, but it does have the advantage of being manifestly relativistically invariant. I shall not go through the mechanics of the gauge transformation, but the result is

$$[\partial \partial^{3} \frac{1}{i} \frac{\delta}{\delta J} - \partial^{2} \frac{1}{i} \frac{\delta}{\delta J} - e^{2}(1 + \partial D\partial)(J + \frac{1}{2} \frac{\delta}{\delta \eta} \alpha q \frac{\delta}{\delta \eta})]G[\eta J] = 0. \quad (4.15)$$

where J is the external current,  $(\frac{1}{2})(\delta/\delta\eta)\alpha q(\delta/\delta\eta)$  corresponds to the physical fermion current  $\frac{1}{2}\psi\alpha q\psi$  and the projection operator  $(1+\partial D\partial)$  ensures charge conservation. We still have to write the transcription for the last equation, corresponding to the choice of gauge. The final gauge equation is

$$\partial \frac{\delta}{\delta J} G[\eta J] = 0.$$
 (4.16)

which says that the Lorentz gauge is the chosen one.

#### J. SCHWINGER

We must now solve the functional equations for the Green's function. Let us first consider the Dirac equation. The variational derivative  $\delta/\delta J$  can be treated as c-numbers (and they behave like c-numbers in the sense that they are commutative). For the moment we shall call them  $i A_{\mu}$ , i.e.

$$\frac{1}{i} \frac{\delta}{\delta J^{\mu}} \rightarrow A_{\mu}. \tag{4.17}$$

The Green's function now obeys an equation in the presence of an external field  $A_{\mu}$ . Treating this field as a parameter we can convert the differential equation

$$\frac{\delta}{\delta \eta} G[\eta J] = -\int dx' G(x, x', A) \eta(x) G[\eta J], \qquad (4.18)$$

where G(x, x', A) is the Green's function for the Dirac equation in the presence of an external potential  $A_{u}$ .

The formal solution of the last equation can immediately be written down. It is, of course, an exponential

$$G[\eta J] = G[J] \exp\left\{-\frac{1}{2}\int dx \, dx' G(x,x',\frac{1}{i}\frac{\delta}{\delta J})\eta(x')\right\}$$
(4.19)

where G[J[ is a constant of integration. We can now transfer this partial solution to the Maxwell equation in order to determine also the J dependence of G:

$$[\partial \partial \frac{1}{i} \frac{\delta}{\delta J} - \partial^2 \frac{\delta}{\delta J} - e^2 (1 + \partial D \partial) (J - \frac{1}{2} \operatorname{Tr} \alpha q G(x, x', \frac{1}{i} \frac{\delta}{\delta J})]G[J] = 0.$$
(4.20)

Taking this equation with the characteristic condition for the Lorentz gauge

$$\partial (\delta G/\delta J) = 0,$$
 (4.21)

we see that the previous equation is equivalent to

$$\left[ (-\partial + e^2/\pi)(1/i)\delta/\delta J - e^2(1+\partial D\partial) J \right] G = 0 \qquad (4.22)$$

in which, use has been made of the known structure of the current

$$(\frac{1}{2})$$
 Tr  $\alpha q G(x, x'; A) = j(A)$  (4.23)

in the case of the external potential, for which the current (in the Lorentz gauge) was proportional to  $A_{\mu}$ . Again, the differential equation can be replaced by an integral functional equation by using the Green's function for the problem:

$$(-\partial^2 + e^2/\pi) G(x, x') = \delta(x, x').$$
 (4.24)

The Green's functional G[J] is therefore given exactly by

$$G[J] = \exp\left[\frac{i}{2}\int dx \, dx' \, J^{\mu}(x) \, G_{\mu\nu}(x, x') \, J^{\nu}(x)\right]$$
(4.25)

102

where

$$G_{\mu\nu} = (1 + \partial D \partial)_{\mu\nu} G. \qquad (4.26)$$

All the physical characteristics are contained in G. The projector  $(1+\partial D\partial)$  assures the fulfilment of the Lorentz gauge condition.

The expansion of the exponential in the solution of the Green's functional produces, in a sense, all the possible states of the system. The coefficients of the expansion refer to the physical propagation of the system giving the multiple Green's functions for them.

At this point it may be instructive to consider one example: This is the comparison of:

(1) The quantum electrodynamic case with its gauge invariance; and

(2) The vector field case with new zero mass and no gauge condition.

# 5. COMPARISON OF THE QUANTUM ELECTRODYNAMIC CASE AND THE VECTOR FIELD CASE

It has been shown that one may, under suitable physical conditions, have a gauge invariant theory, exact conservation laws and yet no zero mass vector particle.

In the two simple theories which have been considered (pure electrodynamics which is gauge invariant and a vector field which already has a mass constant and therefore is not gauge invariant) a distinct difference in the nature of the spectra has been found. In the gauge invariant case one has a particle with a non-zero mass, which depends on the coupling constant, whereas in the non-gauge invariant case one has both a vector particle with non-zero mass and a scalar particle with mass zero.

The complete set of Green's functions, which in principle contain the answers to all possible physical questions, are finite in the electrodynamic case and meet all general requirements in a perfectly reasonable way. With these Green's functions one can go on to discuss scattering and radiation properties of the Fermi particles in interaction with the Bose field.

In the non-gauge invariant case one meets "divergences", which does not mean that anything is infinite, but rather that almost everything is zero. The system does not respond to Fermi excitations in a way that is formally characterized by the vanishing of the Fermi field renormalization constant.

This does not mean that we have just two different theories. We have the choice between one field theory, where everything is finite and reasonable, and another field theory which is unphysical, even though it is "renormalizable".

The mere possibility of renormalizability is not sufficient for physical acceptability if the renormalization constants are zero. Renormalization is part of the process of physical interpretation, not a mathematical means of suppressing divergencies.

The general technique in the investigation of the simle model has been the use of the Green's functional  $G[\eta, J]$  which is the response of the system to elementary disturbances.

The dependence of the Green's functional on the Fermi sources is given by: J. SCHWINGER

$$G[\eta, J] = G[J] \exp\left\{-\frac{1}{2}\int \eta(x) G\left(x, x', \frac{1}{i}\frac{\delta}{\delta J}\right) \eta(x')\right\}.$$
 (5.1)

This formula is completely general. In the simple model G[J] is given by:

$$G[J] = \exp\left\{\frac{ie^2}{2}\int J^{\mu}(x) G_{\mu\nu}(x, x') J^{\nu}(x')\right\}.$$
 (5.2)

It is possible to evaluate all the Fermi Green's functions. In the electrodynamic case they will be finite, whereas in the non-gauge invariant case they all vanish.

The Green's function in the presence of an external electro-magnetic field  $A_{\mu}\left(x\right)$  is:

G (x, x'; A) = G<sup>0</sup> (x, x') exp
$$\left\{ i q[(\phi(x) - \phi(x')] \right\},$$
 (5.3)

where

$$G^{0}(\mathbf{x}, \mathbf{x}') = \begin{cases} \int_{0}^{\infty} \frac{d\mathbf{p}}{2\pi} \exp\left[ip\alpha^{\mu}(\mathbf{x}_{\mu} - \mathbf{x}_{\mu}')\right], & \text{for } \mathbf{x}^{0} > \mathbf{x}'_{0} \\ -\int_{-\infty}^{0} \frac{d\mathbf{p}}{2\pi} \exp\left[ip\alpha^{\mu}(\mathbf{x}_{\mu} - \mathbf{x}_{\mu}')\right], & \text{for } \mathbf{x}^{0} < \mathbf{x}'_{0}. \end{cases}$$
(5.4)

is the Green's function for the non-interacting case. This Green's function corresponds to a Fermi particle with zero mass moving in one dimension. There is an invariant distinction between a particle moving to the right or to the left.

The function  $\Phi(\mathbf{x})$  satisfies the differential equation

$$\alpha^{\mu}\partial_{\mu}\phi(\mathbf{x}) = \alpha^{\mu}A_{\mu}(\mathbf{x})$$
(5.5)

from which we can construct  $\phi(\mathbf{x})$  as a linear functional of  $A_{\mu}(\mathbf{x})$ . Multiplying Eq. (5.5) from the left with the operator  $(\partial_0 - \alpha^1 \partial_1)$  one obtains the second order differential equation:

$$-\partial^2 \phi = (\partial_0 - \alpha^1 \partial_1) \alpha^{\mu} A_{\mu}$$
 (5.6)

This equation has the solution:

$$\Phi(\mathbf{x}) = \int (d\xi) D(\mathbf{x}-\xi) \left(\frac{\partial}{\partial \xi^0} - \alpha^1 \frac{\partial}{\partial \xi^1}\right) \alpha^{\mu} A_{\mu}(\xi).$$
 (5.7)

The exponential in Equ. (3.3) may now be written:

$$\exp iq[\phi(x) - \phi(x')] = \exp i \int (d\xi) A^{\mu}(\xi) J_{\mu}(\xi; x, x')$$
 (5.8)

where  $j_{\mu}(\xi; \mathbf{x}, \mathbf{x}')$  is a definite numerical function, which acts as a current in connection with the vector field. The following explicit expression for the current is obtained from Eqs.(5.7) and (5.8):

$$j_{\mu}(\xi; \mathbf{x}, \mathbf{x}') = q \alpha_{\mu} \left( \alpha^{1} \frac{\partial}{\partial \xi^{1}} - \frac{\partial}{\partial \xi^{0}} \right) \left[ \mathbf{D} \left( \mathbf{x} - \xi \right) - \mathbf{D} \left( \mathbf{x}' - \xi \right) \right] .$$
 (5.9)

The divergence of the current is:

$$\partial_{\mu} j^{\mu}(\xi; x, x') = q [\delta (x - \xi) - \delta (x' - \xi)]. \qquad (5.10)$$

The current is seen to have two sources with opposite signs at x and x' respectively, corresponding to the effect of the Fermi field at these points.

The different Green's functions may be constructed by expanding the Green's functional  $G[\eta, J]$  in powers of the sources  $\eta$  and J. Expanding the Green's functional in powers of the Fermi sources  $\eta$  one obtains:

G[
$$\eta$$
, J] =  $\left[1 - \frac{1}{2}\int \eta \, G \, \eta + \frac{1}{8}\int \eta \eta \, GG\eta\eta + \dots\right] \exp \frac{ie^2}{2}\int J \, G \, J.$  (5.11)

The integrands contain, apart from the Fermi sources, products of the Green's functions. The product in the third term, for example, may be written:

$$G(\mathbf{x}_{1}, \mathbf{x}_{1}') G(\mathbf{x}_{2}, \mathbf{x}_{2}') = G^{0}(\mathbf{x}_{1}, \mathbf{x}_{1}') G^{0}(\mathbf{x}_{2}, \mathbf{x}_{2}')$$
  
$$\cdot \exp\left\{ \int (d\xi) [j(\xi; \mathbf{x}_{1}, \mathbf{x}_{1}') + j(\xi; \mathbf{x}_{2}, \mathbf{x}_{2}')] \frac{\delta}{\delta J(\xi)} \right\}$$
(5.12)

where  $G^0(x_1, x_1')$  and  $G^0(x_2, x_2')$  are the free field Green's functions. The dependence on the external potential is contained in the exponential, which is of the form

$$\left\{ \exp\left[ \int (d\xi) \frac{\delta}{\delta J(\xi)} \right] \right\} f(J) = f(J + j) \quad .$$
 (5.13)

so that the exponentials act as simple displacement operators.

The first purely fermion Green's function is

$$\mathbf{G}(\mathbf{x},\mathbf{x}') = \mathbf{G}^{0}(\mathbf{x},\mathbf{x}') \left\{ \exp\left[ \int j_{\mu}(\boldsymbol{\xi};\mathbf{x},\mathbf{x}') \frac{\delta}{\delta J(\boldsymbol{\xi})} \right] \right\} \left\{ \exp\left[ \frac{\mathbf{i}e^{2}}{2} \int \mathbf{J} \mathbf{G} \mathbf{J} \right] \right\} \left| \int_{\mathbf{J}=0}^{\infty} (5.14) \mathbf{G} \mathbf{J} \right|$$

Performing the variational differentiation, one obtains in the limit  $J \rightarrow 0$ :

$$G(\mathbf{x}, \mathbf{x}') = G^{0}(\mathbf{x}, \mathbf{x}') \exp\left[\frac{ie^{2}}{2} \int j^{\mu}(\xi; \mathbf{x}, \mathbf{x}') G_{\mu\nu}(\xi, \xi') j^{\nu}(\xi'; \mathbf{x}, \mathbf{x}')\right]. \quad (5.15)$$

This expression is still purely formal. The question of existence depends on the specific form of  $G_{\mu\nu}$  which has a tensor structure and may thus be written:

$$G_{\mu\nu}(\xi,\xi') = g_{\mu\nu}G_1(\xi,\xi') + \partial_{\mu}\partial'_{\nu}G_2(\xi,\xi')$$
(5.16)

where  $g_{\mu\nu}$  is the metric tensor and  $G_1$  and  $G_2$  are scalar functions. Substituting Eq. (5.16) in Eq. (5.15) one obtains

$$\mathbf{G}(\mathbf{x},\mathbf{x}') = \mathbf{G}^{0}(\mathbf{x},\mathbf{x}') \exp\left[\frac{\mathrm{i}\mathbf{e}^{2}}{2} \int \mathbf{j}^{\mu}(\boldsymbol{\xi};\mathbf{x},\mathbf{x}')\partial_{\mu}\partial_{\nu}'\mathbf{G}_{2}(\boldsymbol{\xi},\boldsymbol{\xi}') \mathbf{j}^{\nu}(\boldsymbol{\xi}';\mathbf{x},\mathbf{x}')\right] \quad (5.16a)$$

where use has been made of

$$j^{\mu}j_{\mu} = \alpha^{\mu}\alpha_{\mu} \dots = 0 \tag{5.17}$$

After performing partial integrations Eq. (5.16a) becomes

G (x, x') = G<sup>0</sup>(x, x') exp
$$\left[\frac{ie^2}{2}\int \partial_{\mu} j^{\mu}(\xi; x, x') G_2(\xi, \xi') \partial_{\nu} j^{\nu}(\xi; x, x')\right]$$
 (5.17a)

where the divergence of the current is given by Eq. (5.10). What is really involved in the calculation of the Green's function G(x, x') is thus the structure of the scalar function  $G_2(\xi, \xi')$ .

In the electrodynamic case the function  $G_{\mu\nu}$  in the Lorentz gauge is given by

$$G_{\mu\nu} = (1 + \partial D \partial)_{\mu\nu} \cdot G \qquad (5.18)$$

where  $(1 + \partial D \partial)_{\mu\nu}$  is a projection operator and G is a scalar function corresponding to the mass  $\mu = e / \sqrt{\pi}$ . In the non-electrodynamic situation the function  $G_{\mu\nu}$  is:

$$\mathbf{G}_{\mu\nu} = (\mathbf{1} + \partial \mathbf{D} \,\partial)_{\mu\nu} \cdot \mathbf{G} - (\mathbf{1}/\mu_0^2) \,\partial_{\mu}\partial_{\nu} \,\mathbf{D}$$
 (5.19)

where the scalar function G corresponds to the mass  $\sqrt{\mu_0^2 + (e^2/\pi)}$  while the D function is associated with mass zero. In the electromagnetic case the exponential in Eq. (5.15) may be written:

$$\exp\left\{-ie^{2}\int \frac{dp}{(2\pi)^{2}}[1-\exp ip(x-x')]\frac{1}{(p^{2}-i\epsilon)}\frac{1}{(p^{2}+\mu^{2}-i\epsilon)}\right\}.$$
 (5.20)

The integral in the exponent is convergent, i.e. neither ultraviolet nor infrared divergences occur. The simplest Green's function has now been constructed. It is entirely finite and one would now ask for its physical interpretation. The Green's function  $G^0$  has a pole at mass zero. The exponential factor changes this pole into a singular branch point. This corresponds to the fact that we are dealing with particles of mass zero. When the source operates, it may produce one particle with mass zero, but in addition it may produce any number of pairs of particles, i.e.there is a continuous spectrum.

The physical particle with mass zero can be identified only to the extent that one can effectively isolate the initial point from the continuous spectrum. Thus, in ordinary quantum electrodynamics the electron can, strictly speaking, not be uniquely identified. If a charge is created, any number of photons of arbitrary small frequencies may also be created. The identification of the electron is actually the identification of a localized excitation carrying a unit charge and with a certain lattitude in the mass set by the experimental circumstances.

In the one-dimensional situation there is a zero-mass particle superimposed on a continuous background of pairs. This is the approximate physical transcription of the structure of the Green's function in which there appears, not a pole at zero mass, but a singular branch point. The physical interpretation is thus complicated by this quite irrelevant question, as far as the general picture is concerned, of the "infrared problem" which involves the identification of zero-mass particle states, despite the fact that mass zero is not separated by any finite gap from the other masses.

One can now go on to compute all the other Green's functions and to calculate how particles moving along on a line interact with each other and with vector particles of mass  $\mu$ .

For the non-electromagnetic vector field, where a "bare" mass has been inserted, the exponential in Eq.(5.15)may be written:

$$\exp\left\{-ie^{2}\int \frac{dp}{(2\pi)^{2}}\left[1-\exp\left(ip\left(x-x'\right)^{\frac{1}{p^{2}-i\epsilon}-\left(\frac{1}{p^{2}+\mu^{2}-i\epsilon}-\frac{1}{\mu_{0}^{2}}\right)}\right)\right]\right\}$$
(5.21)

This integral is convergent for  $-p^2 \rightarrow 0$ , but logarithmically divergent for  $-p^2 \rightarrow \infty$ . From Eq.(515) it then follows that the Green's function G(x, x') vanishes. This is also true for every Fermi Green's function, i.e. the system cannot be excited as far as Fermi responses are concerned. This contradicts the formal properties of the Green's functions as vacuum expectation values of field products, so that this theory must be rejected despite the fact that the theory would be considered renormalizable.

After this discussion of a simple model we shall turn to some general considerations of which the model can be taken as an example. The onedimensional model is over-simplified in one essential respect, since it contains no critical dependence on the coupling constant. We have two different situations. One is electrodynamic, i.e. a vector field coupled by a gauge invariant mechanism to a charge. In this case there is a zero-mass particle. The other is a hypothetical vector field coupled to a nucleonic charge also by a gauge invariant mechanism. In this case there is no zero-mass particle. In other words, there must be a critical coupling strength such that below this the zero-mass particle remains and above this the zero-mass particle disappears. Now I will give a general discussion of the simplest Green's function, which gives an account of the vector particle spectra. If one has a weak external current  $J^{\mu}$ , the expectation value of  $A^{\mu}(x)$  may be written as

$$\langle A^{\mu}(\mathbf{x}) \rangle = \int G^{\mu\nu}(\mathbf{x} - \mathbf{x}') J_{\nu}(\mathbf{x}') + \dots$$
 (5.22)

where non-linear terms have been omitted since the external current is assumed to be weak.

The Green's function  $G^{\mu\nu}(x-x')$  describes free particles. The Fourier tranform of  $G^{\mu\nu}$  may be written:

$$G_{\mu\nu}$$
 (p) =  $\Pi_{\mu\nu}$  (p) G (p) (5.23)

where  $\Pi_{\mu\nu}$  (p) is a projection operator, which is determined by the choice of gauge. The scalar function G (p) contains the specific propagation properties of the system. We are studying here the response of the system to excitation by an external current. The excitation will in general produce a spectrum of possible states. This spectrum will be represented by the spectral structure of the Green's function. The scalar function in Eq. (5.23) may be represented by

G (p) = 
$$\int \frac{B(m^2) dm^2}{p^2 + m^2 - i\epsilon}$$
 (5.24)

where  $B(m^2) dm^2$  is the probability that the excitation produces a transfer of energy and momentum which is characterized by the mass m. Since  $B(m^2)$ is a probability density it must be non-negative

$$B(m^2) \ge 0 . \tag{5.25}$$

The probability density  $B(m^2)$  is assumed to satisfy the sum rule

$$\int dm^2 B(m^2) = 1.$$
 (5.26)

This assumption may be justified in the following way.

In the Lorentz gauge

$$\Theta_{\mu} A^{\mu}(\mathbf{x}) = 0$$
(5.27)

and the propagation equation is

$$-\partial^2 A = J + j$$
 (5.28)

where J is the external current and j the other physical currents. These currents would of course in turn be determined by suitable fields. I want to insist that the fundamental vector fields shall be observable for very short times (or very high frequencies). The time intervals must be so short, that the interaction effects do not have time to obscur the underlying field. The response of the system to the current of the Green's function must then have the following asymptotic behaviour:

$$G \sim 1/p^2(1 + ...)$$
 for  $-p^2 \to \infty$  (5.29)

where the omitted terms vanish for  $-p^2 \rightarrow \infty$ . The rate at which these terms vanish depends on the dynamics of the system and cannot be asserted in advance. It now follows from Eq. (5.24) that this asymptotic behaviour is only possible if Eq. (5.26) holds. No other sum rule can be stated in general, because that depends on an assumption of how the current behaves, i.e. of the dynamics.

It is of interest to find a representation of G(p) which incorporates the required asymptotic behaviour. Introducing the complex variable z, Eq. (5.24) may be written

G(z) = 
$$\int \frac{B(m^2) dm^2}{m^2 - z}$$
. (5.30)

This function is regular everywhere except on the positive real axis. The singularities correspond to the physical values of m. The boundary value of the function G(z) is G(p) for  $z^2 \rightarrow -p^2 + i\epsilon$ . If z tends to infinity, except along the real axis, one has:

$$G(z) \sim -1/z$$
. (5.31)

For the inverse function we have:

$$G^{-1}(z) \sim -z$$
 (5.32)

Since G (z) has no complex zeros,  $G(z)^{-1}$  will have no complex poles or complex singularities. In addition

$$(1/z)(G^{-1}+z) \to 0$$
. (5.33)

for z tending to infinity.

The function  $(1/z)(G^{-1}+z)$  has only singularities along the positive real axis, which includes a pole at z = 0. Hence

$$(1/z)(G^{-1}+z) = \lambda^2/z - \int dm^2 s(m)^2/(m^2-z)^{-1}$$
 (5.34)

from which we obtain the following representation, on placing  $z = -p^2 + i\epsilon$ :

G (p) = 
$$\left[p^2 - i\epsilon + \lambda^2 + (p^2 - i\epsilon)\int \frac{dm^2 s(m^2)}{p^2 + m^2 - i\epsilon}\right]^{-1}$$
. (5.35)

This representation of G (p) has the correct asymptotic behaviour. Com-

paring Eq. (5.24) and Eq. (5.35) one sees that  $s(m^2)$  must be non-negative since  $B(m^2)$  is non-negative:

$$\mathbf{s}(\mathbf{m}^2) \ge \mathbf{0}. \tag{5.36}$$

From Eq. (5.24) one obtains

G(0) = 
$$\int \frac{\mathrm{dm}^2 B(\mathrm{m}^2)}{\mathrm{m}^2} > 0$$
 (5.37)

whereas from Eq. (5.35) it follows that

$$G(0) = 1/\lambda^2$$
 (5.38)

so that  $\lambda^2 > 0$ .

As far as the physical properties we have inserted are concerned, any non-negative  $\lambda^2$  and any non-negative  $s(m^2)$  for which the integral in Eq.(5.34) exists, will give a possible Green's function. If one requires zero mass to be part of the physical spectrum the parameters  $\lambda^2$  and  $s(m^2)$  can no longer be chosen arbitrarily. If zero mass is in the spectrum it follows from Eq. (5.37) that G(0) is infinite. Comparing with Eq. (5.38) one then obtains  $\lambda=0$ as a necessary condition. For  $\lambda=0$  Eq.(5.35) may be written:

$$G(\mathbf{p}) = \left[1/(\mathbf{p}^2 - i\epsilon)\right] \left[1/\left(1 + \int \frac{d\mathbf{m}^2 \mathbf{s}(\mathbf{m}^2)}{\mathbf{p}^2 + \mathbf{m}^2 - i\epsilon}\right)\right] \quad . \tag{5.39}$$

For  $p^2 \sim 0$  this equation leads to:

$$G(\mathbf{p}) = \left[ 1/(\mathbf{p}^2 - \mathbf{i}\epsilon) \right] \left[ 1/\left(1 + \int \frac{\mathrm{dm}^2 \mathbf{s}(\mathbf{m}^2)}{\mathbf{m}^2} \right) \right] \quad . \tag{5.40}$$

For mass zero to be present in the physical spectrum as an isolated singularity, the residue of the pole in Eq.(40) must not vanish, i.e.

$$\int_{\rightarrow 0} \frac{\mathrm{dm}^2 \mathrm{s} \, (\mathrm{m}^2)}{\mathrm{m}^2} < \infty \ . \tag{5.41}$$

I now want to examine what dynamical changes are necessary in order to go from a situation where these conditions are satisfied to a situation where they cease to be valid. That would be the continuous change from electrodynamics where there is zero-mass particle to a theory where this particle ceases to exist.

We have found the form (see 5.35):

$$G(p) = 1 / \left[ p^2 - i\epsilon + \lambda^2 + (p^2 - i\epsilon) \int_0^\infty \frac{dm^2 s(m^2)}{p^2 + m^2 - i\epsilon} \right]$$
(5.42)

for the gauge independent part of the "photon Green's function", where  $\lambda^2$  and  $s(m^2)$  are non-negative quantities. The (-it) refers to the boundary condition of outgoing waves in time. The sum rule  $\int_0^{\infty} dm^2 B(m^2) = 1$  requires that  $\int_0^{\infty} dm^2 s(m^2) < \infty$ . If we put  $p^2 = 0$ , we obtain  $0 < 1/\lambda^2 = \int_0^{\infty} [B(m^2)/m^2] dm^2$ , assuming the existence of  $\int_0^{\infty} [s(m^2)/m^2] dm^2$ . Now we want to find the necessary conditions for the existence of a physical particle with zero mass, so that we can imagine conditions under which such a particle would cease to exist. Then, we could suppose a continuous variation as we go from the electromagnetic field with its physical photon to the hypothetic vector field associated with nuclear charge which does not possess a zero-mass particle, and investigate how the photon ceases to exist.

If we have a photon, then it is necessary that  $\lambda^2$  = 0. The resultant Green's function is then

G(p) = 
$$[1/(p^2 - i\epsilon)] \left[ 1/(1 + \int_0^\infty \frac{dm^2 s(m^2)}{p^2 + m^2 - i\epsilon}) \right]$$
 (5.43)

and, in the neighbourhood of  $p^2 = 0$ ,

$$G(p) \simeq [1/(p^2 - i\epsilon)] \left[ 1/1 + \int_0^\infty \frac{dm^2 s(m^2)}{m^2} \right].$$
 (5.44)

Then, the residue of the pole,  $B_0$ , is

$$B_0 = 1 / \left[ 1 + \int_0^\infty \frac{dm^2 s(m^2)}{m^2} \right]$$
 (5.45)

which must be greater than zero for the existence of a pole, i.e.

$$\int_0^{\cdot} dm^2 \left[ s(m^2) / m^2 \right] < \infty$$

is the second necessary condition for a physical particle of zero mass. The integral up to infinity certainly exists, since the representation itself was based on the existence of the integral  $\int_0^\infty dm^2 s(m^2)$ . What can vary from one physical situation to another is the integration down to zero. Then we must have

$$\mathbf{s}(\mathbf{m}^2) \xrightarrow[\mathbf{m}^2 \to 0]{} \bullet \tag{5.46}$$

The structure of  $B(m^2)$  when a photon exists is

$$B(m^{2}) = B_{0} \delta(m^{2}) + B_{1} (m^{2})$$
(5.47)

where  $B_1$  (m<sup>2</sup>) is continuous and

$$1 = B_0 + \int_0^\infty dm^2 B_1(m^2), \qquad (5.48)$$

The photon exists only if  $0 < B_0$ . (In the usual language,  $B_0 = z_3$ ). If we calculate the long-range Coulomb interaction, we find that the effective charge is given by

$$e^2 = B_0 e_0^2$$
 (5.49)

• A zero renormalization constant is <u>not</u> to be interpreted as a mathematical problem but as a physical statement of the absence of a particle.

The function  $B(m^2)$  can be interpreted as the probability that a source will produce excitations in the vacuum of mass  $m^2$ . Now, we want to find a physical interpretation for  $s(m^2)$ .  $s(m^2)$  is the measure of excitations by an established field. Consider the vacuum state with an external current. Now, we ask for the probability that the vacuum is maintained, then the relevant quantities

$$\langle \left| \begin{array}{c} \right\rangle^{J} \simeq \exp \left[ \left( \left. i \right/ 2 \right) \int J^{\mu \cdot} G_{\mu \nu} J^{\nu} \right] \right]$$

where we have taken J to be weak, ignoring more complicated processes. We use the exponential to take into account the possibility that many weak processes are occurring all over space. If we use a conserved current,  $\partial_{\mu} J^{\mu} = 0$ , then the  $\pi_{\mu\nu}$  multiplying the scalar Green's function becomes effectively  $g_{\mu\nu}$  and

$$\langle | \rangle^{J} \simeq \exp[(1/2) J^{*\mu}(p) G(p) J_{\mu}(p) dp],$$
 (5.50)

writing the integrals in momentum space. Now, we introduce the vector potential

$$A^{\mu}(p) = G(p) J^{\mu}(p),$$
 (5.51)

thus:

.

$$\langle | \rangle^{J} \simeq \exp \left[ (1/2) \int A^{*\mu}(p) G^{-1*}(p) A_{\mu}(p) dp \right],$$
 (5.52)

the probability of the vacuum's remaining unexcited is:

$$|\langle | \rangle^{J} |^{2} \simeq \exp[-\int dp | A^{\mu}(p) |^{2} \operatorname{Im} G^{-1}(p)],$$
 (5.53)

Im G<sup>-1\*</sup>(p) = 
$$-\pi p^2 \int_0^\infty dm^2 s(m^2) \delta(m^2 + p^2).$$
 (5.54)

We have transferred our attention from the current, which may lie far out-

side the region of interest to the field which lies in the region. In doing so, we find that the inverse Green's function becomes the important quantity and  $s(m^2)$  measures the excitations of the vacuum. The resultant expression is

$$|\langle |\rangle^{J}|^{2} \simeq \exp\left[-\pi \int dp \ dm^{2} \ \delta \ (p^{2} + m^{2}) \ s(m^{2}) \ (-1/2) \left| F^{\mu\nu}(p) \right|^{2} \right].$$
 (5.55)

The expression in the exponential gives a measure of the probability of excitation of mass m by an external field  $F^{\mu\nu}$  .

The condition s(0) = 0 is characteristic of a normal threshold, i.e. (at the beginning of the excitation spectrum) there is a zero probability of exciting the vacuum by an external field. To put it another way, the condition for the existence of a photon is that nothing unusual happens at the zeromass threshold. On the otherhand, if the photon is not to exist, then something must happen at zero mass.

If we have abnormal behaviour, we have two possibilities:

(1) s(0) is finite or singular such that  $g \int dm^2 s(m^2)/(m^2-g) \xrightarrow{\rightarrow 0} dm^2 s(m^2)/(m^2)/(m^2) am^2 s(m^2)/(m^2) am^2 s(m^2)/(m^2)/(m^2) am^2 s(m^2)/(m^2)/(m^2) am^2 s(m^2)/($ 

Then, we have no pole at  $p^2 = 0$ , but m = 0 is still in the spectrum, i.e. we have a branch point at  $p^2 = 0$ , no pole and B(0) has a non-vanishing weight. Then, there is no recognizable particle of mass zero.

(2) The second possibility is that  $s(m^2)$  possesses a delta function singularity at  $m^2 = 0$ :

$$s(m^2) = \lambda^2 \delta(m^2) + s_1 (m^2).$$
 (5.56)

Then  $m^2 = 0$  is not in the spectrum at all. The inverse Green's function is then

$$G^{-1}(p) = p^{2} - i\epsilon + \lambda^{2} + (p^{2} - i\epsilon) \int_{0}^{\infty} \frac{dm^{2} s_{1}(m^{2})}{p^{2} - i\epsilon + m^{2}}, \qquad (5.57)$$

our original form. Now, B(0) = 0.

To see this, we write:

$$B(m^{2}) = (1/\pi) \operatorname{Im} G(p) = (1/\pi) \operatorname{Im} G^{1^{*}}(p) / |G(p)|^{2}$$
$$= m^{2} s_{1} (m^{2}) / R(m^{2}) + [\pi m^{2} s_{1} (m^{2})]^{2}, \qquad (5.58)$$

where

$$R(m^{2}) = m^{2} - \lambda^{2} + m^{2} P \int_{2}^{\infty} \frac{dm'^{2} s(m'^{2})}{m'^{2} m^{2}}$$

then:

B(0) = 
$$\lim_{m^2 \to 0} m^2 s_1(m^2) / \lambda^4 = 0.$$

Now, consider the case where  $s_1 (m^2)$  is zero for  $m^2 < m_0^2$ . In the real world, we might expect that this is never true, but it could be true as an approximation for strong interactions. If  $R(m^2) = 0$ , we will have a stable particle for  $m < m_0$ . At  $-\infty$ ,  $R(m^2) = -\infty$ . Then, if  $R(m_0^2) > 0$ , we have a

#### J. SCHWINGER

stable particle, since R must pass through zero. On the other hand, if  $R(m_0^2) < 0$ , there cannot be a stable particle, since R is a monotonic function for  $m < m_0$ . If there is no stable particle, there must be an unstable particle, since R must pass through zero in the continuum. We would only be able to recognize it as such if the width is sufficiently small. The width is given by

$$\gamma = \pi m^2 s_1 (m^2) / (dR/dm^2). \qquad (5.60)$$

It is possible that R may cross through zero several times, giving more than one resonance. These would emerge from the same Green's function, reflecting the long dynamic chain from the complicated spectrum of the observed particle to the simpler underlying fields.

In the one-dimensional model, we had  $s(m^2) = (e^2/\pi) \delta(m^2)$ . This reflects the possibility of creating pairs of fermions travelling along a line. The function  $s(m^2)$  is an example of the second case, and we find a single stable particle of mass  $e^2/\pi$ . This simplicity depends on two things: the geometry of one dimension and the fact that we only considered zero-mass fermions. The dynamics are not so simple, it is merely the elementary kinematics which allowed us to find solutions which fit the general dynamic framework. The delta function of  $s(m^2)$  at  $m^2=0$  is so because a fermion pair is still a particle of zero mass as opposed to the case in three dimensions, where the pair has a mass spectrum. In three dimensions, the probability of a photon going into three photons goes as something like the eighth power of the available energy assuring us that s(0) = 0 We would expect the photon to disappear as the s(0) becomes an abnormal threshold, i.e. somethreshold moves down to zero mass as the strength of the interaction builds up.

A crude mathematical model might be given by a characteristic resonance function

$$s_0(m^2) = (\lambda^2 / \pi^2) m \Gamma / [(m^2 - m_0^2 K)^2 + m^2 \Gamma^2]$$
(5.61)

where

$$m_0 = 2m_e, K \simeq 1 - \alpha^2 / 2, \Gamma \simeq \alpha^5$$
. (5.62)

In electrodynamics,  $s_0$  certainly consists of such contributions. This could be the positronium contribution, which for  $m^2 < m_0^2$  K, would be the three photon contribution (or virtual positronium). As the coupling increases, K must decrease since the binding energy of positronium increases. At some critical strength, K would become zero and p would also be zero since there is nothing into which the positronium can decay. By then the language is appropriate, since the multiple photon contribution would not be distinguishable from the "positronium", but we shall continue our terminology analytically.

Since the binding energies are so large, we would also expect to find other bound states corresponding to particles or resonances. Such particles as the spin-zero mesons would then appear as a result of the complete strongly interacting set of fields and there would be no need for a separate field. We have not discussed the complete set of particles - there is more than one type of baryon and the list of conservation laws includes such quantities as isotopic spin. This is not an absolute conservation law, however, so we would not insist strongly on a dynamic explanation. But we can ask if a theory with a non-Abelian invariance group can be given a dynamic explanation in terms of what we might call a non-Abelian gauge field. In order to investigate such a theory, we must investigate the mathematical-physical problem of the formulation and quantization of such a theory.

In the case of electrodynamics the field is the dynamic means of manifesting an electrical charge. But the c.m. field itself does not carry a charge. On the other hand, the gravitational field interacts with all energy and momentum, including that which it carries itself. The non-Abelian gauge fields are intermediate in that they carry the quantity of which they are the dynamic manifestations, but this quantity is not a space-time property. Before we can ask physical questions about the theory, we must verify that it fits within the framework of possible quantum mechanical fields. In a theory in which the question of commutation relations is not faced, there is no difficulty in writing down a theory. Similarly, there is no difficulty in assuring appropriate three dimensional invariance properties. The difficulty arises in assuring the consistency of the commutation relations and the Lorentz invariance of the theory. There is a criterion which states in one line a sufficient and, for a certain class of theories, necessary condition for relativistic invariance.

The statement of relativistic invariance means that there exist operators, constructed from the fundamental variables of the theory, whose commutators obey the structure relations in the inhomogeneous Lorentz groups. The entire structure of the theory will then remain invariant under the unitary transformations generated by the operators. What is special about field theories is that these generators are constructed additively from contributions by small regions of space. That is:

$$\mathbf{P}^{\mu} = \int d^3 x \, T^{0\mu} \, (x), \qquad (5.63)$$

$$J^{\mu\nu} = \int d^3 x \left( x^{\mu} T^{0\nu} - x^{\nu} T^{0\mu} \right).$$
 (5.64)

The requirement that  $P^{\mu}$  and  $J^{\mu\nu}$  obey the structure relations of the Lorentz-group imposes restrictions on the commutation relations of the densities. Since the three-dimensional case presents no problems, we assume that we know  $T^{0k}$  and that it gives  $J^{k\ell}$  and  $P^k$  which generate the inhomogeneous rotation group in the correct way.

1

$$[P^k, P^\ell] = 0,$$
 (5.65)

$$[\mathbf{P}^{k}, \mathbf{J}^{0\ell}] = \mathbf{i} \mathbf{P}^{0} \delta^{k\ell}$$
 (5.66)

are assured by the three-dimensional invariance. In order to assure such relations as

$$[P^0, J^{0k}] = iP^k,$$
 (5.67)

$$[J^{0k}, J^{0\ell}] = i J^{k\ell},$$
 (5.68)

the equal-time energy density commutator must have the following form for  $x^0 = x'^0$ :

$$(-i)[T^{00}(x), T^{00}(x')] = (T^{0k}(x) + T^{0k}(x') \partial_k \delta(x - x') + \psi(x, x')$$
 (5.69)

where

$$\psi(\mathbf{x}, \mathbf{x}') = -\psi(\mathbf{x}', \mathbf{x}, \mathbf{y})$$

$$\int d^{3} \mathbf{x} \, \psi(\mathbf{x}, \mathbf{x}') = 0 = \int d^{3} \mathbf{x} \, \mathbf{x}^{k} \, \psi(\mathbf{x}, \mathbf{x}'). \quad (5.70)$$

There is a class of theories for which  $\tau$  vanishes identically, which includes spin 1/2 and spin 1 fields. Within this class then, the relation

$$x^{0} = x'^{0}(-i) [T^{00}(x), T^{00}(x')] = - (T^{0h}(x')) \partial_{k} \delta(x-x')$$

is a necessary and sufficient condition for Lorentz invariance.

#### 6. FUNDAMENTAL COMMUTATION RELATIONS

We are now going to give a general derivation of the fundamental commutation relation which relates the energy densities of a relativistic field theory at various points of space and the same time. We want to see in a general way that there exists a class of physical systems, for which a simple commutation relation relating the energy and momentum densities of a physical system is both necessary and sufficient for relativistic invariance. Much of what we shall be doing will be entirely by analogy to and in parallel with similar considerations referring to the electric current vector.

Let us start with some remarks on the analogy between the electric current vector on the one hand and the tensor of energy and momentum on the other, with regard to the question of equal-time commutation relations. Commutation relations are, of course, interpreted in quantum mechanics as statements of measurability. Measurability is fundamentally a dynamical process and therefore the underlying general dynamic properties that characterize these two sets of operators should be pointed out. We are not talking about any vector or any symmetrical tensor, but about these very special quantities with their dynamical significance. First of all, the vector  $j^{\mu}$  and the symmetrical tensor  $T^{\mu\nu}$  are locally conserved quantities:

$$\partial_{\mu} j^{\mu} = 0, \quad \partial_{\mu} T^{\mu\nu} = 0.$$
 (6.1)

Secondly, these vectors are not just mathematically conserved quantities, but they are also of immediate physical significance, because we understand the electric current vector, for example, to have a dynamical meaning as the source of the electromagnetic field. The operators  $j^{\mu}$  and  $T^{\mu\nu}$  have thus in common the fact that they are the sources of important fields;  $j^{\mu}$  is the source of the electromagnetic field and  $T^{\mu\nu}$  the source of the gravitational

field. That is their essential unique dynamical significance and it is upon these facts that we want to base the theory of their commutation relations. We shall understand what can be called the kinematics of special relativity – the equal-time commutation relations – in terms of the dynamics of somewhat more general systems.

Now, the fact that these physical operators (or sets of operators) are respectively sources of the two fields, electromagnetic and gravitational also gives us the general basis for understanding why they satisfy conservation laws. These are not arbitrary restrictions, they flow from the structure of the field equations, from the requirement of what we shall call generally "gauge" invariance, although this will of course take different forms. It is characteristic of both of these fields that they make use of more field components than are necessary to describe the physical information and there are corresponding freedoms of "gauge" transformations. This corresponds in the electromagnetic case to the usual gauge invariance, while for the gravitational case it is specifically the freedom of coordinate transformations. Under these general "gauge" transformations it follows that the operators which are the sources of the fields must obey certain identities; these are the laws of conservation of electrical charge and energy-momentum in ordinary flat space, respectively. We now want to exploit, not just the fact that these operators are the sources of the fields, but the reciprocal aspect, that these operators are also measures of the response of a given physical system to external fields.

Imagine a given physical system in an external electromagnetic field or an external gravitational field. How do these two basic properties enter? To answer this question, one may think of the action operator \*

$$W = \int (dx) \mathcal{L}(x). \qquad (6.2)$$

Let the external vector potential be  $A_{\mu}$  and  $G_{\mu\nu}$  be the external gravitational potential. Infinitesimal variations of these external potentials produce corresponding variations in the action operator

$$\delta_{A}W = \int (dx) \, \delta \, A_{\mu} j^{\mu} , \qquad (6.3)$$

$$\delta_g W = \int (dx) \sqrt{-g} \frac{1}{2} \delta g_{\mu\nu} T^{\mu\nu}$$
(6.4)

where, as usual,  $g = \det g_{\mu\nu}$ . This is a way of defining the operators  $j^{\mu}$  and  $T^{\mu\nu}$ . Here we are studying the responses of the system to external potentials, which must of course be such that they are consistent with the requirement of general gauge invariance. A gauge transformation is not a physical transformation; if the change of a vector potential is

<sup>\*</sup> All these ideas, of course, are characteristic of the local theory of fields.

$$\delta A_{\mu} = \partial_{\mu} \delta \lambda$$

then the action integral will not change, with appropriate boundary conditions, which implies the conservation law

$$\partial_{\mu} j^{\mu} = 0.$$
 (6.5)

Similarly, an infinitesimal co-ordinate transformation

$$\overline{\mathbf{x}}^{\mu} = \mathbf{x}^{\mu} - \delta \boldsymbol{\xi}^{\mu} \tag{6.6}$$

induces an infinitesimal change in the symmetrical tensor  $g_{uv}$ :

$$\delta g_{\mu\nu} = \delta \xi^{\lambda} \partial_{\lambda} g_{\mu\nu} + \partial_{\mu} \delta \xi^{\lambda} g_{\mu\nu} + \partial_{\nu} \delta \xi^{\lambda} g_{\mu\nu}, \qquad (6.7)$$

and the action integral is invariant under this transformation. Then, upon inserting Eq. (6.7) into Eq. (6.4) and integrating by parts with appropriate boundary conditions, we get:

$$\partial_{\mu} \sqrt{-g} g_{\lambda\nu} T^{\mu\nu} = (1/2) \partial_{\lambda} g_{\mu\nu} T^{\mu\nu} . \qquad (6.8)$$

If we now specialize to the ordinary space time, the left-hand side vanishes and we come back to the conservation law:

$$\partial_{\mu} T^{\mu\nu} = 0.$$
 (6.9)

This expresses the fact that in an external electromagnetic field, charge conservation still has its usual form, whereas Eq. (6.9) takes a slightly different form given by Eq. (6.8) owing to the fact that the gravitational field itself carries energy and momentum. Here we see how the response of the system to an external field is the origin of these conservation laws.

Now we come back to the connection with commutation relations; we want to base the theory of commutation relations for equal time on these conservation laws (Eqs. (6.5) and (6.8)). Both of them are equations of motion of the form

$$\partial_{o} A(\mathbf{x}) = B(\mathbf{x})$$
 (6.10)

which maintains its structure independently of the values of the external parameters (external potentials). The meaning of A and B will, of course, change.

I want to show now that when we have such a situation, it immediately implies an equal-time commutation relation. This is the connection between the dynamics implied in the conservation laws and the commutation relations. To do this we shall first of all use the action principle in the following way. Consider the expression

$$\partial_0 \langle t_1 | A(x) | t_2 \rangle = \langle t_1 | B(x) | t_2 \rangle$$
 (6.11)

where  $t_1 > t > t_2$  the matrix element of Eq. (6.10) between the states at times  $t_1$  and  $t_2$ . Let us now perform an infinitesimal parameter variation. The matrix elements would change for two reasons. First, A(x) and B(x) may be explicit functions of the parameters; we shall denote by  $\delta^{1}A(x)$ , say, the corresponding variation. Then, there would be a change associated with the change in dynamics of the system as a result of this parameter variation. The change in the transformation function will be given by

$$\delta \langle t_1 | t_2 \rangle = i \langle t_1 | \int (dx) \delta^{i} \mathcal{L} | t_2 \rangle.$$
(6.12)

Therefore Eq. (6.11) would change into:

$$\partial_{0} \langle t_{1} | \delta^{1} A(\mathbf{x}) + i \int (d\mathbf{x}^{1}) (A(\mathbf{x}) \delta \mathcal{L}(\mathbf{x}^{1}))_{+} | t_{2} \rangle$$
  
=  $\langle t_{1} | \delta^{1} B(\mathbf{x}) + i \int (d\mathbf{x}^{1}) (B(\mathbf{x}) \delta \mathcal{L}(\mathbf{x}^{1}))_{+} | t_{2} \rangle$  (6.13)

where we have dropped the "dash" on  $\delta$  because this is the only change in the Lagrangian we consider. Now, from Eq. (6.10) and the definition of time ordered products we have

$$\partial_0 (\mathbf{A}(\mathbf{x}) \delta \boldsymbol{\mathcal{I}}(\mathbf{x}^1))_+ = (\mathbf{B}(\mathbf{x}) \delta \boldsymbol{\mathcal{I}}(\mathbf{x}^1))_+ \\ + \delta (\mathbf{x}^0 - \mathbf{x}^{0^1}) [\mathbf{A}(\mathbf{x}), \ \delta \boldsymbol{\mathcal{I}}(\mathbf{x}^1)]$$

and this gives us the equal-time commutation relation, written in operator form as

$$\frac{1}{i} \int (d^3 x') [A(x), \, \delta \, \mathcal{L}(x')]_{x^0 = x^{0'}} = \partial_0 \, \delta' A - \delta' B. \qquad (6.14)$$

Here we have an instrument, whenever we have an equation of motion involving some parameters, to find a commutation relation at equal time between the object that obeys the equation of motion and the measure of the response of the system to the variation of parameters.

An alternative derivation (without using the action principle) can also be given. We have:

$$\partial_0 A = B$$
  
= (1/i)[A, P<sup>0</sup>] + ( $\partial_0 A$ )<sub>exp</sub> (6.15)

where the last term refers to any explicit time dependence.

A change in the parameters will induce the change:

$$\delta^{1}B = (1/i)[\delta^{1}A, P^{0}] + (1/i)[A, \delta^{1}P^{0}] + \delta^{1}(\partial_{0}A)_{exp}$$
(6.16)

but

$$\delta' \mathbf{P}^0 = -\int d^3 \mathbf{x} \ \delta \mathbf{L}(\mathbf{x}) \tag{6.17}$$

and therefore:

$$\frac{1}{i} \left[ \mathbf{A}, \int (\mathbf{d}^3 \mathbf{x}^1) \delta \boldsymbol{\lambda} (\mathbf{x}^1) \right]_{\mathbf{x}^0 = \mathbf{x}^{01}} = \partial_0 \delta^1 \mathbf{A} - \delta^1 \mathbf{B},$$

as established above. As an illustration, let us consider the electromagnetic field. We have

$$\partial_0 j^0 = -\partial_k j^k$$
, (6.18)

i.e.  $A(x) = j^0(x)$  and  $B(x) = -\partial_k j^k(x)$ . The external parameters are the continuous set of values of the components of the external vector potential. Therefore, Eq. (6.14) gives the equal-time commutation relation

$$\frac{1}{i} \int (\mathbf{d} \mathbf{x}^{i}) [j^{0}(\mathbf{x}), j^{\mu}(\mathbf{x}^{i}) \delta \mathbf{A}_{\mu}(\mathbf{x}^{i})]$$
  
=  $\partial_{0} \delta^{i} j^{0}(\mathbf{x}) + \partial_{k} \delta^{i} j^{k}(\mathbf{x}).$  (6.19)

Before evaluating the right-hand side we use first Eqs. (6.12) and (6.3) to obtain

$$\delta_{\mathbf{A}} \langle \mathbf{t}_{1} | \mathbf{t}_{2} \rangle = \mathbf{i} \langle \mathbf{t}_{.1} | \int (\mathbf{d}\mathbf{x}) \mathbf{j}_{\mu} \delta \mathbf{A}^{\mu} | \mathbf{t}_{2} \rangle.$$
 (6.20)

A second variation gives:

$$\delta_{A}^{2} \langle t_{1} | t_{2} \rangle = - \langle t_{1} | \int \int (dx) (dx') \delta A^{\mu} (x) \delta A^{\nu} (x')$$

$$\cdot \left[ (j_{\mu}(x)j_{\nu}(x'))_{+} - i \frac{\delta^{i}j_{\mu}(x)}{\delta A^{\nu}(x')} \right] | t_{2} \rangle. \qquad (6.21)$$

Now, the integral on the right-hand side is a quadratic form, symmetric in x,  $\mu$  and x<sup>1</sup>,  $\nu$ . The first term can also be taken as such. We thus obtain the reciprocity relation:

$$[\delta^{j} j_{\mu}(x)] / [\delta A^{\nu}(x^{j})] = [\delta^{j} j_{\nu}(x^{j})] / [\delta A^{\mu}(x)] . \qquad (6.22)$$

We shall consider a special class of physical systems in which  $j_{\mu}(x)$  is local in time in its explicit dependence on the extreme potential, i.e. in which the current does not depend explicitly upon the time derivative of the external potential,  $\partial_0 A^{\nu}(x)$ . Under this assumption it follows from Eq. (6.19) that the charge density  $j_0(x)$  does not depend explicitly on the potentials at all.

$$\left[\delta^{\dagger}j_{o}(\mathbf{x})\right] / \left[\delta A^{\mu}(\mathbf{x}^{\dagger})\right] = 0$$

and therefore, the reciprocity relation gives:

$$[\delta^{1}j_{k}(x^{1})]/[\delta A^{0}(x)] = 0,$$
 (6.23)

i.e. the spatial current density is not an explicit function of the scalar potential,  $A^0$ . In what follows we shall show that the spatial current must depend explicitly on the spatial part of the vector potential. Therefore  $\delta^{i} j^{k}(x)$ may be written as

$$\delta^{\prime}j^{k}(\mathbf{x}) = \int (d^{3}\mathbf{x}^{\prime}) [\delta^{\prime}j_{k}(\mathbf{x})] / [\delta_{3}A_{\ell}(\mathbf{x}^{\prime})] \delta A_{\ell}(\mathbf{x}^{\prime})$$
(6.24)

where the sub-index 3 indicates a three-dimensional variational derivative. Inserting this into Eq. (6.19), we find the equal-time commutation relation

$$[j^{0}(\mathbf{x}), j^{0}(\mathbf{x}')] = 0$$
 (6.25)

and

$$\frac{1}{i}[j^{0}(\mathbf{x}), j^{\ell}(\mathbf{x}')] = \partial_{k}\left(\frac{\delta^{i}j^{\ell}(\mathbf{x}')}{\delta_{3}A_{k}(\mathbf{x})}\right)$$
(6.26)

where the right-hand side has been rewritten using the reciprocity relation.

The commutator of Eq. (6.26) cannot vanish, because if it were zero, it would violate the physical requirement that there should be a vacuum state. In order to prove this, we take the three-dimensional divergence of Eq. (6.26) and use Eq. (6.18) to obtain:

$$[j^{0}(\mathbf{x}), -i\partial_{0}j^{0}(\mathbf{x}^{*})] = -\partial_{k}\partial_{1}[\delta'j^{1}(\mathbf{x}^{*})] / [\delta_{3}A_{1}(\mathbf{x})]. \qquad (6.27)$$

Now, the commutator of an operator and its derivative is intrinsically positive, as can be seen in the following way. If  $A(x^0)$  is a hermitian local operator, a spatial average of  $j^0(x)$  over an arbitrary test function, then

$$[i\partial_0 A, A] = [[A, P^0], A].$$

Taking the vacuum expectation value of this expression and using the property of the vacuum state of having zero energy, we get.

#### J. SCHWINGER

$$\langle [i\partial_0 A, A] \rangle = 2 \langle AP'A \rangle$$

and, as the operation of A on the vacuum produces higher excited states, the right-hand side is intrinsically positive. Therefore, the commutator of Eq. (6.27) cannot vanish identically and this implies (Eg. (6.26)) that the vector current is an explicit function of the external potentials.

This result apparently contradicts what one knows about the Dirac field, where a vector current is given by

$$j_{\mu} = (1/2) \psi \alpha_{\mu} q \psi = \overline{\psi} \gamma_{\mu} \psi$$

and does not depend explicitly on the external fields. What in fact happens is that this product is not really defined, and can be given a meaning only by separating the points spatially and defining a suitable limiting procedure which must maintain gauge invariance as in paragraph 1. In this limit the dependence on the external potentials will appear.

Let us now give a similar discussion for the case of an external gravitational field. The situation here is somewhat more complicated because the corresponding conservation law (eq. (6.8)) contains explicitly the external potential.

Eq. (6.8) can be rewritten as:

$$\partial_{\mu} \left( g_{\lambda \nu} T^{\mu \nu} \right) = (1/2) T^{\mu \nu} \left( \partial_{\lambda} g_{\mu \nu} - g_{\lambda \nu} g^{\alpha \beta} \partial_{\mu} g_{\alpha \beta} \right) .$$
 (6.28)

We now specialize to a particular gravitational field where

$$g_{k\ell} = \delta_{k\ell}, g_{ok} = 0$$

and  $g_{00}$  is an arbitrary function of x. Eq. (6.28) then reduces, for  $\lambda = 0$ , to

$$\partial ((-g_{00})T^{00}) = -\partial_k ((-g_{00})T^{0k}) + (1/2)T^{0k}\partial_k g_{00}$$
 (6.29)

while, for  $\lambda = k$ , Eq. (6.8) gives

$$\partial_0 \left( \sqrt{-g_{00}} T^{0k} \right) = -\partial_{\ell} \left( \sqrt{-g_{00}} T^{k\ell} \right) + (1/2) \sqrt{-g_{00}} T^{00} \partial_k g_{00}.$$
 (6.30)

We shall use these relations to derive the commutation relation. As in the electromagnetic case we shall consider a special class:  $T^{kf}$  may (in fact, it must ) be an explicit function of  $g_{00}$  at the same time, but it does not depend on  $g_{00}$  at different times, i.e. it does not depend explicitly on the time derivatives (time locality). From this assumption it follows (Eqs. (6.29) and (6.30)) that the combinations ( $g_{00}$ )  $T^{00}$  and  $\sqrt{-g_{00}}$   $T^{0k}$  are not explicit functions of  $g_{00}$  at all. Performing the corresponding variation and using Eq. (6.14), we obtain, after setting  $g_{00} = -1$ :

$$(1/i)[T^{00}(x), T^{00}(x^{i})] = -\partial_{k} \delta_{3}(x - x^{i})(T^{0k}(x) + T^{0k}(x^{i})). \quad (6.31)$$

This is the fundamental commutation relation which the energy density must obey for the assumed class of physical systems. It is also a necessary and sufficient condition to guarantee the relativistic invariance for these systems. Upon integration, we obtain from it the commutation relation for the generators of the Lorentz group.

## 7. CONSTRUCTION OF A RELATIVISTICALLY INVARIANT, CONSISTENT THEORY OF NON-ABELIAN GAUGE FIELDS

We should mention here a few things about gauge invariance, because this will again be the motivating consideration in the construction of such a more general theory. For electromagnetic gauge invariance, we have

$$\begin{split} A_{\mu}(\mathbf{x}) &\to A_{\mu}(\mathbf{x}) + \partial_{\mu} \lambda (\mathbf{x}) , \\ F_{\mu}(\mathbf{x}) &\to F_{\mu\nu}(\mathbf{x}) , \\ & \swarrow (\mathbf{x}) \to e^{iq\lambda(\mathbf{x})} \chi (\mathbf{x}) \end{split}$$
(7.1)

where X(x) is the field carrying an electrical charge. This transformation forms an Abelian group.

Let us imagine a situation in which we have several charge-like properties, for instance the various components of isotopic spin, which are carried by some field but which are also carried by the gauge field itself. Let  $T_a$ ,  $a = 1, \ldots, n$ , be the charge like matrices associated with the field X and t the matrices associated with the gauge field  $\Phi_{\mu a}$  and  $G_{\mu\nu a}$ . Consider the the class of infinitesimal gauge transformations

$$X \rightarrow [1 + i\sum_{a=1}^{n} T_{a} \delta \lambda_{a}(x)] X, \qquad (7.2)$$

$$G_{\mu\nu} \rightarrow [1 + i\sum_{a=1}^{n} t_a \delta \lambda_a(x)] G_{\mu\nu}, \qquad (7.3)$$

$$\Phi_{\mu} \rightarrow [1 + i\sum_{a=1}^{\Sigma} t_{a} \delta \lambda_{a}(\mathbf{x})] \Phi_{\mu} + \partial_{\mu} \delta \lambda(\mathbf{x}).$$
(7.4)

Note that the field  $G_{\mu\nu}$  transforms now according to Eq. (7.3), while in the electromagnetic case the corresponding field strength  $F_{\mu\nu}$  remains unchanged because it does not carry an electrical charge. Also, the transformation of  $\Phi_{\mu}$  (Eq. (7.4)) expresses the fact that it carries a charge and is a gauge field. These transformations must form a group (which we assume to be compact). This requirement implies commutation relations for  $T_a$  and  $t_a$ :

$$[T_{b}, T_{c}] = \sum_{a} T_{a} t_{abc},$$
 (7.5)

and

$$[t_b, t_c] = \sum_a t_a t_{abc}$$
(7.6)

where the  $t_{abc}$  are the structure constants of the groups. Also, for the inhomogenous transformation (Eq. (7.4)) to belong to this group, we must have

$$(t_b)_{ac} \equiv t_{abc}$$
 (7.7)

In order to keep the fields  $G_{\mu\nu}$  hermitian, the finite hermitian matrices t must be imaginary and therefore antisymmetrical. From this last property follows the antisymmetry of the structure constants in the indices a and c. Furthermore, the commutation relations imply their antisymmetry in the indices b and c. Therefore the structure constants are antisymmetrical in all indices. From here follows the important remark that for a group to be non-abelian ( $t_{abc} \neq 0$ ), it must at least be a three-parameter group. In the three-dimensional case  $t_{abc} = i\epsilon_{abc}$ ,  $\epsilon_{abc}$  being the totally antisymmetric unit tensor, and the commutation relations become the familiar angular momentum commutation relations for isotopic spin.

The infinitesimal gauge transformations which characterize a nonabellian gauge field are:

$$\chi \rightarrow (1 + iT \delta \lambda)\chi,$$

$$G_{\mu\nu} \rightarrow (1 + it \delta \lambda)G_{\mu\nu},$$

$$\Phi_{\mu} \rightarrow (1 + it \delta \lambda)\Phi_{\mu} + \partial_{\mu}\delta\lambda.$$
(7.8)

 $\chi$  is a Fermi field. The T's are matrices, and in T $\delta\lambda$  we understand that there is summation over the n gauge functions:

$$T\delta\lambda = \sum_{1}^{n} T_{\alpha}\delta\lambda_{\alpha}$$
 (7.9)

(sometimes, to avoid ambiguity, we shall use the bracket notation  $T\delta\lambda = {}^{*}T\delta\lambda'$ ). In the electromagnetic case the field G is gauge invariant, but here it also undergoes gauge transformation with the characteristic n dimensional matrices t. Thus  $G_{\mu\nu}$  is a vector with the n components  $(G_{\mu\nu})_a$ . The components of the matrices t are given by the set of structure constants  $t_{abc}$  that are characteristic of the group:

$$(t_{\rm b})_{\rm ac} = t_{\rm abc}^{\rm c}$$
 (7.10)

The vector field  $\Phi$  is on the one hand a gauge field - to this property corresponds the term  $\partial_{\mu} \delta \lambda$  in the gauge transformation - and on the other hand carries the internal properties and so responds linearly to gauge transformations in the term  $(1 + it\delta\lambda)\Phi_{\mu}$ .

#### 8. NOTATIONAL DEVELOPMENTS

Suppose that in an n dimensional space we have vectors A, C and matrices  $t_b$  and we form the scalar product At C. This has components corressonding to the n matrices  $t_b$  and we may form its scalar product with a third vector B:

$$(A t C)B = \sum_{a,b,c} A_a t_{abc} C_c B_b.$$
(8.1)

 $t_{abc}$  is totally antisymmetric in the indices a, b, c so  $\Sigma t_{abc} A_a B_b C_c$  is a totally antisymmetric function of the three vectors. The product (AtC) B is unchanged by cyclic permutation of its factors:

$$(AtC)B = (BtA)C$$
 etc., (8.2)

and is changed in sign by anticyclic permutations of its factors:

$$(AtC)B = -(BtC)Aetc., \qquad (8.3)$$

If we remove one of the vectors from these triple scalar products, say we remove A from (AtC)B = -(AtB)C, we get a vector equation which can be expressed in our previous notation as:

tC'B = -tB'C.

Now we return to the gauge transformation for  $\Phi$ :

$$\Phi_{\mu} = \Phi_{\mu} + (\partial_{\mu} - i t \Phi_{\mu}) \delta \lambda. \qquad (8.4)$$

We have used the last result to write 't $\delta\lambda'\Phi_{\mu} = -t\Phi'_{\mu}\delta\lambda$ . The gauge transformation therefore involves, not just a simple gradient, but a sort of extended gradient in which a term involving  $\Phi_{\mu}$  has to be added (cf.electromagnetism, whence one introduces the electromagnetic interactions by replacing  $\partial_{\mu}$  by  $\partial_{\mu} - eA_{\mu}$ ). We may call  $\partial_{\mu}$  -'it  $\Phi'_{\mu}$  the "gauge covariant derivative".

Consider now some properties of the gauge covariant derivative. The gauge transformation can be introduced in the following algebraic way. Let us take  $\partial$ -i't  $\Phi'$  (suppressing all indices) and apply to it an orthogonal transformation in n dimensional space:

$$(1-i t\delta\lambda')(\partial -it\Phi)(1+it\delta\lambda) = \partial -it(\Phi - (\partial -i't\Phi')\delta\lambda).$$
(8.5)

We shall call this an orthogonal transformation because the matrices t are antisymmetric and imaginary. (The  $\delta\lambda$  are a set of n arbitrary functions). In the derivation of this equation we have used the commutation relation

$$[t \Phi', t\delta\lambda'] = -t(\Phi t\delta\lambda). \tag{8.6}$$

We see that the effect of the orthogonal transformation on  $\partial$ -it $\Phi$  is to maintain its structure but to replace  $\Phi$  by the gauge-transformed operator corresponding to the gauge functions  $-\delta\lambda$ . The invariance of  $\partial$ -it  $\Phi$  will thus be maintained under the orthogonal transformation provided that we simultaneously subject  $\Phi$  to a gauge transformation corresponding to the gauge function  $+\delta\lambda$ .

Another important expression involving these gradients is:

$$[\partial_{\mu} - i't \Phi'_{\mu}, \partial_{\nu} - i't \Phi'_{\nu}] = -i't f^2 G'_{\mu\nu} \quad \text{(definition of } f^2 G_{\mu\nu}), \quad (8.7)$$

from which emerges

$$f^2 G_{\mu\nu} = \partial_{\mu} \Phi_{\nu} - \partial_{\nu} \Phi_{\mu} + i(\Phi_{\mu} t \Phi_{\nu}). \qquad (8.8)$$

Note that this commutation relation refers only to the n-dimensional matrices. The commutation relations of the  $\Phi$ 's considered as operators will be treated later.

Let us consider the effect of an orthogonal transformation on  $G_{\mu\nu}$ . Multiplying  $[\partial_{\mu} - i t \Phi'_{\mu}, \partial_{\nu} - i t \Phi'_{\nu}]$  from the left by 1-it $\delta\lambda$  and from the right by 1+it $\delta\lambda$  is equivalent to transforming the  $\Phi$ 's by the gauge transformation  $-\delta\lambda$ . What we find is

$$(1-i't\delta\lambda')t G(1+it'\delta\lambda') = t(1-i't\delta\lambda')G, \qquad (8.9)$$

and the gauge transformation of G under the gauge transformation -  $\delta\lambda$  is

$$G \rightarrow (1-i^{\dagger}t\delta\lambda')G.$$
 (8.10)

Replacing -  $\delta\lambda$  by  $\delta\lambda$ , we see that the transformation law of the G introduced there is the same as that written down at the beginning of this part. Later we shall come to identify our present G with the previous one; but for the moment the result is just that when  $\Phi$  undergoes an inhomogeneous gauge transformation, the structure G undergoes an homogeneous one.

Now we turn from the defining of objects with simple transformation properties and go to dynamics. The dynamics, of course, consist of our Fermi fields which carry a property we may as well call isotopic spin, interacting with the vector fields. First we consider the Fermi field by itself, treating  $\Phi_{\mu}$  effectively as an extended field. Of course,  $\Phi_{\mu}$  is not really an extended field, but we temporarily treat it as such. The Lagrange function is

$$\mathcal{L} = (i/2)\Psi \alpha^{\mu} (\partial_{\mu} - i'T \Phi'_{\mu})\Psi + (i/m)\Psi \beta \Psi$$
(8.11)

which contains the gauge covariant derivative  $\partial_{\mu}$ -i'T  $\Phi'_{\mu}$ . This Lagrange function is invariant under the infinitesimal gauge transformation:

$$\Psi \rightarrow (1+i^{T} \delta \lambda')\Psi,$$

$$\Psi_{\mu} \rightarrow \Phi_{\mu} + (\partial_{\mu} - i^{1}t \Phi_{\mu}')\delta\lambda.$$
(8.12)

Note the two kinds of matrices: T for the spinor field and t for the vector field. However, the charges induced involve the commutators of the matrices and the commutators of the T are given in terms of the t, so that the charge produced by the variation of  $\Psi$  can, and does, cancel that produced by the variation of  $\Phi$ .

Let us consider the charge in the Lagrange function induced by an infinitesimal charge in the vector  $\Phi$ . We can write it

$$\delta_{\Phi} \mathcal{L} = \delta \Phi_{\mu} k^{\mu}$$
 (8.13)

126

where each  $k^{\mu}$  is a vector with n components  $k_{a}^{\mu}$  and

$$k_a^{\mu} = (1/2) \Psi \alpha^{\mu} T_a \Psi. \qquad (8.14)$$

The  $k^{\mu}_{a}$  form a set of currents, since currents are always identified through the effect of a change of potential. It is a great advantage of our way of writing currents that we clearly separate the kinematic vector which is associated with flow from the object that flows. This is usually observed when talking only about an electrical charge because it can be diagonalized; but when there are n non-commuting objects they cannot all be diagonalized.

'We next ask what restrictions are imposed on these currents  $k^{\mu}$  by the requirement of gauge invariance. The action operator of the system is

$$W = \int (dx) \mathcal{L}$$
 (8.15)

and the infinitesimal in the action operator associated with an infinitesimal charge  $\delta \Phi_{\mu}$  in the external field is:

$$\delta \mathbf{W} \approx \int (\mathrm{d}\mathbf{x}) \mathbf{k}^{\mu} \, \delta \, \Phi_{\mu} \bullet \tag{8.16}$$

If the variation  $\delta \, \Phi_\mu$  is chosen to be that trivial charge which is associated with a gauge transformation

$$\delta \Phi_{\mu} \left( \partial_{\mu} - \mathbf{i}^{\prime} \mathbf{t} \, \Phi_{\mu}^{\prime} \right) \delta \lambda \tag{8.17}$$

with appropriate boundary conditions at infinity, then the variation of the action must vanish locally and we find

$$(\partial_{\mu} - i't \Phi'_{\mu})k^{\mu} = 0$$
 (8.18)

which is a kind of generalized conservation equation. Thus the current  $k^{\mu}$  is, strictly speaking, not conserved: there is an analogy here with the stress tensor  $T^{\mu\nu}$ , which is not conserved in an external gravitational field because the gravitational field transports energy and momentum. So here the currents  $k^{\mu}$  of the Fermi field are not conserved because, if you like, they transfer isotopic spin to the Bose field.

Our generalized conservation equation immediately implies commutation relations for the  $k^{\mu}$ . We employ the same device as used in the previous sections to derive the commutation relations for the electrical charge density and for the energy density. We regard the  $\Phi_{\mu}$  as an external property which is entirely consistent for the derivation of the commutation relations for the Fermi fields alone. Proceeding as before, we write down the equation of motion for  $k^0$ :

$$\partial_0 \mathbf{k}^0 = \mathbf{i}^{\mathsf{t}} \mathbf{t} \, \Phi_0^{\mathsf{t}} \, \mathbf{k}^0 - (\partial_\theta - \mathbf{i} \mathbf{t} \, \Phi_\theta) \mathbf{k}^{\mathsf{f}}. \tag{8.19}$$

Now we make use of two things: a parameter  $\Phi_0$  appears in this equation of motion, and the effect upon the equations of motion of a variation of  $\Phi_0$ , which is coupled to  $k^0$  in the Lagrange function, tells one the commutators at the same time between the object,  $k^0$ , which obeys the equation of motion, and the generator of those infinitesimal transformations. The commutation relations can be read off from

$$\sum_{b} \int (d\mathbf{x}')(1/i) [\mathbf{k}_{a}^{0}(\mathbf{x}), \mathbf{k}_{b}^{0}(\mathbf{x}')] \delta \Phi_{0b}(\mathbf{x}') = -i \sum_{b,c} t_{abc} \delta \Phi_{ab}(\mathbf{x}) \mathbf{k}_{c}^{0}(\mathbf{x})$$

which implies, since the  $\delta \Phi$ 's are arbitrary, that at equal times

$$[k_a^0(\mathbf{x}), k_b^0(\mathbf{x}')] = \delta(\mathbf{x} - \mathbf{x}') \Sigma t_{abc} k_c^0(\mathbf{x}).$$
(8.20)

Thus the  $k^0$  at different points commute and at the same point they obey something like the group commutation relations. If we define quantities  $K_a$  by integrating  $k_a^0$  over all space:

$$\mathbf{K}_{a} = \int (\mathbf{d}^{3} \mathbf{x}) \mathbf{k}_{a}^{0}(\mathbf{x})$$
 (8.21)

then the K<sub>a</sub> satisfy

$$[K_a, K_b] = \sum_{c} t_{abc} K_c = \sum_{c} K_c t_{cab}$$
(8.22)

which are just the group commutation relations. Or in other words, the  $K_a$  furnish a representation of the group. But it is important to recognize that the  $K_a$  are not constants of the motion  $\dot{K}_a \neq 0$ . This is because the  $k_a^0$  do not obey conservation equations or, in other words,  $K_a$  is only a part of the total isotopic spin (the vector field carries the rest).

#### 9. DYNAMICS OF THE FULL SYSTEM

Now we turn to the dynamics of the full system. We use the notion of gauge invariance as a guide in writing down a tentative Lagrange function for the whole system. Then we attempt to find the commutation relations of the fundamental operators. Finally, we must ask whether our tentative Lagrange function is really completely satisfactory, in the sense that it produces a Lorentz invariant theory. We will find that the original Lagrange function was ambiguous within a certain class of Lagrange functions and a particular one must be selected if we are to meet the requirement of relativistic invariance. There is no guidance here to be gained from the correspondence principle: the ambiguous terms are of the order of Planck's constant squared and are simply not determined by any requirement other than that of relativistic invariance. We shall have to apply the test we developed in terms of the commutator of the energy density.

The tentative Lagrange function is constructed so as to give first order field equations. It must therefore contain first derivatives

$$\mathcal{L} = -(1/2)G^{\mu\nu}(\partial_{\mu}\Phi_{\nu} - \partial_{\nu}\Phi_{\mu} + i(\Phi_{\mu}t \Phi_{\nu})) +(f^{2}/4)G^{\mu\nu}G_{\mu\nu} + (i/2)\Psi\alpha^{\mu}(\partial_{\mu} - i'T \Phi_{\nu}')\Psi + (i/2)\Psi\beta\Psi m_{0}.$$
(9.1)

f is a characteristic coupling constant (dimensionless in the three-dimen-

sional case). The question of the order of multiplication of operators is of course basic, but we cannot yet usefully discuss it.

Let us now take the Lagrangian function and write down the equations of motion. If we vary  $G_{\mu\nu}$  we obtain:

$$f^{2}G_{\mu\nu} = \partial_{\mu} \Phi_{\nu} - \partial_{\nu} \Phi_{\mu} + (\Phi_{\mu} i t \Phi_{\nu}). \qquad (9.2)$$

It must be said at this point that we are using the variational principle in a formal way. The purpose is to come back later to this point and criticize and rectify what we are doing. At the moment we are applying a classical action principle without imposing any particular order upon the products of operators.

If we vary  $\Phi_{\mu}$  we get :

$$(\partial_n - \mathbf{i} \mathbf{t} \boldsymbol{\Phi}'_n) \mathbf{G}^{\mu\nu} \approx \mathbf{k}^{\mu}.$$

This completes the full set of the vector field equations. We also have the Dirac equation obtained by variations of  $\Psi$ :

$$(\alpha^{\mu} \partial_{\mu} - i^{\prime}T \Phi_{\mu}' + \beta m)\Psi = 0,$$

completing the preliminary set of field equations.

We have always said that the structure of the Maxwell field equations must guarantee as an identity the conservation of charge. The same condition must be imposed here. The structure of the non-abelian vector gauge field must guarantee as an identity the extended conservation equations of the vector current. Observe that if we take the gauge covariant divergence of  $k^{\mu}$  we will have:

$$\begin{aligned} (\partial_{\mu} - i't \, \Phi'_{\mu})k^{\mu} &= (\partial_{\mu} - i't \, \Phi'_{\mu})(\partial_{\nu} - i't \, \Phi'_{\nu})G^{\mu\nu} \\ &= (1/2)[\partial_{\mu} - i't \, \Phi'_{\mu}), (\partial_{\nu} - i't \, \Phi'_{\nu})]G^{\mu\nu} \\ &= (1/2)(i't \, G'_{\mu\nu})G^{\mu\nu} = (i/2)G^{\mu\nu}t \, G_{\mu\nu} = 0. \end{aligned}$$
(9.3)

In the electromagnetic case the term't  $\Phi'$  is absent and the result is evident. Here it follows from the antisymmetry of  $G^{\mu\nu}$  and the fact that t is totally antisymmetrical. The result obtained is so far formal, because it is necessary to take into account the possibility that the different components of  $\Phi$  may not commute. In other words, the question of operator multiplication obscures the simplicity of the derivation and the simple result no longer obviously follows within the framework of operator equations, although it is true in the classical derivation. All this is preliminary to an actual derivation of the identification of the fundamental variables and their basic commutation relations.

We will now introduce source terms in the Lagrangian, to make use of a uniform technique and exploit the device we have been using so far, in which, from equations of motion in the presence of a suitably disturbed system, we infer commutation relations in such a way that we can identify the fundamental variables. We go back to the Lagrangian and introduce there the simple linear source terms

$$\frac{1}{2} \operatorname{G}^{\mu\nu} \operatorname{M}_{\mu\nu}(\Phi) + \Phi^{\mu} \mathcal{I}_{\mu}(\Phi)$$

where  $M_{\mu\nu}$  is the external source for the field intensities  $G_{\mu\nu}$  and  $\mathcal{I}_{\mu}$  is the external current for the potential  $\Phi^{\mu}$ . However, the addition of these terms must not violate the general gauge invariance of the Lagrangian. This means that  $M_{\mu\nu}$  and  $\mathcal{I}_{\mu}$  must respond to the gauge transformations of the vector field and for that reason the sources are functions of  $\Phi^{\mu}$ . Note also that here the situation is more complicated than in the electrodynamic case because  $\Phi$ undergoes an homogeneous as well as an inhomogeneous gauge transformation. But the relation between the sources and the vector field must be simple in order not to destroy the utility of this technique. We have to exhibit, for example,  $M(\Phi)$  in such a way that it responds properly to a gauge transformation but also in such a form that, at least for particular calculations, the  $\Phi$  dependence disappears. That means that in a particular gauge the sources are independent of  $\Phi$ . In other words, we shall not insist upon full gauge invariance for M, but only explicit invariance in the neighbourhood of the specific chosen gauge. We also want the connection between the sources and the field quantities to be instantaneous, i.e. we must impose time locality. All the relations between sources and the vector field must then be local in time,

Let us again write the infinitesimal gauge transformation properties of  $\Phi_{\mu}$ :

$$\Phi_{\mu} \Rightarrow \Phi_{\mu} + (\partial_{\mu} - \mathbf{i}' \mathbf{t} \Phi_{\mu}') \delta \lambda .$$
(9.4)

We see that the time component changes by the time derivative, so that if we want time locality, we must use only the space part and not the time component which carries the time derivative. The spatial part is:

$$\vec{\Phi} \rightarrow \vec{\Phi} + (\nabla - i't \Phi') \delta \lambda. \tag{9.5}$$

We are interested in exhibiting a function of the vector  $\vec{\Phi}$  which finally will depend only on  $\delta\lambda$ . Isolating this dependence to counter the gauge transformation of  $\Phi$ , the gauge variation of the vector  $\vec{\Phi}$  contains the gradient of  $\delta\lambda$ , but we want to construct a scalar equation and, naturally, we take the divergence of  $\vec{\Phi}$ :

$$\nabla. \ \overline{\Phi} \rightarrow \overline{\nabla}. \ \overline{\Phi} + \nabla. (\nabla - \mathbf{i}' \mathbf{t} \ \Phi') \delta. \tag{9.6}$$

Having once done that, the natural gauge about which we have to perform the infinitesimal variation of gauge appears to be the one in which  $\nabla$ .  $\Phi = 0$ , i.e. the radiation gauge. In an infinitesimal neighbourhood of this gauge we have:

$$\nabla_{\bullet} \Phi = \nabla_{\bullet} (\nabla_{\bullet} i^{\dagger} t \Phi') \delta \lambda. \qquad (9.7)$$

The characteristic Green's function for this differential equation satisfies :

$$-\nabla \cdot (\nabla - i't \Phi') \mathcal{D}(\mathbf{x}, \mathbf{x}') = \delta_3(\mathbf{x} - \mathbf{x}')$$
(9.8)

where  $\mathscr{D}$  is real and symmetric

$$\mathcal{D}_{ab}(\mathbf{x}, \mathbf{x}') = \mathcal{D}_{ba}(\mathbf{x}', \mathbf{x}) \tag{9.10}$$

and replaces the Coulomb-gauge function for the electromagnetic case. The solution for  $\delta\lambda$  is

$$\delta \lambda = - \hat{\mathcal{D}}_{\phi} \nabla_{\bullet} \Phi_{\bullet} \tag{9.11}$$

This solution gives the gauge variation  $\delta\lambda$  for an infinitesimal neighbourhood of the radiation gauge  $\nabla \cdot \Phi = 0$ .

We can now write down explicitly what  $M_{\mu\nu}(\Phi)$  must be, not for any gauge but specifically for the consideration of infinitesimal variations about the radiation gauge. It must be such a function of  $\Phi$  that it responds by the counter transformation

$$\mathbf{M}_{\mu\nu}(\Phi) = (1 - \mathrm{it} \, \mathfrak{D}_{\Phi} \nabla \cdot \Phi) \mathbf{M}_{\mu\nu} \tag{9.12}$$

where the  $M_{\mu\nu}$  are just simple numbers. This form for the source satisfies all our three requirements. First of all, these gauge variant sources are related to arbitrary numerical quantities only at the same time. Secondly, in the radiation gauge that dependence disappears and the sources are arbitrary numbers. Thirdly, for infinitesimal variations about the radiation gauge, they vary by the factor  $(1+it\delta\lambda)$  which just compensates the gauge variation of  $G^{\mu\nu}$ . We have achieved gauge invariance in an effective computational form for infinitesimal variations about the radiation gauge. And when we actually work in the radiation gauge the  $\Phi$  dependence disappears.

Let us now construct  $\mathcal{I}_{\mu}(\Phi)$  by the same procedure :

$$\mathcal{I}_{\mu}(\Phi) = (1 - \mathrm{it} \,\,\mathfrak{D}_{\Phi} \nabla \cdot \Phi) \mathcal{I}_{\mu} \,. \tag{9.13}$$

 $\mathcal{I}_{\mu}$  is independent of  $\Phi$  but, since there are two parts to the gauge variation of  $\Phi$ , one is inhomogeneous and for the gauge invariance of this part it must be  $\partial_{\mu} \mathcal{I}^{\mu} = 0$ . Thus we have invariance under infinitesimal gauge transformation about the radiation gauge.

Returning now to the equations of motion and adding the extracontributions from the external sources, we can read all the equal time commutation relations by merely inspecting the structure of the field equations.

The new equations are, first for  $\Phi$ :

$$\partial_{\mu} \Phi_{\nu} - \partial_{\nu} \Phi_{\mu} + i \Phi_{\mu} t \Phi_{\nu} = f^2 G_{\mu\nu} + M_{\mu\nu}. \qquad (9.14)$$

The  $M_{\mu\nu}$  are here numbers independent of  $\Phi$  because we are now working in the radiation gauge. Secondly, the equation for  $G_{\mu\nu}$ :

$$(\partial_{\nu} - it \Phi_{\nu})G^{\mu\nu} = k^{\mu} + \mathcal{I}^{\mu} + \nabla^{\mu} \mathcal{D}_{\Phi} [(1/2)G^{\lambda\nu}it M_{\lambda\nu} + \Phi^{\nu}it \mathcal{I}_{\lambda\nu}]$$
(9.15)

where  $\nabla^{\mu}$  is the four component vector of which the time component is zero. Let us now take the two terms involving  $\mathcal{I}_{\mu}$ :

$$J + \nabla \mathcal{D} (-i t \Phi) \mathcal{I} = [1 + \nabla \mathcal{D} (\partial - i t \Phi)] \mathcal{I}. \qquad (9.16)$$

the derivative  $\partial$  has been added to obtain the "covariant gradient"  $\partial$ -it $\Phi$ . Its contribution is zero because  $\mathcal{I}$  is conserved. This last equation interests us because the operator that acts upon  $\mathcal{I}$  is a projector operator that picks up exactly the right properties of the vector  $\mathcal{I}^{\mu}$ , in the following sense: we observe that the right side, as a current, should be conserved in the extended sens that, applying the gauge covariant divergence, one must obtain zero and this is in fact the case:

$$(\partial - \mathrm{it}\,\Phi) - [1 + \nabla\,\mathbf{D}\,(\partial - \mathrm{it}\,\Phi)] = (\partial - \mathrm{it}\,\Phi) - (\partial - \mathrm{it}\,\Phi) = 0, \qquad (9.17)$$

because

$$(\partial - it \Phi) \cdot \nabla = \nabla_{\bullet} (\nabla - it \Phi) + it \nabla_{\bullet} \Phi,$$
 (9.18)

The first term on the right-hand side is the differential operator defining  $\mathscr{D}$  and in the radiation gauge  $\nabla$ .  $\Phi = 0$ . This is the importance of the projection operator that guarantees charge conservation in the extended sense. The derivative  $\partial_{\mu}$  acting upon  $\mathscr{I}^{\mu}$  may be said to be optional, but if we use the projector in the form we wrote it, then for all variations of  $\mathscr{I}^{\mu}$  the constraint equation  $\partial_{\mu} \mathscr{I}^{\mu} = 0$  needs no longer be considered. Obviously, since the divergence of  $\mathscr{I}^{\mu}$  is equal to zero, not all variations of  $\mathscr{I}^{\mu}$  can be independent. In particular:

$$\partial_0 \delta J^0 = -\nabla \cdot \delta J \tag{9.19}$$

and the variation of the longitudinal part  $\nabla$ .  $\mathcal{I}$  is completely determined by the constraint. But now, the structure:

$$[1+\nabla \mathcal{D} (\partial - it \Phi)] \mathcal{I}$$
(9.20)

does not depend at all upon the longitudinal part of  $\mathcal{I}$ .

$$[1 + \nabla \mathcal{D}(\partial - it \Phi)], \nabla = \nabla - \nabla = 0 \tag{9.21}$$

(an integration by parts is involved in the proof of this equality). So it is not necessary to make use of the constraint equation and we can vary  $\mathcal{I}^{\mu}$  freely.

We will now examine the field equations to see which of them are equations of motion and which of them are only equations of constraint. Let us first write down the field equations which are equations of motion, i.e. equations having time derivatives in it. They are:

$$\partial_0 \Phi_k = (\partial_k - \operatorname{it} \Phi_k) \Phi_0 + f^2 G_{0k} + M_{0k}, \qquad (9.22)$$

132

$$(\partial_0 - it\Phi_0)G^{k0} = -(\partial - it\Phi)G^{\ell k} + k^{\ell} + [1 + \nabla \mathcal{D}(\partial - it\Phi)]\mathcal{I} + \partial^{\ell} (1/2)G^{\lambda \nu}itM_{\lambda \nu}$$

On the other hand, the constraint equations are:

$$f^{2}G_{k\ell} + M_{k\ell} = \partial_{k} \Phi_{\ell} - \partial_{\ell} \Phi_{k} + (\Phi_{k} \text{ it } \Phi_{\ell})$$

$$(9.23)$$

$$(\partial_{\nu} - \text{ it } \Phi_{\nu})G^{0k} = k^{0} + \mathcal{I}^{0}.$$

They tell us of course that neither the divergence of the "electric field"  $G^{ok}$  nor the components of the "magnetic field"  $G_p^{k\ell}$  can be treated as independent variables.

We will now look at the equations of motion and vary the parameters. We then automatically get a commutation relation with the operators associated with the parameters in the Lagrange function. And since we have equations of motion for the fields  $\Phi_k$  and  $G_{0k}$ , we will get commutation relations between these operators and the operators that appear in the action integral.

It should be mentioned that the first equation of motion contains a hidden constraint, because in the radiation gauge  $\nabla \cdot \Phi = 0$  and so, taking the divergence of that equation, the time derivative disappears and we are left with

$$-\nabla_{\bullet} (\nabla - \operatorname{it} \Phi) \Phi_0 = \partial^k (f^2 G_{0k} + M_{0k})$$
(9.24)

which eliminates  $\Phi_0$  as an independent variable. This is an indication that only the transverse part of  $\Phi$  and the transverse part of  $G^{0k}$  can be considered to be the fundamental variables, and all this occurs exactly as in the electromagnetic case.

We will now look at the structure of the equations of motion and simply read off the commutation relations. Let us vary the chosen set of parameters starting with  $\mathcal{I}_k$ . Looking at the equation of motion for  $\Phi_k$  we see that  $\mathcal{I}_k$  does not appear nor is it even hidden in the dependent variable so that the coefficient of the variation of  $\mathcal{I}_k$  is zero:

$$\delta(\partial_0 \varphi_k) = (0) \delta J_k \qquad (9.25)$$

and from this follows the equal-times commutation relation

$$[\Phi_{k}(x), \Phi_{\ell}(x')] = 0. \qquad (9.26)$$

Next, looking at the equation of motion for  $G^{0k}$  and considering the effect of the variation of  $\mathcal{I}_{\ell}$ , which only appears explicitly and multiplied by a projection operator, we see that

$$\delta \partial_0 G^{0k} = -\delta(\operatorname{Projector})\mathcal{I}$$
(9.27)

from which we can read off the commutation relation

$$i[G^{0k}(x), \Phi_{\theta}(x)] = {}^{k}(1 + \nabla \mathcal{D} (\partial - it \Phi'))_{\theta}(x, x'), \qquad (9.28)$$

or, showing explicitly the different components of  $G^{ok}$  and  $\Phi_{\mu}^{i}$ 

$$i[G_a^{ab}(\mathbf{x}), \boldsymbol{\Phi}_{\ell b}(\mathbf{x}')] = \delta_{\ell}^k \delta_{ab} \delta(\mathbf{x} - \mathbf{x}') + \partial^k [\mathcal{D}_{\Phi}(\mathbf{x}, \mathbf{x}')(-\overleftarrow{\partial}_{\ell}' - i \cdot \mathbf{t} \boldsymbol{\Phi}_{\ell}'(\mathbf{x}')]_{ab}. \quad (9.30)$$

Last of all, we should find the commutation relations between the  $G^{0k}$  themselves. Once we know these, the other commutation relations can be computed, knowing the way in which the other fields depend upon the fundamental ones. The last commutation relations can be obtained considering the variations of  $M^{0k}$ . The first equation of motion does not give anything new, it merely repeats the commutation relation just found (showing, of course, that the procedure is consistent). The other equation of motion gives the information we require. It contains  $M^{0k}$  explicitly in the last term and also implicitly in the dependent variable  $\Phi_0$ . Taking into account both dependences we arrive at :

$$\mathbf{i}[\mathbf{G}^{0k}(\mathbf{x}), \mathbf{G}^{0\ell}(\mathbf{x}')] = \partial^{k} \mathcal{D}(\mathbf{x}, \mathbf{x}') \mathbf{i} \mathbf{t} \mathbf{G}^{0\ell}(\mathbf{x}') + \mathbf{i} \mathbf{t} \mathbf{G}^{0k}(\mathbf{x}) \mathcal{D}(\mathbf{x}, \mathbf{x}') \overline{\partial}^{\ell'}. \quad (9.31)$$

These commutation relations seem to be complicated but it must be realized that we have derived them for the full operator  $G^{0k}$  which consists of a dependent longitudinal part and the independent transverse part  $G^{0kT}$  which is the fundamental variable. It is possible to extract the commutation relation for  $G^{0kT}$  only. We can see that the right-hand side of the commutation relation does not contain any purely transverse part and therefore

$$[G^{0kT}(x), G^{0\ell}(x)^{T}] = 0$$
(9.32)

which, together with

$$[\Phi_{k}(\mathbf{x}), \Phi_{\theta}(\mathbf{x}')] = 0 \tag{9.33}$$

and

$$i[G^{0kT}(x), \Phi_{\ell}(x')] = \delta_{\ell}^{k}[\delta(x-x')]^{T}, \qquad (9.34)$$

form the canonical commutation relations between the fundamental field quantities. By comparison with the electromagnetic case, we see that it still contains the essential simplicity which consists of the fact that the fundamental variables are exactly the same transverse parts of the potential and the electric field. The commutation relations have the same appearance except of course for the fact that in the electromagnetic field we have only one such equation and here we have n X n such equations. In other words, the equations just found are really matrix equations in the "internal" vector space.

# STRONG COUPLING IN A GAUGE INVARIANT FIELD THEORY

## K. JOHNSON

# PHYSICS DEPARTMENT, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASS., UNITED STATES OF AMERICA

### STRONG COUPLING IN A GAUGE INVARIANT FIELD THEORY

I would like to discuss some approximations which may be significant in the domain of strong coupling in a field system analogous to quantum electrodynamics. The motivation of this work is the idea that the strong couplings and elementary particle spectrum may be the consequence of the dynamics of a system whose underlying description is in terms of a set of Fermi fields gauge invariantly coupled to a single ("bare") massless neutral vector field. The basis of this gauge invariance would of course be the exact conservation law of baryons or "nucleonic charge". It seems to me that a coupling scheme based on an invariance principle is most attractive if that invariance is an <u>exact</u> one. It would then be nice to try to account for the approximate invariance principles in the same way one would describe "accidental degeneracies" in any quantum system.

As is well known, the outstanding objection against the attempt to account for the strong interactions on the basis of such a mechanism is that it has usually been assumed that a system which is based upon an originally massless vector field coupled to a conserved vector always would be capable of vector excitations of zero mass. However, we have learned from Prof. Schwinger that, in fact, such reasoning is faulty. To put our considerations in the appropriate context let me begin by paraphrasing some of his remarks.

We shall discuss the dynamics of a single Fermi field gauge invariantly coupled to a neutral vector field. Of course the world of the strong interactions must involve more than a single Fermi field but we will simplify the discussion by considering only a single field. Thus, the model we shall consider is perfectly analogous to quantum electrodynamics except that we shall assume that now the coupling is strong rather than weak.

If a weak external current  $J_{\mu}(q)$ , where

$$J_{\mu}(q) = \int \frac{(dx)}{(2\pi)^2} e^{-iqx} J_{\mu}(x),$$

acts in the vacuum it induces a net field,

$$\langle \mathbf{F}_{\mu\nu} \rangle = \mathbf{i}(\mathbf{q}_{\mu}\mathbf{J}_{\nu}(\mathbf{q}) - \mathbf{q}_{\nu}\mathbf{J}_{\mu}(\mathbf{q}))\mathcal{G}(\mathbf{q}^{2})$$
(1)

where  $\mathcal{G}(q^2)$  is the vector Green's function. It is appropriate to relate the total field F to the part of the total field which appears as a consequence of the polarization of the vacuum. Thus,
$$\langle \mathbf{F}_{\mu\nu} \rangle = \langle \mathbf{F}_{\mu\nu}^{\text{pol}} \rangle + \mathbf{F}_{\mu\nu}^{\text{ext}}$$
 (2)

where

$$F_{\mu\nu}^{ext} = i(q_{\mu} J_{\nu} - q_{\nu} J_{\mu})/q^{2}.$$
 (3)

Then,

$$\left\langle \mathbf{F}_{\mu\nu}^{\text{pol}} \right\rangle = \left\langle \mathbf{F}_{\mu\nu} \right\rangle - \mathbf{i} (\mathbf{q}_{\mu} \mathbf{J}_{\nu} - \mathbf{q}_{\nu} \mathbf{J}_{\mu}) / \mathbf{q}^{2}$$

$$= \left\langle \mathbf{F}_{\mu\nu} \right\rangle - \frac{1}{\mathbf{q}^{2} \mathscr{G}(\mathbf{q}^{2})} \left\langle \mathbf{F}_{\mu\nu} \right\rangle$$

$$= - \mathbf{P}(\mathbf{q}^{2}) \left\langle \mathbf{F}_{\mu\nu} \right\rangle$$

$$(4)$$

where the vacuum polarizability is

$$P(q^2) = 1/q^2 \mathscr{G}(q^2) - 1,$$

 $\mathbf{or}$ 

$$\frac{1}{\mathscr{G}} = q^2 \left( 1 + P(q^2) \right).$$
(5)

If we express the  $\underline{current}$  induced in the vacuum in terms of the effective field F, we find since

$$\left\langle j_{\mu\nu}^{\text{pol}} \right\rangle = iq^2 \left\langle F_{\mu\nu}^{\text{pol}} \right\rangle$$
$$= -iq^2 P(q^2) \left\langle F_{\mu\nu} \right\rangle$$

and if we express  $\langle F \rangle$  in terms of the potential which generates it,

$$\langle \mathbf{F}_{\mu\nu} \rangle = \mathrm{iq}_{\mu} \langle \mathbf{A}_{\nu} \rangle - \mathrm{iq}_{\nu} \langle \mathbf{A}_{\mu} \rangle$$

then,

$$\left\langle j_{\mu}^{\text{pol}} \right\rangle = -iq^2 P(q^2) \left( iq_{\mu} \langle A_{\nu} \rangle - iq_{\nu} \langle A_{\mu} \rangle \right)$$
$$= \left( -g_{\mu\nu}q^2 + q_{\mu}q_{\nu} \right) P(q^2) \langle A^{\nu} \rangle$$

Since we have assumed that field external current is weak, this equation can be written as a variational derivative in the form:

$$\frac{\delta \langle \mathbf{j}_{\mu}^{\text{pol}} \rangle}{\delta \langle \mathbf{A}_{\nu} \rangle} |_{\langle \mathbf{A} \rangle = 0} = (-g_{\mu\nu} q^2 + q_{\mu}q_{\nu}) P(q^2).$$
(6)

Thus, the presence of the factor  $q^2$  in front of P in the expression for  $\mathcal{G}$  can be attributed to the conservation law obeyed by the source  $j_{\mu}$ . Further, the "additive  $q^2$ " in the expression can be attributed to the gauge invariance of the Lagrangian. In the normal case, these two factors ensure that as  $q^2 \rightarrow 0$ ,  $\mathcal{G}^{-1} \rightarrow 0$  so that there are massless vector excitations of the coupled

136

system. However, if the coupling becomes sufficiently strong to produce a pole in  $P(q^2)$  at  $q^2 = 0$ , then these excitations will be completely suppressed. It was in not allowing for the possibility that a pole could appear in P that previous treatments of the photon mass question were deficient.

We see therefore, it would be singularly inappropriate to measure the coupling by the usual "renormalized" coupling constant which is defined by assuming that a massless state always occurs. Thus, suppose we examine the coupled chain of equations for the Green's functions. The vacuum current can be expressed in terms of the particles Green's function, calculated in the presence of an external field, or the potential  $\langle A \rangle$  which describes the resultant effective field by the equation

$$\langle j_{pol}^{\mu}(\mathbf{x}) \rangle = i e_0 \hbar \left( \gamma^{\mu} G(\mathbf{x}, \mathbf{x}) \right)$$
 (7)

where naturally, suitable limiting procedures must be applied to ensure the meaningfullness of the above expression. The dependence of G upon a weak effective field  $\langle A \rangle$  is usually expressed in terms of the "vertex function"

$$\Gamma_{\mu} = -\frac{\delta}{\delta(\langle A \rangle \ell_0)} G^{-1} |_{\langle A \rangle = 0}$$
(8)

using (6) and (7) we obtain for P the familiar formula

$$(-g_{\mu\nu} q^{2} + q_{\mu} q_{\nu}) P = ie_{0}^{2} \int \frac{(dp)}{(2\pi)^{4}} \hbar \left\{ \gamma_{\mu} G(p+q/2) \Gamma_{\nu}(p+q/2, p-q/2) G(p-q/2) \right\} (9)$$

Now the equations for  $G \neq \Gamma$  contain the Lagrangian coupling constant only in the combination,  $e_1^2 \mathcal{G}$ . Hence, let us redefine  $\mathcal{G}$  so

 $e_n^2 \mathcal{Q} \to \mathcal{Q}$ 

and at the same time let

 $P \rightarrow e_0^2 P$ .

Then, we obtain the equations

$$1/\mathscr{Q} = q^{2} \left( 1/e_{0}^{2} + P(q^{2}, \mathscr{Q}) \right)$$
(10)

and

$$(-g^{\mu\nu} q^2 + q_{\mu}q_{\nu})P(q^2, \mathscr{G}) = i \int \frac{(dp)}{(2\pi)^4} \hbar (\gamma_{\mu} G\Gamma_{\nu} G)$$
 (11)

where we have written P  $(q^2; \mathcal{G})$  to emphasize that P may be regarded as a functional of  $\mathcal{G}$  since the integral equations for  $G \neq \Gamma$  involve  $\mathcal{G}$ . Thus, the Lagrangian coupling constant  $e_0^2$  now enters only through equation (10) which may be regarded as a functional equation for  $\mathcal{G}$ . The perturbation theory method of solution of (10) is to define the renormalized, or "physical" coupling constant as the weight of the "assumed" pole at  $q^2 = 0$  in  $\mathcal{G}$ , i.e.,

$$1/e^2 \equiv 1/e_0^2 + P(0, \mathcal{G}).$$
 (12)

If we eliminate  $e_0^2$  in favour of  $e^2$  we obtain

$$1/\mathscr{G} = q^{2} (1/e^{2} + [P(q^{2}, \mathscr{G}) - P(0, \mathscr{G})])$$
$$= q^{2} (1/e^{2} + \overline{P}(q^{2}, \mathscr{G})).$$
(13)

This equation is solved in perturbation theory by an expansion in  $e^2$ . However, if P developes a pole at  $q^2 = 0$ , the use of  $e^2$  as a parameter to describe the coupling would clearly be most unsuitable. In fact, the sort of behaviour of  $e^2$  considered as a function of  $e_0^2$  in the situation that is contemplated here can be represented graphically in Fig. 1. That is, we shall attempt to obtain approximate solutions of the above equations assuming that  $e^2 = 0$  for  $e_0^2 > e_c^2$ , where  $e_c^2$  is some critical coupling strength.



| Fig |   | 1 |
|-----|---|---|
|     | ٠ | _ |

Naturally, in order to obtain approximate solutions for  $e_0^2$  large we must make assumptions as to the nature of the system that results as the coupling becomes strong. That is, we must try to use some self-consistent approach. I shall assume that if the above scheme is to be successful, then it must account for the fact that when the fields are coupled strongly there still results a particle spectrum and these particles are phenomenologically strongly coupled. In other words, the spectrum contains sharp and heavily weighted peaks. We shall assume that the peaks in the boson spectra are a consequence of strong "pair correlations" in the Fermi field system. It is not unreasonable to suppose this since a neutral vector field coupled in the way of electrodynamics is just the kind of thing which should produce strong correlations between pairs. In fact, if we could calculate, with high precision, the function  $\mathcal{G}(q^2)$ , in the weak coupling case we should find that it has an approximate pole in the neighborhood of  $q^2 = -(2m - \epsilon)^2$  where  $2m - \epsilon$  is the mass of the  ${}^{3}S_{1}$  state of positronium. This pole in  $\mathscr{G}$  would correspond to a zero in  $1/e^2 + \overline{P}(q^2)$ , which in turn would result from a nearby pole in  $\overline{P}(q^2)$ . This pole would lie at a position between 0 and  $-(2m+\epsilon)^2$ . We shall assume that as the coupling  $e_0^2$  increases that this pole in  $P(q^2)$ (or  $\overline{P}$ ) moves to zero and the associated pole in  $\mathscr{G}(q^2)$  moves to  $q^2 = -\mu^2$ . Further, we shall assume that this pole in  $\mathcal G$  dominates the vector spectrum. Thus, we assume that if  $e_t^2$  is sufficiently large  $P(q^2)$  has the form:

$$P(q^2) = \lambda^2 / q^2 + \int dm^2 \frac{s(m^2)}{q^2 + m^2}.$$
 (14)

We shall further assume that the vector particle is stable or almost so. That is, we shall assume that if, as we increase the coupling, scalar states move

down much more rapidly than the vector state, the vector state nevertheless still remains narrow. Thus, for simplicity we assume that the equation

$$1/9 = 0$$
 for  $q^2 = -\mu^2$ 

i.e.,

$$\mu^{2} = \frac{\lambda^{2}}{1/e_{0}^{2} + \int dm^{2} \frac{s(m^{2})}{m^{2} - \mu^{2}}}$$
(15)

has a solution (i.e., we assume the threshold on s is larger than  $\mu$ ). Further, we shall assume that this state dominates the vector spectrum. Thus as  $q^2 \rightarrow -\mu^2$ 

$$\frac{1}{a_{f}} \longrightarrow (\mathbf{q^2} + \mu^2) \frac{1}{\mathbf{f^2}}$$

 $\mathbf{so}$ 

$$1/f^{2} = 1/e_{0}^{2} + \int dm^{2} \frac{m^{2}}{(m^{2} - \mu^{2})^{2}} s(m^{2}).$$
 (16)

We assume  $f^2$  is large when  $e_0^2$  is large. In terms of the "coupling constant"  $f^2$  we can rewrite (15) in the form

$$\mu^{2} = \lambda^{2} / [1/f^{2} - \mu^{2} \int dm^{2} \cdot s(m^{2}) / (m^{2} - \mu^{2})^{2}]$$
(17)

Equations (14), (16) and (17) are the basis of our approximation scheme, since we have made the fundamental assumption that the state with mass  $\mu^2$  and weight  $f^2$  dominates the spectrum of vector states, we shall solve (10) self-consistently by calculating the vacuum polarization P, assuming for  $\mathscr{G}$  the form

$$\mathcal{L}_{\mathbf{j}} \simeq \mathbf{f}^{\mathbf{2}} / (\mathbf{q}^{\mathbf{2}} + \boldsymbol{\mu}^{\mathbf{2}}).$$

Thus, using this expression for  $\mathcal{G}$ , P and hence,  $\lambda$  and s become functions of  $\mu$  and f. In this way, equations (16) and (17) give two relations between three parameters,  $\mu$ , f and  $e_0$ . The third condition to be imposed is that P have a pole at  $q^2 = 0$ . We expect this to happen for a critical value of  $e_0$  and to continue to be true for  $e_0$  larger than the critical value.

It may also be consistent, and is certainly in the same spirit to replace (17) by the approximate form

$$\mu^2 \cong f^2 \lambda^2 \langle \mu, f \rangle. \tag{18}$$

In this case, it would be possible to obtain two conditions on  $\mu$ , f without an actual calculation of s (m<sup>2</sup>), but only  $\lambda^2$ . (18) expresses one condition, the other would be that f<sup>2</sup> must be such to produce the pole at q<sup>2</sup> = 0.

Finally, if we assume that as long as  $e_t^2 > e_c^2$ , where  $e_c$  is the critical coupling the pole remains at  $q^2 = 0$ , then it would be possible to pass to the limit  $e_t^2 \rightarrow \infty$ . In this case, we can obtain both f and  $\mu$  uniquely by solving (17) and (16) in this limit, namely

K. JOHNSON

$$\frac{1}{f^2} = \int dm^2 \, \frac{m^2}{(m^2 - \mu^2)^2} \, s(m^2)$$
(18a)

In this instance, we would have a so-called "bootstrap" in the sense that the pole in  $\mathcal{G}$  at  $q^2 = -\mu^2$  would be entirely responsible for itself since no free parameters remain in the theory in this limit.

Of course, in order to make further progress it is necessary to make more approximations to obtain P. We shall make these approximations based on the assumption of the dominance of the "pair correlations".

We may first note that it is possible to express the condition that P have a pole at  $q^2 = 0$ , in terms of a corresponding condition on the vertex function  $\Gamma$ . Thus, it is clear from (11) that P will develop a pole at  $q^2 = 0$  only if the vertex  $\Gamma$  developes such a pole as  $q^2 \rightarrow 0$ . Thus, it must be that as  $q^2 \rightarrow 0$ 

$$\Gamma_{\mu}(p + q/2, p - q/2) \rightarrow 1/q^2 \widetilde{\Gamma}_{\mu}(p + q/2, p - q/2).$$

In this case, we obtain for  $\widetilde{\Gamma}$  from the equation for  $\Gamma$  the exact equation

$$\widetilde{\Gamma}_{\mu}(p + q/2, p - q/2) = -i \int \frac{(dp')}{(2\pi)^4} \mathscr{G}((p - p')^2) \gamma_{\lambda} G(p' + q/2) \widetilde{\Gamma}_{\mu}$$
(19)  
×  $(p' + q/2, p' - q/2) G(p^2 - q/2) \Gamma^{\lambda}(p' - q/2, p - q/2)$ 

which is a homogeneous, linear integral equation. Thus, the condition that  $P \text{ develops a pole at } q^2 = 0 \text{ can be rephrased in terms of the requirement}$  that the linear equation (19) have a solution. The first step in our approximation scheme is to replace  $\frac{q}{2}$  by

$$^{a_{j}} \sim f^{2} / [(p - p)^{2} + \mu^{2}]$$

in the kernel for this integral equation. Our second step must be to replace G and  $\Gamma$  by suitable approximations in the above kernel.

Our basic assumption is that the correlations between <u>pairs</u> predominate. Thus, in the equation for the Green's function in a <u>weak</u> external field which is

$$G^{-1}(\langle A \rangle) = G_{0}^{-1} - \gamma^{\mu} \langle A_{\mu} \rangle + i \int \mathcal{G} \gamma_{\lambda} (G + G\Gamma_{\mu} \langle A \rangle^{\mu} G) \Gamma^{\lambda}$$
  
+  $i \int \mathcal{G} \gamma_{\lambda} G\Gamma_{\lambda\mu}^{(2)} \langle A \rangle^{\mu}$  (20)

where

$$\Gamma^{(2)} = \delta \Gamma / \delta \langle A \rangle |_{\langle A \rangle = 0},$$

we shall assume that we can drop the last term which means that we drop a term corresponding to correlations between more than pairs, and <u>also</u> we replace the  $\Gamma$  on the end above by  $\gamma \times [\text{const.}]$ .

The second assumption is not independent of the first since it is the expansion of the vertex on the <u>end</u> in  $\langle A \rangle$  which produces the last term in the above equation. Since the momentum transfer that takes place at the second vertex is p' - q/2 - (p - q/2) = p' - p, if the argument of  $\mathcal{G}$  is p' - p.

140

this transfer is positive  $((p' - p)^2 > 0)$  when  $\mathcal{G}$  produces an attractive force. Thus, the "sign" of the charge that enters and leaves the vertex on the end is the same and therefore we may assume a slow variation of this  $\Gamma$  in the domain of integration which is important in (20). The equation for  $\mathcal{G}$  then takes the form

$$\mathbf{G}^{-1}(\langle \mathbf{A} \rangle) \simeq \mathbf{G}_{\mathbf{0}}^{-1} - \gamma \langle \mathbf{A} \rangle + \mathrm{i} \int \mathscr{G} \gamma_{\lambda} (\mathbf{G} + \mathbf{G} \Gamma \langle \mathbf{A} \rangle \mathbf{G}) \gamma^{\lambda} \left[ \mathrm{const.} \right]$$
(21)

where  $G_0^{-1} = \gamma p + m_0$ ,  $m_0 = bare mass of fermion.$ 

Clearly, we have maintained gauge invariance, if we use this equation to obtain the equation for  $\Gamma,$  thus

$$\Gamma_{\mu} = \gamma_{\mu} - i \int \mathcal{G} \gamma_{\lambda} G \Gamma_{\mu} G \gamma \ [ \text{ const.} ] .$$
 (22)

In fact it can be trivially verified that Ward's identity holds. The equation for G is then

$$G^{-1} = G_{\mathfrak{g}}^{-1} + i \int \mathcal{G} \gamma_{\lambda} G \gamma^{\lambda} [\text{const.}]. \qquad (23)$$

We shall choose the constant in (22) and (23) in such a way that we suppress maximally the contribution of the high energy states in the integrations. Thus, we first write

G = 
$$Z_2 \int dk \frac{A(k)}{\gamma p + k} = Z_2 \vec{G}$$

where  $\overline{G}$  is the renormalized Green's function in the sense that its pole at  $p^2 = -m^2$ , is normalized to unity.

 $Z_2$  is given by

$$1/Z_2 = \int dk A(k),$$

and is the most sensitive parameter to high energy states. We can then write

$$\overline{\mathrm{G}}^{-1} = \mathrm{G}_0^{-1} Z_2 + \mathrm{i} Z_2^2 \text{ const. } \left[ \int \mathscr{G} \gamma_{\lambda} \overline{\mathrm{G}} \gamma^{\lambda} \right]'$$

where now  $\overline{G}_0^{-1} = \gamma p + m$ , where m is the physical mass of the fermion, and the ' denotes a subtraction of the integral at  $\gamma p = -m$ . We then must impose the normalization condition on  $\overline{G}$  and this is equivalent to a second subtraction (denoted by '') so

$$\widetilde{\mathbf{G}}^{-1} = \mathbf{G}_{6}^{-1} + \mathbf{i} \quad Z_{2}^{2} \text{ const.} \left[ \int \mathcal{G} \boldsymbol{\gamma}_{\lambda} \, \widetilde{\mathbf{G}} \, \boldsymbol{\gamma}^{\lambda} \right]^{\prime \prime}.$$

Thus, we can completely suppress the high mass contribution  $(Z_2)$  if we put const. =  $1/(Z_2^2)$ , and hence obtain the equation

$$\overline{G}^{-1} = G_0^{-1} + i \left[ \int \mathcal{Q} \gamma_{\lambda} \overline{G} \gamma^{\lambda} \right]''.$$
(24)

The corresponding equation for the renormalized vertex is

$$\overline{\Gamma}_{\mu} = \gamma_{\mu} Z_{2} - i \int \mathcal{G} \gamma_{\lambda} \overline{G} \overline{\Gamma}_{\mu} \overline{G} \gamma^{\lambda}.$$
(25)

Thus, it is these equations which we may regard as our first approximation. We may view this treatment of the vertex on the end as the first step in an iterative procedure, where the next step would be to solve a similar linear integral equation for the next approximation which would use the vertex gotten from the first step on the end and a Green's function gotten from an equation using that vertex.

If we require  $\overline{\Gamma}$  to develop a pole as  $q^2 \rightarrow 0$ ,

 $\overline{\Gamma}_{\mu} \rightarrow \widetilde{\Gamma}_{\mu} \, / \, \mathrm{q}^2$ 

then

$$\widetilde{\Gamma}_{\mu} = -i \int \mathcal{G} \gamma_{\lambda} \overline{G} \widetilde{\Gamma}_{\mu} \overline{G} \gamma^{\lambda}, \qquad (26)$$

so the approximation to the kernel of (19) involves solving the non-linear equation (24). We note that to obtain the condition that (26) possesses a solution it is not necessary to solve a 4 dimensional integral equation. Thus, the condition that

$$\begin{split} \widetilde{\Gamma}_{\mu} \left( p + q/2, p - q/2 \right) &= -i f^2 \int \frac{(dp')}{(2\pi)^4} \frac{1}{(p - p')^2 + \mu^2} \gamma_{\lambda} \overline{G}(p' + q/2) \\ &\times \widetilde{\Gamma}_{\mu}(p' + q/2, p' - q/2) \overline{G}(p' - q/2) \gamma^{\lambda}, \end{split}$$

have a solution, can be expressed simply in terms of the limiting form of the equation as  $q^{\mu} \rightarrow 0$ . Hence, it comes out to be necessary and sufficient that, as  $q^{\mu} \rightarrow 0$ , an integral equation, which can be reduced to a one dimensional integral equation have a solution. The solution of this equation also allows  $\lambda$  to be computed.

We mention that if the solution of the non-linear integral equation for  $\overline{G}$  is such that  $Z_2 = 0$ , then the integral equation (25) reduces to a homogeneous form,

$$\overline{\Gamma}_{\mu} = -i \int \mathcal{G} \gamma_{\lambda} \overline{G} \overline{\Gamma}_{\mu} \overline{G} \gamma^{\lambda}$$

hence in this case the homogeneous equation will have a solution for all values of  $q^2$ . Thus, in this case it is necessary to require that such a homogeneous equation develop a pole at  $q^2 = 0$ , in order that a corresponding pole in P at  $q^2 = 0$  can exist. This leads to a new set of integral equations which also can be reduced to one-dimensional equations. I shall not further discuss these complications here. I will conclude by saying that the solution of (24) will be computed by reducing (24) to a one-dimensional non-linear integral equation for the spectral weight. Thus, we write

$$\overline{G} = \frac{1}{\gamma p + m} + \int dk \frac{A(k)}{\gamma p + k}$$

where k runs from  $-\infty$  to  $-(m + \mu)$  and from  $(m + \mu)$  to  $\infty$ . If this representation

is inserted into (24) we obtain a non-linear one dimensional integral equation for A(k). If the numerical solution of this equation has the feature that k A(k)  $\rightarrow 0$  as  $k \rightarrow \infty$ , we shall take that to mean  $Z_2$  is finite. If k A(k)  $\rightarrow$  const. as  $k \rightarrow \infty$ , we shall take it to mean  $Z_2 = 0$  in this approximation. We shall then investigate the two possible sets of linear integral equations in these separate instances.

# BOOK II

•

# SYMMETRY PROPERTIES OF ELEMENTARY PARTICLES

## PART I

# EXPERIMENTAL REVIEW OF KN AND KN INTERACTIONS

## KN AND KN INTERACTIONS\*

## R. CAPPS NORTHWESTERN UNIVERSITY, EVANSTON, ILL., UNITED STATES OF AMERICA

#### INTRODUCTION

The general topic under discussion is the strange particles and some of the resonances and interactions of strange particles that are particularly interesting. As is well-known, experimental developments are coming very fast in this field nowadays so there is some advantage in being located near an experimental centre, such as CERN, or near one of the United States experimental centres. However, those from more isolated places who are in this field should not be too discouraged. For one thing, having either been to the CERN conference or talked to many people who have, they are certainly not behind on experimental developments now. Also it is true that several of the very significant theoretical developments in this field have been suggested by experiments over a year old, so that it is not really necessary to be "on top of the new experimental data".

#### TABLE I

| В | s  | I          | Name                    | м      | Г      | J   |
|---|----|------------|-------------------------|--------|--------|-----|
| 0 | 1  | 1 <u>2</u> | к*                      | . 888  | ~50    | 3   |
| 1 | -1 | 0          | <b>Y</b> 0 <sup>*</sup> | 1405   | 50 ?   | ?   |
| 1 | -1 | 0          | Y0**                    | 1520   | 16     | 3/2 |
| 1 | -1 | 0          | ¥0 <sup>***</sup>       | 1815   | -      | ?   |
| 1 | -1 | 1          | Y1*                     | 1385   | ~50    | 3/2 |
| 1 | -2 | ł          | Ξ*                      | ~ 1535 | (7~30) | . 3 |

#### ESTABLISHED RESONANCE WITH S $\neq 0$

Table I is a list I made of the well-established resonances of strangeness unequal to zero, and this paper will include all these resonances and will be divided into four parts:

<sup>\*</sup>Text based on notes by E. Ferreira and G. Wolters.

- (1) The spin and parities of the strange particles;
- (2) The P-wave meson-baryon resonances and their significance;
- (3) The S-wave meson-baryon interactions and their possible significance;
- (4) The 1815 MeV resonance.

The fourth part is very short and is necessary because this resonance does not fit into what will be said in the first three points.

### 1. SPINS AND PARITIES OF THE STRANGE PARTICLES

The parity of the cascade particle is not yet known, but quite a bit happened last year about the parities of the  $\Lambda$  and  $\Sigma$  hyperons.

The usual convention a year ago (and even now) was to assign arbitrarily positive parity to the  $\Lambda$  particle, as to the nucleon, and then have the parities of K and  $\Sigma$  taken from the experiments. This was a natural convention when it was not known whether the neutral and charged K mesons were members of an isotopic spin doublet, having the same parities. It is now reasonably clear that isotopic spin is a good quantum number, that the K is a doublet, the  $\Sigma$  a triplet and so on. So, we adopt here the convention of calling K pseudoscalar and then determining the parities of the  $\Lambda$  and  $\Sigma$  particles from the experiments.

### 1.1. The $\Lambda$ parity

Something new happened last year on the experimental side concerning the  $\Lambda$  parity, although the arguments involved are a couple of years old. When K<sup>-</sup> is absorbed in helium, it can produce, among other things, the  $_{\Lambda}$ He<sup>4</sup> and  $_{\Lambda}$ H<sup>4</sup> hyperfragments:

$$K^{-} + He^{4} \rightarrow He^{4} + \pi^{-}$$
(1)

$$\rightarrow H^4 + \pi^0$$
 (2)

Since the spins of the K<sup>-</sup>, He<sup>4</sup> and  $\pi$  are zero, if the spin J of the hyperfragment is also zero (we assume from charge symmetry that the spins of  $_{\Lambda}H^4$  and  $_{\Lambda}He^4$  are the same), the conservation of angular momentum implies conservation of orbital angular momentum. Hence if J=0 and parity is conserved, the very existence of the interaction implies that the  $\Lambda$  parity is even. This result is independent of the angular momentum  $\ell$  of the state from which the capture takes place.

It is not known whether the ground state or an excited state of the hyperfragments is produced in reactions (1) or (2). If an excited state is produced,  $\gamma$  rays may be emitted before the decay of the hyperfragment takes place; however, the experimentalists have not yet looked carefully to see them. We can discuss our doubt about the spin of the states of the hyperfragments produced in the K<sup>-</sup> capture by considering two possibilities, either of which would invalidate the argument for even  $\Lambda$  parity:

(a) The ground state is produced directly, and its spin is  $J \neq 0$ ;

(b) The ground state may have spin J = 0, but what is actually produced is an excited state with  $J \neq 0$ .

Most of the things that happened during the last year concern the spin of the ground state of the  $_{\Lambda}\mathrm{He}^{4}$  and  $_{\Lambda}\mathrm{H}^{4}$ , and they show that most likely it is J = 0.

DALITZ and LIU [1], assuming that the mechanism of the pionic decay of  ${}_{\Lambda}$ H<sup>4</sup> is the same as that of the free  $\Lambda$ , computed the ratio:

$$R = \left[ {}_{A}H^{4} \rightarrow \pi^{-} + He^{4} \right] / \left[ {}_{A}H^{4} \rightarrow all \pi^{-} modes \right]$$

(an example of one of the other modes is  ${}_{\Lambda}H^4 \rightarrow \pi^- + p + H^3$ ). Calling J the spin of  ${}_{\Lambda}H^4$ , they found that

if J = 0,  $R = 1.41 |S|^2 [1.84 |S|^2 + 0.35 |P|^2]^{-1}$ and

if J = 1,  $R = 0.76 |P|^2 [0.43 |S|^2 + 1.12 |P|^2]^{-1}$ , where S and P are the magnitudes of the S and P-wave amplitudes for the  $\Lambda \rightarrow \pi^- + p$  decay. The point then is to measure experimentally the ratio R and the ratio P/S and see which of these formulae fits better.

The ratio R has been measured by AMMAR et al.[2] in nuclear emulsions. They found R = 0.66 ± 0.06, a rather high value. Using the formulae above this implies that for J = 0 one should have  $|P/S| \leq 1.5$ , and for J = 1 there should be a large amount of P-waves, with  $|P/S| \gtrsim 1.2$ . A measurement of the polarization of the protons in the decay  $\Lambda \rightarrow p + \pi^-$  may give information on the ratio P/S. The parameters that are usually referred to are

$$\alpha = \frac{2 \operatorname{Re} (S^* p)}{|S|^2 + |P|^2} , \quad \beta = \frac{2 \operatorname{Im} (S^* p)}{|S|^2 + |P|^2} , \quad \gamma = \frac{|S|^2 - |P|^2}{|S|^2 + |P|^2}$$

By measuring the polarization of the emitted protons, BEALL <u>et al.</u>[3] have recently obtained

 $\alpha = -0.67 \pm 0.2$  ,  $\gamma = +0.74 \begin{array}{c} +0.13 \\ -0.32 \end{array}$ 

By combining their results with those of Ammar et al. and taking as basis the calculations of Dalitz and Liu, they found that the assignment of J = 0 to the ground state of  ${}_{\Lambda}H^4$  is strongly favoured.

Thus we can imagine that what happens is that the  $\Lambda$  decay goes essentially through S-waves and there is no need of spin flip in the  $\Lambda \rightarrow p + \pi^-$  decay that occurs inside the hyperfragment, since both the initial  ${}_{\Lambda}H^4$  and the residual H<sup>4</sup> have spin zero. The  $\pi^-$  just goes off in an S-wave. This simple mechanism is perhaps what makes the  ${}_{\Lambda}H^4 \rightarrow He + \pi^-$  so predominant among the other modes of decay into  $\pi^-$ .

There is still another experiment that has been done concerning the determination of the spin of the  $_{\Lambda}H^4$ . BLOCK <u>et al.</u> [4] made the absorption of K<sup>\*</sup> at rest in helium and then looked at the angular distribution of

the products of the two-body decay modes of  ${}_{\Lambda}H^4$  with respect to the direction of the  $\pi^0$  first produced (produced in the capture process):

Let us consider that the capture of K<sup>-</sup> occurs from an S-wave (this will be discussed later). The spins and the total angular momentum in the lefthand side of the above reaction are all zero, so that if  ${}_{\Lambda}$ H<sup>4</sup> had spin 1, the  $\pi^0$  and  ${}_{\Lambda}$ H<sup>4</sup> would have to be produced in a state of relative orbital angular momentum  $\ell = 1$ . If we quantize along the direction of the originally produced  $\pi^0$ , the z value of the  ${}_{\Lambda}$ H<sup>4</sup> spin must be zero. When the  ${}_{\Lambda}$ H<sup>4</sup> decays, the  $\pi^-(\text{or }\pi^0)$  and He<sup>4</sup> (or H<sup>4</sup>) then produced, both having spin zero, would be in a state of relative orbital angular momentum 1, with component zero along the direction of the first  $\pi^0$ . The angular correlation between the  $\pi^0$ produced in the capture and the  $\pi^-(\text{or }\pi^0)$  emitted in the decay would then be of the form  $\cos^2 \theta$ . Actually with about 50 events it seems that there is isotropy in the angular correlation. The statistical data are not overwhelming but do give some support to the assignment of spin zero to the ground state of  ${}_{\Lambda}$ H<sup>4</sup>.

Let us now look at possibility (b) where the capture process may produce an excited state with quantum numbers that are not known and that cannot be studied by looking at the decays of the ground state. No theorist can tell whether excited states of these hyperfragments exist or not. If such an excited state exists, it is probably not very weakly bound since the hyperfragment formation probablilty here is above that expected from a binding energy of about 2 MeV (which is the binding energy in the ground state). Experimentalists will have to look for v rays carefully to try to plug this loophole in the  $\Lambda$  parity argument; it seems probable, however, that the  $\Lambda$  parity is even.

Something should be said here about the orbital angular momentum state in the K<sup>-</sup> capture, because this is important for several arguments to be made later. This will be based essentially on the theory of DAY et al. [5] which was produced three years ago and which was one of the major theoretical contributions to strange particle physics that particular year, even though it had little to do with the strong interactions. Only a very simpleminded explanation of the argument will be given here. This is one of those things that is very complicated in detail but very simple in effect. The question concerns, what happens to a K<sup>-</sup>meson caught in a high coulomb orbit: what is the angular momentum of the state from which it is captured? Let us suppose the K<sup>-</sup> meson is in an S-state orbit of some principal quantum number. We can ask the question of how long it will live before being captured. Knowing the probability for the K meson in such a coulomb orbit being found at the origin and also knowing the strength of the S-wave capture interaction from doing experiments on the capture of K<sup>-</sup> in flight by nucleons, we can have an idea of the K<sup>-</sup> lifetime in any S-wave orbit. We can also estimate the lifetime of the meson in a P-wave orbit. This is a rather rough estimate because, in the available data for K<sup>-</sup> nucleon capture cross-section in flight,

the P-wave contribution seems to come in very slowly, but an upper limit for the lifetime can be obtained. By doing this, we find that for a given principal quantum number the S-state lifetime is much shorter than the P-state lifetime. This is a result of the fact that the range of the strong interaction force responsible for the capture is much shorter than the radius of the coulomb orbit. In a P-state the probability of the K<sup>-</sup> being within the force range is very small compared with that in the S-state. The ratio of the ranges essentially gets cubed in the expression for the rates of the processes, so that there is in fact a  $10^5$  or  $10^6$  difference, the S-wave being much more powerful in capturing the K<sup>-</sup> meson than the P-wave. This might make one think immediately that the K<sup>-</sup> is always captured in the S-wave, and that in fact would be the case, unless for some reason the P-states have a tremendous head start in the race to capture the K. What worried the physicists for quite a while was that the P-wave might have that head start, because it was believed that the mechanism necessary for the K<sup>-</sup> to change from one coulomb orbit to another was simply radiative transitions. A particle reaches a P-state before an S-state by cascading down from a state of high l value. It can also be argued that the P-state reached this way is almost always the 2P-state, and unfortunately the lifetime for radiative transition in this 2P-state is of the same order of magnitude as the lifetime for capture. What Day, Snow and Sucher did was simply to show that in a liquid the K<sup>\*</sup> molecules going near the electric fields of the other nuclei would be subject to a strong Stark effect, which causes transitions between the several  $\ell$ -states. The mechanism of these transitions is quite complicated, but the essential point is that the transitions caused by this Stark effect occur in much shorter times than the lifetime for P-wave capture. Thus, even if a P-wave is reached first, it is most likely that there will soon be both P-and S-waves, and the natural power of capture from S-waves will assure that the capture will almost always be from an S-state. In detail this argument is certainly more valid for hydrogen (or deuterium) than for helium, but it is probably valid in all these cases.

### 1.2. The $\Sigma$ -parity

There are no stationary states of total spin zero in which the  $\Sigma$ -hyperon is bound. Thus the method used in order to determine the  $\Lambda$  parity cannot be applied here. The reaction

$$\mathbf{K}^{-} + \mathbf{p} \rightarrow \Sigma + \pi \tag{3}$$

is a simple as we can find to study the  $\Sigma$  [6]. The existence of the interaction does not indicate the  $\Sigma$ -parity because of the possibility of spin-flip.

It is well established that for K<sup>-</sup> momentum (lab) < 250 MeV/c the angular distribution for all three final charge states in (3), as well as for the elastic and charge exchange process, are essentially isotropic[7]. However, at 400 MeV/c a strong forward-backward peaking is observed. We now know that these are the result of a J = 3/2 resonance (called the  $Y_0^{33}$ )[8]. How-ever, we cannot distinguish among four possibilities from the angular distribution measurements in the resonance energy region. The resonance

could be in any of the amplitudes  $P_{3/2} \rightarrow P_{3/2}$ ,  $D_{3/2} \rightarrow D_{3/2}$ ,  $P_{3/2} \rightarrow D_{3/2}$  or  $D_{3/2 \rightarrow} P_{3/2}$ , where the first symbol represents the  $K^-$ -P state and the second symbol the  $\pi$ - $\Sigma$  state.

The ambiguity is reduced to two-fold by the following argument. The large isotropic cross-section below 250 MeV/c follows the 1/v law and therefore must result from an S-wave of the  $\overline{K}$  - N system. The absence of odd terms in  $\cos \theta$  in the angular distribution anywhere in or below the resonance region then indicates that the resonance has the same parity as this low energy amplitude and must result from a  $\overline{K}$  -N D-wave. The two possibilities for the two important amplitudes are

$$S_{1/2} \rightarrow S_{1/2}$$
  $S_{1/2} \rightarrow P_{1/2}$   
or  
and  $D_{3/2} \rightarrow D_{3/2}$  and  $D_{3/2} \rightarrow P_{3/2}$ .

The angular dependence of the polarization in the  $\Sigma^+ + \pi^-$  events, measured later, supports this assumption of two strong amplitudes of the same parity.

The remaining ambiguity is a generalization of the Minami-ambiguity for  $\pi N$  scattering. In this particular case it says that if the angular distribution and polarization data can be described in terms of the transitions,

$$S_{1/2} \rightarrow S_{1/2}$$

$$D_{3/2} \rightarrow D_{3/2} ,$$
(4)

1

then an equivalent description can be obtained by replacing these amplitudes by the amplitudes,

$$\begin{array}{ll} P_{1/2}^{*} \rightarrow & P_{1/2}^{*} \\ P_{3/2}^{*} \rightarrow & P_{3/2}^{*} \end{array} \tag{5}$$

where the  $\Sigma$ -parity has now beenchanged. The asterisk indicates complex conjugate amplitudes. This ambiguity must be resolved if the  $\Sigma$ -parity is to be determined. A distinction between (4) and (5) is possible because the Wigner theorem [9] applied to the phase shift of a resonant state with narrow width has the form:

$$d\eta/dt > 0.$$
 (6)

The CM energy is called t.

If one also makes the reasonable assumption that the phase of the large non-resonant amplitude is changing less rapidly than that of the resonant amplitude, then the sign of the change in the relative phase is predicted, and this can be used to eliminate either possibility (4) or (5). The radius of interaction must not be too large for (6) to be valid. As a consequence of the narrow width of  $\sim 16$  MeV the upper bound is as much as 15 fermi.

One can prove (6) by considering the amplitude for (3) as an analytic function of t in the upper half plane of the complex energy plane. The pole, corresponding to the resonance, lies (in the unphysical sheet) just below the branch line in the t-plane which is along the real axis. (Causality forbids a pole above the branch line.) It is easy to see that a pole just below the real axis leads to a positive energy derivative of the phase, provided this pole is the dominant singularity.

If one wants to apply the Wigner theorem for the two possible cases (4) and (5), it is necessary to know something about the interference between the J = 1/2 and the J = 3/2 transitions in each case. This can be done by considering the polar-equatorial ratio:

$$\rho = (p - E)/(p + E)$$
.

p and E stand for number of events, for which  $|\cos \theta| > \frac{1}{2}$ , resp.  $< \frac{1}{2}$ . Here  $\theta$  is the polar angle in the CM system.

The measured values of  $\rho$  are as follows:

| Energy<br>(MeV) | 370  | 390  | 410  |
|-----------------|------|------|------|
| ρ               | 0.36 | 0.50 | 0.36 |

for the  $\Sigma^+ \pi^-$  events. This is very large. In fact even for a pure Y = 3/2 transition,  $\rho$  is only 0.375; and when one considers that the resonance bump here is smaller than the non-resonant background, one would expect a  $\rho$  of only about 0.15, if there is no interference. This suggests that the interference is very important in the resonance region. The interference term in the angular distribution is given by

$$(6 \cos^2 \theta - 2) f_{1/2} f_{3/2} \cos \eta . \tag{7}$$

The magnitudes of the 1/2 and 3/2 amplitudes are represented by  $f_{1/2}$  and  $f_{3/2}$ , whereas their relative phase equals  $\eta$ . From the large measured value of  $\rho$  one can conclude that

$$\cos \eta > 0 \tag{8}$$

in the resonance region.

Finally, the  $\Sigma$ -polarization will also contain a term analogous to (7). This term is proportional to

$$\mp f_{1/2} f_{3/2} \sin \eta$$
. (9)

This differs from (7) in that the angular dependence is left out here. It is not necessary to the argument, although it has to be taken into account in the experiment [10]. Moreover, a  $\pm$  sign stands in front, which corresponds to the cases (4) and (5) respectively.

Assuming that the magnitudes f do not change rapidly with t near the resonance peak, the energy derivative of (7) is

$$\mp f_{1/2} f_{3/2} \cos \eta \, (d\eta/dt). \tag{10}$$

It can be seen by using (6), (8) and (10) that increase of (9) corresponds to odd  $\Sigma$ -parity, and decrease corresponds to even  $\Sigma$ -parity. One expects the polarization to change rapidly with t going through the resonance energy and this will enable one to distinguish experimentally whether (9) increases or decreases.

The experimental result is in agreement with even  $\Sigma$ -parity only [10]. Of course, the evidence obtained has still to be confirmed by independent determination of the  $\Sigma$ -parity. The experiment on the  $\Sigma^0$  decay into Dalitz pair,

$$\Sigma^0 \rightarrow \Lambda^0 + e^+ + e^-,$$

seems to be the most promising attempt.

1.3.Ξ -spin

Several years ago the spins of the baryons N,  $\Lambda$  and  $\Sigma$  were all determined as 1/2; but only limited evidence exists for the spin of  $\Xi$  excluding spins  $\geq 5/2$ . SAMIOS <u>et al.[11]</u> and TICHO <u>et al.[12]</u> have studied the chain of reactions:

$$K^{-} + p \rightarrow \Xi^{-} + K + \pi, \qquad (11)$$

$$K^{-} + p \rightarrow \Xi^{0} + K + \pi, \qquad (11a)$$

$$\Xi \rightarrow \Lambda + \pi$$
, (12)

$$\Lambda \rightarrow p + \pi^{-}.$$
 (13)

The cascade particle will be polarized perpendicularly to the production plane. One can measure the up-down asymmetry in the decay process (13). This asymmetry will depend on the product of the parity-mixing parameters  $\alpha_{\rm E} \cdot \alpha_{\rm A}$ . The results obtained are

$$\alpha_{\Xi} \cdot \alpha_{\Lambda} = \begin{cases} -0.63 \pm 0.20, & \text{SAMIOS et al.[11]} \\ -0.30 \pm 0.8, & \text{TICHO et al.[12]} \end{cases}$$

For  $\Lambda$  separately one has [3]

$$\alpha_{\Lambda} = 0.62 \pm 0.07$$
.

This indicates that most probably the value of  $\alpha_{\Xi}$  lies in between ~ -1 and ~-0.4.

The test by LEE and YANG[13] can serve to eliminate values of the spin  $J \ge 3/2$  only if the asymmetry  $|\alpha \bar{p}|$ ,  $\bar{p}$  = average polarization, is > 1/3.

As  $\alpha \bar{p}$  in the case of optimal polarization ( $\bar{p}=1$ ) still may have a value ~ 1/3, it is not possible to exclude J = 3/2 on the basis of the present data. However, J = 1/2 seems to be more likely than J = 3/2.

### 2. THE P-WAVE INTERACTION

Let us now discuss the strong P-state interactions. I shall try to be objective about the experimental data, but the grouping together of certain resonances under the title of P-wave resonance is rather subjective. That is, not everything discussed here is necessarily a P-wave resonance, but it is hoped that there will be no difficulty in distinguishing the subjective statements from the others.

The first P-wave resonant interaction known was of course the 3-3 pion nucleon resonance. Global symmetry predicts two spin 3/2 pion-hyperon resonances, one with isotopic spin I=1, the other with I=2.

A pion-lambda resonance, which we are now calling  $Y_1^*$ , was discovered two years ago by ALSTON, GOOD, ALVAREZ et al.[14] and reported at the 1960 Rochester Conference [15]. Even though a lot of experimental – work has already been done in studying this resonance, its parity and spin are not yet known. This is an example of all the pain and struggle that are sometimes necessary to determine whether a little number is 1/2 or 3/2. Alston et al. looked at the reaction

$$K^{-} + p \rightarrow \Lambda + \pi^{+} + \pi^{-}$$
(14)

produced by a beam of  $1.15 \text{ GeV/c K}^-$  in a hydrogen bubble chamber and studied the energy distributions of the two pions in the K<sup>-</sup>p centre-of-mass system. They found sharp peaks in these distributions and tried to interpret them in terms of the possible mechanisms of the reaction (14). They found that these peaks are those expected if first a two-body system is formed:

$$K^{-} + p \rightarrow Y_{1}^{*\pm} + \pi^{\mp}, \qquad (14a)$$

where  $Y_1^*$  has a quasi-well defined mass of about 1385 MeV and decays into a lambda and a pion:

$$Y_1^{*\pm} \rightarrow \Lambda + \pi^{\pm} + 130 \text{ MeV}$$
 (14b)

The isotopic spin of the  $Y_1^{*\pm}$  state is, of course, 1, since it decomposes into a  $\Lambda$  and a  $\pi$ . Then the question of determining the spin of this particle arose. The first problem was whether J = 1/2 or  $J \ge 3/2$ . Several kinds of aniso-

tropies and angular correlations between the direction of production of the  $Y_1^*$  and the direction of emission of its decay could possibly be observed for  $J \ge 3/2$ , thus making a distinction between J = 1/2 and  $J \ge 1/2$  possible. Thus, for spin J = 3/2 it is expected that the  $\Lambda$ 's will have an angular distribution of the form  $A + B \cos^2 \eta$ , where  $\eta$  is the angle between the direction of emission of the  $\Lambda$  and the normal to the plane in which  $Y_1^*$  and  $\pi$  are produced. Thus, the polar-to-equatorial ratio about the normal to the production plane may be different from 1 for  $J \ge 3/2$  but must be equal to 1 for J = 1/2.

At that time, with limited statistics, Alston <u>et al.</u> thought they had seen a definite polar-to-equatorial ratio different from 1, thus giving an indication of spin 3/2 to the  $Y_1^*$ . But shortly after that, a lot of experiments were made applying several kinds of analysis of angular correlations, for example the famous ADAIR analysis [16], and they seem to have rather favoured spin 1/2. BLOCK [17] produced  $Y_1^*$  in He and also seemed to get arguments for spin 1/2. But then a few people began to point out that a lot of these experiments were not very significant. Particularly DALITZ and MILLER [18] showed that, because they neglected the effects of the symmetrization of the two pions to be introduced to account for Bose statistics, most of these experiments did not say anything about the spin of the  $Y_1^*$ . More recently in the last CERN Conference, Block presented new data, but still nothing-conclusive could be extracted from them [4].

There is one experiment, however, which is fairly significant, though not conclusive. That is the experiment by ELY <u>et al.[19]</u>, with 1.11 GeV/c K<sup>-</sup> mesons in a propane bubble chamber. They looked at the distribution of the  $\Lambda$ 's with respect to the normal to the production plane and found that the best fit for the law

$$1 + a \cos^2(\vec{\Lambda}, \vec{K} \times \vec{Y}_1^*)$$

is obtained with

 $a = 1.5 \pm 0.4$ .

This result favours  $J \ge 3/2$  but is not conclusive.

So it is still a matter of opinion what the  $Y_1^*$  spin is, but there is some evidence in favour of J = 3/2. One of the reasons why people tend to believe in this is that things are fitting together. For example, this fits our arguments that the  $\Lambda$  - and  $\Sigma$  -parities are probably positive, since in this case global symmetry predicts such a J = 3/2 resonance. Also, there is no longer any reason to expect that the resonance might be a J = 1/2 resonance of the Dalitz-Tuan type, since the most recent analysis of low energy  $\overline{K}$ -N data does not yield a solution consistent with such a resonance in the I=1 state. This S-wave analysis will be discussed later.

The next resonance we shall discuss is one which has not been discovered, and thus it is not known if it exists or not. This is the isotopic spin 2,  $P_{3/2}$ ,  $\pi \Sigma$  resonance. This resonance is important because it is predicted by global symmetry.

There are some hints of the existence of this resonance which were reported about a year ago at the Aix-en-Provence Conference, but nothing has happened since then and it seems that nobody has really seen it. The Alston group at Berkeley, who saw the  $Y_1^*$  resonance, has a large number of events of the type:

$$K^{-} + p \rightarrow \Sigma^{+} + \pi^{+} + \pi^{-} + \pi^{-}$$

and the same with all the charges in the final state reversed; the  $I = 2, \pi \Sigma$  resonance could show up in the analysis of these events. Therefore it begins to appear that the thing may not exist, although we cannot be sure since we do not know what would be the cross-section for producing it in this particular process.

Now let us discuss one more resonance, the  $\Xi^*$ , I = 1/2 resonance. It is very subjective to group this together with the P 3/2 resonances, since its spin has not been measured.

This new resonance has been discovered by both TICHO et al.[12] and the Syracuse-Brookhaven collaboration group and reported at the last conference in CERN. The two groups found about the same mass of 1535 MeV (80 MeV above the  $\pi \Xi$  threshold) but very different widths: Ticho et al. found 7 MeV and the other found 30 MeV. The surprising thing is that experiments seem to have indicated isotopic spin 1/2 for this resonance. Global symmetry believers again expected that there should appear an I = 3/2,  $\pi \Xi$  resonance analogous to the  $\pi$  N one, because  $\Xi$  and N are both isotopic spin doublets.

Now, let us examine some numbers which were given by Samios at the CERN Conference. (The only reason for giving these numbers rather than those of Ticho et al. is that Samios talked first and the author was wide enough awake to write down his numbers.) They absorbed a beam of 2 GeV/c K<sup>\*</sup> mesons in a hydrogen bubble chamber, producing the reaction:

$$K^{-} + p \rightarrow \Xi + \pi + K.$$

They found by kinematic analysis of the final products that there should be an intermediate two-body system,

$$K^{+} p \rightarrow \Xi^{+} + K$$

with immediate decay of the  $\Xi^*$  into  $\Xi$  and  $\pi$ . They looked at the charge state  $\Xi^{*0} + K^0$ , measured the ratio,

$$R_1 = (\Xi^{0*} \rightarrow \Xi^- + \pi^+) / (\Xi^{0*} \rightarrow \Xi^0 + \pi^0),$$

and found 5/0. Then they looked at events producing the charge state  $\Xi^{*-} + K^{+}$  and by observing the final products, they measured

$$\mathbf{R}_2 = (\Xi^{*-} \rightarrow \Xi^0 + \pi^-)/(\Xi^{*-} \rightarrow \Xi^- + \pi^0) = 3/2.$$

By just writing Clebsch-Gordan coefficients, we find that, if the  $\Xi^*$  isotopic spin is I = 1/2, we obtain  $R_1 = R_2 = 2$ , and if I = 3/2, we obtain  $R_1 = R_2 = 1/2$ . The experimental results given above seem to indicate more the value 2 than the value 1/2. So we shall accept I = 1/2.

In the processes studied here there are only four possible charge states, two in the K<sup>-</sup>p system and two in the  $\Xi^*$  K system, so that we can form only three independent ratios, the  $\Xi^*/\Xi^{*0}$  ratio being independent of the two above. Now, if  $\Xi^*$  is an I = 3/2 object, then with the K produced it can form either isospin 2 or 1. If  $\Xi^*$  has I = 1/2, then the final state may have isotopic spin 1 or 0. But in the initial state we have 1 or 0, so that in the case of  $\Xi^*$  being isospin 3/2 the only possibility is that of total isotopic spin 1, and, since one has only one amplitude, the  $\Xi^*/\Xi^{*0}$  ratio can be predicted under the conditions and turns out to be 1. The experimental value for this ratio is about 1. Of course no definite prediction can be made for the case of isotopic spin 1/2 assignment to  $\Xi^*$ , since then two amplitudes are involved. Thus this third ratio does not say anything.

Essentially, this is the evidence, which is meagre but supported by two independent groups, the evidence of Ticho et al. being similar to the above except they have more events. Very little can be said about the spin. Samios has reported that by measuring the polar-to-equatorial ratio with respect to the normal to the production plane of the final  $\Xi$ 's, they found P/E = 15/5, which is still meagre evidence in favour of spin 3/2.

We group this, resonance together with the P 3/2 resonances only because experiments slightly suggest it and because nobody has a theory which predicts an S-wave  $\pi \Xi$  resonance. There might be something like a Dalitz-Tuan type resonance, but in this case the resonance energy is quite a way below the K<sub>1</sub> threshold, so that it looks as if this cannot be so. It has also been suggested that this might be the second  $\pi \Xi$  resonance and not the first one. But it seems probable that, if this is the second resonance, the first one should have been seen in the same experiment. So we shall group the  $\Xi$  \* with the P 3/2 resonances just because this is how some people have expected it to be. This resonance seems to fit into a multiplet which is predicted by the ten-fold representation of the unitary symmetry.

Let us now very briefly examine the significance of this and in order to understand these things it is very important to know about both dispersion relations and group theory. Even if one prefers the former, something should be known about the latter. If one predicts something by group theory, a knowledge of dispersion relations - though maybe not much more than Chew-Low equations- is essential to check the prediction by experiments, because means of this some relations between widths of resonances and coupling constants, and so forth can be obtained. One can see from unitary symmetry that the  $\pi$  is analogous to the K's and the  $\eta$ , but they have quite different masses and this makes quite a difference; and these differences can best be seen when poles and dispersion relations are written down. Also if the reason for these mass differences is not known, the coupling constants, for the  $\eta$  let us say, cannot really be known by comparison with the coupling constants for the  $\pi$  and K. This must be seen from the data.

On the other hand, if one has to operate with dispersion relations, there are certainly advantages in knowing in what symmetries to believe. For instance, consider the problem of the  $\pi N$  resonances as it was first discussed by Chew and Low. They could write down a dispersion equation and solve it essentially by the N/D method. They could not predict the position of the resonance, since it depended upon the radius of arbitrary cut-off they put in. But knowing the position, they could predict the width in terms of the coupling constant. This method was improved by Frautschi and Walecka, who made it relativistic and at the same time put in some other forces. By making it relativistic, they did not have to use an arbitrary cut-off, but one can see that the convergence obtained came about at energies in the integral of the order of the nucleon mass, so that essentially in an attempt to predict the position of the resonance, forces coming in at higher energies, i.e. interms of configuration space forces of short range, are important and nobody knows what the short range forces are. This is very physical of course. It is well known that both long-range and short-range forces are important for determing whether particles are bound or where a resonance is. But something that has to do more with the details of the shape, like the width, may depend more particularly on the long-range forces, i.e. on the close singularities in the energy plane. So one cannot really predict the resonance position. On the other hand, people who write down formulae in group theory write down these magic mass formulae and say at what masses they expect resonances to exist, so it is worthwhile asking if there is any sense in these formulae, which there should seem to be. If only one resonance is being discussed, merely guessing about the high-energy region is a little wild, although it may be worthwhile. But if one has two resonances which belong to the same symmetry multiplet, it might be a little more reasonable to assume that the high-energy contributions might be the same for both. One might complain about this and point out that the long-range forces are very different for different members of the same symmetry multiplet, because  $\pi$  and  $\eta$  and so forth have different masses, but it could be that the shortrange forces being made of many different contributions might be the same, or nearly the same. In trying to predict the position of the things, it might be that much can be learnt by comparing the different resonances which are at the same symmetry multiplet.

Here are a few speculative remarks about the forces that might be important in predicting the P-wave resonances. People like to believe that in the  $\pi N$  resonance the  $\pi$  poles which are close act as the main forces that produce the resonance. This is a hope, made because things are simpler if it is true than if it is not true.

Let us suppose then that it is true that in this whole family of resonances the poles which are caused by the interchange of the pseudoscalar mesons  $(\pi, K, \eta)$  are the main forces which cause the resonances. What can we learn from this? Which resonances exist, and which do not exist? We shall be concerned with the J = 3/2 resonances only because they seem to be the most important ones. Let us write the simplest diagrams, as in Fig. 1, making the static approximation. If we do not worry about isotopic spin factors and the signs they introduce, a diagram like that in Fig. 1a, with intermediate state of



less energy than the initial state, gives rise to a repulsive force; and a diagram as in Fig. 1b, with intermediate state with higher energy, contributes an attractive force. Both these diagrams contribute to a J = 1/2 amplitude. But for a J = 3/2 amplitude only the attractive graph exists. When we include the isospin factors, Fig. 1(b) may not always be attractive; but it is still true that the J = 3/2 is the most attractive in general.

Now let us look at some states that can be produced by some pairs of particles. Let us first consider those with hypercharge 2, which can be produced by a KN system. We know that in the K<sup>+</sup>p system, which is a pure I = 1 state, no P-wave resonance occurs. Let us see what the pole terms would be, that is, what we would expect to come from diagrams of the above type. For the I = 1 state, neglecting mass factors and the  $\Lambda\Sigma$  mass difference (because we do not know how big the coupling constants are anyway), the residue is proportional to

 $\frac{2}{3}$  (G<sup>2</sup><sub>K\SigmaN</sub> + G<sup>2</sup><sub>KAN</sub>).

This gives an attractive force, but we know from experiment that it is not strong enough to produce a resonance.

Now let us look at the Y = -2 states, which can be created by the KE system. A resonance is predicted for I = 0 by unitary symmetry. In this state the residue of the pole is proportional to

$$\frac{2}{3}$$
 (3 G<sup>2</sup><sub>K= $\Sigma$</sub>  ~ G<sup>2</sup><sub>K= $\Lambda$</sub>  ).

This is of particular interest because, if this pole term is strong enough to produce a resonance, this might be an evidence for a strong K meson interaction. Till now there has been no indication that the pseudoscalar K-meson baryon interactions are strong, and this would be the first evidence of it.

Now, if we look at the other states, we run into the difficulty that we

may have many pairs of particles that can produce them and so many channels open. For example, for hypercharge Y = -1, we have the following systems:  $\pi \Xi$ ,  $\overline{K}\Sigma$ ,  $\overline{K}\Lambda$ ,  $\eta \Xi$ . In these cases the correspondence between poles and graphs is not so simple. If we use some N/D many-channel dispersion relation, we get a denominator that starts out with 1, includes terms that are quadratic in the coupling constants and have an energy dependence of the form  $(W-W_0)$ , and then follow terms which are of the fourth order in the coupling constants, and so forth. If there is only one channel open, then there are only the terms that are quadratic in the coupling constants. Where the denominator is zero we say that we have a resonance. The quadratic terms are what we have when we look at the elastic scattering diagrams separately, not worrying about their being coupled. To get an idea of what could happen. let us assume that a resonance can occur only if one of the elastic scattering diagrams is strongly attractive. This does not mean that the resonance would have to show up only in that particular state, since the states are all coupled. In other words, what we want to assume is that we need to have a strongly attractive term in one of the elastic diagrams to obtain a resonance in one or more of the coupled states.

Now for the Y = -1 systems in the I = 1/2 (the state of the recently discovered  $\Xi^*$ ) state we have the following factors in the lowest order elastic diagrams:

 $\begin{aligned} \pi \Xi &: -(2/3) \ \mathrm{G}_{\Xi\Xi\pi}^2 \ , \\ \widetilde{\mathrm{K}}\Sigma &: -(2/3) \ \mathrm{G}_{\mathrm{N}\Sigma\mathrm{K}}^2 \ , \\ \widetilde{\mathrm{K}}\Lambda &: (2/3) \ \mathrm{G}_{\mathrm{N}\Lambda\mathrm{K}}^2 \ , \\ \eta \Xi &: (2/3) \ \mathrm{G}_{\mathrm{n}\Xi\Xi}^2 \ . \end{aligned}$ 

The first term, being strongly negative, is not able to produce a resonance in our model. The KEN and the KAN interactions were not strong enough to produce a resonance in the Y = +2 states, and so we may assume that they are not strong enough here either, although we have to admit that the poles are a little closer to the physical region here.

In the  $\eta \equiv \pm$  interaction we have the sign and perhaps the strength to produce a resonance. If this resonance occurs at an energy below  $\overline{K}\Lambda$ ,  $\overline{K}\Sigma$  and  $\eta \equiv$  thresholds, it will decay into the only open channel,  $\pi \equiv$ .

Now let us look at the Y = 0 states. There we have, for I = 2, only the  $\pi \Sigma$  channel, with a factor  $\frac{2}{3} (G_{\pi\Lambda\Sigma}^2 + G_{\pi\Sigma\Sigma}^2)$ . Since this is attractive, there arises the same question we had in the I = 1 state of the KN system: why is there no resonance? Perhaps, again, the coupling constants are not strong enough.

For I = 1, we have

 $\pi \Sigma : (2/3) (G_{\pi\Sigma\Sigma}^2 - G_{\pi\Lambda\Sigma}^2),$  $\pi \Lambda : (2/3) G_{\pi\Lambda\Sigma}^2 ,$ 

$$\overline{K}$$
N, KE : 0,  
 $\eta \Sigma$  : (2/3)  $G_{\eta \Sigma\Sigma}^2$ 

Here the  $\eta \Sigma$  interaction might well be partly responsible for the appearance of a resonance in one of the above states. Hence strong  $\eta$  interactions are one possible explanation for the fact that a  $Y_1^*$  exists, while a  $Y_2^*$  does not seem to exist.

Thus in the future, with more and more data coming in, we shall perhaps have to start being worried with the  $\eta$  and K interactions. Particularly, we shall perhaps have to see whether they are coupled to resonances that may be found in future experiments.

Let us look at one more argument. If the  $\pi \Lambda \Sigma$  coupling is strong enough to produce a resonance, what happens to the analysis of the  $\Lambda$ -nucleon forces? De Swart and Iddings have analysed the  $\Lambda N$  interaction in terms of a few simple diagrams, and from that they constructed a potential to describe the  $\Lambda N$  interaction. From a comparison of these results with a potential obtained from hyperfragment data they conclude that the strength of the  $\pi \Lambda \Sigma$  coupling is of the order of the  $\pi NN$  coupling:

$$f_{\pi\Lambda\Sigma}^2 \approx 0.08$$
.

A rough argument for remembering this result is the following: We know that the  $\Lambda$  in a hypernucleus is bound less deeply than a nucleon in a normal nucleus. But this does not mean a large difference in potentials because, in a three-body nucleus, a reduction of the depth of the potential by a factor of two causes the nucleons to fly apart, the binding energy being so small. And in fact the hyperfragment data indicate that the  $\Lambda$ -N potential is about 2/3 as strong as the N-N potential. On the other hand, the  $\Lambda$  does not have the one-pion-exchange diagram. If we take out the one-pion-exchange term in the nucleon force, the depth of the potential is reduced by about 1/3 (this really depends on the spin state; in the deuteron this number is about true). So here in the  $\Lambda$ -N case, as there is no one-pion-exchange and the potential depth is just about 2/3 of the nuclear potential, we may have the other things about equal, which implies  $f_{\pi\Lambda\Sigma}^2 \approx f_{\pi NN}^2$ . But if this is so, why does no resonance occur? Maybe it is because  $G_{\pi\Sigma\Sigma}$  is small, because the resonances are produced by other singularities, because this analysis is wrong or because  $\eta$  and K's are important in the  $\Lambda$  nucleon potential too.

### 3. THE S-WAVE INTERACTIONS

The strangeness +1 system will be first considered: the  $K^+$  p interaction in the pure T = 1 state. No P-wave resonances are present; in fact the elastic cross-section is isotropic to 640 MeV/c  $K^+$  momentum. Some new data have recently emerged. GOLDHABER, GOLDHABER <u>et al.</u>[20] have made an analysis for scattering length. From data, for momenta up to 355 MeV/c, they get  $a_1 = -0.29 \pm 0.2$  fermi.

The effective range could not be measured very well, and the result obtained was

 $r_1 = 0.6 \pm 0.6$  fermi.

The entire momentum range up to 640 MeV/c yields

$$a_1 = -0.29 \pm 0.2$$
 fermi

and

$$r_1 = 0.5 \pm 0.15$$
 fermi.

The phase shift  $\eta_1$  goes up to -36°. This is just what is expected from a repulsive core interaction.

The T = 0 interaction for the strangeness +1 system cannot be measured as simply as the T = 1 interaction. All that is known is that it is very much weaker than the T = 1 interaction, and probably one has for the scattering length

$$a_0 < 0.10$$
 fermi.

There has been a hint from optical model analysis that there might be some P-wave interaction here. Also the Ticho group has found some indication that the P-wave might be as important as the S-wave for T = 0, but the strength of the P-wave effect is not known. Anyway, in the T = 1 channel there certainly is not appreciable P-wave interaction.

Next we will discuss the strangeness -1 KN system. Large S-wave interaction was observed in meson-baryon systems for the first time in this system.

One can associate a resonance with the large S-wave  $\overline{K}N$  interaction, as first suggested by DALITZ and TUAN [21], and at the moment the only resonance one can think of in this connection is  $Y_0^*$  at 1405 MeV and width  $\sim 50$  MeV. However, till now there is really no strong evidence that spin  $Y_0^* = 1/2$ . ALEXANDER et al. [22] have observed that the resonance peak is cut off more abruptly at the high-energy side than at the low-energy side, and this must be expected for a Dalitz-Tuan type of resonance not far below threshold. But as this data is sparse, this effect cannot be considered as strong evidence. The main reason why the  $Y_0^*$ -spin is thought to be 1/2 is that this fits with other experimental evidence on the  $\overline{K}N$  system, as will be explained shortly. However, there is at least one argument in favour of the assignment  $P_{3/2}$  to the  $Y_0^*$ . That is, assuming that pion couplings are predominant in pion-hyperon interactions, one mechanism that might explain why the  $Y_1^*$  at 1305 MeV disintegrates almost completely in  $\Lambda + \pi$  (the ratio  $\Sigma \pi / \Lambda \pi$  is less than 3%) is that  $f_{\pi \Lambda \Sigma}^2 \gg f_{\pi \Sigma \Sigma}^2$ . The residue of the Chew-Low pole term for T = 0,  $\pi\Sigma$  scattering in the P 3/2 state, is essentially

 $\frac{2}{3}$  (f<sup>2</sup><sub> $\pi\Lambda\Sigma$ </sub> -2f<sup>2</sup><sub> $\pi\Sigma\Sigma$ </sub>).

#### R. CAPPS

Clearly, if  $f_{\pi\Lambda\Sigma}^2 \gg f_{\pi\Sigma\Sigma}^2$ , this would give attractive interaction in the  $P_{3/2}$  state, and a resonance  $Y_0^*$  is therefore expected in the  $P_{3/2}$  state.

Recently some progress in understanding the S-wave interaction has been made. But before describing it let me review a little. We consider the absorption processes:

$$K^{-} + p \rightarrow \Sigma^{+} + \pi^{-} , \qquad (15)$$

$$K^- + p \rightarrow \Sigma^0 + \pi^0 , \qquad (16)$$

$$K^{-} + p \rightarrow \Sigma^{-} + \pi^{+} , \qquad (17)$$

which can be described in terms of two amplitudes for isospin 0 and 1:  $T_0$  and  $T_1$  respectively.

At threshold, experimental data give  $|T_0| \gg |T_1|$  and the relative phase  $\phi_t = \phi_0 - \phi_1$  between these amplitudes can be determined in magnitude:

$$\phi_{r} = \pm \ 60^{\circ}, \tag{18}$$

At 175 MeV/c K<sup>-</sup> momentum (lab), one finds about equal cross-sections for (15) and (17). Thus at this momentum the interference term between  $T_0$  and  $T_1$  vanishes, and therefore

$$\phi_{\rm r} = \pm 90^{\circ}. \tag{19}$$

One expects that the positive phases at threshold and at 175 MeV/c belong together and similarly the negative phases, because the phase should not change too rapidly. In order to get information about  $\phi_r$  at an intermediary energy the  $\overline{K}^0n$  system is considered. The threshold is here 5.3 MeV higher. The  $\overline{K}^0n - K^-p$  mass difference causes a cusp, as can be deduced using the Dalitz-Tuan zero range approximation [22]. Because the T = 0 absorption is much bigger than the T = 1 absorption, one finds that the phase  $\phi_r$  must increase between the  $\overline{K}p$  and  $\overline{K}^0n$  thresholds. So one has at least the two sets of phases:

| $\widetilde{\mathbf{K}}$ p threshold | $\overline{K}^{0}$ n threshold | 175 MeV/c |
|--------------------------------------|--------------------------------|-----------|
| 60°                                  | ~ 80°                          | 90°       |
| - 60°                                | ∼ - 50°                        | -90°      |

Several years ago it was observed that K<sup>-</sup> capture on deuterium gives the value of  $\phi_r$  below threshold [21]. Assuming that the K<sup>-</sup> is caught from an S atomic state, in accordance with the Day, Snow, Sucher argument, it was deduced from equal  $\Sigma^+$  and  $\Sigma^-$  ratios in

 $K^- + d \rightarrow \Sigma^+ + \pi^- + n,$  $K^- + d \rightarrow \Sigma^- + \pi^+ + n$ 

that  $\phi_r = \pm 90^\circ$  at  $E \sim -10$  MeV (below K<sup>-</sup>p threshold). The  $\Sigma + \pi$  energy in the deuterium experiment is that much below the hydrogen experiment because of the deuterium binding energy and the energies of recoil of the neutron and the  $\Sigma + \pi$  pair. It was assumed in [23] that a strong dependence on the  $\Sigma + \pi$  energy was responsible for the difference in the H and D experiments.

Now, in the zero range approximation, one cannot have  $\phi_r$  first decreasing and then increasing around threshold. This therefore rules the positive set of phases out.

However, the fact that  $\phi_r$  changes so rapidly below the K p threshold suggests a resonance  $Y_0^*$ . In fact the  $Y_0^*$  was predicted in this way [23].

Additional information has been obtained recently [24]. The  $\Sigma^+$  and  $\Sigma^-$  ratios for the  $Y_0^{**}$  are such as to give  $\phi_r = \pm 110^\circ$ . The sign can be obtained by studying the interference between the S-waves and the resonating  $D_2^3$  wave amplitude. The conclusion is that  $\phi_r = -110^\circ$  there, strongly suggesting the negative sign at lower energies as well. It is notices that this also indirectly supports the point of view that  $Y_0^*$  is an S-wave resonance.

Last year Humphrey and Ross determined two solutions in a zero-range approximation for low energy  $\overline{K} N$  interaction. The solutions I and II correspond to the mentioned two possibilities of sign of  $\phi_r$ .

One has complex scattering length in this analysis because there is absorption into the  $\Sigma\pi$  channels, even at threshold A = a + ib. All the amplitudes have the energy-dependent factor 1/(1-iRA) above threshold.

Below threshold one has to replace k by +i|k|. If a < 0, then one might have a pole below threshold in the lower half plane which corresponds to a bound state of the  $\overline{K}N$  system. If there were no connection with the  $\Sigma\pi$  channel, b=0, which then gives a bound state pole.

Solution I seems to be ruled out, because for this solution  $a_0 \approx a_1 \approx 0$ . So again this confirms  $\phi_t < 0$ . Solution II reads as follows:

> $a_0 \approx -0.6 \text{ fermi}$ ,  $a_1 \approx 1.2 \text{ fermi}$ .

This is the most acceptable solution. However, the negative value of  $a_0$  is not too well determined. It depends on the assumption that the effective range is very small. It may be that  $a_0 \approx -1.2$  f, as predicted by Schult and Capps. A large negative  $a_0$  can give a resonance of the Dalitz-Tuan type, and this may be the  $Y_0^*$ .

The nature of the forces leading to S-wave resonances is not too well understood. There is one model which tends to predict the signs of things correctly. This model, based on the exchange of vector mesons  $\rho$  and  $\omega$ , has been discussed by Sakurai. If the model is correct, the fact that the  $\overline{KN}$  system is coupled to the  $\Sigma \pi$  system suggests that the graph in Fig. 2 is important where the intermediate line is a vector meson of strangeness 1 or -1. It would be hard to apply



the model if one only had zero strangeness vector mesons. Some years ago the Alston, Ticho group discovered the  $\pi$ -K resonance K<sup>\*</sup>. However, its spin is not, yet known, so we shall now discuss the evidence for the spin of K<sup>\*</sup>.

Two kinds of measurements have been done on the  $K^*$  spin, and the results were presented at the CERN Conference, 1962. The Ticho group studied the processes

$$K^{*} + p \rightarrow K^{*} + p$$
$$K^{*} \rightarrow K + \pi$$

by looking for any sort of asymmetry in the  $K^*$  decay. If spin  $K^{*=} 0$ , no asymmetry can occur, but asymmetry can exist for spin 1. No asymmetry was found.

The second measurement is the experiment on  $\bar{p}p$  annihilation by Armenteros et al. at CERN. The analysis requires knowledge of the orbital state from which  $\bar{p}$  is caught. If the Day, Snow and Sucher argument applied, this would be an S-state. There is, however, a better argument in favour of an S-state. Consider the process  $\bar{p} + p \rightarrow K^0 + \bar{K}^0$ . The  $K^0$  and  $\bar{K}^0$  being mixtures of the eigenstates  $K_1^0$  and  $K_2^0$  of Cp, one expects to see the decay modes of the following combinations:

 $K_1^0 \quad K_1^0$  (20)

$$K_2^0 \quad K_2^0$$
 (21)

 $K_1^0 \quad K_2^0$  (22)

Suppose the initial state is an S-state. Because  $\bar{p}$  and p have opposite parities whereas  $K^0$  and  $\bar{K}^0$  have the same parity, the final state has odd orbital angular momentum. It has to be a p-state, and the initial state is  ${}^3S_1$ . This state is odd under charge conjugation. Because in the final state only  $K_1^0$  K<sub>2</sub><sup>0</sup> is odd under C, only the decay modes of (22) should occur. The experiment gives

$$(20)$$
 :  $(22)$  = 0 : 54

The Padua group has seen one event.

If the initial state is a P-state, one finds in a similar way that type (1) is allowed. The experiment therefore gives strong evidence for annihilation in an S-state.

Before a discussion of the measurement on the  $K^*$  spin, it is noticed that the experimental result excluding (20) and (21) seems to contradict the Sakata model. In this model  $K^0$  and  $\overline{K}^0$  are composite particles ( $N\overline{\Lambda}$ ) and ( $\overline{N}\Lambda$ ) respectively. As p, n and  $\Lambda$  form the basic triplet of the model, there is symmetry for interchange of neutron and  $\Lambda$  and hence of  $K^0$  and  $\overline{K}^0$ . In particular the amplitude for the process, in which  $K^0$  moves in a given direction, is equal to the amplitude for  $\overline{K}^0$  going the same way. This is not the case if the final state is a P-state.

After having determined that the  $\mathbf{p} + \mathbf{p}$  capture occurs in the S-state, Armenteros <u>et al.</u> complete the argument about the K\* spin in the following way: They look at the following type events:

$$\overline{\mathbf{p}} + \mathbf{p} \rightarrow \mathbf{K}_1^0 + \mathbf{K}_0^*$$
 and  
 $\rightarrow \mathbf{K}_1^0 + \overline{\mathbf{K}}_0^*.$ 

They detect these events from the  $\pi^+ + \pi^-$  decay of the  $K_1^0$  and consider even events where the  $K_1^0$  energy corresponds to K\* formation. Let us assume now that the K\*spin (and  $\overline{K}$ \*spin) is 0. Since the K\* decays into a  $\pi + K$ , and the  $\pi$  is pseudoscalar, the K\*must then have the opposite parity from the K, so the intrinsic parities of  $\overline{p}$  + p and  $\overline{K}$  + K\* (or K +  $\overline{K}$ \*) are equal. Hence the K +  $\overline{K}$ \* must occur in the S-states, and angular momentum conservation implies that the initial state is a singlet, i.e.  ${}^{1}S_{0}$ .

The  ${}^{1}S_{0}$  state is even under charge conjugation so the final state must also be even under charge conjugation. This, together with the fact that the observed K produced with the K\* is a K<sub>1</sub><sup>0</sup>, implies that the neutral decay mode of the K\*(or K\*) must be K<sub>1</sub><sup>0</sup> +  $\pi^{0}$ . No K<sub>2</sub><sup>0</sup> +  $\pi^{0}$  is allowed. Armenteros <u>et al.</u> measure the ratio,

$$R = \frac{K_1^0 + [K_1(visible) + \pi^0]}{K_1^0 + [K(invisible) + \pi^0]}$$

where the K in the square brackets is the K from the  $K^*$  decay. A  $K_2^0$  will live so long as to be invisible and the  $\pi^0 + \pi^0$  decay mode of the  $K_1^0$  will be invisible. Now, if  $J(K^*) = 0$ , the K of the  $K^*$  decay is always the  $K_1^0$ ; this ratio will be simply the ratio of the  $(\pi^{+} + \pi^{-})/(\pi^0 + \pi^0)$  decay rates of the  $K_1^0$ , i.e.2. R cannot be measured exactly by experiment because of the difficulty of separating  $K^*$  events from the non-resonant background, but the data (with limited statistics) clearly shows that R is appreciably less than one. Hence, it is reasoned, the  $J(K^*) = 0$  assignment must be wrong.

This measurement is, of course, a long way from conclusive; and since the California data favours  $J(K^*) = 0$ , we must conclude that the spin of the  $K^*$  is not yet known.

### 4. THE 1815 MeV RESONANCE

The 1815 resonance has baryon number one and strangeness -1 and lies above the  $\overline{K}$  + N threshold, so it may be produced without accompanying particles in interactions such as  $\overline{K}$  + N  $\rightarrow \overline{K}$  + N and  $\overline{K}$  + N  $\rightarrow \pi$  +  $\Sigma$  [25]. The I-spin of this resonance was previously known to be zero. Keefe et al. examined angular distribution in the resonance region carefully and concluded that, if waves of J  $\geq$  7/2 can be neglected, the resonance must have angular momentum 5/2. Perhaps this resonance is part of a family that includes the F 5/2  $\pi$ -N resonance.

In conclusion, three of the most interesting questions about strange particle physics will be repeated since much theoretical and experimental work should be done on these questions in the next few years:

(1) Is the exchange of vector mesons the source of the large S-wave meson-baryon interactions? If not, what is?

(2) Are the pseudoscalar  $\eta$ -baryon interactions strong and is the  $\eta$  a brother to the pion in some symmetry scheme such as the unitarity-symmetry?

(3) Are the pseudoscalar K-meson baryon interactions strong? The fact that certain processes involving K mesons are strong clearly shows that some K meson interactions are strong, but there is still no very good evidence that the pseudoscalar K baryon interactions are large at all. They may be nearly zero. According to unitary symmetry, of course, they are large.

As has been demonstrated, it would seem that the  $\Sigma$  and  $\Lambda$  parities are even. In any case it is clear that the rapid development of strange particle physics is not going to slow down in the next year or two.

#### REFERENCES

- [1] DALITZ, R. H. and LIU, L., Phys. Rev. 116 (1959) 1312.
- [2] AMMAR, R.G. et al., Nuovo Cim. 19 (1961) 20.
- [3] BEALL, E.F., CORK, B., KEEFE, D., MURPHY, P.G. and WENZEL, W.A., Phys. Rev. Letters 8 (1962) 75.
- [4] BLOCK, M. M. et al., Proc. Conf. on High Energy Physics, CERN (1962).
- [5] DAY, T.B., SNOW, G.A. and SUCHER, J., Phys. Rev. Letters <u>3</u> (1959) 61.
- [6] CAPPS, R.H., Phys. Rev. Letters 6 (1961) 375; Phys. Rev. 126 (1962) 1574.
- [7] ALVAREZ, L. W., Ninth Int. Conf. on High Energy Physics, Kiev (1959).
- [8] FERRO-LUZZI, M., TRIPP, R.D. and WATSON, M.B., Phys. Rev. Letters 8 (1962) 28.
- [9] WIGNER, E. P., Phys. Rev. 98 (1955) 145.
- [10] TRIPP, R. D., WATSON, M. B. and FERRO-LUZZI, M., Phys. Rev. Letters 8 (1962) 175.
- [11] SAMIOS et al., Proc. Conf. on High Energy Physics, CERN (1962).
- [12] TICHO, H. K. et al., Proc. Conf. on High Energy Physics, CERN (1962).
- [13] LEE, T. D. and YANG, C. N., Phys. Rev. 109 (1958) 1755.
- [14] ALSTON, M. et al., Phys. Rev. Letters 5 (1960) 520.
- [15] Ibid., Preliminary rpt in Proc. Int. Conf. on High Energy Physics, Rochester (1960) 445.
- [16] ADAIR, R.K., Phys. Rev. 100 (1955) 1540.
- [17] BLOCK, M. M. et al., Nuovo Cim. 20 (1961) 724.
- [18] DALITZ, R.H. and MILLER, D.H., Phys. Rev. Letters 6 (1961) 562,
- [19] ELY, R. P. et al., Phys. Rev. Letters 7 (1961) 461.

- [20] GOLDHABER, GOLDHABER et al., Proc. Conf. on High Energy Physics, CERN (1962).
- [21] DALITZ, R.H. and TUAN, S.F., Ann. Phys. <u>8</u> (1959) 100; <u>10</u> (1960) 307; Phys. Rev. Letters <u>2</u> (1959) 425.
- [22] ALEXANDER, G. et al., Phys. Rev. Letter 8 (1962) 447.
- [23] SCHULT, R. L. and CAPPS, R. H., Phys. Rev. 122 (1961) 1659; Nuovo Cim. 23 (1962) 416.

١

- [24] AKIBA, T. and CAPPS, R. H., Phys. Rev. Letters 8 (1962) 457.
- [25] KEEFE, D. et al., Proc. Conf. on High Energy Physics, CERN (1962).

# BOOK II

# SYMMETRY PROPERTIES OF ELEMENTARY PARTICLES

## PART II

SYMMETRIES OF INTERACTIONS
## THE FORMALISM OF LIE GROUPS

### A. SALAM IMPERIAL COLLEGE OF SCIENCE AND TECHNOLOGY, LONDON, UNITED KINGDON

#### 1. INTRODUCTION

Throughout the history of quantum theory, a battle has raged between the amateurs and professional group theorists. The amateurs have maintained that everything one needs in the theory of groups can be discovered by the light of nature provided one knows how to multiply two matrices. In support of this claim, they of course, justifiably, point to the successes of that prince of amateurs in this field, Dirac, particularly with the spinor representations of the Lorentz group.

As an amateur myself, I strongly believe in the truth of the non-professionalist creed. I think perhaps there is not much one has to learn in the way of methodology from the group theorists except caution. But this does not mean one should not be aware of the riches which have been amassed over the course of years particularly in that most highly developed of all mathematical disciplines - the theory of Lie groups.

My lectures then are an amateur's attempt to gather some of the fascinating results for compact simple Lie groups which are likely to be of physical interest. I shall state theorems; and with a physicist's typical unconcern rarely, if ever, shall I prove these. Throughout, the emphasis will be to show the close similarity of these general groups with that most familiar of all groups, the group of rotations in three dimensions.

In 1951 I had the good fortune to listen to Prof. Racah lecture on Lie groups at Princeton. After attending these lectures I thought this is really too hard; I cannot learn this; one is hardly ever likely to need all this complicated matter. I was completely wrong. Eleven years later the wheel has gone full cycle and it is my turn to lecture on this subject. I am sure many of you will feel after these lectures that all this is too damned hard and unphysical. The only thing I can say is: I do very much hope and wish you do not have to learn this beautiful theory eleven years too late.

#### 2. SOURCES

A word about the sources [1] and the scheme I wish to follow. The chief sources in this theory are the famous thesis of Cartan in which most of this subject was created Hermann Weyl and his classical text on "Classical Groups" and Racah's Princeton lectures [2]. However, I believe conceptually the most concise existing treatment of the subject is in the works of DYNKIN [3]. Dynkin's paper has a magnificent appendix which gives a review of the known results and this appendix is my major source. From the point of view of a physicist working on symmetry problems perhaps the best reference is to the review paper of BEHRENDS, LEE, FRONSDAL and DREITLEIN [4]. I have checked with Lee that apparently while these authors knew of Dynkin's work they did not have it accessible when they were writing their review. Thus their treatment of the fundamentals resembles Cartan and Racah more closely rather than Dynkin. Another excellent paper for physicists is SPEISER and TARSKI [5]. For a fuller exposition of Dynkin, reference may also be made to two Imperial College theses - those of NE'EMAN [6] and IONIDES [7].

#### 3. DEFINITIONS

The general theory of Lie groups follows closely the pattern of the one group we are all thoroughly familiar with, the theory of the three-dimensional rotation group  $O_3$ . It is indeed a matter of deep regret that the elementary expositions of this familiar case do not employ the same terminology as that of the general theory. Half the conceptual difficulties of the subject would simply disappear if this had consistently been done in our undergraduate courses. To illustrate and to anticipate notation we summarize known facts about the rotation group  $O_3$ . (All statements made here will be formalized later.) We know that this group is completely determined by three infinitesimal generators:

$$J^{\pm} = 1/\sqrt{2} (J_1 \pm i J_2), J_3$$

and their commutation relations:

$$[J^+, J_2] = J^+, [J^-, J_2] = -J^-, [J^+, J^-] = J_3,$$

The commutation relations tell us that

(i) The number of operators (out of these three) which can be diagonalized is one  $(J_3)$ . Call this number the "rank" of the group. Thus the rank of  $O_3 = 1$ .

(ii) Call the eigenvalues of  $J_3$  (i.e. the magnetic quantum numbers) by the name "weights". The highest eigenvalues j of  $J_3$  uniquely labels a representation. We shall call this "the highest weight".

(iii) The commutation relations tell us (from  $[J^{\pm}, J_3] = \pm J_3$ ) that, irrespective of what the weights are, the difference of two consecutive weights is  $\pm 1$ . These numbers  $\pm 1$  which are characteristic of the commutation relations of the group and not of any particular representation are called "roots". In the subsequent general study of Lie groups these three concepts, "rank" of the group, "roots" of the group and "weights" (and particularly the highest weight) will be generalized and will play crucial roles.

(iv) Another way of labelling the representations of  $O_3$  is to use the operator  $\underline{J}^2$ . This operator commutes with all other operators and thus for a given representation equals a constant multiple of unity. If j is the highest weight,  $\underline{J}^2 = j(j + 1) \underline{I}$ . This operator is called the "Casimir operator". We shall find that the concept of a general "Casimir operator" is not as highly developed, and for this reason we shall treat this concept at an early stage (section 5) and then not mention it at all later.

#### 4. MATHEMATICAL PRELIMINARIES

4.1. A group G is a set of elements a, b.... with a composition law (multiplication) such that the following conditions are fulfilled:

(i) if a and b are elements of the set, then also the product c = ab belongs to the set.

(ii) the composition is associative: a (b c) = (a b) c,

(iii) the set contains a unit element e such that a e = ea = a,

(iv) to any element a of the set, there exists one and only one element  $a^{-1}$  of the set such that  $a^{-1}a = a a^{-1} = e$ .

The definition of a group does not imply that the two elements ab and ba are equal; i.e., the composition is not necessarily commutative. A group in which all elements commute is called <u>abelian</u>.

A sub-group H of a group G is a sub-set of elements of G, which again fulfils the group postulates. G and the group consisting of the unit element, e, are called trivial sub-groups of G. A sub-group N is called an <u>invariant</u> <u>sub-group</u> of G if for any element n of N ( $n\in N$ ),  $sns^{-1}$  is again an element of N where s is any element of  $G(s\in G)$ .

A group is called <u>simple</u> if it contains no non-trivial invariant sub-groups, except possibly discrete ones.

A group is called <u>semi-simple</u> if it contains no non-trivial invariant abelian sub-groups, except possibly discrete ones.

4.2. A representation of a group G is a mapping of the group into a set of linear transformations D of a vector space R such that

if then ab = c D(a) D(b) = D(c),  $D(a^{-1}) = D^{-1}(a),$ D(e) = I,

where I is the unit operator.

Ŧ

A representation is <u>reducible</u> if it leaves a sub-space of R invariant. Then every transformation matrix can be brought into form:

|  | A  | B |
|--|----|---|
|  | _0 | D |

A representation is <u>fully reducible</u> if every transformation matrix can be written as

 $\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}$ 

4.3. A Lie group is a group whose elements form an analytic manifold in such a way that the composition ab = c is an analytic mapping of the manifold  $G \times G$  into G and the inverse  $a \rightarrow a^{-1}$  is an analytic mapping of G into G. A Lie group can thus be viewed from an algebraic, topological or analytical

point of view. The topological concepts of importance are connectedness, compactness and invariant integral on the group (see SPEISER and TARKSI [5]).

A group G is compact if every infinite sequence in G has a limit point in G. For a compact group one can define a <u>finite</u> total volume which is invariant under the group.

For example, the group of rotation in three dimensions  $O_3$  without reflections is a connected and compact group. The proper Lorentz group is connected but not compact and the improper Lorentz group is neither connected nor compact.

The study of simple groups is important because every semi-simple connected group is essentially a direct product of simple groups, and any connected compact Lie group is essentially a product of a semi-simple and a one-parameter (abelian) compact group.

Ex.  $O_4 \approx O_3 \times O_3$ ;  $O_3$  simple;  $O_4$  semi-simple.

The symbol  $\approx$  means locally isomorphic. From now on we consider only simple compact Lie groups.

#### 5. SIMPLE COMPACT LIE GROUPS

So far as a physicist is concerned, a Lie group is a group of transformation of variables which depend analytically on a finite set of N parameters. The fundamental idea of Lie was to consider not the whole group but that part of it which lies close to the identity consisting of the so-called infinitesimal transformations. To formalize this, we have Theorem I.

#### Theorem 1

Ś

Every representation of a compact Lie group is equivalent to a unitary representation and is fully reducible (RACAH, WEYL [2]). Thus, since the matrices D(g) can be taken as unitary, they can be put into the form:

D = exp(
$$i\epsilon^{\alpha}X_{\alpha}$$
),

where  $\dot{X}_{\alpha}$  are constant hermitian matrices ( $X_{\alpha}^{+} = X_{\alpha}$ ), which are called infinitesimal generators of the group.  $\epsilon^{\alpha}(\alpha = 1, 2..., N)$  are N real parameters on which the set of transformations D depend.

The group is called <u>unimodular</u> if for any D(S), det[D(s)] = 1. Then tr X = 0.

#### Theorem 2

Fundamental Theorem of Lie

The local structure of a Lie group is completely specified by the commutation relations between the operators  $X_{\alpha}$ :

$$[X_{\alpha}, X_{\beta}] = C_{\alpha\beta}^{\gamma} X_{\gamma} ; \alpha, \beta, \gamma = 1, 2..., N, \qquad (5.1)$$

where the coefficients  $C^{\gamma}_{\alpha\beta}$  which are independent of the representations of

the group are numbers (called the structure constants of the group). These numbers satisfy two requirements:

(a) antisymmetry in the two lower indices

$$C^{\gamma}_{\alpha\beta} = -C^{\gamma}_{\beta\alpha},$$

(b)

$$C_{\alpha\beta}^{\delta} C_{\delta\gamma}^{\epsilon} + C_{\gamma\alpha}^{\delta} C_{\delta\beta}^{\epsilon} + C_{\beta\gamma}^{\delta} C_{\delta\alpha}^{\epsilon} = 0.$$

Note that conditions (a) and (b) are equivalent to the antisymmetry of the Commutator bracket  $[X_{\alpha}, X_{\beta}]$  and the Jacobi identity:

$$[[X_{\alpha}, X_{\beta}], X_{\gamma}] + [[X_{\gamma}, X_{\alpha}], X_{\beta}] + [[X_{\beta}, X_{\gamma}], X_{\alpha}] = 0.$$

Rewrite (b) in the form:

$$(\mathbf{C}_{\alpha})_{\delta}^{\epsilon} (\mathbf{C}_{\beta})_{\gamma}^{\delta} - (\mathbf{C}_{\beta})_{\delta}^{\epsilon} (\mathbf{C}_{\alpha})_{\gamma}^{\delta} = \mathbf{C}_{\alpha\beta}^{\delta} (\mathbf{C}_{\delta})_{\gamma}^{\epsilon}.$$

Thus, we have shown the following:

#### Theorem 3

The N matrices  $C_{\alpha}$  with matrix elements  $(C_{\alpha})_{\gamma}^{\epsilon}$  form the so-called regular or adjoint representation of the Lie algebra<sup>\*</sup>.

The problem of classification of Lie groups is the problem of finding the numbers c's which satisfy (a) and (b) and then of finding N constant matrices which satisfy the fundamental commutation relation of Theorem 1. This problem was completely solved by Cartan in 1913. Before however we state Cartan's results, we first wish to recast the fundamental commutation relation (5.1) in a "canonical" form and also get over a number of auxiliary results connected with Casimir operators.

#### 6. CASIMIR OPERATORS

From the structure constants we can define a metric tensor:

$$g_{\mu\nu} = C^{\beta}_{\mu\alpha} C^{\alpha}_{\nu\beta}$$
.

Theorem 4

The necessary and sufficient condition for a Lie group to be semi-simple is that

<sup>\*</sup> The set of N matrices  $X_{\alpha}$  span a linear vector space over the field of complex numbers and define a Lie Algebra; the sum of two matrices is an element of the algebra and so is their commutator. Lie algebras and Lie groups possess a one-one correspondence, and it is possible to go freely from Lie groups to Lie algebras. The study of Lie algebras (first introduced by Weyl) is in effect the study of the infinitesimal aspect of Lie group theory. Even though it is galling to bring in a new concept (of a Lie algebra) at this stage, this apparently improves the mathematical rigour of the statements made in these lectures!

# det $[g_{uv}] \neq 0$ (Cartan).

Thus for a semi-simple group we can define an inverse metric  $g^{\mu\nu}$  such that

$$g^{\mu\nu}g_{\nu\rho} = \delta^{\mu}_{\rho}$$
,

and we can use the metric tensors for raising and lowering indices.

Now define an operator  $F = g_{\alpha_1,\alpha_2} X^{\alpha_1} X^{\alpha_2}$ . This is called the <u>Casimir</u> operator and has the property that it commutes with all the generators of the group:

 $[\mathbf{F}, \mathbf{X}_{\alpha}] = \mathbf{0}.$ 

The proof of the result is trivial. The significance of the Casimir operator lies in recalling that by Schur's Lemma any operator which commutes with all the generators of the group must be a multiple of the identity.

For  $O_3$  this operator is the total angular momentum  $J^2$ . One can define generalized Casimir operators:

$$\mathbf{F}^{\mathbf{n}} = \mathbf{C}_{\alpha_1\beta_1}^{\beta_2} \mathbf{C}_{\alpha_2\beta_2}^{\beta_3} \dots \mathbf{C}_{\alpha_m\beta_m}^{\beta_1} \mathbf{X}^{\alpha_1} \mathbf{X}^{\alpha_2} \dots \mathbf{X}^{\alpha_n}.$$

It is easy to see that all these commute with  $X^{\alpha}$ .

For O<sub>3</sub> all inequivalent irreducible representations can be characterized by giving different values of  $\lambda$  where  $\lambda I = J^2$ . The question arises if this is true in general. Racah gives the following partial answer: Write the set  $\{\lambda^k\}$  defined by  $\lambda^k I = F^k$ . For simple groups if the representation D and  $(D^{-1})^T$ are equivalent representations, then the set  $\{\lambda^k\}$  gives an unequivocal characterization of all the inequivalent representations.

#### 7. CANONICAL FORMS OF THE COMMUTATION RELATIONS AND RANK OF A GROUP

#### Theorem 6 (P. Ionides)

By a suitable choice of linear combination of the X's, the  $C^{\alpha}_{\beta\gamma}$  can be made antisymmetric in all three indices and pure imaginary; i.e. one can write the commutation relations in the form:

$$[X_{\alpha}, X_{\beta}] = i f_{\alpha\beta\gamma} X_{\gamma},$$

with  $f_{\alpha\beta\gamma}$  purely antisymmetric and real.

In the usual theory of angular momentum, the first step is to rewrite (the Ionides type of) commutation relations,

$$[\mathbf{J}_{\alpha}, \mathbf{J}_{\beta}] = \mathbf{i} \epsilon_{\alpha\beta\gamma} \mathbf{J}_{\gamma} , \quad \alpha, \beta, \gamma = 1, 2, 3, \qquad (7, 1)$$

in the so-called "canonical form". Defining the non-hermitian operators,

$$J_{+} = (J_{1} \pm i J_{2}) / \sqrt{2},$$

we rewrite (7.1) as

$$[J_{\pm}, J_{3}] = \pm J_{\pm},$$
  
 $[J_{+}, J_{-}] = J_{3}.$  (7.2)

There are two virtues of this canonical form:

(1) If  $J_3$  is diagonalized  $(J_3 | m) = m | m)$ , we infer from (7.2) that the operators  $J_{\pm}$  act as "creation" and "annihilation" operators.

(2) (7.2) shows that the consecutive eigenvalues m of  $J_3$  differ by ±1. Our first task is to cast the commutation relations (5.1) in the "canonical form".

Assume that among the N generators, there are l which mutually commute and can thus be simultaneously diagonalized. This number l is called the <u>rank</u>, and we shall designate these l (hermitian) operators as H<sub>1</sub>, H<sub>2</sub>... ... H<sub>l</sub>. (For O<sub>3</sub>, l = 1). These operators have a direct physical meaning since their eigenvalues for any representation provide us the quantum numbers.

Let us consider  $H_1$ ,  $H_2$ ...,  $H_\ell$  as the components of an  $\ell$ -dimensional operator-valued vector <u>H</u>. The components of <u>H</u> clearly satisfy the commutation relations:

$$[H_i, H_i] = 0$$
 for i, j = 1, 2, ....,  $k$ .

If the dimension of the algebra is N (i.e. the number of parameters of the corresponding group is N), we still need  $(N - \ell)$  elements to complete a basis of the algebra. A suitable choice of these is provided by the following:

#### Theorem 7

There exists a basis of the Lie algebra consisting of the elements  $H_1$ ,  $H_2$ ...,  $H_l$ ;  $E_{\pm 1}$ ,  $E_{\pm 2}$ ... $E_{\pm (N-l)/2}$  such that the following commutation relations hold:

$$[\underline{\mathbf{H}}, \mathbf{E}_{\alpha}] = \underline{\mathbf{r}} (\alpha) \mathbf{E}_{\alpha}, \qquad (7.3)$$

$$[\mathbf{E}_{\alpha},\mathbf{E}_{\alpha}] = \underline{r} \quad (\alpha) \mathbf{H} , \qquad (7.4)$$

$$[\mathbf{E}_{\alpha}, \mathbf{E}_{\beta}] = \mathbf{N}_{\alpha\beta} \mathbf{E}_{\gamma} \text{ for } \alpha \neq -\beta , \qquad (7.5)$$

with  $\alpha$ ,  $\beta = \pm 1, \pm 2, \dots \pm (N-\ell)/2$ . E's are non-hermitian matrices and <u>r</u> ( $\alpha$ ) are real vectors in an  $\ell$ -dimensional space. The <u>r</u>'s are called <u>roots</u> of the algebra; they have the property that

 $\underline{\mathbf{r}}(\alpha) = -\underline{\mathbf{r}}(-\alpha). \tag{7.6}$ 

Clearly the total number of the roots is  $(N - \ell)$ .

The scalar product appearing in (7.4) is the usual Euclidean scalar product provided the H's are chosen in such a way that the following normalization conditions hold:

$$\sum_{\alpha} \mathbf{r}_{i}(\alpha) \mathbf{r}_{j}(\alpha) = \mathbf{R} \,\delta_{ij} \,; \, ij = 1, 2 \dots \ell \,, \qquad (7.7)$$

with an arbitrary scale constant. Finally,  $N_{\alpha\beta}$  are real numbers which are different from zero if and only if  $\underline{r}(\alpha) + \underline{r}(\beta)$  is also a root.

The roots, being essentially our old friends the structure constants, specify completely the group (at least in the local sense). They possess a twin role in the theory. First, as may be inferred from (7.3), the roots are the differences of the eigenvalues of <u>H</u>. Second and more important for our present purposes, the roots allow us to classify Lie groups. In terms of the roots we can state Cartan's solution of the problem of finding all simple Lie groups. The crucial theorem here is Theorem 8 which lists further properties of the roots and in terms of these gives a complete classification of Lie groups.

#### 8. CLASSIFICATION OF LIE GROUPS

A root is said to be <u>positive</u> if its first non-vanishing component (in an arbitrary basis) is positive. A root is called <u>simple</u> if it is a positive root and in addition it cannot be decomposed into the sum of two positive roots.

#### Theorem 8

(i) For a simple group of rank  $\ell$  there exist  $\ell$  simple roots and they are all linearly independent. (We shall call the set of simple roots the *π*-system.)

(ii) Every positive non-simple root can be expressed as a linear combination  $\sum_{n(\alpha)\in\pi} R_{\alpha} \underline{r}(\alpha)$  where  $R_{\alpha}$  are non-negative integers.

(iii) If  $r(\alpha)$  and  $r(\beta)$  are two simple roots, the angle  $\theta_{\alpha\beta}$  between these can take only the following values:

90° 120° 135° and 150°,

so that  $2\underline{r}(\alpha) \cdot \underline{r}(\beta)/\underline{r}(\alpha) \cdot \underline{r}(\alpha)$  and  $2\underline{r}(\alpha) \cdot \underline{r}(\beta)/\underline{r}(\beta) \cdot \underline{r}(\beta)$  are both integers.

(iv) For every simple group, all the simple roots either have the same length or their length ratios assume simple values. More explicitly one has

 $\frac{|\mathbf{r} (\alpha)|^2}{|\mathbf{r} (\beta)|^2} = 2 \quad \text{if} \quad \theta_{\alpha\beta} = 135^\circ$  $3 \quad \text{if} \quad \theta_{\alpha\beta} = 150^\circ.$ 

If  $\theta_{\alpha\beta} = 90^\circ$ , the ratio of lengths is undetermined.

#### Dynkin diagrams

As we shall see in a moment, the geometrical properties of the simple roots in the  $\pi$ -system characterize in a unique manner the corresponding Lie groups. Therefore it is most convenient to incorporate them in a schematic diagram. These diagrams (the so-called Schouten-Dynkin diagrams) are drawn in Fig. 1.

From Theorem 8, the lengths of the simple roots of a given simple Lie group can assume at most two different values. This fact together with the



Cartan solution of all possible single Lie groups.

properties about the angles enumerated above can be symbolically described by associating with each simple root a small circle. For the roots of greatest length the circle is marked in black. If the angle between two consecutive simple roots is equal to  $120^\circ$ ,  $135^\circ$  or  $150^\circ$ , the corresponding circles are joined by simple, double or triple lines respectively. If the angle is  $90^\circ$ , the circles are not joined. For a group of rank l there are l simple roots and therefore l circles (black or white).

In terms of these diagrams we give now the Cartan solution of all possible simple Lie groups. Broadly these fall into two categories: the socalled "classical groups" and the five "exceptional groups".

To anticipate we shall find that the classical Lie groups are some of the well known objects:

 $A_{\ell}$  is the group of unitary unimodular matrices in complex space of  $(\ell + 1)$  dimensions  $(SU_{\ell+1})$ .

 $B_{\ell}$  and  $D_{\ell}$  are groups of orthogonal transformations (rotations) in real spaces of  $2\ell + 1$  and  $2\ell$  dimensions respectively ( $O_{2\ell+1}$  and  $O_{2\ell}$ ).

 $C_{\ell}$  is the group of unitary matrices U in complex space of  $2\lambda$  dimensions which fulfil the condition  $U^{T} J U = J$  where J is a non-singular antisymmetric matrix (the symplectic group)\*.

\* Note from the Dynkin diagrams:

(i)

 $= B_2$ 

Also

(ii)  $C_2 = 0$ 

```
i.e. O5 ~ C2.
```

i.e. O<sub>6</sub> ~ SU<sub>4</sub>.

To take simple examples of root structures:

For l = 1 (i.e. group O<sub>3</sub>) there is just one simple root + 1. The space spanned by simple roots (the  $\pi$ -space) is {1}. For  $\ell = 2$ , the space is a plane, the relevant groups being

| A <sub>2</sub> : | 00           | Two simple roots of equal length, and the angle between them is $120^{\circ}$ . |
|------------------|--------------|---|
| В <sub>2</sub> : |              | Two simple roots. Their length ratio is 2.                                      |
| C <sub>2</sub> : |              | The angle between them is 135°.   |
| G <sub>2</sub> : |              | Two simple roots with length ratio equal to 3, and angle 150°.                  |
| D <sub>2</sub> : | o<br>15 semi | -simple, $D_2 \approx A_1 \times A_1$   |

Summarizing this section then, from the Dynkin diagrams we read off immediately the rank  $\ell$  of the group, the lengths of the simple roots and their mutual angles (and of course the dimensionality of the Euclidean space  $(\pi)$ spanned by these  $\ell$  independent vectors)\*\*\*. The simple roots r(1).  $\underline{r}(2),\ldots,\underline{r}(\ell)$ , are given by the following formulae:

\* It is perhaps worthwhile to make the reminder at this stage that not all roots are simple. In fact the total number of roots is  $(N-\ell)$ , the distinct ones being  $(N-\ell)/2$  in virtue of  $r(\alpha) = -r(-\alpha)$ ,  $\alpha = 1, 2, ..., (N-\ell)/2$ . The remaining  $(N-3\ell)/2$  distinct non-simple roots can easily be constructed, and in Footnote \*\* we give a complete ansatz for drawing a complete root diagram (for l = 2 for example in a plane; for  $l = 3 \ln \{3\}$ space and so on). Personnally, I consider these diagrams pointless. However, to satisfy current prejudice the root diagrams for  $A_2$ ,  $B_2$  and  $G_2$  are reproduced in Fig. 2.



Root diagrams for A,, B, and G,

\*\* The following scheme incorporates all the requirements about angles and lengths of simples roots specified by the diagrams.

For  $A_{0}$  define the following vectors:

o

$$\begin{split} \lambda_1, \lambda_2, \dots, \lambda_{\ell+1} \\ \\ \underline{\lambda}_1 + \underline{\lambda}_2 + \dots + \underline{\lambda}_{\ell+1} &= 0, \\ \\ \underline{\lambda}_1^2 &= \underline{\lambda}_2^2 = \dots = \underline{\lambda}_{\ell+1}^2 = \ell_A, \\ \\ \underline{\lambda}_D, \lambda_D &= -A, p \neq q = 1, 2, \dots, \ell+1. \end{split}$$

by the conditions

$$\underline{\underline{r}}(\ell) = \underline{\lambda}_{\ell} - \underline{\lambda}_{\ell+1},$$
  

$$\underline{\underline{r}}(\ell-1) = \underline{\lambda}_{\ell-1} - \underline{\lambda}_{\ell},$$
  

$$\vdots$$
(8.1)

$$\underline{\underline{r}}(1) = \underline{\lambda}_1 - \lambda_2,$$

$$\underbrace{O}_{\ell} \underbrace{O}_{\ell-1} \underbrace{O}_{\ell} \underbrace{O}_{\ell-1} \underbrace{O}_{\ell} \underbrace{O}_{\ell-1} \underbrace{O}_{\ell} \underbrace{$$

For  $B_{0}$ : the simple root structure is as follows:

| $\underline{r}(\ell) = \underline{\lambda}_{\ell}$ , (This is the smallest root)             |       |
|--|-------|
| $\underline{\mathbf{r}}(\ell-1)=\underline{\lambda}_{\ell-1}-\underline{\lambda}_{\ell},$    | (8.2) |
| $\underline{\mathbf{r}}\left(1\right) = \underline{\lambda}_{1} - \underline{\lambda}_{2} ,$ |       |

where

$$\frac{\lambda_1^2}{\ell_1} = \frac{\lambda_2^2}{\ell_2} = \dots + \frac{\lambda_\ell^2}{\ell_\ell} = A,$$

$$\frac{\lambda_p}{\ell_q} = 0, p \neq q,$$

$$(8.3)$$

$$\underbrace{\bullet}_{\ell \neq 1} = \underbrace{\bullet}_{\ell \to 1} = \underbrace{\bullet$$

For  $C_{\theta}$ : the simple roots are given by:

$$\underline{r}(\ell) = 2 \underline{\lambda}_{\ell}, \quad \text{(This is the greatest root.)}$$

$$\underline{r}(\ell-1) = \underline{\lambda}_{\ell-1} - \underline{\lambda}_{\ell-2}, \quad (8, 4)$$

$$\vdots$$

$$\underline{r}(1) = \underline{\lambda}_{2} - \underline{\lambda}_{1}, \quad (6, 4)$$

$$\underline{r}(1) = \underline{\lambda}_{2} - \underline{\lambda}_{1}, \quad (6, 4)$$

where the  $\lambda$  's satisfy (8.3).

For  $D_{\ell}$ : the simple roots are given by:

$$\underline{\mathbf{r}}(\ell) = \underline{\lambda}_{\ell-1} + \underline{\lambda}_{\ell},$$

$$\underline{\mathbf{r}}(\ell-1) = \underline{\lambda}_{\ell-1} - \underline{\lambda}_{\ell}$$

$$\vdots$$

$$\underline{\mathbf{r}}(1) = \underline{\lambda}_{\mathbf{p}} - \underline{\lambda}_{\mathbf{2}}$$

$$\overset{\ell}{\underset{\ell=1}{\overset{\ell-2}{\longrightarrow}}} - \underbrace{-\mathbf{0}}_{1}$$
(8.5)

The  $\lambda$ 's satisfy (8.3). So much for simple roots. All roots are given for the classical groups by the following expressions:

A<sub>1</sub>: 
$$(\lambda_{p} - \lambda_{q})$$
; p, q = 1, 2, ....,  $l + 1$ 

$$B_{\ell}: \pm \underline{\lambda}_{p}, \pm \underline{\lambda}_{p} \pm \underline{\lambda}_{q} ; p, q = 1, 2, \dots, \ell$$

$$C_{\ell}: \pm 2\underline{\lambda}_{p}, \pm \underline{\lambda}_{p} \pm \underline{\lambda}_{q} ; p, q = 1, 2, \dots, \ell$$

$$D: \pm \underline{\lambda}_{p} \pm \lambda_{q} ; p, q = 1, 2, \dots, \ell.$$

The  $\pm$  signs are to be taken in arbitrary combinations.

Similar expressions can be given for the exceptional groups. Also one can give a full correspondence between the "canonical" expressions for the commutation relations and the more familiar manner in which one writes the commutation relations for the orthogonal, symplectic groups, etc.

Thus, for the orthogonal group in  $(2\ell + 1)$  dimensions which leaves invariant the quadratic form

$$\sum_{p=-\ell}^{\ell} X^{p} X^{-p}$$

one may write the infinitesimal operators:

$$X_{pq} = -X_{qp} = X^{p} \frac{\delta}{\delta X^{-q}} - X^{q} \frac{\delta}{\delta X^{-p}},$$

with the commutation relations:

$$[X_{ik}, X_{mn}] = \delta_{k+m} X_{in} - \delta_{k+n} - X_{im} - \delta_{\ell+m} X_{kn} - \delta_{\ell+n} X_{km}$$

where  $\delta q = 1$  if q = 0 and zero otherwise. These operators correspond to the E's and the H's of B<sub>0</sub> if we make the following identifications:

$$X_{p-p} \equiv H_p, X_{\pm p\pm q} \equiv E_{\pm \lambda p\pm \lambda q}, X_{0\pm p} \equiv E_{\pm \lambda p}; p, q > 0.$$

Similar correspondence can be stated for  $A_{\ell}$ ,  $C_{\ell}$ ,  $D_{\ell}$  etc. (Racah's notes).

#### 9. REPRESENTATIONS OF LIE GROUPS: WEIGHTS

9.1. Now we come to physically the most important problem of all - the problem of finding representations of the group, i.e. the matrices corresponding to  $\underline{H}$  and  $\underline{E}_{\alpha}$ .

Consider a representation of dimension (or degree) d. Since  $H_1, H_2, ..., H_l$  are hermitian matrices, and since they commute with each other, we can simultaneously diagonalize these. Let  $|m\rangle$  be a simultaneous eigenket:

$$\underline{H} |\mathbf{m}\rangle = \underline{m} |\mathbf{m}\rangle. \tag{9.1}$$

Since H's are  $d \times d$  matrices, the total number of such eigenkets  $|m\rangle$  is d. The m's in Eq. (9.1) are real numbers and are called "weights". They

form *l*-dimensional vectors in a Euclidean space for whose basis one may take the  $\pi$ -space of the group (the space spanned by the *l* simple roots). Summarizing, for the case of a group of rank *l* and for a given representation of dimensionality d, there are

*l* : simple root vectors

#### $(N-3\mathcal{A})/2$ : distinct non-simple root vectors

d : weight vectors (provided we count each weight vector as many times as its multiplicity indicates, the multiplicity being defined as the number of independent eigenkets  $|m\rangle$  corresponding to a given weight m).

Note that root vectors are characteristic of the group. They are really the structure constants. The weight vectors on the other hand are characteristic of the representation. There are only  $\ell$  linearly independent roots (simple roots). There are also only  $\ell$  linearly independent weight vectors. The simplest (oblique axis) basis for the weight vectors is that provided by the simple root vectors.

All this intertwining of weights and roots is exciting enough, but still further and the more exciting result comes when we look for the analogue of the result in  $O_3$  that all weights are either integers or half-integers. The analogous result is Theorem 9, which gives the "component" of any weight-vector along a simple root-vector.

#### Theorem 9

For every weight <u>m</u>, the number  $\underline{m} \cdot \underline{r}(\alpha)/\underline{r}(\alpha) \cdot \underline{r}(\alpha)$ , where  $\underline{r}(\alpha) \in \pi$ , is an integer or a half-integer,  $\ge 0$ .

Theorem 9 provides the justification for Dynkin's insistence on simple roots as the primary entities on which all conceptual emphasis should be placed. Dynkin cares neither for the non-simple roots nor for the weight vectors. Given the simple roots, Theorem 9 tells us what the weights look like through the simplest possible generalization of the familiar results for the {3} rotation group\*. In this insistence on simple roots possibly lies the superiority of Dynkin's presentation of Lie group theory.

#### 10. IRREDUCIBLE REPRESENTATIONS AND THEIR DIMENSIONALITY

Definition: A weight <u>m</u> is said to be <u>higher</u> than <u>m</u>' if <u>m</u>-<u>m</u>' has a positive number for its first non-vanishing component in an arbitrary basis. The weight <u>A</u> which is higher than all the others is called the <u>highest</u> (or <u>greatest</u>) weight.

#### Theorem 10

A representation is uniquely characterized by its highest weight  $\underline{\Lambda}$ , and the highest weight always has multiplicity one.

$$[E_{\alpha}, \underline{H}] = \underline{I}(\alpha)E_{\alpha}$$

Note the role of  $E_{\alpha}$  as a creation operator.

<sup>\*</sup> Earlier it was mentioned that roots are differences of weights. The formal result is: If  $|\underline{m}\rangle$  is an eigenket of  $\underline{H}$  corresponding to a weight  $\underline{m}$ ,  $\underline{E}_{\alpha}|\underline{m}\rangle$  is also an eigenket with weight  $\underline{m} + \underline{r}(\alpha)$ . The result follows from

#### Theorem 11

In order that a vector  $\underline{\Lambda}$  be the highest weight of some irreducible representation, it is necessary and sufficient that  $j_{\alpha}$ , defined as  $j_{\alpha} = \underline{\Lambda} \cdot \underline{r} (\alpha) / \underline{r} (\alpha) \cdot r(\alpha)$ , is a non-negative integer or half-integer.

Thus to get the irreducible representations of any Lie group, we should mark each circle in the Dynkin diagram with a non-negative integer or halfinteger  $j_{\alpha}$ . These numbers characterize uniquely the irreducible representation with  $\underline{\Lambda}$  as its highest weight, the "components"  $\underline{\Lambda} \cdot \underline{\mathbf{r}}(\alpha)/\underline{\mathbf{r}}(\alpha) \cdot \underline{\mathbf{r}}(\alpha)$  of  $\underline{\Lambda}$  being just  $(\mathbf{j}_1, \mathbf{j}_2, \dots)$ . The dimensionality of this representation is given by the following theorem of Weyl:

#### Weyl's Theorem: Theorem 12

Let  $\Sigma_{+}$  be the system of all positive roots of a semi-simple Lie algebra, and let an irreducible representation be uniquely characterized by the highest weight  $\Lambda$ . Then its dimensionality d is given by the formula:

$$\mathbf{d} = \prod_{\underline{\mathbf{r}}} \left[ \mathbf{\alpha} \right] \in \Sigma_{+} \left[ \mathbf{1} + \underline{\Lambda} \cdot \underline{\mathbf{r}} \left( \boldsymbol{\alpha} \right) \right] / \underline{\mathbf{g}} \cdot \underline{\mathbf{r}} \left( \boldsymbol{\alpha} \right),$$

where

$$\underline{g} = \frac{1}{2} \sum_{\underline{r}(\beta) \in \Sigma_{+}} \underline{r}(\beta)$$
.

If one writes the vectors  $\underline{\Lambda}$  and  $\underline{g}$  in terms of the auxiliary quantities  $\lambda$ 's previously introduced in the third footnote of section 8,

$$\frac{\Lambda}{g} = \sum_{i=1}^{\infty} f_{i} \frac{\lambda}{\lambda_{i}},$$

The Weyl formula above gives the explicit expressions listed in Table I.

As examples consider some of the interesting physical cases, namely, the case of rank l = 2. In this case the number of commuting matrices in the algebra is two, and we can associate them, for example, with the third component of the isotopic spin and the hypercharge. The only simple compact Lie groups of rank 2 are  $A_2$ ,  $B_2$ ,  $C_2$  and  $G_2$ . Any irreducible representation of these groups can be labelled by means of two non-negative integers  $j_1$ ,  $j_2$ . The formulae for the dimensionality given in Table I can be written explicitly in a simple way and is shown in Table II.

For instance, for the simplest choices of the arrays  $j_1$ ,  $j_2$  one gets the following dimensions:

| $A_2$ : | .d(0,0) =                      | 1  | $\mathbf{B}_2 (\approx \mathbf{C}_2 \approx \mathbf{O}_5)$ | : | d(0, 0) = 1                       | G <sub>2</sub> : | d(0,0) = 1               |
|---------|--------------------------------|----|--|---|-----------------------------------|------------------|--------------------------|
|         | d(1,0) =                       | 3  | e a ta   |   | $d(\frac{1}{2}, 0) = 4$           |                  | $d(\frac{1}{2}, 0) = 7$  |
|         | $d(0, \frac{1}{2}) =$          | 3  |  |   | $d(0, \frac{1}{2}) = 5$           |                  | $d(0, \frac{1}{2}) = 14$ |
|         | d(1,0) =                       | 6  |  |   | d(1,0) = 10                       |                  | d(1,0) = 27              |
|         | $d(\frac{1}{2},\frac{1}{2}) =$ | 8  |  |   | d(0,1) = 14                       |                  |                          |
|         | $d(1, \frac{1}{2}) =$          | 15 | `  |   | $d(\frac{1}{2},\frac{1}{2}) = 16$ |                  |                          |
|         | d(1,1) =                       | 27 |  |   |                                   |                  |                          |

 $(j_i)$ 's are non-negative integers or half-integers)

| Group | N<br>number of<br>parameters | Dynkin diagrams                                 | Dimension of the irred                            | . represent.   | Expressions<br>for f and g   |
|-------|------------------------------|---|---|--|--|
|       |                              |   |   | where<br>$\alpha_{pq} = \frac{f_p - f_q}{g_p - g_q}$ | $f_{\mathbf{k}} = f_{\ell+1} + 2 \sum_{\substack{i=k \\ \ell+1}}^{\ell} j_i$ $f_{\ell+1} = -\frac{4}{\ell+1} \sum_{\substack{i=1 \\ \ell+1}}^{\ell} j_i$   |
| Ag    | g <sup>2</sup> + 2g          | ig ig-1 i1<br>0-0-00                            | π(1+ α <sub>pq</sub> )<br>p. q                    | $\beta_{pq} = \frac{f_p + f_q}{g_p + g_q}$           | $g_k = \frac{-\ell}{2} + (\ell - k + 1)$<br>$g_{\ell+1} = -\ell/2$   |
|       |                              |   | · · ·   | $\gamma_{\rm p} = f_{\rm p}/g_{\rm p}$               | Note $f_1 + \ldots + f_{\ell+1} = 0$   |
| ₿ĝ    | 2¢² + ¢                      | j <sub>g</sub> j <sub>g-1</sub> j <sub>1</sub>  | $\pi (1+\gamma_p)(1+a_{pq})(1+\beta_{pq})$<br>p.q |  | $f_{k} = j_{\ell} + 2 \frac{\ell - 1}{i = k} j_{i}$ $g_{k} = (\ell - k + \frac{1}{2})$   |
| Cg    | 2¢² + ¢                      | jg jg-1 j1                                      | n   | "  | $f_{k} = 2 \sum_{i=k}^{\ell} j_{i}$ $g_{k} = (\ell - k + \frac{1}{2})$   |
| Dg    | 2g <sup>2</sup> - g          | $j_{\ell}$<br>$j_{\ell-1}$ $j_{\ell-2}$ $j_{1}$ | $\pi (1 + \gamma_p) (1 + a_{pq})$<br>p,q          |  | $f_{\mathbf{k}} = \mathbf{j}_{\ell-1} + \mathbf{j}_{\ell} + 2 \sum_{\mathbf{k}}^{\ell-2} \mathbf{j}_{\mathbf{i}}$ $f_{\ell} = \mathbf{j}_{\ell-1} - \mathbf{j}_{\ell}, \ \mathbf{g}_{\ell} = 0, \ \mathbf{g}_{\mathbf{k}} = \ell - \mathbf{k} \text{ for } \mathbf{k} \leq \ell - 1$ |

The products here range over all possible values of p and q; the indices denoted by distinct letters must have distinct values, and of all sets of values obtained from one another by permutations of indices only one must be chosen.

187

| TABLE | [] |
|-------|----|
|-------|----|

| Group   | Number of<br>parameters N | Dimension of the $i\pi$ . rep.  |
|---|---------------------------|---|
| A <sub>2</sub>  | 8                         | $\frac{1}{27}$ (J <sub>1</sub> ) (J <sub>2</sub> ) [J <sub>1</sub> + J <sub>2</sub> ]                                     |
| $\left. \begin{array}{c} B_2 \\ C_2 \end{array} \right\}$ | 10                        | $\frac{1}{3!}$ (J <sub>1</sub> ) (J <sub>2</sub> ) [J <sub>1</sub> + J <sub>2</sub> ] (2J <sub>2</sub> + J <sub>1</sub> ) |
| G2  | 14                        | $\frac{1}{51} (J_1) (J_2) (J_1 + J_2) [2J_2 + J_1] \times [3J_2 + J_1] [3J_2 + 2J_2]$                                     |
|   |                           |   |

{ Note: Here  $J_1 = (2j_1 + 1)$  and  $J_2 = (2j_2 + 1)$  }

These numbers  $d(j_{1}, j_{2})^{*}$  represent the number of particles which can be accomodated in any given multiplet in physical applications.

The <u>adjoint</u> (or <u>regular</u>) representation R plays a very important role in vector meson theories. For the case of l = 2, these representations are the following:

> $A_2 : d_R = d(\frac{1}{2}, \frac{1}{2}) = 8,$   $B_2(C_2): d_R = d(1, 0) = 10,$  $G_2 : d_R = d(0, \frac{1}{2}) = 14.$

These groups, therefore, can accommodate 8, 10 and 14 vector gauge mesons respectively if these mesons correspond to the adjoint representation.

# 11. COMPUTATION OF ALL WEIGHTS OF A GIVEN IRREDUCIBLE REPRESENTATION

Notwithstanding the fact that the greatest weight uniquely characterizes an irreducible representation, it is important for physical applications to be able to compute all the weights of an irreducible representation. Later we shall construct weight diagrams for some irreducible representation of low dimensionality for the case of rank 2 groups ( $A_2$ ,  $B_2$ ,  $C_2$ ,  $G_2$ ). In contrast to the root diagrams, the weight diagrams are directly of physical interest.

An explicit method to calculate all the weights in terms of the highest weight and the simple roots is given by the next theorem. We have learnt earlier that the roots equal differences of weights.

<sup>\*</sup> I have introduced a small change of notation in the labelling of representations. Dynkin and Behrends et al. label irreducible representations with numbers  $a_1, a_2, \ldots, a_\ell$  where  $a_i$  are (non-negative) integers. I have used for labelling the numbers  $j_1, j_2, \ldots, j_\ell$  where the j's are (non-negative) integers or half-integers. The new notation possibly brings out still more the fact that a general Lie group of rank  $\ell$  is a simple "generalization" of  $O_3$  and has  $\ell$  distinct "angular momenta"  $j_1, j_2, \ldots, j_\ell$  rather than just one  $(j_1)$ .

Let  $\underline{\Lambda}$  and W be the highest weight and the set of all weights respectively of a given irreducible representation.

An element  $\underline{m} \in W$  is said to belong to the layer  $\Delta^{(k)}$  if it can be obtained by subtracting K simple roots from  $\underline{\Lambda}$ . Clearly  $\Delta^{(0)}$  consists only of  $\Lambda$ , and

$$W = \Delta^{(0)} \cup \Delta^{(1)} \cup \Delta^{(2)} \dots$$

Note that all the layers are disjointed.

Theorem 13

Every element  $m^{(k)} \in \Delta^{(k)}$  can be expressed as

$$m^{(k)} = m^{(k-1)} - r(\alpha),$$

where

 $m^{(k-1)} \in \Delta^{(k-1)}$ 

and

$$\underline{\mathbf{r}}(\alpha) \in \pi$$
.

However, if  $\underline{\mathbf{m}^{(k-1)}}$  belongs to  $\Delta^{(k)}$  and  $\underline{\mathbf{r}}(\alpha)$  is an arbitrary simple root, the difference  $\underline{\mathbf{m}^{(k-1)}} - \underline{\mathbf{r}}(\alpha) \in \Delta^{(k)}$  if and only if the following condition is satisfied:

$$2 m^{(k-1)} \cdot r(\alpha)/r(\alpha) \cdot r(\alpha) + Q > 0,$$

where the number Q is defined by the requirements:

$$\underline{\mathbf{m}}^{(k-1)} + q \underline{\mathbf{r}}(\alpha) \in \mathbb{W} \text{ for } q \leq \mathbf{Q},$$
$$\mathbf{m}^{(k-1)} + q \mathbf{r}(\alpha) \in \mathbb{W} \text{ for } q = \mathbf{Q} + 1.$$

Example:

Perhaps the best way to show that the theorem is actually quite harmless and simple in practice is to construct the weights for a specific case. Consider the group  $A_2 \approx SU_3$  for which  $\ell = 2$ . The Dynkin diagram is O—O. The  $\pi$ -space is two-dimensional; and if we call the roots  $\alpha$  and  $\beta$ , the diagram tells us that their lengths are equal ( $|\alpha|^2 = |\beta|^2$ ) and the angle between them is 120° so that

$$\underline{\alpha} \cdot \beta / \underline{\alpha} \cdot \underline{\alpha} = -\frac{1}{2}.$$

Consider now the regular representation  $(\frac{1}{2}, \frac{1}{2})$ . The dimensionality in this case is d = 8, so that the representation could accommodate 8 particles. The "components" of the highest weight  $\underline{\Lambda}$  (ie)j<sub> $\alpha$ </sub>, j<sub> $\beta$ </sub> are given by

$$j_{\alpha} = \underline{\Lambda} \cdot \underline{\alpha} / \underline{\alpha} \cdot \underline{\alpha} = \frac{1}{2}, \qquad (11.1)$$

$$j_{\beta} = \underline{\Lambda} \cdot \underline{\beta} / \underline{\beta} \cdot \underline{\beta} = \frac{1}{2}.$$
(11.2)

Noticing that  $\underline{\alpha}$  and  $\underline{\beta}$  do not form an orthogonal basis, we find from (11.1) and (11.2) that

$$\underline{\Lambda} = \underline{\alpha} + \beta.$$

Now using Theorem 13, if we are given an arbitrary weight M and we wish to know whether  $\underline{M} - \underline{\alpha}$  is a possible weight or not, we proceed as follows:

Write the series  $\underline{M}$ ,  $\underline{M} + \alpha$ ,  $\underline{M} + 2\alpha$ ,  $\dots$ ,  $\underline{M} + (Q+1)\alpha$  where Q is an integer. The series terminates for a Q defined by the requirement that while  $\underline{M}$ ,  $\underline{M} + \alpha$ ,  $\dots$ ,  $\underline{M} + Q\alpha$  are weights,  $\underline{M} + (Q+1)\alpha$  is not a weight. Now compute the number,

$$\mathbf{Q} + \mathbf{M}_{\alpha}$$
 where  $\mathbf{M}_{\alpha} = 2 \mathbf{M} \cdot \alpha / \alpha \cdot \alpha$ .

If  $M_{\alpha} + Q > 0$ , then  $\underline{M}-\underline{\alpha}$  is a weight; otherwise it is not. In starting this procedure the crucial point to remember is that  $\underline{\Lambda} + \underline{\alpha}$  where  $\underline{\alpha}$  is a simple root is never a possible weight.

Consider now the case when  $\underline{M} = \underline{\Lambda}$ . Since  $\underline{\Lambda} + \underline{\alpha}$  is not a weight, Q = 0. Since

$$\Lambda_{\alpha} = \underline{\Lambda} \cdot \underline{\alpha} / \underline{\alpha} \cdot \underline{\alpha} = \mathbf{j}_{\alpha} > 0, \qquad (11.3)$$

we see from (11.3) that  $\underline{\Lambda} - \underline{\alpha}$  is indeed a weight. Likewise, since  $j_{\beta} > 0$ ,  $\Lambda - \beta$  is also a weight.

We can now start with  $(\underline{\Lambda} - \underline{\alpha})$  and test if  $(\underline{\Lambda} - \underline{\alpha}) - \underline{\alpha}$  and  $(\underline{\Lambda} - \underline{\alpha}) - \underline{\beta}$  are possible weights or not. It is easy to see that  $\underline{\Lambda} - 2\underline{\alpha}$  is not a weight, but  $\underline{\Lambda} - \underline{\alpha} - \beta$  is. Proceeding in this fashion, we find that all possible weights are given by the diagram shown in Fig. 3.



Fig. 3

Notice that the weight  $\underline{\Lambda} - \underline{\alpha} - \underline{\beta}$  is of multiplicity two. The diagram does not further fan out, and we obtain a totality of eight weights. Writing  $\Lambda = \underline{\alpha} + \underline{\beta}$ , we have the following system of weights:

$$\underline{\alpha} + \beta, \ \underline{\alpha}, \ \overline{\beta}, \ 0, \ 0, \ -\beta, \ -\underline{\alpha}, \ -(\underline{\alpha} + \beta).$$
 (11.4)

The multiplicities are spindle-shaped: they increase, come to a maximum and decrease again. (The weight zero has multiplicity two.) This is a general result which will not be discussed further.



Euclidean diagrams

Fig.4 gives the Euclidean diagram of these weights. The two rings in the centre indicate the two zero weights. A tentative identification of the stable baryons with the appropriate weights has also been made in the figure, provided we identify

$$m_1 = I_3,$$
  
 $m_2 = (2/\sqrt{3})U_1$ 

where  $\underline{\mathbf{m}} = \binom{\mathbf{m}_{1}}{\mathbf{m}_{1}}$  in a Euclidean basis.

For illustrative purposes, here are some more weight diagrams corresponding to the representations [4] shown in Fig. 5.

Before concluding this section we state one important theorem and make one final remark.

#### Theorem 14

For the adjoint representation, the root vectors and the non-zero weight vectors coincide. The weight zero occurs with a multiplicity equal to the rank of the group.

An illustration of this theorem is given by the weight diagram of the  $(\frac{1}{2}, \frac{1}{2})$  representation of SU<sub>3</sub> computed earlier in this section. Because of



Fig. 5

this rather remarkable property clearly the adjoint representation has a greater claim to attention than any other. Remark

In  $O_3$ , the eigenvalues of  $J_3$  (the weights) are non-degenerate for any given representation and hence suffice to label the representation. For general Lie groups, except for the highest weight, all others may possess multiplicities of > 1 (compare the weight (0, 0) for  $SU_3$  which has multiplicity 2). If the multiplicity is > 1 we need additional operators all commuting with each other and with the  $H^1$ s, whose eigenvalues will enable us to remove the degeneracy and label uniquely the eigenvectors of the Hs, belonging to the same given weight. (A Casimir operator which has the same eigenvalue for all vectors of a given representation is clearly useless for this purpose.) The number of extra operators needed can be shown to equal  $(N-\ell)/2-\ell$ =  $(N-3\ell)/2$ . For O<sub>3</sub>, N = 3,  $\ell$  = 1 so that no extra operator is needed to characterize all the eigenkets of  $J_3$  in a representation specified (uniquely) by the highest weight j. For SU<sub>3</sub>, however, N = 8, l = 2 so that we need one more operator besides  $I_3$  and U to label uniquely the eigenkets of  $I_3$  and U. It is not hard to show that in this case such an operator is given by  $I^2$ . For  $C_2$ , (N-3l)/2 = 2. Thus, even additional to  $L^2$  (and U and  $I_3$ ), one more quantum number is needed to form a complete set of commuting observables. For  $G_{0}$ ,  $(N-3\ell)/2 = 4$ .

#### 12. REDUCIBLE REPRESENTATIONS

Let us take stock of the situation. For a physicist working in symmetry problems, the information necessary for progress is the following:

- (i) Classification of irreducible representation for a group of rank  $\ell$ . We possess a complete solution of this problem.
- (ii) The eigenvalues of the commuting operators  $H_1, \ldots, H_l$ . This is the same problem as the problem of determination of weights. Again we possess a complete solution of this.
- (iii) Determination of the extra  $(N-3\ell)/2$  operators to enable a unique labelling of the eigenkets of  $H_1, \ldots, H_\ell$ . For groups like  $A_2, B_2$ ,  $C_2$ ,  $D_2$  we know how to construct such operators but a general systematic procedure apparently is not known.
- (iv) The reduction of a reducible representation into the direct sum of irreducible representations. There are two parts of this problem: first, finding out which irreducible representations make their appearance in this direct sum; second, to find the Clebsch-Gordon coefficients. Theorem 15 will give the procedure for solving the first problem. The second problem will be dealt with by Ruegg and Goldberg in their lectures for some special (fortunately for the physicist, extremely important) cases. No general solution however exists.

First, some obvious definitions:

#### Kronecker products

If  $R_1$ ,  $R_2$ ,  $R_3$  are three linear spaces of dimensions m, n and mn respectively, we shall say  $R_3$  is the Kronecker product of  $R_1$  and  $R_2$  ( $R_3$ =  $R_1 \times R_2$ ) provided to every vector  $|\xi_1 \rangle \in R_1$ ,  $|\xi_2 \rangle \in R_2$ , there corresponds a vector  $|\xi_3 \rangle \in R_3$  (notation  $|\xi_3 \rangle = |\xi_1 \rangle \times |\xi_2 \rangle$ ) such that: (i) The operation  $|\xi_1 \rangle \times |\xi_2 \rangle$  is linear in each argument;

(ii)  $R_3$  is spanned by vectors of the form  $|\xi_1\rangle \times |\xi_2\rangle$ . If  $\phi_1$  and  $\phi_2$  are linear representations of a Lie algebra operating in  $R_1$ and  $R_2$ , the representation  $\phi_3$  defined in  $R_1 \times R_2$  by the formula,

 $\phi_{2} \left\{ \left| \mathbf{\xi}_{1} \right\rangle \times \left| \mathbf{\xi}_{2} \right\rangle \right\} = \left\{ \phi_{1} \left| \mathbf{\xi}_{1} \right\rangle \right\} \times \left| \mathbf{\xi}_{2} \right\rangle + \left| \mathbf{\xi}_{1} \right\rangle \times \left\{ \phi_{2} \left| \mathbf{\xi}_{2} \right\rangle \right\},$ 

is called the Kronecker product of  $\phi_1$  and  $\phi_2$  and will be denoted as

$$\phi_3 = \phi_1 \times \phi_2 .$$

Theorem 15

- (i) Addition of weights
  - If  $\Delta_{\varphi_i}$  is the weight space of  $\phi_1$  and  $\Delta_{\varphi_i}$  is the weight space of the
- representation  $\phi_2$ , then  $\Delta_{\phi_3} = \Delta_{\phi_1} + \overline{\Delta}_{\phi_2}$ . (ii) If  $\underline{\Lambda}_1$  and  $\underline{\Lambda}_2$  are the greatest weights of  $\phi_1$  and  $\phi_2$ , the greatest weight of  $\phi_3$  is  $\underline{\Lambda}_1 + \underline{\Lambda}_2$ .

This theorem is an obvious generalization of the addition theorem for angular momenta in  $O_3$  which we consider in detail. If  $j_1$  and  $j_2$  are the highest weights of two irreducible representations  $\phi(j_1)$  and  $\phi(j_2)$ , the (reducible) product representation has the highest weight  $j_1 + j_2$ . Also the totality of its weights is given by

Weight 
$$\rightarrow j_1 + j_2, j_1 + j_2 - 1, j_1 + j_2 - 2, ..., - j_1 - j_2$$
  
multi-  
plicity  $\rightarrow 1, 2, 3, ..., 1$ 

The multiplicities are easily deduced. For example,  $j_1 + j_2 - 1$  arises in two ways: either as the sum  $j_1 + (j_2 - 1)$  or equally as the sum of the weights  $(j_1-1) + j_2$ . The usual procedure to find the irreducible representations contained in  $\phi$  (j<sub>1</sub>)  $\times \phi$  (j<sub>2</sub>) can be stated thus: Take away from the totality of weights those which belong to the representation  $\phi$  (j<sub>1</sub> + j<sub>2</sub>). Among the remaining weights occurs the weight  $j_1 + j_2 - 1$  with unit multiplicity. Clearly this must be the highest weight of the representation  $\phi$  (j<sub>1</sub> + j<sub>2</sub> -1) which therefore must also be contained in  $\phi$   $(j_1) \times \phi$   $(j_2)$ . Taking away all the weights belonging to  $\phi$  (j<sub>1</sub>+j<sub>2</sub>-1), we next identify the occurrence of  $\phi$  (j<sub>1</sub> + j<sub>2</sub> - 2) in the direct sum from the fact that the highest weight left is  $(j_1 + j_2 - 2)$ . This procedure is continued till we reach  $\phi$  ( $|j_1 - j_2|$ ). At this stage all weights are exhausted, leading to the inference that

$$\phi(j_1) \times \phi(j_2) = \phi(j_1 + j_2) + \phi(j_1 + j_2 - 1) + \dots + \phi(|j_1 - j_2|).$$

The procedure is obviously completely general. Its only drawback is that in order to apply it we need to know all the weights. A simpler version has been developed by Racah, Speiser and Ruegg where, if  $j_1 \ge j_2$ , one adds all weights belonging to the representation  $\phi(j_2)(i.e.j_2, j_2-1, ..., -j_2)$  to the highest weight  $j_1$  of  $\phi(j_1)$ . For O<sub>3</sub>, the resulting weights are clearly the highest weights of the irreducible representations contained in  $\phi(j_1) \times \phi(j_2)$ . For the more general cases this sum may lead to a certain number of negative weights which certainly cannot qualify as highest weights. These then have to be excluded, and the procedure for this is explained in Ruegg's lecture.

#### Cartan composition

If  $\phi_1$  and  $\phi_2$  are two irreducible representations, the Kronecker product  $\phi_1 imes \phi_2$  is in general a reducible representation. Consider its greatest com-

194

ponent,  $\overline{\phi_1 \times \phi_2}$ . This is an irreducible representation with the highest weight  $\underline{\Lambda_1} + \underline{\Lambda_2}$ . The operation of Kronecker multiplication of two irreducible representations followed by the operation of isolating the greatest component lead to the formation of a new irreducible representation  $(\overline{\phi_1 \times \phi_2})$  and is called the cartan composition of irreducible representations.

Those irreducible representations of an algebra which cannot be obtained from other irreducible representations are called basic representations by Cartan. These representations are characterized by the fact that their highest weights cannot be split into the sums of two elements that are themselves highest weights. Clearly a representation  $\phi$  is basic if, and only if, all the labelling numbers  $j_1$ ,  $j_2$ , ...,  $j_\ell$  are zero except one which equals  $\frac{1}{2}$ . Thus every simple algebra of rank  $\ell$  has  $\ell$  basic representations.

One can go further and show that all basic representations themselves can be constituted from a few so-called elementary representations by Kronecker multiplications followed by an antisymmetrization procedure which is somewhat familiar in ordinary tensor theory and will not be described here in detail. For  $A_{\ell}$  and  $B_{\ell}$  there are just two elementary representations.  $C_{\ell}$  has one elementary representation and  $D_{\ell}$  has three. One of the elementary representations  $\phi$  of  $A_{\ell}$  is realized as the group  $SL(\ell+1)$  of all matrices of order  $\ell+1$  with determinant + 1, the other being given by

$$\phi' = - \left[\phi_1\right]^{\mathrm{T}}.$$

For  $B_{\ell}$ , one of the elementary representations is obtained by considering the group  $O(2\ell + 1)$  of all unimodular orthogonal transformations of the  $(2\ell+1)$  dimensional space, while the second elementary representation is the so-called spinor representation. The realization of the group  $C_{\ell}$  in the form of the group Sp(2n) of the symplectic matrices of order  $2\ell$  gives its elementary representation, while for  $D_{\ell}$  ( $\ell \ge 5$ ) one elementary representation is given by the group of unimodular orthogonal matrices of order  $2\ell$ and in addition there are two distinct spinor representations. For the elementary representations of the exceptional groups reference may be made to Dynkin.

This brief description of the results in representation theory does not even touch the practical problem of reduction of representation in the manner the physicist wants it solved. For this we must fall back on our amateur methods, multiplying matrices, symmetrizing and antisymmetrizing tensor indices, though perhaps somewhat emboldened by the knowledge that this is also the entire, and when I say entire - I mean entire, stock-in-trade of the professional group theorist.

#### REFERENCES

- [1] PAULI W., "Continuous groups in quantum mechanics", lecture notes (CERN-31).
- [2] RACAH, G., "Group theory and spectroscopy", Institute for Advanced Studies lecture notes (1951)(reprinted; CERN 61-8).
- [3] DYNKIN, E.B., Amer. Math. Soc., Transl. II Vol.6, "Maximal sub-groups of the classical groups", Appendix.

#### A. SALAM

- [4] BEHRENDS, R.E., DREITLEIN, J., FRONSDAL, C. and LEE, B.W., Rev. Mod. Phys. 34 (1962) 1.
- [5] SPEISER, D. and TARKSI, J., "Possible schemes for global symmetry", Princeton preprint (1961).
- [6] NE'EMAN, Y., "Gauges group and an invariant theory of strong interactions" (1961); Thesis, Imperial College, London (1962).
- [7] IONIDES, P., Thesis, Imperial College, London (1962).

# CONSEQUENCES OF HIGHER SYMMETRY FOR ELECTROMAGNETIC AND FOR WEAK TRANSITIONS

#### R. GATTO

#### NATIONAL LABORATORY OF FRASCATI, FRASCATI, ITALY

In my lecture I shall review some work that has been done on the consequences of higher symmetry schemes for electromagnetic and for weak transitions. The higher symmetry schemes I shall consider will be those based on the simple compact Lie groups of rank two. The work on the subject that I shall review has been done by RUEGG [1], by COLEMAN and GLASHOW [2] and by CABIBBO and GATTO [3]. For the group theory concepts that will be used here we refer to SALAM's lectures [4] and to the article by BEHRENDS, DREITLEIM, FRONSDAL and LEE [5].

#### 1. GROUPS

We shall confine our attention to models based on the simple compact Lie groups of rank two. From the theory of Lie groups we know that there are four such groups (two of which are isomorphic to each other). They are: - SU(3), the special unitary group in 3 dimensions. It is called  $A_2$  in Cartan's notations. It is the group of all unitary unimodular matrices in complex 3-dimensional space. The order of SU(3) is 8.

-  $G_2$  is one of the so-called exceptional groups. It is a subgroup of  $O_7$  (for its characterization see reference [5], p. 26). The order of  $G_2$  is 14. -  $B_2$  is the orthogonal group  $O_5$  in five dimensions. Its order is 10.

-  $C_2$  is isomorphic to  $B_2$ . It is the group of unitary matrices in 4 dimensions that leave a non-singular antisymmetric matrix invariant. Its order is 10.

To define a model one has to decide on the assignment of the various particles (baryons and mesons) to particular representations of the group. Thus many different models can, in principle, be constructed for each of the above group, depending on the way one assigns the particles to the representations of the group.

We now review the simplest models one can construct.

#### 2. MODELS

(a) SU(3). We first consider the simplest models based on SU(3), viz. the Sakata model and the Gell-Mann-Ne'eman model.

The Sakata Model - based on the following assignment

D<sup>3</sup>(1,0): p, n, Λ, D<sup>8</sup>(1,1): mesons (π, K,  $\chi$ ).

[The irreducible representations are, as usual, denoted by  $D^n$  (a<sub>1</sub>, a<sub>2</sub>) where

n is the dimension of the representation and  $a_1$ ,  $a_2$  are non-negative integers such that the highest weight of the representation can be written as a linear combination with coefficients  $a_1$  and  $a_2$  of the two fundamental dominant weights of the group.] The weight diagram for the representation  $D^3$  (1,0) is shown in Fig.1.



Fig. 1

Weight diagram for the representation D<sup>3</sup>(1,0)

In this diagram H<sub>1</sub> and H<sub>2</sub> are the two commuting infinitesimal generators of the group. Each vector in the weight diagram is a simultaneous eigenvector of the commuting operators  $H_1$  and  $H_2$  with eigenvalues as indicated by the co-ordinates of its end-point. A convenient change of scale has been made by reporting on the co-ordinate axes the eigenvalues of  $\sqrt{3}$  H<sub>1</sub> and of 2 H<sub>2</sub> rather than those of H<sub>1</sub> and H<sub>2</sub>. In this way one sees that  $\sqrt{3}$  H<sub>1</sub> can be identified with I<sub>3</sub> (the third component of isotopic spin) and particles can be assigned to each weight vector in a definite way. One then checks that  $2H_2$  can be related by hyperchange Y by the relation  $2H_2 = Y - 2/3$ . In this way one finds the relation, valid for the particular model, between the two conserved quantum numbers of the theory,  $I_3$  and Y, and the two commuting operators  $H_1$  and  $H_2$  of the (rank two) group. Note that Y and  $I_3$  are not, in the Sakata model that we are considering, simply multiples of  $H_1$  and  $H_2$ , but the relation between Y and  $H_2$  is inhomogeneous. This circumstance is quite peculiar of the Sakata model and will not occur in the models we shall consider in the following. The relation between  $I_3$ , Y and  $H_1$ ,  $H_2$ , can only be homogeneous if  $\Lambda^0$  and  $\Sigma^{\circ}$  (which have Y = I<sub>3</sub> = 0) both lie at the centres of the weight diagrams for the representations to which they belong. In the Sakata model mesons belong to D<sup>8</sup>(1,1), whose weight diagram we shall discuss in connection with the eight-fold way. We also report the multiplication rules:

> $3 \times 3 = 1 + 8$ ,  $8 \times 8 = 1 + 8 + 8 + 10 + 10 + 27$ .

We have put in a little square the regular representation. The eightfold way of Gell-Mann and Ne'eman is based on the assignments

- $D^8(1,1)$  : baryons  $(N,\Lambda,\Sigma,\Xi)$ ,
- $D^{8}(1,1)$  : mesons  $(\pi, K, x)$ .

The weight diagram for  $D^8$  (1,1) is shown in Fig. 2.



Weight diagram for the representation D<sup>8</sup> (1,1)

On the side of each vector we have reported the corresponding baryon and in brackets the corresponding meson. The multiplication rule is, of course, as before

$$8 \times 8 = 1 + 8 + 8 + 10 + 10 + 27$$
.

It is important to note that the regular representation 8 occurs twice in  $8 \times 8$ .

(b).  $G_2$  The model based on  $G_2$  that we shall consider is based on the following assignment :

| $D^{2}(0,0)$         | : | Λ,                  |
|----------------------|---|---------------------|
| D <sup>7</sup> (1,0) | : | Ν,Σ,Ξ,              |
| $D^{7}(1,0)$         | : | mesons $(\pi, K)$ . |

The weight diagram for  $D^7$  (1,0) is shown in Fig. 3.

4



Weight diagram for the representation D7(1,0)

The multiplication rules are

$$1 \times 1 = 1$$
,  $1 \times 7 = 7$ ,  
 $7 \times 7 = 1 + 7 + 14 + 27$ .

In the product  $7 \times 7$  the regular representation  $\mathbb{I}_{4}$  occurs only once. It does not occur in the other products.

(c)  $B_2$ . The model based on  $B_2$  is the following:

D<sup>1</sup> (0,0) :  $\Lambda$ , D<sup>4</sup> (1,0) : N  $\Xi$ , D<sup>5</sup> (0,1) :  $\Sigma \chi$  (a new baryon), D<sup>4</sup> (1,0) : K, D<sup>5</sup> (0,1) :  $\pi \times$  (a new meson).

The weight diagram for  $D^4(1,0)$  is shown in Fig. 4, and that for  $D^5(0,1)$ 





Weight diagram for the representation D4(1,0)

is shown in Fig. 5. The multiplication rules that one needs are

$$1 \times 4 = 4,$$
  
 $1 \times 5 = 5,$   
 $4 \times 4 = 1 + 5 + 10,$   
 $4 \times 5 = 4 + 16,$   
 $5 \times 5 = 1 + 10 + 14.$ 

We see that the regular representation is only contained once in  $4\times4$  and  $5\times5.$ 

(d) C<sub>2</sub>. We consider two models based on C<sub>2</sub>. We call the first model  $[C_2]_I$  and the second  $[C_2]_{II}$ .

200



Weight diagram for the representation  $D^{5}(1,0)$ 

 $[C_2]_I$  is based on the following assignment:

| $D^{5}(0,1)$          | : | Λ, Ν, Ξ,                    |
|-----------------------|---|-----------------------------|
| $D^{10}(2,0)$         | : | $\Sigma$ + other baryons,   |
| D <sup>10</sup> (2,0) | : | $\pi$ , K, D (a new meson). |

The weight diagram for  $D^5(0,1)$  is shown in Fig.6. The weight diagram for





Weight diagram for the representation D<sup>5</sup>(0,1)

 $D^{10}(2,0)$  is shown in Fig. 7. The relevant multiplication rules are

$$5 \times 5 = 1 + 10 + 14,$$
  
 $5 \times 10 = 5 + 10 + 35,$   
 $10 \times 10 = 1 + 5 + 10 + 14 + 35 + 35'.$ 

Again the regular representation is only contained once.



Weight diagram for the representation D10(2,0)

 $[C_2]_{\pi}$  is instead based on the following assignment:

| $D^{10}(2,0)$   | : | baryons, |
|-----------------|---|----------|
| $D^{(10)}(2,0)$ | : | mesons.  |

The assignment of baryons to  $D^{(10)}$  (2,0) is obvious and can be read from the weight diagram that we gave for  $D^{(10)}$  (2,0). Two new baryons are required to fit the scheme. The relevant multiplication rule is 10 × 10 given above.

#### 3. ELECTROMAGNETIC FORM FACTORS

The electromagnetic vertex  $\langle A | j_{\mu} | A \rangle$  where A is a baryon or a meson and  $j_{\mu}$  is the electromagnetic current operator which can be expressed in terms of the form factors of A:

 $\langle A | j_{\mu} | A \rangle = \overline{u}_{A} [F_{1} (k^{2}) \gamma_{\mu} \Omega + ...] u_{B'}$ 

The matrix  $\Omega$  in the above equation depends on the particular group-theoretical model.

For all the models that we have discussed, except for those based on SU(3), it is very easy to derive the conditions that the group symmetry imposes on the form factors. In fact for the models that we called  $G_2$ ,  $B_2$ ,  $[C_2]_1$  and  $[C_2]_{II}$  the following circumstance holds:  $I_3$  and Y are multiples of  $H_2$  and  $H_1$ . Therefore the charge  $Q = I_3 + \frac{1}{2}$  Y is a linear homogeneous function of  $H_1$  and  $H_2$ . Therefore a realization of  $\Omega$  is Q itself. But it is also the only possible realization since for  $G_2$ ,  $B_2$ ,  $[C_2]_1$  and  $[C_2]_{II}$  the regular representation is only contained once (at most) in the product  $D^n \times D^n$  where  $D^n$  is one of the representations employed to describe the particles. It follows that all positively charged particles have the same form factor, all negatively charged particles the same form factor (equal and opposite to that of the

positively charged ones) and all neutral particles have form factors of zero. It is easy to generalize these remarks by including also transition matrix elements such as, for instance, that responsible for  $\Sigma^0 \rightarrow \Lambda^0 + \gamma$ . Then in  $G_2$  such a matrix element is zero, in  $B_2$  it is also zero, while in  $[C_2]_1$  it can be different from zero (and in fact it will in general be different from zero) and in  $[C_2]_{II}$  is again zero by the general argument given before.

We next discuss the models based on SU(3) which need a more detailed discussion. It is known from the general theory of Lie algebras that there exists a choice of the group generators  $F_m$  such that commutation relations are

$$[F_m, F_n] = i f_{mn} F_n$$

with  $f_{mn\ell}$  real and completely antisymmetric. For SU(3) the  $f_{mn\ell}$  are as follows [6]:

$$f_{123} = 1,$$
  

$$f_{147} = f_{246} = f_{257} = f_{345} = f_{516} = f_{367} = 1/2,$$
  

$$f_{458} = f_{678} = \sqrt{3}/2.$$

and the remaining ones can be obtained from the antisymmetry requirement. Since  $f_{22} = 1$ , one has the commutation relations

$$[F_1, F_2] = i F_3,$$
  
 $[F_2, F_3] = i F_1,$   
 $[F_3, F_1] = i F_2,$ 

suggesting that  $F_1$ ,  $F_2$ ,  $F_3$  are to be interpreted as  $I_1$ ,  $I_2$  and  $I_3$ . Next one looks for  $F_n$  that commutes with  $F_3$ . From

$$[F_3, F_n] = if_{3n}, F_n = 0$$

and from the values of the  $f_{mn\ell}$  it follows that only  $F_8$  commutes with  $F_3$ . Thus  $F_3$  and  $F_8$  are the two commuting elements of the Lie algebra. For the physical interpretation of  $F_8$  one has to specify the model.

We first consider the eight-fold way. From the weight diagram for  $D^{8}(1,1)$  of SU(3) (see section 2) we learn that

$$\sqrt{3} H_1 = F_3 = I_3, 2 H_2 = (2/\sqrt{3}) F_8 = Y.$$

In analogy with

$$A = I_{q} + \frac{1}{2} Y$$

the electromagnetic current is given by

$$j = j_3 + (1/\sqrt{3}) j_2$$

where the currents j<sub>m</sub> satisfy

$$[F_{m}, j_{n}(x)] = i f_{mn\ell} j_{\ell}(x),$$

i.e. they belong to the 8-dimensional (regular) representation. The group generator  $F_m$  are the space integrals of  $j_m$ :

$$\mathbf{F}_{\mathbf{m}} = \int d\sigma_{\mu} j_{\mathbf{m}}^{(\mu)}(\mathbf{x}).$$

Now it is easy to see that  $F_3$ ,  $F_8$ ,  $F_6$  and  $F_7$  commute with j. It is obvious that  $F_3$  and  $F_8$  commute with j, and the only physical implication of this fact is that j conserves both  $I_3$  and Y. It can also easily be seen that

$$[F_6, j] = [F_6, j_3] + 1/\sqrt{3} [F_6, j_8] = 0,$$

using the values for  $f_{mn\ell}$ ; and similarly

 $[F_7, j] = 0.$ 

The physical implications of the conservation of  $F_7$  are the same as for the conservation of  $F_6$ , so we shall only consider these last ones. From

$$[F_{e}, J(j)] = 0,$$

where J(j) is any function of j, such as for instance a retarded product etc., we have

$$\langle A | [F_6, J(j)] | B \rangle = 0$$

or

$$\langle 0 | A[F_6, J(j)] B^+ | 0 \rangle = 0,$$

where  $A^{+}$  and  $B^{+}$  are the creation operators of the states  $|A\rangle$  and  $|B\rangle$ .

Now for the vacuum

 $F_m \mid 0 > = 0$ 

since the vacuum state is assumed to be invariant under the group. So we can write the above equation as

$$\langle 0 | [A, F_6] J(j) | B \rangle = \langle A | J(j) [F_6, B^+] | 0 \rangle.$$

We specialize A and B to be one-particle states (baryons or mesons). In the eight-fold way both baryons and mesons belong to the regular eightdimensional representation. By a suitable choice of the A's one thus has

$$[F_m, A] = if_{mAC} C.$$

Thus we find

 $f_{AbC} \langle C \mid J(j) \mid B \rangle = f_{BbC} \langle A \mid J(j) \mid C \rangle,$ 

which constitute a set of identities to be satisfied by the matrix elements of J (j) between one-particle states.

By specializing the above result one finds a number of consequences of which we list some:

For mesons:

The  $K^0$  (or  $\overline{K}^0$ ) form factor is identically zero; The  $K^4$  (K<sup>-</sup>) form factor is identical to the  $\pi^+$  ( $\pi^-$ ) form factor; The Compton effect matrix elements satisfy

 $\langle \mathbf{K}^{+} | \mathbf{j}(\mathbf{x}) \mathbf{j}(\mathbf{x}') | \mathbf{K}^{+} \rangle = \langle \pi^{+} | \mathbf{j}(\mathbf{x}) \mathbf{j}(\mathbf{x}') | \pi^{+} \rangle,$ 

 $\sqrt{3}\langle \mathbf{K}^{0} | \mathbf{j}(\mathbf{x}) \mathbf{j}(\mathbf{x}') | \mathbf{K}^{0} \rangle = \langle \mathbf{X}^{0} | \mathbf{j}(\mathbf{x}) \mathbf{j}(\mathbf{x}') | \pi^{0} \rangle - \sqrt{3} \langle \mathbf{X}^{0} | \mathbf{j}(\mathbf{x}) \mathbf{j}(\mathbf{x}') | \mathbf{X}^{0} \rangle,$ 

 $-\langle \mathbf{K}^{0} | \mathbf{j}(\mathbf{x}) \mathbf{j}(\mathbf{x}') | \mathbf{K}^{0} \rangle = \langle \pi^{0} | \mathbf{j}(\mathbf{x}) \mathbf{j}(\mathbf{x}') | \pi^{0} \rangle - \sqrt{3} \langle \pi^{0} | \mathbf{j}(\mathbf{x}) \mathbf{j}(\mathbf{x}') | \chi^{0} \rangle.$ 

For the  $2\gamma$  decay modes:

(amplitude for 
$$\chi^0 \rightarrow 2\gamma$$
) =  $(1/\sqrt{3})$  (amplitude for  $\pi^0 \rightarrow 2\gamma$ )

This last equation will probably be useful in connection with the recent experiments that indicate a large branching ratio for  $\chi^0 \rightarrow 2\gamma$ .

For baryons:

$$\begin{array}{c|c} \langle \Sigma^{+} \mid j \dots j \mid \Sigma^{+} \rangle = \langle \mathbf{p} \mid j \dots \mid \mathbf{p} \rangle, \\ \langle \Sigma^{-} \mid j \dots j \mid \Sigma^{-} \rangle = \langle \Xi^{-} \mid j \dots j \mid \Xi^{-} \rangle, \\ \langle \Xi^{0} \mid j \dots j \mid \Xi^{0} \rangle = \langle \mathbf{n} \mid j \dots j \mid \mathbf{n} \rangle, \\ - 1/\sqrt{3} \langle \Sigma^{0} \mid j \dots j \mid \Lambda \rangle = \langle \mathbf{n} \mid j \dots j \mid \mathbf{n} \rangle - \langle \Lambda \mid j \dots j \mid \Lambda \rangle, \\ - \sqrt{3} \langle \Lambda \mid j \dots j \mid \Sigma^{0} \rangle = \langle \mathbf{n} \mid j \dots j \mid \mathbf{n} \rangle - \langle \Sigma^{0} \mid j \dots j \mid \Sigma^{0} \rangle, \end{array}$$

where we have denoted briefly by  $j \dots j$  a product  $j(x) j(x') \dots j(x^{(n)})$ .

From the last equations we find the relation between the electromagnetic mass splittings

$$\delta m_{p} - \delta m_{p} = \delta m_p - \delta m_n + \delta m_{p} - \delta m_{p+1}$$

(just by adding the first three equations).

The only information used up to this point has been the commutativity of the electromagnetic current  $j(\boldsymbol{x})$  with  $F_6$ . However we know more directly that

$$j(x) = j_3(x) + \frac{1}{\sqrt{3}} j_8(x)$$

so that if we have to compute a matrix element of the simple form

 $\langle A | j(x) | B \rangle$  we can make use of the explicit form of j and express it interms of a few reduced matrix elements. The procedure is quite analogous to the use of the Wigner-Eckart theorem, very common in problems involving the three-dimensional rotation group. Essentially since A and B belong to the 8-dimensional representation of SU(3) and j(x) also belongs to the regular representation, one has to extract from the direct product  $8 \times 8$ = 1 + 8 + 8 + 10 + 10 + 27 the regular representation which is contained there twice. So one has two reduced matrix elements. The reduction formula is  $\langle A|j_m|B \rangle = i_{ABm}\theta + d_{ABm}\delta$ , where  $f_{ABm}$  has already been reported,  $d_{ABm}$  is a completely symmetrical tensor (see later), and are reduced matrix elements (corresponding to the double occurrence of 8 in the product  $8 \times 8$ , one time as 8-antisymmetrical and one time as 8 symmetrical). We shall sketch here an inelegant proof of this reduction formula by specializing to the threedimensional representation of SU(3) (the simplest non-trivial one). In such a representation one represents the elements of the Lie algebra  $F_m$  by matrices  $\frac{1}{2}\lambda_{\rm m}$ , which satisfy

$$[\lambda_i, \lambda_i] = 2 \operatorname{if}_{iik} \lambda_k.$$

An explicit choice is the following:

$$\lambda_{1,2,3} = \begin{bmatrix} \sigma_{1,2,3} & 0 \\ 0 & 0 \end{bmatrix} \qquad \lambda_{4} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \lambda_{5} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$
$$\lambda_{6} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \lambda_{7} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \lambda_{7} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

The matrices  $\lambda$  are traceless and satisfy

$$\left\{\lambda_{i},\lambda_{j}\right\} = 2d_{ijk}\lambda_{k} + \frac{4}{3}\delta_{ij},$$

where d<sub>ijk</sub> is a completely symmetric tensor with components

$$d_{118} = d_{228} = d_{338} = -d_{888} = 1/\sqrt{3},$$
  

$$d_{146} = d_{157} = -d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377} = 1/2,$$
  

$$d_{448} = d_{558} = d_{668} = d_{778} = -i/2\sqrt{3}.$$

The matrices are normalized such that

$$Tr (\lambda_i \lambda_j) = 2 \delta_{ij}.$$

Now, from the trilinear product  $\lambda_A \lambda_m \lambda_B$  I can form two invariants, namely Tr  $(\lambda_A \lambda_m \dot{\lambda}_B)$  and Tr  $(\lambda_B \lambda_m \lambda_A)$ , or better, using the commutator and the anticommutator, Tr  $([\lambda_A, \lambda_B] \lambda_m)$  and Tr  $([\lambda_A, \lambda_B] \lambda_m)$ . But

$$\operatorname{Tr} \left( \left[ \lambda_{A}, \lambda_{B} \right] \lambda_{m} \right) = 2 \operatorname{if}_{ABK} \operatorname{Tr} \left( \lambda_{K} \lambda_{m} \right) = 2 \operatorname{if}_{ABK} \cdot 2 \delta_{Km} = 4 \operatorname{if}_{ABm},$$

$$\operatorname{Tr} \left(\{\lambda_{A}, \lambda_{B}\}\lambda_{m}\right) = 2\operatorname{id}_{ABK} \operatorname{Tr} \left(\lambda_{K}\lambda_{m}\right) + (4/3)\delta_{AB} \operatorname{Tr} \lambda_{m} = 4\operatorname{id}_{ABm},$$

from which one gets the reduction formula. Applying the reduction formula to the electromagnetic current j one has

$$\langle A | j | B \rangle = i (f_{AB3} + (1/\sqrt{3}) f_{AB8}) \theta + (d_{AB3} + (1/\sqrt{3}) d_{AB8}) \delta.$$

In this way one finds:

For the mesons : the amplitudes for transitions

vector meson  $\rightarrow$  pseudoscal meson +  $\gamma$ 

are related by  $(\rho^+ \rightarrow \pi^+ \gamma) = (K'^+ \rightarrow K^+ \gamma)$ ,

 $(\rho^0 \rightarrow \eta \gamma) = (\rho^0 \rightarrow \pi^0 \gamma) = -(1/\sqrt{3} (\omega^0 \rightarrow \eta \gamma) = (1/3) (\omega^0 \rightarrow \pi^0 \gamma) = -(2/\sqrt{3}) (K^0 \rightarrow K^0 \gamma).$ 

For the baryons : one has the explicit expression of the form factors in terms of the two independent matrix elements  $\theta$  and  $\xi$ 

$$\langle \Sigma^{0} | \mathbf{j} | \Sigma^{0} \rangle = (1/3)^{*}, \qquad \langle \Xi^{0} | \mathbf{j} | \Xi^{0} \rangle = -(2/3)^{*},$$

$$\langle \Lambda^{0} | \mathbf{j} | \Lambda^{0} \rangle = -(1/3)^{*}, \qquad \langle \mathbf{p} | \mathbf{j} | \mathbf{p} \rangle = (1/3)^{*}, \qquad \langle \mathbf{p} | \mathbf{j} | \mathbf{p} \rangle = (1/3)^{*}, \qquad \langle \Sigma^{0} | \mathbf{j} | \Lambda^{0} \rangle = -(1/\sqrt{3})^{*}, \qquad \langle \mathbf{n} | \mathbf{j} | \mathbf{n} \rangle = -(2/3)^{*}, \qquad \langle \Sigma^{-1} | \mathbf{j} | \Sigma^{-} \rangle = (1/3)^{*}, \qquad \langle \Sigma^{+1} | \mathbf{j} | \Sigma^{+} \rangle = (1/3)^{*}, \qquad \langle \Xi^{-1} | \mathbf{j} | \Xi^{-} \rangle = (1/3)^{*}, \qquad \langle \Sigma^{+1} | \mathbf{j} | \Sigma^{+} \rangle = (1/3)^{*}, \qquad \langle \Xi^{-1} | \mathbf{j} | \Xi^{-} \rangle = (1/3)^{*}, \qquad \langle \Xi^{-1} | \mathbf{j} | \Xi^{-} \rangle = (1/3)^{*}, \qquad \langle \Xi^{-1} | \mathbf{j} | \Xi^{-} \rangle = (1/3)^{*}.$$

We now discuss the electromagnetic form factors in the Sakata model. We have already derived the connection between  $I_3$  and Y and the two commuting generators of  $\mathrm{SU}_3$ 

$$I_3 = \sqrt{3} H_1$$
,  
Y = (2/3) + 2 H<sub>2</sub>.

In terms of the generators F

$$I_3 = F_3$$
,  
Y = 2/3 + (2/\sqrt{3})F\_8

Therefore the current is

$$j = (j_3 + (1/\sqrt{3})j_8) + (1/3)j_0$$

where the current  $j_0$  is associated to the phase transformation that, added to SU(3), produces U(3). In reducing an electromagnetic vertex  $\langle A | j | B \rangle$  for the three fundamental p, n, A, we use the multiplication rule 3 × 3 = 1 + 8.
So there is a contribution from 1 (corresponding to  $j_0$ ) and a contribution from 8 (corresponding to  $j_3$  and  $j_8$ ). In the eight-fold way we had two reduced matrix elements because 8 was contained twice in 8 × 8. Here we have again two reduced matrix elements but for a different reason (the appearance of a term  $j_0$  in j). If we denote  $M_8$  the matrix element for 8 and  $M_1$  that for 1, we have

$$\langle A | j | B \rangle = (Q - 1/3) M_8 + (1/3) M_1$$

or explicitly

$$\langle \mathbf{p} | \mathbf{j} | \mathbf{p} \rangle = (2/3)\mathbf{M}_8 + (1/3)\mathbf{M}_1,$$
  
 $\langle \mathbf{n} | \mathbf{j} | \mathbf{n} \rangle = -(1/3)\mathbf{M}_8 + (1/3)\mathbf{M}_1,$   
 $\langle \Lambda | \mathbf{j} | \Lambda \rangle = -(1/3)\mathbf{M}_8 + (1/3)\mathbf{M}_1.$ 

In particular we expect for the Sakata model that n and X have the same anomalous moment. On the other hand the eight-fold way gives for the anomalous  $\Lambda$  magnetic moment one-half of that of n. Experiments are still uncertain to decide in favour of one of the two alternatives.

## 4. WEAK INTERACTIONS

One is tempted to assume that weak interactions are also expressed in terms of currents belonging to the regular representation. The  $\Delta S = 0$  vector current is then given by  $g(j_1 + i j_2)$  for  $\Delta Q = + 1$  and  $g(j_1 - i j_2)$  for  $\Delta Q = -1$ ; similarly for  $\Delta S = + 1$  one has  $g'(j_4 + i j_5)$  for  $\Delta Q = + 1$  and for  $\Delta S = -1$   $g'(j_4 - i j_5)$  for  $\Delta Q = -1$ . The weak constants g and g'are presumably not the same. If they were, one would expect, if string perturbations are to be excluded, a much faster rate for hyperon  $\beta$ -decay than that observed. With the same reasonings of the previous sections one derives easily

$$\begin{split} g &\langle \Xi^{-} | j_{4} + i j_{5} | \Lambda \rangle = (1/\sqrt{2}) (\sqrt{3} \theta' - (1/\sqrt{3} \xi'), \\ g &\langle \Sigma^{-} | j_{4} + i j_{5} | n \rangle = - \theta' + \xi', \\ g &\langle \Sigma^{0} | j_{4} + i j_{5} | p \rangle = (1/\sqrt{2})(-\theta' + \xi'), \\ g &\langle \Lambda | j_{4} + i j_{5} | p \rangle = (1/\sqrt{2})(\sqrt{3}\theta' + \xi'/\sqrt{3}), \\ g &\langle \Xi^{-} | j_{4} + i j_{5} | \Sigma^{0} \rangle = (1/\sqrt{2})(\theta' + \xi'), \\ g &\langle \Xi^{0} | j_{4} + i j_{5} | \Sigma^{+} \rangle = \theta' + \xi'. \end{split}$$

Unfortunately, currents with  $\Delta S/\Delta Q = -1$  seem to be present and the point of view considered here of choosing the weak currents as belonging to the regular representation does not allow for currents with  $\Delta S/\Delta Q = -1$ . Currents with  $\Delta S/\Delta Q = -1$  would require  $\Delta T = 3/2$ . One can try to relax the con-

208

dition that the weak currents belong to the regular representation. This might seem unpleasant in some respects (i.e. strangeness conserving vector part) but would allow some more freedom. So we shall look in the following for the simplest way in any of the models we are discussing to have currents with T = 1, 1/2, 3/2 and which could originate the observed decay modes. In SU(3) the simplest (lowest dimensionality) representations containing 3/2 are D<sup>10</sup>(3,0) and D<sup>10</sup>(0,3), called briefly 10 and  $\overline{10}$ . We give the weight diagram of D<sup>10</sup>(3,0) in Fig. 8.





So one can obtain currents with  $\Delta S = 1$ ,  $\Delta T = 3/2$ ,  $\Delta S = 0$ ,  $\Delta T = 1$ ,  $\Delta T = 1/2$ ,  $\Delta T = 0$ ,  $\Delta S = -2$ . For symmetry reasons one would then have to introduce  $\overline{10}$  also. Among the difficulties of such a scheme one is the presence of  $\Delta S = \pm 2$ currents, which would lead for instance to  $\Xi \rightarrow N + \ell + \nu$ . All the amplitudes would be related in the eight-fold way, whereas in the Sakata model, since  $3 \times 3$  does not contain 10 or  $\overline{10}$ , there would be no  $\beta$ -decay of baryons and the model would be inconvenient.

Passing now to  $G_2$ , one needs currents belonging to  $D^7$  (1,0) for  $\Lambda \rightarrow N + e + \nu$  (1 × 7 = 7) while for  $\beta$ -decay one has 7 × 7 = 27 + 14 + 7 + 1. However  $D^7(1,0)$  does not contain T = 3/2 (isotopic content of  $D^7(10)$  is 1/2, 1/2, 1). The simplest representation containing T = 3/2 is  $D^{14}(0,1)$ . So one is lead to a superposition of  $D^7$  and  $D^{14}$ . The isotopic content of  $D^{14}$  is 0, 0, 0, 1 3/2. Note that it does not contain 1/2. If one wants 1/2 and 3/2 in the same representation, one has to use  $D^{27}(2,0)$  with isotopic content 0, 1/2, 1/2, 1, 1, 1, 3/2, 3/2, 2. But in this case one has to invert a particular treatment  $\Lambda\beta$ -decay which only goes through  $D^7$ . With a superposition of  $D^{14}$  and  $D^7$ one has two reduced matrix elements (7 × 7 = 27 + 14 + 7 + 1) for leptonic decays, with  $D^{27}$  only one reduced matrix elements.

Things do not get much more appealing with the other models. With  $B_2$  it is typical that there are no representations that contain both integer and semi-integer spins. The simplest choice containing T = 1 is  $D^5(0,1)$ ; the simplest containing 1/2 and 3/2 is  $D^{16}(1,1)$ . The isotopic contents are: for  $D^5(0,1)$  T = 0, 0, 1; for  $D^{16}(1,1)$  T = 1/2, 1/2, 1/2, 1/2, 3/2, 3/2. With such choice  $\Lambda_\beta$  decay (1 × 4 = 4) would require a separate explanation.  $\Sigma_\beta$ -

decay and N  $\beta$ -decay would be completely unrelated. One would need two reduced matrix elements and a separate explanation for  $\Lambda\beta$ -decay.

In  $[C_2]_I$  we need for decay some representation in the product  $5 \times 5 = 14 + 10 + 1$ , for  $\Sigma \beta$ -decay some of  $10 \times 5 = 35' + 10 + 5$ , and for mesondecay some of  $10 \times 10 = 35 + 35' + 14 + 10 + 5 + 1$ . The regular representation 10 is contained in all of the products but does not contain T = 3/2 (its isotopic content is 0, 0, 0, 1/2, 1). The simplest to have a complete isotopic content is  $D^{20}(3,0)$  (isotopic content 0, 0, 0, 0, 1/2, 1/2, 1/2, 1, 3/2), but it is not contained in any of the products so it is of no use.  $D^{35}(2,1) \equiv 35'$ has a sufficient isotopic content but does not lead to nucleon  $\beta$ -decay. A possible way out would be a superposition of 10 and 35 or of 10 and 35'(the second choice (10 + 35') would allow  $\Sigma \beta$ -decay with  $\Delta T = 3/2$  while the first one would not).

Finally in  $[C_2]_{II}$  we have only to choose one representation in the product  $10 \times 10 = 35 + 35' + 14 + 10 + 5 + 1$  that has a complete isotopic content (T = 1, 1/2 and 3/2). Both 35 and 35' can do it (35 has T = 0, 0, 0, 0, 0, 0, 1/2, 1/2, 1/2, 1/2, 1, 1, 1, 3/2, 3/2, 2; 35' has <math>T = 0, 0, 0, 1/2, 1/2, 1/2, 1/2, 1/2, 1, 1, 1, 3/2, 3/2). By choosing 35' (no T = 2) one would have only one reduced matrix element for all the leptonic decays of baryons.

### REFERENCES

- [1] RUEGG, H., Nuovo Cimento 24 (1962) 461.
- [2] COLEMAN, S. and GLASHOW, S. L., Phys. Rev. Letters 6 (1961) 1423.
- [3] CABIBBO, N. and GATTO, R., Nuovo Cimento 21 (1961) 872.
- [4] SALAM, A., Lectures on the theory of Lie groups (in these Proceedings).
- [5] BEHRENDS, R.E., DREITLING, J., FRONJDAL, C. and LEE, W., Rev. mod. Phys. 32 (1962) 1.
- [6] GELL-MANN, M., Phys. Rev. 125 (1962) 1067.

# EXPERIMENTAL TESTS OF SU<sub>3</sub> AND G<sub>2</sub> SYMMETRIES IN STRONG INTERACTIONS

## H. RUEGG UNIVERSITY OF GENEVA, GENEVA, SWITZERLAND

#### 1. INTRODUCTION

One knows that isospin invariance entails relations amongst the scattering amplitudes. The most simple of them are equalities or triangular inequalities. For example the pion-nucleon scattering

$$N + \pi \rightarrow N' + \pi'$$

would be described by 8 independent amplitudes if there was no symmetry, but only two if charge independence holds, because N and  $\pi$  can combine only into isospin T = 1/2 and T = 3/2. Therefore one gets six relations amongst the scattering amplitudes. Similarly, if the strong interactions are invariant under a larger group than the isospin group (SU<sub>2</sub>), then one gets new relations.

At this point we should emphasize that, whereas the isospin invariance is violated by known and relatively weak interactions (i. e. electromagnetic and weak interactions), the higher symmetry is violated by unknown but surely strong interactions, since the mass differences between particles belonging to the same multiplet are quite large. Therefore, one can only express the hope that at high energies and high momentum transfers, where the mass differences should not play a great role, the violations are negligible. But this question is not yet solved in a satisfactory way.

If one neglects the interactions which violate the symmetry, one gets rigorous relations.

We shall consider as examples a baryon-meson scattering and a baryonantibaryon annihilation into two mesons. Processes like production of one vector meson, for example  $B + M \rightarrow B' + V$ , are quite similar from the mathematical point of view. As applications, we shall discuss the Pomeranchuk conjecture and the Pomeranchuk theorem.

## 2. BARYON-MESON SCATTERING $B + M \rightarrow B' + M'$

#### 2.1. General considerations

Suppose B and B' belong to an irreducible representation  $D^1$ , and M and M' to the representation  $D^2$  of the group considered. In order to find the independent amplitudes one has to study how the direct product

 $D^1 \otimes D^2$ 

decomposes into irreducible representations. In the example of isospin given above:

$$N + \pi \rightarrow N' + \pi'$$

$$\Gamma = 1/2 \quad 1 \quad 1/2 \quad 1$$

$$D^{1/2} \bigotimes D^1 = D^{1/2} \bigoplus D^{3/2}$$

So there are two possible amplitudes:

$$A_{1/2}$$
: T = 1/2 → T = 1/2,  
 $A_{3/2}$ : T = 3/2 → T = 3/2.

Therefore we have to study how, in the different models, the product of the two representations reduces [1].

2.2. Sakata model  $(SU_3)$  [2]

Sakaton 
$$\begin{pmatrix} p \\ n \\ \Lambda \end{pmatrix}$$
 belongs to representation 3.

Mesons  $\begin{pmatrix} \pi \\ K \\ \chi \end{pmatrix}$  belong to representation 8.

A T = 0, Y = 0 meson denoted by  $\chi$  is predicted, which could be the  $\eta$  meson. The decomposition of the direct product into irreducible representations

$$3 \otimes 8 = 15 \oplus 6 \oplus 3$$

gives 3 independent amplitudes.

2.3. Octet model (SU<sub>3</sub>) [3]

In this case 8 baryons belong to representation 8 and 8 mesons belong to representation 8. The meson family is the same as in the Sakata model. We have the following decomposition of the direct product into irreducible representations

$$8 \otimes 8 = 1 \oplus 8 \oplus 8 \oplus 10 \oplus 10 * \oplus 27$$

where  $8_s$  and  $8_a$  are respectively symmetric and antisymmetric combinations of B and M. 10 and 10<sup>\*</sup> are inequivalent representations.

There are 8 independent amplitudes but, because of time reversal invariance,  $A(8_s \rightarrow 8_a) = A(8_a \rightarrow 8_s)$ , so there are only seven left.

212

## 2.4. Model of Behrends and Sielen $(G_2)[4]$

In this model N,  $\Sigma$  and  $\Xi$  belong to representation 7 and  $\pi$  and K belong to representation 7.

The decomposition of the direct product into irreducible representations

$$7 \otimes 7 = 27 \bigoplus 14 \bigoplus 7 \bigoplus 1$$

gives 4 amplitudes.

Now that we have found the number of independent invariant amplitudes we must find how a physical process can be expressed in terms of these amplitudes. This amounts to calculating the Clebsch-Gordan coefficients [5,6]. For example, consider the following scattering process in the octet model

pK 
$$\rightarrow \Lambda \pi^0$$
.

Let us call  $\psi_1^{\mu}$  an eigenvector of SU<sub>3</sub>, where n designates the representation of SU<sub>3</sub> to which it belongs, and T the isospin. For example  $\psi_1^{27}$  is an eigenvector of SU belonging to representation 27, and whose isospin is T = 1. Then

$$\begin{split} \mathbf{p}\mathbf{K}^{-} &= -\frac{1}{\sqrt{10}} \psi_{1}^{27} + \frac{1}{\sqrt{12}} \left(\psi_{1}^{10} - \psi_{1}^{10}^{*}\right) + \frac{\sqrt{3}}{2\sqrt{5}} \psi_{1}^{8} + \frac{1}{2\sqrt{3}} \psi_{1}^{8a} \\ &+ \frac{3}{\sqrt{20}} \psi_{0}^{27} - \frac{1}{2\sqrt{5}} \psi_{0}^{8a} + \frac{1}{2} \psi_{0}^{8a} + \frac{1}{\sqrt{8}} \psi_{0}^{1}, \\ \Lambda \pi^{0} &= \sqrt{\frac{3}{10}} \psi_{1}^{27} - \frac{1}{2} \left(\psi_{1}^{10} + \psi_{1}^{10}^{*}\right) + \frac{1}{\sqrt{5}} \psi_{1}^{8s}. \end{split}$$

Calling the invariant amplitudes  $A_{27}$ , etc. we are allowed to write:

$$A (pK^- \to \Lambda \pi^0) = -\frac{\sqrt{3}}{10} A_{27} - \frac{1}{2\sqrt{12}} (A_{10} - A_{10} *) + \frac{\sqrt{3}}{10} A_{8_{ss}} + \frac{1}{2\sqrt{15}} A_{8_{as}}$$

Of course transitions like  $\psi_0^{8_s} \rightarrow \psi_1^{8_s}$  are forbidden by isospin conservation.

2.5. Baryon-antibaryon annihilation  $B + \overline{B}' \rightarrow \overline{M} + M'$ 

This can immediately be found from the crossed channel (Fig. 1). The number of independent amplitudes is the same. This is a group-theoretical fact and has nothing to do with analytic continuation [7].



#### H. RUEGG

#### 2.6. Production of vector mesons $B + M \rightarrow B' + V$

In the two models based on  $SU_3$ , both the vector mesons and the pseudoscalar mesons belong to the representation 8, so, from the mathematical point of view, there is no difference between this problem and the previous one, except that we now have two different particles M and V in the initial and final states. The results can be found in [6,8,9,10]. Amongst the tremendous number of relations one gets, we give a few typical ones.

## 3. BARYON-ANTIBARYON ANNIHILATION INTO TWO MESONS [10]

We start with this example for the following reasons:

- (1) Due to the large baryon rest mass, the kinetic energy available for the mesons is much larger than the kaon rest mass.
- (2) The predictions of the Sakata model are in contradiction with experiment.

The Sakata model predicts (amongst other relations)

$$\sigma(p\overline{p} \rightarrow \pi^+\pi^-) = \sigma(p\overline{p} \rightarrow K^+K^-),$$

and forbids  $p\overline{p} \to K_1^0 K_2^0$ . This last process is experimentally observed with a rate comparable to  $p\overline{p} \to K^+ K^-$  whereas  $\sigma(p\overline{p} \to \pi^+\pi^-)$  is larger by a factor of about 3. One can easily deduce this equality and the selection rule predicted by the Sakata model if one considers the symmetry properties of the weight diagrams (Fig. 2). Reflection with respect to the axis {1}, {2}, {3}, are operations of SU<sub>3</sub>. The above relations are obtained by reflection with re-



spect to {1} and by noting that  $K_{1}^{0}K_{2}^{0}$  in antisymmetric with respect to  $K^{0}K^{0}$ .

In the <u>octet</u> model of  $SU_3$ , no such simple relations hold, except if one assumes the further invariance under the R operation defined by

$$p \leftrightarrow \Xi^{-} n_{\prime} \leftrightarrow \Xi^{0} \Sigma^{+} \leftrightarrow \Sigma^{-} \Sigma^{0} \leftrightarrow \Sigma^{0} \Lambda \leftrightarrow \Lambda$$
$$K^{+} \leftrightarrow K^{-} K^{0} \leftrightarrow \overline{K}^{0} \pi^{+} \leftrightarrow \pi^{-} \pi^{0} \leftrightarrow \pi^{0} \chi \leftrightarrow \chi$$

However, R is not an operation of  $SU_3$ , so it does not necessarily hold.

G, model

The weight diagram for the representation 7 is shown in Fig. 3. Here there are 6 reflections axis, the angle between two adjacent axis being 30°C. Note also that the weight (0,0) occurs only once (it corresponds to  $\Sigma^0$ , respectively  $\pi^{0}$ ).

p and  $\overline{p}$  lie on the same reflection axis (5), therefore

214



$$\sigma(\overline{p} + p \rightarrow \pi^{+} + \pi^{-} + x\pi^{0}) = \sigma(\overline{p} + p \rightarrow K^{0} + \overline{K}^{0} + x\pi^{0})$$

for an arbitrary x.

### 4. MESON-BARYON SCATTERING

We give only relations where the target is the proton and the projectile is  $\pi^{\pm}$ , K<sup>±</sup>, eventually K<sup>0</sup>.

Sakata model (relations for the corresponding amplitudes in which K  $p_{ef}$  means K  $p \rightarrow K p$  etc.)

etc.

$$\mathbf{K}^{\mathbf{p}}_{e\,\boldsymbol{\ell}} = \pi^{\mathbf{p}}_{e\,\boldsymbol{\ell}},\tag{1}$$

$$K^{\dagger}p_{e\ell} = \pi^{\dagger}p_{e\ell}, \qquad (2)$$

$$(\mathbf{K}^{-}\mathbf{p} \to \mathbf{\overline{K}}^{0}\mathbf{n}) = -(\pi^{*}\mathbf{p} \to \mathbf{K}^{0}\Lambda), \qquad (3)$$

Octet model SU<sub>3</sub>

$$(\mathbf{K}^{-}\mathbf{p} \to \Xi^{0}\mathbf{K}^{0}) = (\mathbf{K}^{-}\mathbf{p} \to \Sigma^{-}\pi^{+}), \qquad (4)$$

$$\pi^{-}\mathbf{p}_{e\,\ell} = \mathbf{K}^{-}\mathbf{p}_{e\,\ell} - (\mathbf{K}^{-}\mathbf{p} \to \Sigma^{+}\pi^{-}), \tag{5}$$

$$\pi^{+} \mathbf{p}_{e\ell} = \mathbf{K}^{+} \mathbf{p}_{e\ell} - (\pi^{+} \mathbf{p} \to \mathbf{K}^{+} \Sigma^{+}), \tag{6}$$

$$(\pi^{-}p \to K^{+}\Sigma) = (K^{-}p \to \Xi^{-}K^{+}) + (K^{-}p \to \Sigma^{-}\pi^{+}), \tag{7}$$

G<sub>2</sub> model

All predictions of the octet model of  $SU_3$ , which do not involve  $\Lambda$ , are also given by  $G_2$ . Besides these one gets many more relations like

etc.

$$p_{e\ell} \pi^+_{e\ell} = p K^0_{e\ell}, \tag{8}$$

$$p \pi_{e\ell} = p \overline{K}_{e\ell}^0, \qquad (9)$$

$$(p\pi \to n\pi^0) = (p\pi \to \Sigma^0 K^0), \qquad (10)$$

$$(pK^{-} \to n \overline{K}^{0}) = (pK^{-} \to \Sigma^{+} \pi^{-}), \qquad (11)$$

$$p\pi_{e,\theta}^{-} = p\pi_{e,\theta}^{+} + \sqrt{2} \ (p\pi^{-} \rightarrow \Sigma^{0} K^{0}), \tag{12}$$

$$pK_{e\ell}^{\dagger} = p\pi_{e\ell}^{\dagger} + (p\pi \rightarrow \Sigma^{\dagger}K^{\dagger}), \qquad (13)$$

$$p\pi_{e\ell}^{+} = pK_{e\ell}^{-} + (pK^{-} \rightarrow \Xi^{0}K^{0}), \qquad (14)$$

I

and relations (4) to (7) etc.

The relations (3), (4), (7), (10), (11) could help to discriminate between the 3 models, but measurements are very difficult.

One gets more interesting conclusions if one assumes that the higher symmetry schemes are valid in the forward direction, because one then gets a relation to the <u>Pomeranchuk conjecture</u> and the <u>Pomeranchuk theorem</u>.

#### 5. POMERANCHUK CONJECTURE AND POMERANCHUK THEOREM

I. The Pomeranchuk conjecture says that at high energy in the forward direction the processes are negligible where charge or hypercharge is exchanged.

For example

$$p + \pi^{-} \rightarrow n + \pi^{0}$$

or

$$p + \pi^- \rightarrow \Sigma^- + K^+$$

which are small against the absorptive part of

$$p + \pi^{-} \rightarrow p + \pi^{-}$$
.

II. The theorem says that  $\sigma_{tot}$  (AT) =  $\sigma_{tot}$  (AT) at high energy.

For example

$$\sigma_{tot} (\pi^- p) = \sigma_{tot} (\pi^+ p)$$
.

By the optical theorem, Pomeranchuk's theorem gives relations between the imaginary part of the amplitudes in the forward direction.

In particular cases, I and II are identical because the forward scattering amplitudes are essentially imaginary.

Example:  $N\pi$  scattering

According to I,  $A(p + \pi^- \rightarrow n + \pi^0) = 0$  gives the relation  $A_{1/2} = A_{3/2}$ , and according to II,  $A(p + \pi^- \rightarrow p + \pi^-) = A(p + \pi^+ \rightarrow p + \pi^+)$  gives the relation  $A_{1/2} = A_{3/2}$ . In general the two statements are not equivalent.

Example: NK scattering

For the NK scattering, we have the amplitudes  $A_1$  and  $A_0$  whereas for the NK scattering, we have the amplitudes  $A'_1$  and  $A'_0$ .

According to I,  $A_1 = A_0$ ;  $A'_1 = A'_0$ , and according to II,  $A_1 = 1/2(A'_1 + A'_0)$ and  $A'_1 = 1/2(A_1 + A_0)$ . In the second case the difference between the relations based on I and II comes from the fact that K and  $\overline{K}$  are not in the same isomultiplet.

As a consequence of Pomeranchuk's conjecture, one sees that the absorptive amplitudes in the forward direction become independent of isospin. Therefore, all the absorptive elastic amplitudes of  $A + B \rightarrow A' + B'$  are equal for all A, A' belonging to some representation of  $SU_2$ , and B, B' to some other. Hence, by the optical theorem, the total cross-sections are also equal.

This consequence is not peculiar to  $SU_2$ , but is true for any simple compact Lie group [11].

If only the Pomeranchuk theorem II is used, one gets less stringent restrictions [7] (see Table. 1).

| Group  | Meson-meson<br>scattering | Meson-baryon<br>scattering |  |  |
|--|---------------------------|----------------------------|--|--|
|  | N Np                      | N N <sub>p</sub>           |  |  |
| $\int Sakata; (3,8)$   | 64                        | 3 2                        |  |  |
| <sup>SU</sup> 3 Octet: <b>(</b> 8,8)   | 64                        | 74                         |  |  |
| G <sub>2</sub> (7,7)   | 4 2                       | 4 2                        |  |  |
| N is the number of independent amplitudes without theorem $N_p$ is the number of independent amplitudes with theorem |                           |                            |  |  |

| TABLE | I |
|-------|---|
|-------|---|

This result can easily be obtained by looking at the theorem for the crossed channel

$$(M_1 + \overline{M}_2 \rightarrow B_3 + \overline{B}_4) = (\overline{M}_1 + M_2 \rightarrow B_3 + \overline{B}_4),$$

which shows that the T-matrix is symmetrical with respect to the interchange of M and  $\overline{M}$ .

By looking at the symmetry properties of  $D(M) \bigotimes D(\overline{M})$  one immediately finds the result.

#### APPENDIX

We give here a simple rule, due to D. Speiser and H. Ruegg, for finding the irreducible representations  $D^{(i)}$  contained in the product of two irreducible representations D and D' of a simple Lie group.

Let  $\Lambda$ ,  $\Lambda'$ ,  $\Lambda^{(i)}$  be the highest weight vectors of D, D', D<sup>(i)</sup>; let R be half the sum of the positive roots, and  $K = R + \Lambda$ .

Suppose  $\Lambda' \geq \Lambda$ . Then add all the weight vectors of D to the vector K. The resulting vectors belong to the girdles of the representations D(i)- They have positive or negative weights if they can be transformed into a certain vector K(i) by an even or odd number of reflections of the Weyl group (see [1]). These vectors determine uniquely the representations D(i) appearing in the product  $D \otimes D'$ , but they should be counted with their respective positive or negative weight. Of course, due to some compensations, only positive weights will appear in the final formula.

As an example, we give the explicit prescription for  $SU_3$ . An irreducible representation of  $SU_3$  is characterized by two non-negative integers  $\lambda_1, \lambda_2$ ) which are the components of  $\Lambda$ , in a suitable co-ordinate system.

We want to find the irreducible representations contained in  $(\lambda_1, \lambda_2) \otimes (\lambda_1, \lambda_2)$ .

Suppose  $\Lambda \ge \Lambda$ . By adding all the weight vectors of the representation  $(\lambda_1, \lambda_2)$  to the vector  $\Lambda'$  with components  $(\lambda'_1, \lambda'_2)$  one gets a set of vectors with components,  $\lambda'_1, \lambda'_2$ . (We work here with  $\Lambda$  instead of K since they differ only by the constant vector R and since we know explicitly the reflections of the Weyl group.)

Four possibilities may happen:

(a) If  $\lambda_1^{(i)} \ge 0$ ,  $\lambda_2^i \ge 0$  the vector  $(\lambda_1^i, \lambda_2^i)$  is the highest weight  $\Lambda^{(i)}$  of one of the representations which may appear in the product. See however (c).

(b) If  $\lambda_{1}^{i} = -1$  or  $\lambda_{2}^{i} = -1$ , the corresponding vector lies on a reflection axis and corresponds to no representation (we could say that its weight is zero).

(c) If  $\lambda_1^i + \lambda_2^i + 2 > 0$  and  $\lambda_2^i + 1 < 0$  perform the transformation  $\mu_1^i = \lambda_1^i + \lambda_2^i + 1$ ;  $\mu_2^i = -\lambda_2^i - 2$ . This is precisely <u>one</u> operation of the Weyl group and ensures that  $\mu_1^i \ge 0$  and  $\mu_2^i \ge 0$ . One then gets a vector  $(\mu_1^i, \mu_2^i)$  which must be subtracted (negative weight.) from a vector with the same components appearing in (a). In other words, not all representations which appear in (a) have to be counted, but some of them are compensated by representations which appear in (c) with "negative weight".

(d) If  $\lambda_1^i + \lambda_2^i + 2 > 0$  and  $\lambda_1^i + 1 < 0$ , define  $\mu_1^i = -\lambda_1^i - 2$  and  $\mu_2^i = \lambda_1^i + \lambda_2^i + 1$ , which is again a reflection of the Weyl group. Then proceed like in (c). \*

Example: (1, 1)  $\otimes$  (3, 0) (i.e. 8  $\otimes$  10).

In the oblique co-ordinate system we have chosen, the weights of the fundamental representations are: For the representation 3(1,0);(1,0);(0,-1);(-1,1).

For the representation  $3^{(0,1)}(0,1)(-1,0)(1,-1)$ .

Therefore, the weights of the representation  $\underline{8}$  (1,1) are ( $\underline{3} \otimes \underline{3^*} = \underline{8} + \underline{1}$ ):(1,1): (2,-1): (-1,2): (0,0): (0,0): (-2,1): (1,-2): (-1,-1)

Adding these weights to (3,0) one gets:

(4, 1); (5, -1); (2, 2); (3, 0); (3, 0); (1, 1); (4, -2); (2, -1)

case (a) (b) (a) (a) (a) (a) (c) (b)

There is one vector belonging to case (c):

$$\lambda_1 = 4, \lambda_2 = -2,$$
  
 $\mu_1 = \lambda_1 + \lambda_2 + 1 = 3; \ \mu_2 = -\lambda_2 - 2 = 0,$ 

 $(\mu_1, \mu_2) = (3, 0)$ .

This representation has to be "subtracted", and therefore one gets the final result:

 $(1, 1) \otimes (3, 0) = (4, 1) \oplus (2, 2) \oplus (3, 0) \oplus (1, 1)$ 

or

8 S 10 = 35 G 27 D 10 D 8

#### REFERENCES

- BEHRENDS, R. E., DREITLEIN, J., FRONSDAL, C. and LEE, B. W., Rev. Mod. Phys. <u>34</u> (1962) 1. See also appendix of this lecture.
- [2] IKEDA, M., OYAWA, S. and OHNUKI, Y., Progr. Theor. Phys. 22 (1959) 715.
- [3] GELL-MANN, M., Caltech Report CISL-20 (1961).
   NEEMAN, Y., Nucl. Phys. <u>26</u> (1961) 222.
- [4] BEHRENDS, R.E. and SIELIN, A., Phys. Rev. <u>121</u> (1961) 324.
- [5] EDMONDS, A.R., Preprint, Imperial College of London (1962).
- [6] FREUND, P.G.O., MORALES, A., RUEGG, H. and SPEISER, D., Nuovo Cim. 25 (1962) 307.
- [7] AMATI, D., PRENTKI, L and STANGHELLINI, A., CERN Preprint 4176/Th. (1962) 279.
- [8] LEVINSON, C. A., LIPKIN, H. J. and MESHKOV, S., Nuovo Cim. 23 (1962) 236.
- [9] ibid. Phys. Letters 1 (1962) 44.
- [10] LEVINSON, C. A., LIPKIN, H. J., MESHKOV, S., SALAM, A. and MUNIR, R., Phys. Letters <u>1</u> (1962) 125.
- [11] GALINDO, A., MORALES, A. and RUEGG, H., These proceedings.

<sup>\*</sup> If one had chosen  $\Lambda' < \Lambda$ , the procedure would be more complicated. One would have to consider 8 different cases

# THE DECOMPOSITION OF DIRECT PRODUCTS OF IRREDUCIBLE REPRESENTATIONS OF SU (3)

## H. GOLDBERG ISRAEL A TOMIC ENERGY COMMISSION, REHOVOTH, ISRAËL

### INTRODUCTION

In what follows, most of the information will be brought without proofs. Although the methods are exposed for cases relevant to the octet model only, they may be generalized for any semi-simple group in quite a simple fashion.

### 1. THE ADJOINT REPRESENTATION OF SU(3)

This group is generated by the following infinitesimal operators:

$$\begin{split} &H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\ &E_3^1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ E_3^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ E_1^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ &E_1^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ E_2^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ E_1^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{split}$$

The operators  $T_z = (1/2)H_1$ ,  $T_+ = E_2^1$  and  $T_- = E_1^2$  generate the isospin group. The H<sup>s</sup>s and the E<sup>s</sup>s operate on covariant as well as on contravariant vectors. Let  $x^1$ ,  $x^2$  and  $x^3$  be the basic contravariant vectors and  $y_1$ ,  $y_2$  and  $y_3$  the basic covariant ones. The results are shown in Table I.

The results of all other operations are 0.

The representations of SU(3) on the x's and the y's are contravariant to each other.

Consider the direct product of the x-space and the y-space; this constitutes a basis for another representation of SU(3) which is, however, reducible. In order to carry on the reduction one has to know the way infinitesimal operators act on a product. The rule is:

$$O \cdot (v_1 v_2) = (O \cdot v_1) v_2 + v_1 (O \cdot v_2).$$
(1)

O an infinitesimal operator similar to the operation of a derivative; the extension to products with any number of factors is obvious. Choose the following basis:

$$I = \sqrt{(1/3)}(x^{1}y_{1} + x^{2}y_{2} + x^{3}y_{3});$$
  

$$a_{1} = x^{1}y_{3}; \quad a_{2} = x^{2}y_{3}; \quad a_{3} = x^{1}y_{2}; \quad a_{4} = \sqrt{(1/6)}(x^{1}y_{1} + x^{2}y_{2} - 2x^{3}y_{3}); \quad (2)$$
  

$$a_{5} = \sqrt{(1/2)}(x^{1}y_{1} - x^{2}y_{2}); \quad a_{6} = x^{2}y_{1}; \quad a_{7} = x^{3}y_{2}; \quad a_{8} = x^{3}y_{1}.$$

TABLE I

EFFECT OF THE OPERATORS H AND E ON VECTORS x AND y

|                          | •                         |                  |                      |
|--------------------------|---------------------------|------------------|----------------------|
| $E_3^T x^3 = x^T$        | $E_{3}^{1}y_{1} = -y_{3}$ | $H_1 x^1 = x^1$  | $H_1 y_1 = -y_1$     |
| $E_3^2 x^3 = x^2$        | $E_{3}^{2}y_{2} = -y_{3}$ | $H_1x^2 = -x^2$  | $H_{1}y_{2} = y_{2}$ |
| $E_2^1 x^2 = x^1$        | $E_2^1 y_1 = -y_2$        | $H_2 x^2 = x^2$  | $H_2y_2 = -y_2$      |
| $E_1^2 x^1 = x^2$        | $E_{1}^{2}y_{2} = -y_{1}$ | $H_2 x^3 = -x^3$ | $H_2 y_3 = y_3$      |
| $E_2^3 x^2 = x^3$        | $E_{2}^{3}y_{3} = -y_{2}$ |                  |                      |
| $E_{1}^{3}x^{1} = x^{3}$ | $E_{1}^{3}y_{3} = -y_{1}$ |                  |                      |
|                          |                           |                  |                      |

It follows from (1) that the vector I is invariant under SU(3). Similarly, the vectors  $a_i$  (1  $\leq i \leq 8$ ) span a space which is irreducible under SU(3). The effect of the H's and the E's in this 8-dimensional space is shown in Table II.

TABLE II

EFFECT OF THE OPERATORS H AND E IN A 8-DIMENSIONAL SPACE

 $E_3^2 a_3 \rightarrow -a_1$  $E_2^1 a_2 \rightarrow a_1$  $E_8^1 \quad a_4 \rightarrow -\sqrt{(3/2)} a_1$  $a_4 \rightarrow -\sqrt{(3/2)} a_2$  $a_5 \rightarrow -\sqrt{2}a_3$  $a_5 \rightarrow -\sqrt{(1/2)} a_1$ a<sub>6</sub> → -a<sub>2</sub>  $a_5 \rightarrow \sqrt{(1/2)} a_2$ a<sub>6</sub> → √2a<sub>5</sub>  $a_7 \rightarrow \sqrt{(3/2)} a_4 - \sqrt{(1/2)} a_5$  $a_8 \rightarrow -a_7$  $a_7 \rightarrow a_3$  $a_8 \rightarrow \sqrt{(3/2)} a_4 + \sqrt{(1/2)} a_5$ a<sub>8</sub> → a<sub>6</sub>  $E_1^3 \quad a_1 \rightarrow -\sqrt{(3/2)} a_4 - \sqrt{(1/2)} a_5 \qquad E_2^3 \quad a_1 \rightarrow -a_3$  $E_1^2 \quad a_1 \rightarrow a_2$ a<sub>2</sub> → -a<sub>6</sub>  $a_2 \rightarrow -\sqrt{(3/2)} a_4 + \sqrt{(1/2)} a_5$ a3 → -√2a5  $a_4 \rightarrow \sqrt{(3/2)} a_7$ a<sub>5</sub> → √2a<sub>6</sub>  $a_3 \rightarrow a_7$  $a_5 \rightarrow -\sqrt{(1/2)} a_7$  $a_4 \rightarrow \sqrt{(3/2)} a_8$  $a_7 \rightarrow -a_8$  $a_5 \rightarrow \sqrt{(1/2)} a_8$  $a_6 \rightarrow a_8$  $H_1 \quad a_1 \rightarrow a_1; \quad a_2 \rightarrow \neg a_2; \quad a_3 \rightarrow 2a_3; \quad a_6 \rightarrow \neg 2a_6; \quad a_7 \rightarrow a_7 \quad ; \quad a_8 \rightarrow \neg a_8$  $H_2 = a_1 \rightarrow a_1; a_2 \rightarrow 2a_2; a_3 \rightarrow -a_3; a_6 \rightarrow a_6; a_7 \rightarrow -2a_7; a_8 \rightarrow -a_8$ 

The results of all other operations are 0.

By the definition of weights, the correspondence between weights and vectors of this representation is:

 $a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 (1 1) (2 - 1) (-1 2) (0 0) (0 0) (1 - 2) (-2 1) (-1 - 1)$ 

The representation is called "the adjoint representation" because its nonvanishing weights are the roots of the group SU(3). They correspond to the E's in the following way:

## 2. WEIGHT DIAGRAMS OF IRREDUCIBLE REPRESENTATIONS OF SU(3); MULTIPLICITY

Let  $\mathcal{L}$  be the lattice of all points  $(\lambda, \mu)$  in plane, where  $\lambda$  and  $\mu$  are integers and  $\lambda-\mu$  is divisible by 3. If both  $\lambda$  and  $\mu$  are non-negative, there is an irreducible representation of SU(3), the highest weight\* of which is  $(\lambda, \mu)$ , and this representation appears in the reduction of a certain product of the form:

$$(1 \ 1) \times (1 \ 1) \times \ldots \times (1 \ 1).$$

Conversely, if an irreducible representation appears in the decomposition of such a product, its highest weight  $(\lambda, \mu)$  is such that  $\lambda, \mu \ge 0$  and  $\lambda-\mu$  is divisible by 3.

Given  $(\lambda, \mu)$ , draw the hexagon defined by the points:

$$(\lambda,\mu), (\lambda + \mu, -\mu), (\mu, -\lambda - \mu), (-\lambda, \lambda + \mu), (-\lambda - \mu, \lambda), (-\mu - \lambda).$$

It is readily seen that all these points belong to  $\mathcal{L}$ . Every point of  $\mathcal{L}$  lying on the sides or inside the hexagon is a weight of the representation  $(\lambda, \mu)$ ; no weight of the representation lies outside the hexagon. The hexagon shrinks into a triangle when either  $\lambda = 0$  or  $\mu = 0$ ; yet the statement remains true.

Different vectors of an irreducible representation may correspond to the same weight; e.g.  $a_4$  and  $a_5$  in the adjoint representation correspond both to (0 0). The number of independent vectors corresponding to a weight is called the multiplicity of this weight.

Suppose  $\lambda \ge \mu > 0$ , and consider the set weights:

$$(\lambda, \mu), (\lambda - 1, \mu - 1), (\lambda - 2, \mu - 2), \dots, (\lambda - \mu, 0).$$

One may draw for each such weight a hexagon in a way similar to the original one; e.g.,  $(\lambda - 1, \mu - 1)$  determines the six points:

$$(\lambda - 1, \mu - 1), (\lambda + \mu - 2, -\mu + 1), (-\lambda + 1, \lambda + \mu - 2), (\mu - 1, -\lambda - \mu + 2),$$
  
 $(-\lambda - 1\mu + 2, \lambda - 1), (-\mu + 1, -\lambda + 1).$ 

221

<sup>\*</sup> A weight  $(\alpha, \beta)$  is positive when either  $\alpha + \beta > 0$  or  $\alpha + \beta = 0$ ,  $\beta > 0$ .  $(\alpha, \beta)$  is higher than  $(\gamma, \delta)$  when  $(\alpha - \gamma, \beta - \delta)$  is positive.

The rule is that all weights lying on the  $(\lambda - k, \mu - k)$  hexagon have multiplicity k + 1; and those lying on the innermost triangle <u>or inside it</u> have the multiplicity  $\mu + 1$ .

When  $\mu \geq \lambda > 0$  we deal with a set of weights:

$$(\lambda, \mu)$$
,  $(\lambda - 1, \mu - 1)$ ,...  $(0, \mu - \lambda)$ ;

yet the rule remains unchanged. Similarly, when either  $\lambda = 0$  or  $\mu = 0$ , we have a triangle and the multiplicity of each weight is 1.

Example: (5, 2) defines a hexagon the vertices of which are (5, 2), (7, -2), (2, -7), (-5, 7), (-7, 5), (-2, -5). (Its weights diagram is drawn in Fig.1).





The weights diagram of the representation (2 2)

### 3. THE CALCULATION OF THE REPRESENTATIONS

#### (a) Example

As weights have in general multiplicity > 1, one has to use additional quantum numbers in order to specify uniquely a vector corresponding to a given weight. It was shown by Racah that one needs  $(N - 3\ell)/2$  such additional quantum numbers for a group of order N and rank  $\ell$ . In the case of SU(3) we have  $(N - 3\ell)/2 = 1$ , and the labeling according to T<sup>2</sup> solves the problem.

The following lemma is important for calculation: If the vector  $|m\rangle$  corresponds to the weight  $\underline{m}$ , the vector  $E_{\alpha} |m\rangle$  corresponds to the weight  $\underline{m} + \alpha$ .

Proof: 
$$H_i E_{\alpha} | m > = [H_i E_{\alpha}] | m > + E_{\alpha} H_i | m > = (\alpha_i + m_i) E_{\alpha} | m > Q. E. D.$$

It follows that knowing a vector of the representation corresponding to a given weight, we may "walk" all over the diagram with the aid of the E's and get vectors corresponding to all other weights of this representation. The method seems to be best explained by an example. We have:

$$(1 \ 1) \times (1 \ 1) = (2 \ 2) + (3 \ 0) + (0 \ 3) + (1 \ 1) + (1 \ 1) + (0 \ 0) \dots$$
 (3)

Let  $a_i$  and  $b_k$  (1  $\leq i$ ,  $k \leq 8$ ) span the bases of the two representations appearing on the left and suppose one has to calculate the basis of (2 2). The vectors belonging to this basis are combinations of products of the form  $a_i b_k$  in such a way that each combination corresponds to a definite weight and has a definite T. It is readily seen that the weight corresponding to a product is the sum of the weights which correspond to the factors. Hence  $a_1 b_1$  is the only product which corresponds to (2 2).

By Table II and using (1):

$$T_{-}(a_1 b_1) = a_2 b_1 + a_1 b_2;$$
  $T_{-}(a_2 b_1 + a_1 b_2) = 2 a_2 b_2.$ 

 $\sqrt{(1/2)}(a_1b_2 + a_2b_1)$  corresponds to (3 0),  $a_2b_2$  to (4 -2). It can be seen from Fig.2 that together with  $a_1b_1$  they form an isospin triplet, because



Fig. 2 The weights diagram of the representation (2 2)

The vector

$$\sqrt{(1/2)} E_2^3(a_1b_1) = \sqrt{(1/2)}(a_1b_3 + a_3b_1)$$

corresponds to (0 3). Together with:

$$-\sqrt{(1/3)}T_{\sqrt{(1/2)}}(a_{1}b_{3} + a_{3}b_{1}) = \sqrt{(1/3)}(a_{1}b_{5} + a_{5}b_{1}) - \sqrt{(1/6)}(a_{2}b_{3} + a_{3}b_{2}),$$
  
$$\sqrt{(1/2)}T_{\sqrt{(1/3)}}(a_{1}b_{5} + a_{5}b_{1}) - \sqrt{(1/6)}(a_{2}b_{3} + a_{3}b_{2})] = \sqrt{(1/6)}(a_{1}b_{6} + a_{6}b_{1})$$
  
$$+\sqrt{(1/3)}(a_{2}b_{5} + a_{5}b_{2})$$

 $\sqrt{(1/3)} T_{\bullet} \left[ \sqrt{(1/6)} (a_1 b_6 + a_6 b_1) + \sqrt{(1/3)} (a_2 b_5 + a_5 b_2) \right] = \sqrt{(1/2)} (a_2 b_6 + a_6 b_2),$ 

which correspond to  $(1 \ 1)$ ,  $(2 \ -1)$  and  $(3 \ -3)$  respectively, they form an isospin quartet. However, the multiplicity of  $(1 \ 1)$  (and of  $(2 \ -1)$ ) is 2; i.e.  $(1 \ 1)$  corresponds to another vector with a different T. Certainly, this T is 1/2. The vector

223

$$-E_1^3(a_1b_1) = \sqrt{(3/2)}(a_1b_4 + a_4b_1) + \sqrt{(1/2)}(a_1b_5 + a_5b_1)$$

corresponds also to (1 1), and therefore it is a combination of the T = 3/2 and T = 1/2 vectors corresponding to (1 1). By the Grahm-Schmidt procedure one finds the T = 1/2 vector, which is orthogonal to the T = 3/2 one:

$$\sqrt{(1/30)} \{ \sqrt{(3/2)}(a_1 b_4 + a_4 b_1) + \sqrt{(1/2)}(a_1 b_5 + a_5 b_1) + (a_2 b_3 + a_3 b_2) \}$$

Operating on this by T. we get the second member of the doublet:

$$\sqrt{(1/30)} \{ (a_1 b_6 + a_6 b_1) + 3\sqrt{(3/2)} (a_2 b_4 + a_4 b_2) - \sqrt{(1/2)} (a_2 b_5 + a_5 b_2) \}$$

which is the second vector corresponding to (2 - 1). The vector

$$-\sqrt{(1/2)}E_2^3\sqrt{(1/2)}(a_1b_3 + a_3b_1) = a_3b_3$$

corresponds to (-2, 4). Together with  $N^{(i)} T^i(a_3 b_3)$  where i = 1, 2, 3, 4 and where  $N^{(i)}$  are normalization factors we get an isospin quintet. The T = 1 vector which corresponds to (-1, 2) is obtained by operating with  $E_2^3$  on the T = 1/2 vector corresponding to (1 1)\* ;the other members of the triplet are obtained with the aid of T.

Operating with  $E_1^3$  on the T = 1/2 vector corresponding to (1 1) we get a combination of the T = 1 and T = 0 vectors which correspond to (0 0). By the Grahm-Schmidt method we again pick out the T = 0 vector.

The continuation of the procedure is obvious (Fig. 3).



Fig. 3 The root diagram.

(b) The general method

In order to perform the decomposition of  $(\lambda_1 \lambda_2) \times (\mu_1 \mu_2)$  one needs the following information:

(1) The representations  $(\lambda_1 \lambda_2)$  and  $(\mu_1 \mu_2)$  of the E's. The bases of these representations consist of polynomials in baryons, antibaryons, mesons and vector-mesons. Therefore rule (1) may be used to get these representations.\*\*

224

<sup>\*</sup> The reason is that  $E_2^3$  (as well as  $E_1^3$ ,  $E_3^2$ ,  $E_3^1$ ) can change T only by  $\ddagger$ .

<sup>\*\*</sup> However, G. Racah calculated explicit formulae for the matrix-elements of the E's in any irreduble representation (private communication).

(2) Which representations appear in the decomposition. Define: P(x) = (1/2)(|x| + x)

$$A = \mu_{in} \{\lambda_1 (1/3)[(\lambda_1 - \lambda_2) + (\mu_1 + 2\mu_2) - (\nu_1 - \nu_2)], (1/3)[(2\lambda_1 + \lambda_2) + (2\mu_1 + \mu_2) - (2\nu_1 + \nu_2)]\},\$$
  

$$B = \mu_{o-x} \{P[(\lambda_1 - \nu_1) + (1/2)P((1/3)|(\lambda_1 + 2\lambda_2) - 2(\mu_1 - \mu_2) - (\nu_1 + 2\nu_2)| + (\nu_1 - \lambda_1))], (1/3)[2(\lambda_1 - \lambda_2) + (2\mu_1 + \mu_2) - (2\nu_1 + \nu_2)](1/3)[(\lambda_1 - \lambda_2)]\}$$

$$+(\mu_1 + 2\mu_2) - (\nu_1 + 2\nu_2)]$$

The number of times  $(\nu_1 \nu_2)$  appears in  $(\lambda_1 \lambda_2) \times (\mu_1 \mu_2)$  is P(A - B + 1).

(3) The vector corresponding to the maximal weight of such a representation. Let  $\underline{\ell}$  be a weight of  $(\lambda_1 \lambda_2)$  and  $\underline{m}$  a weight of  $(\mu_1 \mu_2)$  such that  $\underline{\ell} + \underline{m} = (\nu_1 \nu_2)$ . If  $|\underline{\ell}|$ ,  $T_1 > \text{is a vector of the basis of } (\lambda_1 \lambda_2)$  corresponding to  $\underline{\ell}$ , and  $|\underline{m}|$ ,  $T_2 > \text{a vector of the basis of } (\mu_1 \mu_2)$  corresponding to  $\underline{m}$ , then the product  $|\underline{\ell}|$ ,  $T_1 > |\underline{m}|$ ,  $T_2 > \text{corresponds to } (\nu_1 \nu_2)$ .

We look for a linear combination of such products which is the vector corresponding to the highest weight of  $(\nu_1\nu_2)$ . Such a combination must vanish under  $E_3^1$ ,  $E_3^2$  and  $E_2^1$ , as follows from the lemma of part (a) (since  $(\nu_1 + 1, \nu_2 + 1)$ ,  $(\nu_1 + 2, \nu_2 - 1)$  and  $(\nu_1 - 1, \nu_2 + 2)$  are not weights of the representation  $(\nu_1\nu_2)$ . The combination will also vanish under  $E_1^2$ , if  $\nu_2 = 0$ , and under  $E_2^3$  if  $\nu_1 = 0$ . Considering the coefficients of the combination, we get for them a set of homogeneous equations when  $E_3^1$ ,  $E_3^2$  and  $E_2^1$  (may be  $E_1^2$ , or  $E_2^3$  also) are applied. If the solution is not unique, the representation  $(\nu_1\nu_2)$  occurs several times, and one chooses an appropriate basis arbitrarily. Example: The vector corresponding to the highest weight of (1 1) in the decomposition of (1 1) × (1 1). This vector is of the form:

$$\alpha(a_1 b_4) + \beta(a_1 b_5) + \gamma(a_2 b_3) + \delta(a_4 b_1) + \epsilon(a_5 b_1) + \zeta(a_3 b_2).$$

It will vanish under the operations of  $E_3^1$ ,  $E_3^2$  and  $E_2^1$ . Hence:

$$\begin{split} E_3^1 &: -\sqrt{(3/2)}(\alpha + \delta)(a_1 b_1) - \sqrt{(1/2)}(\beta + \epsilon)(a_1 b_1) = 0, \\ E_3^2 &: -\sqrt{(3/2)}\alpha(a_1 b_2) + \sqrt{(1/2)}\beta(a_1 b_2) - \gamma(a_2 b_1) - \sqrt{(3/2)}\delta(a_2 b_1), \\ &+ \sqrt{(1/2)}\epsilon(a_2 b_1) - \zeta(a_1 b_2) = 0, \end{split}$$

$$E_{2}^{1}: -\sqrt{2} \beta(a_{1} b_{3}) + \gamma(a_{1} b_{3}) - \sqrt{2} \epsilon(a_{3} b_{1}) + \zeta(a_{3} b_{1}) = 0.$$

From these equations we get:

$$\begin{split} &\sqrt{(3/2)}(\alpha + \delta) + \sqrt{(1/2)}(\beta + \epsilon) = 0, \\ &-\sqrt{(3/2)}\alpha + \sqrt{(1/2)}\beta - \zeta = 0, \end{split}$$

$$-\gamma - \sqrt{(3/2)}\delta + \sqrt{(1/2)}\epsilon = 0,$$
  
$$-\sqrt{2}\beta + \gamma = 0,$$
  
$$-\sqrt{2}\epsilon + \xi = 0.$$

The set of solutions is two dimensional. Adding the condition  $\gamma = \zeta$  or  $\gamma = -\zeta$  one gets two mutually orthogonal solutions.

#### 4. REMARKS

(1) According to the usual exposition of the Lie theory one should write  $E_{(1 \ 1)}$  instead of  $E_3^1$ ,  $E_{(2 \ 1)}$  instead of  $E_3^2$  etc.. The present notation emphasizes the fact that the E's operate in a 3-dimensional vector space.

(2) The highest weights  $(\lambda,\mu)$  are associated with Young-schemes. An irreducible representation of SU(3) may be characterized by a Young-scheme of not more than 3 rows. Denoting this scheme by  $[a_1 a_2 a_3]$  (where  $a_i$  is the length of the i<sup>s</sup>th row) we have  $\lambda = a_1 - a_2$ ,  $\mu = a_2 - a_3$ .

(3) As  $T_+$  and  $T_-$  commute with the hypercharge, it follows that vectors belonging to the same isospin multiplet have the same hypercharge.

(4) The decomposition procedure described above does not determine any general phase convention.

## VECTOR MESONS AND UNITARY SYMMETRY

## J. J. SAKURAI ENRICO FERMI INSTITUTE OF NUCLEAR STUDIES UNIVERSITY OF CHICAGO, CHICAGO, ILL. UNITED STATES OF AMERICA

## INTRODUCTION

I am going to talk about the vector theory of strong interactions, the universality of the vector meson couplings and then about the various decay modes of the  $\omega$  meson. The last part of this paper will be concerned with unitary symmetry, especially F and D type couplings and the mysterious mass formula which seems to work rather well.

#### 1. VECTOR THEORY (GAUGE THEORY) OF STRONG INTERACTIONS

The basic philosophy behind the vector theory or gauge theory of strong interactions can be summarized in the following way. It is essentially an attempt to construct a theory of strong interactions in analogy with electromagnetism. We know that, from a certain point of view, quantum electrodynamics is remarkably simple and elegant. The notions of conserved current, universality and what we might call the principle of minimal electromagnetic couplings play important roles. Similarly, in the realm of weak interactions, it has become apparent that the weak interactions are also vectorial, apart from parity non-conservation, and there have been speculations on the divergencelessness of the currents involved in weak processes. Moreover, we know that the notion of universality has been successfully applied to some domains of weak interactions of non-strange particles. Finally, there are conjectures on the possible existence of spin-one particles (W particles) which mediate various weak processes.

If we now turn our attention to the strong interactions, the following questions very naturally arise. Why are the strong interactions also vectorial? Why do we not have a universal theory of strong interactions based on conserved currents? The vector theory of strong interactions is an attempt to answer these questions by constructing a theory of strong interactions which shares the various elegant features of the electromagnetic and weak interactions.

Now let us go back to some speculations made by WIGNER [1] many years ago. He noted that there are essentially two ways to determine the electric charge of a particle.

First, electric charge is regarded as a pure number - a purely additive number - which is conserved in any reaction. For instance, take the reaction

$$\mu^+ \rightarrow e^+ + \overline{\nu}_{\mu} + \nu_e.$$

If we know from some other experiments that the electric charge of the posi-

#### J.J. SAKURAI

tron is plus one and that the electric charges of the neutrinos are zero, then, by conservation of charge, the electric charge of the  $\mu^+$  is determined to be plus one. But the meaning of electric charge is more than that. We can place a beam of charged particles in an electric field and see how much the beam deflects. So electric charge is not only countable but also measurable, and it is in this second sense that we say that the charge of the electron is equal in magnitude to the charge of the proton to a fantastic degree of accuracy, to a few parts in  $10^{19}$ . (This charge equality is one of the most remarkable equalities in modern physics. Quantum electrodynamics says that, if the bare charges are equal, then the corresponding renormalized charges are also equal. Yet nobody can explain the equality of bare charges!).

Wigner argues in the following way. Both the electron and the proton are highly stable. The stability of the electron can be attributed to the conservation law of electric charge since the electron is the least massive particle that bears electric charge. Similarly, the stability of the proton can be attributed to the conservation law of what we might call "baryonic charge" since the proton is the least massive particle with baryon number one. Nobody understands the deep reason for the existence of the conservation laws of electric charges and of baryonic charges, but, says Wigner, let us assume that the two conservation laws have similar causes, and these causes have similar consequences. With this in mind, let us ask what we mean by "baryonic charge"? Take, for instance, the reaction

 $\Lambda \rightarrow p + \pi$ -.

If the baryonic charge of the proton is one and that of the pion is zero, then we argue that the baryonic charge of the  $\Lambda$  hyperon must also be one. This is how we determine the baryonic charge of a particle. So we are using the notion that baryonic charge is some additive number which is conserved in any reaction. The point to be emphasized is that in the conventional theory there is nothing analogous to Wigner's second way of measuring the charge of a particle, i.e. the notion of coupling constant is completely missing. So although the electric charge and the baryonic charge are similar in the sense that they are both conserved to a fantastically high degree of accuracy (the proton lifetime  $> 10^{24}$  yr, the electron lifetime  $> 10^{19}$  yr), they are quite dissimilar because in one case the "charge" means both conserved additive number and coupling constant, whereas in the other case the "charge" means just conserved additive number. This asymmetry is quite ugly and disturbing.

The asymmetry between baryonic charge and electric charge can be seen from a somewhat more formal point of view as follows. In the electromagnetic case the charge conservation is an immediate consequence of Maxwell's equations in the sense that the continuity equation

$$(\partial/\partial \mathbf{x}_{\mu})\mathbf{j}_{\mu} = \vec{\nabla} \cdot \vec{\mathbf{j}} + (\partial \mathbf{P}/\partial t) = 0$$

follows from

$$\vec{\nabla} \cdot \vec{B} - (\partial \vec{E} / \partial t) = \vec{j}, \vec{\nabla} \cdot \vec{E} = P.$$

228

In the baryonic case, however, baryon conservation stands by itself, so to speak.

Historically, Wigner tried to remove the asymmetry between electric charge and baryonic charge by postulating that the pion is coupled universally to the various baryons. This is the origin of "global symmetry". This analogy, however, is rather superficial, and it cannot be pursued much further. The reason is that the quantity to which the photon field is coupled is a conserved current density, whereas the quantity to which the pion is coupled is a pseudoscalar density which has little to do with baryon conservation.

A much more natural way is to assume that there is a vector meson coupled universally to the baryon current just as the photon is coupled universally to the electric charge current. If the mass of the vector meson were zero, we would get into difficulties because there would be a kind of long - range, anti - gravity effect (analogous to the Coulomb repulsion) between two macroscopic objects, which has been discussed by LEE and YANG [2].

Such an effect, if it exists at all, can be shown to be much weaker than the gravitational interaction; in any case it would have nothing to do with the strong interactions. So we assume that the vector meson coupled to the baryon current is massive.

We may naturally generalize this idea of associating a vector meson to a conserved current to other conserved currents of the strong interactions. For every conserved quantity we postulate the existence of a vector meson coupled linearly to the appropriate conserved current in question. This is the basic idea of the vector theory of strong interactions.

Historically, a number of people have tried to "justify" the vector theory on the basis of what we might call the gauge principle. The requirement . that the gauge transformation associated with the conservation law of baryonic charge, etc. be local (space-time dependent) in character demands the existence of a vector field with zero bare mass coupled universally to the baryon current. We can argue endlessly whether or not such an approach makes sense, because the physical mass of the vector meson associated with the vector field must be finite in order that we have a physically interesting theory of strong interactions. But I shall not discuss this very important problem.

From a practical point of view there are a few important points. First, is the idea that for every conserved current there exists a strongly interacting vector meson right? If so, are the vector mesons coupled universally to the appropriate conserved currents in the same sense that the electromagnetic field is coupled universally to the electric charge current? How can we test the universality principle?

I should emphasize at this moment that, given a symmetry of conserved operators, the number and the nature of the vector mesons are determined. If you are just concerned with the exactly conserved currents of the strong interactions, then there are only three - the isospin current, the baryon current and the hypercharge current. Of course, we may take any linear combination of the strangeness current and the baryon current instead of the hypercharge current, but in higher symmetry models, such as the unitary symmetry model or any model in which there is some symmetry between N and  $\Xi$ , it is natural to take the hypercharge current, as we shall show later. We can easily verify that the isospin current is isovector and even under G conjugation, whereas the baryon current and the hypercharge current are isoscalar odd under G. So we are led to conjecture that there exist one T=1, even G vector meson and two T=0, odd G vector mesons.

So far we have considered only the exact symmetries of the strong interactions. Perhaps there are hidden symmetries which are approximate. If there are, there may be more currents which are conserved, but only to the extent that this mass difference between the nucleon and the  $\Lambda$  etc. can be ignored. Indeed, in the unitary symmetry model to be discussed later, there is a strangeness changing current with isospin 1/2 which is approximately conserved. So we may conjecture on the existence of a T = 1/2 vector meson coupled to the quasi-conserved strangeness changing current.

Before proceeding, I would like to give credit to the people who are involved in this line of thinking. The first suggestion that there ought to be a vector meson coupled to the isospin current was made by YANG and MILLS [3] as early as in 1954. It was FUJII [4] who first suggested that there should be a strongly interacting vector meson coupled to the baryon current. Subsequently I formulated a theory in which the vector mesons coupled to the baryon current, isospin current and hypercharge current play vital roles in the physics of strong interactions [5]. For the currents generated by gauge transformations of unitary symmetry based on the Sakata triplet, SALAM and WARD [6] have shown that we must have an octet of vector mesons. There is another version of the unitary symmetry model where we again have an octet of vector mesons as shown by GELL-MANN and NE'EMAN [7].

When this kind of theory was proposed, there was no direct experimental evidence for or against the existence of strongly interacting vector mesons. As is well known, there are now two vector mesons whose existence has been firmly established by numerous experiments - the  $\rho$  meson with mass  $\approx$  750 MeV with T = 1, G = +1 decaying into three pions. The  $\rho$  meson can be identified with the vector meson coupled to the isospin current whereas the  $\omega$  meson can be one of the candidates to the two T = 0, G = -1 vector mesons proposed by the vector theory of strong interactions. If one subscribes to the philosophy that for every conserved current there should be a vector meson, it would be better to have another T = 0, G = -1 vector meson. In spite of their similarity the two T = 0 vector mesons are quite distinct because the baryon current is very different from the hypercharge current. For instance, the one coupled to the hypercharge current would not be emitted or absorbed by  $\Lambda$  since the  $\Lambda$  has hypercharge = zero, whereas the  $\Lambda$  can emit or absorb the vector meson coupled to the baryon current. This distinction also becomes apparent in the octet version of the unitary symmetry model to be discussed later; the one coupled to the baryon current is an unitary singlet whereas the one coupled to the hypercharge current is a member of a unitary octet. In any case I would like to urge the experimentalists to look for another T = 0, G = -1 vector meson. Perhaps it is relevant to mention that if the conjectured T = 0 meson has mass greater than 2 m<sub>K</sub>, then its main decay mode may be  $K^+ + K^-$  and  $K^0 + \overline{K}^0$ . Since

the  $K_1^0$   $K_1^0$  mode and the  $K_1^o$   $K_2^0$  mode are forbidden by Bose statistics and also by G conjugation invariance, we should see a bump in the  $K_1^0$ ,  $K_2^0$  Q value distribution but not in the  $K_1^0$   $K_1^0$  distribution. The conjectured meson may be looked for in the reactions  $K + p \rightarrow K_1^0 + K_2^0 + \Lambda$ ,  $K^+ + K^- + \Lambda$ .

In the unitary symmetry model there is room for a vector meson with  $T = \frac{1}{2}$ ,  $S = \pm 1$  which may be identified with the 880 MeV K. There is some preliminary evidence from  $p\bar{p}$  annihilation experiments carried out by the CERN - Collège de France group that the spin of K\* is likely to be one.

|                     | Isospin       | G          | Hypercharge or<br>strangeness | Unitary symmetry classification |
|---------------------|---------------|------------|-------------------------------|---------------------------------|
| Isospin current     | 1             | +          | 0                             | Member of unitary octet         |
| Hypercharge current | 0             | -          | 0                             | Member of unitary octet         |
| Baryon current      | 0             | -          | 0                             | Unitary singlet                 |
| S changing current  | $\frac{1}{2}$ | no meaning | ± 1                           | Member of unitary octet         |

|  | ΤA | BLE | I |
|--|----|-----|---|
|--|----|-----|---|

The predictions of the vector theory are summarized in Table I. The existence of  $\rho = \omega$  and K<sup>\*</sup> is gratifying especially if we recall that when the theory was proposed there was no direct evidence for any of these mesons. There are, however, two predictions that have not yet been checked:

- (i) The spin of  $K^*$  must be one (for which there is some evidence); and
- (ii) There must exist another T = 0,  $J = 1^-$ , G = -1 vector meson whose major decay modes may well be  $K_1^0 + K_2^0$  and  $K^+ + K^-$  (but not  $K_1 + K_1^0$ ,  $K_2^0 + K_2^0$ ).

### 2. UNIVERSALITY

From the quantitative point of view the most important question in the vector theory of strong interactions is the one of the universality of the interactions between the vector mesons and the baryon and meson currents. In the old-fashioned way the interactions of the  $\rho$  meson with the nucleon and pion can be written as

$$L_{I} = f_{\rho \eta \pi} \rho_{\mu} (\vec{\pi} \times \partial_{\mu} \vec{\pi}) + f_{\rho N} \rho_{\mu} (i \overline{N} \gamma_{\mu} (\vec{\tau}/2) N)$$
(2.1)

Universality means

$$f_{\rho \pi\pi} = f_{\rho NN}$$

Now we may argue endlessly about whether this kind of equality is supposed to hold at zero momentum transfer, as in the case of the electromagnetism, or on the mass-shell momentum transfer. In this elementary discussion we shall leave aside this question.

In order to test the universality hypothesis we shall calculate from the experimental data  $f_{\rho\pi\pi}$  and  $f_{\rhoNN}$  From the width for the decay  $\rho \rightarrow 2\pi$ , we can obtain  $f_{\rho\pi\pi}^2/4\pi$ . This procedure is good if the width of the  $\rho$  particle is very narrow ( $f_{\rho\pi\pi}$  is very small). Actually  $\rho$  manifests itself as a resonance in

J = 1, T = 1,  $\pi\pi$  scattering, so we would like to know how to determine  $f_{\rho\pi\pi}^2/4\pi$  from the  $\pi\pi$  scattering amplitude.

Let us recall how we usually define the coupling constant for the interaction between two pions and a stable particle, which for simplicity we suppose to be scalar. Let us denote this particle by  $\sigma$  and suppose that its mass is smaller than twice  $m_{\pi}$ . Near the  $\sigma$  pole the T matrix can be written as:

$$T_{\simeq} g^2/s - m_{\sigma}^2$$

 $\mathbf{so}$ 

 $\left|\frac{\mathrm{d}}{\mathrm{ds}}\left(\frac{1}{\mathrm{T}}\right)\right|_{\mathrm{S}} = \mathrm{m}_{a}^{2} \overset{\approx}{\mathrm{g}}^{-2} \qquad (2.2)$ 

The rate at which 1/T varies with the energy square d near the (mass)<sup>2</sup> of the intermediate particle measures the coupling constant. Note that (1/T) vanishes at s =  $m_{\sigma}^2$ .

In the unstable  $\sigma$  case (i.e.  $m_{\sigma} > 2m_{\pi}$ ), the  $\sigma$  meson manifests itself as a resonance in s-wave  $\pi\pi$  scattering.

Since

$$\frac{e^{i\delta}\sin\delta}{k} = \frac{1}{k\cot\delta - ik}$$

and

$$\operatorname{Re}\left(\frac{1}{T}\right) = -\frac{\operatorname{k}\cot\delta}{8\pi\sqrt{s}},$$

the real part of (1/T) goes through zero near the resonance just as 1/T goes through zero in the stable  $\sigma$  particle case. This suggests the definition

$$\frac{\mathrm{d}}{\mathrm{ds}}\left[\mathrm{Re}\left(\frac{1}{\mathrm{T}}\right)\right] = \mathrm{g}^{-2}.$$
(2.3)

Now, for a sufficiently narrow resonance, the phase shift is given by

$$e^{i\delta}\sin\delta = -\frac{m_{\sigma}\Gamma}{(s-m_{\sigma}^2)+im_{\sigma}\Gamma}, s \approx m_{\sigma}^2,$$

so we obtain

$$\frac{1}{g^2} = \frac{k_{\text{res.}}}{8\pi m_{\text{cl}}^2 \Gamma}$$

or

$$\Gamma = \left(\frac{g^2}{4\pi}\right) \frac{1}{4} \frac{\left(m_{\sigma}^2 - 4m_{\pi}^2\right)^{\frac{1}{2}}}{m_{\sigma}^2}.$$

It is important to note that exactly the same expression can be obtained by computing the life-time or the decay width by perturbation theory using the effective Lagrangian  $g_{\sigma\pi\pi}$ . So we see that the well-known formal identity between the pole terms in the sense of dispersion theory and the renormalized Born terms in the sense of perturbation theory can be extended to the case of unstable particles [8].

In a similar manner we can compute the decay width for  $\rho \rightarrow 2\pi$ . We obtain for the width

$$\Gamma_{\rho} = \frac{2}{3} \cdot \frac{f_{\rho\pi\pi}^2}{4\pi} \cdot \frac{p^3}{m^2_p} = \frac{1}{12} \cdot \frac{f_{\rho\pi\pi}^2}{4\pi} \left(1 - \frac{4m^2\pi}{m^2p}\right)^{\frac{3}{2}} m_p \quad , \qquad (2.4)$$

where p is the pion momentum in the rest system of the decaying  $\rho$  particle, and  $\Gamma_{\rho}$  is the full width. Experimentally  $\Gamma_{\rho}$  is 100-125 MeV; then

$$f_{0\pi\pi}^2/4\pi \simeq 2.0 - 2.5.$$
 (2.5)

How do we get the  $f_{\rho NN}$  coupling constant? One possibility should be through nuclear forces, but the related calculations would be very complicated. The potential is best known for p-p scattering but in this case it is not possible to discriminate between the  $\rho$  and  $\omega$  contributions. They appear in the same way. Additional complications come from an anomalous magnetic moment-like term, and, what is more important, we do not have a reliable calculational method for the other contributions (e.g. contributions as a result of the exchange of an uncorrelated pair of pions). The best we can do is to look at  $\pi$ -N scattering. Let us see the contribution from Fig. 1, which gives the product  $f_{\rho\pi\pi} f_{\rho NN}$ .



The effect of the  $\rho$  meson on low energy  $\pi$ N scattering has been estimated by many people. Let us first do the most naive thing, i.e., to see the contribution of the above Born graph in the s-wave  $\pi$ N scattering amplitude.

Fortunately the anomalous magnetic moment term does not contribute to s-wave at low energy, and we get something like this:

$$\frac{\tan \delta_3}{k} - \frac{\tan \delta_1}{k} = \frac{3}{4} \frac{f_{\rho \pi \pi} - f_{\rho NN}}{4\pi} \cdot \frac{\omega m N}{W} \cdot \frac{1}{k^2} \log\left(1 + \frac{4k^2}{m_0^2}\right), \quad (2.6)$$

where  $\delta_3$  is the phase shift of the isospin 3/2 s-wave amplitude and  $\delta_1$  the phase shift for isospin 1/2; W is the total energy in the C. M. system.  $\omega$  is the energy of the pion in the C. M. system. The log term comes from the partial wave projection of the  $\rho$  meson propagator,  $1/[2k^2(1 - \cos \theta) + m_0^2]$ .

If we assume that the  $\rho$  meson exchange dominates as  $k \rightarrow 0$ , Eq. (2.6) gives the difference between the two scattering lengths  $a_3 - a_1$ , from which it follows:

$$f_{\rho\pi\pi} \cdot f_{\rho NN} / 4\pi \simeq 2.5.$$
 (2.7)

#### J. J. SAKURAI

In a more sophisticated approach HAMILTON, SPEARMAN and WOOLCOCK [9] tried to fit the energy dependence of the phase shift instead of scattering length, and in the notation of Bowckok et al. they obtained:

$$\ell_1 = -\frac{1}{3} \frac{f_{\rho \pi \pi} f_{\rho NN}}{4\pi} = -0.7 \pm 0.1,$$

or

$$f_{\alpha\pi\pi}/4\pi \simeq 2.1 \pm 0.3,$$
 (2.8)

which agrees with (2.7). This shows that  $f_{\rho \pi\pi} \simeq f_{\rho NN}$ , as required for universality.

It would be nice to test the universality hypothesis in other reactions, for example, in KN and  $\overline{\text{KN}}$  scattering. To isolate the  $\rho$  contribution in these reactions is a very difficult task since we do not know how to calculate other contributions.

We can, however, make an interesting speculation. Whenever the one pion exchange is forbidden by symmetry considerations, then the isospin dependent amplitude for any low energy scattering is dominated by the exchange of a  $\rho$  particle coupled universally to the isospin current. This hypothesis can readily be shown to imply the simple rule: the  $\rho$ -exchange force is attractive when isospins are antiparallel, repulsive when they are parallel.

It is amusing to notice that this rule works nicely in five cases. Thus in the  $\pi N$ , T = 1/2 state we have attraction; in the T = 3/2 state, repulsion. In KN,T = 1 repulsion is very strong as it is verified in the K<sup>+</sup>p scattering experiments.

For KN nucleon scattering, the T = 0 state is more attractive than the T = 1 state since the  $Y_0^*$  resonance of 1405 MeV is most likely an s-wave (KN) bound state whereas the 1385 MeV  $Y_1^*$  resonance is not likely to be related to the s-wave KN channel. Also there seems to be an attractive s-wave interaction in T = 0 KK scattering. Finally, in the  $\pi\pi$  case, T = 0 is more attractive than T = 2.

#### 2.1. $\omega$ meson

Let us assume for the sake of argument that the  $\omega$  meson is coupled to the hypercharge current. And let us do the same kind of thing for the KN, KN. K and N have hypercharges + 1, K has hypercharge - 1. If in the low energy domain the idea that the  $\omega$  exchange dominates is correct, then on the average KN is repulsive and KN is attractive. This follows because the exchange of a  $\omega$  meson coupled to the hypercharge current generates Coulomb like interaction at short distances; i.e., for similar hypercharges we have repulsion, for opposite hypercharges, attraction. Then the "potentials" for KN and KN can be written in the following form if we take into account only  $\rho$  and  $\omega$  exchange:

> KN:  $V_{\omega} + V_{\rho} \vec{\tau}_{K} \cdot \vec{\tau}_{N}$ KN:  $-V_{\omega} + V_{\rho} \vec{\tau}_{K} \cdot \vec{\tau}_{N}$

where

$$\vec{\tau}_{K} \cdot \vec{\tau}_{N} = \begin{cases} -3 \text{ for } T = 0\\ 1 \text{ for } T = 1 \end{cases}$$

The signs of  $V_{\omega}$  and  $V_{\rho}$  are determined to be positive in the vector theory based on the universality principle (but are arbitrary in any other theory). There is some experimental evidence that the simple description given here corresponds to reality. With the  $\omega$  meson we may hope to understand nuclear forces at short distances. Again in N-N interaction the exchange of a T = 0 vector meson gives a Coulomb-like repulsion at small distances. This might give rise to the phenomenological hard core in nucleon-nucleon scattering.

To end this section, I would like to show how the universality principle might be formulated on the basis of dispersion theory.

Let us go back to the  $\rho$  meson. Suppose that the  $\rho$  meson dominates the charge form factors. This means that the nucleon or pion form factors could be approximated by the Fig. 2.



Fig. 2

If we are far from the resonance we can essentially ignore the complications due to the instability of the  $\rho$  meson [8]. We then have:

$$F_{\pi} (q^2) = \gamma_{\gamma \rho} f_{\rho \pi \pi} / (q^2 + m_{\rho}^2)$$

while for the isovector nucleon charge form factor F(v),

$$F_{N}^{(v)}(q^{2}) = (\gamma_{\gamma\rho} f_{\rho NN}^{2}/2)/(q^{2} + M_{\rho}^{2}),$$

where  $\gamma_{\gamma\rho}$  is the coupling constant of the  $\rho$  particle to the photon. Now for  $q^2 \rightarrow 0$ , we have  $F_{\pi} \rightarrow \rho$  and  $F_N^{(\chi)} \rightarrow \rho/2$ , because at zero momentum transfer <u>electric</u> charges are universal. Thus  $f_{\rho NN} = f_{\rho \pi \pi}$ , which agrees with the concept of universality.

Now there is another final point which is extremely interesting. We have here

$$\gamma_{\gamma\rho} = \mathrm{em}_{\rho}^{2} / \mathrm{f}_{\rho\mathrm{NN}} = \mathrm{em}_{\rho}^{2} / \mathrm{f}_{\rho\pi\pi}. \tag{2.9}$$

 $\gamma\gamma\rho$  is <u>inversely</u> proportional to the strong interaction constant, a very different result from the perturbation result. In perturbation theory we consider

$$\gamma \rightarrow \left\{ \begin{matrix} \pi & + \ \pi \\ \mathrm{N} + \mathrm{N} \end{matrix} \right\} \ \rightarrow$$

#### J.J. SAKURAI

so we would expect  $\gamma_{\gamma\rho}$  proportional to the strong interaction constant  $(f_{\rho\pi\pi}, f_{\rho NN})$ . The fact that  $\gamma_{\gamma\rho}$  is inversely proportional to the strong coupling constant is analogous to the well-known Goldberger-Treiman relation. There again the pion decay rate is inversely proportional to the strong coupling constant.

### 3. DECAY MODES OF THE $\omega$ MESON

A particle with the quantum numbers  $T = 0, J = 1^-$  was predicted by Nambu in 1957 to explain the isoscalar electromagnetic form factor of the nucleon. This means that Fig. 3 must be important.



Then, through the photon which is coupled to any pair of charged particles the  $\omega$  meson can decay in the following ways:

$$\omega \rightarrow \gamma \rightarrow \left\{ \begin{array}{c} \mathrm{e^+e^-} \\ \mu^+\mu^- \\ \pi^+\pi^- \end{array} \right.$$

If the  $\omega$  contribution dominates the isoscalar form factor, we can readily show that

$$\bar{\gamma}_{\gamma\omega} = \frac{\mathrm{e}\,\mathrm{m}^2_{\,\omega}}{2\,\mathrm{f}_{\,\mathrm{cl}}} \,\,, \tag{3.1}$$

in complete analogy to what we did for the  $\rho$  meson.

The constant  $f_{\omega}$  is defined through the interaction

$$L_{I} = f_{\omega}\omega_{\mu}(i\overline{N}\gamma_{\mu}N + \dots). \qquad (3.2)$$

We can give the following simple rule. Whenever  $\omega$  dominates, then we insert in the corresponding diagram the factor  $\gamma_{\gamma\omega} = e m_{\omega}^2 / 2 f_{\omega}$  between  $\gamma$  and  $\omega$ ;  $\rho$  occurs, we insert  $\gamma_{\gamma\rho} = e m_{\rho}^2 / f_{\rho}$ . So the decay rate for  $\omega \rightarrow e^+ + e^-$ ,  $\mu^+ + \mu^$ is given by

$$\Gamma\left(\omega \rightarrow \frac{e^+}{\mu^+} \frac{e^-}{\mu}\right) = \left(\frac{1}{137}\right)^2 \cdot \frac{1}{12} \cdot \frac{m_\omega}{f_\omega^2/4\pi} \cdot \left(1 - \frac{4m_\rho^2 u}{m_\omega^2}\right)^{\frac{1}{2}}, \quad (3.3)$$

which is inversely proportional to the strong coupling constant  $f_{\omega}^2$ . Numerically we get:

$$\Gamma (\omega \to e^+ e^-) \simeq \Gamma (\omega \to \mu^+ + \mu^-) = \frac{3 \text{ keV}}{f_{\omega}^{\rho} / 4\pi} , \qquad (3.4)$$

as shown by NAMBU and SAKURAI [10].

If  $\omega$  does not dominate, we have to multiply the above expression by  $|\alpha_{\omega}|^2$  where  $\alpha_{\omega}$  is the coefficient of the  $\omega$  contribution to the isoscalar charge form factor.

$$\Gamma_{1}^{(s)}(q^{2}) = \frac{m_{\omega}^{2} \alpha_{\omega}}{q^{2} + m_{\omega}^{2}} + 1 - \alpha_{\omega}.$$
 (3.5)

If the muon had an anomalous interaction, then the  $\omega \rightarrow \mu^+ + \mu^-$  and  $\omega \rightarrow e^+ + e^-$  ration would not be equal. If, for example, there exists a meson which interacts somewhat strongly with the muon but not with the electron, then we expect a very different result for the branching ratio

$$[\Gamma(\omega \to \mu + \mu^{-})] / [\Gamma(\omega \to e^{+}e^{-})].$$

This would be a sensitive test of the idea that the muon is a pure Dirac particle. So far there is no experiment on leptonic decays of the  $\omega$  meson.

Now the situation is somewhat more complicated for the  $\omega \rightarrow 2\pi$  decay. It is a process where we have violation of T conservation and G invariance. So  $\omega$  can decay into  $2\pi$  via electromagnetic interaction:

$$\omega \rightarrow \gamma \rightarrow 2\pi$$
.

In calculating the decay rate, one must take into account the final state interaction between the two pions, and this can be expressed by Fig. 4, where  $F_{\pi}$  is the pion form factor. The result is



$$\Gamma(\omega \to 2\pi) \simeq \frac{0.7}{f_{\omega}^2/4\pi} \text{ keV} \left| \mathbf{F}_{\pi} \left( \mathbf{q}_z^2 = \mathbf{m}_{\omega}^2 \right) \right|^2, \qquad (3.6)$$

where the pion form factor  $F_{\pi}$  is given by

$$\mathbf{F}_{\pi} \left( \mathbf{q}^2 \right) \simeq \frac{\mathbf{m}_{\rho}^2}{\left( \mathbf{q}^2 + \mathbf{m}_{\rho}^2 \right) \, \mathrm{i} \, \Gamma_{\rho} \mathbf{m}_{\rho}}, \text{ for } \mathbf{q}^2 \boldsymbol{\approx} - \mathbf{m}_{\rho}^2,$$

if we assume that only the  $\rho$  meson contributes to the pion form factor. Since the  $\rho$  and  $\omega$  masses are very near, the enhancement factor  $|F_{\pi}|^2$  in (3.6) can be very large - something like 50.

Let us next consider decays of the  $\omega$  meson which have been observed. For the  $\omega \to \pi^0 \gamma$  decay, we assume that the dominant graph is as in Fig. 5. Although we do not know the  $\omega \rho \pi$  coupling constant, we can compare this process with the  $\omega \to 3\pi$  decay which we suppose to be dominated by the diagram as shown in Fig. 6. In this way, in the branching ratio  $[\Gamma(\omega \to \pi^0 \gamma)/\Gamma(\omega \to 3\pi)]$  the unknown



 $\omega \rho \pi$  coupling constant is cancelled out, and only known quantities remain. For  $\rho \to \gamma$  we use  $\ell m_{\rho}^2/2f_{\rho}$  and for  $\rho \to 2\pi$  we use  $f_{\rho\pi\pi}^2/4\pi \simeq 2.0$ , corresponding to  $\Gamma_{\rho} \simeq 100$  MeV.

We then get:

$$\Gamma(\omega \rightarrow \pi^0 \gamma) / \Gamma(\omega \rightarrow 3\pi) \simeq 17\%$$
 [11].

Experimentally, both the CERN-Paris group  $(\vec{p} + p \rightarrow K^0 + \vec{K}^0 + \omega)$  and the Berkeley group  $(K+p \rightarrow \Lambda + \omega)$  give for the above ratio 15-20% in excellent agreement with the  $\rho$  dominance model. The  $\omega \rho \pi$  coupling constant can be calculated if we assume that in the  $\pi^0 \rightarrow 2\gamma$  decay the dominant graph is as shown in Fig. 7.



Fig. 7

This was first pointed out by GELL-MANN and ZACHARIASEN [8]. We obtain the absolute value for the decay rate of  $\omega \rightarrow 3\pi$ , which comes out to be approximately 400 keV, if we assume  $f_{\omega}^2/4\pi \simeq 1.5$ , as suggested by the unitary symmetry.

Let us consider a more direct method for measuring the  $\omega$  decay rate. For example,  $\Gamma(\omega \to \pi^0 \gamma)$  can be obtained from  $\pi^0$  photo-production. Some preliminary study of  $\gamma + p \rightarrow p + \pi^0$  has revealed a peculiar angular distribution which cannot be explained with usual phenomenological terms like a reasonable number of powers of  $\cos \theta$ . There is some evidence that the angular distribution at  $E_j \sim 1.1$  BeV is completely dominated by the diagram shown in Fig. 8.

This experiment is still in progress at the California Institute of Technology. Note that the above graph gives the product of  $f_{\omega}$  and  $f_{\omega\gamma\pi}$ .



Fig. 8

It was shown experimentally that the process  $\pi^{\pm} + p \rightarrow p + \omega + \pi^{\pm}$  is very strong so we might try the approach in Fig.9, and in this way we could obtain the  $\omega - 3\pi$  vertex. Unfortunately, the experimental results at  $p_{\pi}^{(lab)} \sim \text{GeV/c}$  do not give any indication of the importance of this one-pion exchange graph. Experimentally, the above processes are dominated by  $\omega + N^*$ . Perhaps at much higher energies the one-pion-exchange mechanism will become important.



Fig. 9

Finally, there is an experiment proposed by CABIBBO and GATTO in which we study  $\omega$  production in electron-positron colliding beam annihilation [12]. Then the total cross-section for the process  $e^++e^- \rightarrow$  final state f, where the intermediate state is  $\omega$ , is given by

$$\sigma \left( e^+ e^- \to f \right) = \frac{3 \pi \lambda^2 \Gamma \left( \omega \to 2e \right) \Gamma \left( \omega \to f \right) / 4}{\left( E - m_{\omega} \right)^2 + \Gamma^2 / 4}, \qquad (3.7)$$

where  $\Gamma(\omega \rightarrow f)$  is the partial width for  $\omega \rightarrow f$  in al state in question and  $\Gamma(\omega \rightarrow 2e)$  is the partial width for the  $\omega \rightarrow e^+ + e^-$  decay. What one experimentally measures is not the peak but rather the cross-section averaged over some energy interval.

$$\overline{\sigma} = \frac{1}{2\Delta E} \int_{m_{\omega^{-}}\Delta E}^{m_{\omega^{+}}\Delta E} \sigma(E) dE. \qquad (3.8)$$

Numerically  $\overline{\sigma} (e^+e^- \rightarrow 3\pi) \approx 6.5 \ \mu B$  if we assume  $\Delta E = 10$  MeV,  $\Gamma = 500$  keV and the branching ratio  $[\Gamma(\omega \rightarrow 2e)] / \Gamma_{tot} \approx 1\%$ . This is much larger than the usual electrodynamic cross-sections.

## 4. UNITARY SYMMETRY

Let us start with some familiar concepts: charge conservation and isospin conservation. In the charge conservation case, we have a unitary group with one parameter which corresponds to the gauge transformations  $e^{i\alpha}$ . For the isospin conservation, we have a three-parameter unitary group which corresponds to rotations in a three-dimensional Euclidean space. It is just an accident that the isospin rotation is usually discussed in analogy with the Euclidean rotation; we can approach isospin rotation in an entirely different way. We first consider two primitive objects, the proton and the neutron:

$$\left( \begin{array}{c} p\\ n \end{array} \right)$$

Any isospin rotation can be completely characterized by its effects on the two primitive objects p, n. The usual way of writing the isospin rotation is as

$$\exp\left(i\left[\tau_e \theta_e/2\right]\right) {p \choose n}, \qquad (4.1)$$

where  $\tau_{\rm e}$  are the usual Pauli matrices and  $\theta_{\rm e}$  specify the rotation. But we can also write

$$\exp\left(i\left[\tau_{n}\theta_{n}/2\right]\right) = \begin{pmatrix} \alpha & \beta \\ -\beta^{*} & \alpha^{*} \end{pmatrix}$$
(4.2)

with  $|\alpha|^2 + |\beta|^2 = 1$ . Note that we again have 3 <u>independent</u> parameters (two complex numbers and one constraint).

The above matrix generates a unitary and unimodular (det = 1) transformation in two-dimensional space. The group of the unitary unimodular  $2\times 2$  matrices is denoted by SU<sub>2</sub> (S stands for simple and U<sub>2</sub> for unitary and unimodular).

Then, instead of considering the group of the transformations  $O_3$  (Euclidean rotation in real 3-dimensional space), we may as well consider the equivalent group  $SU_2$ . We may note that (4.2) is not the most general unitary two dimensional matrix, but it is the most general unitary unimodular matrix. The most general unitary matrix is obtained by multiplying (4.2) by the one-parameter gauge transformation  $e^{i\alpha}$  with real  $\alpha$ .

More complicated objects like the pions can be built up from the outer product of  $(\overline{pn})$  and  $\binom{p}{h}$ :

$$\left(\begin{array}{cc} \overline{p}p & \overline{n}p \\ \overline{p}n & \overline{n}\overline{n} \end{array}\right).$$
(4.3)

But this has mixed properties under isospin rotations or equivalently under  $SU_2$ : the reason is that the trace part transforms like a singlet. To obtain the triplet we subtract the trace:

= . = .

$$\vec{\pi} = \begin{pmatrix} \vec{p}p & \vec{n}p \\ \vec{p}n & \vec{n}n \end{pmatrix} - \begin{pmatrix} \vec{p}p + \vec{n}n \\ 2 & 0 \\ \vec{p}p + \vec{n}n \end{pmatrix}$$
$$= \begin{pmatrix} \vec{p}p - \vec{n}n \\ 2 & \vec{n}p \\ \vec{p}n & -\frac{\vec{p}p - \vec{n}n}{2} \end{pmatrix}, = \begin{pmatrix} \pi^{0} & \pi^{+} \\ \sqrt{2} & \pi^{0} \\ \pi^{-} & \sqrt{2} \end{pmatrix}.$$
(4.4)

Although we can construct all non-strange particles in similar ways, we can never build up strange particles by starting with the primitive objects p and n, which are both non-strange. If we work with SU(2), it is impossible to incorporate the degree of freedom that corresponds to the gauge transformation for strangeness or hypercharge conservation.

Now SU<sub>3</sub> is a slight generalization of SU<sub>2</sub>. We can again consider infinitesimal generators  $1+i\lambda_i(\delta\theta_i/2)$  instead of  $1+i\tau_i(\delta\theta_i/2)$  of SU<sub>2</sub>.  $\lambda_i$  here are 3×3 traceless matrices: their representation can be found, for example, in GELL-MANN's paper [7].

For  $SU_3$  we have three primitive objects. For example, in the Sakata model, they are the p, n and A. This model was extensively studied especially by Ikeda, Ohnuki and Ogawa. In this approach  $\Sigma$  and  $\Xi$  belong to different representations from p, n,  $\Lambda$ . If the  $\Lambda \Sigma$  parity were odd, then this would be very promising. There are now good indications that the  $\Sigma\Lambda$ parity is even, so it is natural to put  $\Sigma$  and  $\Lambda$  together. Another thing is that in this model the most likely assignment for the cascade spin is 3/2. So far, there is no argument against this assignment, but some preliminary experiments indicate that there is an asymmetry in the  $\Xi$  decay with respect to the normal to the production plane; if this persists, the spin 3/2 assignment will be ruled out by the so-called Lee and Yang test. (If we have a higher spin object, there will be more tendency for any decay  $\Lambda$  products to be emitted in the production plane rather than in the direction normal to the production plane.) There are also some predictions on the decay of  $\overline{pp} \rightarrow 2\pi$ , KK etc., and some experiments on this seem to contradict the Sakata model as will be discussed in other papers.

We shall discuss the octet model the "eight-foldway" introduced by Gell-Mann and Neéman independently. Here the primitive objects are hidden. For pedagogical purposes, we shall introduce a mathematical lepton multiplet  $l = \begin{pmatrix} \mu \\ \mu \end{pmatrix}$  where  $\nu e^-$  form a doublet and  $\mu^-$  a singlet with baryon number B = 0. Let us now introduce also a mathematical boson multiplet with baryon number equal to one (B=+1):  $L = (D^0, D^+, S^+)$  where  $D^0, D^+$ form a doublet and  $S^+$  is a singlet. Isotopically this multiplet transforms like an antilepton multiplet. In the preceding case of SU<sub>2</sub> where we constructed the pion out of the nucleon-antinucleon doublets, we considered the outer product  $(\bar{p}, \bar{n}) \times (pn)$  out of the nucleon-antinucleon doublets. Let us do the same kind of thing taking the outer product of  $\bar{L}$  and 1. We then obtain the matrix:

Note that the trace of the above matrix is invariant under unitary transformations. If we subtract from the above matrix the corresponding trace which is a unitary singlet,

$$\begin{array}{c|cccc} \frac{D^{0}\nu + D^{+}e^{-} + S^{+}\mu^{-}}{3} & 0 & 0\\ 0 & \frac{(idem)}{3} & 0\\ 0 & 0 & \frac{(idem)}{3} \end{array}$$
(4.5)

we obtain

$$\frac{2D^{0}\nu - D^{+}e^{-} - S^{+}\mu^{-}}{3} \qquad D^{+}\nu \qquad S^{+}\nu$$

$$D^{0}e^{-} \qquad \frac{-D^{0}\nu + 2D^{+}e^{-} - S^{+}\mu^{-}}{3} \qquad S^{+}e^{-}$$

$$D^{0}\mu^{-} \qquad D^{+}\mu^{-} \qquad \frac{-D^{0}\nu - D^{0}e^{-} + 2S^{+}\mu^{-}}{3}$$
(4.6)

Now one can easily identify the various elements of this matrix with stable baryons

$$\begin{array}{c} \mathbf{S}^+ \boldsymbol{\nu} = \mathbf{p} \\ \mathbf{S}^+ \mathbf{e}^- = \mathbf{n} \end{array} \right\} \quad \begin{array}{c} \mathbf{D}^0 \,\boldsymbol{\mu}^- = \,\Xi^- \\ \mathbf{D}^+ \boldsymbol{\mu}^- = \,\Xi^0 \end{array} \right\}$$

since they form an isospin doublet; also  $\Sigma^+ = D^+\nu$ ,  $\Sigma^- = D^0e^-$  and by charge independence we identify the neutral member

$$\Sigma^0 = (D^0 \nu - D^+ e^-) / \sqrt{2}$$
.

What is left over must be an isosinglet:

$$\Lambda^0 = (D^0 \nu + D^+ e^- - 2S^+ \mu^-)/\sqrt{6}$$
.

Let us note that we obtain the right strangeness for the above particles if we put  $S(e, \nu, S^+) = 0$  and  $S(D^0, D^+, \mu^-) = -1$ , where S is the strangeness. As a result we obtain the baryon octet:

$$B = \begin{vmatrix} \frac{\Sigma^{0}}{\sqrt{2}} + \frac{\Lambda^{0}}{\sqrt{6}} & \Sigma^{+} & p \\ \Sigma^{-} & -\frac{\Sigma^{0}}{\sqrt{2}} + \frac{\Lambda^{0}}{\sqrt{6}} & n \\ \Xi^{-} & \Xi^{0} & -\frac{2}{\sqrt{6}} \Lambda^{0} \end{vmatrix}$$
(4.7)

We have used 1 and  $\overline{L}$  only as a device to keep track of transformation properties. Once you have obtained the unitary octet, you can forget about them. Any unitary octet must have the same structure.

Now, the observed pseudoscalar mesons also form an octet.

$$\mathbf{m} = \begin{vmatrix} \frac{\pi^{0}}{\sqrt{2}} + \frac{\eta^{0}}{\sqrt{6}} & \pi^{+} & \mathbf{K}^{+} \\ \pi^{-} & \frac{-\pi^{0}}{\sqrt{2}} + \frac{\eta^{0}}{\sqrt{6}} & \mathbf{K}^{0} \\ \mathbf{K}^{-} & \mathbf{\overline{K}}^{0} & -\frac{2}{\sqrt{6}} \eta^{0} \end{vmatrix}, \qquad (4.8)$$

where  $\eta^0$  is an isosinglet as the  $\Lambda^0$ , which may be identified as the 560 MeV object. Similarly, the vector mesons might be put in a unitary octet u, and they can be obtained from m by the following substitution:

$$u\Sigma m \begin{pmatrix} \pi \to \rho \\ K \to n \\ \eta \to \omega \end{pmatrix}.$$
 (4.9)

In this model K\* has spin 1 which may be the 880 MeV K $\pi$  resonance.

## 5. INTERACTIONS

Now we can write the unitary symmetry Yukawa type interactions. This can readily be done if we recall that the trace of a matrix product is invariant under unitary transformations. Omitting  $\gamma$  matrices, we have the traces:

 $Tr(\overline{B}mB)$  and  $Tr(\overline{B}Bm)$ 

$$\overline{\mathbf{B}} = \begin{vmatrix} \overline{\overline{\Lambda}}^{0} + \overline{\overline{\Sigma}}^{0} & \overline{\Sigma}^{-} & \overline{\Xi}^{0} \\ \overline{\sqrt{6}} + \overline{\sqrt{2}}^{0} & \overline{\Sigma}^{-} & \overline{\Xi}^{0} \\ \overline{\Sigma}^{+} & - \overline{\sqrt{2}}^{0} + \overline{\sqrt{6}}^{0} & \overline{\Xi}^{0} \\ \vdots \\ \overline{p} & \overline{n} & - \frac{2}{\sqrt{6}} \overline{\Lambda}^{0} \end{vmatrix}$$

Let us observe that  $\overline{B}$  was chosen in such a way that it is obtained from B by transposing and taking the bar. Otherwise, we do not have conservation of electric charge.

We can consider two types of unitary symmetric interaction:

These types of coupling are also invariant under the following discrete operations:

where
D type is invariant under  $B \rightarrow B^T$ ,  $m \rightarrow m^T$  and

F type is invariant under  $B \rightarrow B^T$ ,  $m \rightarrow -m^T$ 

(where the superscript T stands for "transposed".)

So the most general interaction is a linear combination of the above. To choose an interaction of D or F type means to impose invariance under some discrete operation called R (or hypercharge reflection). The R operation corresponds essentially to interchange N and  $\Xi$  and charge conjugate the mesons. The D type couplings have the following properties:

$$g_{\pi \Lambda \Sigma}^2 = (4/3) g_{\pi NN}^2$$
;  $g_{\pi \Sigma \Sigma}^2 = 0.$  (4.11)

In the F type couplings the pion is coupled to the pseudo-scalar density that transforms like isospin. Then  $g_{\pi\Sigma\Sigma}^2 \neq 0$  and  $g_{\pi\Lambda\Sigma}^2 = 0$ . In choosing between the two types of coupling, the D type is probable more reasonable because there is some evidence from hypernuclei that  $g_{\pi\Lambda\Sigma}^2 = 0$  gets into difficulty. The  $\Lambda N$  forces seem to require some sizable  $g_{\pi\Lambda\Sigma}^2$ .

In the D type coupling we have  $g_{\pi NN} = -g_{n\mathbb{Z}\Sigma}$  while in the F type coupling  $g_{\pi NN} = g_{n\mathbb{Z}\Sigma}$ . Then if our interaction is an equal mixture of D and F types, the  $\Xi$  would not interact strongly with the pions. This point might be of some interest in the dynamical approach to the recently discovered  $\Xi$  resonance.

Whether one has the pure D type, the pure F type or a mixture of both, it is impossible to have the K couplings much weaker than the pion couplings. If one compares pseudoscalar constants, there is some evidence from photoproduction of K mesons that the KAN and KEN couplings are weaker than the  $\pi$ N couplings. But it is known that unitary symmetry is broken by large mass ratio, e.g.  $m_x$ ,  $m_{\pi} = 3.5$ . Now, if one uses a pseudovector coupling, this mass ratio is exactly compensated and the pseudovector coupling constants are practically equal for  $\pi$  and K interactions.

Let us now consider the couplings of the vector mesons. Again we have two possible linearly independent couplings:

D type : Trace (
$$\overline{B}vB + \overline{B}Bv$$
)

F type : Trace  $(\overline{B}vB + \overline{B}Bv)$ .

The D type couplings bear no resemblance whatsoever to the vector theory (or gauge theory) discussed earlier in which the vector mesons are coupled to the various conserved currents of the strong interactions. On the other hand, with pure F type couplings the vector mesons are coupled to the currents generated by the gauge tranformations of unitary symmetry:

$$1+i\lambda_i (\delta \theta_i/2)$$
.

More precisely, the  $\rho$  is coupled to the isospin current and the  $\omega$  is coupled to the hypercharge current. Moreover, we also have the K\* which is coupled to the quasi-conserved strangeness changing current:

$$\left(-\frac{1}{\sqrt{3}}\,\overline{\mathrm{N}}\,\Lambda+\overline{\mathrm{N}}\,\vec{\tau}\cdot\vec{\Sigma}+\ldots\right)\mathrm{K}^{*}.$$

As is well known, it might be possible to detect this kind of interaction in associated production experiments provided that reactions such as  $\pi^- + p \rightarrow \Lambda^0 + K^0$  are dominated by the exchange of  $K^*$ .

In the unitary symmetry scheme with F type couplings of the vector mesons, there is a relation between the coupling constants  $f_{\omega}$  and  $f_{\rho}$ :

$$f_{\mu}^{2}/4\pi = (3/4)(f_{0}^{2}/4\pi), \qquad (4.12)$$

which does not appear in the usual vector theory without unitary symmetry. From the width of the  $\rho$  meson, we have

$$f_0^2/4\pi \approx 2.0$$
 (for  $\Gamma_0 \approx 100 \,\mathrm{MeV}$ ),

which leads to

$$f_{\omega}^2/4\pi \approx 1.5.$$

Recently performed nuclear force calculations seem to give a larger value for this coupling constant. This discrepancy might be due to the possible existence of another T = 0 vector meson discussed earlier.

It is also interesting to note that the couplings of the vector mesons to the pseudoscalar mesons of the form vmm must be of the F type. For instance, the  $\rho$  must be coupled universally to the sum of the  $\pi$  meson isospin current and the K meson isospin current. If the D type couplings were assumed, there would be terms like  $\rho^0 \eta^0 \pi^0$  which would not be invariant under charge conjugation.

There are two couplings of the vector mesons which may "directly" be observed:  $\rho \rightarrow 2\pi$  and  $K^* \rightarrow K^+\pi$ . Using (mvm - mmv), we can readily obtain

$$\Gamma(K^*)/\Gamma(\rho) = (3/4)[p_{K\pi}^3/m_{K^*}^2]/[p_{\pi\pi}^3/m_{\rho}^2].$$

If the K<sup>\*</sup> mass is assumed to be 880 MeV, then from  $\Gamma_{\rho} = 100$  MeV we obtain  $\Gamma(K^*) = 30$  MeV, which is not far from the observed K<sup>\*</sup> width ( $\Gamma_{exp}$ , 47 MeV).

In the unitary symmetry model there is an open possibility for a vector meson coupled to the baryon current. This will be a unitary singlet vector meson since the baryon current is of the form

$$(\overline{pp} + \overline{nn} + \overline{\Lambda} \Lambda + \dots) = Trace (\overline{BB}),$$

which is obviously a unitary singlet. Note also that it is impossible to construct a vector current that transforms like a unitary singlet with pseudoscalar mesons (nor with vector mesons). For instance, we cannot construct a vector current bilinear in the  $\eta$  meson. So if there exists a unitary singlet vector meson, it must necessarily be the kind coupled to the baryon current.

Let us now summarize the predictions of the unitary symmetry model based on the Gell-Mann - Neéman octet. First of all, all members of a unitary symmetry multiplet must have the same spin-parity. From this point of view the most crucial test is the spin of  $\Xi$ , which in the octet model must be 1/2. Also the K\* spin must be 1 (or else there must be some other K $\pi$  resonance with spin 1).

As for the parities of the baryons and mesons, once we define the parity i.e.  $\Gamma \Sigma$  even, and N $\Xi$  even. The K meson must be pseudoscalar with reto be even by convention, then all baryons must have the same parity, i.e.  $\Lambda\Sigma$  even, and N $\Xi$  even. The K meson must be pseudoscalar with respect to both N $\Lambda$  and N $\Sigma$ . When the octet model was proposed, the parity was not known to be odd, nor was there any evidence for the T = 0 pseudescalar  $\eta$ meson.

The second prediction of the unitary symmetry model is that all members of a unitary symmetry multiplet must have the same mass. Experimentally, we know that this "prediction" is not fulfilled (otherwise unitary symmetry would have been discovered many years ago). However, if unitary symmetry is broken only in lowest order, there are many interesting mass relations that can be checked experimentally.

Let us go back to our "mathematical" model of baryons in which the baryons are composed of  $\ell$  and I particles. If unitary symmetry is broken, the D-S mass difference and the  $(e\nu)-\mu$  mass difference need not be zero. But let us assume that the forces that bind  $\ell$  and I are independent of strangeness and isospin; otherwise, we would be considering higher order violations of unitary symmetry. Assuming for simplicity that the binding energies are zero, we have

$$m_N = m_s + m_e$$
  
 $m_\Lambda = (2/6)(m_D + m_e) + (4/6) (m_s + m_\mu)$   
 $m_\Sigma = m_D + m_e$   
 $m_Z = m_D + m_\mu$ .

From these relations, it follows:

$$(m_N + m_Z)/2 = (3m_A + m_\Sigma)/4,$$
 (4.13)

as first noted by Gell-Mann. Experimentally, the left-hand side gives 1127 MeV while the right-hand side is equal to 1134 MeV.

A similar relation holds for the pseudoscalar octet, but, as suggested by Feynman, it is better to work with  $(mass)^2$ .

$$m_{K}^{2} = (3m_{\eta}^{2} + m_{\pi}^{2})/4.$$
 (4.14)

Experimentally, for the left-hand side we have  $(495 \text{ MeV})^2$ ; for the right-hand side,  $(480 \text{ MeV})^2$ .

The mass relation is not so good for the vector mesons. The mass formula with the observed  $\rho$  and K\* mass predicts the T = 0 member of the octet at 920 MeV (rather than at 780 MeV). Perhaps the observed  $\omega$  meson is a unitary singlet and a second T = 0 vector meson yet to be discovered is the T = 0 member of the vector meson octet. Or else it may well be that the two T = 0 vector mesons get mixed up in a complicated way; perhaps this kind of mixing is responsible for breakdown of unitary symmetry. But all this is very speculative.

If we take unitary symmetry seriously, baryon isobars must also be classified according to various unitary symmetry multiplets. The representation 8 is obviously inadequate to describe the 3-3 resonance. It is possible to build up multiplets with higher dimensions by decomposing  $8 \times 8$  just as we decomposed  $3 \times 8$  into 1 + 3. It can be shown that

$$8 \times 8 = 1 + 8 + 8 + 10 + \overline{10} + 27$$
.

Each representation can further be decomposed into various ordinary multiplets with hypercharge and isospin. This is summarized in Table II.

| 1  | Y ≈ 0,           | T = 0        |
|----|------------------|--------------|
| 8  | Y ≈ 1            | T = 1/2      |
|    | $\mathbf{Y} = 0$ | T = 0, 1     |
|    | Y = -1           | T = 1/2      |
| 10 | <b>Y</b> ≈ 1     | T = 3/2      |
|    | $\mathbf{Y} = 0$ | T = 1        |
|    | Y = -1           | T = 1/2      |
|    | <b>Y</b> = -2    | T = 0        |
| 10 | <b>Y</b> = 2     | T = 0        |
|    | Y = 1            | T = 1/2      |
|    | $\Upsilon = 0$   | T = 1        |
|    | Y = -1           | T = 3/2      |
| 27 | <b>Y</b> = 2     | T = 1        |
|    | <b>Y</b> = 1     | T = 1/2, 3/2 |
|    | $\mathbf{Y} = 0$ | T = 0, 1, 2  |
|    | Y = -1           | T = 1/2, 3/2 |
|    | <b>Y</b> = -2    | T = 1        |
|    |                  |              |

TABLE II

Various excited baryons can be discussed within the framework of the representations listed above [13]. If we use the following pieces of information taken from experiments:

- (i) The  $Y_1^*$  (1385) spin is most likely 3/2;
- (ii) The Y<sup>\*</sup><sub>d</sub> (1405) is probably an s-wave KN bound state,
   ∴ Y<sup>\*</sup><sub>1</sub> spin ≠ Y<sup>\*</sup><sub>d</sub> spin;
- (iii) There is no resonance in K<sup>+</sup>p scattering,

then it is natural to let the 3-3 resonance and the 1385 MeV  $Y_1^{\pm}$  belong to the representation 10. Note that we then predict a  $T = \frac{1}{2}$ ,  $Y = -1\pi \Xi$  resonance. This may be identified with the recently discovered  $\Xi_{\pm}^{\pm}$  at 1530 MeV.

Meanwhile Okubo was able to generalize the Gell-Mann mass formula to any unitary symmetry multiplet as follows:

$$m = m_0 \{ 1 + aY + b [T (T + 1) - (Y^2/4)] \}.$$
 (4.15)

For the representation 8, the formula reduces to (4.13) and (4.14). But for the representation 10, because of the linear relation T = 1 + Y/2, the quadratic terms in (4.15) cancel each other. So we are led to the "equal-spacing rule" (emphasized by Gell-Mann at the CERN conference):

$$m = m'_0 (1 + a'Y).$$

Experimentally;

 N<sup>\*</sup><sub>3/2</sub>
 1238 MeV

 Y<sup>\*</sup><sub>1</sub>
 1385 MeV

 Ξ<sup>\*</sup><sub>1/2</sub>
 1535 MeV

in fantastic agreement with the mass formula. (Moreover, if we assume that the parameters a and b are common for the baryon octet and the 10 isobars, then even the spacing parameter is correctly predicted.) If we take the mass formula seriously, there sould be a Y = -2 (strangeness = -3) singlet at ~ 1685 MeV. But the predicted mass of this object (denoted by Z<sup>-</sup>) is below the  $\overline{KE}$  threshold. Therefore Z<sup>-</sup> should be stable against decay via strong interactions.

It may be produced via

$$\begin{array}{l} \mathbf{K}^{-} + \mathbf{p} \rightarrow \mathbf{Z}^{-} + \mathbf{K}^{0} + \mathbf{K}^{+} \\ \Xi^{-} + \mathbf{p} \rightarrow \mathbf{Z}^{-} + \mathbf{p} + \mathbf{K}^{0} \\ \overline{\mathbf{p}} + \mathbf{p} \rightarrow \mathbf{Z}^{-} + \overline{\mathbf{Z}}^{-} \text{ etc.} \end{array}$$

It is expected to decay into

 $Z^- \rightarrow \pi + \Xi, \overline{K} + \Lambda, \overline{K} + \Sigma$ 

via weak interactions (long lifetime). Should the Z<sup>-</sup> be found experimentally, our confidence in unitary symmetry would grow by an order of magnitude.

#### REFERENCES

- [1] WIGNER, E.P., Proc. Natl. Acad. of Sci. 38 (1952) 449.
- [2] LEE, T.D. and YANG, C.N., Phys. Rev. 98 (1955) 1501.
- [3] YANG, C.N. and MILLS, R.L., Phys. Rev. 96 (1954) 191.
- [4] FUJII, Y., Progr. Theoret. Phys. 21 (1959) 232
- [5] SAKURAI, J.J., Ann. Phys. 11 (1960) 1.

- [6] SALAM, A. and WARD, J.C., Nuovo Cim. 20 (1961) 419.
- [7] GELL-MANN, M., Phys. Rev. 125 (1962) 1067; NE'EMAN, Y., Nucl. Phys. 26 (1961) 222.
- [8] GELL-MANN, M. and ZAHARIASEN, F., Phys. Rev. 124 (1961) 953.
- [9] HAMILTON, J., SPEARMAN, T.D. and WOOLCOCK, W.S., Ann. Phys. 17 (1962) 1.
- [10] NAMBUR, Y. and SAKURAI, J.J., Phys. Rev. Letters 8 (1962) 79.
- [11] GELL-MANN, M., SHARP, D. and WAGNER, W.G., Phys. Rev. Letters 8 (1962) 261.
- [12] CABIBBO, N. and GATTO, R., Phys. Rev. 124 (1961) 1517.
- [13] GLASHOW, J. and SAKURAI, J.J., Nuovo Cim. 25 (1962) 837.

## OCTONIONS AND SUPER-GLOBAL SYMMETRY

#### J. TIOMNO

### BRAZILIAN CENTRE OF RESEARCH IN PHYSICS, RIO DE JANEIRO, BRAZIL

#### INTRODUCTION

It has been recently pointed out by PAIS [1] that the use of the octonions (Cayley numbers) for the description of the baryonic and mesonic fields leads to the interaction with the 7-dimensional orthogonal symmetry  $(0_7)$  proposed a few years ago [2] and which is called G<sub>b</sub> by Pais.

Although this "super-global" interaction has too many symmetries, thus leading to some unwanted selection rules as well as all the needed ones, it may be present as part of the strong interactions and may play some role in weak interactions.

This paper will consider properties of the octonion algebra, its relation to the  $G_{-}$  interaction and some useful tricks in playing with doublet symmetry.

### 1. QUATERNIONS AND GLOBAL SYMMETRY

It is well known that quaternions may be used to describe spin and that their use for building strong interactions leads to global symmetry.

A quaternion is a hypercomplex number of the form

$$X = x_0 + x_i e_i$$
 (1)

(summation over i = 1, 2, 3 is here understood), where the  $e_i$ 's have the following multiplication property :

$$\mathbf{e}_{i} \mathbf{e}_{j} = -\delta_{ii} + \Sigma_{ijk} \mathbf{e}_{k}$$
(2)

with  $\Sigma_{ijk}$  totally antisymmetric ( $\Sigma_{123}$  = 1).

For real quaternions  $(x_0, x_i \text{ real})$  the following property, true also for real and complex numbers, holds:

$$N(AB) = N(A) N(B), \qquad (3)$$

where  $N(X) = x_0^2 + x_i x_i = x^2$  is called the norm of the quaternion. If we define the scalar of a quaternion by  $S(X) = x_0$  we have

$$N(X) = S(\overline{X} X) = x_0^+ x_0^+ x_i^+ x_i^+, \qquad (4)$$

where  $\overline{X}$  is the adjoint of X:

$$\overline{\mathbf{X}} = \mathbf{x}_0^+ - \mathbf{e}_i \mathbf{x}_i^+, \tag{5}$$

 $x_0^+$ ,  $x_i^+$  representing the complex conjugates of  $x_0$  and  $x_i$  for c-quaternions  $(x_0^+ = x_0, x_i^+ = x_i$  for real quaternions) and the hermitian conjugates for q-quaternions (i.e. quaternions having q-number fields as components).

From the product rule (2) it follows that

$$\mathbf{e}_{i}(\mathbf{e}_{j} \mathbf{e}_{k}) = (\mathbf{e}_{i} \mathbf{e}_{j}) \mathbf{e}_{k}; \tag{6}$$

and thus quaternion multiplication is associative :

$$A(BC) = (AB)C.$$

The quaternion algebra forms a group and is isomorphic to the algebra of Pauli matrices (of square -1).

Also, as the substitution

$$e_i \rightarrow e'_i = a_{ij} e_i$$

leaves the product rule invariant:

$$\mathbf{e}_{i}' \mathbf{e}_{j}' = -\delta_{ij} + \Sigma_{ijk} \mathbf{e}_{k}'$$

if  $(a_{ij})$  are orthogonal  $(3 \times 3)$  matrices, we find that the continuous automorphism of  $e_i$  generates the group  $0_3$ .

Let now

$$B = b_0 - e_i b_i; b_0 = \Lambda, b_i = + i \Sigma_i,$$

$$\Phi = \Phi_0 - i \Phi_i e_i; \Phi_0 = \eta_0 \Phi_i = \pi_i.$$
(7)

The equation

$$DB \equiv (i \not a + m) B = i g \gamma_5 \Phi B$$
(8)

can be expanded accordingly to (2) as

$$\begin{split} & \mathrm{D}\,\mathbf{b}_0 \,=\, \mathbf{g}\,(\Phi_0 \,\,\mathbf{b}_0 \,-\, \mathrm{i}\,\Phi_i \,\,\mathbf{b}_i) \,\rightarrow\, \mathrm{D}\Lambda \,=\, \mathbf{g}(\eta_0\,\Lambda \,+\, \overrightarrow{\pi} \,\cdot\, \overrightarrow{\Sigma}) \\ & \mathrm{D}\,\mathbf{b}_{\mathbf{k}} \,=\, \mathbf{g}\,(\Phi_0 \,\,\mathbf{b}_{\mathbf{k}} \,+\, \mathrm{i}\,\mathbf{b}_0\,\Phi_{\mathbf{k}} \,+\, \mathrm{i}\,\Sigma_{ijk}\,\,\Phi_i \,\,\mathbf{b}_j) \,\rightarrow\, \mathrm{D}\,\overrightarrow{\Sigma} \,=\, \mathbf{g}(\eta_0\overrightarrow{\Sigma} \,+\, \overrightarrow{\pi}\Lambda \,+\, \mathrm{i}\,\overrightarrow{\pi}\times\,\overrightarrow{\Sigma}). \end{split}$$

These equations may also be written

$$Db_{\mu} = g(\Phi_0 \ b_{\mu} - i\Phi_1 \ F_{i\mu\nu} \ b_{\nu}); \qquad \mu, \nu = 0, 1, 2, 3,$$

where

$$F_{iik} = \Sigma_{iik};$$
  $F_{i0k} = -F_{ik0} = \delta_{ik};$   $D_{i00} = 0$ 

or, in matrix notation,

$$Db = g(\Phi_0 - iF_i \Phi_i)b = g(\Phi_0 + \vec{\theta} \cdot \vec{\Phi})b, \qquad (9)$$

where

$$\mathbf{b} = \begin{pmatrix} \mathbf{b}_0 \\ \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{pmatrix} ; \quad (\mathbf{F}_i)_{uv} = \mathbf{F}_{i\mu v}.$$

The  $\theta_i$  matrices are

$$\theta_{i} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}; \quad \theta_{2} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}; \quad \theta_{3} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix};$$

that is,

$$\theta_i = 1 \times \sigma_2; \quad \theta_2 = \sigma_2 \times \sigma_3; \quad \theta_3 = \sigma_2 \times \sigma_1$$
(10)

(the  $\sigma^{*}s$  being the Pauli matrices). From (10) and the properties of the inner product,

$$(\mathbf{Q} \times \mathbf{R})(\mathbf{Q}' \times \mathbf{R}') = \mathbf{Q}\mathbf{Q}' \times \mathbf{R}\mathbf{R}'$$
$$\mathbf{Q} = \gamma_0 + \sigma_i \gamma_i; \mathbf{R} = \gamma_0 + \sigma_i \gamma_i, \text{ etc.},$$

and of the Pauli matrices, it follows:

$$\theta_{i} \theta_{j} = \delta_{ij} + i \Sigma_{ijk} \theta_{k}.$$
 (11)

Notice that in (7), B and  $\Phi$  are written differently. This was done in order to obtain the simpler representation (10) for  $\theta_i$ . Now if we write

$$B' = b'_0 - i\Phi'_i \cdot e_i; \quad b'_0 = \Lambda, \quad b'_i = \Sigma_i, \quad (7a)$$

we get, instead of (9),

$$Db' = g(\Phi_0 + \vec{\theta'} \cdot \vec{\Phi})b'$$
 (9a)

with

$$\mathbf{b'} = \mathbf{U} \mathbf{b}; \quad \mathbf{U} = \begin{pmatrix} 1 & & \\ & -i & \\ & & -i \end{pmatrix}$$

As b and b' in (9) and (9a) are connected by a unitary transformation, also  $\theta_i'$  satisfies the multiplication rule.

$$\theta_i' \theta_j' = \delta_{ij} + i \Sigma_{ijk} \theta_k'.$$
 (11a)

#### J. TIOMNO

The advantage of using first the representation (7) was that (11) followed straightforwardly from (10). Here it would be almost as simple to prove (11a) directly from the matrices  $\theta_i$ '. This would be, however, cumbersome in the case of octonions, in which case the simpler representation  $\theta_i$  will lead immediately to the multiplication properties.

We now consider the doublet representation. Due to (11a), it is possible to find a unitary transformation:

$$\Psi = T b \tag{12}$$

such that

$$T \theta_{i}' T^{-1} = T U \theta_{i} U^{-1} T^{-1} = \tau_{i} = 1 \times \sigma_{i}.$$
(13)

Now we can easily show that

$$\mathbf{b} = (\mathbf{b}_0 + \vec{\mathbf{b}} \cdot \vec{\theta}) \mathbf{w}_0 , \quad \mathbf{w}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus as b' = U b; U  $w_0 = w_0$ , we also have

$$\mathbf{b}' = (\mathbf{b}_0' + \vec{\mathbf{b}}' \cdot \vec{\theta}') \mathbf{w}_0$$

and, finally

$$\Psi = (\mathbf{b}_0' + \vec{\mathbf{b}}' \cdot \vec{\tau}) \Omega_0 \cdot \Omega_0 = \mathbf{T} \omega_0 = \mathbf{T} \mathbf{U} \omega_0.$$
(14)

We have therefore

$$D\Psi = i g \gamma_5 (\Phi_0 + \vec{\tau} \cdot \vec{\Phi}) \Psi.$$
 (9b)

In order to determine  $\Omega_0$  we write

$$\theta_i = M_i + N_i; \quad i = 1, 2, 3,$$
 (15a)

where  $M_i$  is obtained from  $\theta_i$  by making all elements of the first row and first column vanish. We see that

$$x_i = M_i - N_i; \quad j = 1, 2, 3,$$
 (15b)

commutes with all  $\theta_i$  and also satisfies

$$\mathbf{x}_i \mathbf{x}_j = \delta_{ij} + i \Sigma_{ijk} \mathbf{x}_k$$

Thus  $x_i$  must be transformed as

$$T U x_i U^{-1} T^{-1} = \eta_i = \sigma_i \times 1.$$

Finally, as  $\vec{M} w_0 = 0$ , we get

$$(\vec{\tau} + \vec{\eta})\Omega_0 = 0. \tag{16}$$

Thus one finds that

$$\Omega_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0\\1\\-1\\0 \end{pmatrix}$$
(17)

This could be found, of course, directly from TU, which transforms  $\vec{\theta}$  into  $\vec{\tau}$ . Also, as  $\overline{w}_0 \ \vec{\theta} \ w_0 = 0$ , we get

$$\vec{\Omega}_0 \vec{\tau} \Omega_0 = 0. \tag{18}$$

Now we can build the doublet  $\Psi$  from (14) and (16):

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} b_1 - i b_2 \\ b_0 - b_3 \\ b_3 + b_0 \\ b_1 + i b_2 \end{pmatrix} = \begin{pmatrix} \Sigma_+ \\ Y_0 \\ Z_0 \\ \Sigma_- \end{pmatrix}; \quad Y_0 = \frac{\Lambda - \Sigma_0}{\sqrt{2}}, \quad Z_0 = \frac{\Lambda + \Sigma_0}{\sqrt{2}}. \quad (19)$$

It should be mentioned that from (14), (16) and (18) we obtain, without having to use (19) explicitly,

$$\overline{\Psi}\Psi = \overline{\Lambda}\Lambda + \overline{\Sigma}_i \Sigma_i = \overline{b}b, \qquad (19a)$$

$$(1/2)\overline{\Psi}(\vec{\tau} + \vec{\eta})\Psi = i\vec{\Sigma} \times \vec{\Sigma}, \qquad (19b)$$

$$(1/2)\overline{\Psi}(\vec{\tau} - \vec{\eta})\Psi = \vec{\Sigma}\Lambda + \vec{\Lambda}\vec{\Sigma}, \qquad (19c)$$

$$\overline{\Psi}\vec{\tau}\Psi = i\vec{\Sigma}\times\vec{\Sigma} + \vec{\Sigma}\Lambda + \overline{\Lambda}\Sigma = \vec{b}\vec{\theta}b; \qquad (19d)$$

## (19a) and (19d) are also easily obtained starting from b or b' representations. Thus we have for the interaction Hamiltonian density :

$$H_{in} = ig \overline{\Psi} \beta \gamma_5(\Phi_0 + \vec{\Phi} \cdot \vec{\tau}) \Psi = ig \overline{b} \beta \gamma_5(\Phi_0 + \vec{\Phi} \cdot \vec{\theta}) b = ig S(\overline{B} \beta \gamma_5 \Phi B).$$
(20)

This is invariant in  $0_3$  as it is obvious in quaternion notation:

$$\mathbf{B} = \mathbf{b}_0 + \mathbf{e}_i \mathbf{b}_i = \mathbf{b}_0 + \mathbf{e}'_i \widetilde{\mathbf{b}}_i,$$

with

$$\mathbf{e}_i' = \mathbf{a}_{ij} \mathbf{e}_j, \quad \widetilde{\mathbf{b}}_i = \mathbf{a}_{ik} \mathbf{b}_k, \quad \mathbf{a}_{ij} \mathbf{a}_{ik} = \delta_{jk}.$$

Thus b transforms as:

$$b \longrightarrow \exp(i \vec{M} \cdot \vec{\phi})b,$$
 (21a)

and  $\Psi$  as

$$\Psi \longrightarrow \exp\left(i\frac{\vec{\tau} + \vec{\eta}}{2} \cdot \vec{\varphi}\right)\Psi.$$
 (21b)

However, it is also invariant in  $0_4$ , in view of the additional transformation:

$$b \longrightarrow \exp(i\vec{N} \cdot \vec{x})b \quad ; \quad \Psi \longrightarrow \exp\left(i\frac{\vec{\tau} \cdot \vec{\eta}}{2} \cdot \vec{x}\right)\Psi.$$
 (22)

In (21) and (22)  $\Lambda$ ,  $\overrightarrow{\Sigma}$  transform as a 4-vector,  $\overrightarrow{\Phi}$  being a self-dual tensor and  $\Phi_0$  a scalar [3].

 ${\rm H_{in}}$  is then also invariant in S  ${\rm U_2}$  X S  ${\rm U_2}$  , first by the i-spin rotation :

$$\Psi \longrightarrow \exp\left(i\frac{\vec{\tau}}{2}\cdot\vec{\varphi}\right)\Psi, \qquad (23)$$

for which  $\Psi_a = \begin{pmatrix} \Psi_1 \\ \Psi_4 \end{pmatrix}$  and  $\Psi_b = \begin{pmatrix} \Psi_3 \\ \Psi_1 \end{pmatrix}$  transform as two spinors, and by the transformation

$$\Psi \longrightarrow \exp\left(i\frac{\vec{\eta}}{2}\cdot\vec{\phi}\right)\Psi, \qquad (24)$$

which  $\min \Psi_a$  and  $\Psi_b$ .

Thus if interaction (20) holds both for  $(\Lambda, \vec{\Sigma})$  and for  $(N, \Xi)$ (global symmetry), the i-spin properties of baryons would be undetermined [2, 3, 4]. Additional less symmetrical interactions would make the choice of (21) for  $(\Lambda, \vec{\Sigma})$  and (22) for  $(N, \Xi)$ .

#### 2. OCTONIONS

An octonion is a hypercomplex number

٦

$$X = x_0 + e_i x_i \tag{25}$$

(summation over i = 1, ....7) with themultiplication property

$$\mathbf{e}_{\mathbf{i}} \, \mathbf{e}_{\mathbf{j}} = - \, \delta_{\mathbf{i}\mathbf{j}} + \, \Sigma_{\mathbf{i}\mathbf{j}\mathbf{k}} \, \mathbf{e}_{\mathbf{k}} \tag{26}$$

with  $\Sigma_{ijk}$  totally antisymmetric, with values  $\pm 1$  and 0 and with only one non-vanishing  $\Sigma_{ijk}$  for a given pair (ij). They have the additional property

$$e_i(e_i A) + e_i(e_i A) = -2\delta_{ii} A,$$
 (27)

which assures the norm property :

$$N(A B) = N(A) N(B)$$
(28)

for real octonions (notation is the same as for quaternions except for the fact that summations are now from 1 to 7).

In order to prove (28) we notice first that

$$\mathbf{e}_{i}(\mathbf{e}_{i} \mathbf{e}_{k}) - (\mathbf{e}_{i} \mathbf{e}_{i}) \mathbf{e}_{k} = \mathbf{f}_{ijkr} \mathbf{e}_{r}.$$
(29)

So that

$$S[e_i(e_je_k)] = S[(e_ie_j)e_k]$$

 $\mathbf{or}$ 

$$S[A(B C)] = S[(A B)C].$$

(30)

Therefore we get

$$N(A B) = S[(\overline{B}\overline{A})(A B)] = S[\overline{B}(\overline{A}(A B))]$$
$$= a^{2} S(\overline{B} B) = a^{2} b^{2} = N(A) N(A).$$

Here we used  $\overline{A}(A B) = a^2 B$  (for real octonions A) which results from (27), thus showing that (27) implies (28). It should be noted that (30) is correct even if the octonion multiplication is not associative, as we shall find ( $f_{ijkr}$  in (29) is not zero, as for quaternions).

Now we proceed to build the multiplication table for octonions. From the multiplication rule (26) the product of two octonion unities is another one, except for a sign which may in some cases be absorbed in the unity. So we pick two unities  $e_1$  and  $e_2$  and define

$$e_3 = e_1 e_2.$$
 (31)

Now we pick another unity  $e_4$  different from  $e_1$ ,  $e_2$  and  $e_3$  and define

$$e_5 = e_4 e_1 , e_6 = e_4 e_2 , e_7 = e_4 e_3.$$
 (32)

Indeed, from (26) these products should give hypercomplex unities and they should be all different. We now obtain the other multiplication rules from (31) and (32) by the use of (27):

$$e_{1} e_{6} = e_{1}(e_{4} e_{2}) = -e_{4}(e_{1} e_{2}) = -e_{4} e_{3} = -e_{7},$$

$$e_{2} e_{5} = e_{2}(e_{4} e_{1}) = -e_{1}(e_{2} e_{1}) = -e_{4} e_{3} = -e_{7},$$

$$e_{3} e_{5} = e_{3}(e_{4} e_{1}) = -e_{4}(e_{3} e_{1}) = -e_{4} e_{2} = -e_{6}.$$
(33)

J. TIOMNO

Thus we have found the non-vanishing  $\Sigma_{ijk}$ 's:

$$\Sigma_{123} = \Sigma_{415} = \Sigma_{426} = \Sigma_{437} = \Sigma_{176} = \Sigma_{257} = \Sigma_{365} = 1.$$
(34)

All other products give the same result as (31 - 33) if the antisymmetric properties of  $\Sigma_{ijk}$  are taken into account. The multiplication coefficients (34) are different from those of CAYLEY [5] but reduce to them by the change of the sign of the basic unit  $e_4(e_4 \rightarrow -e_4)$ . They differ also from those given by PAIS [1] but become identical by the substitution

 $e_4 \rightleftharpoons e_5 = e_6 \rightleftarrows e_7$ .

The non-associativity of octonion multiplication can be shown immediately. Thus we find

$$(\mathbf{e}_{i} \ \mathbf{e}_{i}) \mathbf{e}_{k} = \mathbf{e}_{i}(\mathbf{e}_{i} \ \mathbf{e}_{k}) = -\Sigma_{iik}$$

if (ijk) belongs to a triad represented in (34) or if two of the indices are equal, and

$$(\mathbf{e}_i \ \mathbf{e}_j) \mathbf{e}_k = - \mathbf{e}_i (\mathbf{e}_j \ \mathbf{e}_k)$$

for all other triads.

Although the octonion algebra does not form a group because of the nonassociativity of multiplication, it generates the group  $G_2$ . Indeed if

$$e_{i} \longrightarrow e_{i}' = a_{ij}e_{j},$$
$$e_{i}' e_{j}' \longrightarrow -\delta_{ij} + \Sigma_{ijk} e_{k}',$$

we must have

$$a_{ij} a_{ik} = \delta_{jk}$$

$$a_{2i} a_{sj} \Sigma_{ijk} a_{kt} = \Sigma_{2st}$$
(35)

# 3. NON-EXISTENCE OF OTHER HYPERCOMPLEX ALGEBRAE WITH THE PROPERTY (3)

If we try to generalize the octonions including additional units but keeping (25 - 27), we run immediately into contradiction. Thus if we have more than seven units of square (-1), we build  $e_1 \dots e_7$  by the same procedure as before, thus obtaining the multiplication coefficients (34). Now, calling  $e_8$  a unity different from these, we form

$$e_8 e_i = e_{8+i}$$
;  $i = 1, \dots, 7$ .

Thus we shall have 16 hypercomplex unities. In particular

 $e_8 e_4 = e_{12}$   $e_{12}(e_8 e_4) = -1.$ 

We now form

$$\lambda = e_1\{e_2(e_3[e_{12}(e_8 e_4)])\} = -e_1[e_2(e_3 \cdot 1)] = -e_1^2 = +1.$$

Now, using successively (27) we find, however,

$$\lambda = -e_{12} \{e_8(e_4[e_1(e_2 e_3)])\} = e_{12}(e_8 e_4) = -1.$$

Thus we cannot generalize the octonions if we keep relations (26) with the condition that the product of two units  $e_i e_j$  gives another unit  $e_k$  (except for a sign) as well as condition (27).

Thus, instead of (26), we take the more general relation,

$$\mathbf{e}_{i} \mathbf{e}_{j} = -\delta_{ij} + \mathbf{f}_{ijk} \mathbf{e}_{k}$$
(36)

with  $f_{ijk}$  totally antisymmetric but not vanishing for several k's for a given (i j) pair.

We also take instead of (27) the norm condition (28) for

$$A = a_0 + a_i e_i, \qquad B = b_0 + b_i e_i$$

with  $a_0$ ,  $a_i$ ,  $b_0$ ,  $b_i$  real.

We shall prove first that even in this more general case (28) implies (27). Indeed,

$$\mathbf{C} = \mathbf{A} \mathbf{B} = (\mathbf{a} \cdot \mathbf{b}) + (\mathbf{a}_0 \mathbf{b}_i + \mathbf{a}_i \mathbf{b}_0) \mathbf{e}_i + \mathbf{f}_{ijk} \mathbf{a}_i \mathbf{b}_j \mathbf{e}_k$$

with  $(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a}_0 \mathbf{b}_0 - \mathbf{a}_i \mathbf{b}_i$ .

If we write (28) and expand, we find

$$\mathbf{a}_{i} \mathbf{a}_{r} \mathbf{b}_{i} \mathbf{b}_{s} \mathbf{Q}_{ir;is} = \mathbf{0}, \tag{37}$$

where

$$Q_{ir;js} = Q_{ri;js} = Q_{ir;sj} = f_{ijk} f_{rsk} + f_{isk} f_{rjk} + \delta_{ij} \delta_{rs} + \delta_{is} \delta_{jr} - 2 \delta_{ir} \delta_{js}.$$
(38)

Thus, as (37) must vanish for arbitrary  $a_i$ ,  $b_i$ , (28) is equivalent to

$$Q_{ir,is} = 0.$$
 (39)

But since (27) is true for  $A = a_0$ , it is enough to prove it for  $A = e_r(r \text{ arbi-trary})$ . Expanding it, we find

 $e_{i} \delta_{jr} + e_{j} \delta_{ir} + f_{iks} f_{jrk} e_{s} + f_{jks} f_{irk} e_{s} = -2 \delta_{ij} e_{r}$ 

This may be written

$$Q_{iirs} e_s = 0$$

which is true if (39) holds. Thus (27) is a consequence of (28). Therefore the only real change from the previous scheme was in the substitution

 $\Sigma_{ijk} \longrightarrow f_{ijk}$ .

Now we shall prove that (38) and (39) exclude any hypercomplex number with n - 1 anticommuting unities except for n = 2 (complex numbers), 4 (quaternions) and 8 (octonions)[5]. Indeed, if we consider the antisymmetric matrices,

$$(F_i)_{\mu\nu}$$
;  $\mu, \nu = 0.1, \ldots n-1$ ,

defined by

 $(F_i)_{ik} = f_{ijk}$ ;  $(F_i)_{0j} = \delta_{ij}$ ;  $(F_i)_{j0} = -\delta_{ij}$ 

then (38) - (39) may be written

$$(\mathbf{F}_{\mathbf{i}} \ \mathbf{F}_{\mathbf{s}} + \mathbf{F}_{\mathbf{s}} \ \mathbf{F}_{\mathbf{i}})_{\mathbf{i}\mathbf{r}} = -2 \,\delta_{\mathbf{i}\mathbf{s}} \ \delta_{\mathbf{i}\mathbf{r}} \,.$$

Also, as

$$(F_{i} F_{s} + F_{s} F_{i})_{00} = -\delta_{js}$$
 ;  $(F_{i} F_{s} + F_{s} F_{i})_{0i} = 0$ 

we have

$$F_{i} F_{s} + F_{s} F_{i} = -2\delta_{si}$$
 (40)

Thus the  $F_j$ 's form a set of anticommuting  $n \times n$  matrices of square -1, all antisymmetric. Thus n is even.

Considering all antisymmetric products of two or more  $F_i$  matrices, we obtain  $2^{n-1}$  independent matrices (including the unit matrix) if  $n = 0 \pmod{2}$ , and  $2^{n-2}$  if  $n = 0 \pmod{4}$ . The reduction by one-half in the last case comes from the fact that  $F_1$   $F_2$  ....  $F_{n-1}$  is a multiple of unity. Now as there can be only  $n^2$  independent matrices of rank n, we have

$$2^{n-1} \le n^2$$
,  $n \ge 0 \pmod{2}$   
 $2^{n-2} \le n^2$ ,  $n \ge 0 \pmod{4}$ .

In the first case we get  $n \le 6$ . In the second case  $n \le 8$ . Thus, besides the possibilities  $n \ge 2$ , 4, 8 mentioned, we might have also  $n \ge 6$ . This is, however, excluded because we would then have 16 independent antisymmetric matrices (5  $F_i$ 's, 10  $[F_i, F_j]$ 's and the matrix  $F_1$   $F_2$   $F_3$   $F_4$   $F_5$ ). This is not possible because there are only 15 antisymmetric matrices of rank 6.

#### 4. OCTONION FIELD INTERACTIONS

We may proceed as in the quaternion case and take the equation

$$(i \not \partial + m)B = i g \gamma_5 \Phi B \qquad (41)$$

with

$$\mathbf{B} = \mathbf{b}_0 - \mathbf{e}_i \mathbf{b}_i \quad ; \quad \Phi = \Phi_0 - i\Phi_i \mathbf{e}_i. \tag{42}$$

Then we get

$$(i \not a + m)b = g(\Phi_0 - i F_i \Phi_i)b$$
 (43)

with

1

If we build the -  $i F_i$  matrices, we find

$$-iF_2 = \sigma_3 \times 1 \times \sigma_2 ; -iF_5 = \sigma_1 \times \sigma_3 \times \sigma_2$$
  

$$-iF_2 = \sigma_3 \times \sigma_2 \times \sigma_3 ; -iF_6 = \sigma_1 \times \sigma_2 \times 1$$
  

$$-iF_3 = \sigma_3 \times \sigma_2 \times \sigma_1 ; -iF_7 = \sigma_1 \times \sigma_1 \times \sigma_2$$
  

$$-iF_4 = \sigma_2 .$$

We thus easily verify that

$$\{F_{i}, F_{i}\} = -2\delta_{ii},$$

which we know is a consequence of (27). Also if we write

$$-iF_i = \sigma_3 \times \theta_i$$
,  $-i\Gamma_{i+4} = -\sigma_i \times X_i$ ;  $i = 1, 2, 3$ ,

where  $\theta_i$  are given by (10) and

$$\chi_1 = -\sigma_3 \times \sigma_2$$
;  $\chi_2 = -\sigma_2 \times 1$ ;  $\chi_3 = -\sigma_1 \times \sigma_2$ .

The  $X_j$  matrices are the same given by (15b). Thus we can find a canonical transformation which takes (43) into

$$(i \not a + m) \Psi = ig(\Phi_0 + \Gamma \Phi_i) \gamma_5 \Psi$$
(44)

with

$$\Gamma_{i} = \sigma_{3} \times \tau_{i} ; \ \Gamma_{4} = \sigma_{2} \times 1 ; \ \Gamma_{4+i} = \sigma_{1} \times \eta_{1} \quad (i = 1, 2, 3).$$
(45)

These are the same matrices  $\Gamma_{\mu}$  used in [2], Eq.(44) being the same then obtained if the scalar interaction  $i \overline{\Psi} \gamma_5 \Psi \Phi$  is added.

Another interesting way of obtaining directly equation (44) is the following. Although the octonion algebra is not associative and the  $\Gamma_i$  algebra is, there is an isomorphism between products from the left of octonions and projected products of linear combinations of  $\Gamma_i^{\bullet}s$ . Indeed, if we make the correspondence among the quantities

$$A = a_0 + e_i a_i \xrightarrow{\longrightarrow} \mathcal{A} = a_0 + i \Gamma_i e_i,$$

we shall also have

$$A(BC) = AD = E \longrightarrow \mathcal{A} \mathcal{B} \mathcal{C} \mathcal{P} = \mathcal{A} \mathcal{D} \mathcal{P} = \mathcal{L} \mathcal{P}$$

if a P exists with the property

$$(1/2) [\Gamma_{i}, \Gamma_{j}] P = -i \Sigma_{ijk} \Gamma_{k} P \qquad (i, j, k = 1, ..., 7).$$
(46)

We find immediately that (46) is true for  $\Gamma_i$  given by (45) if

$$\mathbf{e}_{3}\mathbf{P} = (\sigma_{3} \times \mathbf{1} \times \mathbf{1}) \mathbf{P} = -\mathbf{P},$$
  
$$(\vec{\tau} + \vec{\eta})_{8} = [\mathbf{1} \times (\vec{\tau} + \vec{\eta})_{4}] \mathbf{P} = \mathbf{0}.$$

Thus P is given by

$$P = [(1 + e_3)/2] [(1 + \vec{\tau} \cdot \vec{\eta})/4] ; P^2 = P.$$
(47)

where we used for short  $1 \times \vec{\tau} \longrightarrow \vec{\tau}$ ,  $1 \times \vec{\eta} \longrightarrow \vec{\eta}$ . The relation

$$A(BC) + B(AC) = (a \cdot b)c$$
;  $a \cdot b = a_0b_0 - a_ib_i$ 

corresponding to (27) also leads to

$$[\mathcal{A} \mathcal{B} \mathcal{G} + \mathcal{B} \mathcal{A} \mathcal{G}] \mathcal{P} = (a \cdot b) \mathcal{G} \mathcal{P},$$

this being a consequence of the anticommutation property of  $\Gamma_i$ 's. Thus if

$$B = b_0 - i e_i b_i ; \Phi = \Phi_0 - i \Phi_i e_i$$
$$B = b_0 + \Gamma_i b_i ; \Phi = \Phi_0 + \Gamma_i \Phi_i.$$

Eq. (41) leads to

$$(i \mathcal{J} + m) \mathcal{B}P = ig \gamma_{\varepsilon} \mathcal{B}P.$$

If we now take

$$\Omega = \mathbf{P}\mu,$$

$$\mu = \begin{pmatrix} 0\\0\\0\\-\\1\\0\\0\\0 \end{pmatrix} ; \quad \Omega = \frac{1}{\sqrt{2}} \begin{pmatrix} 0\\0\\0\\-\\-\\0\\1\\-1\\0 \end{pmatrix} = \left(\frac{0}{\Omega_0}\right) ;$$

$$\Psi = \mathcal{B} \ \Omega_0,$$

we obtain (44) with

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}; \Psi_1 = \begin{pmatrix} \mathbf{P} \\ \mathbf{N} \\ \Xi_0^0 \\ \Xi \end{pmatrix} = \frac{-1}{\sqrt{2}} \begin{pmatrix} \mathbf{b}_5 + \mathbf{i} \mathbf{b}_6 \\ -\mathbf{b}_7 + \mathbf{i} \mathbf{b}_4 \\ \mathbf{b}_7 + \mathbf{i} \mathbf{b}_4 \\ \mathbf{b}_5 - \mathbf{i} \mathbf{b}_6 \end{pmatrix} ; \Psi_2 = \begin{pmatrix} \Sigma^+ \\ \mathbf{Y}^0 \\ \mathbf{Z}^0 \\ \Sigma^- \end{pmatrix}$$

Notice that

$$\sum_{4}^{7} \Gamma_{i} \mathbf{b}_{i} \Omega = - (+ i \mathbf{b}_{4} + \vec{\tau} \cdot \vec{\mathbf{b}}) \Omega'; \quad \Omega' = + i \Gamma_{2} \Omega = \begin{pmatrix} \Omega_{0} \\ 0 \end{pmatrix},$$

where

$$\vec{b} = (b_5, b_6, b_7).$$

Also

$$\begin{split} & \overset{3}{\overset{\Sigma}{\overset{\Sigma}{\underset{i=0}{\sum}}} \overline{\Psi} \ \Gamma_{i} \ \overrightarrow{\phi_{i}} \Psi = \overline{\Psi_{i}} \overrightarrow{\phi} \cdot \overrightarrow{\tau} \Psi_{1} - \overline{\Psi_{2}} \overrightarrow{\phi} \cdot \overrightarrow{\tau} \Psi_{2} \\ & \overset{7}{\overset{\Sigma}{\overset{\Sigma}{\underset{i=4}{\sum}}} \overline{\Psi} \ \Gamma_{i} \ \phi_{i} \Psi = \overline{\Psi_{1}} (-i\kappa_{4} + \overrightarrow{\eta} \cdot \overrightarrow{\kappa}) \Psi_{2} + \text{h.c.} \\ & = \overline{\Psi_{1}} (-i\kappa_{4} + \overrightarrow{\eta} \cdot \overrightarrow{\kappa}) (\Lambda + \overrightarrow{\tau} \cdot \overrightarrow{\Sigma}) \Omega_{0} + \text{h.c.} \\ & = \overline{\Psi_{1}} (\Lambda + \overrightarrow{\tau} \cdot \overrightarrow{\Sigma}) \kappa + \text{h.c.} , \end{split}$$

where

$$\kappa = (-i\kappa_4 + \vec{\eta} \cdot \vec{\kappa})\Omega_0 = -(i\kappa_4 + \vec{\tau} \cdot \vec{\kappa})\Omega_0$$

$$-\kappa = \frac{1}{\sqrt{2}} \qquad \begin{pmatrix} \kappa_1 + i\kappa_2 \\ -\kappa_3 + i\kappa_4 \\ \kappa_3 + i\kappa_4 \\ \kappa_1 - i\kappa_2 \end{pmatrix}$$

Thus

$$\begin{split} & \overline{\Psi} \, \Gamma_{i} \Phi_{i} \Psi = \overline{\Psi_{1}} \vec{\Phi} \cdot \vec{\tau} \, \Psi_{1} \, - \, i \, \vec{\vec{\Sigma}} \times \vec{\vec{\Sigma}} \cdot \vec{\Phi} \\ & - \vec{\vec{\Sigma}} \, \Lambda \, \vec{\Phi} \, - \, \Lambda \, \vec{\vec{\Sigma}} \cdot \vec{\Phi} \, + \overline{\Psi_{1}} \, (\Lambda + \vec{\tau} \cdot \vec{\vec{\Sigma}}) \kappa \, + \, \overline{\kappa} (\overline{\Lambda} + \vec{\tau} \cdot \vec{\vec{\Sigma}}) \Psi_{1} \\ & \overline{\Psi} \Psi \Phi_{0} \, = \, \Phi_{0} \, (\overline{\Psi_{1}} \, \Psi_{1} \, + \, \overline{\vec{\Sigma}} \cdot \vec{\Sigma} \, + \, \overline{\Lambda} \Lambda). \end{split}$$

Thus only using the properties of  $\Omega$  and  $\Omega_0$  we have been able to write the super-global interaction in the Gell-Mann Nishijima multiplet notation.

#### REFERENCES

- [1] PAIS, A., Phys. Rev. Letters 7 (1961) 291.
- [2] TIOMNO, J., Proc. Conf. on High-Energy Physics, Rochester (1957).
- [3] SCHWINGER, J., Proc. Conf. on High-Energy Physics, Rochester (1957).
- [4] GELL-MANN, M., Proc. Conf. on High-Energy Physics, Rochester (1957).
- [5] DICKSON, Ann. Math. 20 (1918) 155.

# POMERANCHUK CONJECTURE AND SYMMETRY SCHEMES

# A. GALINDO, A. MORALES AND H. RUEGG JUNTA DE ENERGÍA NUCLEAR, MADRID, SPAIN EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH, GENEVA, SWITZERLAND UNIVERSITY OF GENEVA, GENEVA, SWITZERLAND

#### 1. INTRODUCTION

Pomeranchuk has conjectured that the cross-sections for charge-exchange processes vanish asymptotically as the energy tends to infinity. (By "charge" it is meant any internal quantum number, like electric charge, hypercharge, ...). It has been stated by several people [1] that this conjecture implies equalities among the total cross-sections whenever any symmetry scheme is invoked for the strong interactions. But to our knowledge no explicit general proof of this statement has been given so far. We want to give this proof for any compact Lie group\*. We also prove, under certain assumptions, that the equality of the total cross-sections implies that s<sup>-f</sup> times the charge-exchange forward scattering absorptive amplitudes tend to zero as  $s \to \infty$ .

The Pomeranchuk conjecture is usually stated in the following way:

$$\sigma_{ch. ex.}(s) \rightarrow 0 \quad as \quad s \rightarrow \infty$$
 (1)

For our purpose, however, we have to sharpen this either by assuming for the absorptive amplitudes in the forward direction that

$$A_{ch, ex}(s, 0) = 0(A_{ah, el}(s, 0)) (s \to \infty)$$
 (2)

or by stating that

$$A_{ch} = (s, 0) / s \rightarrow 0 \quad as \quad s \rightarrow \infty$$
 (3)

(A  $_{ab.\,el.}$  stands for the absolute elastic amplitudes, without charge exchange and without spin flip). It is well-known that both behaviours (2) and (3) are predicted by some dynamical models [2].

We prove first that for the two-body process

$$A + B \rightarrow A' + B' \tag{4}$$

where A, A' belong to the same irreducible multiplet of the symmetry group G and likewise B, B' to another one, either (2) or (3) imply that

<sup>\*</sup> Actually this can be proved for any compact continuous group; examples can be given which show that for a discrete symmetry group this may not be true. On the other hand, it is essential for our purpose to assume that the symmetry is valid also for scattering in the forward direction.

#### A. GALINDO, A. MORALES AND H. RUEGG

$$\sigma_{\text{tot. (A B)}}(s) \approx \sigma_{\text{tot. (A' B')}}(s) \quad (s \to \infty) .$$
(5)

Secondly, we prove that, under certain hypotheses, (5) implies (3).

Although we derive these results by using only group-theoretical arguments, we would like to remark that if one sticks to the Chew-Frautschi interpretation of high energy phenomena, then the assumption that there exists a vacuum pole scalar under G which dominates reactions (4) leads immediately to (1) and (5). Conversely, if one assumes (1), an underlying symmetry group and the existence of dominating Regge poles for each reaction (4), then it follows easily that all these poles must be scalar and identical.

Section 2 is devoted to establishing some necessary mathematical theorems. Finally, in Section 3, we prove the above statements.

#### 2. MATHEMATICS

Let G be a compact Lie group, and let  $\Phi$  and  $\Phi'$  be any two irreducible unitary finite representations of G. Denote by  $R_{\Phi}$  the representation space of  $\Phi$ , and let  $d_{(\Phi)}$  be its dimensionality. Let us call the "physical" basis  $B_{\Phi}$  of  $R_{\Phi}$  the set of normalized vectors in  $R_{\Phi}$  with definite quantum numbers associated with G. (If some weight vector is degenerate, one can always add to the Cartan sub-algebra, which provides us with the first  $\ell$  ( $\ell$ = rank of G) quantum numbers, further observables so as to have a complete set of commuting matrices). Finally let  $\underline{\Lambda}$ ,  $\underline{\Lambda}'$  be the highest weights of  $\Phi$ ,  $\Phi'$  respectively.

Theorem 1. Let

$$\Phi \otimes \Phi' = \sum_{s=1}^{r} \oplus \nu_{s} \Phi^{(s)} = \Phi^{(1)} \oplus \Phi^{(2)}_{1} \oplus \ldots \oplus \Phi^{(2)}_{\nu_{2}} \oplus \Phi^{(3)}_{1} \oplus \ldots$$
(6)

be the direct orthogonal decomposition of the Kronecker product  $\Phi \otimes \Phi'$  into irreducible representations  $\Phi_i^{(s)}$ , where  $\Phi_i^{(s)} \sim \Phi_j^{(s)}$  and  $\Phi_i^{(s)} \not\sim \Phi_j^{(t)}$ .  $(\Phi^{(1)})$  is the irreducible representation with highest weight A + A'). Let  $|a \rangle$ ,  $|b \rangle \in \mathbb{R}_{\Phi}$  and  $|a' \rangle$ ,  $|b' \rangle \in \mathbb{R}_{\Phi'}$ . If  $\mathcal{V}(aa')$  denotes the minimal subspace containing  $|aa' \rangle \equiv |a\rangle |a' \rangle$  and invariant under G, then  $|bb' \rangle$  has a non-vanishing projection on  $\mathcal{V}(aa')$ .

Proof: The irreducibility of  $\Phi$  implies the existence of some  $Y \,{\in}\, G$  such that

$$\langle \mathbf{b} | \mathbf{Y}_{\Phi} | \mathbf{a} \rangle \neq \mathbf{0}.$$
 (7)

Write  $|c\rangle = Y_{\Phi}|a\rangle$ ,  $|c'\rangle = Y_{\Phi'}|a'\rangle$  and  $|cc'\rangle = Y_{\Phi \oplus \Phi'}|aa'\rangle$ . Clearly  $|cc'\rangle \in (aa') \forall and \langle b|c \rangle \neq 0$ . Let us suppose now that  $|bb'\rangle \perp \forall (aa')$ . Then, a fortiori,  $|bb'\rangle \perp \forall (cc')$  and hence

$$\langle bb' | X_{\Phi \otimes \Phi} | cc \rangle = 0 \text{ for all } X \in G$$
 (8)

i.e.

$$\langle b | X_{\Phi} | c \rangle \langle b' | X_{\Phi'} | c' \rangle = 0 \text{ for all } X \in G.$$
 (8.a)

Since  $\langle b | c \rangle \neq 0$  and the representation  $\Phi$  is continuous, we infer the existence of a neighbourhood N of the identity of G such that

$$\langle b|X_{\Phi}|c \neq 0 \text{ for all } X \in \mathbb{N}.$$
 (9)

From (8.a) and (9) it follows

$$\langle b' | X_{\Phi'} | c' \rangle = 0$$
 for all  $X \in \mathbb{N}$ 

and therefore,

$$\left\langle \mathbf{b}' \left| \mathbf{A}_{\Phi'} \right| \mathbf{c}' \right\rangle = 0 \tag{10}$$

for all A's belonging to the Lie algebra of G. This result is in clear contradiction to the irreducibility of  $\Phi'$ . We conclude therefore that  $|bb'\rangle$  cannot be orthogonal to  $\mathcal{V}$  (aa'), Q.E.D.

<u>Corollary 1.</u> Let  $\underline{\Lambda}_s$  be the highest weight of  $\Phi_i^{(s)}$  and let  $|\lambda\rangle$ ,  $|\lambda'\rangle$ ,  $|\lambda\lambda'\rangle$  be the vectors in  $\mathbf{R}_{\Phi}$ ,  $\mathbf{R}_{\Phi'}$ ,  $\mathbf{R}_{\Phi \otimes \Phi'}$ , having the quantum numbers specified by  $\underline{\Lambda}$ ,  $\underline{\Lambda}'$ ,  $\underline{\Lambda}_1(=\underline{\Lambda}+\underline{\Lambda}')$ . Then

$$\mathcal{V}(aa') \supset \mathcal{V}(\lambda \lambda') \tag{11}$$

To prove (11) it suffices to note that  $\Lambda_1$  is simple and  $\mathcal{V}(\lambda\lambda') = \mathbb{R}_{\Phi(1)}$ . Since Theorem 1 implies in particular that  $\mathcal{V}$  (aa') and  $(\lambda\lambda')$  cannot be orthogonal,  $\mathcal{V}$  (aa') must contain  $\mathcal{V}(\lambda\lambda')^*$ .

<u>Corollary 2.</u> "All the weight diagrams  $W_s$  of  $\Phi_i^{(s)}(s=2,\ldots,r)$  are contained (as point sets) in  $W_1$ ." Consider just any weight vector  $M_s \in W_s$  and take any state |aa'> having as its first  $\ell$  quantum numbers those given by  $M_s$ . Theorem 1 implies that |aa'> has a non-vanishing projection on  $R_{\Phi}(i)$  and hence one concludes that  $M_s$  must belong to  $W_1$ . (The multiplicity, however, of  $M_s$  as an element of  $W_s$  has not to be necessarily lesser than or equal to its multiplicity as a weight vector of  $\Phi^{(1)}$ ).

<u>Theorem 2.</u> Let T be an operator in  $R_{\Phi \otimes \Phi}$ ; given by a matrix  $(T_{ij})$  in the tensorial basis  $B_{\Phi} \otimes B_{\Phi}$ , and suppose that T commutes with all  $X_{\Phi \otimes \Phi}$ . Then

$$T_{ii} = 0$$
 (i = j) (12)

implies that

$$\mathbf{T}_{ii} = \mathbf{T}_{ii} \,. \tag{13}$$

<sup>\*</sup> Note that the projector associated with  $\mathcal{V}(\lambda, \lambda^{*})$  belongs to the group algebra  $L_{2}(G)$ .

Proof: Associated with the decomposition (6) one can choose an orthonormal basis B in  $R_{\Phi \notin \Phi}$ , consisting of elements

$$|\alpha_{ij}^{(s)}\rangle$$
; (s=1...r; i=1... $\nu_{s}$ ; j=1...d( $\Phi_{i}^{(s)}$ )) (14)

such that

$$\left| \alpha_{ij}^{(s)} \right\rangle \in \mathbb{R}_{\Phi_{i}^{(s)}} ; \quad \left| \alpha_{ij}^{(s)} \right\rangle = \mathbb{U}_{k \to i} \left| \alpha_{kj}^{(s)} \right\rangle$$

$$(15)$$

where  $U_{k\,\to\,i}$  is the matrix intertwining  $\Phi_i^{(s)}$  and  $\Phi_k^{(s)}$  . The invariance of T under G implies

$$\left\langle \alpha_{ij}^{(s)} \mid \mathbf{T} \mid \alpha_{k\,\ell}^{(t)} \right\rangle = \mathbf{T}_{ik}^{(s)} \delta_{st} \delta_{j\,\ell}$$
(16)

where  $T_{1k}^{(s)}$  are the so-called invariant T-matrix elements. Let  $V_s$  denote the subset of  $B_{\Phi} \otimes B_{\Phi}$ , whose elements are eigenvectors of  $H \equiv (H_1, \ldots, H_{\ell})$ with eigenvalue  $\Lambda_s \cdot (H_1, \ldots, H_{\ell}$  is a normalized basis of the Cartan subalgebra of G). Finally, let us order the representations appearing in (6) in such a wa that

$$A_1 > A_2 > A_3 > \dots > A_r$$
 (17)

and let  $P_i^{(s)}$  denote the orthogonal perjector onto  $R_{\Phi_i(s)}$ .

Take now  $|v_2\rangle \in V_2$ . From Theorem 1 and Corollary 2 we know that

$$\mathbf{P}^{(1)} \left| \mathbf{v}_{2} \right\rangle \neq \mathbf{0}. \tag{18}$$

"

On the other hand

$$|v_{2}\rangle = P^{(1)}|v_{2}\rangle + P^{(2)}|v_{2}\rangle, \quad (P^{(s)} = \sum_{1=1}^{v_{s}} P_{i}^{(s)})$$
 (19)

and hence,

$$T | v_2 \rangle = T P^{(1)} | v_2 \rangle + T P^{(2)} | v_2 \rangle.$$
 (20)

But since  $\nu_1 = 1$ , it follows from (16) that

$$T P^{(1)} |v_2\rangle = T_{11}^{(1)} P^{(1)} |v_2\rangle.$$
 (21)

By comparing (12), (18), (19), (20) and (21) one gets

$$T P^{(2)} |v_2\rangle = T_{11}^{(1)} P^{(2)} |v_2\rangle.$$
 (22)

As (22) is true for any  $|v_2\rangle \in V_2$ , and  $[T, X_{\Phi \otimes \Phi'}] = 0$ , for all  $X \in G$ , we can conclude that

$$T P^{(2)} |v\rangle = T_{11}^{(1)} P^{(2)} |v\rangle$$
 (23)

for any  $|v\rangle$  belonging to the linear closure of the set {  $X_{\Phi \otimes \Phi'} |v_2\rangle$ }, (X  $\in$  G,  $|v_2\rangle \in V_2$ ), and hence

$$\Gamma P^{(2)} = T_{11}^{(1)} P^{(2)},$$
 (24)

i.e.

$$T_{ik}^{(2)} = T_{11}^{(1)} \delta_{ik}$$
 (25)

Suppose now, in general, that we have proved that

$$T_{ik}^{(t)} = T_{11}^{(1)} \delta_{ik}$$
,  $(t = 1, 2, \dots s - 1)$ . (26)

Then, take any  $|v_s\rangle \in V_s$ . Once again  $P^{(1)} |v_s\rangle \neq 0$  and by using (26) we get

$$|\mathbf{v}_{s}\rangle = \mathbf{P}^{(1)}|\mathbf{v}_{s}\rangle + \ldots + \mathbf{P}^{(s)}|\mathbf{v}_{s}\rangle,$$
 (27)

$$T |v_{s}\rangle = T P^{(1)} |v_{s}\rangle + ... + T P^{(s)} |v_{s}\rangle$$
  
=  $T_{11}^{(1)} (P^{(1)} |v_{s}\rangle + ... + P^{(s-1)} |v_{s}\rangle) + T P^{(s)} |v_{s}\rangle$  (28)

and the same argument used above allows us to conclude that

$$TP^{(s)} = T_{11}^{(1)}P^{(s)}$$
 (29)

i.e.

$$\mathbf{T}_{ik}^{(s)} = \mathbf{T}_{11}^{(1)} \, \delta_{ik} \,. \tag{30}$$

We have therefore proved by induction that

$$T = T^{1}_{(11)} I.$$
 (31)

Theorem 3. With the same notation as in Theorem 2, we claim that the equalities (13) imply (12), provided that all  $\nu_s = 1$ . Proof: Take  $|v_1\rangle \in V_1$ . Then, from (16)

$$\mathbf{T} |\mathbf{v}_{1}\rangle = \mathbf{T}_{11}^{(1)} |\mathbf{v}_{1}\rangle$$
(32)

and therefore,

$$\langle \mathbf{v}_1 | \mathbf{T} | \mathbf{v}_1 \rangle = \mathbf{T}_{11}^{(1)}$$
 (33)

Take now  $|v_2\rangle \in V_2$ ; from (16), 20), (33) and  $v_2 = 1$  one gets

$$\langle \mathbf{v}_{2} | \mathbf{T} | \mathbf{v}_{2} \rangle = \mathbf{T}_{11}^{(1)} \langle \mathbf{v}_{2} | \mathbf{P}^{(1)} | \mathbf{v}_{2} \rangle + \mathbf{T}_{11}^{(2)} \langle \mathbf{v}_{2} | \mathbf{P}^{(2)} | \mathbf{v}_{2} \rangle.$$
 (34)

On the other hand

$$1 = \langle \mathbf{v}_2 | \mathbf{v}_2 \rangle = \langle \mathbf{v}_2 | \mathbf{P}^{(1)} | \mathbf{v}_2 \rangle + \langle \mathbf{v}_2 | \mathbf{P}^{(2)} | \mathbf{v}_2 \rangle.$$
(35)

A simple manipulation of (33), (34), (35) and  $\langle v_1 | T | v_1 \rangle = \langle v_2 | T | v_2 \rangle$  (see (13)) leads to

$$T_{11}^{(1)} = T_{11}^{(2)}$$
 (36)

By repeating this argument one proves by induction that

$$T = T_{11}^{(1)} I.$$
 (37)

One should note that Theorem 3 is not, in general, the converse of Theorem 2, since the assumption is made therein that all  $\Phi^{(s)}$  appear only once in  $\Phi \Phi \Phi'$ . It is known that this is always the case for simply reducible groups like  $A_1 = SU_2$ . For  $B_2$ ,  $C_2$  and  $G_2$  the above assumption is satisfied provided that the degrees of  $\phi$  and  $\phi'$  are reasonably low. The classical groups  $SU_m(m \ge 2)$  are not simply reducible and it is a well-known fact that the representation  $\underline{8}$  of  $SU_3$  is such that  $\underline{8} \otimes \underline{8}$  contains  $\underline{8}$  twice.

Nevertheless it is possible to prove Theorem 3 without the assumption  $\nu_s = 1$  provided that some new requirements are made. These are essentially necessary since one can see that whenever

$$T_{ii}^{(s)} = T_{11}^{(1)} \qquad (i = 1...\nu_{s}; s = 1...r)$$
(38)  
$$T_{ik}^{(s)} + T_{ki}^{(s)} = 0 \qquad (i \neq k)$$

one has  $T_{ii} = T_{jj}$  even though  $T_{ij}$  may be different from zero. However, if we assume that there exist  $\nu_s$  vectors in  $V_s$  (s=z...r) such that:

(i) they have non-vanishing projections on  $P^{(s)} = R_{\Phi \otimes \Phi}$ , which are mutually orthogonal;

(ii) for each couple of these  $\nu_s$  vectors there exists some vector in  $V_s$  the projection of which on P<sup>(s)</sup>  $R_{\Phi\otimes\Phi}$  is a linear combination of them with coefficients which are both different from zero; and

(iii) T is a symmetric matrix in  $B_{\Phi} \otimes B_{\Phi'}$ ,

then it can be proved with just slightly different arguments that (13) implies (12).

#### 3. PHYSICS

We want now to apply these mathematical results to the physical problems stated in the Introduction.

First of all it is clear that Theorem 2 means that

$$T_{ii} - T_{jj}$$
;  $T_{kk}^{(s)} - T_{\ell\ell}^{(t)}$ ; and  $T_{\ell k}^{(s)}$  ( $\ell \neq k$ )

are linear combinations of  $T_{mn}$ 's  $(m \neq n)$ .

Secondly, Theorem 3 implies, under the conditions stated therein, that each  $T_{mn}$  (m  $\neq$  n) is a linear combination of the differences  $T_{ii}$  -  $T_{jj}$ , and similarly that each  $T_{kk}^{(s)}$  is a linear combination of  $T_{ii}$ 's (the unitary invariance of the trace is involved here).

Let us now identify  $T_{ij}$  ( $i \neq j$ ) with the charge-exchange absorptive amplitudes in the forward direction A <sub>ch. ex.</sub> (s, 0) for the two-body reaction (4)

where the particles AA', BB' are members of irreducible multiplets  $\Phi, \Phi'$ of an arbitrary symmetry group G. Similarly  $T_{ii}$  will be the absolute elastic absorptive amplitude A<sub>ab. el.</sub>(s, o) for the corresponding processes

$$A + B \rightarrow A + B$$
.

Therefore if relations (2) are satisfied the above statements prove that

$$\frac{A_{ab. el.}(s, 0)}{s} - \frac{A'_{ab. el.}(s, 0)}{s} = 0 \left( \max \frac{A''_{ab. el.}(s, 0)}{s} \right) \text{ for } (s \to \infty) \quad (39)$$

and hence, by applying the optical theorem,

$$\sigma_{\text{tot. (AB)}}(s) - \sigma_{\text{tot. (A'B')}}(s) = 0 \left( \max \sigma_{\text{tot. (A''B'')}}(s) \right) \text{ for } (s \to \infty).$$
(40)

$$\sigma_{\text{tot. (A B)}}(s) \approx \sigma_{\text{tot. (A'B')}}(s) \text{ for } (s \to \infty)$$
(5)

It can be proved in a similar way that (3) implies (5). An analogous application of Theorem 3 shows that (5) implies (3) whenever the product of the two multiplets  $\Phi$  and  $\Phi'$  is simply reducible, i. e. all  $\nu_s = 1$ . In the case  $\underline{8} \otimes \underline{8}$  of SU<sub>3</sub> one can check that the requirements a) and b) appearing in the comments following Theorem 3 are fulfilled. For instance, if we consider baryonmeson scattering in the octect model, the physical vector  $|\Lambda \pi^+\rangle$  has no component on  $\underline{8}_A$  but it has a non-vanishing projection on  $\underline{8}_s$ . The situation is reversed for  $|\Sigma^0\pi^+\rangle$ . Finally,  $|\Xi^0K^+\rangle$  has non-vanishing projections on both  $\underline{8}_A$  and  $\underline{8}_s$ . On the other hand, requirement (c) is automatically fulfilled whenever time reversal invariance is assumed. Therefore (5) implies (3) in this case.

This work has been performed during our stay in Trieste as participants in the IAEA Seminar on Theoretical Physics. It is a pleasure for us to thank Professor Salam and the IAEA for their hospitality. One of us (A.G.) thanks the IAEA for its financial support. (A.M.) and (H.R.) are indebted to CERN for the same reason.

#### REFERENCES

- [1] See, for instance, WEINBERG, S., Phys. Rev. 124 (1961) 2049.
- [2] AMATI, D., FUBINI, S. and STANGHELLINI, S., Preprint CERN 1962 (to be published); see also Phys. Lett. <u>1</u> (1962).

•

# BOOK III

# COMPLEX ANGULAR MOMENTA AND MANDELSTAM REPRESENTATION

# MATHEMATICAL THEORY OF POTENTIAL SCATTERING

#### T. REGGE

#### ISTITUTO DI-FISICA DELL' UNIVERSITA DI TORINO, TORINO, ITALY

#### INTRODUCTION

Before we go into a detailed discussion of the potential scattering we would like to spend a few words on the reason potential scattering is interesting. We think that one of the main reasons of success of the potential model is that we can discuss it quite rigorously and that at the same time it gives a fairly intuitive picture of the scattering process and it provides in a way the language for a fully relativistic theory. We do not think that the potential model has been particularly satisfactory in explaining quantitatively the known experimental data, for instance the nucleon-nucleon scattering; yet we have good reasons to believe that at low energy any field theory will ultimately yield some sort of spin-dependent potential, containing spin orbit coupling and exchange terms. How this can be done and how far one has gone in this direction has nothing to do with the subject of these lectures which are merely concerned with the discussion of the solutions of the Schroedinger equation for a given class of potentials. That is, we assume from the very beginning that a potential exists although we do not know it or we know only broad features like the range and its analytic properties as function of the distance. For simplicity we do not deal with spin or exchange terms although they can be taken care of with little modifications. We just want to find those features of potential scattering which are to a large extent independent of the particular selection of the potential.

In so doing we shall need a large mathematical apparatus in order to derive those properties of the scattering amplitude which have been suggested by the general field theory, like dispersion relations. Unfortunately although it has not been possible to eliminate entirely from these lectures this apparatus, we have tried however to use as much as possible standard mathematical tricks and we have endeavoured to cover them with the largest amount of proofs. There are different mathematical approaches to the theory of potential scattering. Historically the first approach was developed by Heisenberg in his first attempts to create a theory of the S-matrix. But the most rigorous and extensive results on this particular subject were actually found by JOST and BARGMANN [1]. The starting point of their approach is the partial wave expansion of the wave function and of the scattering amplitude. Actually they did not derive any interesting feature of the full amplitude, but rather of the partial phase shifts only. The amount of work afterwards done on the properties of phase shifts as function of the energy has been considerable, and it has clarified the role of the potential in determining them.

This was not however the end of the story. When the first dispersion relations for fixed transmitted momentum were discovered in field theory, it was a natural question to ask whether these properties had a counterpart in potential scattering. This was found to be true by KHURI [2]. The paper of Khuri avoids entirely the use of partial waves and uses Fredholm's theory on the Green integral form of the Schroedinger equation written in full threedimensional formalism. Alternative and simpler proofs then appeared in the literature [3, 4, 6]. The reason the partial wave expansion is totally unsuitable for this purpose is that it fails to converge in the interesting region where we want to prove analyticity in the energy. The advent of the Mandelstam conjecture of the double dispersion relation raised the question as to whether these relations were true for potential scattering. Mandelstam representation can be proved today for a special class of potentials (super-position of Yukawa potentials).

A proof of GOLDBERGER et al.[3] uses the pertubative expansion of the scattering amplitude as written in momentum space (as derived from the Lippman-Schwinger equation). They prove that each term of the expansion satisfies the Mandelstam representation, and they also succeeded in going around the question of uniform convergence. Incidentally, an incomplete proof, without uniform convergence, was given first by Bowcock and Martin. A paper by KLEIN also deals with this subject [4]. The partial wave expansion however can be used successfully in providing analytic properties in the momentum transfer for fixed energy. The usual form of it is apparently unsuitable for the job, but fortunately about fifty years ago WATSON [5] found a method of transforming it into an integral which is a highly flexible tool in these kinds of problem. With some care the Watson integral can be used to prove almost all of the analytic properties of the scattering amplitude, including those of Khuri's paper. It is for this reason that we decided to rest the whole theory on the partial wave expansion in the Watson form because we feel that in this way the whole structure of the lectures will be more homogeneous.

# 1. THE FORMALISM OF POTENTIAL SCATTERING. ELEMENTARY THEORY

The starting point of the theory is the Schroedinger equation:

$$\Delta \Psi(\vec{r}) + E \Psi(\vec{r}) = V \Psi(\vec{r}). \tag{1.1}$$

In this equation  $\vec{r}$  is the position vector of the scattered particle, r its length,  $\vec{r}$  has components x, y, z. We use natural units h = c = 1 and 2M = 1, where M is the mass of the scattered particle. The scattering of two particles of different mass  $m_1$ ,  $m_2$  can be treated by the same equation where M is now the reduced mass  $m_1m_2/(m_1 + m_2)$  of the system. In our units the energy has the dimension of an area<sup>-1</sup>. The local potential V(r) depends on r only.

V(r) is supposed to be a short range potential; that is, we suppose it to decrease exponentially. Truly this is a rather restricted hypothesis; but if we have in mind a comparison with the field theoretical results, all in-

teresting potentials satisfy this criterion apart from the Coulomb potential. We shall not examine here Coulomb-like potentials because there is no extensive and deep work done on this subject. Under these conditions [7] we may define the (total) scattering amplitude  $f(E, \theta)$  once we know the solution of eq. (1.1) with the following asymptotic behaviour  $(r \rightarrow \infty)$ :

$$\Psi \sim e^{\vec{i}\vec{k}\cdot\vec{r}} + f(E,\theta) \frac{e^{ikr}}{r}.$$
 (1.2)

This wave function represents a three-dimensional scattering process of a plane wave against a fixed scatterer. The plane wave is given by  $e^{i\vec{k}\cdot\vec{r}}$ , where  $\vec{k}$  is the ingoing momentum. We have  $(\vec{k})^2 = E$ . The second contribution comes from the scattered waves and depends of course on the potential. The angle  $\theta$  is the angle between  $\vec{k}$  and the direction in which we take the asymptotic limit  $r \rightarrow \infty$ . In other words, we put  $\vec{r} = r \vec{n}$  into  $\Psi(\vec{r})$  and we let  $r \rightarrow \infty$  while  $\vec{n}$  is a fixed unit vector. Then  $\vec{k} \cdot \vec{n} = k \cos \theta$ .  $d\Omega |\vec{f}(E, \theta)|^2$ is then the probability of finding the particle scattered in the solid angle  $d\Omega$  with the outgoing momentum  $\vec{k} = k \vec{n}$ .

There is no potential of the class considered by us for which eq. (1.1) is explicitly solvable. For any practical purpose of numerical evaluation one solves instead (1.1) with the method of the separation of variables due to D'Alembert. One tries to find the solution of (1.1) of the form

$$\Psi = \frac{\phi(\mathbf{r})}{\mathbf{r}} \quad Q(\theta, \phi). \tag{1.3}$$

It is well known then that Q has to be a spherical harmonic,

$$Q(\theta,\phi) = Y_{\theta}^{m}(\theta,\phi), \qquad (1.4)$$

and that  $\phi$  satisfies the ordinary differential equation

$$\phi_{\ell}^{\prime\prime} + E \phi_{\ell} - \frac{\ell(\ell+1)}{r^2} \phi_{\ell} - V \phi_{\ell} = 0.$$
 (1.5)

(1.5) depends on  $\ell$  only and not on m.  $\phi_{\ell}$  must also satisfy the boundary condition of vanishing at the origin. More precisely, the analysis of (1.5) according to the Fuchsian classification of singularities shows that any solution of (1.5) behaves when r is small like

$$\phi_{\ell} \approx \alpha r^{\ell+1} + \beta r^{-\ell} \tag{1.6}$$

under some restrictive hypothesis on the potential to be examined closely later. If we want to avoid singularities at r = 0, we are forced to choose  $\beta = 0$ . In this case the  $\ell$ -th partial wave function vanishes rapidly for small r. Physically we may interpret this fact as due to the repulsive centrifugal barrier  $\ell(\ell + 1)/r^2$  which becomes very large when the orbital momentum  $\ell$  is also large. This barrier keeps the particle from approaching the origin.

#### T. REGGE

This boundary condition defines each partial wave apart from a multiplicative factor. Take now r large. We have good reasons now to suppose that both V(r) and  $\ell(\ell + 1)/r^2$  can be neglected in comparison with E so that (1.5) becomes

$$\phi_{\ell}^{11} + \mathbf{E} \ \phi_{\ell} = 0. \tag{1.7}$$

This equation is trivially solved by oscillating exponentials (E  $\geq$  0) and the corresponding asymptotic behaviour of  $\phi_{\ell}$  will be of the form

$$\phi_{\ell} \sim C_{\ell} \sin \left[ kr - \frac{\ell \pi}{2} + \delta_{\ell} (k) \right]. \qquad (1.8).$$

We have introduced on purpose the term  $\ell \pi/2$  in this asymptotic behaviour. Indeed, when V = 0, eq. (1.5) can be solved exactly in terms of Bessel functions of semi-integer order and the asymptotic behaviour at infinity explicitly evaluated. This behaviour corresponds to having  $\delta_{\ell}(k) = 0$ . The phase shift  $\delta_{\ell}(k)$  therefore describes a cumulative effect of the potential on the wave function in the whole interval  $0 \dots \infty$ . A large part of these lectures will be devoted to the investigation of the properties of  $\delta_{\ell}(k)$ . The importance of  $\delta_{\ell}$  is evident from the well-known Rayleigh-Faxen formula:

$$f(E, \theta) = \frac{1}{2ik} \sum_{\ell=0}^{\infty} (e^{2i\delta_{\ell}(k)} - 1)(2\ell + 1) P_{\ell}(\cos \theta). \qquad (1.9)$$

We shall refer to this fundamental formula as the expansion of the scattering amplitude in partial waves or more concisely as the RF expansion. A full account of (1.9) is contained in any elementary textbook on quantum mechanics and we shall not go into this matter further.

In (1.9) the functions  $P_{\ell}$  (cos  $\theta$ ) are Legendre polynomials which form an orthogonal set normalized as follows:

$$\int_{-1}^{1} P_{\ell}(x) P_{m}(x) dx = \frac{2}{2\ell + 1} \delta_{m\ell}$$

The total cross-section is given by

$$\sigma(\mathbf{E}) = \int d\Omega \left| \mathbf{f}(\mathbf{E},\theta) \right|^2 = \frac{4\pi}{\mathbf{E}} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2 \delta_{\ell}.$$

#### 2. THE SWAVE

The S wave scattering is the first that has been extensively discussed, and this is for the simple reason that the mathematics of it is considerably simpler than that of the higher waves. A number of potentials have been

produced which are explicitly solvable for the S wave and which give quite a number of clues concerning the general dependence of  $\delta$  on the energy E. One such potential is of course the square well potential defined as

$$V(\mathbf{r}) = A$$
 if  $\mathbf{r} < R$ ,  
 $V(\mathbf{r}) = 0$  if  $\mathbf{r} > R$ ; we put  $k_1 = \sqrt{E - A}$ .

It is simple exercise to find the corresponding phase shift:

$$e^{2i\hat{o}(k)} = S(k) = e^{-2ikR} \frac{\cos k_1 R + (ik/k_1) \sin k_1 R}{\cos k_1 R - (ik/k_1) \sin k_1 R}.$$
 (2.1)

Some features of this formula are the following:

(1) the formula is also valid when E is not positive. S(k) is the ratio of two holomorphic functions of k and is therefore meromorphic.

(2)  $S(-k) = S^{-1}(k)$ ; that is,  $\delta$  is an odd function of k. It is more convenient to study S(k) instead of  $\delta$  because  $\delta$  has logarithmic singularities at every pole or zero of S(k).

(3)  $[S(k^*)]^* = S^{-1}(k)$ . This implies that  $\delta$  is real when k is real. We refer to this property as unitarity.

(4)  $\lim S(k) = 1$  or  $\lim \delta = 0$  when  $k \rightarrow \infty$ . How do we understand this result? If k (or E) is very large, the speed of the incoming particle, which in our units is given by 2k, also increases. The time of transit of the particle inside the potential well is of the order of R/k. Presumably the interaction is proportional to the transit time and the phase shift will be also of the order of magnitude of R/k or rather of the dimensionless parameter AR/k. Indeed, for large k we have from (2.1)

$$\delta \sim - \frac{AR}{2k} = - \frac{1}{2k} \cdot \int_{0}^{\infty} V(\mathbf{r}) d\mathbf{r}. \qquad (2.2)$$

This result is naturally false in the relativistic region, and it is already different for the Dirac or Klein-Gordon equation. The limit is much more complicated if we move to infinity along any direction of the complex k plane. It must be pointed out that although  $k_1$  is a two valued function of k it does not matter which value we use in (2.1). If A > 0, the potential is repulsive and pushes out the wave function. We expect  $\delta$  to be negative in agreement with the asymptotic behaviour (2.2).

Take now eq. (1.5) when E is negative (and l = 0):

$$\phi^{\,\mathbf{n}} + \mathbf{E} \,\phi - \mathbf{V} \,\phi = 0, \qquad (2.3)$$

This equation does not describe any scattering state and it will have solutions which are bounded at infinity and at the origin only for special values of E. Putting  $E = -b^2$  where b is real, we have for large r
$$\phi \sim \mu(b) e^{-br} + \nu(b) e^{br}$$
.

If b is not restricted, we shall have an exploding exponential term at large distances. If however for a particular value of b we have  $\nu(b) = 0$ , the solution becomes square integrable and represents a bound state of the system. Great progress in the theory of bound states was achieved when it became clear that bound states correspond to poles of S(k); i.e. if a bound state of binding energy  $-B^2$  occurs, there is a pole of S(k) in k = iB. Unfortunately it is not in general true that all poles of S(k) correspond to bound states. This makes it difficult to deduce the bound states from the analytic continuation of S(k), or at least it made it difficult before the advent of the modern ideas of dispersion theory. Before going into a detailed discussion of this connection, we point out that our statement can be verified directly on the explicit formula which we have just given for the square well potential. We leave this as an exercise for the reader. Other examples of soluble potential can be found in [1]. Jost defines a particular solution of eq. (2.3) with the boundary condition (the Jost solution)

$$f''(k, r) + E f(k, r) - V f(k, r) = 0,$$
  
$$f(k, r) \sim e^{-ikr}, r \rightarrow \infty.$$
 (2.4)

This solution will not satisfy in general the boundary conditions in r = 0; that is,  $f(k, 0) \neq 0$ . Let us define the Jost function as f(k) = f(k, 0). If f(k) = 0, the Jost solution is regular in r = 0. Besides f(k, r), f(-k, r) also is a solution of (2.4); and since the Wronskian of these two functions does not depend on rand equals -2ik, they form a pair of independent solutions of (2.4). Take now the "regular" solution  $\phi(k, r)$  defined by the boundary condition in r = 0:

$$\phi(\mathbf{k}, 0) = 0; \phi'(\mathbf{k}, 0) = 1.$$
 (2.5)

 $\phi$  is not linearly independent of f(k, r) and f(-k, r) so that we have with some coefficients C, D

$$\phi$$
 (k, r) = C f(k, r) + D f(-k, r).

Now  $W(\phi, f) = \phi' f - f' \phi$  is independent of r and we calculate it for r = 0:

$$W(\phi, f) = \phi'(k, 0) f(k, 0) - \phi(k, 0) f'(k, 0) = f(k).$$

On the other hand,

$$W(\phi, f) = C W [f(k, r), f(k, r)] + D W [f(-k, r), f(k, r)]$$
  
= D W [f(-k, r), f(k, r)] = 2ikD = f(k),

so that D = f(k)/2ik. Similarly C = -f(-k)/2ik. It follows that

$$\phi(k, r) = [f(k) f(-k, r) - f(-k) f(k, r)]/2ik = \phi(-k, r). \quad (2.6)$$

The asymptotic behaviour of  $\phi$  is then

$$\phi$$
 (k, r) ~ [e<sup>ikr</sup> f(k) - e<sup>-ikr</sup> f(-k)]/2ik.

But from the definition of phase shift we have

 $\phi \sim \text{const. sin}(kr + \delta) = \text{const. } (e^{ikr} e^{i\delta} - e^{-ikr} e^{-i\delta}).$ 

By comparison we get

$$e^{2i\delta(k)} = f(k)/f(-k).$$
 (2.7)

If V = 0, then  $f(k, r) = e^{-ikr}$ ,  $\phi(k, r) = (1/k) \sin kr$ , f(k) = 1. If V is the already defined square well potential, we have

$$f(k) = e^{-ikR} (\cos k_1 R + i(k/k_1) \sin k_1 R).$$
 (2.8)

In this case f(k) turns out to be the entire function of k. Bargmann has investigated the general behaviour of f(k) as a function of the complex variable k, paying special attention to the role of the range of the potential. His starting point is the integral equation for f(k, r):

$$f(k, r) = e^{-ikr} + \frac{1}{k} \int_{r}^{\infty} V(x \sin k(x-r) f(k, x) dx)$$

We shall prove and discuss this equation in the next section.

## 3. THE ANALYTIC PROPERTIES OF JOST'S FUNCTION

In the last section we examined the integral equation for f(k, r):

$$f(k, r) = e^{-ikr} + \frac{1}{k} \int_{r}^{\infty} V(x) \sin k(x-r) f(k, x) dx. \qquad (3.1)$$

This equation can be proved as follows: Clearly we have

$$f'' + k^2 f = V f,$$

$$\frac{d^2}{dx^2} \sin k(x-r) + k^2 \sin k(x-r) = 0. \qquad (3.2)$$

Therefore,

$$Vf(k, x) \sin k(x-r) = f''(k, r) \sin k(x-r) - f(k, x) \frac{d^2}{dx^2} \sin k(x-r)$$
$$= \frac{d}{dx} [f'(k, x) \sin k(x-r) - f(k, x) \frac{d}{dx} \sin k(x-r)].$$

If we use the above form of the integrand in (3.1), the integration can be carried out explicitly and the result is

$$\int_{r}^{\infty} V(x) \sin k(x-r) f(k, x) dx = \int_{r}^{\infty} \frac{d}{dx} [f'(k, x) \sin k(x-r) -f(k, x)] \frac{d}{dx} \sin k(x-r) ] dx = k [f(k, r) - e^{-ikr}].$$
  
OED

We regard (3.1) as the proper definition of the Jost solution because it implies both the differential equation and the appropriate boundary conditions. Putting  $f(k, r) e^{ikr} = g(k, r)$ , we find

$$g(k, r) = 1 + \frac{1}{k} \int_{r}^{\infty} V(x) \left[ \frac{1 - e^{-2ik(x-r)}}{2i} \right] g(k, r) dx.$$
 (3.3)

A formal solution of (3.3) is given by the perturbative expansion:

$$g(k, r) = \sum_{n} g_{n}(k, r); g_{0}(k, r) = 1,$$
 (3.4)

where

$$g_{n+1}(k, r) = \frac{1}{k} \int_{r}^{\infty} \frac{1 - e^{-2ik(x-r)}}{2i} g_{n}(k, x) V(x) dx.$$
 (3.5)

This expansion defines a solution of (3.3) only when it converges. In order to decide whether it really does so we need some preliminary bound on the kernel of (3.3). There is no real complication and much to be gained in supposing k complex. We put Im k = b. The proof and also the result are quite different for the cases b > 0 and b < 0. Let us first suppose b < 0, but  $k \neq 0$ . We have the bound (remember that x > r):

$$\left|\frac{1-e^{-2ik}(x-r)}{2ik}\right| < \left|\frac{1}{2ik}\right| + \left|\frac{e^{-2b}(x-r)}{2ik}\right| < \frac{1}{|k|}.$$
 (3.6)

Obviously,

$$|g_{n+1}(k, r)| < \frac{1}{|k|} \int_{r}^{\infty} |V(x)| |g_{n}(k, x)| dx,$$
  
 $|g_{1}(k, r)| < \frac{1}{|k|} \int_{r}^{\infty} |V(x)| dx = \frac{M(r)}{|k|}.$ 

A second iteration yields:

$$\left|g_{2}(k,r)\right| < \frac{1}{|k|^{2}} \int_{r}^{\infty} \left|V(x)\right| dx M(x) = -\frac{1}{|k|^{2}} \int_{r}^{\infty} \frac{dM(x)}{dx} M(x) dx = \frac{M^{2}(r)}{2|k|^{2}}.$$

This suggests that we have the following inequality for the general term:

$$\left| \mathbf{g}_{n}(\mathbf{k},\mathbf{r}) \right| \leq \frac{\mathbf{M}^{n}(\mathbf{r})}{n \left| \mathbf{k} \right|^{n}}$$
 (3.7)

We prove it with the induction method; that is, it is supposed to be true for  $g_n(k, r)$  and we deduce the result for  $g_{n+1}(k, r)$ . We have

$$\left| g_{n+1}(k, r) \right| < \frac{-1}{|k|^{n+1} n!} \int_{r}^{\infty} \frac{d M(x)}{dx} M^{n}(x) dx$$
$$= \left[ M^{n+1}(r) \right] / |k|^{n+1} (n+1)!$$
(3.8)

QED.

By summing up all these inequalities we find

$$|g - 1| < e^{\frac{M(t)}{|k|}} - 1$$
. (3.9)

What is the outcome of (3.9)? We have proved at least the following results: (1) A solution exists for b < 0,  $k \neq 0$  if  $\int_{1}^{\infty} |V(x)| dx < \infty$ , because the per-

turbative expansion converges.

(2) Each term of the expansion is analytic in k as long as the corresponding integral converges; this is true by the above proof in b < 0,  $k \neq 0$ . The sum is therefore also analytic because we have uniform convergence.

(3) We have the limit  $g(k, r) \rightarrow 1$  when  $k \rightarrow \infty$  in any direction in the lower half plane of k and along the real axis.

(4) Since clearly g(k, 0) = f(k), points (1), (2), (3) also hold for the Jost function if  $M(0) < \infty$ .

A different condition can be obtained if we replace (3.6) by

$$\left|\frac{1-e^{-2ik}(x-r)}{2ik}\right| = \left|\int_{0}^{x-r} e^{-2ik\eta} d\eta \right| < |x-r| < x.$$
(3.10)

There is of course no difficulty in repeating the proof with the new bound and we find in lieu of (3.9)

$$|g - 1| < e^{N(r)} - 1; N(r) = \int_{r}^{\infty} x |V(x)| dx.$$
 (3.11)

This last evaluation implies a slightly more stringent condition on V(r) for large r, but it includes k = 0 and it relaxes the condition on V(r) for small r. To this purpose we notice that for all short-ranged potentials both M(r) and N(r) exist but M(0) diverges for the Yukawa potential.

We turn now to the case b > 0. Here we cannot use (3.6) or (3.10) but rather

$$\left| \left[ 1 - e^{-2ik(x-r)} \right] / 2ik \right| < e^{2b(x-r)} / |k|$$
. (3.12)

We have correspondingly

$$|g_{1}(\mathbf{k},\mathbf{r})| < \frac{1}{|\mathbf{k}|} e^{-2b\mathbf{r}} P(\mathbf{r}); P(\mathbf{r}) = \int_{\mathbf{r}}^{\infty} |V(\mathbf{x})| e^{2b\mathbf{x}} d\mathbf{x}.$$

By induction we can similarly check that

$$|g_{n}(k, r)| < P(r) \frac{M^{n-1}(r)}{(n-1)!} \frac{1}{|k|^{n}} e^{-2br}.$$
 (3.13)

This implies again analyticity in k if  $P(\mathbf{r}) < \infty$ . (M converges if P converges.) This is by no means trivially satisfied, as we had before for M and N. If V(r) decreases exponentially, we can always choose b large enough to have P diverging. If V(r) ~  $e^{-mr}/r$ , we find b < m/2. If V is a Gaussian potential or a square well, then we have unrestricted convergence. But the interesting potentials are usually superposition of Yukawa potentials, and therefore we expect f(k) to have singularities in the upper half-plane. With a slight modification of the proof the origin can be included in the analyticity domain.

Concluding: f(k) is analytic in k in the half-plane b < m/2. Therefore S(k) =f(k)/f(-k) is meromorphic in the strip |b| < m/2. This is BARGMANN's result[1]. In the above Bargmann's strip S(k) can have poles only when f(-k) vanishes. We shall discuss the significance of the poles of S(k) in the next section. Here we just wish to give some kind of pictorial view of the analyticity of f(k). As we said, f(k, r) is that solution which behaves like  $e^{-ikr}$ for large r. As long as k is real, this is perfectly sufficient to define f(k, r)from a physical point of view: if k > 0 (< 0), f(k, r) represents a sink (source) in r = 0 which absorbs (emits) a set of stationary purely ingoing (outgoing) waves. If b < 0 the waves are damped at infinity. f(-k, r) waves are exploding; there is no way of having f(-k, r) waves accidentally mixed with f(k, r), because for large r they would violently predominate. A damped wave is therefore guite uniquely determined. This in turn corresponds to the full solvability of the integral equation. If instead we take b > 0, there is apparently no safe way of defining an exploding wave because we are entitled to add to it any damped wave without disturbing the behaviour at infinity. It is possible to get round part of the difficulty by defining as a purely exploding wave  $f(\mathbf{k}, \mathbf{r})$ . b > 0 in such a way that  $f(k, r) - e^{-ikr}$  decreases faster than  $e^{ikr}$ . It is quite possible to do so for the potential well; in fact, there we have  $f(k, r) - e^{-ikr} = 0$ identically outside the potential. But in general this procedure will meet some difficulty because the potential tail perturbs the exploding wave by roughly the amount e<sup>-mr</sup> e<sup>-ikr</sup>. If this part is already larger than the damped wave, we have little chance of going further. The condition  $|e^{-mr} - ikr| \ll |e^{ikr}|$ for large r is precisely b < m/2. This is Bargmann's condition. We went into some detail of this pictorial view of the analyticity proof because with this kind of reasoning one often anticipates the final analyticity domain and paves the way to a rigorous proof.

## 4. POLOLOGY OF S(k)

We want now to discuss in detail the physical meaning of S(k). If V is a real function (we wish to point out that Bargmann's proof holds even if V is not real), we have the following hermiticity properties:

$$f(k, r)^{*} = f(-k^{*}, r),$$
  

$$f(k)^{*} = f(-k^{*}),$$
  

$$S(k^{*})^{*} = S(k)^{-1}.$$
(4.1)

These properties can be broadly referred to as unitarity. They follow from the fact that  $f(-k^*, r)^*$  satisfies exactly the same integral equation as f(k, r).

Suppose now  $f(-k_0) = 0$  within the Bargmann strip. From (2.6) we have  $\phi(k_0, r) = f(k_0) f(-k_0, r)/2ik_0$ .  $f(-k_0, r)$  is therefore regular in r = 0. If  $k_0 = ib$ , b real > 0, f(-k, r) behaves like  $e^{br}$  for larger r and is the wave function of a bound state. Therefore poles of S(k) occuring on k = ib, b > 0 correspond to bound states.

The restriction of the Bargmann strip is essential; otherwise a pole of S(k) could arise from a singularity of f(k) and not from a zero of f(-k).

#### T. REGGE

This was regarded as a serious objection to the theory in the early times, and there were quite a number of attempts toward the elimination of these false poles. (Actually they discussed the zeros of S(k), but this is just the same by S(k) S(-k) = 1.) What about the other poles not lying on  $k_0 = ib, b > 0$ ? If there is a pole in  $k_0 = h + ib$ , b > 0, we must have a pole in  $-k_0^* = -h + ib$  by unitarity. By the same discussion used above both f(h - ib, r) and f(-h-ib, r) are square integrable solutions of our differential equation corresponding to different eigenvalues of the energy  $E = (h \pm ib)^2$ . They are orthogonal. This implies

$$\int_{0}^{\infty} dr f (h - ib, r) f (-h - ib, r) = 0.$$
 (4.2)

This is clearly impossible because f(-h-ib, r) is the conjugate of f(h-ib, r)and the above integral is positive. Therefore, if b > 0, the only way out is h = 0. This proof is the usual quantum mechanical proof that a hermitian operator has real eigenvalues. The same proof breaks down if b < 0 because then the wave function is no longer square integrable. The b < 0 poles of S(k) occur either on k = ib, b > 0 or in pairs of conjugate poles. There is no commonly accepted name for the purely imaginary poles; either antibound states or virtual states have been used, and we suggest the first one. Numerical investigation on solvable examples [4] shows that they actually occur for reasonable choices of potentials. Experimentally they have no outstanding identity like the bound states; but, as we shall see, they can be seen as rather indirect effects on the low-energy cross-section. Indeed, suppose that an antibound state occurs with a small value of b. If k is small, we can expand f(-k)in powers of k-ib. We have

$$f(-k) \approx iC (k-ib)$$
.

C here is real because of unitarity. It follows

$$S(k) = e^{2i\delta} = f(k)/f(-k) \approx -(k+ib)/(k-ib).$$
 (4.3)

At low energies the cross-section is almost entirely due to S-waves:

$$\sigma(\mathbf{E}) = 4\pi \sin^2 \delta / \mathbf{E}.$$

In our approximation we have

$$\sigma(E) = 4 \pi / (E + b^2). \qquad (4.4)$$

If b is small, the cross-section should be abnormally large at E = 0. This

is what we see in the singlet state of the proton-neutron system where we know that there is no bound state. Of course, since b is squared in (4.4), there is no way of telling from the cross-section whether we have a bound or antibound state. The pairs of conjugate poles are named (in [9] there is some disagreement with our convention) resonances. The reason is that they are quite visible in the cross-section if their b is small. Incidentally, we cannot have b = 0, because then also  $f(k_0)$  would vanish and therefore also  $\phi(k_0, r)$  and  $\phi'(k_0, 0) = 1$ , and this is contradictory. In order to see how the cross-section behaves near a resonance we calculate the phase shift for an energy which is very close to the location of the poles. If f(-k) has a zero in k = h + ib, b < 0, f(k) will have a zero in k = h - ib. Taking into account unitarity, we see that  $\delta$  can be represented, when k is close to h, by the formula  $(E_0 = h^2, \Gamma = -4bh)$ :

$$\delta \approx \eta + \arctan tg \frac{\Gamma/2}{E_0 - E}$$
 (4.5)

Suppose for simplicity  $\eta = 0$ . The cross-section will be given by

$$\sigma(\mathbf{E}) = \frac{4\pi}{\mathbf{E}} \sin^2 \delta = \frac{4\pi}{\mathbf{E}} \frac{\Gamma^2/4}{(\mathbf{E} - \mathbf{E}_0)^2 + \Gamma^2/4}.$$
 (4.6)

If we plot the phase shift as function of E in the neighbourhood of  $E_0$ , we find that it starts from the value  $\eta$  if  $E_0 - E \gg \Gamma/2$  and it rapidly jumps up to  $\eta + \pi/2$ when E passes through the value  $E_0$ . If  $\eta = 0$ ,  $\delta$  takes the value  $\pi/2$  when  $E = E_0$ ; this corresponds to a maximum of the cross-section because  $\sin^2 \delta$ takes then the maximum value 1. The same behaviour is evident from (4.6) and shows up as a sharp peak in the plot of the cross-section. If of course b is not so small,the peak broadens and loses its identity by mixing up with nearby peaks. Eq. (4.6) is a simplified version of the Breit-Wigner one level formula. Correspondingly the wave function for an energy close to  $E_0$  is very small outside the range of the potential. This we see from (2.6). Indeed, if k = h, we know that both f(k) and f(-k) are nearly vanishing. As  $\phi(k, r)$  inside the region of interaction is reasonably large, i.e.  $\phi(k, r) \approx r$ , if we normalize the solution from the asymptotic behaviour for large r by choosing a unit flux of ingoing and outgoing particles the amplitude inside the potential will in turn become abnormally large. (Incidentally we notice that  $\phi(\mathbf{k}, \mathbf{r})$  is normalized in the origin.) We may picture the process as follows: the incoming particles spend a long time inside the potential well before coming out. Their interaction is therefore quite strong and this explains the occurrence of large cross-sections. Resonances are often called metastable states, and in several ways they can be approximately considered as states in the usual quantum mechanical sense, like bound states.

#### 5. YUKAWIAN POTENTIALS. THE RESTRICTED CASE OF S WAVES

A potential will be named Yukawian if it can be written in the form:

$$V(\mathbf{r}) = \int_{m}^{\infty} \sigma(\mu) \frac{e^{-\mu r}}{r} d\mu, \qquad (5.1)$$

where  $\sigma(\mu)$  is a suitable weight distribution. Yukawian potentials can be continued for complex values of r in the half-plane Re(r) > 0. This follows from the properties of Laplace transforms which are analytic in the halfplane of convergence. If a potential is Yukawian, then the Jost function has remarkable analyticity properties. The standard theory of differential equations tells us that, if the potential is analytic in some domain, then the wave function is also analytic in the same domain. The Jost solution can be continued then in the complex r domain Re(r) > 0. Take now  $\rho$  as a new variable in eq. (1.5) where  $r = \rho e^{i\sigma}$  and  $\sigma$  is a fixed angle,  $|\sigma| < \pi/2$ . We have  $(\ell = 0)$ 

$$\frac{d^{2}\psi}{d\rho^{2}} + e^{2i\sigma} E\psi = e^{2i\sigma} V(\rho e^{i\sigma})\psi.$$
 (5.2)

This equation looks formally the same as (1.5) with a new distance  $\rho$ , a new wave function  $\psi$ , a new energy  $E_1 = E e^{2i\sigma}$ , a new (complex) potential  $V_1(\rho) = V(\rho e^{i\sigma}) e^{2i\sigma}$ . We are still able to define a new Jost solution  $f_1(k_1, \rho)$  such that  $f_1$  satisfies (5.2) and  $f_1 \sim e^{-ik_1\rho}$  for large  $\rho$ . But  $f(k, \rho e^{i\sigma})$  also satisfies (5.2) with the same boundary conditions<sup>\*</sup> so that

$$f_1(k_1, \rho) = f(k, \rho e^{i\sigma}), k_1 = k e^{i\sigma}.$$

Now,  $f_1$  is analytic in  $k_1$  in the Bargmann domain Im  $k_1 < m_1/2$  where  $m_1$  is related to the range of  $V_1(\rho)$  just as m is related to the range of V(r). If V is given by (5.1), the lower limit in this integral has already been chosen to yield m as the correct value for the Bargmann proof; that is, P(r) of eq. (3.13) converges if b < m/2. Now, if r is large, the main contribution to V(r) from (5.1) is of the kind

$$V(\mathbf{r}) \sim \sigma(\mathbf{m}) e^{-\mathbf{m}\mathbf{r}}/\mathbf{r}^2$$
.

It follows that

$$|V_1(\rho)| \sim \sigma(m) e^{-m\rho\cos\theta}/\rho^2$$
.

Clearly the correct value for  $m_1$  is  $m \cos \sigma \cdot f_1$ , and therefore f is analytic in Im  $k_1 < (m \cos \sigma)/2$ . This domain is different from the original Bargmann domain of f(k, r). The union of all these domains for all  $|\sigma| < \pi/2$  is the k plane with the cut k = ib, where  $m/2 < b < \infty$ .

S(k) is therefore analytic in the k plane with two cuts k = ib, with  $M/2 < |b| < \infty$ . There are of course different and more interesting ways of de-

<sup>\*</sup> This result implies that the analytic continuation of the asymptotic behaviour of f coincides with the asymptotic behaviour of the analytic continuation of f. This seems to be evident but it is not, and it has to be justified It can best be proved using the Phragmen+Lindeloef lemma See CARTWRIGHT [8].

riving the same result. We did it here just because the result was very cheaply obtained. There is one case of Yukawian potential which can be solved exactly [9]: the Bethe potential  $V(r) = -V_0 e^{-mr}$ ,

$$\psi'' + E\psi + V_0 e^{-mr} \psi = 0.$$
 (5.3)

This equation can be reduced to the standard Bessel equation by going to the variable  $\xi = 2(V_0^{1/2} / m)e^{-mr/2}$ . We obtain for the Jost function

$$f(k) = e^{-(ik/m)ln(V_0/m^2)} \Gamma(1 + 2ik/m)J_{2ik/m} (2V_0^{1/2}/m).$$
 (5.4)

This example was considered by Jost in his discussion of the false poles. Our Jost function has namely an infinite set of false poles in the points k = inm/2, n integer > 1, these poles being what remains of the cut along the imaginary axis of k. For the pure Yukawa potential however there is a logarithmic singularity in k = im/2 and more complicated ones farther on. It is, however, much simpler to study these singularities with Martin's method, which shows the very interesting fact that higher perturbation terms produce singularities moving farther and farther away with increasing order of the term.

Martin's method works as follows: he defines

$$g(k, r) = e^{ikr} f(k, r)$$

and starts from the following Ansatz:

$$g(\mathbf{k},\mathbf{r}) = 1 + \int_{\mathbf{m}}^{\infty} \rho(\mathbf{k}\cdot\boldsymbol{\alpha}) e^{-\alpha \mathbf{r}} d\boldsymbol{\alpha}. \qquad (5.5)$$

Inserting g(k, r) into the Schroedinger equation, one finds for it the differential equation.

$$g''(k \cdot r) - 2ikg'(k \cdot r) - V(r)g(k \cdot r) = 0.$$

In this equation we replace g(k, r) by its integral representation (5.5), and we use for V(r) an expansion of the kind

$$\mathbf{V}(\mathbf{r}) = \int_{\mathrm{m}}^{\infty} \mathbf{C}(\mu) \, \mathrm{e}^{-\mu r} \, \mathrm{d}\mu.$$

Of course this representation for V(r) is just equivalent to (5.1) provided

$$\frac{\mathrm{d}}{\mathrm{d}\mu} \mathrm{C}(\mu) = -\sigma(\mu).$$

We get then the integral equation,

$$\mu (\mu + 2ik) \rho (k \cdot \mu) = C(\mu) + \int_{m}^{\mu-m} C(\mu - \alpha) \rho (k \cdot \alpha) d\alpha.$$
 (5.6)

The main point about eq. (5.6) is that the value of  $\rho(\mathbf{k},\mu)$  in a given interval  $nm \le \mu \le (n+1)m$  can be calculated from the knowledge of the values of  $\rho(\mathbf{k},\mu)$  when  $\mu \le nm$ . This provides an interesting method of construction of  $\rho(\mathbf{k},\mu)$  since we know already that  $\rho(\mathbf{k},\mu) = 0$ ,  $\mu \le m$  and  $\rho(\mathbf{k},\mu) = C(\mu)/\mu(\mu + 2i\mathbf{k})$  for  $m \le \mu \le 2m$ . One can see the above situation also by saying that for values of  $\mu$  lying in the interval  $nm \ldots (n+1)n$  the (n+1)m perturbation term and the following one vanish identically so that the perturbation expansion always terminates. It is clear then that this also means that the support of the n-th terms moves away with increasing n. Martin has carefully examined this expansion, and a detailed account can be found in the Hercegnovi lectures.

## 6. THE HIGHER WAVES

All the results that we have so far derived for S waves can be extended to higher waves [9]. There is no simple method for doing this like the one we have for S waves. The reason is that the Green integral functions, which are used in order to define particular solutions of the wave equation, contain Bessel functions in their kernels and these are clumsy to handle.

We intend to quote here the corresponding results and we also give a list of the most important functions used in the formalism of higher waves. The proof of these results actually does not teach anything newer than what we already know for S waves. A fairly complete review of this subject is in [9]. The reason we skip these lengthy mathematical proofs is that for Yukawian potentials Martin has much simpler methods.

Here follows a list of the most important functions of the theory:

(1) The Jost solution. It can be defined with an integral equation similar to (3.1) (see App. 1):

$$f_{\ell}(k, r) = e^{-ikr} + \frac{1}{k} \int_{r}^{\infty} \sin k(x - r) \left[ V(x) + \frac{\ell(\ell+1)}{x^2} \right] f_{\ell}(k, x) dx. \quad (6.1)$$

If we put V = 0, we have,

$$f_{\ell}^{0}(\mathbf{k},\mathbf{r}) = e^{-i\pi(\ell+1)/2} (\pi kr/2)^{1/2} H_{\ell+1/2}^{(2)}(\mathbf{kr}).$$
 (6.2)

We have also the equation (see App. I)

$$f_{\ell}(k, r) = f_{\ell}^{0}(k, r) - i \frac{\pi}{4} \sqrt{r} \int_{r}^{\infty} \sqrt{\xi} [H_{\ell+1/2}^{(1)}(k\xi) H_{\ell+1/2}^{(2)}(kr)]$$

$$- H_{\ell+1/2}^{(2)}(\mathbf{k}\,\xi) H_{\ell+1/2}^{(1)}(\mathbf{k}\mathbf{r}) ] \nabla(\xi) f_{\ell}(\mathbf{k},\xi) d\xi.$$
 (6.3)

(2) The Jost function is defined as

$$f_{\ell}(k) = \lim_{r \to 0} r^{\ell} f_{\ell}(k, r).$$
 (6.4)

If V = Q we have the free Jost function,

$$f_{\ell}^{0}(k) = \pi^{-1/2} e^{-i\ell\pi/2} \Gamma(\ell+1/2) (2/k)^{\ell}.$$
 (6.5)

The regular solution is defined as

$$\phi_{\ell}(\mathbf{k},\mathbf{r}) = \frac{f_{\ell}(\mathbf{k}) f_{\ell}(-\mathbf{k},\mathbf{r}) - (f_{\ell})(-\mathbf{k}) f_{\ell}(\mathbf{k},\mathbf{r})}{2 i \mathbf{k}},$$
  
$$\phi_{\ell}(\mathbf{k},\mathbf{r}) = \phi_{\ell}(-\mathbf{k},\mathbf{r}); \ \phi_{\ell}(\mathbf{k},\mathbf{r}) \approx \mathbf{r}^{\ell+1}, \ \mathbf{r} \rightarrow 0.$$
(6.6)

Comparing the asymptotic behaviour of this solution with the definition of phase shift, we find the formula,

$$S_{\ell}(k) = e^{2i\delta_{\ell}} = e^{i\pi \ell} \frac{f_{\ell}(k)}{f_{\ell}(-k)}$$
 (6.7)

We quote here some results concerning the analyticity domain of these functions. All these analyticity proofs run exactly in the same way as for S waves (see App. II); that is, we place upper bounds on the perturbative expansion of  $f_{\ell}$  (k, r), and we show that it converges uniformly in the Bargmann domain and that each term has the prescribed analytic properties:

(1)  $k^{\ell} f_{\ell}(k, r)$  is analytic in  $b \le m/2$ . If the potential is Yukawian the possible singularities lie on the cut  $m/2 \le b \le \infty$ , with k = ib.

(2) The same result holds for  $k^{\ell} f_{\ell}(k)$ . For large k in the lower halfplane we have  $\lim_{k \to \infty} f_{\ell}(k) / f_{\ell}^{0}(k) = 1$ .

(3)  $S_{\ell}(k)$  is analytic in the cut k plane:  $k = ib, m/2 < |b| < \infty$ . The discussion of the poles of S(k) is exactly the same as the one we gave for  $\ell = 0$ .

(4) The only striking difference between higher waves and the S wave regards the behaviour of the phase shift at low energies. This property is linked with the so-called scattering length approximation. It asserts the validity of the expansion,

$$k^{2\ell+1} \operatorname{ctg} \delta_{\ell}(k) = a_{\ell} + a_{1}k^{2} + \dots$$
 (6.8)

This expansion will be a byproduct of the complete theory of the properties of  $S_{\ell}(k)$  as a function of both k and of (complex)  $\ell$  which will be worked out in the next sections. Physically (6.8) has its origin in the existence of a repulsive centrifugal barrier which pushes the wave function out of the region of interaction. A parameter which decides the order of magnitude of the phase shift is the impact parameter (distance of closest classical approach)  $T = \ell/k$ . If k decreases while  $\ell$  is kept constant, the wave function will scan the potential at increasing distances and the interaction will become negligible when  $\ell/k \gg 1/m$ .

(5) If we let  $\ell$  increase while we keep k constant, we provide another mechanism which increases T and decreases the phase shift. The phase shift can be estimated for large  $\ell$  with the following argument: We know that if T  $\gg 1/m$  the bulk of the wave function lies almost totally outside the potential, and it is a good guess that the wave function is only slightly perturbed by the potential. We take now the exact formula,

$$\sin \delta_{\ell}(\mathbf{k}) = -\mathbf{k} \int_{0}^{\infty} \mathbf{V}(\mathbf{r}) \frac{\phi_{\ell}(\mathbf{k}, \mathbf{r})}{|f_{\ell}(\mathbf{k})|} \frac{\phi_{\ell}^{0}(\mathbf{k}, \mathbf{r})}{|f_{\ell}^{0}(\mathbf{k})|} d\mathbf{r},$$

and we replace  $\phi_{\ell}(\mathbf{k}, \mathbf{r})/|\mathbf{f}_{\ell}(\mathbf{k})|$  by  $\phi_{\ell}^{0}(\mathbf{k}, \mathbf{r})/|\mathbf{f}_{\ell}^{0}(\mathbf{k})|$ . This yields the socalled Born approximation. The general reliability of the Born approximation has been repeatedly questioned, and now it is agreed that it gives at most only the order of magnitude of the scattering amplitude if blindly applied to low waves and it increases in accuracy at high energies. Anyway, if  $T \gg 1/m$  and k is large, we can confidently use it. To us it is interesting just because it gives a reliable estimate of the phase shift for large  $\ell$ , and we need it in order to discuss the convergence of the Rayleigh-Faxen expansion outside the physical range of  $\cos \theta$ . This argument can be made somehow more rigorous, but it then becomes so dull that we prefer not to interrupt our main flow of ideas with insipid mathematics. Anyway the general theory which we shall work out in the next lectures will bring new arguments to support our conclusions. We would just like to mention that Carter in an unfortunately unpublished thesis has proved rigorous equivalent results. (See [9], p. 333 or [3].) He states that for  $\ell \to \infty$  the bound holds:

$$\left| \delta_{\ell} \right| < C \left| \delta_{\ell} \right|_{Born}$$
, (6.9)

where C > 1 is some constant. This is enough for our purposes.

#### 7. LEVISON'S THEOREMS

There is a class of very elegant theorems which relate the number of bound states for a given partial wave to the total variation of the phase shift in the interval  $0 \le k \le \infty$ .

We know that for all reasonable potentials  $\lim_{k\to\infty} S_t(k) = 1$  (k real). At infinity we can always choose

$$\delta_{\sigma}(\infty) = 0.$$

Even within the Bargmann strip  $\delta_{\ell}(k)$  is not analytic, because in general it has logarithmic branch points wherever  $S_{\ell}(k)$  has poles or zeros. We define  $\delta_{\ell}(0)$  as the value we get by continuing  $\delta_{\ell}(k)$  analytically along the real k axis from  $k = +\infty$ . We know that, unless there is a bound state at k = 0, which we exclude for simplicity,  $\sin \delta_{\ell}(0) = 0$  so that  $\delta_{\ell}(0)$  is a multiple of  $\pi$ . Levinson's theorem then states that

$$\delta_{\theta}(0) - \delta_{\theta}(\infty) = n_{\theta}\pi, \qquad (7.1)$$

when  $n_{\ell}$  is the number of bound states of angular momentum  $\ell$ . The proof we prefer here has been somewhat shortened (for a full discussion see [9], p.332). Take the function

$$g_{a}(k) = f_{a}(k)/f_{a}^{0}(k).$$
 (7.2)

(Incidentally  $g_1(k)$  is named Jost function in [9] and written  $f_{\ell}(k)$ .) We know that in the lower half-plane of k we have  $\lim_{k\to\infty} g_{\ell}(k) = 1$ . By unitarity it is obvious that for real k

$$\delta_{\ell}(k) = \arg g_{\ell}(k)$$
.

Moreover  $\delta_{\ell}(k)$  is an odd function of k. This semicircle in b < 0 (b = Im k) is indented on the real axis of k. This semicircle encloses all zeros of  $g_{\ell}(k)$  which correspond to bound states. We define arg  $g_{\ell}(0 + \epsilon) = \delta_{\ell}(0) = n\pi$ , where n is an integer which we do not identify yet with the number of bound states. We now move along the real axis of k until we meet the semicircle, here by definition arg  $g_{\ell}(k) = 0$ . On the whole semicircle we also have arg  $g_{\ell}(k) = 0$ . We move along the semicircle until we arrive on the real negative axis. If we move now toward  $k = 0 = \epsilon$ , we have the relation  $\delta_{\ell}(-k) = -\delta_{\ell}(k)$ . When we arrive at k = 0, we have  $\delta_{\ell}(0 - \epsilon) = -n\pi$  and  $\delta_{\ell}(k)$  is clearly discontinuous in k = 0. The discontinuity arises from the fact that we have enclosed the bound states in the semicircle and we have gone clockwise around the zeros of  $g_{\ell}(k)$ . During the trip arg  $g_{\ell}(k)$  inside the contour. But  $\delta_{\ell}(k)$  has just decreased by  $\delta_{\ell}(0 + \epsilon) - \delta_{\ell}(0 - \epsilon) = n\pi - (-n\pi) = 2n\pi$ . Therefore  $n = n_{\ell}$ . QED.

# 8. THE TECHNIQUES OF COMPLEX ANGULAR MOMENTA

In the other lectures we have discussed the scattering amplitude for integer values of  $\ell$ . This is easily understood because we cannot associate any direct physical meaning to unrestricted values of  $\ell$ ;  $\ell$  came from the expansion in partial waves, and integer values of  $\ell$  are a natural consequence of the quantization of angular momentum. Moreover, we apparently need to consider  $\delta_{\ell}(\mathbf{k})$  when  $\ell$  is integer only in order to know the scattering amplitude.

We want to oppose this general attitude and the reasons are the following:

(a)  $\ell$  is quantized because spherical harmonics are considered on the sphere, that is for  $|\cos \theta| < 1$ , where  $\theta$  is the scattering angle. Truly one can make experiments only when  $|\cos \theta| < 1$ , however, the crossing properties implied by the relativistic Mandelstam representation also mean that, for instance, the pion-nucleon scattering is directly related to the nucleonantinucleon annihilation into two pions. In a way, therefore, the process,  $N + \overline{N} \rightarrow \pi + \pi$  is simply the process  $\pi + N \rightarrow \pi + N$  viewed in a region considered unphysical before. In other words, if we measure the first process we actually measure the second for  $|\cos \theta| > 1$ . Now the natural way of expanding a function of a hyperbolic angle is to use the set  $P_{i\mu-1/2}(\cos \theta)$  which is the corresponding harmonics for a Lorentz invariant hyperboloid in an indefinite metric. Therefore, Mandelstam's representation is naturally associated with non-integer angular momenta. The potential scattering retains part of the full information of the original relativistic scattering, and there should be no surprise if unphysical angular momenta turn up.

(b) Even without the previous argument the technique has been used for years in the discussion of diffraction phenomena; a typical problem in this field was the theory of the rainbow or the theory of propagation of waves around the earth [5]. It is therefore a highly successful tool in a wide range of problems.

The basic idea of the technique arises from a transformation, due to Watson, of the Rayleigh-Faxen formula:

$$f(E, \theta) = \frac{1}{2ik} \sum_{\ell=0}^{\infty} (2\ell + 1) [S_{\ell}(k) - 1] P_{\ell}(\cos\theta). \quad (8.1)$$

This transformation is successful only if one succeeds in proving the existence of an analytic function  $S(\lambda, k)$  of the complex variable  $\lambda$  which takes the values  $S_{\ell}(k)$  of (8.1) when  $\lambda = \ell + 1/2$ . We use the variable  $\lambda$  because in the following it will have a more symmetrical role than  $\ell$  and corresponds more closely to the classical angular momentum than  $\ell$ . In this hypothesis (8.1) can be transformed into

$$f(E, \theta) = -\frac{1}{2k} \int_{C} \frac{\lambda d \lambda}{\cos \pi \lambda} P_{\lambda-1/2} (-\cos \theta) [S(\lambda, k) - 1]. \qquad (8.2)$$

The path C of integration encloses all the positive zeros of  $\cos \pi \lambda$  but avoids the singularities of  $S(\lambda, k)$ . (See Fig. (1).) If we calculate the integral (8.2) with the contour method, we find the expansion (8.1).





The path of integration for the Watson integral

Eq. (8.2) contains all the information of eq. (8.1) and it has additional features of its own. Indeed, in constructing it we use properties of  $S(\lambda, k)$  which depend rather critically on the potential. The path C of integration can be deformed in accordance with the analytic properties of  $S(\lambda, k)$ . The ensuing convergence domain of (8.2) depends on  $P_{\lambda-1/2}$ , and this allows us to extend the analytic properties beyond the Lehmann ellipse.

The necessary steps which we have to carry out in order to establish the validity of Watson's transformation are the following:

(1) Definition of  $S(\lambda, k)$  for general values of  $\lambda$  and k. Analytic properties and asymptotic behaviour of  $S(\lambda, k)$  for  $\lambda$  large. All these properties will be derived for the restricted case of Yukawian potentials.

(2) The convergence of (8.2) is investigated for the specific case of Yukawian potentials. The Mandelstam representation then follows for the total amplitude.

In order to achieve our goal we shall enlarge the definitions which we have used so far for S waves and higher waves. We think that those definitions are self-evident if one keeps an eye on the previous section.

The starting point of our theory is the partial wave Schroedinger equation:

$$\psi''(z) + k^2 \psi(z) - (\lambda^2 - \frac{1}{4}) z^{-2} \psi(z) - V(z) \psi(z) = 0.$$
 (8.3)

We shall use the attribute "physical" for the variables  $\lambda = \ell + 1/2$  and k when is integer and k is real.

Moreover we assume the following conditions for the potential:

(a) V(z) has the representation

$$V(z) = \int_{m>0} \sigma(\mu) \frac{e^{-\mu z}}{z} d\mu,$$

with a suitable weight distribution  $\sigma(\mu)$ .

(b) V(z) can therefore be continued into the half-plane Re z > 0.

c) On any ray arg z = 
$$\sigma$$
,  $|\sigma| < \pi/2$ , we have

$$\int_{-\infty}^{\infty} \left| V(z) z \right| ds \le M \le \infty, ds = \left| dz \right|.$$

We exclude the value  $\sigma = \pi/2$  because the last condition would rule out the interesting case of the Yukawa potential. More refined assumptions will be made in order to derive special results if needed.

We intend to study eq. (8.3) and the associated quantities when  $\lambda$  and k are both complex. This programme has been partly carried out in previous papers [3,4,10,11], and we may group previous results into two classes:

(a) Analyticity in k when  $\lambda$  is physical;

(b) Analyticity in  $\lambda$  when k is physical.

We repeat here some of the already known definitions and formulas which will be used extensively in the lectures. Most of these definitions are purely formal since there are involved, for instance, variables defined through solutions of an integral equation, whose existence has only been proved in the cases (a) and (b). The proofs will be given in the next sections, and the formulas listed should be regarded rather as a framework for the parts to come. There are two ways of defining particular solutions of eq. (8.3).

(i) We define  $\phi(\lambda, \mathbf{k}, \mathbf{z})$  as that solution which behaves like  $z^{\lambda + 1/2}$  when z is small. More rigorously, we define  $\phi$  through the integral equation,

$$\phi(\lambda, \mathbf{k}, \mathbf{z}) = z^{\lambda+1/2} - \frac{1}{2\lambda} \int_{0}^{z} \frac{\xi^{\lambda+1/2}}{z^{\lambda-1/2}} - \frac{z^{\lambda+1/2}}{\xi^{\lambda-1/2}} \left[ V(\xi) - \mathbf{k}^{2} \right] \phi(\lambda, \mathbf{k}, \xi) \, \mathrm{d}\xi \,.$$
(8.4)

If  $\phi_0$  is the solution of eq. (8.3) when V = 0 (free solution), we have

$$\phi_{0}(\lambda, k, z) = \Gamma(\lambda + 1) (2/k)^{\lambda} z^{1/2} J_{\lambda}(k z) \approx z^{\lambda + 1/2}, z \to 0.$$
 (8.5)

Similarly,

$$\phi(\lambda, \mathbf{k}, \mathbf{z}) = \mathbf{z}^{\lambda+1/2} - \frac{\pi\sqrt{z}}{2\sin(\pi\lambda)} \int_{0}^{z} \sqrt{\xi} \left[ \mathbf{J}_{\lambda}(\mathbf{k}\xi) \mathbf{J}_{-\lambda}(\mathbf{k}z) - \mathbf{J}_{-\lambda}(\mathbf{k}\xi) \mathbf{J}_{\lambda}(\mathbf{k}z) \right]$$

 $\times V(\xi)\phi(\lambda,k,\xi) d\xi. \qquad (8.6)$ 

The derivation of these equations is quite simple (see App. I). Clearly  $\phi(\lambda, k, z) = \phi(\lambda, -k, z)$ . However,  $\phi(-\lambda, k, z)$  is a new solution. If Re  $\lambda > 0$ ,  $\phi(+\lambda, k, z)$  will be regular at the origin, and any other independent solution will be irregular. On the line Re  $\lambda = 0$ ,  $\phi(\lambda, k, z)$  and  $\phi(-\lambda, k, z)$  exchange their regularity roles and both have an oscillatory character. It is evident from this and other features of (8.3) that  $\lambda$  dominates the behaviour in the origin while k determines the behaviour at infinity. So far we have not committed ourselves to any theorem of existence of these solutions. In fact, unless one makes a special hypothesis on the potential, the region where both  $\phi(\lambda, k, z)$  and  $\phi(-\lambda, k, z)$  exist and are analytic is in general very limited. The line Re  $\lambda = 0$  will be seen to belong to this region. Of some use is the Wronskian:

W[
$$\phi(\lambda, k, z), \phi(-\lambda, k, z)$$
]

$$= \phi(\lambda, k, z) \phi'(-\lambda, k, z) - \phi(-\lambda, k, z) \phi'(\lambda, k, z) = -2\lambda.$$
(8.7)

(ii) The second class of solutions is defined through the boundary conditions at infinity. Such a class of solutions was first introduced by Jost for S waves. We define  $f(\lambda, k, z)$  as that solution which behaves like  $e^{-ikz}$  for large z. More rigorously.

$$f(\lambda, k, z) = e^{-ikz} + \frac{1}{k} \int_{z}^{\infty} \sin k(\xi - z) \left[ V(\xi) + \frac{\lambda^{2} - 1/4}{\xi^{2}} \right] f(\lambda, k, \xi) d\xi. \quad (8.8)$$

If  $f_0(\lambda, k, z)$  is the free solution, we have

$$f_0(\lambda, k, z) = e^{i\pi(\lambda+1/2)/2} (\pi kz/2)^{1/2} H_{\lambda}^{(2)}(kz) \sim e^{-ikz}.$$
 (8.9)

Similarly,

$$f(\lambda, k, z) = f_0(\lambda, k, z) - i \frac{\pi}{4} \sqrt{z} \int_z^{\infty} \sqrt{\xi} [H_{\lambda}^{(1)}(k\xi) H_{\lambda}^{(2)}(kz) - H_{\lambda}^{(2)}(k\xi) H_{\lambda}^{(1)}(kz)] V(\xi) f(\lambda, k, \xi) d\xi. \qquad (8.10)$$

We can also define  $f(\lambda, ke^{-i\pi}, z)$ . However, in general  $f(\lambda, k, z)$  has a branch point in k = 0 and  $f(\lambda, ke^{-i\pi}, z)$  will be different from  $f(\lambda, k, z)$ . This already happens for free solutions. For instance,

$$f_0(\lambda, ke^{-i\pi}, z) = e^{i\pi(\lambda + 1/2)/2} (\pi kz/2)^{1/2} H_{\lambda}^{(1)}(kz) \sim e^{ikz}$$
. (8.11)

The Wronskian is uniquely defined:

$$W[f(\lambda, k, z), f(\lambda, ke^{-i\pi}, z)] = 2ik.$$
 (8.12)

From the general theory of differential equations we know that the analyticity domains in z of  $\phi(\lambda, k, z)$ ,  $f(\lambda, k, z)$  and V(z) are the same. If we take the conjugate of each of the previously written equations, we find (if V(z) is real on the real positive axis of z):

$$\phi(\lambda, k, z) = \phi^{*}(\lambda^{*}, k^{*}, z), \qquad f(\lambda, k, z) = f^{*}(\lambda^{*}, -k^{*}, z). \quad (8.13)$$

The hermiticity requirement on the Hamiltonian needed for the above results will not be used in the proofs on the convergence of the perturbation expansions which we shall derive in the next section. This we do, not in view of possible application to absorbing potentials, but just as a mathematical artifice in order to extend the analytic properties. This will be apparent in the following.

# 9. THE JOST FUNCTIONS AND ANALYTIC PROPERTIES OF THE PAR-TIAL WAVE FUNCTIONS

Once we have defined the functions  $\phi(\pm\lambda, \mathbf{k}, z)$  and  $f(\lambda, \pm \mathbf{k}, z)$ , we possess four solutions of the same differential equation. The Wronskian of any two solutions is of course a constant. We have already given such a Wronskian between two  $\phi$ 's and two f's in (8.7), (8.12); these two Wronskians do not carry any information about the potential, and they are therefore useful but trivial. A more useful quantity (the so-called Jost function) is

$$W[f(\lambda, k, z), \phi(\lambda, k, z)] = f(\lambda, k).$$
(9.1)

Besides  $f(\lambda, k)$ , we consider  $f(-\lambda, k)$ ,  $f(\lambda, -k)$  and  $f(-\lambda, -k)$  too. The Jost function is interesting because, as we shall see, it is directly related to the scattering matrix. In order to show this let us first notice that, according to general principles, there is always a linear relation between any three solutions of (8.3). In particular, we must have

$$\phi(\lambda, \mathbf{k}, \mathbf{z}) = \mathrm{Af}(\lambda, \mathbf{k}, \mathbf{z}) + \mathrm{Bf}(\lambda, -\mathbf{k}, \mathbf{z}),$$

$$\phi(-\lambda, \mathbf{k}, \mathbf{z}) = \mathrm{Cf}(\lambda, \mathbf{k}, \mathbf{z}) + \mathrm{Df}(\lambda, -\mathbf{k}, \mathbf{z}).$$
(9.2)

Here A, B, C, D are independent of z, but they are expected to be functions

of  $\lambda$  and k. In order to evaluate them we introduce the formula (9.2) for  $\phi(\lambda, k, z)$  into (9.1), thus finding:

$$2ikA = -f(\lambda, -k).$$
 (9.3)

Similarly,

$$2ikB = f(\lambda, k), \ 2ikC = -f(-\lambda, -k), \ 2ikD = f(-\lambda, k).$$

These values can be reintroduced into (9.2), and we find

$$\phi(\lambda, \mathbf{k}, \mathbf{z}) = \left[ f(\lambda, \mathbf{k}) f(\lambda, -\mathbf{k}, \mathbf{z}) - f(\lambda, -\mathbf{k}) f(\lambda, \mathbf{k}, \mathbf{z}) \right] / 2i\mathbf{k}.$$
(9.4)

Finally we calculate the Wronskian W[ $\phi(\lambda, k, z)$ ,  $\phi(-\lambda, k, z)$ ], using eq. (9.4), and we compare it with the known value,

$$f(\lambda, -k) f(-\lambda, k) - f(\lambda, k) f(-\lambda, -k) = 4i\lambda k.$$
(9.5)

This is an important identity\*. From (9.5) and (9.4) we can find easily

$$f(\lambda, k, z) = [f(-\lambda, k)\phi(\lambda, k, z) - f(\lambda, k)\phi(-\lambda, k, z)]/2\lambda,$$

$$(9.6)$$

$$f(\lambda, -k, z) = [f(-\lambda, -k)\phi(\lambda, k, z) - f(\lambda, -k)\phi(-\lambda, k, z)]/2\lambda.$$

The free Jost functions are given by the formula,

$$f_{0}(\lambda, k) = (2/\pi)^{1/2} 2^{\lambda} \Gamma(\lambda+1) k^{-\lambda+1/2} e^{-i\pi(\lambda-1/2)/2}$$
(9.7)

They are multivalued in k.

We now proceed to find the connection between the Jost functions and the scattering phase shifts. It is almost unnecessary to point out that what we shall define is actually a function which interpolates for unphysical values of  $\lambda$  (and k) the known and measurable phase shifts. It is also clear that there could be no other interpolations. The one we select is convenient merely because it retains part of the properties of the physical phases. Our definition starts from the known behaviour of the "regular" free solution  $\phi_0(\lambda, k, z)$ at infinity:

$$\phi_0(\lambda, \mathbf{k}, \mathbf{z}) \sim e^{i\pi(\lambda - 1/2)/2} \frac{1}{k} f_0(\lambda, \mathbf{k}) \sin[kz - \pi(\lambda - 1/2)/2].$$
 (9.8)

\* This is identity (1.8) of [10]. The functions  $C(\lambda)$  and  $S(\lambda)$  are linear combinations of Jost functions.

This theory follows immediately from the theory of Bessel functions. We compare it with the behaviour of the perturbed regular solution (9.4):

$$\phi(\lambda, \mathbf{k}, \mathbf{z}) \sim [f(\lambda, \mathbf{k})e^{i\mathbf{k}\mathbf{z}} - f(\lambda - \mathbf{k})e^{-i\mathbf{k}\mathbf{z}}]/2i\mathbf{k}$$
  
=  $e^{i\pi(\lambda - 1/2)}e^{-i\delta(\lambda, \mathbf{k})} \frac{1}{\mathbf{k}}f(\lambda, \mathbf{k})\sin[\mathbf{k}\mathbf{z} - \pi(\lambda - 1/2)/2 + \delta(\lambda, \mathbf{k})],$  (9.9)

where we have defined

$$S(\lambda, k) = e^{2i\delta(\lambda, k)} = [f(\lambda, k)/f(\lambda, -k)]e^{i\pi(\lambda - 1/2)}.$$
(9.10)

This formula we retain even when the comparison is no longer valid, in particular when one deals with exploding expotentials (k not real). So far this definition is purely formal, since we know very little about the existence and the analyticity of the Jost function when both k and  $\lambda$  are not physical. This will be discussed in the next sections.

We give here, for completeness, a relation that will be used later,

$$e^{i\delta(\lambda, k)} \sin \delta(\lambda, k) = -k \int_{0}^{\infty} V(z) \frac{\phi_{0}(\lambda, k, z)}{f_{0}(\lambda, k)} - \frac{\phi(\lambda, k, z)}{f(\lambda, -k)} dz, \quad (9.11)$$

and that has been deduced from

$$(\phi'\phi_0 - \phi'_0\phi) \int_0^\infty = -e^{i\pi(\lambda-1/2)} e^{-i\delta(\lambda,k)} \frac{1}{k} f_0(\lambda,k) f(\lambda,k) \sin \delta(\lambda,k)$$

$$= \int_{0}^{\infty} V(z)\phi_{0}(\lambda, k, z)\phi(\lambda, k, z) dz.$$

The existence theorems and the analytic properties of the partial wave functions are usually derived by the following method:

(a) We iterate the defining integral equation, and we define a formal perturbation expansion.

(b) The analytic properties of each term of the expansion must then be examined in order to find the analyticity domain of the solution.

(c) Bounds are placed on the solution in such a way that the series are seen to converge uniformly inside the analyticity domain. We give the practical calculations in Appendix II, and we merely state here the most important results:

(i)  $\phi(\lambda, k, z)$  and  $\phi'(\lambda, k, z)$  are integral functions of k (i.e., regular for all k with the exception of an essential singularity at  $k = \infty$ ) and are analytic in  $\lambda$  for  $\operatorname{Re} \lambda > 0$ ; the expansion also converges for  $\operatorname{Re} \lambda = 0$  (in fact, we think it is possible to show that the analyticity region can be pushed inside the region  $\operatorname{Re} \lambda < 0$  under very special assumptions on the potential).  $\phi(\lambda, k, z)$  is analytic in both variables in the topological product of the k plane ( $k = \infty$  excluded) with the half-plane  $\operatorname{Re} \lambda > 0$  (and continuous for  $\operatorname{Re} \lambda = 0$ ).

(ii)  $f(\lambda, k, z)$  is analytic in the pair of variables  $\lambda$ , k in the topological product of the whole  $\lambda$  plane ( $\lambda = \infty$  excluded) with the half-plane Im  $\hat{k} < 0$  (and continuous for Im k = 0). This allows one to define  $f(\lambda, -k, z)$  as  $f(\lambda, ke^{-i\pi}, z)$  unambiguously when k is real; in order to avoid confusion we shall retain the clearer notation  $f(\lambda, ke^{-i\pi}, z)$ . Correspondingly in (8.13) we have  $f(\lambda, k, z) = f^*(\lambda^*, k^*e^{-i\pi}, z)$ . It follows that  $f(\lambda, k)$  is analytic in  $\lambda$ , k in the product on the half-planes Re  $\lambda > 0$ , Im k < 0 and is continuous on the boundaries Re  $\lambda = 0$ , Im k = 0. The branch point at k = 0 will be discussed later.

Under the stated assumptions on the potential it is possible to enlarge the analyticity domain of  $f(\lambda, k, z)$  and consequently that of  $f(\lambda, k)$ . for this purpose let us consider eq. (8.3) along a prescribed direction in the complex z = x + iy plane. Let therefore  $z = \rho e^{i\sigma}$ , where  $\sigma$  is a constant angle  $|\sigma| < \pi/2$ . Eq. (8.3) can be written in the variable  $\rho$ :

$$\frac{d^2\psi}{d\rho^2} - \frac{\lambda^2 - 1/4}{\rho^2}\psi + k^2 e^{2i\sigma}\psi - e^{2i\sigma}V(\rho e^{i\sigma})\psi = 0.$$
(9.12)

This equation is still of the same kind as eq.(8.3), with a new wave number  $k_1 = ke^{i\pi}$  and a new(complex) potential  $V_1 = V(\rho e^{i\sigma})e^{2i\sigma}$ . The previous analysis can be carried out on the new equation, and we shall arrive at a new set of wave functions  $\phi_1(\lambda, k_1, \rho)$ ,  $f_1(\lambda, k_1, \rho)$  and at a new Jost function  $f_1(\lambda, k_1)$ .

The Jost solution  $f_1(\lambda, k_1, \rho)$  is defined as the solution with the following behaviour:

$$f_1(\lambda, k_1, \rho) \sim e^{-ik_1\rho}$$

for any value of  $\sigma$ . On the other hand, the Jost solution  $f(\lambda, k, z)$ , already defined for z real, may be continued analytically in the half-plane Re z > 0 with the same boundary condition because of the conditions on the potential V(z). So the analytic continuation of  $f(\lambda, k, z)$  coincides with  $f_1(\lambda, k_1, \rho)$ , and we have

$$f_1(\lambda, k_1, \rho) = f(\lambda, k, z), \phi(\lambda, k_1, \rho) = e^{-i\sigma(\lambda+1/2)} \phi(\lambda, k, z),$$

$$f_1(\lambda, k_1) = e^{-i\sigma(\lambda+1/2)} f(\lambda, k)$$

But the same general analysis used before for the variable z (for real values), if used for the variable  $\rho$ , implies that the new Jost function is analytic in Im  $k_1 < 0$  and Re  $\lambda > 0$  and that the old Jost function is also analytic in this domain, in view of the above relation. This domain depends on  $\sigma$ , where  $|\sigma| < \pi/2$ . The Jost function is therefore analytic in the union of all domains of the kind Im (kei $\sigma$ ) < 0; this union is simply the k plane cut along the upper imaginary axis k = in ( $\eta > 0$ ). Previous results (see Appendix II) actually state that, when  $\lambda$  is physical, the cut starts at  $\eta = m/2$ , m being the lower limit of integration in the integral defining V(z).

Similarly,  $f(\lambda, ke^{-i\pi})$  is holomorphic in the topological product of the whole k plane, cut along the lower imaginary axis (when i is integer, the cut starts at  $\eta = -m/2$ ), with the half-plane Re  $\lambda > 0$ .

Finally we discuss the branch point of the Jost functions at k = 0. From (8.9) it follows that

$$f_{0}(\lambda, ke^{-2i\pi}, z) = f_{0}(\lambda, k, z) + a(\lambda) f_{0}(\lambda, ke^{-i\pi}, z),$$

$$f_{0}(\lambda, ke^{-3i\pi}, z) = [1 + a^{2}(\lambda)]f_{0}(\lambda, ke^{-i\pi}, z) + a(\lambda) f_{0}(\lambda, k, z),$$

$$a(\lambda) = -2i \cos (\pi \lambda).$$

Introducing this relation now in definition (9.1), we have that the result of a circuit around the origin can be written as follows:

$$f_{0}(\lambda, ke^{-2i\pi}) = f_{0}(\lambda, k) + a(\lambda) f_{0}(\lambda, ke^{-i\pi}),$$

$$(9.13)$$

$$f_{0}(\lambda, ke^{-3i\pi}) = [1 + a^{2}(\lambda)] f_{0}(\lambda, ke^{-i\pi}) + a(\lambda) f_{0}(\lambda, k).$$

If we think of eq. (8.10) written for  $f(\lambda, ke^{-i\pi}, z)$  and make the linear combination  $f(\lambda, k, z) + a(\lambda) f(\lambda, ke^{-i\pi}, z)$ , we find that, if one follows a path which, without crossing the  $m/2 \le \eta \le \infty$  cut, encircles the origin, then, when  $\lambda$  is real,  $f(\lambda, k)$  has exactly the same law of transformation as  $f_0(\lambda, k)$ .

Later on it proves convenient to use the function  $F(\lambda, k) = f(\lambda, k)/f_0(\lambda, k)$ . In terms of  $F(\lambda, k)$  one writes

$$\mathbf{F}(\lambda, \mathbf{k} e^{-2i\pi}) = 2 \cos \pi \lambda e^{-i\pi\lambda} \mathbf{F}(\lambda, \mathbf{k} e^{-i\pi}) - e^{-2i\pi\lambda} \mathbf{F}(\lambda, \mathbf{k})$$

 $\mathbf{or}$ 

$$\mathbf{F}(\lambda, \mathbf{k} \mathbf{e}^{-2i\pi}) - \mathbf{F}(\lambda, \mathbf{k} \mathbf{e}^{-i\pi}) = \mathbf{e}^{-2i\pi\lambda} \left[ \mathbf{F}(\lambda, \mathbf{k} \mathbf{e}^{-i\pi}) - \mathbf{F}(\lambda, \mathbf{k}) \right].$$

## MATHEMATICAL THEORY OF POTENTIAL SCATTERING

One could argue that it would be easier to represent everything with a single cut starting from the origin. This is not true since we would lose the information that the branching properties at the origin do not depend on the potential and are purely kinematical. On the contrary, the other cut depends critically on the potential, and it is useful to separate the contributions.

For the S matrix (9.13) gives \*

$$S(\lambda, ke^{-2i\pi}) = \frac{S(\lambda, k) - 2\cos(\pi\lambda)e^{i\pi\lambda}}{[1 - 4\cos^2(\pi\lambda)] + 2\cos(\pi\lambda)e^{-i\pi\lambda}S(\lambda k)},$$
(9.14)

or

$$S(\lambda, k^{-i\pi}) = \frac{e^{2i\pi(\lambda - 1/2)}}{S(\lambda, k) - 2i\cos(\pi\lambda) e^{i\pi(\lambda - 1/2)}}.$$
 (9.15)

It is useful to introduce a new function,

$$Z(\lambda, \mathbf{k}) = i \mathbf{k}^{2\lambda} [S(\lambda, \mathbf{k}) - \mathbf{e}^{2i\pi\lambda}] / [S(\lambda, \mathbf{k}) - 1].$$
(9.16)

The function Z can be linked to the so-called scattering length expansion. This expansion represents  $k^{2\ell+1}$  ctg  $\delta(\ell, k)$  at low energies as a power series. From this expansion it is evident that  $\delta(\ell, k)$  tends to vanish, like  $\approx k^{2\ell+1}$ , when  $k \rightarrow 0$  in the  $\ell^{th}$  wave. Now, if  $\lambda$  is physical ( $\lambda = \ell + 1/2$ ), we have  $Z(\lambda, k) = k^{2\ell+1}$  ctg  $\delta(\ell, k)$ . This shows that  $Z(\lambda, k)$  is the natural generalization of  $k^{2\ell+1}$  ctg  $\delta(\ell, k)$  because it retains the property of admitting a power series expansion in a neighbourhood of the origin. It must be noticed that  $Z(\lambda, k)$  is not only regular in k = 0 but also an even function of k; its meromorphy domain is the same as that of  $S(\lambda, k)$ .

The following formula is also useful:

$$S(\lambda, \mathbf{k}) = [Z(\lambda, \mathbf{k}) - i\mathbf{k}^{2\lambda} e^{2i\pi\lambda}] / [Z(\lambda, \mathbf{k}) - i\mathbf{k}^{2\lambda}]. \qquad (9.17)$$

Finally, we wish to point out that eq. (9.5) implies

$$e^{-i\pi\lambda}S(\lambda, k) - e^{i\pi\lambda}S(-\lambda, k) = -4k\lambda/[f(\lambda, ke^{-i\pi}) f(-\lambda, ke^{-i\pi})]. \qquad (9.18)$$

This equation only holds when  $\lambda$  is imaginary; otherwise one of the two functions  $f(\lambda, ke^{-i\pi})$ ,  $f(-\lambda, ke^{-i\pi})$  is not defined. We also have from eqs. (8.13)

\* In the following when we write  $f(\lambda, k)$  we mean the Jost function on the sheet:  $-3\pi/2 < \arg k < \pi/2$ .

$$f(\lambda, k) = f^{*}(\lambda^{*}, k^{*}e^{-i\pi}), S^{*}(\lambda, k) = S^{-1}(\lambda^{*}, k^{*}e^{-i\pi}).$$
 (9.19)

If we define  $Z(\lambda, k)$ , when  $\lambda$  is not real, by eq. (9.16), we find that it is meromorphic in half-planes Re k > 0 and Re k < 0. There is at the moment no way of joining the left and right domains of  $Z(\lambda, k)$ , because there is no gap through the cut of either  $f(\lambda, k)$  or  $f(\lambda, ke^{-i\pi})$  unless  $\lambda$  is real. From Appendix II we can prove that actually the result holds for any real positive  $\lambda$ . Indeed, if  $\lambda$  is real positive,

$$\lim_{\epsilon \to 0} Z(\lambda, i\xi - \epsilon) - Z(\lambda, i\xi + \epsilon) = D(\lambda) = 0, |\xi| < m/2.$$

But we know that in general  $D(\lambda)$  is analytic in  $\lambda$  for Re  $\lambda > 0$ . It follows that  $D(\lambda) = 0$  for Re  $\lambda > 0$  and  $|\xi| < m/2$ . This result enables us to join the right and left domains of meromorphy of  $Z(\lambda, k)$  (and of course of  $S(\lambda, k)$  and of related functions) through the gap  $|\xi| < m/2$ . This shows that actually the branch point of S in k = 0 is a purely kinematical one: that is, it does not depend on the potential.

## 10. THE ASYMPTOTIC BEHAVIOUR OF THE PHASE SHIFT

The behaviour for large values of  $\lambda$  and k of the phase shift can best be investigated with the help of the WKB method. In the current practice the use of this method has been limited for obvious reasons to the physical values of k and  $\lambda$ . We wish to point out, however, that the extension to the unphysical range of these variables does not add anything essentially new to the method and that the only difficulty is an increased complexity and variety in the classification and behaviour of the turning points. The most rigorous paper on this subject is certainly KEMBLE's paper [12], and we could almost quote his results with obvious changes. As Kemble's analysis is in some cases incomplete for our purposes or it becomes too complicated, it will not be reported here. A more realistic view of the situation has suggested that these details should be published elsewhere [16] and that we should discuss here the final results only.

The general idea of the WKB method is that of constructing a differential equation, which is very close to the Schroedinger equation, and whose solutions are well known. Such an equation is satisfied by the functions:

$$\frac{1}{\sqrt{p(x)}} \exp\left(\pm i \int^{x} p(z) dz\right),$$

(10, 1)

$$p^{2}(z) = k^{2} - \lambda^{2}/z^{2} - V(z),$$
  $p_{0}^{2}(z) = k^{2} - \lambda^{2}/z^{2}.$ 

The approximation is generally good on the whole complex z plane except in the neighbourhood of the points where p(z) vanishes. These points are usually named turning points T. If k and  $\lambda$  are very large, there is only one turning point in the domain Re z > 0 and this occurs very close to  $T_0$  or  $-T_0$ ,  $T_0 = \lambda/k$ , which are exactly the two turning points when V = 0. The choice between  $T_0$  and  $-T_0$  is dictated by the fact that only one of these points is on the good side Re z > 0 where V(z) is analytic. The turning points are branch points of p(z).

The main problem of the WKB is to connect the solution (10.1), which is good approximation at large distances. These solutions cannot in general be represented by the same formula because the approximation scheme fails near the turning point. An appropriate connection formula can be found in the literature [7].

The result of the above analysis is that when k and  $\lambda$  are large we have the following asymptotic formulas:

$$f(\lambda, k) \sim f_{0}(\lambda, k) \exp\left(-i \int_{0, \Gamma_{f}}^{\infty} [p_{0}(z) - p(z)] dz\right),$$
(10.2)  
$$f(\lambda, ke^{-i\pi}) \sim f_{0}(\lambda, ke^{-i\pi}) \exp\left(i \int_{0, \Gamma_{h}}^{\infty} [p_{0}(z) - p(z)] dz\right).$$

The integration paths  $\Gamma_{\ell}$  and  $\Gamma_{h}$  connect the origin with the infinity in the half-plane Re z > 0.  $\Gamma_{\ell}$  passes below and  $\Gamma_{h}$  above T(see Figs. 2 and 3). The



Diagram for the asymptotic formulas

Diagram for the asymptotic formulas

proposed formulas are valid under the restrictions that paths satisfying these criteria actually exist. If Re  $T_0 > 0$ , then we can obviously trace both paths. Suppose now that we let arg  $T_0$  gradually increase toward  $\pi/2$ . When arg  $T_0 = \pi/2$ , the high path gets pinched between the turning point and the

#### T. REGGE

imaginary axis. For arg  $T_0 > \pi/2$  the corresponding second formula is no longer valid. Only one formula therefore remains because it is still possible to define the low paths (see Fig. 3). Clearly, however, these low paths run high with respect to  $-T_0$ , which is now in the Re z > 0 plane. If in the formula we now replace k by ke<sup>-iπ</sup>, we see that the second formula has been replaced by the first. Conversely, if we let arg  $T_0$  decrease toward  $-\pi/2$ , we find that the first formula is now meaningless and that the second one takes its place. An important complement to these formulas is that p(z) is made single-valued in Re z > 0 by cutting the z plane with a cut which joins T to the origin. On the opposite sides of this cut p(z) takes opposite values.

Let us now evaluate the asymptotic formula for the S function. We insert the expressions (10.2) into (9.10) and use (9.7). Thus we obtain

$$S(\lambda, k) \sim \exp\left(-i\left(\int_{0, \Gamma_{0}}^{\infty} + \int_{0, \Gamma_{h}}^{\infty}\right)\left[p_{0}(z) - p(z)\right]dz\right)\right).$$
 (10.3)

It is obvious that the sum of a high and a low integral can be reduced to a single complex integral which comes from infinity, passes across the cut of p(z) and goes back to infinity on the other sheet of the function p(z) after having encircled the point T. After this has been understood, it is clear that the WKB formula for the phase shift is just the one we already know from more elementary treatments:

$$\delta(\lambda, \mathbf{k}) \sim - \int_{T \approx T_{\alpha}}^{\infty} [\mathbf{p}_{0}(z) - \mathbf{p}(z)] dz. \qquad (10.4)$$

The domain of validity of this formula is the intersection of the validity domain of the formulas (10.2); that is, Re  $T_0 > 0$  (see Fig. 4). There is no





The regions of validity of the WKB method (shaded) and of the bounds (10.5) (unshaded)

point anyway in trying to use (10.4) when Re  $T_0 < 0$ , because the corresponding integral is in general meaningless (it implies the knowledge of V(z) when z is equal or at least very close to T, but V(z) is defined only when Re z > 0). From this formula it is apparent that  $\lim S(\lambda, k) = 1$  when  $|\lambda|, |k| \rightarrow \infty$ , under the quoted restrictions. If Re T < 0, we have no proof of the validity of the above limit and we actually consider it not to be true. For our discussion it is necessary to know some upper bound on  $S(\lambda, k)$ . These bounds are derived in a paper by BOTTINO, LONGONI and REGGE [16]. They refer to the behaviour when  $\lambda$  is large and k is constant. (If  $\lambda$  is kept constant and k is large,  $\lim S(\lambda, k) = 1$ .) The desired bounds are

$$|S(\lambda, k)| < \mu(\lambda, k) e^{2(\arg T_0 + \pi/2) \operatorname{Im} \lambda}, \arg T_0 < -\pi/2, \operatorname{Im} \lambda < 0$$

$$(10.5)$$

$$|S(\lambda, k)^{-1}| < \mu(\lambda, k) e^{2(\arg T_0 - \pi/2) \operatorname{Im} \lambda}, \operatorname{Im} \lambda > 0, \arg T_0 > \pi/2.$$

The indicated domains of validity of these two bounds are the two unshaded regions in Fig. 4.  $\mu(\lambda, k)$  is here a function which is bounded above by a constant independent of k and  $\lambda$ . Both bounds are equivalent to each other through the use of unitarity.

In using (10.4), one must always be aware that there is an error associated with it. If  $\delta$  vanishes very rapidly with  $\lambda$ , the above formula becomes meaningless, because it can easily happen that the error, although small, is still larger than  $\delta$ . The usefulness of the WKB method here is that it yields a proof that  $\delta$  vanishes for large  $\lambda$  whenever Re  $T_0 > 0$ . This is already enough to obtain results concerning the analyticity in the variables  $s = k^2$  and t (momentum transfer). Besides these asymptotic evaluations, we want to quote a more precise result which states that for large angular momenta the Born approximation (see (9, 11)),

$$\delta \sim \delta_{\rm B} = -\mathbf{k} \, \mathrm{e}^{-\mathrm{i} \pi (\lambda - 1/2)} \int_{0}^{\infty} V(z) \left( \frac{\phi_0(\lambda, \mathbf{k}, z)}{f_0(\lambda, \mathbf{k})} \right)^2 \mathrm{d}z, \qquad (10.6)$$

is a very reliable one. The reason for this is that the wave function for large  $\lambda$  lies totally outside the potential and is practically unaffected by it. Therefore,  $\phi \sim \phi_0$  for large  $\lambda$ . This can be shown more exactly from the WKB analysis. For z fixed we find  $\lim(\phi/\phi_0) = 1$ . We would have to prove uniform convergence in order to derive  $\lim(\delta/\delta_B) = 1$ .

We do not want to cram the paper with an uninteresting proof\*.

It is well known that the Born formula can be integrated for the class of Yukawa potentials and yields

$$\delta_{B} = -\frac{1}{2k} \int_{m}^{\infty} Q_{\lambda - \frac{1}{2}} (1 + \mu^{2}/2k^{2}) \sigma(\mu) d\mu.$$

<sup>\*</sup> In [3] it is stated that an equivalent rigorous proof has been obtained by D.S. Carter (Princeton thesis), but unfortunately this proof has not been published. Of course, a proof follows from the three-dimensional formalism and from the existence of the small Lehmann ellipse.

If  $\lambda$  is large, the asymptotic behaviour of  $\delta$  is

$$\delta \sim -(\pi/2)^{1/2} \frac{\sigma(m)}{2m} k(\sin h a)^{1/2} \frac{e^{-a\lambda}}{\lambda^{3/2}}; \cos h a = 1 + m^2/2k^2.$$
(10.7)

The standard WKB method yields instead

$$\delta = 0(e^{-m\lambda/k}).$$

The last evaluation is for our purposes too optimistic at low energies but becomes reliable at large energies.

# 11. THE POLES OF $S(\lambda, k)$

Earlier analysis of the poles of  $S(\lambda, k)$  have been carried out in the following cases:

- (1)  $\lambda$  physical, k complex. The current names given to these poles are
  - (a) bound states if  $k = i \eta$  ( $\eta$  real > 0),
  - (b) anti-bound states or virtual states if  $k = -i\eta$ ,
  - (c) resonances if Im k < 0.

The resonances occur in pairs of conjugate poles. Except for bound states, the region Im k > 0 is forbidden to poles. It is evident from the existing literature that the anti-bound states and metastable states (resonances) are not states in the accepted frame of definition of quantum mechanics because their wave functions are not square integrable. However, they share many of the properties of ordinary states.

(2) k physical,  $\lambda$  complex. The poles occur only when  $\lambda > 0$ . They have been named shadow states in [11] \*. In the full complex domain of k and  $\lambda$ shadow states and resonances are particular intersections of the same singular surface of S( $\lambda$ , k). For we remember that analytic functions of two variables are never singular on isolated points but always on analytic surfaces (of dimension 2). In [11] a number of inequalities was derived concerning the distribution of the shadow states.

The discussion will now be extended to complex  $\lambda$  and k. Roughly speaking, there are two kinds of limitations on the position of the poles: the first follows from the equation of continuity and applies equally well, under very weak conditions, to any kind of potential; the second uses special properties of V(x) like limitations on the depth and width of V(x) and analyticity.

The continuity equation can be used as follows: We suppose that, for a particular set of values of  $\lambda$  and k,  $\lambda = \lambda_0$ ,  $k = k_0$ , inside its meromorphy domain,  $S(\lambda, k)$  has a simple pole. Then clearly  $f(\lambda_0, k_0 e^{-i\pi}) = 0$ . Under this hypothesis,

<sup>\*</sup> Recently we have found a paper [13] where the name "spiralling states" has been adopted.

$$\phi(\lambda_0, k_0, z) = \frac{f(\lambda_0, k_0)}{2ik_0} f(\lambda_0, k_0 e^{-i\pi}, z). \qquad (11.1)$$

If Re  $\lambda_0 > 0$  and Im  $k_0 > 0$ , the above function vanishes as a function of real z at zero and at infinity. Its complex conjugate  $\phi^*$  will also vanish in the same points;  $\phi^*$  satisfies the conjugate equation (z real):

$$\phi^{*''} + k_0^{*2} \phi^{*} - \frac{\lambda_0^{*2} - 1/4}{z^2} \phi^{*} - V(z) \phi^{*} = 0.$$
 (11.2)

It follows that:

$$(\phi'\phi^* - \phi^{*'}\phi)' = (k_0^{*2} - k_0^2) |\phi|^2 - (\lambda_0^{*2} - \lambda_0^2) \frac{|\phi|^2}{z^2}.$$
(11.3)

This identity can be integrated from zero to infinity. The contribution of the first term vanishes with  $\phi$  and  $\phi^*$  at both ends. What is left yields the equation,

$$\operatorname{Im} k_{0} \operatorname{Re} k_{0} \int_{0}^{\infty} \left| \phi \right|^{2} dz - \operatorname{Im} \lambda_{0} \operatorname{Re} \lambda_{0} \int_{0}^{\infty} \frac{\left| \phi \right|^{2}}{z^{2}} dz = 0. \quad (11.4)$$

From (11.4) it is clear that, where Re  $k_0$  and Im  $\lambda_0$  have opposite signs, poles do not occur. Therefore we obtain two domains of holomorphy of

$$\begin{cases} \operatorname{Re} k_0 > 0 \\ \operatorname{Im} \lambda_0 < 0 \end{cases} \qquad \qquad \begin{cases} \operatorname{Re} k_0 < 0 \\ \operatorname{Im} \lambda_0 > 0 \end{cases} \qquad (11.5)$$

having a common boundary where Re  $k_0 = 0$ , Im  $\lambda_0 = 0$ .

A complete discussion of domain of analyticity beyond what is stated in (11.5) is contained in a paper by BOTTINO and LONGONI [17]. A preliminary discussion can be found in [11]. We just notice that, while (11.5) holds for any of the potentials considered by us, any other inequalities will contain some more detailed information on V(r). Particularly interesting are the upper bounds on  $\text{Re}\lambda_0$  when k is real, because they insure a finite number of subtractions in the scattering amplitude. If, for instance,

$$|V(i\gamma)| < \frac{M}{|\gamma|}$$
, then  $\operatorname{Re} \lambda < \frac{M}{k}$ .

# 12. THE TOTAL AMPLITUDE AND THE LEHMANN ELLIPSE

We have recalled so far a number of properties of the partial wave amplitudes. The next task is to relate them to the properties of the total scattering amplitude. After Mandelstam's work it has become fashionable to use the notations s = E and  $t = -\Delta^2 = -2E(-\cos \theta)$ . We define f(s, t) through (1.9) or the equivalent transforms.

The property of the total amplitude which we shall discuss is the existence of the so-called small Lehmann ellipse.

The mathematical theory of Legendre polynomials teaches us that any expansion in these polynomials;

$$F(\cos \theta) = \sum_{\ell=0}^{\infty} a_{\ell} P_{\ell} (\cos \theta), \qquad (12.1)$$

converges in the  $\cos \theta$  plane within an ellipse of foci  $\pm 1$ . It may happen that the ellipse of convergence reduces to the segment joining -1 to 1. It always happens that the function represented by (12.1) is analytic within the convergence region. This is quite analogous to the corresponding theorem for power series where we have circles instead of ellipses. The magnitude of the ellipse of convergence must be such that the sum of the expansion does not have singularities inside the ellipse. Therefore the singularities which are nearest to the foci are those which dominate the convergence. Without an attempt to make our arguments rigorous but only the suggestion that they are reasonable, all the above results can be understood from the asymptotic behaviour of  $P_{\ell}$  (cos  $\theta$ ) when  $\ell$  is large and fixed. This behaviour is of the kind

$$\mathbf{P}_{\ell} (\cos \theta) \sim \left(\frac{2}{\pi \,\ell \, \sin \theta}\right)^{1/2} \cos \left[(\ell + 1/2)\theta - \pi/4\right]. \tag{12.2}$$

If  $\cos \theta$  is complex and  $\ell$  is large and real,  $P_0(\cos \theta)$  will be dominated by Im  $\theta$ .  $P_{\ell}(\cos \theta)$  is therefore always exploding for high real  $\ell$ , unless  $\theta$  is real, in which case it is oscillating.

If we consider the expansion (12, 1) and we suppose it to be convergent for a given value of  $\theta$ , it follows that the general term of it must vanish for large  $\ell$ :

$$\lim_{\ell \to \infty} \mathbf{a}_{\ell} \mathbf{e}^{\ell |\operatorname{Im} \theta|} = 0, \text{ or } \mathbf{a}_{\ell} < \operatorname{Ce}^{-\ell |\operatorname{Im} \theta|}.$$
(12.3)

The general term is therefore dominated by a decreasing geometric progression. Clearly the expansion also converges for smaller values of Im  $\theta$ , and it represents there an analytic function because it is a uniform convergent series of analytic functions.

In the  $\cos \theta$  plane the curve Im  $\theta$  = const. is an ellipse. Suppose namely that  $z = \cos \theta = x + iy$  and  $\theta = \sigma + i\mu$ . We have

$$x = \cos \sigma \cosh \mu,$$
 (12.4)  
 
$$y = -\sin \sigma \sinh \mu.$$

From these equations we deduce easily

$$(x^{2}/\cosh^{2}\mu) + (y^{2}/\sinh^{2}\mu) = 1; \quad (x^{2}/\cos^{2}\sigma) - (y^{2}/\sin^{2}\sigma) = 1.$$
  
(12.5)

The first of these equations does not depend on  $\sigma$  and represents the locus of all points in the z plane which have the same Im  $\theta = \mu$ . This locus is evidently an ellipse. The other equation is the locus of the points where Re  $\theta = \sigma = \text{const.}$ 

This locus is obviously a hyperbola with foci  $\pm 1$ . The sets of ellipses and hyperbolas are mutually orthogonal. The hyperbola which corresponds to  $\sigma = 0$  degenerates into the upper and lower limit of the  $\cos \theta > 1$ , the one with  $\sigma = \pi$  into the line  $\cos \theta < -1$ . Any value of  $\sigma$  between these extremes corresponds to half a hyperbola; the other half obviously comes from  $\pi$  -  $\sigma$ . The whole z plane can be mapped into the strip  $0 < \sigma < \pi$  of the  $\theta$  plane. However, it is better to map it into  $-\pi < \sigma < \pi$  and  $\mu > 0$ . A given value of  $\sigma$  is then associated with a quarter of a hyperbola. By taking all the combinations  $\pm \sigma$  and  $\pm \pi \pm \sigma$  within the interval (- $\pi$ ,  $\pi$ ), we get all quarters of the hyperbola. The line  $\mu$  = const is then a full ellipse. This kind of mapping is very similar to the usual polar co-ordinates where  $\mu$  plays the role of a radius and  $\sigma$  the role of the polar angle. We prefer this mapping also because it is the natural one when we want the asymptotic behaviour of the Legendre functions when the index  $\ell$  is large. As long as  $\ell$  remains an integer, there is no doubt about the meaning of (12.2) because it is unessential which determination we take of Re  $\theta$  =  $\sigma$  + n  $\pi$  when cos  $\theta$  = z is given. But if  $\ell$  is no longer an integer, we are forced to specify the value of n. This turns out to be the one of our mapping. This fact is very important when used with Watson's integral.

The size of the ellipse can now be estimated for large  $\ell$  with the help of (10.7) and (12.3). The partial wave expansion clearly converges if  $\text{Im}\theta = \mu < a$ , where  $\cosh a = 1 + m^2/2k^2$ . We refer to the ellipse  $\mu = a$  as to the small Lehmann ellipse.

The term large Lehmann ellipse is commonly used instead for the analytic continuation of the imaginary part of f(s, t). We define it in the physical region as

$$F(s, t) = Im f(s, t).$$
 (12.6)

We take s real and 0 < -t < 4s. We consider then the analytic continuation of F(s, t) when s is kept fixed and t is complex. People refer to F(s, t) somewhat improperly as the imaginary part of f(s, t), but this is true only under the stated conditions. The partial wave expansion of F(s, t) is then

$$F(s, t) = \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell + 1) \sin^{2} \delta_{\ell} (k) P_{\ell} (\cos \theta). \qquad (12.7)$$

This expansion converges in an ellipse which is larger than the small Lehmann ellipse, because the general term contains  $\sin^2 \delta$  and vanishes more rapidly. This new ellipse is given by  $\mu = 2$  a. This fact could have been deduced from unitarity directly if the corresponding result for f(s, t) were known, without passing through the partial wave expansion for F(s, t).

## 13. ANALYTICITY IN t FOR FIXED s

In this section we want to explore with new techniques the full domain of analyticity of f(s, t) in the t plane. We already know of the existence of Lehmann s ellipse, but we must go much further in order to prove the analogue of the Mandelstam representation for potential scattering. For the sake of simplicity we work on the assumption that for real positive k

$$\lim_{\lambda \to \infty} [S(\lambda, k) - 1] = 0$$
(13.1)

in every direction of the Re  $\lambda > 0$  half plane, including (and this is really an additional hypothesis) the imaginary axis of  $\lambda$ . We know that for Yukawian potentials the result holds in any direction within the above region; we know also that little can be said when  $\lambda = ia$ . The proof which follows could be carried out without this additional hypothesis, but there is nothing interesting to be gained and the formal machinery would be much more complicated. Under this simplification we apply Watson's transform and we obtain the formula:

$$f(s, t) = -\frac{1}{2k} \int_{-i\infty}^{i\infty} \frac{e^{2i\delta(\lambda k)} - 1}{\cos \pi \lambda} P_{\lambda - 1/2} (-\cos \theta) \lambda d\lambda$$
$$+ i \frac{\pi}{k} \sum_{n} S_{n} P_{\ell_{n}} (-\cos \theta) \frac{2_{n}^{\ell} + 1}{\sin \pi \ell_{n}} . \qquad (13.2)$$

The path C has now been deformed into the line  $\lambda = ia$ . The extra terms arise from the poles of  $S(\lambda, k)$  which we know to exist in the upper half-plane of  $\lambda$  only. We examine, separately, the contributions of the integral and of the poles.  $S_n$  is the residue of  $S(\lambda, k)$  at the pole  $\lambda = \ell_n + 1/2$ . The convergence of the integral is now determined uniquely by  $\cos \theta$ . If  $\lambda = ia$  is large, we have

$$\left|\frac{1}{\cos \pi \lambda}\right| \sim 2 e^{-\pi |a|}, P_{\lambda^{-1/2}}(\cos \theta) \sim 0(e^{|a||\pi^{-d}}).$$
(13.3)

The integral therefore converges if  $|\pi - \sigma| < \pi$ . The asymptotic behaviour of  $P_{\lambda-1/2}(-\cos \theta) = P_{\lambda-1/2}[\cos(\pi - \theta)]$  for large  $\lambda$  has been evaluated by keeping the condition  $|\pi - \sigma| < \pi$  in accordance with the discussion of section 12, so that  $0 < \sigma < 2\pi$  is the range of  $\sigma$ . This includes the whole z plane with the cut z real > 1. The terms  $P_{in}(-\cos \theta)$  have the same cut. The cut in the z plane actually starts outside the small Lehmann ellipse at the point  $z = 1 + m^2/2k^2$ . (This concerns the cut of f(s, t) which also includes the cut:

$$m^2 < t < \infty. \tag{13.4}$$

This is actually the full result to be expected from the Mandelstam representation. Our discussion obviously holds also when  $[s(\lambda, k)-1]$  does not vanish along  $\lambda$  = ia but grows at most like a power of a.

What about the behaviour when t or z is large? The usual partial wave expansion is really unsuitable, because it breaks down long before we need to use it and anyway its accuracy decreases with  $\mu$ . Eq. (13.2) can still be used and yields the interesting result that this behaviour is actually controlled by the poles of  $s(\lambda, k)$ . Indeed if we now consider  $P_{\lambda-1/2}(-\cos \theta)$  when  $\lambda$ is fixed and  $\cos \theta$  is now variable and large, we find

$$P_{\lambda-1/2}(-\cos \theta) \sim 0(z^{\lambda-1/2}).$$
 (13.5)

This term is growing provided Re  $\lambda > 1/2$  and is at the same time oscillating if  $\lambda$  is complex, as expected. If z is very large, then what counts is the pole with the larger Re  $\lambda$ . What about the integral? This is easily disposed of because it is the superposition of decreasing terms with strongly oscillating factors when  $|a| = |\lambda|$  is large. We expect it to vanish for large z. Concluding, we are led to the behaviour:

. .

$$f(s, t) \sim O(t^{a(s)}),$$
 (13.6)

where  $a(s) = \ell_{s}(s)$ ,  $\ell_{s}(s)$  being the  $\ell_{n}$  with the largest real part. This behaviour is energy-dependent.

What is the physical interpretation of these poles? We expect a(s) to be an analytic function of s in some region which we do not need to specify now in detail. We suppose such a pole to exist for  $s = s_0$  with a small Im  $\lambda$  and Re  $\lambda$  almost half-integral (physical). This means that for some value  $s_0$  of s

$$\mathbf{a}(\mathbf{s}_0) = \ell + \epsilon(\mathbf{s}_0) + \mathbf{i}\eta(\mathbf{s}_0); \ \epsilon \ll \ell, \ \eta \ll \ell.$$
(13.7)

If we now exploit the fact that a(s) is analytic in s in a sufficiently large region around  $s_0$ , we can expand a(s) in a power series in s -  $s_0$ :

$$a(s) = \ell + \epsilon(s_0) + i \eta(s_0) + (s - s_0) \frac{da}{ds} |_{s = s_0} + \dots$$
 (13.8)

We can choose however s equal to

$$\mathbf{s} = \mathbf{s}_0^{-} \left( \epsilon(\mathbf{s}_0) + i\eta(\mathbf{s}_0) \right) \left( \frac{\mathrm{d}\mathbf{s}}{\mathrm{d}\mathbf{a}} \mid \mathbf{s} = \mathbf{s}_0 \right)^{-1}$$
(13.9)

in order to make a(s) = 1. It is clear now that, if there is a shadow pole, we expect a pole to appear when l is integer and s is almost real, this pole being the same complex singularity in the variables  $\lambda$  and k (or s) intersecting the many-fold  $\lambda - 1/2$  = integer. This pole can only be interpreted as a resonance according to the discussion of section 4 or section 12. Resonances are therefore responsible for the high t behaviour of f(s, t).

In [11] quite a number of inequalities has been derived for a(s) for a large class of potentials, including the pure Yukawa potential. We wish to point out that it is not at all impossible to choose potentials such that there is an infinite set of shadow poles and, even worse, such that there is no upper bound on  $\operatorname{Re} \ell_n$ . f(s, t) in this case shows an extremely complex behaviour for large t, and one needs an infinite number of subtractions in order to write the Mandelstam representation. It is a good feature that we can rule out this trouble for the most interesting potentials, i.e. those we can form by choosing for  $\sigma(\mu)$  in (5. 1) a distribution with no higher singularities than Dirac's functions (positively no derivatives of it).

#### 14. THE RESULTS OF KHURI

In the previous section we have investigated the analytic properties of f(s, t) when s was held fixed and t was varying. A more difficult task in our formalism is to prove analytic properties in s when t is fixed. We now keep t fixed and real negative. None of the previously proposed representations for f(s, t) seems to be working now because they all diverge. We now use instead

$$\mathbf{f}(\mathbf{s},\mathbf{t}) = \frac{1}{2\mathbf{k}} \int_{C} \frac{\mathrm{e}^{-\mathrm{i}\pi(\lambda+1/2)}}{\cos\pi\lambda} [S(\lambda,\mathbf{k}) - 1] P_{\lambda-1/2}(\cos\theta)\lambda d\lambda. \qquad (14.1)$$

The integration path C is the same as in Fig. 1. The validity of (14.1) can

be first of all proved when z < 1 or inside the small Lehmann ellipse. In particular, if -1 < z < 1, then t is negative: 0 < -t < 4s.

Here the WKB method holds: we have  $S(\lambda, k) - 1 \rightarrow 0$  when  $\lambda \rightarrow \infty$ . Secondly,  $P_{\lambda-1/2}(\cos \theta) \sim 0(e^{\pm i\lambda \theta})$ , whichever choice is larger. If

Im 
$$\lambda \to +\infty$$
,  $\frac{e^{-i\pi(\lambda+1/2)}}{\cos \pi \lambda} \to -2i$  and if Im  $\lambda \to -\infty$ ,  $\left| \frac{e^{-\pi(\lambda+1/2)}}{\cos \pi \lambda} \right| \to e^{-2|Im\lambda|\pi}$ 

In this last case the above factor provides a strong cut-off which makes the integral easily converging for  $\operatorname{Im} \lambda \rightarrow -\infty$ . If  $\operatorname{Im} \lambda < 0$ , we can move the path C along the lower imaginary axis of  $\lambda$ . In so doing, even if  $P_{\lambda-1/2}(\cos \theta)$  now diverges like  $e^{|\operatorname{Im} \lambda||\sigma|}$ , we still have convergence since  $|\sigma| < \pi$ .

We now move k into the domain  $R_e k > 0$ , Im k > 0. Now the WKB formula breaks down for  $\lambda = ia$ ,  $a \rightarrow -\infty$ , but there we have no trouble since by the formula (10.5)  $S(\lambda, k) - 1$  is bounded in this domain by  $e^{|Im \lambda|\pi}$ . When  $\lambda \rightarrow +\infty$ , we have to be careful. The factors here which decide the convergence are  $P_{\lambda-1/2}(\cos\theta) \sim 0(e^{\pm i\lambda\theta})$  and  $s - 1 \rightarrow 0$ . Recalling now that  $\theta = 1 + t/2s$ and that t is real and < 0, we see that, if s is complex, then  $\cos \theta$  and  $\theta$  are also complex. We expect  $e^{\pm i\lambda\theta}$  to diverge in any direction of the  $\lambda$  plane with the sole exception of arg  $\lambda = n\pi$  - arg  $\theta$  where n is integer. Is it possible to choose  $\arg \lambda$  in  $0 < \arg \lambda < \pi/2$  such that this happens? The answer is yes because, when k is moved from the real axis to the imaginary axis,  $\sigma$  and  $\mu$  vary in the range  $\pi < \sigma < 0$ ,  $\mu > 0$ . Arg  $\theta$  is therefore always in the range  $\pi/2 < \arg \theta < \pi$ . We get the desired result by taking  $\arg \lambda = \pi - \arg \theta$ . Our integral representation is convergent in the upper quadrant Re k > 0. If Re k < 0, we simply use the fact that, if k is real and t real negative, then  $f^{*}(s+i\epsilon, t) = f(s-i\epsilon, t)$  so that by analytic continuation we have in the whole cut s plane  $f^*(s, t) = f(s^*, t)$ . This cut plane maps into the upper half plane of k. This equality is quite adequate for definition of an analytic continuation of f(s, t) in the quadrant Re k < 0, Im k > 0.

We are left with the points of the imaginary k axis (negative s axis). Here apparently a new singularity appears, which is not caused by any failure of (14.1) to converge but rather by the fact that  $S(\lambda, k)$  has singularities along the imaginary axis of k. However, when we are close to the imaginary axis of k,the WKB formula holds along  $\lambda = ia$ , a > 0. We can deform C into the imaginary axis of the  $\lambda$  plane, because  $\lambda$  and k are imaginary and the integral converges. Now,

 $\frac{\lambda}{\cos \pi \lambda} \mathbf{P}_{\lambda-1/2}(\cos \theta)$ 

is an odd function of  $\lambda$ , and therefore what counts in the integral is only the odd part of  $e^{-i\pi(\lambda+1/2)}[S(\lambda, k) - 1]$ . But if we use identity (9.18), this odd part can be written as

$$\sin \pi \lambda \left( 1 - \frac{f_0(\lambda, -k) f_0(-\lambda, -k)}{f(\lambda, k) f(-\lambda, -k)} \right).$$
(14.2)
Upon substitution into (14.1) we find

$$f(s,t) = \frac{1}{2k} \int_{-i\infty}^{i\infty} \left( 1 - \frac{f(\lambda, -k) f(-\lambda, -k)}{f(\lambda, -k) f(-\lambda, -k)} \right) tg \pi \lambda P_{\lambda - 1/2}(\cos \theta) \lambda d\lambda \quad (14.3)$$

+ the contribution of poles. But now the function (14.2) is analytic in the whole upper half k-plane, and there is no discontinuity associated with  $S(\lambda, k)$  on the dynamical cut k = ib,  $m/2 < b < \infty$ . This happens because  $e^{i\pi\lambda}S(\lambda, k)$  and  $e^{i\pi\lambda}S(-\lambda, k)$  have the same discontinuity and when the odd part is taken, it disappears. (14.3) can therefore be used in defining f(s, t) in a region containing the imaginary axis of k. We have now joined the right and left part of Im k > 0, because the f(s, t) defined in (14.3) clearly satisfies f\*(s, t) = f(s\*, t). Indeed,

$$f^{*}(s,t) = \frac{1}{2k^{*}} \int_{-\infty}^{\infty} \left(1 - \frac{f_{0}(\lambda, -k) f_{0}(-\lambda, -k)}{f(\lambda, -k) f(-\lambda, -k)}\right)^{*} tg \pi \lambda^{*} P_{\lambda^{*}-1/2}(1 + \frac{t}{2s^{*}})\lambda^{*} d\lambda^{*}$$

+ (the contribution of poles)\*. But  $\lambda^* = -\lambda$ ,  $P_{-\lambda-1/2}(\cos \theta) = P_{\lambda-1/2}(\cos \theta)$  and  $f(\lambda, -k)^* = f(-\lambda, k^*)$  so that  $f^*(S(k), t) = f(S(-k^*), t) = f(s^*, t)$ . Clearly k and  $-k^*$  are both in the upper half-plane. Formula (14.3) therefore defines an analytic function of s in the neighbourhood of the real negative axis of s (apart from the contribution of the poles, which we shall discuss later). For we notice that according to the WKB formula

$$1 - \frac{f_0(\lambda, -k) f_0(-\lambda, -k)}{f(\lambda, -k) f(-\lambda, -k)}$$

decreases exponentially for large  $\lambda$ . This is necessary in order to have analyticity in a neighbourbood of the imaginary axis of the k plane rather than convergence on a line only. The actual size and form of this domain is unimportant once we have the full analyticity domain.

We now give some approximate argument about the behaviour of f(s, t)when t is held fixed and negative and  $|s| \rightarrow \infty$  in the cut s plane which maps into the upper half k-plane. We use the WKB formula for  $f(\lambda, k)$  and eq.(14.3). We put

$$\cos \theta = 1 - \Delta^2/2k^2$$
,  $k = i\xi$ ,  $\lambda = i\eta$ ,  $t = -\Delta^2$ , (14.4)

and we obtain

$$f(s, t) = -\frac{1}{\xi} \int_{0}^{\infty} \left( 1 - \frac{f_{0}(i_{\eta}, -i\xi) f_{0}(-i\eta, -i\xi)}{f(i\eta, -i\xi) f(-i\eta, -i\xi)} \right) tgh(\pi\eta) P_{i\eta - 1/2} \left( 1 + \frac{\Delta^{2}}{2\xi^{2}} \right) \eta d\eta.$$

Using the formula  $P_{\ell}(\cos \theta) \approx J_0[(\ell + 1/2)\theta] = J_0(\lambda \theta)$ , which is valid for large  $\ell$ ,  $\theta \ll 1$ , and taking into account that  $\cos \theta \approx 1 - \theta^2/2$ , we have

$$P_{i\eta-1/2}(1 + \Delta^2/2\xi^2) \approx J_0(T_0\Delta), T_0 = \lambda/k.$$

The WKB formulas (10.2) give us

$$\frac{f_0^{(\lambda, -k)} f_0^{(-\lambda, -k)}}{f(\lambda, -k) f(-\lambda, -k)} \sim e^{\frac{2i}{T_0} \int_0^\infty p_0^{-p_0^{-1}} dz}.$$

If  $\lambda$ , k are large, we deduce approximately

$$1 - e^{\frac{2i\int (p-p_0) dz}{T_0}} \sim \frac{1}{\xi} \int_{T_0}^{\infty} \frac{V(z) dz}{(1 - T_0^2/z^2)^{1/2}}$$

It follows that

$$f(s, t) \sim \int_{0}^{\infty} T dT J_{0} (T\Delta) \int_{T}^{\infty} \frac{V(z) dz}{\left(1 - T^{2}/z^{2}\right)^{1/2}} \int_{0}^{\infty} dz V(z) \int_{0}^{z} \frac{T J_{0} (T\Delta) dT}{\left(1 - T^{2}/z^{2}\right)^{1/2}}$$

Putting T =  $z \sin \phi$ , dT =  $\cos \phi d\phi$ , we obtain [14]

$$\int_{0}^{z} \frac{TJ_{0}(T\Delta)}{\left(1-T^{2}/z^{2}\right)^{1/2}} dT = z^{2} \int_{0}^{\pi/2} J_{0}(z\Delta \sin \phi) \sin \phi d\phi = \sin (z\Delta).$$

Finally we get the Born approximation:

$$f(s, t) \sim \frac{1}{\Delta} \int_{0}^{\infty} z \sin(z\Delta) V(z) dz.$$

#### T. REGGE

This result is independent of s and can be obtained directly from KHURI's approach [2]. We frankly admit that the above argument is not rigorous. However, there is no point in being choosy about it, because rigorous proofs exist abundantly and whoever wants them has only to look for them in the quoted literature. Here we show it just for completeness.

What about the contribution of the poles of  $S(\lambda, k)$  in the formula (14.3)? They give extra contributions to f(s, t) of the sort;

$$\sum_{n} \frac{C_{n}(s)}{\sin \pi \ell_{n}(s)} \quad \mathbf{R}_{n} (\cos \theta) \, e^{i \pi \ell_{n}(s)}$$

where  $C_n(s)$  are some s-dependent constants. This contribution has a singularity when some of  $\ell_n(s)$  become integral. This happens on the upper imaginary k axis when Im k > 0, according to our general discussion in sections 4 and 6, and these poles represent bound states. f(s,t) is therefore analytic in Im k > 0 with the exception of a finite number of bound state poles. All these properties can be condensed into the single formula:

$$f(s, t) = f(t) + \frac{1}{\pi} \int_{0}^{\infty} \frac{\operatorname{Im} f(s', t)}{s' \cdot s} ds' + \sum_{n} \frac{C_{n}(t)}{s \cdot s_{n}},$$

where  $s_n > 0$  are the binding energies of the bound states, f(t) is the Born approximation.  $C_n(t)$  are polynomials in t. This result is due to KHURI [2].

# 15. EXTENSIONS AND GENERALIZATION OF THE THEORY OF COMPLEX ANGULAR MOMENTA

A number of papers dealing with an interesting generalization and application of the idea of complex angular moments has appeared since the first draft of these notes was first published. Remaining in the frame of potential scattering, one has tried to do away with potentials bounded by a power  $A/r^{2+\epsilon}$ ,  $\epsilon > 0$  in the neighbourhood of the origin. In particular, one has allowed V(r) to have a strong repulsive core at small r. As is well known, attractive cores require very disturbing boundary conditions, and it is generally agreed that, if anything can be called physics in the frame of potential scattering, this has nothing to do with attractive cores, which produce systems where there are for instance no ground states but there are states of arbitrarily low energies.

With repulsive cores, however, FIVEL and others [18, 19, 20, 21] have shown that a peculiar fact occurs in the angular momentum plane, that is, that the scattering amplitude can be continued in the Re  $\lambda < 0$  plane by virtue of a simple reflection property:

 $e^{-\pi\lambda}S(\lambda, k) = e^{i\pi\lambda}S(-\lambda, k).$ 

This property formally follows from (9.18) when the Jost function is allowed to be infinite. As a matter of fact, this is in a way to be expected; because, if we try to calculate  $f(\lambda, k)$  with the usual perturbation expansion, we find diverging integrals. The analyticity of (15.1) makes it natural to ask whether we can postulate it in field theory. So far we have no evidence either in favour of or against it apart from its logical simplicity.

Other work has been carried out on the many channel problems, mainly by CHARAP and SQUIRES [21, 22]. They show that, as far as we are concerned with angular momentum properties, all previous results extend in a straightforward manner. Particularly interesting, however, is the extension of Clebsch-Gordan coefficients for the composition of angular momenta to complex values of the indices. I feel that we shall hear more of these properties in the future as soon as the necessity of studying more complicated systems urges us. In fact, just the interaction of a resonance with an elementary particle (if there are any) or with another resonance is already confronting us with such a problem. They also produce some results on the wave functions of the symmetrical top, and this is natural because they adopt in their second paper the helicity formalism of Jacob and Wick. Incidentally, properties of the many channel amplitudes as functions of the energy and transmitted momentum were discussed in [23].

Particularly interesting in regard to its immediate application to field theory is the so-called factorization theorem for the many channel problem. This theorem was first suggested by Gell-Mann and proved by Charap-Squires.

It states, that barring accidental degeneracy, the residuum of the scattering amplitude matrix at a pole in the angular momentum is a matrix  $\Omega_{a\beta}$  of characteristics zero; that is, all minors of the determinant of the matrix vanish. This implies that  $\Omega_{a\beta}$  factorizes as

$$\Omega_{a,\beta} = A_a B_{\beta},$$

where  $\alpha, \beta$  label the channels. This of course happens for resonances in the energy variable.

Another type of problem which has excited the phantasy of many, me included, is how to continue the amplitude for Re  $\lambda < 0$ . My personal philosophy is in favour of course of the symmetry (15.1), but there are some who would like to see what happens for ordinary potentials. Well, this problem has been completely solved by two papers by Froissart and Mandelstam. Froissart solves it for all potentials, and he finds indeed a lot of singularities; in particular, there are singularities about any time the analytic continuation of the Mellin transform of V;

$$\mathbf{M}(\lambda) = \int_{0}^{1} \mathbf{r}^{2\lambda} \mathbf{V}(\mathbf{r}) d\mathbf{r}, \qquad (15.2)$$

is singular in  $\lambda$ . There are other sources of singularities, but we stick to (15.2) just to exemplify. Clearly we can produce almost anything by a ju-

dicious choice of V(r), including a natural boundary of Re  $\lambda = 0$ . Moreover, small variations in V do not correspond to small variations in  $M(\lambda)$ , and in fact  $M(\lambda)$  is completely unstable in Re  $\lambda < 0$ . So no definite V-independent conclusion can be deduced from this analysis. Mandelstam solves the Yukawa potentials in a very elegant way, which is used later by Lovelace in order to carry out numerical calculations on the trajectories, that is, on the function  $\lambda_0(s)$ . The Mandelstam method reduces to the time-honoured Schroedinger method of solving the hydrogen atom where the Yukawa potential reduces to a Coulomb potential.

Numerical calculations have been performed in large amounts, but unfortunately much effort has been wasted in calculating trajectories for negative Re  $\lambda$ , where, as stated, their physical interpretation is doubtful and where in fact they do crazy things. These calculations show a definite pattern in Re  $\lambda > 0$  which can be sketched as follows: We know that for negative real energies the trajectories lie on the real axis and move forward with increasing energies. Where E = 0, the pole leaves the real axis forward if in that point  $\lambda > \frac{1}{2}$ , at  $\frac{\pi}{2}$  angle if  $\lambda = \frac{1}{2}$  (s waves) and backwards if  $\lambda < \frac{1}{2}$ . The pole then eventually swings backwards into the Re  $\lambda < 0$  region.

If we let the range  $m^{-1}$  of the Yukawa potential grow to infinity, that is, we carry out the transition to Coulomb potential, the pole leaves the real axis at very large angular momenta. Therefore, it crosses the integer values several times, and many bound states arise. The swing-back loop is then very large, and in the limit m = 0 it plunges into infinity. We have then an infinite number of bound states.





Swing back loop

#### APPENDIX I

In this appendix we deduce all the integral equations appearing in these lecture notes. The scheme by which they can be derived is summarized in Table I.

| D   | $\frac{d^2}{dz^2} - \frac{\lambda^2 - 1/4}{z^2}$ | $\frac{d^2}{dz^2} - \frac{\lambda^2 - 1/4}{z^2} + k^2$                  | $\frac{d^2}{dz^2} + k^2$             | $\frac{d^2}{dz^2} - \frac{\lambda^2 - 1/4}{z^2} + k^2$                               |
|---|--|---|--------------------------------------|--|
| h   | $V(z)-k^2$                                       | V(z)  | $V(z) + \frac{\lambda^2 - 1/4}{z^2}$ | V(z)   |
| Behaviour<br>of $\psi$ at<br>$z \rightarrow z_1 = 0$      | z <sup>\lambda+1/2</sup>                         | z <sup> \\+ 1/2</sup>   |                                      |  |
| Behaviour<br>of $\psi$ at<br>$z \rightarrow z_1 = \infty$ |  |   | e <sup>-ikz</sup>                    | e <sup>-ikz</sup>  |
| $\psi_1$  | $z^{\lambda+1/2}$                                | $(2/k)^{\lambda} \Gamma(\lambda+1)$<br>$\times z^{1/2} J_{\lambda}(kz)$ | e <sup>ikz</sup>                     | $\times \left(\frac{e^{i\pi(\lambda+1/2)/2}}{2}\right)^{1/2} H_{\lambda}^{(1)} (kz)$ |
| $\psi_2$  | $z^{-\lambda+1/2}$                               | $(2/k)^{\lambda} \Gamma(-\lambda+1) \ 	imes z^{1/2} J  (kz)$            | $\bar{e}^{ikz}$                      | $\times \left(\frac{\pi kz}{2}\right)^{1/2} H_{\lambda}^{(2)} (kz)$                  |
| $\mathbf{W}\left[\psi_{1},\psi_{2}\right]$                | -2λ  | -2λ   | -2ik                                 | -2ik   |

TABLE I

Let us consider a differential equation of this kind:

$$D(\lambda, k, z)\psi(\lambda, k, z) \equiv \left[\frac{d^2}{dz^2} + g(\lambda, k, z)\right]\psi(\lambda, k, z) = h(\lambda, k, z)\psi(\lambda, k, z).$$
(A1.1)

.

As is well known, 'the integral equation equivalent to (A 1.1) is

$$\psi(\lambda,\mathbf{k},\mathbf{z}) = \lim_{\mathbf{z}\to\mathbf{z}_1} \psi(\lambda,\mathbf{k},\mathbf{z}) + \frac{1}{\mathbf{W}[\psi_1\psi_2]} \int_{z_1}^{z} [\psi_1(z')\psi_2(z) - \psi_2(z')\psi_1(z)] h(\lambda,\mathbf{k},z')\psi(\lambda,\mathbf{k},z')dz',$$

where  $\psi_1$  and  $\psi_2$  are two independent solutions of the "free" equation

 $D(\lambda, k, z)\psi(\lambda, k, z) = 0,$ 

and

$$W[\psi_{1}, \psi_{2}] = \psi_{1}\psi_{2}' - \psi_{1}'\psi_{2}.$$

#### T. REGGE

## APPENDIX II

Here we want to give the majorizations of the integral equations deduced in App. I in order to deduce analytic properties of the functions  $\phi(\lambda, k, z)$ and  $f(\lambda, k, z)$ . The integral equations we are dealing with can be written in this general form:

$$g(\lambda, k, z) = g_0(\lambda, k, z) + \int_{z_1}^{z} L(\lambda, k, z') g(\lambda, k, z') dz'.$$
 (A2.1)

Then

$$|g(\lambda, k, z)| \leq |g_0(\lambda, k, z)| + \int_{z_1}^{z} |L(\lambda, k, z')g(\lambda, k, z')dz'|.$$

It is useful to introduce the notations

$$|g_0(\lambda, k, z)| \le M(\lambda, k, z), \quad G(\lambda, k, z) = \frac{g(\lambda, k, z)}{M(\lambda, k, z)}$$

in order to get

$$|G(\lambda, k, z)| \leq 1 + \int_{z_1}^{Z} |K(\lambda, k, z') G(\lambda, k, z')dz'|,$$

where

$$K(\lambda, k, z') = \frac{M(\lambda, k, z')}{M(\lambda, k, z)} L(\lambda, k, z').$$

By using TITCHMARSCH's lemma [15], we obtain

$$|g(\lambda, k, z)| \leq M(\lambda, k, z) \exp\left(\int_{z_1}^{z} |K(\lambda, k, z')dz'|\right).$$

Let us write the solution of (A 2.1) in the following way:

$$\mathbf{g}(\lambda, \mathbf{k}, \mathbf{z}) = \sum_{n=0}^{\infty} \mathbf{g}_{n}(\lambda, \mathbf{k}, \mathbf{z}).$$

Then Titchmarsch's lemma assures the convergence of this series if we put an upper bound to the integral:

$$\int_{z_1}^{z} \left[ K(\lambda, k, z') dz' \right].$$
(A2.2)

The common region of analyticity of all terms  $g_n$  represents the analyticity domain of  $g(\lambda, k, z)$ . We give in the following the majorizations of the integral (A 2.2) for the integral equations previously written:

$$\begin{split} \phi(\lambda, \mathbf{k}, \mathbf{z}) &= z^{\lambda+1/2} - \frac{1}{2\lambda} \int_{0}^{z} \underbrace{\frac{z'^{\lambda+1/2}}{z^{\lambda-1/2}} - \frac{z^{\lambda+1/2}}{z'^{\lambda+1/2}}}_{\mathbf{z}'^{\lambda+1/2}} [V(z') - \mathbf{k}^{2}] \phi(\lambda, \mathbf{k}, \mathbf{z}') dz', \\ &|g_{0}(\lambda, \mathbf{k}, \mathbf{z})| = |z^{\lambda+1/2}| \equiv M(\lambda, \mathbf{k}, \mathbf{z}), \\ &\left| \frac{z'^{\lambda+1/2}}{z^{\lambda+1/2}} \right| \left| \frac{z'^{\lambda+1/2}}{z^{\lambda-1/2}} = \frac{z^{\lambda+1/2}}{z'^{\lambda-1/2}} \right| \leq \left| \frac{z'^{2\lambda+1}}{z^{2\lambda}} - z' \right| \leq 2z', \text{ Re } \lambda \leq 0, \\ &|V(z') - \mathbf{k}^{2}| \leq \mathrm{Hz'}^{\epsilon-2} + \mathrm{N} \leq \mathrm{Rz'}^{\epsilon-2}, \end{split}$$

where k is from any finite domain of the k-plane, where the upper limit of  $k^2$  is N. H and R are constants.

$$\int_{0}^{z} |K(\lambda, k, z')dz'| = \frac{1}{2|\lambda|} \int_{0}^{z} \left| \frac{z'^{\lambda+1/2}}{z^{\lambda+1/2}} \right| \left| \frac{z'^{\lambda+1/2}}{z^{\lambda+1/2}} - \frac{z^{\lambda+1/2}}{z'^{\lambda-1/2}} \right| |V(z') - k^{2}| dz'$$

$$\leq \frac{R}{|\lambda|} \int_{0}^{z} z'^{\epsilon-2} \cdot z' dz' \leq \frac{R}{\epsilon |\lambda|}.$$

It is now apparent that  $\phi(\lambda, k, z)$  is an integral function of k, holomorphic in the half-plane Re  $\lambda > 0$  (continuous for Re  $\lambda = 0$ ).

$$\begin{split} f(\lambda, k, z) &= e^{-ikz} + \frac{1}{2ik} \int_{z}^{\infty} \left[ e^{ik(z'-z)} - e^{-ik(z'-z)} \right] \left[ V(z') + \frac{\lambda^{2} - 1/4}{z'^{2}} \right] f(\lambda, k, z') dz', \\ &\qquad \left| g_{0}(\lambda, k, z) \right| = \left| e^{-ikz} \right| \equiv M(\lambda, k, z), \\ \int_{z}^{\infty} \left| K(z') dz' \right| &= \frac{1}{z \mid k \mid} \int_{z}^{\infty} \left| e^{-ik(z'-z)} \mid \left| e^{ik(z'-z)} - e^{-ik(z'-z)} \mid \right| V(z') + \frac{\lambda^{2} - 1/4}{z'^{2}} \right| dz', \\ &\qquad \left| 1 - e^{-2ik(z'-z)} \mid \leq N = \text{const.}, \text{ Im } k \leq 0. \end{split}$$

Then

$$\int_{z}^{\infty} \left| K(z') dz' \right| \leq \frac{N}{2|k|} \int_{z}^{\infty} \left| \frac{H}{z'^{2-\epsilon}} + \frac{\lambda^{2} - 1/2}{z'^{2}} \right| dz' \leq \text{Const.}$$

Therefore,  $f(\lambda, k, z)$  is an integral function of  $\lambda$ , holomorphic in the half-plane Im  $k \le m/2$  (continuous for Im k = 0).

For real  $\lambda$  and Yukawian potentials this analyticity domain of  $f(\lambda, k, z)$  can be extended to Im  $k \le m/2$ , and it is continuous for Im k = m/2. This can be shown by treating the integral equation

$$f(\lambda, k, z) = f_{\alpha}(\lambda, k, z)$$

$$-\frac{i\pi}{4} z^{1/2} \int_{z}^{\infty} z'^{1/2} \left( H_{\lambda}^{(2)}(kz') H_{\lambda}^{(2)}(kz) - H_{\lambda}^{(2)}(kz') H_{\lambda}^{(2)}(kz) \right) V(z') f(\lambda, k, z') dz',$$

in a way similar to that used before.

This method could have been used to derive the same analytic properties for the prime derivatives of the solutions considered.

#### APPENDIX III

We know from standard textbooks the most important properties of Legendre functions. It is well known that Legendre functions are particular cases of hypergeometric functions with singularities located at  $\pm 1$  and  $\infty$ . Therefore, the only singularities of  $P_t(x)$  and  $Q_t(x)$  lie on  $\pm 1$  or  $\infty$ .

From the general theory of Legendre equations one finds out at once that in  $\pm 1$  the solutions either are regular or have a logarithmic singularity. It is always possible, however, to choose the parameters in the general integral of the equations in such a way as to make the solution regular in a given point. In particular,  $P_{\ell}(x)$  is regular in x = 1 and  $P_{\ell}(1) = 1$  and  $Q_{\ell}(x)$ is regular at  $x = \infty$  provided Re  $(\ell) + 1/2 \ge 0$ . Since  $\ell$  enters in the differential equation under the form  $\ell(\ell+1)$  and since the boundary conditions for  $P_{\ell}(x)$  are  $\ell$ -independent, it follows from a general theorem of Poincaré that  $P_{\ell}(x)$  is an entire function of  $\ell$  for x fixed and that  $P_{\ell-1}(x) = P_{\ell}(x)$  because  $\ell(\ell+1)$  is invariant under the substitution  $\ell \rightarrow -\ell - 1$ . Also, if  $\lambda = \ell + 1/2$ ,

$$P_{\lambda - 1/2}(x) = P_{-\lambda - 1/2}(x).$$
 (A 3.1)

 $P_{g}$  (z) has a cut between -1 and -  $\infty$ . It is otherwise regular in z. Its asymptotic behaviour for large  $\lambda$  is given by

$$P_{\lambda-1/2}(\cosh \alpha) \simeq \frac{1}{\sqrt{2\pi\lambda}} \frac{1}{\sinh \alpha} \left( e^{\alpha \lambda} + i e^{-\alpha \lambda} \right) \left[ 1 + 0 \left( \frac{1}{\lambda} \right) \right]. \quad (A3.2)$$

For large  $z P_{\ell}(z) = 0$  ( $z^{\ell}$ ).  $Q_{\ell}(z)$  is instead defined through its behaviour for large z; that is,

$$Q_{\lambda^{-1}/2}(z) = O(z^{-\ell-1}).$$
 (A 3.3)

If Re l + 1/2 > 0, this is the only solution which does so apart from a multiplicative factor. We have also for large  $\lambda$ 

$$Q_{\lambda 1/2}(\cos h \alpha) = \sqrt{\frac{\pi \sinh \alpha}{2 \lambda}} e^{-\alpha \lambda} \left[ 1 + 0 \left( \frac{1}{\lambda} \right) \right]. \quad (A 3.4)$$

 $Q_{\mu}(z)$  has singularities in both 1 and -1. Moreover,

$$Q_{-\lambda 1/2}(z) = Q_{\lambda 1/2}(z) + \pi \operatorname{tg} \pi \lambda P_{\lambda 1/2}(z).$$
 (A3.5)

This relation says that  $Q_{\lambda 1/2}(z)$  has poles in Re  $\ell < 0$  at the negative halfinteger points. In these points the residuum of  $Q_{\lambda-1/2}$  is given by the corresponding  $P_{\lambda 1/2}$ , which turn out to be polynomials. From the pre-existing literature one knows already that  $Q_{\lambda-1/2}$  is regular in Re  $\ell > 0$ .

We have already listed the symmetries arising from the reflection  $\lambda \rightarrow -\lambda$  or  $\ell \rightarrow -\ell -1$ . But the Legendre equation turns out also to be symmetric under the exchange  $z \rightarrow = z$ . The consequences of this fact are

$$Q_{\lambda-1/2}(z e^{\mp i\pi}) = \pm i e^{\pm i\pi\lambda} Q_{\lambda-1/2}(z).$$
 (A3.6)

There is ambiguity in taking  $e^{\pm i\pi}$  because  $Q_{\lambda 1/2}(z)$  has a cut  $-1 \ge z \ge -\infty$ . It has to be remarked that, in encircling anticlockwise both points  $\pm 1$ ,  $Q_{\lambda - 1/2}(z)$  is multiplied by the factor  $e^{-2i\pi(\lambda + 1/2)}$  so that  $Q_{\lambda - 1/2}(z) z^{\lambda + 1/2}$  is left unaffected.

We also have:

$$P_{\lambda^{-1/2}}(e^{\mp i\pi}z) = \pm i e^{\mp i\pi\lambda} P_{\lambda^{-1/2}}(z) + \frac{2}{\pi}\cos \pi\lambda Q_{\lambda^{-1/2}}(z); \qquad (A3.7)$$

if  $\lambda$  is half integer, it reduces simply to

$$P_{\ell}(-x) = (-)^{\ell} P_{\ell}(x).$$
 (A 3.8)

Mehler has found the following interesting inversion formula when  $\lambda$  is imaginary (conical functions): If

$$\mathbf{F}(\lambda) = \operatorname{tg} \pi \lambda \int_{1}^{\infty} \mathbf{P}_{\lambda-1/2}(\omega) \mathbf{G}(\omega) d\omega, \qquad (A3.9)$$

then

$$G(\omega) = i \int_{0}^{1\infty} \lambda d\lambda P_{\lambda-1/2}(\omega) F(\lambda),$$

valid under conditions similar to those of the Fourier transform. They can be written as

$$\int_{-i\infty}^{i\infty} \lambda \, \mathrm{d} \lambda \, \mathrm{tg} \, \pi \lambda \, \mathbf{P}_{\lambda^{-1/2}}(\xi) \, \mathbf{P}_{\lambda^{-1/2}}(\eta) = -2i \, \delta(\xi - \eta). \tag{A3.10}$$

.

This is the prototype of many integrals to be derived. From (A3.5) we have

$$\int_{-i\infty}^{i\infty} \lambda d\lambda P_{\lambda-1/2}(\xi) Q_{\pm \lambda-1/2}(\eta) = \pm i \pi \delta(\xi - \eta).$$
(A3.11)

We notice that easily

$$\int_{-i\infty}^{i\infty} \lambda d\lambda \ Q_{\pm \lambda 1/2}(z_1) \ Q_{\pm \lambda - 1/2}(z_2) \ \dots \ Q_{\pm \lambda - 1/2}(z_n) = 0, \qquad (A \ 3.12)$$

where all  $\pm$  are correlated.

(A 3.11) follows from the fact that the integrand is analytic in the right (left) hand plane and it vanishes there at large distances.

Take now Heine's formula:

÷ ---

$$\sum_{\ell} (2 \,\ell + 1) \, Q_{\ell}(\xi) \, P_{\ell}(\eta) = 1/(\xi - \eta), \qquad (A \, 3 \, . \, 13)$$

which holds for  $\text{Im } \alpha > \text{Im } \beta$  where  $\cos \alpha = \xi$ ,  $\cos \beta = \eta$ . Take  $\xi$ ,  $\eta$  real > 1 and apply to it the Watson-Sommerfeld transform. We get

$$\int_{-i\infty}^{i\infty} \lambda d \lambda Q_{\lambda-1/2}(\xi) P_{\lambda-1/2}(\eta) t g \pi \lambda = i/(\xi - \eta); \qquad (A3.14)$$

and using (A3.5),

.

$$\int_{-i\infty}^{1\infty} \lambda \, d \, \lambda \, Q_{\lambda - 1/2}(\xi) \, Q_{-\lambda - 1/2}(\eta) = i\pi / (\xi - \eta). \tag{A 3.15}$$

We can have more complicated identities as follows: Take the addition theorem for Legendre functions (linteger):

$$P_{\ell}(x) P_{\ell}(y) = P_{\ell}(xy + \sqrt{1 - x^{2}}\sqrt{1 - y^{2}}\cos t)$$
  
+ 
$$2\sum_{m=1}^{\infty} (-)^{m} \frac{\Gamma(\ell - m + 1)}{\Gamma(\ell + m + 1)} P_{\ell}^{m}(x) P_{\ell}^{m}(y) \cos m t; \qquad (A3.16)$$

Re x > 0; Re y > 0;  $| \arg (x - 1) | < \pi; | \arg (y - 1) | < \pi.$ 

Let us integrate this on  $\psi$  between 0 and  $\pi.$  All terms containing  $\cos m \, \psi$  vanish,and we have

$$P_{\ell}(x) P_{\ell}(y) = \frac{1}{\pi} \int_{0}^{\pi} P_{\ell}(xy + \sqrt{1 - x^{2}} \sqrt{1 - y^{2}} \cos \psi) d\psi.$$
 (A3.17)

Let

$$z = xy + \sqrt{1 - x^2} \sqrt{1 - y^2} \cos \psi$$

be a new variable instead of  $\psi$ . We have

$$d\psi = \frac{d\psi}{dz} dz = dz \frac{1}{dz/d\psi} = -\frac{dz}{\sqrt{1-x^2}\sqrt{1-y^2}} \sin \psi$$

but

$$\sin \psi = \sqrt{1 - \cos^2 \psi} = \sqrt{1 - (z - xy)^2 / (1 - x^2)(1 - y^2)}$$
$$= \sqrt{1 - z^2 - x^2 - y^2 + 2xyz} / \sqrt{(1 - x^2)(1 - y^2)}$$

so that

$$d\psi = - dz / \sqrt{1 - z^2 - x^2 - y^2 + 2xyz}.$$

It is easily seen that the limits of integration in z are the points where  $1 - z^2 - x^2 - y^2 + 2xyz$  vanishes. It follows that.

$$\mathbf{P}_{\boldsymbol{\ell}}(\mathbf{x}) \ \mathbf{P}_{\boldsymbol{\ell}}(\mathbf{y}) = \frac{1}{\pi} \int_{-1}^{1} \frac{dz \ge (1 - z^2 - x^2 - y^2 + 2xyz)}{\sqrt{1 - z^2 - x^2 - y^2 + 2xyz}} \ \mathbf{P}_{\boldsymbol{\ell}}(z). \quad (A3.18)$$

From this it is evident that (x, y, z < 1 and real)

$$\sum_{\ell=0}^{\infty} (2\ell+1) P_{\ell}(x) P_{\ell}(y) P_{\ell}(z)$$
  
=  $(2/\pi) \Theta (1 - z^2 - x^2 - y^2 + 2xyz) / \sqrt{1 - z^2 - x^2 - y^2 + 2xyz}$   
=  $(2/\pi) K_1(x, y, z)$  (A 3.19)

| T VOLG II | ТΑ | BLE | Ц |
|-----------|----|-----|---|
|-----------|----|-----|---|

.

| $f(\lambda)_{a,\beta,\gamma\geq 1}$   | $\int_{-\infty}^{\infty} F(\lambda) \lambda d\lambda$  |                  |
|---|--|------------------|
| $Q_{\pm\lambda-1/2}(\alpha) Q_{\pm\lambda-1/2}(\beta)$  | 0  | (B2.30)          |
| $Q_{\lambda-1/2}(\alpha) Q_{-\lambda-1/2}(\beta)$   | $i\pi(\alpha-\beta)^{-1}$  | (B2.31)          |
| $Q_{\pm \lambda - 1/2}(\alpha) P_{\lambda - 1/2}(\beta)$  | $\pm i\pi\delta(\alpha-\beta)$   | (B2.32)          |
| $Q_{\pm\lambda-1/2}(\alpha) P_{\lambda-1/2}(\beta) tg\pi\lambda$  | $i(\alpha-\beta)^{-1}$   | (B2.33)          |
| $P_{\lambda-1/2}$ (α) $P_{\lambda-1/2}$ (β) tgπλ  | - 2iδ(α-β)   | (B2.34)          |
| $Q_{\pm\lambda-1/2}(\alpha) Q_{\pm\lambda-1/2}(\beta) Q_{\pm\lambda-1/2}(\gamma)$                         | 0  | (B2.35)          |
| $Q_{\pm\lambda-1/2}(\alpha) Q_{\pm\lambda-1/2}(\beta) Q_{\pm\lambda-1/2}(\gamma)$                         | $\pm i\pi \int_{1}^{\infty} d\omega K(\omega; \alpha, \beta) (\omega - \gamma)^{-1}$   | (B2.36)          |
| $\mathbf{Q}_{\pm\lambda-1/2}(\alpha) \mathbf{Q}_{\pm\lambda-1/2}(\beta) \mathbf{P}_{\lambda-1/2}(\gamma)$ | $\pm i\pi K(\gamma; \alpha, \beta)$  | (B2.37)          |
| $Q_{\lambda-1/2}(\alpha) Q_{\lambda-1/2}(\beta) P_{\lambda-1/2}(\gamma)$                                  | $i\pi \left\{ K(\alpha;\beta,\gamma) - K(\beta;\alpha,\gamma) \right\}$  | (B2. <b>3</b> 8) |
| $Q_{\pm\lambda-1/2}(\alpha) Q_{\pm\lambda-1/2}(\beta) P_{\lambda-1/2}(\gamma)tg\pi\lambda$                | $i\int_{1}^{\infty} d\omega K(\omega; \alpha, \beta) (\omega - \gamma)^{-1}$   | (B2.39)          |
| $Q_{\lambda-1/2}(\alpha) Q_{-\lambda-1/2}(\beta) P_{\lambda-1/2}(\gamma) tg \pi \lambda$                  | $-i\int_{1}^{\infty} \left( \frac{K(\omega;\alpha,\gamma)}{\omega-\beta} - \frac{K(\omega;\beta,\gamma)}{\omega-\alpha} \right) d\omega$ | (B2.40)          |
| $Q_{\pm\lambda^{-1/2}}(\alpha) P_{\lambda^{-1/2}}(\beta) P_{\lambda^{-1/2}}(\gamma)$                      | $\pm i K_{1}(\alpha, \beta, \gamma)$   | (B2.41)          |
| $Q_{\pm\lambda-1/2}(\alpha) P_{\lambda-1/2}(\beta) P_{\lambda-1/2}(\gamma)tg_{\pi\lambda}$                | i H(α; β, γ)   | (B2.42)          |
| $P_{\lambda-1/2}(\alpha)$ $P_{\lambda-1/2}(\beta)$ $P_{\lambda-1/2}(\gamma)tg\pi\lambda$                  | $(2/\pi i) K_{1}(\alpha,\beta,\gamma)$   | (B2.43)          |

by definition. This remarkable formula came to our knowledge through Prof. Goldberger and does not seem to appear anywhere in the literature. It generalizes the usual

$$\sum_{0}^{\infty} (2 \ell + 1) P_{\ell}(x) P_{\ell}(y) = 2 \delta (x - y).$$
 (A3.20)

We could try the Watson transform directly on (A3.18), but it would be of no use because it does not converge. A better way is to multiply (A3.18)by  $1/(\xi - x)$  and integrate on x between  $\pm 1$ . The result is

$$\sum_{\ell} (2\ell+1) Q_{\ell}(\xi) P_{\ell}(y) P_{\ell}(z) = 1/\sqrt{\xi^2 + y^2 + z^2 - 2yz\xi - 1}.$$
 (A3.21)

(A 3.21) is valid also when  $\xi$ , y, z are complex, while in (A 3.19) they had to be real. The only condition is that, if  $\cos \beta = y$ ,  $\cos \gamma = z$ ,  $\cos \alpha = \xi$ , then  $\operatorname{Im} \xi > \operatorname{Im} \beta + \operatorname{Im} \gamma$ . Applying to (A 3.21) the Watson transform, we get

$$\int_{-i\infty}^{1\infty} \lambda \, \mathrm{d} \, \lambda \, \mathrm{tg} \, \pi \lambda \, \mathrm{Q}_{\lambda - 1/2}(\xi) \, \mathrm{P}_{\lambda - 1/2}(\eta) \, \mathrm{P}_{\lambda - 1/2}(\zeta)$$
$$= i / \sqrt{\xi^2 + \eta^2 + \zeta^2 - 2\zeta \eta \xi + 1}.$$

Using identity (A 3.11), (A 3.5) repeatedly, one arrives at a large number of integrals. We skip here a detailed proof and limit ourselves to giving a table of them (Table II). Here  $K_1$  is defined by (A 3.19) and

K(
$$\xi$$
;  $\eta$ ,  $\zeta$ ) =  $\Theta(\xi - \xi >) / \sqrt{\xi^2 + \zeta^2 + \eta^2 - 2\eta\zeta\xi - 1}$ ,

where  $\xi_{>}$  is the largest root of the denominator;

$$H(\xi; \eta, \zeta) = K(\xi; \eta, \zeta) - K(\eta; \zeta, \xi) - K(\zeta; \xi, \eta).$$

Many other identities can be written, but they would take much more space and we refer the reader to a coming paper by V. de Alfaro, T. Regge and G. Rossetti to be published in Nuovo Cimento.

## REFERENCES

- [1] JOST, R., Helv. Phys. Acta 20 (1947) 256; BARGMANN, V., Rev. Mod. Phys. 21 (1949) 488.
- [2] KHURI, N., Phys. Rev. 107 (1957).
- [3] BLANKENBECL, R., GOLDBERGER, M. L., KHURI, N. and TREIMAN, S. B., Ann. of Phys. 10 (1960) 62.
- [4] KLEIN, A., J. Math. Phys. 17(1960) 41.
- [5] SOMMERFELD, A., Partielle Differential Gleichungen der Physik, Leipzig (1943) 285.

- [6] KLEIN, A. and ZEMACH, C., Ann. of Phys. 7 (1959) 440: Nuovo Cimento 10 (1958) 1078.
- [7] SCHIFF, L., Quantum Mechanics, McGraw-Hill, New York.
- [8] CARTWRIGHT, M.L., Integral Functions (1956) 28.
- [9] NEWTON, R.G., J. Math. Phys. 1 (1960) 319.
- [10] REGGE, T., Nuovo Cimento 14 (1959) 951.
- [11] REGGE, T., Nuovo Cimento 18 (1960) 947.
- [12] KEMBLE, E. C., Phys. Rev. <u>48</u> (1935) 549; KEMBLE, E. C., Fundamental Principles of Quantum Mechanics, McGraw-Hill (1937).
- [13] FORD, K. W. and WHEELER, J. A., Ann. of Phys. 7 (1959) 259.
- [14] Higher Transcendental Functions (Bateman Manuscript Project) II, McGraw-Hill, New York (1953) 46.
- [15] TITCHMARSCH, E.C. Eigenfunction Expansions, Clarendon Press, Oxford (1946) 97.
- [16] BOTTINO, A., LONGONI, A. M. and REGGE, T., Nuovo Cimento 23 (1962) 954.
- [17] BOTTINO, A. and LONGONI, A. M., Nuovo Cimento 24 (1962) 353.
- [18] JOST, R. and PAIS, A., Phys. Rev. 82 (1951) 840.
- [19] FIVEL, D., Phys. Rev. 125 (1962) 1085.
- [20] PREDAZZI, E. and REGGE, T., Nuovo Cimento 24 (1962) 518.
- [21] CHARAP, J. M. and SQUIRES, reprint UCRL 10209 10115 10209 10138.
- [22] FAVELLA, L. and REINERI, M. T., Nuovo Cimento 23 (1962) 616.
- [23] FONDA, L., RADICATI, L. and REGGE, T., Ann. Phys. 12 (1961) 68.

# DETERMINANTAL METHOD FOR COMPLEX ANGULAR MOMENTA IN POTENTIAL SCATTERING

## B. W. LEE

# UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, Pa., UNITED STATES OF AMERICA

In this paper I would like do describe a formulation of the complex angular momenta in potential scattering based on the Lippmann-Schwinger integral equation rather than on the Schrödinger differential equation. This is intended as a preliminary to the paper by SAWYER [1] on the Regge poles and high energy limits in field theory (Bethe-Salpeter amplitudes), where the integral formulation is definitely more advantageous than the differential formulation.

For integral l, the scattering wave function  $\Psi_{l}(k, r)$  satisfies

$$\Psi_{\boldsymbol{\ell}}(\mathbf{k},\mathbf{r}) = \mathbf{j}_{\boldsymbol{\ell}}(\mathbf{k}\mathbf{r}) + \int_{0}^{\infty} \mathbf{r}^{\prime 2} \, \mathrm{d}\mathbf{r}^{\prime} \, \mathbf{G}_{\boldsymbol{\ell}}(\mathbf{k};\mathbf{r},\mathbf{r}^{\prime}) \, \mathbf{V}(\mathbf{r}^{\prime}) \, \Psi(\mathbf{k},\mathbf{r}^{\prime})$$
(1)

where the (outgoing) Green's function  $G_{\emptyset}(k \cap, r')$  is defined as

$$G_{\ell}(\mathbf{k}; \mathbf{r}, \mathbf{r}') = \mathbf{k} \mathbf{j}_{\ell}(\mathbf{k}\mathbf{r}_{<}) \mathbf{h}_{\ell}^{(1)}(\mathbf{k}\mathbf{r}_{>})$$
$$= \mathbf{i}(\pi/2)(\mathbf{r}\,\mathbf{r}')^{-\frac{1}{2}} \mathbf{J}_{\ell+\frac{1}{2}}(\mathbf{k}\mathbf{r}_{<}) \mathbf{J}_{\ell+\frac{1}{2}}^{(1)}(\mathbf{k}\mathbf{r}_{>}), \qquad (2)$$

where  $r_{<} = \min r$  and  $r_{>} = \max r$ .

The extension of the  $\Psi_{\ell}$  in complex  $\ell$  is then achieved simply by extending the subscripts  $\ell$  of the Bessel functions into complex values.

The Green's function  $G_{\ell}(k; r, r')$  has the following spectral representation [2]:

$$G_{\ell}(k; r, r') = \int_{0}^{\infty} \frac{k' dk'}{k'^{2} - k^{2} - i\epsilon} J_{\nu}(kr) J_{\nu}(kr') \frac{1}{\sqrt{rr'}}$$
(3)

where  $\nu = \ell + \frac{1}{2}$ .

The partial wave scattering amplitude  $T_{\ell}(k, k)$  is defined by

$$T_{\ell}(k',k) = k' \int_{0}^{\infty} r^{2} dr j_{\ell}(k'r) V(r) \Psi_{\ell}(kr)$$
 (4)

and satisfies the integral equation

$$T_{\ell}(k',k) = V_{\ell}(k',k) + \frac{2}{\pi} \int_{0}^{\infty} \frac{k'' dk''}{k''^{2} - k^{2} - i\epsilon} V_{\ell}(k',k'') T_{\ell}(k'',k).$$
(5)

 $V_{\ell}(k',k)$  is defined as

$$V_{g}(k',k) = k' \int_{0}^{\infty} r^{2} dr j_{g}(k'r) V_{g}(r) j_{g}(kr), \qquad (6)$$

and, for a Yukawa potential

$$V(r) = g r^{-1} e^{-\mu r}$$

it has the form [3]

$$V_{\ell}(\mathbf{k}',\mathbf{k}) = g \frac{\pi}{2} \sqrt{\frac{\mathbf{k}'}{\mathbf{k}}} \int_{0}^{\infty} \ell^{\mu \mathbf{r}} J_{\ell+\frac{1}{2}}(\mathbf{k}\mathbf{r}) J_{\ell+\frac{1}{2}}(\mathbf{k}'\mathbf{r}) d\mathbf{r}$$
$$= (g/2)(1/\mathbf{k}) Q_{\ell}[(\mathbf{k}^{2} + \mathbf{k}'^{2} + \mu^{2})/2\mathbf{k}\mathbf{k}'].$$
(7)

Since  $J_{\nu}(z)$  behaves as

$$J_{\nu}(z) \sim (\Gamma (\nu + \frac{1}{2}))^{-1} (z/2)^{\nu}$$

for small z, the integral

$$\int_{0}^{\infty} d\mathbf{r} \ e^{-\mu t} \ J_{\nu}(\mathbf{kr}) \ J_{\nu}(\mathbf{k'r}) = (1/\pi \sqrt{\mathbf{k}} \ \mathbf{k'}) \ Q_{\rho}[(\mathbf{k}^{2} + \mathbf{k'}^{2} + \mu^{2})/2\mathbf{kk'}]$$

diverges for Re  $\nu \leq -\frac{1}{2}$ , i.e. Re  $\ell \leq -1$ . However, the right hand side is meromorphic in the whole complex  $\ell$  plane and agrees with the left hand side for Re  $\ell > -1$ , and, therefore, is the analytic continuation of the left hand side to the whole  $\ell$  plane. The Legendre function of the second kind  $Q_{\ell}(z)$  has poles at negative integer values of  $\ell$ . We record some properties of  $Q_{\ell}(z)$ .

$$\begin{aligned} & Q_{\nu}(z) \quad \frac{1}{z \to 1} \quad -\frac{1}{2} \ln \frac{z-1}{2} - \gamma - \Psi(\nu+1), \\ & Q_{\nu}(z) \quad \frac{1}{z \to \infty} \sqrt{\pi} \, \Gamma(\nu+1) (2z)^{-\nu-1} / \, \Gamma(\nu+\frac{3}{2}), \\ & Q_{\nu}(z) \quad \frac{1}{\nu \to -1} \quad \frac{1}{\nu+1} - \ln \frac{z+1}{2} + 0(\nu+1). \end{aligned}$$

We shall concentrate upon the Yukawa potential for the time being; we shall consider the superposition of Yukawa potentials later. Eq. (5) can be written as in this case:

$$T_{\ell}(p,\sqrt{s}) = \frac{g}{2\sqrt{s}} Q_{\ell} \left(\frac{p^2 + s + \mu^2}{2p\sqrt{s}}\right) + \frac{g}{\pi} \int_{0}^{\infty} Q_{\ell} \left(\frac{p^2 + q^2 + \mu^2}{2pq}\right) \frac{dq}{(q^2 - s)} T_{\ell}(q,\sqrt{s}).$$
(8)

This is an integral equation for  $T_\ell(p,\!\sqrt{s})$  with s as a parameter, and the kernel is

$$K_{\ell}(p,q;s) = (g/\pi)[1/(q^2-s)] Q_{\ell}[(p^2+q^2+\mu^2)/2pq].$$
(9)



Fig.1

Equation (8) can also be derived more simply by considering the full Lippmann-Schwinger integral equation (Fig. 1):

$$T(\vec{p}, \vec{k}) = g \frac{1}{(\vec{p} - \vec{k})^2 + \mu^2} + \int \frac{d^3q}{(2\pi)^3} \frac{1}{(q^2 - k^2)} \cdot \frac{g}{(\vec{p} - \vec{q})^2 + \mu^2} T(\vec{q}, \vec{k})$$

By decomposing the potential

$$\frac{g}{(\vec{k}-\vec{p})^2+\mu^2} = \frac{g}{2kp} \sum_{\ell} (2\ell+1) Q_{\ell} \left(\frac{k^2+p^2+\mu^2}{2kp}\right) P_{\ell}(\hat{p},\hat{k})$$

and the T-matrix

$$T(\vec{p}, \vec{k}) = (\pi/p) \sum_{\ell} (2\ell+1) \hat{T}_{\ell}(p, k) P_{\ell}(\hat{p}, \hat{k}),$$

and performing the angular integrations, we obtain Eq. (8) for integer l. The "correct" extension to complex l is achieved simply by regarding the subscript l of  $Q_l$  as complex. The point of the first derivation of Eq. (8) is to show that the "prolongation" to the complex l adopted is indeed the same as that of Regge.

The formal solution to Eq. (8) can be written as

$$T_{\ell}(p,\sqrt{s}) = [N_{\ell}(p,\sqrt{s})]/[D_{\ell}(s)]$$

where  $D_{\ell}$  is the Fredholm determinant of the operator  $K_{\ell}(p, q; s)$ :

$$D_{\ell}(\mathbf{s}) = \det_{(\mathbf{p}, \mathbf{q})} \{ \delta_{\mathbf{p}, \mathbf{q}} - K_{\ell} [\mathbf{p}, \mathbf{q}; (\sqrt{\mathbf{s}})^{2}] \}$$

$$= \exp \{ \hbar \ln (1 - K_{\ell}) \}$$

$$= 1 - \frac{g}{2\pi} \int_{0}^{\infty} \frac{ds'}{(s' - s)\sqrt{s'}} Q_{\ell}(1 + \mu^{2}/2s')$$

$$- \left( \frac{g}{2\pi} \right)^{2} \left\{ \int_{0}^{\infty} \frac{ds'}{(s' - s)\sqrt{s'}} \int_{0}^{\infty} \frac{ds''}{(s'' - s)\sqrt{s''}} Q_{\ell}^{2} \left( \frac{s' + s'' + \mu^{2}}{2\sqrt{s's''}} \right) - \left[ \int_{0}^{\infty} \frac{ds'}{(s' - s)\sqrt{s'}} Q_{\ell}(1 + \mu^{2}/2s') \right]^{2} \right\}$$

$$+ \dots \qquad (10)$$

and

$$N_{\varrho}(\mathbf{p}, \sqrt{\mathbf{s}}) = \int_{0}^{\infty} B_{\varrho}(\mathbf{p}, \mathbf{q}) \frac{\delta}{\delta K_{\varrho}(\sqrt{\mathbf{s}}, \mathbf{q})} D_{\varrho}(\mathbf{s}),$$

$$B_{\varrho}(\mathbf{p}, \mathbf{q}) = \left(\frac{g}{2 \mathbf{q}}\right) Q_{\varrho}\left(\frac{p^{2} + q^{2} + \mu^{2}}{2pq}\right).$$
(11)

The above solution is only formal in so far as we have not established the convergence of  $D_{\ell}$  and  $N_{\ell}$ . To prove the convergence, we first transform the kernel of Eq. (9) into a symmetric one  $\overline{K}$  which will reproduce the same Fredholm determinant:

$$\widetilde{K}_{\ell}(p,q;s) = \frac{g}{\pi} \frac{1}{(p^2 - s)^{\frac{1}{2}}} Q_{\ell} \left( \frac{p^2 + q^2 + \mu^2}{2pq} \right) \frac{1}{(q^2 - s)^{\frac{1}{2}}}$$
(12)

and make changes of variable of the form

$$p = f(x), q = f(y)$$

The kernel  $\overline{K}_{\ell}$  is now to be replaced by  $H_{\ell}(x, y; s)$ 

$$H_{\ell}(x, y; s) = [f'(x)]^{\frac{1}{2}} \overline{K}_{\ell}[f'(y)]^{\frac{1}{2}}$$

the function f is to be so chosen that

- (1) the range of integration becomes finite,
- (2) the kernel becomes bounded.

Once the above two conditions are satisfied, the convergence of the determinantal expansion follows trivially, Hadamard's lemma being applicable. A transformation which accomplishes these ends is

$$x = (1 - (1 + p)^{-\epsilon})^{\epsilon}, \quad \epsilon > 0.$$

One can show by a straightforward computation that  $H_{\zeta}(x, y; s)$  is bounded in the region Re  $\ell > \frac{3}{2}$ . It can be shown further that

$$\lim_{|s| \to \infty} \{ D_{\ell}(s) - 1 \} = 0 (1/\sqrt{s}).$$
 (13)

It is also evident from the construction, Eq. (10), that  $D_{\ell}(s)$  is analytic in the cut s-plane, the cut running from 0 to  $\infty$ . This shows that  $D_{\ell}(s)$  is just the Jost function for complex  $\ell$  and the S-matrix can be written as

$$S_{\ell}(s) = \exp \left[2i \delta_{\ell}(s)\right] = D_{\ell}(s - i\epsilon)/D_{\ell}(s + i\epsilon).$$

The function  $Q_{\ell}$  has poles at negative integer values of  $\ell$ . Since each term in the determinantal expansion of  $D_{\ell}$  is meromorphic in  $\ell$  for  $\operatorname{Re} \ell \ge -\frac{3}{2}$ ,  $D_{\ell}(s)$  is meromorphic in  $\ell$  in the same region. At the first glance, it may appear that the term of order  $g^{n}$  in  $D_{\ell}(s)$  might have a pole of the type  $(\ell + 1)^{-n}$  and  $D_{\ell}(s)$  might have an essential singularity at  $\ell = 1$ . (Similar things can be said about other negative integers which however, are outside the proven domain of meromorphy.) This, however, is not the case. The poles of higher orders cancel. To show this more explicitly, we decompose the kernel  $K_{\ell}$  into two parts:

$$K_{\rho}(p, q; s) = [1/(p^2 - s)][1/(\ell+1)] + R_{\rho}(p, q; s)$$

where  $R_{\ell}$  is analytic in the neighbourhood of  $\ell = -1$ . Now  $D_{\ell}(s)$  may be written as

$$\det (1 - K_{0}) = \det (1 - K_{0} + R_{0}) \det [1 - (1 - K_{0} + R_{0})^{-1} R_{0}].$$

The first determinant can be evaluated explicitly since the kernel is separable:

det 
$$(1 - K_{\ell} + R_{\ell}) = 1 - (g/2\pi)[1/(\ell+1)] \int_{0}^{\infty} dp/(p^2 - s)$$
  
= 1 -  $(ig/2\sqrt{s})[1/(\ell+1)].$ 

The second determinant is analytic near  $\ell = -1$  and, therefore,  $D_{\ell}(s)$  has a simple pole at  $\ell = -1$ .

Since the  $D_{\ell}(s)$  is convergent for all values of g, we may try to compute the zeroes of  $D_{\ell}(s)$  (Regge trajectories) from the lowest approximation to  $D_{\ell}(s)$ .

$$D_{\ell}^{(1)}(s) = 1 - \frac{g}{2\pi} \int_{0}^{z} \frac{ds'}{(s'-s)\sqrt{s}'} Q_{\ell}(1 + \mu^{2}/2s').$$

To obtain the expression asymptotic in s for the leading (right-most) Regge trajectory, we note

$$\lim_{|s| \to \infty} \{ D_{\ell}^{(1)}(s) - 1 \} = 0 \begin{pmatrix} 1 \\ \sqrt{s} \end{pmatrix},$$

so that in order to satisfy  $D_{\ell}^{(1)}$  (s) = 0,  $\ell$  must approach to -1 in this limit ( $|s| \rightarrow \infty$ ). Separating the singular and non-singular parts of  $Q_{\ell}$  at  $\ell = -1$ , we obtain

$$D_{\rho}^{(1)}$$
 (s) = 1 - (ig/2 $\sqrt{s}$ ) [1/( $\ell$ +1)] +  $\rho$  (s,  $\ell$ )

where  $\rho(s, \ell)$  is analytic in  $\ell$  in the neighbourhood of  $\ell = -1$ . Therefore, setting  $D_{\ell}^{(1)}(s) = 0$ , we get

$$\ell = -1 + (ig/2\sqrt{s})[1/(1 + \rho(s, \ell))].$$
(14)

Hence to order g, the Regge trajectory is given by

$$\ell'(s) = -1 + (ig/2\sqrt{s})$$
 (15)

since  $\rho(s, \ell)$  contributes to order  $g^2$  because of its regularity in  $\ell$ . The expansion of the  $\ell'(s)$  in g is apparently valid except near the threshold, s = 0. A little more detailed investigation shows that the above way of expansion is really an expansion in the parameters like  $(g/\sqrt{s})$  and  $(\mu/\sqrt{s})$  and necessarily fails near s = 0. From works of Lovelace and Masson, and others, we know that for attractive potentials at any rate  $\ell(s)$  reaches at least to the right of  $-\frac{1}{2}$  at s = 0 and this is the source of the difficulty associated with expanding  $\ell(s)$  near -1 in powers of g.

We will briefly consider the case of superposition of Yukawa potentials:

$$V(r) = \int_{\mu_0}^{\infty} d\mu^2 \, \sigma(\mu^2) \, \underline{e^{-\mu r}}_{r}.$$
 (16)

The most interesting case is

$$\lim_{\mu^2 \to \infty} \sigma(\mu^2) = \text{Const.} \times \mu^{2(-1-\eta)}$$
(17)

corresponding to the potentials of the form  $(1/r^{1+\eta})$  at the origin.

The kernel for the superposition of the Yukawa potentials is

$$K_{\ell}(p, q; s) = \frac{1}{p^2 - s} \int_{\mu_0^2}^{\infty} d\mu^2 \sigma(\mu^2) Q_{\ell}\left(\frac{p^2 + q^2 + \mu^2}{2pq}\right)$$
(18)

and for the spectral function of Eq. (17), we see that there is a pole at  $\ell = -1 + \eta$ , and the leading Regge pole will begin at  $\ell = -1 + \eta$ , at  $s = \pm \infty$ . For  $\eta = 1$ , the integral in Eq. (18) no longer converges. If  $\eta = 1$ , the potential of Eq. (16) will have the  $r^{-2}$  singularity at the origin, and we know that there is a branch cut in the  $\ell$ -plane in this case. For  $\eta \leq 0$ , the first Regge trajectory begins at  $\ell = -1$ .

Finally, I wish to rémark briefly on the Regge behaviour of the perturbation series. The lowest order term is

$$g/(t-\mu^2) \rightarrow g/t.$$

The second order term goes as, for large t,

$$g^2(ig/2\sqrt{s})(lnt/t)$$

as one can easily check from the exact second order calculation. Using a mathematical induction, we can show that the n<sup>th</sup> order term goes as asymptotically as

$$(1/(n-1)!) g [ig/2 \sqrt{s}]^{n-1} [ln^{n-1} t/t].$$

So, if we sum the most singular terms of the series we obtain

$$\frac{g}{t} \left\{ 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{ig}{2\sqrt{s}} \right)^n \ln^n t \right\}$$

=  $(g/t \exp(ig/2\sqrt{s}) \ln t = gt[^{-1+}(ig/2\sqrt{s})]$ 

and we again obtain

$$\ell^{(1)}(s) = -1 + (ig/2\sqrt{s}).$$

#### REFERENCES

[1] SAWYER, R., These proceedings.

[2] WATSON, G. N., "Theory of Bessel Functions", Cambridge (1922) 429.

[3] WATSON, G. N., "Theory of Bessel Functions", Cambridge (1922) 389.

337

m

• .

## **REGGE POLES IN FIELD THEORY**

# R. SAWYER UNIVERSITY OF WISCONSIN, MADISON, WIS., UNITED STATES OF AMERICA

The investigation of the properties of field theoretic scattering amplitudes in the complex angular momentum plane probably can be approached at present only within the framework of an approximation scheme. It is doubtful whether general considerations can provide much information on the nature and location of the singularities in the  $\ell$ -plane, though the assumption of a Mandelstam representation provides a certain domain of analyticity. The present paper is concerned with the properties in the  $\ell$ -plane of the Bethe-Salpeter scattering amplitude in the ladder approximation, and is based on work by B.W. Lee and the author. This amplitude, though closely related to a potential scattering amplitude, has several features of field theory which are not contained in potential scattering:

- (a) I+ is fully relativistic;
- (b) I<sub>+</sub> satisfies a unitary relation to which states of more than two particles contribute, i.e., it takes account of possibility of particle production;
- (c) The crossed ladder graphs represent a set of Feynman diagrams for another scattering process, so that one may hope to interpret the Regge limit,  $t \rightarrow \infty$ , as the high-energy limit of a genuine scattering amplitude.

In terms of graphs, our aim is to discuss the behaviour in the l-plane of the sum shown in Fig.1



Fig.1

and to investigate the connection with the high-energy limit of the graphs shown in Fig. 2



For simplicity we consider the scattering of two J = 0 bosons of mass m through exchanges of a scalar boson of mass  $\mu$ . We write the relevant Bethe-Salpeter equation for the scattering amplitude in momentum space as:

$$\langle \vec{\mathbf{p}}, \boldsymbol{\omega} | \mathbf{T}(\mathbf{s}) | \vec{\mathbf{p}}, \boldsymbol{\omega}' \rangle = \langle \vec{\mathbf{p}}, \boldsymbol{\omega} | \mathbf{B} | \vec{\mathbf{p}}, \boldsymbol{\omega}' \rangle$$
$$+ \int \langle \vec{\mathbf{p}}, \boldsymbol{\omega} | \mathbf{T}(\mathbf{s}) | \vec{\mathbf{p}}', \boldsymbol{\omega}'' \rangle \langle \vec{\mathbf{p}}'', \boldsymbol{\omega}'^{t} | \mathbf{K}(\mathbf{s}) | \vec{\mathbf{p}}, \boldsymbol{\omega}' \rangle \mathbf{d}^{3} \mathbf{p}'' \mathbf{d} \boldsymbol{\omega}'' \tag{1}$$

where

$$\langle \vec{\mathbf{p}}, \omega | \mathbf{B} | \vec{\mathbf{p}}', \omega' \rangle = [g^2/(2\pi)^4] [(\vec{\mathbf{p}} - \vec{\mathbf{p}})^2 + \mu^2 - \mathbf{i} \in -(\omega - \omega')^2]^{-1},$$

$$\langle \vec{\mathbf{p}}, \omega | \mathbf{K}(\mathbf{s}) | \vec{\mathbf{p}}', \omega' \rangle = -\mathbf{i} \mathbf{F}^{-1}(\vec{\mathbf{p}}, \omega, \mathbf{s}) \langle \vec{\mathbf{p}}, \omega, | \mathbf{B} | \vec{\mathbf{p}}', \omega' \rangle,$$

$$\mathbf{F}(\vec{\mathbf{p}}, \omega, \mathbf{s}) = [\vec{\mathbf{p}}^2 + \mathbf{m}^2 - (\sqrt{\mathbf{s}}/2 + \omega)^2] [\vec{\mathbf{p}}^2 + \mathbf{m}^2 - (\sqrt{\mathbf{s}}/2 - \omega)^2] .$$

These equations define a T matrix off the mass shell. s is the square of the total energy in the centre-of-mass system. The solution to Eq.(1) for physical scattering is to be evaluated at

$$\begin{split} \omega &= \omega' = 0, \\ \overrightarrow{p} &= \overrightarrow{u_f} \sqrt{(s/4) - m^2}, \\ \overrightarrow{p} &= \overrightarrow{u_i} \sqrt{(s/4) - m^2}. \end{split}$$

The partial wave projection of Eq. (1) is of the form

$$T_{\ell}(s) = B_{\ell} + T_{\ell}(s)K_{\ell}(s)$$
<sup>(2)</sup>

where

$$\langle \mathbf{p}, \omega | \mathbf{B}_{\ell} | \mathbf{p}', \omega' \rangle = [g^{2}/(2\pi)^{3}] \mathbf{Q}_{\ell} \{ [\mathbf{p}^{2} + \mathbf{p}'^{2} + \mu^{2} - (\omega - \omega')^{2}]/2 \mathbf{p} \mathbf{p}' \},$$

$$\langle \mathbf{p}, \omega | \mathbf{K}_{\ell}(\mathbf{s}) | \mathbf{p}', \omega' \rangle = -\mathbf{i} \mathbf{F}^{-1}(\mathbf{p}, \omega, \mathbf{s}) \langle \mathbf{p}, \omega, | \mathbf{B}_{\ell} | \mathbf{p}', \omega' \rangle.$$

Operator products are defined by

$$\langle \mathbf{p}, \omega | \mathbf{A} \mathbf{B} | \mathbf{p}', \omega' \rangle = \int_{0}^{\infty} d\mathbf{p}' \int_{-\infty}^{+\infty} d\omega \langle \mathbf{p}, \omega | \mathbf{A} | \mathbf{p}', \omega' \rangle \langle \mathbf{p}', \omega' | \mathbf{B} | \mathbf{p}', \omega' \rangle$$

We make the extension to non-integral l from Eq. (2). We note that in the iteration solution to Eq. (2) l enters only in the function  $Q_l$ . The function  $Q_l$  is analytic in the entire l-plane except for fixed poles at the negative integers. Wherever the iteration solution converges we may immediately conclude that the scattering amplitude is analytic in the l plane. The Regge poles, of course, reflect the divergence of this series. To prove a domain of meromorphy in the l-plane we use instead the development of the solution of Eq. (2) as the ratio of two series via the Fredholm method. We write formally

$$T_{\rho} = B_{\rho} (1 - K_{\rho})^{-1}$$
(3)

and use the identity

$$(1 - K_{\theta})^{-1} = -(\delta / \delta K^{T})D_{\theta} / D_{\theta}$$

where

$$D_{\ell} = D\ell + (1 - K_{\ell}) = \exp[\operatorname{Tr} \log(1 - K_{\ell})].$$
(4)

This will be a useful representation wherever the perturbation series for  $D_{\ell}$  converges. As in the case of potential scattering, it can be shown that for Re  $\ell > -3/2$  the integrals of the form Tr  $K_{\ell}^{n}$  in the expansion for D converge and also that the perturbation series for D (and for N) converges absolutely. The proof, which is somewhat lengthy and will not be presented here, involves finding changes of variable which reduce the integral equation (3) to one with a bounded kernel and a finite range of integration. Standard methods are now applicable to show the convergence of  $N_{\ell}$  and  $D_{\ell}$  in this region, Re  $\ell > -3/2$ .

It follows that  $T_{\ell}$  is meromorphic in the half-plane Re  $\ell > -3/2$ . The Regge poles are zeros of  $D_{\ell}$ . Note that  $N_{\ell}$  and  $D_{\ell}$  may have fixed singularities at  $\ell = -1$ . These are both, in fact, simple poles. The proof is analogous to that in the case of potential scattering.

Before proceeding to the Regge trajectories and to their applications, we note some properties of our  $N D^{-1}$  factorization. First we discuss the singularities of  $D_{\ell}$  as a function of s.

A typical trace in the evaluation of  $D_{\ell}$  has the form :

$$\operatorname{Tr} K^{n} = \int dp_{1} d\omega_{1} F^{-1}(p_{1},\omega_{1},s)Q_{\ell} \left[ \frac{p_{1}^{2} + p_{2}^{2} + k^{2} - (\omega_{1} - \omega_{2})^{2}}{2 p_{1} p_{2}} \right]$$
$$\cdot dp_{2} d\omega_{2} F^{-1}(p_{2},\omega_{2},s) \ldots$$
$$\cdot dp_{n} d\omega_{n} F^{-1}(p_{n},\omega_{n},s) Q_{\ell} \left[ \frac{p_{n}^{2} + p_{1}^{2} + k^{2} - (\omega_{n} - \omega_{1})^{2}}{2 p_{n} p_{1}} \right].$$

First we consider the region Re  $\sqrt{s} |< \alpha m$ . In this region we may perform a transformation, following Wick, which considerably simplifies the problem. In this region the zeros of F(p, $\omega$ , s) in the  $\omega$  plane (giving the masses a negative imaginary part) are located as shown in Fig. 3.



Fig.3

#### R. SAWYER

The Wick trick is now to rotate all the  $\omega_i$  integration contours simultaneously counterclockwise into the imaginary axis. No singularities of the  $Q_\ell$  functions in (5) are encountered in this rotation. Now we look at the new denominators in Eq. (5),  $F(p_i, i\omega_i, s)$  and see that for real  $p_i$  and  $\omega_i$ , they are non-vanishing in the region  $|\text{Re }\sqrt{s}| < 2m$ .  $D_\ell(s)$  is therefore free of singularities in this region of the s-plane. We anticipate that  $D_\ell(s)$  has only the right hand cut beginning at  $s = 4 \text{ m}^2$ . We must still show, however, that no spurious singularities at complex s were introduced by our N D<sup>-1</sup> factorization. This can be shown by some further distortions of contour which will not be gone into here.

For integral  $\ell$ ,  $T_{\ell}(s)$  is defined by Eq.(1) with a left-hand cut beginning at  $s = 4 m^2 \cdot k^2$  and a right-hand cut beginning at  $s = 4 m^2$ , with branch points at the production thresholds  $s = (2 m + nk)^2$ . For non-integral  $\ell$  there is a further kinematic cut which can be renewed by factoring out a factor( $s-4m^2$ )<sup> $\ell$ </sup>. Let us define  $n_{\ell}(s)$  by  $N_{\ell}(s) = (s-4 m^2)^{\ell} n_{\ell}(s)$  and investigate the singularities of  $n_{\ell}(s)$ . What can be shown is that  $n_{\ell}(s)$  has no cut from  $s = 4 m^2$  to the first inelastic threshold,  $s = (2 m + k)^2$ . Thus  $n_{\ell}(s)$  has a left hand cut and a right hand cut beginning from  $s = (2 m + k)^2$ . The proof can be done in a compact notation by proving the relation

$$[D_{\ell}(s - i \epsilon)] / [D_{\ell}(s + i \epsilon)] = \exp[2i\delta(s, \ell)]$$
(6)

in the region  $4 \text{ m} < s < (2 \text{ m} + \text{k})^2$ .

Using a property of determinants we write:

$$\operatorname{Det}\left[\frac{1-K_{\ell}(s-i\,\epsilon)}{1-K_{\ell}(s+i\,\epsilon)}\right] = \operatorname{Det}\left\{1+\left[K_{\ell}(s+i\,\epsilon)-K_{\ell}(s-i\,\epsilon)\right]\left[1-K_{\ell}(s+i\,\epsilon)\right]^{-1}\right\}.$$
(7)

In the region  $4 \text{ m}^2 < s < (2 \text{ m} + k)^2$  Cutkosky's method for evaluating discontinuities is equivalent to the replacement:

$$\langle \mathbf{p}, \omega | \mathbf{K}_{\ell}(\mathbf{s} + \mathbf{i} \epsilon) - \mathbf{K}_{\ell}(\mathbf{s} - \mathbf{i} \epsilon) | \mathbf{p}', \omega' \rangle$$

$$\longrightarrow [g^{2}/\mathbf{i}(2\pi)^{3}] \delta[\mathbf{p}^{2} + \mathbf{m}^{2} - (\omega - (1/2)\sqrt{\mathbf{s}})^{2}] \delta[\mathbf{p}^{2} + \mathbf{m}^{2} - (\omega - (1/2)\sqrt{\mathbf{s}})]$$

$$\cdot \mathbf{Q}_{\ell} \left[ \frac{\mathbf{p}^{2} + \mathbf{p}'^{2} + \mathbf{k}^{2} - (\omega - \omega')^{2}}{2\mathbf{p}\mathbf{p}'} \right]. \tag{8}$$

We may then write Eq. (7) as

$$D(s - i \epsilon) / D(s + i \epsilon) = 1 + \left[ i \pi^2 / \sqrt{s[(1/4)s - m^2]} \right] < p, 0 | B_{\ell}(1 - K_{\ell})^{-1} | p, 0 > (9)$$

where the infinite determinant was evaluated by noting that aside from the ones along the diagonal, there is only one non-vanishing column of the matrix of which the determinant is being taken, as indicated by the  $\delta$ -functions in Eq. (8). With our definitions the right hand side of the Eq. (9) is the S matrix element. Thus Eq. (6) is proved and it is furthermore clear that  $n_{\ell}(s)$  has no cut beginning at  $s = 4 \text{ m}^2$ .

The Regge trajectories are given by the roots of the function

$$D(\ell, s) \equiv D_{\ell}(s).$$

The two properties of the trajectories which we shall now discuss are:

(1) The asymptote ( $\lim s \rightarrow \infty$ ) of the leading trajectory;

(2) The development of the trajectory,  $\alpha$  (s), in a perturbation series.

We shall need two properties of  $D_{\ell}(s)$  which can be proved :

(a)  $D_{\ell}(s)$  has a simple pole at  $\ell = -1$ . This follows, as in potential scattering, from the dependence of the singular (at  $\ell = -1$ ) part of the kernel,  $K_{\ell}$ , on only the first indices:

$$\langle \mathbf{p}, \boldsymbol{\omega} | \mathbf{K}_{sing} | \mathbf{p}', \boldsymbol{\omega}' \rangle = - [i/(\ell + 1)] \mathbf{F}^{-1}(\mathbf{p}, \boldsymbol{\omega}, \mathbf{s})$$

(b) For  $\ell \neq -1$ ,  $D_{\ell} \longrightarrow 1$  as  $s \longrightarrow \infty$  with the remainder terms approaching zero at least as fast as  $s^{-1/2}$ . This follows from the transformed form of the kernel,  $\tilde{K}_{\ell}$ , which we referred to above, in which  $\tilde{K}_{\ell}$  is bounded and the range of indices finite. We find, for this new kernel,  $\tilde{K}_{\ell} < (1/\sqrt{s})n$ , where n is some finite constant. The details are in [1].

Now  $D_{\ell}(s)$  may be written in the form

$$D_{\ell}(s) = 1 + [f(s)/(\ell + 1)] + g(s,\ell)$$
(10)

where f(s) and  $g(s, \ell)$  approach zero at infinite s and  $g(s, \ell)$  is regular in  $\ell$  for Re  $\ell > -3/2$ . For large s there may be a root,  $D_{\ell}(s) = 0$ , only near  $\ell = -1$ . The Regger trajectory is given by the solution to

$$\ell = -1 f(s) - g(s, \ell)(\ell + 1)$$
(11)

and the asymptote is clearly l = -1.

The lowest order Regge trajectory follows from computing the lowest order f(s) (which we call  $f_1(s)$ ) in Eq. (10). Using the expansion of  $D_{\ell}(s)$ :

$$D_{\ell}(s) = 1 - Tr K + \frac{(Tr K)^2 - Tr K^2}{2} + \dots$$

we see that  $[f_1(s)/(l+1)]$  is given by the singular part of -TrK,

$$-\operatorname{Tr} K_{\operatorname{sing}} = \frac{\operatorname{i} g^2}{(2\pi)^3 (\ell+1)} \int \frac{\mathrm{d} p \mathrm{d} \omega}{F(p,\omega,s)} = \frac{g^2}{8\pi^2 (\ell+1)} \int \frac{\mathrm{d} s'}{(s'-s-i\epsilon) \sqrt{s'(s'-4m'^2)}} ds'$$

Therefore  $\alpha(s)$  is given in lowest order by :

$$\alpha(s) = -1 + \frac{g^2}{8\pi^2} \int \frac{ds'}{(s'-s-i\epsilon)\sqrt{s'(s'-4m^2)^2}} 0(g4) \dots (12)$$

This is a valid expansion for sufficiently weak coupling except near threshold,  $s = 4 m^2$ . It is straightforward but tedious to work out the next order in the expansion of  $\alpha(s)$  and we have not done it. In the next order the formula should show some effect of the production threshold.

#### R. SAWYER

Now we shall use our first order expression for  $\alpha(s)$  to determine the asymptotic behaviour of the sum of the graphs (Fig. 4) in the  $\lambda c^3$  theory.



To be able to make this connection we need, of course, to open up the Watson contour as in the work of Regge. That is, we need convergence of an integral along a line Re  $\ell = \ell_0$  and the vanishing of an integral along an infinite semicircle. For our amplitude we have proved these properties only for  $\ell_0 > -1/2$ , but we shall need them for  $\ell_0$  slightly less than -1. So let us simp-'y assume for the moment that everything is all right. In addition we require a modification of the Regge formula, due to Mandelstam. to take account of poles to the left of  $\ell = -1/2$ . The outcome is simply that the leading asymptotic term in the sum of the above graphs is simply :

$$\widetilde{T}(s,t) \xrightarrow{} \beta(t) s^{a(t)}$$
(13)

where  $\widetilde{T}(s,t)$  is the T-matrix element for the above "crossed ladder graphs",  $\alpha(t)$  is the trajectory we just computed (s and t were interchanged when the graphs were crossed).

We write:

$$\alpha(t) = -1 + g^2 \alpha_1(t) + g^4 \alpha_2(t) + \dots$$
$$\beta(t) = g^2 \beta_1(t) + g^4 \beta_2(t) + \dots$$

According to our previous calculation:

$$\alpha_1(t) = \frac{1}{8\pi^2} \int_{4m^2}^{\infty} \frac{dt'}{(t'-t)\sqrt{t'(t'-4m^2)}}$$
 (14)

It turns out that the lowest order  $\beta$  is given by :

$$\beta_1 = (2\pi)^{-4}$$

Now expanding Eq. (13) in powers of  $g^2$  we obtain:

$$T_{s}(s,t) = \frac{g^{2}\beta_{1}}{s} + \frac{g^{4}\beta_{1}\alpha_{1}(t)\log s}{s} + \frac{g^{4}\beta_{2}(t)}{s} + 0(g^{n}).$$
(15)

Note that the fourth order term which goes as log s/s' can be computed exactly in terms of the second order  $\alpha$  and  $\beta$ . We have checked this connection

345

by calculating the logs term in the fourth order box diagram directly. We see also the term in the second order diagram of order  $g^{2n}(\log s)^{n}s^{-1}$  can also be given in terms of the lowest order  $\beta_1$  and  $\alpha_1$  functions. It is simply:

$$g^{2n}\beta_1 (\alpha_1(t))^n (\log s)^n / s.$$

Thus it is seen that the Regge idea coupled with perturbation theory provides a very powerful technique for summing the most divergent parts (as  $s \rightarrow \infty$ ) of sets of Feynman graphs. This technique may be useful in field theory whether the entire scattering amplitude is an analytic function of  $\ell$  or not.

One trivial generalization of our model is the inclusion of a mass spectrum for the exchanged particle. We consider a scattering amplitude derived from replacing  $B_{\ell}$  in Eq. (2) by :

$$\int Q_{\ell} \left( \frac{p^2 + p'^2 + y - (\omega - \omega')^2}{2 p p'} \right) \sigma(y) \, dy \, .$$

The interesting change in the previous results occurs when  $\sigma(y)$ , the mass spectral function, goes to zero more slowly than y<sup>-2</sup>. Let us assume a behaviour:

 $\sigma(\mathbf{y}) \longrightarrow \mathbf{y}^{-\eta}$ 

as

where

,

The asymptote of the leading Regge trajectory is now,  $l = -\eta$ . This is the generalization of a result for potential scattering with a potential given by:

$$V(\mathbf{r}) = \int_{\mathbf{y}_0}^{\infty} \frac{e^{-fy}}{\mathbf{r}} dy \sigma(y).$$

There is one difference; our method in potential scattering fails when  $\eta \le \pm 1/2$ , that is, when the potential is more singular than  $r^{-2}$  at the origin. In the relativistic theory all  $\eta > 0$  are allowed. In relativistic theory, therefore, the asymptote may move as far to the right as  $\ell = 0$  in the limit  $\eta \rightarrow 0$ . This would be the case for the  $\lambda \phi^4$  theory, in the sum of the diagrams shown in Fig. 5.



$$0 < \eta < 1$$

Here the basic bubble exchange has a mass spectral function which tends to a constant at infinity.

We see that the asymptotes of the trajectories are extremely dependent on the details of the short range force. It is this that makes me pessimistic about the possibility of doing calculations of trajectories in a realistic model using present day techniques. One surely must include, in addition to exchange of pions, nucleon exchanges, hyperon exchanges and exchanges of everything else if one sets out to calculate the asymptotic parts of the trajectories (which would be useful in interpreting high-energy scattering at large momentum transfers). Is there, nevertheless, a reason why the long range terms alone should dominate the trajectories near  $\sigma = 0$  (the diffraction region)? Probably there is not. Note that what I am discussing here is not



Fig.6

peripheralism. In the high-energy diagram shown in Fig.6, it is not a question of what the masses of the horizontal lines should be (the peripheral question). It is rather a question of whether the immensely massive intermediate state,  $(m_1, m_2, \ldots, m_n)$ , should consist of many particles of low mass or of somewhat fewer particles, some of which are quite massive. The first possibility corresponds to considering only the long range force in the crossed (ladder) channel; the second to including shorter range effects as well.

# ASYMPTOTIC LIMITS IN POTENTIAL SCATTERING AND IN FIELD THEORY

# S. FUBINI INSTITUTE OF PHYSICS, TURIN, ITALY

## INTRODUCTION

A great deal of interest, both theoretical and experimental, has been devoted recently to the study of high-energy interactions. This interest has been greatly stimulated by the valuable application of the results obtained by Regge in potential scattering to high-energy elastic scattering. This paper will discuss an approach to high-energy physics which is to some extent complementary to the one based on Regge poles and has the following features:

(1) It is based on a model for high-energy interactions obtained as the natural generalization of the peripheral model which has been successful in understanding many features of interactions between  $1 \sim 3$  GeV. This model describes the high-energy collisions as the result of the combination of a large number of low-energy interactions.

(2) The techniques used to evaluate the asymptotic limits are based on the direct study of the linear integral equations of the model and do not involve the use of analytic continuation in the angular momentum explicitly.

(3) The model allows an estimate of the main features both of elastic diffraction scattering and of multiple production. The results concerning elastic scattering are closely analogous to the ones obtained on the basis of Regge poles.

In order to explain the mathematical techniques in the simplest possible manner we have discussed in detail (Section 1) their application to potential scattering which is a very useful "laboratory" for theoretical physicists. Section 2 deals with the prediction of the model for elastic diffraction scattering and with the relativistic two-body equation. Finally in section 3 the different predictions concerning multiple production are discussed.

The paper by Stroffolini<sup>\*</sup> is complementary to the present one since the mathematical techniques which can be used in order to evaluate the different "trajectories" both in potential scattering and in field theory are discussed there.

## 1. POTENTIAL SCATTERING

We shall consider the case of potential scattering first. Consider the transition matrix element  $\Phi(\vec{k},\vec{k}_0)$  satisfying the Lippmann-Schwinger equation

\* These proceedings.

S. FUBINI

$$\Phi(\vec{k},\vec{k}_{0}) = V(\vec{k},\vec{k}_{0}) + \frac{1}{(2\pi)^{3}} \int V(\vec{k},\vec{k}') \frac{1}{\vec{k}^{2} - \vec{k}_{0}^{2} - i\eta} \Phi(\vec{k}',\vec{k}_{0}) d^{3}k'$$
(1.1)

where  $V(\vec{k}\,,\vec{k}_0)$  represents the Fourier transform of the static potential. We recall that

$$\Phi(\vec{k}, \vec{k}_0) = \langle \vec{k} | V | \vec{k}_0 \rangle$$

where  $|\vec{k}\rangle$  is an eigenstate of the free Hamiltonian and  $|\vec{k}_0\rangle$  an eigenstate of the total Hamiltonian corresponding to momenta  $\vec{k}$  and  $\vec{k}_0$  respectively. We shall define:

$$u = k^{2}, s = k_{0}^{2}, t = -(\vec{k} - \vec{k}_{0})^{2}.$$
 (1.3)

On the mass shell, i.e. for  $|\vec{k}|^2 = |\vec{k}_0|^2$ , we obtain the usual scattering amplitude

$$f(k^{2}\cos\theta) = \Phi_{\vec{k}|^{2} = |\vec{k_{0}}|^{2}} (\vec{k_{0}}, \vec{k}).$$
 (1.4)

Assume now that the potential V is given by a superposition of Yukawa potentials:

$$V(k, k') = \int_{\mu^2}^{\infty} \frac{v(t_0)dt_0}{t_0 + (\vec{k} - \vec{k}')^2} .$$
 (1.5)

We insert in the right-hand side of Eq. (1.1) the following "Ansatz":

$$\Phi(\vec{k}, \vec{k}_{0}) = \frac{1}{\pi} \int \frac{\phi(k^{2}, k_{0}^{2}, t')dt'}{t' + (\vec{k} - \vec{k}_{0})^{2}} = \frac{1}{\pi} \int \frac{\phi(u, s, t')dt'}{t' - t}$$
(1.6)

We insert Eqs. (1.5) and (1.6) into Eq. (1.1). The integral on  $d^3k'$  can be separated into a radial and an angular part giving

$$\iiint dt_0 \mathbf{v}(t_0) dt' du' \varphi (u', t', s) \int \frac{d^3 \mathbf{k}' \, \delta(\mathbf{k'}^2 - \mathbf{u})}{\left[t_0 + (\vec{k} - \vec{k}')^2\right] \left[t' + (\vec{k}' - \vec{k}_0)^2\right]}$$

The integral on d<sup>3</sup>k' can be performed by standard techniques giving

$$\int \frac{d^{3}k' \,\delta(k'^{2} - u)}{[t_{0} + (\vec{k} - \vec{k'})^{2}][t' + (\vec{k'} - \vec{k_{0}})^{2}]} = \pi \int \frac{K(t, t', t_{0}, u, u', s)}{t' - t} dt'$$

where

$$K = \frac{\theta(\sqrt{t} - \sqrt{t}' - \sqrt{t_0})\theta(\Delta)}{\sqrt{\Delta}}$$
(1.7)

and

$$\Delta = - \begin{vmatrix} u & \frac{t_0 + u + u'}{2} & \frac{t + s + u}{2} \\ \frac{t_0 + u + u'}{2} & u' & \frac{t' + s + u'}{2} \\ \frac{t + s + u}{2} & \frac{t' + s' + u'}{2} & s \end{vmatrix}$$
(1.8)

so that we finally get the following integral equation for the spectral function  $\boldsymbol{\phi}:$ 

$$\varphi(u, t) = v(t) + \iint Q(u, t; u', t') \frac{\varphi(u't') du'dt'}{u' - s - i\eta}$$
(1.9)

where

$$Q(u, t; u', t') = \frac{1}{8\pi^2} \int K_{t_0}(u, t; u't'; t_0) v(t_0) dt_0.$$
 (1.10)

Let us now discuss the properties of Eq. (1.9) (which has been obtained from (1.1) through the transformation (1.6)).

We see that the role of the kernel K is essentially to fix, (through the  $\theta$  functions) the boundaries of the phase space in which the integration variables t, u can vary, for given values of t and u. In particular, the equation  $\Delta = 0$  is a quadratic equation in u, whose solutions are the minimum and maximum value which u' can attain for fixed t, u and t'. The limitation  $\sqrt{t} > \sqrt{t_0} + \sqrt{t'}$  has a very important effect on the structure

The limitation  $\sqrt{t} > \sqrt{t_0} + \sqrt{t}'$  has a very important effect on the structure of the equation. If we solve Eq. (1.9) by means of an iteration procedure:

$$\varphi(\mathbf{u},t) = \sum_{n} \varphi_{n}(\mathbf{u},t), \qquad (1.11)$$

$$\varphi_{n+1}(t) = \iint Q_s(u, t; u't') \varphi_n(u', t') du'dt'.$$
 (1.12.a)

#### S. FUBINI

This limitation has the consequence that for each finite value of t only a finite number  $\varphi_n$  will be different from zero. So for each finite t the perturbation expansion for  $\varphi(t)$  not only converges but also stops after a finite number of terms. This might at first look rather paradoxical, since we know that the perturbation series for the transition matrix element

$$\Phi(t) = \int_{\mu^2} \frac{\Phi(t)dt'}{t'-t}$$
(1.13)

is indeed divergent in all cases in which bound states are present. The reason of this paradox is easy to understand: in the spectral integral (1.13) the integration goes until infinity and the number of  $\varphi_n$  also becomes infinite. This means that for finite values of t the perturbation series for  $\Phi(t)$  contains an infinite number of terms.

These arguments suggest that the behaviour of  $\varphi(t)$  when t- $\infty$  must be very interesting and is in some way connected with the presence of bound states or resonances, since it is the only possible cause for the divergence of the perturbation series of the S matrix. We shall therefore concentrate our attention on the problem of finding the limit of

$$\varphi(t) = \sum_{n} \varphi_{n}(t) \qquad (1.14)$$

when  $t \rightarrow \infty$  and the number of terms of the series likewise goes to infinity.

It will be seen in the next section that the relativistic analogues of the  $\Phi_n(t)$  have a very important physical meaning. In order to obtain this asymptotic limit we consider the form taken from Eqs. (1.7), (1.8), (1.9) and (1.10) for large values of t. First of all we note that, for convergent forms of the potential (1.5), the integration on u' is dominated by values of u' of the order of  $\mu^2$ ; this can be checked directly on each term of the iteration series. So, for very large values of t we can disregard u, u' and s as compared to t in the determinant  $\Delta$ . We thus get for  $\Delta$  the simplified form:

$$\Delta \approx - \begin{vmatrix} u & \frac{t_0 + u + u'}{2} & \frac{t}{2} \\ \frac{t_0 + u + u'}{2} & u' & \frac{t'}{2} \\ \frac{t}{2} & \frac{t'}{2} & 0 \end{vmatrix}$$

Note that we have not disregarded t' as compared to t since the ratio x = t'/t can indeed be of the order of 1. Moreover, we have assumed the spectral function v(t) to tend to zero for  $t_{r'} \infty$ (in order to give a convergent integral (1.5)) so that for large t the contribution of this term to the r.h. s of Eq. (1.9) will be negligible. Thus we are led to the following asymptotic equation:

ASYMPTOTIC LIMITS

$$\Phi_{as}(u,t) = \frac{1}{t} \iint Q_{as}(x,u,u') \frac{\Phi_{as}(u',t')}{u'-s-i\eta} dt' \qquad (1.15)$$

where

$$Q_{as}(x, u, u') = \frac{1}{8\pi^2} \int \frac{\theta[u' - ux - t_0 x/(1 - x)]r(t_0)dt_0}{(1 - x)^{1/2}[u' - ux - t_0 x/(1 - x)]^{1/2}}$$
(1.16)

and where x = t'/t. The asymptotic equation [Eq. (1.15)] satisfies a very important property; it is invariant under the dilatation

$$t \rightarrow Const. t,$$
  
 $t' \rightarrow Const. t'.$  (1.17)

This property will be common to all the asymptotic equations we shall be considering and enables us to obtain a solution of (1.15) in the form:

$$\varphi(\mathbf{u}, \mathbf{t}) = \mathbf{f}_{\alpha}(\mathbf{u}) \mathbf{t}^{\alpha} \tag{1.18}$$

where  $f_{\alpha}(u)$  satisfies the equation

$$f_{\alpha}(u) = \int R_{\alpha}(u, u') \frac{f_{\alpha}(u')du'}{u' - s - i\eta}$$
(1.19)

where

$$R_{\alpha}(u, u') = \frac{1}{8\pi^2} \int v(t_0) dt_0 \int_0^1 \frac{dx x^{\alpha}}{(1-x)^{1/2}} \frac{\theta[u' - ux - t_0 x/(1-x)]}{[u' - ux - t_0 x/(1-x)]^{1/2}} (1.20)$$

Equation (1.19) is an homogeneous linear integral equation of the Fredholm kind giving rise to an eigenvalue problem. For a fixed value of the total energy s,the equation is satisfied only in correspondence with well defined values of  $\alpha$ . For s > 0, i.e. in the scattering region, the presence of the u' - s - in denominator will lead to complex values of  $\alpha$ . For s < 0 the denominator u - s cannot vanish and so the eigenvalues will be real. The eigenvalue of  $\alpha$  having the largest real part is of particular interest since this gives rise to the dominating term as  $t \to \infty$ .

Let us summarize the results obtained. We have started from the usual Lippman-Schwinger equation [Eq. (1.1)] and we have transformed it into Eq. (1.9) for the spectral function  $\varphi(t)$ . We have then considered the "reduced" [Eq. (1.15)] obtained by taking the large t limit on Eq. (1.9). Finally the solution of Eq. (1.15) leads to the asymptotic form (1.18) where the fundamental exponent  $\alpha$  is determined by the homogeneous equation (1.20).
We wish to emphasize the heuristic character of the derivation of Eq. (1.18) since the procedure of taking the asymptotic form of an equation in order to obtain the asymptotic solution, although frequently used by physicists, is not a rigorous one. We shall, however, show that the use of this procedure is indeed justified in our case and that Eqs. (1.18) and (1.19) lead to the correct asymptotic limit of  $\varphi(t)$ . We shall now turn to the problem of determining the asymptotic limit of the scattering amplitude  $\Phi$  itself, related to  $\varphi$  by the dispersion integral (1.6). For convergence reasons this dispersion relation has actually to be written down with m subtractions, where m is the minimum integer greater than Re  $\alpha(s)$ 

$$\Phi(t) = P_{m-1} + \frac{t^{m}}{\pi} \int_{\mu^{2} t^{m}(t'-t)}^{\infty} dt'$$
 (1.21)

where  $P_{m-1}(t)$  is a polynomial in t with maximum power (t-1).

From Eq. (1.21) we obtain the asymptotic form for  $\Phi(t)$  by making the following approximations: first of all we substitute for  $\Phi(t)$  its asymptotic form (1.18) arguing that terms whose asymptotic form is smaller than (1.18) cannot contribute to the asymptotic form of  $\Phi(t)$ . We then extend the integration range between 0 and  $\infty$  since the contribution between 0 and  $\mu$  is negligible. Finally we neglect in Eq. (1.21) the subtraction polynomial (whose maximum power is m-1) and get:

$$\Phi_{as}(t) = f_{\alpha}(u) \quad \frac{t^{m}}{\pi} \int_{0}^{\infty} \frac{dt'}{(t')^{m-\alpha}(t'-t)} = f_{\alpha}(u)t^{\alpha} \int_{0}^{\infty} \frac{dz}{(z)^{m-\alpha}(z-1)}$$

The integral in z is a well-known one (see the theory of the  $\Gamma$  function) and we finally obtain:

$$\Phi_{as}(s,t,u) = f_{\alpha}(u) \quad \frac{e^{i\pi\alpha(s)}}{\sin\pi\alpha(s)} t^{\alpha(s)}$$
 (1.22)

The asymptotic form (1.22) coincides completely with result of Regge, based on theory of continuation in the angular momentum variable.

Equation (1.22) now clearly shows the relation between the asymptotic behaviour of  $\varphi$ (t) and the bound state problem. Indeed we see that the amplitude  $\Phi_{as}$  has poles in s in correspondence to values of s for which

 $\ell$  being any positive integer. Those poles correspond to bound states or resonances (depending on whether they correspond to real or complex s) in states of angular momentum  $\ell$ . (this is because the coefficient t<sup> $\ell$ </sup> represents the asymptotic limit of P<sub> $\ell$ </sub> (cos  $\theta$ )). This means that Eq. (1.19), which determines  $\alpha$ , also leads through Eq. (1.23) to a determination of the

bound states and resonances of the problem. Stroffolini has shown that for entire values of  $\alpha$ , Eq.(1.19) is just a different form of the Schroedinger equation for bound states and particularly suited for continuation in the complex angular momentum.

This result fully confirms the validity of the whole procedure which has led to the asymptotic limit (1.22). Indeed a scattering amplitude has the same poles in s independently on the value of t and hence also in the limit  $t \rightarrow \infty$ . So the fact that for  $\alpha$  entire, Eq. (1.19) coincides with the <u>exact</u> Schroedinger equation for bound states confirms that Eq. (1.22) gives the correct asymptotic limit of  $\Phi(s, t, u)$ .

# 2. RELATIVISTIC TWO-BODY PROBLEM

The simplest relativistic generalization of the potential model discussed in the previous section is the Bethe-Salpeter equation in the ladder approximation. This equation is summing the series of graphs shown in Fig. 1 which represents elastic scattering

$$A_1 + A_2 \rightarrow A_1 + A_2 \tag{2.1}$$

where  $q_1, q_2, n_1, n_2$  are the initial and final momenta.

We define

$$(q_1 + q_2)/2 = (n_1 + n_2)/2 = \Delta,$$
  
 $(q_1 - q_2)/2 = Q,$  (2.2)  
 $(n_1 - n_2)/2 = N.$ 

where  $2 \Delta$  is the total momentum of the system and Q and N are the relative momenta in the initial and final states respectively. The Bethe-Salpeter equation has the form

$$\Phi(\mathbf{Q}, \Delta, \mathbf{N}) = \mathbf{V}(\mathbf{Q}, \mathbf{N}) + \frac{2}{(2\pi)^4} \int \frac{\mathbf{k} \cdot \mathbf{V}(\mathbf{Q}, \mathbf{\phi}) \Phi(\mathbf{\phi}, \Delta, \mathbf{N}) d^4 \mathbf{Q}}{\left[(\Delta + \mathbf{\phi})^2 - \mu^2\right] \left[(\Delta - \mathbf{\phi}')^2 - \mu^2\right]}$$
(2.3)

where the "potential" V is the propagator of the systems exchanged between  $A_1$  and  $A_2$ . If what is exchanged is a single particle:

$$V = g^2 / [(Q - Q')^2 + m^2].$$

If a system of particles is exchanged, then V is represented by a weighted sum of propagators.

$$V(Q, Q') = \int \frac{v_0(s_0) ds_0}{(Q - Q')^2 - s_0} . \qquad (2.4)$$

Finally  $\mu$  represents the masses of A<sub>1</sub> A<sub>2</sub> (which for simplicity are assumed equal). The analogy between the relativistic four-dimensional scattering equation, [Eq. (2.3)], and the non relativistic one discussed in the previous section is quite evident. So the method we shall use to treat both equations will be closely analogous.

There is, however, a very important difference because of the field theoretical nature of Eq. (2.3). The same amplitude  $\Phi$  describes at the same time the reactions

(I); 
$$A_1 + A_2 \rightarrow A_1 + A_2$$

(II); 
$$A_1 + \overline{A}_1 \rightarrow A_2 + \overline{A}_2$$
.

The initial momenta for this second reaction are  $q_1$ , -  $n_1$  and the final momenta -  $q_2$ ,  $n_2$  so that in this new channel 2  $\Delta$  now represents the momentum transfer and Q + N =  $q_1$  +  $n_1$  =  $q_2$  +  $n_2$  the total momentum.

If we define

$$4 \Delta^2 = -t \text{ and } (Q+N)^2 = s$$
 (2.5)

we have:

In channel I: t is the square of the CM energy, s the momentum transfer;

In channel II: s is the square of the CM energy, t the momentum transfer. (The notation here is adapted to channel II). The existence of the substitution rule is of the utmost importance for the physical interpretation of the asymptotic limit of the B.S. equation. Indeed scattering in channel I has a strong analogy with potential scattering so that we shall find the asymptotic limit for small energy t and for momentum transfer  $s \to \infty$ . On the other hand, s plays the role of energy in channel II and therefore the asymptotic result can be interpreted as limit of the  $A_1 + \overline{A_1} \to A_2 + \overline{A_2}$  amplitude for large values of energy s and small momentum transfer t.

Let us now discuss the asymptotic solution of Eq. (2.3). We shall only sketch the main points, since it is very analogous to potential scattering. We define the virtual "masses" of  $q_1$ ,  $q_2$  as:

$$q_1^2 = (Q + \Delta)^2 = -u_1,$$
  
 $q_2^2 = (Q - \Delta)^2 = -u_2$ 
  
(2.5a)

and we introduce for  $\Phi$  the ansatz:

$$\Phi(s, u_1, u_2, t) = \frac{1}{\pi} \int \frac{\Phi(s', u_1, u_2, t)}{s' - s} ds'.$$
 (2.6)

The integral equation obtained by substituting Eq. (2.6) into Eq. (2.3) is:

ASYMPTOTIC LIMITS



The analogy with the corresponding potential Eqs. (2.8), (2.9) and (2.10) is quite striking. The greater complication of the new equations is naturally as result of the four-dimensional nature of the relativistic problem.

The kernel K vanishes for  $\sqrt{s} < \sqrt{s}' + \sqrt{s_0}$ , ensuring that for a finite value of s the series obtained by iterating Eq. (2.7) stops after a finite number of terms. The terms of the iteration series can be represented by the graphs of Fig.1 in which each exchanged particle propagators  $1/[(\Phi - \Phi')^2 - s_0]$  are substituted by  $2\pi\delta[(\Phi - \Phi')^2 - s_0]$ . In other words, in the iteration series of  $\phi$  all exchanged particles are taken on the mass shell. The physical meaning of this important fact will be discussed in the next section.



We turn now to the problem of obtaining the asymptotic limit of  $\varphi(s, t, u_1, u_2)$ . Here there will also be nothing new, as compared with the preceding section. We can disregard, in the determinant  $\Delta$ ,  $u_1$ ,  $u_2$ ,  $u_1'$ ,  $u_2'$ , t,  $s_0$  as compared to s. We then get:

$$\frac{\Delta}{s^2} = \begin{vmatrix} -\frac{t}{2} & \frac{u_1' - u_2'}{2} & \frac{u_1 - u_2}{2} & 0 \\ \frac{u_1' - u_2'}{2} & u_1' + u_2' + \frac{t}{2} & \frac{u_1 + u_2 + u_1' + u_2' + t}{2} + s_0 & x \\ \frac{u_1 - u_2}{2} & \frac{u_1 + u_2 + u_1' + u_2' + t}{2} + s_0 & u_1 + u_2 + \frac{t}{2} & 1 \\ 0 & x & 1 & 0 \end{vmatrix}$$
(2.11)

where x = s'/s. Moreover we can neglect the contribution of  $v(t_0)$  for large s so that we get the asymptotic form of the equation

$$\varphi_{as}(s, u_1, u_2) = \frac{1}{s} \iiint Q_{as}(x, u_1, u_2; u_1' u_2') \cdot \frac{\varphi_{as}(u_1', u_2') du_1' du_2'}{(u_1' + \mu^2)(u_2' + \mu^2)} \quad (2.12)$$

where

$$Q_{as} = \frac{1}{2(2\pi)^{4}} \int v(s_{0}) ds_{0} \frac{\theta [H(z, z_{1}, z_{2})]}{[H(z, z_{1}, z_{2})]^{1/2}}$$
(2.13)

where

$$H(z, z_1, z_2) = -(1/4)[z^2 + z_1^2 + z_2^2 - 2zz_1 - 2zz_2 - 2z_1z_2]$$
(2.14)

$$z = -t(1 - x),$$
  

$$z_{1} = u_{1}' - u_{1}x - s_{0}x/(1 - x),$$
  

$$z_{2} = u_{2}' - u_{2}x - s_{0}x/(1 - x).$$
(2.15)

We recall from elementary geometry that  $(1/2)\sqrt{H}$  represents the area of the plane triangle whose sides are  $\sqrt{z}$ ,  $\sqrt{z_1}$ ,  $\sqrt{z_2}$ . Also here we find the remarkable invariance of the equation under the transformation:

$$s \rightarrow e s$$
 and  $s' \rightarrow e s'$ , (2.16)

which allows to factorize  $\varphi$  in the form

$$\varphi_{as}(s, u_1, u_2, t) = s^{\alpha(t)} f_{\alpha}(u_1, u_2, t)$$
 (2.17)

where f satisfies the homogeneous equation

$$f_{\alpha}(u_{1}, u_{2}) = \int R_{\alpha}(u_{1}, u_{2}, u_{1}', u_{2}') \frac{f_{\alpha}(u_{1}', u_{2}') du_{1}' du_{2}'}{(u_{1}' + \mu^{2})(u_{2}' - \mu^{2})}, \qquad (2.18)$$

$$R_{\alpha}(u_{1}, u_{2}, u_{1}', u_{2}') = \frac{1}{2(2\pi)^{4}} \int v(s_{0}) ds_{0} \int_{0}^{1} dx x^{\alpha} \frac{\theta[H(z, z_{1}, z_{2})]}{[H(z, z_{1}, z_{2})]^{1/2}}.$$
(2.19)

The eigenvalue Eq. (2.13) determines the exponent  $\alpha$  as a function of t. For fixed values of t such equations have a discrete spectrum of eigenvalues. Eq. (2.13) is identical with the corresponding result in potential scattering and coincides with that obtained by extending the Regge results in relativistic theory. The use of the optical theorem gives for the total cross-section:

$$\sigma = f(-\mu^2 - \mu_0^2) S^{\alpha(0) - 1}. \qquad (2.20)$$

The experimental evidence for the high energy total cross-section indicates that the actual value of  $\alpha(0)$  is not very different from 1.

# 3. THE MULTIPERIPHERAL MODEL

We shall discuss in more detail the physical meaning of the ladder graphs treated in the last section. This will enable us to gain a deeper understanding of the significance of the formulae for elastic scattering obtained and at the same time to derive a general model for the inelastic processes taking place at high energy.

We shall consider the ladder graphs of Fig. 1 and we explicitly refer to channel II appropriate to high energy low momentum transfer scattering. Consider the amplitude  $\varphi$  in the forward direction whose structure is shown in Fig. 2. In channel II, the amplitude  $\varphi$  is just the absorptive part of the full amplitude  $\varphi$  so that the forward elastic amplitude

$$\varphi(s, 0, u, u) = A(s, u)$$
 (3.1)

is related to the total cross-section by the well known optical relation:

$$A(s_1 - \mu^2) = \sigma_T / 2q\sqrt{s} = \sigma_T / s \qquad (3.2)$$

where  $q = (1/2)\sqrt{s} - u\mu^2$  is the CM momentum of the incoming particle.



Eq. (3.2) shows that if one makes a model for high-energy elastic scattering, one also implicitly constructs a model for the different production processes whose usm gives rise to the total cross-section appearing in Eq. (3.2). This is quite clear physically, since we know that at high energy, elastic scattering

whose usm gives rise to the total cross-section appearing in Eq. (3.2). This is quite clear physically, since we know that at high energy, elastic scattering is essentially shadow scattering so that the form of the diffraction peak depends essentially on the multiple production processes responsible for the absorption. If one looks at the graph in Fig. 2 one sees that the production graphs giving rise to the diffraction pattern discussed in the last section are the ones shown in Fig. 3. We are therefore led to a model for multiple production which is the generalization to very high-energy of the peripheral model of Chew-Low, Drell and Saltzman. The external outgoing lines represent groups of particles whose mass distribution is given by the spectral function  $v(s_0)$  which can be related to low energy cross-sections.



The absorptive amplitude A(s, u) is obtained from  $(s, u_1, u_2, t)$  by making  $u_1 = u_2$  and t = 0. So that in the high-energy limit the energy for A(s, u) is particularly simple (see Fig. 4):

$$A(s, u) \equiv \frac{1}{16\pi^{3}s} \int v(s_{0}) ds_{0} \int_{0}^{s} ds' \int_{ux} \frac{s_{0}x}{1-x} du' \frac{A(s, u')}{(u' + \mu^{2})^{2}}.$$
 (3.3)

Fig.4

So, by applying the usual factorization:

$$A(s, u) = s^{\alpha(0)} f_{\alpha}(u)$$
 (3.4)

one obtains:

$$f_{\alpha}(u) = \frac{1}{16\pi^{3}} \int v(s_{0}) ds_{0} \int_{0}^{1} x^{\alpha} dx \int_{ux^{+} \frac{s_{0}x}{1-x}}^{\infty} du' \frac{f_{\alpha}(u') du'}{(u'+\mu^{2})^{2}}.$$
 (3.4a)

Estimates of the exponent  $\alpha$  using Eq. (3.4) with a "potential" suggested by low-energy cross-sections lead to values which are not inconsistent with the experimental value  $\alpha \cong 1$ .

Let us now discuss some of the main trends of high-energy collisions which can be predicted on the basis of the multiperipheral model. First of all, let us consider the multiplicity of secondaries. This will be proportional to the average number of "blubs" in the graph of Fig. 3 since the number of particles coming from each blub is, in our model, constant. The evaluation of the behaviour of such "blub multiplicity" is very easy. We write the "potential" in the form

$$\mathbf{v}(\mathbf{s}_{0}) = \lambda \mathbf{u}(\mathbf{s}_{0}) \tag{3.5}$$

where  $u(s_0)$  is normalized to  $+ \int u(s_0) ds_0 = 1$ . Then we write the multiperipheral series exhibiting explicitly the  $\lambda$  dependence

$$A(s) \approx \sum \lambda^{n} a_{n}(s) \qquad (3.6)$$

where  $\lambda n a_n/A$  is the probability for production of n blubs. Thus the multiplicity can be written as

$$< N > = \lambda \left(\frac{\delta}{\delta} \frac{A}{\lambda}\right) / A = \frac{\Sigma n \lambda^{n} a_{n}(s)}{\Sigma \lambda^{n} a_{n}(s)}$$
 (3.7)

But now the forward on-mass shell amplitude is (see Eq. (3.4))

$$A(s_1 - \mu^2) = C(\lambda) s^{a(\lambda)}$$
(3.8)

where the parameter  $\lambda$  enters through the functions  $C(\lambda), \alpha(\lambda)$ , so that using Eq. (3.7) we get:

$$< N > = \lambda \frac{d\alpha}{d\lambda} \log s + \frac{\lambda}{c} \left( \frac{d C}{d \lambda} \right) = \lambda \left( \frac{d \alpha}{d \lambda} \right) \log \frac{s}{s}.$$
 (3.9)

We have obtained the important result that the multiplicity of secondaries grows with the logarithm of the incoming energy. This is not inconsistent with experiment, though the present data are still too rough to distinguish between a logarithmic or a slow power ( $\sim s^{1/4}$ )behaviour.

Let us now consider the average spectra of the produced particles. Let us first look at the spectrum of that final line which is directly connected with the incident particle (first line in the multiperipheral chain). The laboratory energy of this system (see Fig. 1) is given (in the high-energy limit) by:

$$E = (s - s)/2\mu$$
. (3.10)

The energy distribution of this particle is simply obtained by adding in the integrand on the r, u, s of Eq. (3.3) an extra  $\delta(E' - (s - s')/2\mu)$ :

$$\frac{d\sigma}{dt'} \approx s \frac{dA(s - \mu^2)}{dE}$$

$$= \frac{1}{16\pi^{3}} \int \mathbf{v}(\mathbf{s}_{0}) \, \mathrm{d}\mathbf{s}_{0} \int_{0}^{s} \mathrm{d}\mathbf{s}' \int_{-\mu^{2}\mathbf{x}^{+}}^{\infty} \frac{\mathrm{d}\mathbf{u}' \,\delta\left(\mathbf{E}' - \left[\mathbf{s} - \mathbf{s}'\right]/2\,\mu\right) \,\mathbf{A}(\mathbf{s}',\mathbf{u}')}{\left(\mathbf{u}' + \mu^{2}\right)^{2}}$$
(3.11)

and using the asymptotic form (3.4) for A:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\epsilon} = \frac{\mathrm{s}^{a^{-1}}}{16\pi^3} \int \mathbf{v}(\mathbf{s}_0) \,\mathrm{d}\mathbf{s}_0 \int_0^1 \mathbf{x}^a \,\mathrm{d}\mathbf{x} \int_{\mu^2 + \frac{\mathrm{s}_0 \mathbf{x}}{1-\mathbf{x}}}^\infty \frac{\mathrm{d}\mathbf{u}'\mathbf{f}(\mathbf{u}')}{\left(\mathbf{u}' + \mu\right)^2} \,\delta(\epsilon - 1 - \mathbf{x}) \quad (3.12)$$

where

$$\epsilon = \frac{E}{E}$$

is the ratio between the secondary energy E and the primary energy  $E = s/2\mu$ . So we have the very simple result:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\epsilon}\right)/\sigma_{\mathrm{T}} = \mathrm{F}(\epsilon).$$
 (3.13)

The shape of the energy specturm of the first secondary is completely independent of the value of the primary energy. In particular the inelasticity, i.e. the fraction of energy taken by the first secondary, is independent on the primary energy. The spectrum we are studying is particularly easy to measure when the incident particle is distinguishable from the secondary pions. This is the case for nucleon collisions, which have been extensively studied either with accelerators or cosmic rays. The analysis of high-energy jets yields that the average energy carried away by the nucleon in the lab system is nearly a constant fraction of the incident energy, which is just the prediction of our model.

### ASYMPTOTIC LIMITS

It is also easy to study the spectrum of secondaries, regardless of their position in the multiperipheral chain. We shall not report here the calculation which does not offer any new difficulty and only give the results.

Let us call  $k_L$  and  $k_L$  the longitudinal and transverse momenta of the secondary. If  $k_L \ll E$ , the spectrum can be written in the form

$$F(k_{T}^{2}) dk_{L} / k_{L}$$
 (3.14)

where  $F(k_T^2)$  is a universal function independent both of E and of  $k_L$  and strongly peaked for small values of  $k_T$ . These results, especially the separability of the transverse and longitudinal spectra, are not inconsistent with present experimental data.

#### 4. CONCLUSIONS

We wish now to summarize briefly the different results and their physical meaning. We have discussed the predictions for the different high-energy processes obtained on the basis of the multiperpheral model. It has been possible to sum the whole series of multiperipheral graphs by means of a linear integral equation for the off-mass shell absorptive amplitude  $\varphi(s, u_1, u_2, t)$ . The kernel of this integral equation depends on the low-energy amplitude  $v(s_0)$ . The knowledge of this amplitude is sufficient to allow the computation both of the elastic scattering and of multiple production. The on-mass shell amplitude  $\varphi(s, -\mu^2, -\mu^2, t)$  leads to the elastic diffraction cross-sections, while we can evaluate the average distributions of particles in multiple production on the basis of the forward off-mass shell amplitude  $\varphi(s, u_1, u_2, 0)$ .

The asymptotic behaviour of the amplitude is obtained by considering the high-energy limit of the integral equation. In this limit, the integral equation shows a very remarkable feature which is independent of the specific form of the amplitude  $A^R$ . The kernel depends only on the ratio s'/s, so that the equation is invariant under the transformation  $s \rightarrow cs$ ,  $s' \rightarrow cs'$ This allows us to factorize the s dependence of the amplitude in the simple form:

$$\varphi(\mathbf{s}, \mathbf{u}_1, \mathbf{u}_2, t) = \mathbf{s}^{\alpha(t)} f(\mathbf{u}_1, \mathbf{u}_2, t).$$
 (4.1)

The problem is then reduced to the solution of an homogeneous integral equation for  $f(u_1, u_2, t)$ , whose solution determines both the exponent  $\alpha(t)$  and the eigenfunction  $f(u_1, u_2, t)$ . As already pointed out, both eigenvalues and eigenfunctions have a physical meaning: the eigenvalue gives the well-known shrinking of the diffraction peak, whereas the eigenfunction is connected with the average properties of multiple production. A form of the scattering amplitude analogous to Eq. (4.1) has been obtained by many people by adapting the results of Regge in potential theory to high-energy scattering. This analogy can be understood by considering that the multiperipheral graphs observed in the crossed channel are the relativistic analogues of the different iterations of the potential model used by Regge.

The predictions obtained by means of the model can be divided into two categories:

(a) Many general trends of high-energy collisions depend only on the transformation property of the integral equation which is a consequence of the topology of the multiperipheral graphs. These general trends do not in fact depend on any special choice of the low-energy amplitude  $v(s_0)$ :

(b) The specific numerical answers (as, for example, the value of the total cross-sections) do, of course, depend on the choice of  $v(s_0)$  and on the manner in which  $v(s_0)$  is continued off the mass shell.

We shall now summarize the different conclusions obtained on the basis of the multiperipheral model including those which have not been discussed in this paper.

## (1) Elastic amplitude

The high-energy behaviour of the scattering amplitude  $\Phi(s, t)$  is

$$\Phi_{j}(s,t) = s^{a_{j}(t)} C_{j}(t) [-\cot g(\pi \alpha_{j}(t)/2) + i]$$
(4.2)

for symmetric amplitudes under crossing  $s \leftrightarrow \overline{s}$ , as, for instance, absolute elastic scattering, and

$$\boldsymbol{\Phi}_{j}(s,t) = s^{a_{j}(t)} C_{j}(t) [tg(\pi\alpha_{j}(t)/2) + i]$$
(4.3)

for antisymmetric amplitudes under the crossing. We obtain  $d\alpha/dt > 0$ . The exponent for the charge exchange amplitude is always smaller than the one for the purely elastic one. Eqs. (4.2) and (4.3) turn out to be independent of the scattering particles, apart from the value of C(t). The C(t) can be factorized in such a manner that the relation between different amplitudes (dominated by the same pole) is the following:

$$\Phi_{xy}(s,t)/\Phi_{zy}(s,t) = \Phi_{xw}(s,t)/\Phi_{zw}(s,t)$$

where x, y, z and w represent any kind of particles.

## (2) Inelastic scattering

The average properties of inelastic scattering are also very simple and depend only on the general form of Eq. (4.4) The multiplicity grows with the logarithm of the energy, and the inelasticity is energy independent. The spectra of the secondary particles are given by

N(k) 
$$d^4k = F(k^2, \vec{k}_T^2) d\vec{k}_T dk^2 (dk_L/k_L)$$

for  $k_L \ll$  the initial energy.  $k_L (k_T \text{ and } k_L \text{ being the transverse and longitudinal momenta}), where <math>F(k^2, k^2_T)$  is a universal function independent both of s and  $k_1$ , and strongly peaked for  $k_T^2 \leq \mu^2$ .

These results, especially the separability of the transverse and longitudinal spectra, are not inconsistent with present experimental data.

# ASYMPTOTIC LIMITS

## BIBLIOGRAPHY

## Potential scattering BLANCKENBECLER, R., GOLDBERGER, M. L., KHURI, N. N. and TREIMAN, S. B., Ann. Phys. <u>10</u> (1960) 62. BERTOCOHI, L., CEOLIN, C. and TONIN, M., Nuovo Cim. <u>18</u> (1960) 770. REGGE, T., Nuovo Cim. <u>14</u> (1959); <u>8</u> (1960) 947.

 The multiperipheral model AMATI, D., FUBINI, S., STANGHELLINI, A. and TONIN, M., Nuovo Cim. <u>22</u>(1961) 569.
 AMATI, D., FUBINI, S. and STANGHELLINI, A., Nuovo Cim. in press; Phys. Letters <u>1</u>(1962) 29.
 CEOLIN, C., DUIMIO, F., FUBINI, S. and STROFFOLINI, R., Nuovo Cim. in press.
 BERTOCCHI, L., FUBINI, S. and TONIN, M., Nuovo Cim. (1962).

3. The application of Regge poles

CHEW, G. and FRAUTSCHI, S., Phys. Rev. Letters 7 (1961) 349; 8 (1962) 41. FRAUTSCHI, S., GELL-MANN, and ZACHARIASEN, F., Phys. Rev. in press. DRELL, S., Proc. Int. Conf. on High-Energy Physics, CERN (1962). POMERANCHUK, A., Proc. Int. Conf. on High-Energy Physics, CERN (1962). GELL-MANN, M., Proc. Int. Conf. on High-Energy Physics, CERN (1962).

# ON THE EQUATION FOR THE REGGE TRAJECTORIES

# R. STROFFOLINI INSTITUTE OF THEORETICAL PHYSICS, MOSTRA D'OLTREMARE, NAPLES, ITALY

~

# ON THE EQUATION FOR THE REGGE TRAJECTORIES

It is the purpose of this lecture to discuss in detail some general methods which can be used to study the Regge trajectories both in non-relativistic and relativistic potential theory.

The equation for the trajectories was discussed in the Fubini lectures<sup>\*</sup> on the basis of the asymptotic limit for large momentum transfer of the scattering amplitude. We shall show that an equation can be obtained by transforming the usual Schroedinger equation into a form which is better suited to analytic continuation in the angular momentum variable.

The equation for the trajectory obtained in this manner is in a form which is a very good starting point for actual evaluation of the trajectory  $\alpha(s)$ .

We shall first introduce the very simple result which is obtained in the weak coupling limit and discuss its physical significance. We shall then develop a general treatment of those equations based on a convergent expansion of the integral equation. Similar results are obtained in the framework of the relativistic two body equation. In particular, study of the weak coupling limit will be very useful in order to understand the physical reason of the shrinking of the diffraction peak in the framework of the multiperipheral model.

# 1. ASYMPTOTIC LIMIT OF SCATTERING AMPLITUDE

We will recall briefly some of the results obtained in I. The transition matrix element  $\Phi(\mathbf{k}, \mathbf{k}_0)$  satisfying the equation

$$\Phi(\mathbf{k}, \mathbf{k}_0) = V(\mathbf{k}, \mathbf{k}_0) + \frac{1}{(2\pi)^3} \int \frac{V(\mathbf{k}, \mathbf{k}') \Phi(\mathbf{k}', \mathbf{k}_0) d^3 \mathbf{k}'}{(\mathbf{k}'^2 - \mathbf{k}_0^2 - i\eta)}$$
(1)

has been represented into the form

$$\Phi(\mathbf{k}, \mathbf{k}_0) = \frac{1}{\pi} \int \frac{\phi(\mathbf{u}, \mathbf{s}, \mathbf{t}')}{(\mathbf{t}' - \mathbf{t})} \, \mathrm{d}\mathbf{u}$$
(2)

where  $s = k_0^2$ ,  $u = k^2$  and  $t = -(k-k_0)^2$ , and our integral equation was derived

<sup>·</sup> Hereafter referred to as I.

for  $\varphi(u, s, t)$ , by substituting (2) in (1). It was shown that in the asymptotic limit,  $t \to \infty$ , the equation for  $\varphi(u, s, t)$  admits solutions of the form

$$\varphi(\mathbf{u}, \mathbf{s}, \mathbf{t}) = \mathbf{t}^{\alpha} \mathbf{f}_{\alpha}(\mathbf{u}, \mathbf{s}) \tag{3}$$

where  $f_{\alpha}(s, u)$  satisfies the equation

$$f_{\alpha}(u, s) = \int \frac{K_{\alpha}(u, u') f_{\alpha}(u', s) du'}{(u' - s - i\eta)}$$
(4)

$$k(u, u') = \frac{1}{(2\pi)^3} \int v(t_0) dt_0 \int_0^1 dx \frac{x^{\alpha} \theta [u' - ux - t_0 x/(1-x)]}{(1-x)^{1/2} [u' - ux - t_0 x/(1-x)]^{1/2}}$$

By using (2) and (3) the asymptotic behaviour of  $\Phi(\mathbf{k}, \mathbf{k}_0)$  was obtained:

$$\Phi(\mathbf{u},\mathbf{s},\mathbf{t}) = \frac{\mathbf{t}^{\alpha(\mathbf{s})} e^{i\pi\alpha(\mathbf{s})}}{\sin \pi\alpha(\mathbf{s})} \mathbf{f}_{\alpha}(\mathbf{u},\mathbf{s}).$$
(5)

This shows that the condition  $\alpha(s) = \ell$  gives the poles of the scattering amplitude, that is,  $\alpha(s)$  gives a Regge trajectory. The function  $\alpha(s)$  is obtained from the relation between  $\alpha$  and s which must be satisfied for the homogeneous integral equation (4) in order to have a solution. (Notice that in general for each value of s, Eq. (4) admits solutions for several values of  $\alpha$ ; this will give us several Regge trajectories.

It is necessary to know whether such a determination of  $\alpha(s)$  is an exact one. In reply to this question we recall that, according to JOST and PAIS [1], the scattering amplitude can be expressed into the form

$$\Phi(\mathbf{s}, \mathbf{t}) = \mathbf{N}(\mathbf{s}, \mathbf{t}) / \mathbf{D}(\mathbf{s})$$
(6)

and the poles of  $\Phi(s, t)$  are only given by the zeros of the denominator D(s). Eq. (6) shows that the asymptotic limit  $t \to \infty$  modifies only the numerator function N(s, t) leaving D(s) unchanged. The position of the poles will be therefore unchanged so that we expect that Eq. (4) together with

$$\alpha(s) = \ell \text{ and } \ell = \text{integer number}$$
(7)

leads to the exact values of the energies of bound states or resonances. This fact will be directly checked in the following arguments since we will show that for  $\alpha(s) = \ell$ , Eq. (4) coincides with the Schroedinger equation for bound states of angular momentum  $\ell$ .

# 2. THE SCHROEDINGER EQUATION FOR COMPLEX ANGULAR MOMENTUM

Let us consider the homogeneous integral equation for a bound state

$$\Psi(\mathbf{k}, \mathbf{k}_{0}) = \frac{1}{(2\pi)^{3}} \int \frac{\Psi(\mathbf{k}, \mathbf{k}') \Psi(\mathbf{k}', \mathbf{k}_{0}) d^{3}\mathbf{k}'}{(\mathbf{k}'^{2} - \mathbf{k}_{0}^{2})}$$
(8)

with

where now  $k_0^2 < 0$ .

For  $\ell$  positive integer we can put

$$\Psi(\mathbf{k}, \mathbf{k}_0) = \mathbf{f}_{\ell}(\mathbf{k}^2) \mathbf{Y}_{\ell}^{\mathrm{m}}(\mathbf{k})$$
(9)

where  $Y_{\ell}^{m}(k)$  is a spherical harmonic, function of the component of the vector k. By substituting (9) in (8) and with the definitions  $k^{2} = u$ ,  $k^{\prime 2} = u^{\prime}$ ,  $k_{0}^{2} = s$ , we have:

$$f_{\ell}(u)Y_{\ell}^{m}(k) = \frac{1}{(2\pi)^{3}} \int dt_{0} \frac{v}{\pi}(t_{0}) \int \frac{du'f_{\ell}(u')}{(u'-s_{0})} \int \frac{d^{3}k' \,\delta(k'^{2}-u')\,Y_{\ell}^{m}(k')}{[(k-k')^{2}+t_{0}]} \quad (10)$$

In order to transform Eq. (10) into a more manageable form, we shall use the identity

$$\int \frac{d^{3}k' \,\delta(k'^{2}-u')Y_{\ell}^{m}(k')}{[(k-k')^{2}+t_{0}]} = Y_{\ell}^{m}(k) \,\pi \int_{0}^{1} \frac{x^{\ell} \,dx \,\theta[u'-ux-t_{0}x/(1-x)]}{(1-x)^{1/2} [u'-ux-t_{0}x/(1-x)]^{1/2}}$$
(11)

This identity can be proved in the following manner:

$$\int \frac{\mathrm{d}^{3}\mathbf{k}' \Upsilon_{\ell}^{\mathrm{m}}(\mathbf{k}') \,\delta(\mathbf{k}'^{2} - \mathbf{u}')}{\left[(\mathbf{k} - \mathbf{k}')^{2} + \mathbf{t}_{0}\right]} = \frac{1}{2\pi \mathrm{i}} \lim_{\epsilon \to 0} \left[ \mathrm{I}(\mathbf{u}' + \mathrm{i}\epsilon) - \mathrm{I}(\mathbf{u}' - \mathrm{i}\epsilon) \right]$$

where

$$\begin{split} I(u') &= \int \frac{d^3k' Y_{\ell}^m(k')}{[(k-k')^2 + t_0](k'^2 - u')} = \int_0^1 dx \int \frac{d^3k' Y_{\ell}^m(k')}{[k'^2 - 2k \cdot k' x + t_0 x - u'(1-x)]^2} \\ &= \int_0^1 dx \int \frac{d^3k' Y_{\ell}^m(k')}{[(k'-kx)^2 + ux(1-x) + t_0 x - u'(1-x)]^2} \end{split}$$

By introducing t = k' - kx we have

$$I(u') = \int_{0}^{1} dx \int \frac{d^{3}t Y_{\ell}^{m}(t+kx)}{(t^{2}+A)^{2}} = 4\pi \int_{0}^{1} dx Y_{\ell}^{m}(kx) \int \frac{t^{2} dt}{(t^{2}+A)^{2}}$$
$$= 4\pi Y_{\ell}^{m}(k) \int_{0}^{1} dx x^{\ell} \int \frac{t^{2} dt}{(t^{2}+A)^{2}}$$

because  $Y_{\ell}^{m}(kx)$  is an homogeneous polynomial of degree  $\ell$  in the cartesian components of the vector kx. So we finally obtain

$$I(u') = \pi \cdot Y_{\ell}^{m}(k) \int_{0}^{1} \frac{dx \ x^{\ell}}{\sqrt{ux \ (1-x) + t_{0} \ x - u'(1-x)}}$$

and by evaluating the jump across the cut in u' we have

$$\frac{1}{2\pi i} \lim_{\epsilon \to 0} \left[ I(u'+i\epsilon) - I(u'-i\epsilon) \right] = \pi Y_{\ell}^{m}(k) \int_{0}^{1} \frac{dx \ x^{\ell} \theta[u'-ux - t_{0} x/(1-x)]}{(1-x)^{1/2} \ [u'-ux - t_{0} x/(1-x)]^{1/2}}$$

Finally introducing Eq. (11) into (10) we get

$$f_{\ell}(u, s) = \frac{1}{(2\pi)^3} \int v(t_0) dt_0 \int du' \int_0^1 dx \, x^{\ell} \, \frac{dx \, x^{\ell} \, \theta[u' - ux - t_0 \, x/(1-x)]}{(1-x)^{1/2} \, [u' - ux - t_0 \, x/(1-x)]^{1/2}} \, f_{\ell}(u', s).$$
(12)

If one takes  $\ell = \alpha$ , Eq. (12) coincides with Eq. (4) as we could have expected on the basis of the preceding arguments. Looking at Eq. (12) on a different but essentially equivalent point of view we see that this equation is a good starting point for analytic continuation in the angular momentum variable. This continuation can indeed be carried out for all values of  $\ell$  for which integrations on the different variables converge, i.e. for Re  $\ell > -1$ .

Summarizing all the preceding discussions, we can say that Eq. (4) is the analytic continuation in  $\ell$  of Eq. (12).

#### 3. THE WEAK COUPLING APPROXIMATION

Let us now discuss briefly the simple relation that is obtained in the weak coupling limit. We recall that the kernel of Eq. (4) becomes singular tor  $\alpha = -1$  so that we can have solutions only for  $\alpha > -1$ . Let us try to obtain the function  $\alpha(s)$  in the limit  $\alpha \to -1$ . It is convenient to rewrite the r.h.s. of Eq. (4) in the following way:

$$\int \frac{K(u, u')}{(u'-s)} f_{\alpha}(u', s) du' \approx \int_{0}^{1} dx \cdot x^{\alpha} \int \frac{du'\chi(x, u, u')}{(u'-s)} f_{\alpha}(u', s) du'$$
(13)

$$= \int du' \int_{0}^{1} \frac{d(x^{\alpha+1}-1)}{(\alpha+1)} \frac{\chi(x, u, u')}{(u'-s)} f_{\alpha}(u', s) du'$$

$$= \frac{1}{\alpha+1} \int du' \frac{\chi(o, u, u')}{(u'-s)} f_{\alpha}(u', s) du' + \int du' \int_{0}^{1} dx \frac{(1-x^{\alpha+1})\chi'(x, u, u')}{(\alpha+1)(u'-s)} f_{\alpha}(u', s).$$

Now in the limit  $\alpha + 1 \rightarrow 0$  the first term will dominate (we note that  $\lim_{\alpha + 1 \rightarrow 0} \frac{1 - x^{\alpha + 1}}{\alpha + 1}$ = -ln x which is integrable in the interval (0, 1)). Therefore in this limit our equation becomes:

# EQUATION FOR REGGE TRAJECTORIES

$$f_{\alpha}(u, s) = \frac{1}{(\alpha+1)} \frac{1}{(2\pi)^3} \int v(t_0) dt_0 \int \frac{du'\theta(u')}{\sqrt{u'(u'-s)}} f_{\alpha}(u', s) du'.$$
(14)

Eq. (14) could be simply obtained by taking in Eq. (4) the lower limit on the integration variable  $u': u'+t_0x/(1-x) \rightarrow 0$ . In this reduced equation the two variables u and u' are completely decoupled so that we get the trivial solution  $f_{\alpha}(u, s)$  independent from u. As a consequence the function  $\alpha(s)$  can be easily evaluated: in fact we find that  $\alpha(s)$  is given by

$$\alpha = -1 + \frac{1}{(2\pi)^3} \int v(t_0) dt_0 \int_0^\infty \frac{du'}{\sqrt{u'}(u'-s)}$$
$$= -1 + \frac{i \int v(t_0) dt_0}{8\pi^2 \sqrt{s}}$$

for Yukawa potential  $V(r) = g e^{-\mu r} / r$ . With our notation,  $v(t_0) = (2\pi)^2 \delta(t_0 - \mu^2)$ , so that

$$\alpha = -1 + ig/2\sqrt{s} \quad . \tag{15}$$

This means that  $\alpha \rightarrow -1$  in the weak coupling limit  $g \rightarrow 0$ .

The weak coupling limit can be also calculated by evaluating the asymptotic limit in the variable t of each term of the perturbation series solution Eq. (1) and then summing-up the series.

The asymptotic behaviour of the nth term runs as follows:

$$\frac{1}{n-1} g\left(\frac{ig}{2\sqrt{s}}\right)^{n-1} \frac{(\ln t)^{n-1}}{t}$$

If we sum-up the series we have

$$\frac{g}{t} e^{\frac{ig}{2\sqrt{s}} \ln t} = gt^{\left(1 + \frac{ig}{2\sqrt{s}}\right)}.$$

In this way we also obtain the weak coupling limit Eq. (15).

# 4. THE SPECTRAL METHOD OF SOLUTION

We now turn our attention to the general method of solution of Eq. (4). Obviously this equation could be resolved by the Fredholm method. We prefer to use a different method which has the advantage of giving a convergent expansion for the wave-function, the convergence being due to the fact that higher order terms depend only on small range effects.

It is convenient to introduce the function  $\overline{\phi}(u) = \phi(u)/(u+s)$ ; so we have:

$$(u+s)\overline{\phi}(u) = \frac{g}{(2\pi)^3} \int \overline{v}(t_0) dt_0 \int_0^{\infty} du' \int_0^1 \frac{x^{\alpha} dx}{(1-x)^{1/2}} \frac{\theta[u'-ux-t_0x/(1-x)]\overline{\phi}(u')}{[u'-ux-t_0x/(1-x)]^{1/2}}$$

$$g = \int v(t_0) dt_0$$
(16)

Let us introduce the "ansatz"

$$\overline{\varphi}(u) = \int \frac{\rho(a)}{(u+a)} da$$
(17)

$$(u+s)\int \frac{\rho(a)}{(u+a)} da$$

$$= \frac{g}{(2\pi)^3} \int \overline{v}(t_0) dt_0 \int \rho(a') da' \int_0^1 \frac{dx \, x^{\alpha}}{[a' - ux - t_0 \, x/(1 - x)]^{1/2}}$$
$$= \frac{g}{(2\pi)^3} \int \overline{v}(t_0) dt_0 \int \rho(a') da' \int \frac{d\overline{a} \, \theta[\overline{a}x - a' - t_0 \, x/(1 - x)]}{(1 - x)^{1/2} \, [ax - a' - t_0 \, x/(1 - x)]^{1/2} \, (u + \overline{a})}$$
$$= \int \frac{K(\overline{a}, a') \, \rho(a')}{(u + \overline{a})} \, da' \, d\overline{a}$$

By decomposing  $\frac{1}{(u+s)} \frac{1}{(u+\overline{a})}$  in elementary fractions we obtain

$$\int \frac{\rho(\mathbf{a})}{(\mathbf{u}+\mathbf{a})} d\mathbf{a} = \int \frac{d\mathbf{a}}{(\mathbf{u}+\mathbf{a})} \left[ -\delta(\mathbf{a}-\mathbf{s}) \int \frac{K(\overline{\mathbf{a}},\mathbf{a'})}{(\overline{\mathbf{a}}-\mathbf{s})} \rho(\mathbf{a'}) d\mathbf{a'} d\overline{\mathbf{a}} + \frac{1}{(\mathbf{a}-\mathbf{s})} \int K(\mathbf{a},\mathbf{a'}) \rho(\mathbf{a'}) d\mathbf{a'} \right] \cdot$$
(18)

In order that this equation may be satisfied it is sufficient that  $\rho(a)$  obeys.

$$\rho(a) = N \delta(a - s) + \frac{1}{(a - s)} \int K(a, a') \rho(a') da'$$
 (19)

with

$$N = -\int \frac{K(\bar{a}, a')}{(\bar{a} - s)} \rho(a') da' d\bar{a} . \qquad (20)$$

As a consequence of the definition of N, it follows that

$$\int \rho(\mathbf{a}) \, \mathrm{d}\mathbf{a} = 0 \, . \tag{21}$$

Therefore the equation

$$\overline{\rho}(a) = \delta(a-s) + \frac{1}{(a-s)} \int K(a,a') \overline{\rho}(a') da' \qquad (22)$$

with the condition

$$\int \overline{\rho}(a) da = 0$$
 (23)

is equivalent to the Eqs. (19, 20).

Eq. (22) is of the Volterra type; this property ensures that iterative solutions exist for every value of the coupling constant g. The condition  $D(\alpha, g, s) = \int \rho(a) da = 0$  furnishes an equation for the determination of the eigenvalues.

For a given value of the coupling constant the condition  $D(\alpha, g, s) = 0$ gives an implicit relation between  $\alpha$  and s from which  $\alpha(s)$  can be calculated. Moreover, as a consequence of the property of the kernel K(a, a')for which K = 0 for  $a > (\sqrt{a'} + \sqrt{t'_0})^2$ , we obtain the result that n iterations determine exactly the function  $\rho(a)$  for  $a < (n\sqrt{t'_0} + \sqrt{s})^2$ .

To understand the physical significance of this property better, let us consider the wave function in coordinate space. For the s-state we have

$$\Psi(\mathbf{r}) = \int e^{i\mathbf{k}\cdot\mathbf{r}} f_0(\mathbf{k}^2) d^3\mathbf{k} = \int \rho(\mathbf{a}) d\mathbf{a} \int \frac{\mathbf{j}_0(\mathbf{k}\mathbf{r}) \mathbf{r}^2 d\mathbf{r}}{(\mathbf{k}^2 + \mathbf{a})}$$
$$= \int \rho(\mathbf{a}) d\mathbf{a} \ e^{-\sqrt{\mathbf{a}} \mathbf{r}}.$$

So our method of solution corresponds in the coordinate space to the method of Laplace transform used by MARTIN [2], and BERTOCCHI, CEOLIN and TONIN [3].

We see that the higher the order of the term in the expansion for  $\rho(a)$ , the smaller the radium of the spherical region within which such term can influence significantly the evaluation of the wave function. This shows that the good convergence of the iterative series depends essentially on the short range of the potential V(r).

## 5. THE RELATIVISTIC EQUATION

We wish now to generalize on the preceding discussion and to take the case of the relativistic two body equation. The problem of the asymptotic

## R. STROFFOLINI

limit of the scattering amplitude has been treated in detail in I and follows a path closely parallel to that of the Schroedinger scattering theory [4, 5, 6]. We shall therefore concentrate our attention on the problem of deriving an integral equation for the Regge trajectory, which exhibits explicitly the dependence on  $\alpha$  in a way analogous to Eq. (15). In this case such an equation will also coincide with the one obtained by taking the asymptotic limit of the scattering amplitude. The relativistic equation for a bound state reads:

$$\psi(\mathbf{q}) = \frac{\mathbf{i}}{(2\pi)^4} \int \frac{V(\mathbf{q}, \mathbf{q}')\psi(\mathbf{q}')\,\mathrm{d}^4\,\mathbf{q}'}{[(\Delta + \mathbf{q})^2 - \mu^2]\,[(\Delta - \mathbf{q})^2 - \mu^2]} \tag{24}$$

where

$$V(q, q') = \frac{1}{\pi} \int \frac{v(s_0) ds_0}{[(q - q')^2 - s_0]} .$$
 (25)

 $2\Delta_{\mu}$  is the total four momentum, which, in the centre-of-mass system, has the form (E, 0, 0, 0). Eq. (24) is invariant under the three-dimensional rotation group in the space orthogonal to the four vector  $\Delta_{\mu}$  (i. e. the usual threedimensional space if we stay in the centre-of-mass system). Therefore the wave function will be written by using the irreducible representation of the three-dimensional rotation group, i. e. the usual spherical harmonics, function of the three space components of the vector

$$\overline{\mathbf{q}} = \mathbf{q} - \frac{(\mathbf{q}\Delta)}{\Delta^2} \Delta \tag{26}$$

which is the projection of q in the three-dimensional space orthogonal to  $\Delta$  (the time component vanishes in the centre-of-mass system). We write

$$\Psi(\mathbf{q}) = \Upsilon_{\boldsymbol{\ell}}^{\mathrm{m}}(\bar{\mathbf{q}}) f_{\boldsymbol{\ell}}\left((\Delta + \mathbf{q})^2, \ (\Delta - \mathbf{q})^2\right)$$
(27)

where  $\ell$  and m represent the usual quantum number of the angular momentum. We introduce the definitions  $(\Delta + q)^2 = -u_1$  and  $(\Delta - q)^2 = -u_2$ ; so we have

$$f_{\ell}(u_{1}, u_{2})Y_{\ell}^{m}(\overline{q}) = \int v(s_{0}) ds_{0} \int \frac{du_{1}' du_{2}'}{(u_{1}'^{2} + \mu^{2})(u_{2}'^{2} + \mu^{2})} f(u_{1}', u_{2}') \int \frac{d^{4}q' Y_{\ell}^{m}(\overline{q}')}{[(q - q')^{2} - s_{0})]} \\ \cdot \delta \Big( (\Delta + q')^{2} + u_{1}' \Big) \delta \Big( (\Delta - q')^{2} + u_{2}' \Big) .$$
(28)

Let us now study the integral in the r.h.s.

$$J(u'_{1}, u'_{2}) = \int \frac{d^{4}q' Y_{\ell}^{m}(\tilde{q}')}{[(q - q')^{2} - s_{0}]} \delta[(\Delta + q')^{2} + u'_{1}] \delta[(\Delta - q')^{2} + u'_{2}] .$$
(29)

It can be evaluated by using the identity

EQUATION FOR REGGE TRAJECTORIES

$$J(u'_{1}, u'_{2}) = \lim_{\epsilon_{2} \to 0, \epsilon_{1} \to 0} \left\{ [I(u'_{1} + i\epsilon_{1}, u'_{2} + i\epsilon_{2}) - I(u'_{1} - i\epsilon_{1}, u'_{2} + i\epsilon_{2})] - [I(u'_{1} + i\epsilon_{1}, u'_{2} - i\epsilon_{2}) - I(u'_{1} - i\epsilon_{1}, u'_{2} - i\epsilon_{2})] \right\}$$
(30)

where

$$I(u'_{1}, u'_{2}) = i \int \frac{d^{4}q' Y_{\ell}^{m}(\bar{q}')}{[(q - q')^{2} - s_{0}] [(\Delta + q')^{2} + u'_{1}] [(\Delta - q')^{2} + u'_{2}]}$$
(31)

By parametrizing we have

$$I(u'_{1}, u'_{2}) = \frac{i}{2} \int_{-1}^{1} dz \int \frac{d^{4}q' Y_{\ell}^{m}(\bar{q}')}{[q'^{2} + 2(\Delta \cdot q')z + \Delta^{2} + u'_{z}]^{2} [(q - q')^{2} - s_{0}]}$$
(32)

where

$$u'_{z} = u'_{1}(1+z)/2 + u'_{2}(1-z)/2$$

Let us define  $q + z\Delta = k$  and  $q' + z\Delta = k'$ . We have

$$I(u'_{1}, u'_{2}) = \frac{i}{2} \int_{-1}^{1} dz \int \frac{d^{4}k' Y_{\ell}^{m}(\vec{k}')}{[k'^{2} + \Delta^{2}(1 - z^{2}) + u'_{2}]^{2} [(k - k')^{2} - s_{0}]}$$
(33)

Notice that from the definition (26) it follows that  $\overline{k}' = \overline{q}'$ .

By means of another parametrization we obtain

$$I(u'_{1}, u'_{2}) = i \int_{-1}^{1} dz \int_{0}^{1} dx(1-x) \int \frac{d^{4}k' Y_{\ell}^{m}(\overline{k'} + \overline{k}x)}{\left[k'^{2} + k^{2}x(1-x) - s_{0}x + [u'_{2} + \Delta^{2}(1-z^{2})](1-x)\right]^{3}}$$

$$= -\frac{\pi^2}{2} Y_{\ell}^{m}(\bar{k}) \int_{-1}^{1} dz \int_{0}^{1} \frac{dx x^{\ell}}{[(q+z\Delta)^2 x - s_0 x/(1-x) + u_{z}' + \Delta^2 (1-z^2)]}$$
(34)

We now have to obtain the function  $J(u_1^\prime,u_2^\prime).~$  In order to take this step we rewrite  $I(u_1^\prime,u_2^\prime)$  in the form

$$I(u'_{1}, u'_{2}) = \frac{\pi^{2}}{2} Y_{\ell}^{m}(\overline{q}) \int_{-1}^{1} dz \int_{0}^{1} \frac{dx x^{\ell}}{[A_{z} - \Delta^{2}(1 - z^{2})(1 - x)]}$$
(35)

where

$$A_{z} = A_{1}(1+z)/2 + A_{2}(1-z)/2,$$

$$A_{1} = -u_{1}' - (q+\Delta)^{2}x + s_{0}x/(1-x) = -u_{1}' + u_{1}x + s_{0}x/(1-x),$$

$$A_{2} = -u_{2}' - (q-\Delta)^{2}x + s_{0}x/(1-x) = -u_{2}' + u_{2}x + s_{0}x/(1-x),$$
(36)

and we use the identity

.

$$\int_{-1}^{1} \frac{dz}{[A_{z} - \Delta^{2}(1 - x)(1 - z^{2})]} = \frac{2}{\pi} \int \frac{d^{2}p}{[(\Delta' + p)^{2} + A_{1}][(\Delta' - p)^{2} + A_{2}]}$$
(37)

where  $\Delta'^2 = -\Delta^2(1-x)$ . P is a vector in a pseudo-Euclidian bi-dimensional space.

This identity can be easily checked by using a Feynman parametrization in order to pass from the right-hand side to the left-hand side of Eq. (37). So we find that

$$J(u'_{1}, u'_{2}) = \pi Y^{m}_{\ell}(\vec{q}) \int_{0}^{1} dx \ x^{\ell} \int d^{2} p \ \delta[(\Delta' + p)^{2} + A_{1}] \delta[(\Delta' - p)^{2} + A_{2}] .$$
(38)

By substituting this expression by Eq. (28) we obtain the equation for the function  $f_{\ell}(u_1 u_2)$ :

$$f_{\ell}(u_1, u_2) = \int \frac{du'_1 du'_2 f(u_1', u_2')}{(u'_1^2 + \mu^2)(u'_2^2 + \mu^2)} K(u_1, u_2, u'_1, u'_2)$$
(39)

where

$$K(u_1, u_2, u_1', u_2') = \frac{1}{(2\pi)^4} \int v(s_0) ds_0 \int_0^1 x^\ell dx \int d^2 p \, \delta[(\Delta' + p)^2 + u_1 x + s_0 x/(1 - x) - u_1']$$

$$\delta[(\Delta' - p)^2 + u_2 x + s_0 x / (1 - x) - u_2'].$$
(40)

If one takes  $l = \alpha$  Eq. (39, 40) coincides with Eqs. (18, 19) of I. One may therefore easily verify that

$$\int d^{2}q \, \delta[(\Delta' + p)^{2} + u_{1}x + s_{0}x/(1 - x) - u_{1}'] \, \delta[(\Delta' - p)^{2} + u_{2}x + s_{0}x/(1 - x) - u_{2}']$$

$$= \frac{1}{4} \frac{\theta[H(z, z_{1}, z_{2})]}{H(z, z_{1}, z_{2})}$$
(41)

where the quantities which appear in the r.h.s. are defined by formulas (14, 15) of I. This, as in the case of potential scattering, confirms the con-

sistency of the whole procedure since the equation found in I determining the exponent  $\alpha$  coincides for  $\alpha = \ell$  with the exact equation for bound states.

# 6. THE WEAK COUPLING LIMIT FOR THE RELATIVISTIC EQUATION

Let us now discuss the behaviour of the function  $\alpha(t)$  for  $\alpha(t)$  very near to -1. We rewrite the kernel (40) in the form

$$K(u_{1}, u_{2}, u_{1}', u_{2}') = \int_{0}^{1} dx \ x^{\alpha} X(x, u_{1}, u_{2}, u_{1}', u_{2}')$$
$$= \frac{1}{(\alpha+1)} X(0, u_{1}, u_{2}, u_{1}', u_{2}') - \int_{0}^{1} \frac{x^{\alpha+1} - 1}{(\alpha+1)} \ \frac{d X(x, u_{1}, u_{2}, u_{1}', u_{2}')}{dx}$$
(42)

which explicitly shows that in the limit  $\alpha + 1 \rightarrow 0$  the dominant term is the first one on the r.h.s.

Actually,

$$X(0, u_1, u_2, u_1', u_2') = \frac{1}{(2\pi)^4} \int ds_0 v(s_0) \int \delta \left( (\Delta + p)^2 - u_1' \right) \cdot \delta \left( (\Delta - p)^2 - u_2' \right) d^2 p$$

so that the kernel is independent from  $u_1$  and  $u_2$ . Therefore in the limit  $\alpha + 1 \rightarrow 0$ , the solution  $f_{\alpha}$  of the equation (39) does not depend on  $u_1$  and  $u_2$ . The function  $\alpha(t)$  can be explicitly evaluated.

Now we have

$$\alpha(t) + 1 = \frac{1}{(2\pi)^4} \int v(s_0) ds_0 \int \frac{d^2 p}{[(\Delta + p)^2 + \mu^2] [(\Delta - p)^2 + \mu^2]}$$
$$= \frac{G^2}{16\pi^2 \mu^2} F\left(\frac{t}{4\mu^2}\right)$$
(43)

where  $4\Delta^2 = -t$  and where

$$\mathbf{G}^2 = \frac{1}{\pi} \int \mathbf{v}(\mathbf{s}_0) \, \mathrm{d}\mathbf{s}_0 \tag{44}$$

and

$$F\left(\frac{t}{4\mu^2}\right) = 2\mu^2 \int_{4\mu^2}^{\infty} \frac{dt'}{[t'(t'-4\mu^2)]^{1/2}(t'-t-i\eta)}$$
(45)

This shows that  $\alpha \rightarrow -1$  in the weak coupling limit  $G^2 \rightarrow 0$ . From Eq. (45) one sees that for  $t < 4\mu^2$ ,  $F(t/4\mu^2)$  is real and is explicitly given by

F(a) = [(1-a) 
$$|a|$$
]<sup>-1/2</sup> ln [(1-a)<sup>1/2</sup> + ( $|a|$ )<sup>1/2</sup>]; a = t/4 $\mu^2$ .

For t > 4 $\mu^2$ , F(t/4 $\mu^2$ ) becomes complex as it had to be expected on general grounds.

We now wish to verify that the expression we have found for  $\alpha(t)$ , by solving the asymptotic integral equation, coincides in the weak coupling limit with the expression which can be obtained by evaluating the asymptotic limit  $s \rightarrow \infty$  of each term of the perturbative series for the absorptive part of the scattering amplitude in the s channel and then summing the series. In fact the first iteration in Eq. (7) of I yields, by disregarding  $u_1, u_2, u'_1, u'_2, t, s_0$  as compared to s:

$$\varphi_1(s, u_1, u_2, t) = \frac{1}{16 \pi \mu^2 s} F(x, u_1, u_2, t)$$

where

$$F(x, u_1, u_2, t) = \frac{\mu^2}{\pi^3} \int v(s_0) ds_0 \int v(s'_0) ds'_0 \int \frac{d^2 p}{[(\Delta + p)^2 + u_1 x + s_0 x/(1 - x) + \mu^2]}$$
$$\cdot \frac{1}{[(\Delta - p)^2 + u_2 x + s_0 x/(1 - x) + \mu^2]}$$

and  $x = s_0'/s$ . In the limit  $s \to \infty$ , we have

$$F(x, u_1, u_2, t) \Longrightarrow F\left(\frac{t}{4\mu^2}\right) \cdot \frac{G^4}{16 \pi \mu^2 s}$$

By proceeding in analogous way we have for  $s \rightarrow \infty$  (for detail see [8])

$$\phi_{n}(s, u_{1}, u_{2}, t) \rightarrow \phi_{n}^{a}(s, t)$$

$$= \frac{G^{4} F\left(\frac{t}{4\mu^{2}}\right)}{16 \pi \mu^{2} s} \left[ \frac{G^{2} F\left(\frac{t}{4\mu^{2}}\right) \ln s}{(n-1)!} \right]^{n-1}$$

Notice that each term of the perturbative series has in the asymptotic limit a classical diffraction peak, i.e. a factorized s and t dependence. As the diffraction peak of the nth term is  $[F(t/4\mu^2)]^n$ , it is narrower than the preceding one.

Moveover the weight of the contribution of the nth order term increases with the energy as  $(In s)^{n-1}$ . As a consequence of the appearance of many different contributions to the absorptive amplitude, which have narrower and narrower diffraction patterns, the total diffraction peak will have a continuous shrinking as the energy increases. The asymptotic behaviour of the absorption amplitude can be explicitely evaluated by summing the series  $\Sigma \phi_n^a$ . We obtain

$$\varphi^{a}(s,t) = \pi G^{2} \lambda(t) s^{\lambda(t)-1}$$

where

$$\lambda(t) = \frac{G^2 F(t/4\mu^2)}{16 \pi^2 \mu^2} .$$

Thus we obtain when  $\alpha = \lambda(t)-1$ , the expression (43). The weak coupling limit discussed here is very analogous to the results obtained independently by LEE and SAWYER [9], LEVY [10], GELL-MANN and GOLBERGER [11], ARBUSOV et al. [12].

In the case of strong coupling a spectral method analogous to the one introduced for the Schroedinger equation can also be used in this case by means of the "ansatz"

$$f_{\alpha}(u_1, u_2) = \int \frac{\rho(a, b)}{(u_1 + a) (u_2 + b)} dadb.$$

This leads to an integral equation for  $\rho(a, b)$  whose kernel exhibits spectral properties similar to the one of potential scattering (Eq. 22). However the two-dimensional nature of this equation makes all attempts of practical evaluation hard and cumbersome (for detail see [7]).

In conclusion, we have seen that the general method developed for the Schroedinger equation has been easily adapted to the relativistic case. We think that further investigation along those lines can lead to a very useful tool for the understanding of the Regge trajectories in many cases of physical interest.

#### REFERENCES

- [1] JOST, R. and PAIS, A., Phys. Rev., 82 (1951) 840.
- [2] MARTIN, A., Nuovo Cim., 14 (1950) 403; 15 (1960) 99.
- [3] BERTOCCHI, L., COELIN, C. and TONIN, M., Nuovo Cim. 18 (1960) 770.
- [4] AMATI, D., FUBINI, S., STANGHELLINI, A. and TONIN, M., Nuovo Cim. 22 (1961) 596.
- [5] AMATI, D., FUBINI, S. and STANGHELLINI, A., Nuovo Cimento (in press) and Physics Letters 1 (1962) 29.
- [6] CEOLIN, C., DUIMIO, F. and FUBINI, S. and STROFFOLINI, R., Nuovo Cim. (in press).
- [7] BERTOCCHI, L., FUBINI, S. and TONIN, M., Nuovo Cim. (1962).
- [8] BERTOCCHI, L., PREDAZZI, E., TONIN, M. and STANGHELLINI, A., Nuovo Cim. (in press).
- [9] LEE, B. W. and SAWYER, R. F., Institute for Advanced Study Princeton, preprint.
- [10] LEVY, M., Stanford University, preprint.
- [11] GELL-MANN, M. and GOLDBERGER, M. L., Phys. Rev. Letters 9 (1962) 275.
- [12] ARBUSOV, B. A., LOGUNOV, A. A., TAVKELIDZE, A. N. and FAUSTOV, R. N., Phys. Rev. Letters 2 (1962) 150.

# HIGH-ENERGY PROPERTIES OF THE MANDELSTAM REPRESENTATION

M. FROISSART CENTRE D'ÉTUDES NUCLÉAIRES, SACLAY, FRANCE

# 1. INTRODUCTION

The general theme of this paper is fairly difficult to define, as it is made up of bits and pieces of information which do not appear to have much in common. This illustrates fairly well the present situation in this branch of physics called, until recently, physics of elementary particles. In consequence, this is an attempt to develop a picture of the evolution of the ideas as they developed during the past two years. The order of presentation is therefore more or less historical though not rigidly so. There will be no attempt to give a complete picture of the present situation, as this would be very dull; an up-to-date report will appear in the Proceedings of the 1962 High-Energy Conference held in Geneva and the later issues of the standard reviews.

Having thus set up the plan of this paper as being the historical order, an exception will be made of the comparison between theory and experiment which will be postponed to the end. The general scheme is therefore as follows: There will be first a summary of the situation of the theory before the introduction of Regge poles. Then there will be an explanation of how it came to a deadlock, with more and more paradoxes developing the impossibility of having stable particles of spin greater than one and the impossibility of having cross-sections going to constants at infinite energy.

At that point, Chew had the idea of generalizing certain features of Regge's work on potential theory to the relativistic theory based upon the Mandelstam representation. Then all these paradoxes vanished.

Having thus formulated the basic hypothesis of Chew, which had practically no logical support when it was first proposed, we shall discuss the pros and cons from a mere theoretical standpoint, and finally, we shall examine the predictions and the experimental verifications. By that time, the logical weaknesses of the theory should have been sufficiently exposed and the reader left wondering why the meager experimental results seem to confirm it so well.

## 2. THE PRE-REGGE DEADLOCK [1]

#### 2.1. Pre-Regge postulates

In the summer of 1960 the last word in elementary particle physics was the Mandelstam representation. It is assumed that it is thoroughly familiar, however, it will be briefly gone over, if only to define the notation. To simplify matters, only spin zero particles and practically always equally massive particles, the mass being taken as unit of mass will be considered. Furthermore h = 1, c = 1. The basic assumptions are as follows:

(a) The invariant amplitudes for scattering or production, expressed as functions of the external momenta, possess the same analyticity properties as the formal sum of corresponding Feynman graphs, where all possible intermediate particles are taken into account, regardless of whether they are elementary or composite. This is at least true on one sheet (the "physical" sheet) of the Riemann surface thus defined (Landau).

(b) The discontinuities of the amplitudes across the cuts are given as certain (non-linear) functionals of the amplitudes which generalize the physical unitarity condition (Cutkosky).

(c) In addition, the amplitudes behave at infinite values of the external momentum variables no worse than polynomials, at least in the physical sheet (Mandelstam).

These three assumptions are implied by the Mandelstam representation. Consider a world where there is only one type of particle. The scattering of two particles with momenta  $p_1$  and  $p_2$  into particles with momenta  $-p_3$  and  $-p_4$  depends only upon two variables

$$s = (p_1 + p_2)^2,$$
  

$$t = (p_1 + p_3)^2, \qquad s + t + u = 4,$$
  

$$u = (p_1 + p_4)^2, \qquad p_k^2 = 1.$$

The scattering amplitude then, as a function of s, t, u, is analytic, (a), except for cuts at s, t or u real, greater than 4. The jump across the cut is then directly given by unitarity, without any Cutkosky generalization, (b), and the amplitude behaves at infinity in the (s, t, u) space at worst like a polynomial, (c).

We shall write the Mandelstam representation

$$A(s, t, u) = \frac{1}{\pi^2} s^N t^N \int \int \frac{\rho(s', t') ds' dt'}{s^N t'^N (s'-s)(t'-t)} + P_{s, t, u} + \sum_{p=0}^{M} t^p \frac{s^M}{\pi} \int \frac{\rho_p(s') ds'}{s'^M (s'-s)} + P_{s, t, u} + \sum_{p, q}^{L} t^p s^q \rho_{p, q}.$$
 (1)

where L, M, N are some integers and  $P_{s,t,u}$  indicates the two terms obtained from the term written just before by circular permutation of (s, t, u).

We shall call  $\rho(s,t)$  double spectral function,  $\rho_p(s)$  single spectral function of degree p and  $\rho_{p,q}$  coefficient of the residual polynomial of degree p+q. (Strictly speaking these "functions" may be distributions). All these quantities are linked together, and to the production amplitudes by an infinite set of non-linear equations which express the unitarity requirement.

If we consider a theory with a more complicated spectrum, i.e. several types of particles, the number of independent amplitudes becomes very large, also the number of spectral functions and even the number of terms may increase in order to include contributions from complex singularities. However, we shall reason only on the very simplified case just mentioned and shall hope that all we say generalizes to more complicated cases. In par-

ticular, we shall not consider the question of whether there are complex singularities or not, it being understood that complex singularities are only supposed to bring in more terms. It is often believed that the representation (1) and the representations of an analogous type for the other amplitudes, combined with Cutkosky's rules which insure unitarity, are sufficient basis for a dynamical theory.

#### 2.2. First troubles

Of course, it is very difficult, even with fast computers, to solve these equations expressing axioms a, b and c even without insisting on quantitative predictions. The tendency until 1960 had been to try and do everything feasible by using only functions of one variable: the double spectral functions were neglected altogether (Cini, Fubini), and approximate systems of equations involving single spectral functions of lowest degree were solved (Chew, Mandelstam). By the summer of 1960, all calculations on this scheme had been carried out, at least all of those which did not lead to divergences. Mandelstam<sup>\*</sup>has described how this procedure worked and how one ran into great difficulty as soon as P-wave resonances entered the game. Thus people started to contemplate the double spectral function, thinking that they might help somehow. The ideas that they had at that time were fairly simple and they thought that a fair model of what a double spectral function might look like were, for example, the double spectral functions as they appear in Feynman graphs, quite smooth and without much of a structure.

Something did not seem to fit into this picture very well, however, and that was the occurrence of stable particles with high spins. Indeed, whenever there is a particle with spin j and mass m, some of the amplitudes have a pole of the form  $P_j(\cos\theta)/(s-m^2)$  where s is an energy squared variable and  $\cos\theta$  is in general proportional to a momentum transfer squared. Such a pole, therefore, fits in the Mandelstam representation (1), under the condition that M be not less than j. This indicates that (1) is valid only with subtractions at least up to a degree equal to the highest spin of the stable intermediate particles. One might then wonder if this could not lead to very large cross-sections at high energy, increasing polynomially with the energy. But, of course, the experiments, even with the most energetic cosmic rays, indicate that it is not so, that the cross-sections behave in a way consistent with a constant within the experimental accuracy.

The model one had at that time for scattering at high energy was that suggested by Pomeranchuk: whenever the particles interact, they have no chance of recombining to scatter elastically because of the competition of the many inelastic channels open at high energy. Therefore, the scattering amplitude becomes purely absorptive and the elastic scattering is simply diffraction scattering. At very high energies, this diffraction can be treated classically, given an absorption coefficient which reflects the distribution of matter in the clouds of virtual particles. One then gets a constant crosssection, a constant diffraction scattering peak (measured in momentum transfer) in the elastic amplitude and a constant elastic cross-section. This picture was more or less in agreement with the experiments which were not very precise and which, for some reason, were in general fitted by comparison with classical diffraction scattering by a uniformly gray disc, or sphere, but never with a more sophisticated distribution of the absorption coefficient.

# 2.3. Gribov's paradox

This picture was shown by Gribov to be inconsistent with the Mandelstam representation. The essence of Gribov's idea was the following: the mathematical expression of Pomeranchuk's model is that the amplitude A(s, t) in the physical region, for large positive s and small negative t has the asymptotic form

$$A(s,t) \approx sf(t)$$
 (2)

where f(t) determines the shape of the diffraction pattern, and the factor s is there to cancel kinematic coefficients in order to have a constant total cross-section. f(t) in these circumstances has the t-cut of the Mandelstam representation, and we assume that this asymptotic form is valid for t positive, at least up to some value greater than 4. Unitarity in the t channel reads, according to Mandelstam:

$$\rho(\mathbf{s}, \mathbf{t}) = \frac{1}{\pi \sqrt{\mathbf{t}}} \int \int \frac{\mathrm{d}\mathbf{s}_1 \mathrm{d}\mathbf{s}_2 \, \mathbf{A}_{\mathbf{s}} \, (\mathbf{s}_1, \mathbf{t}) \mathbf{A}_{\mathbf{s}}^{*}(\mathbf{s}_2, \mathbf{t})}{\mathrm{K}^{1/2}(\mathbf{t}; \, \mathbf{s}, \, \mathbf{s}_1, \, \mathbf{s}_2)} \, \theta(\mathrm{K}), \tag{3}$$

$$K(t; s, s_1, s_2) = (t-4)(s^2 + s_1^2 + s_2^2 - 2s_1s_2 - 2s_1 - 2s_2) - 4s_1s_2.$$
(4)

 $A_s$  is the absorptive part of A with respect to the s-channel. Substituting the asymptotic form (2), taking into account that f(t) is purely imaginary,  $iA_s(s, t) = A(s, t)$ :

$$\rho(s,t) = \frac{1}{\pi\sqrt{t}} \int \int \frac{ds_1 ds_2 s_1 s_2 |f(t)|^2}{K^{1/2}(t; s, s_1, s_2)} \theta(K) .$$
 (5)

Change variables, putting  $s_1 = u_1 s^{1/2}$ ,  $s_2 = u_2 s^{1/2}$ :

$$\dot{\rho}(\mathbf{s},t) = \frac{1}{\pi\sqrt{t}} \int \int \frac{s^2 du_1 du_2 u_1 u_2 |f(t)|^2}{(t-4)[s^2 + 0(s^{3/2}) - 4s^2 u_1 u_2]^{1/2}} \theta(\mathbf{K}).$$
(6)

The integration takes place in the hyperbola  $u_1 u_2 < (t-4)/4$ , asymptotically and diverges logarithmically, except that the term  $0 (s^{3/2})$ cuts off the integration, thereby introducing a ln sterm which we cannot calculate exactly as it involves non-asymptotic regions. We therefore have the behaviour  $\rho(s, t) \approx \text{const.s In } s|f(t)|^2$ . This is incompatible with  $\rho(s, t) = s \cdot \text{Im } f(t)$ . We shall see later in a much more transparent fashion the deep reasons for this paradox.

One may generalize this reasoning and one finds that the paradox takes place for any asymptotic behaviour of the form

$$A(s,t) = s^{\alpha} \ell n^{\beta}(s) \cdot f(t), \qquad (7)$$

if  $\alpha$  is real and Re  $\beta \ge -1$ .

Gribov's suggestion was to take  $\alpha = 1$ ,  $\beta < -1$ , but this is already a little difficult as it implies cross-sections which go to zero at infinity and again, the problem of how to accommodate particles with large spins stays there as the behaviour of the poles corresponds to  $\alpha$  real and  $\beta = 0$ .

2.4. Bound on the asymptotic behaviour in the physical region

Another difficulty arose in this connection, when the author proved that the cross-sections in the framework of the Mandelstam representation cannot increase faster at infinity than  $ln^2s$ . The intuitive basis for the theorem is the following: consider the Pomeranchuk model classically. We may very well suppose that the absorption coefficient changes with energy. However, the distribution of matter in the cloud of virtual particles falls off essentially exponentially, the range of the exponential being given by the mass of the lightest virtual particle. Therefore, all we may expect is an absorption coefficient of the form  $g e^{-K_1}$  where g may vary with energy. If the impact parameter b of a collision is such that  $g e^{-Kb} <<1$ , there is practically no effect. If  $g e^{-Kb} >>1$  there is complete absorption. The cross-section is determined then by the value a of the impact parameter so that  $g e^{-Ka} \cong 1$  or  $a \cong (1/K) \ln g$ ,  $\sigma_{tot} \cong \pi/K^2 (\ln g)^2$ . Even if we assume that g grows polynomially with the energy, a increases only logarithmically and the cross-section

A very elegant derivation of this theorem was given by Martin. The only assumption of Martin is that the Legendre polynomial expansion of the amplitude converges for  $s > s_0$  up to some positive value  $t_0$  of t and that, at that value of t, the asymptotic behaviour of A(s, t) is polynomial in s. This is automatically guaranteed by the Mandelstam representation[1]. The reasoning of Martin uses the fact that the imaginary part of partial waves is positive and bounded and that the Legendre polynomials  $P_l(z)$  are positive increasing functions of  $\ell$  for z real, z > 1. Let us then write the absorption part of the amplitude:

$$A_{s}(s,t) \approx \sum_{\ell} (2\ell+1) \operatorname{Im} a_{\ell}(s) P_{\ell}(\cos \theta), \qquad (8)$$

$$0 \leq \operatorname{Im} \mathbf{a}_{\theta}(\mathbf{s}) \leq \sqrt{\mathbf{s}(\mathbf{s}-4)} , \quad \cos \theta = 1 + 2t/(\mathbf{s}-4). \tag{9}$$

If one wants to maximize  $A_s(s, 0)$ , holding  $A_s(s, t_0)$  fixed, as  $P_{\ell}[1+2t_0/(s-4)]$  is an increasing function of  $\ell$  whereas  $P_{\ell}(1) = 1$ , one has to take as small values of  $\ell$  as possible:

Im 
$$a_{\ell}(s) = 0$$
,  $\ell > \ell_0$  and Im  $a_{\ell}(s) = \sqrt{s/(s-4)}$ ;  $\ell < \ell_0$ 

The value of  $\ell_0$  is determined by

$$A_{s}(s, t_{0}) = \sum_{\ell=0}^{\nu_{0}} (2\ell+1)\sqrt{s/(s-4)} P_{\ell} [1+2t_{0}/(s-4)] = \sqrt{s/(s-4)} (P_{\ell_{0}+1}' + P_{\ell_{0}}').$$

Assume that  $A_s(s, t_0) \approx C s^{\alpha}$ ; Re  $\alpha > 1$ . We use the following estimate

$$P_{\ell_0} [1 + 2t_0/(s-4)] \approx I_0(\ell_0\sqrt{2t_0/(s-4)}) \qquad (modified Bessel function)$$

and therefore:

$$P'_{\ell_0} = \ell_0 \sqrt{(s-4)/8t_0} I_1(\ell_0 \sqrt{2t_0/(s-4)}).$$

We thus determine  $\ell_0$  by

$$\ell_0 \sqrt{(s-4)/2 t_0} I_1(\ell_0 \sqrt{2 t_0/(s-4)} = C s^{\alpha}.$$

For Re  $\alpha > 1$  the solution of this equation is asymptotically for large s:

$$\ell_0 \sqrt{2 t_0/(s-4)} \approx (\alpha - 1) \ln s$$
 or:  $\ell_0 \approx (\alpha - 1) s^{1/2} \ln(s/\sqrt{2 t_0})$ 

which gives

$$A(s, 0) = \sum_{0}^{\ell_{0}} (2\ell + 1) \sqrt{s/(s-4)} \approx \ell_{0}^{2} \approx (\alpha - 1)^{2} \operatorname{sln}^{2}(s/2t_{0})$$

This corresponds to the classical picture given above. One may also estimate the asymptotic behaviour of the amplitude in the physical region, either at fixed momentum transfer, or at fixed angle. The results are the following:

$$|A(s,t)| < M(t) s \ln^2 s$$
,  $t < 0$ , (10a)

$$|A(s, (s-4)(\cos \theta - 1))| < N(\theta) s^{3/4} \ln^{3/2} s \quad 0 < \theta < \pi.$$
(10b)

## 2.5. Independence of the single spectral functions

An interesting question about the Mandelstam representation was, besides how many subtractions are to be made, whether or not it was possible to change the content of the theory by making more subtractions. This was very interesting particularly in view of the well-known CDD ambiguity which arises when one tries to enforce unitarity on the single spectral functions. It could be that by making more and more subtractions, it becomes possible to introduce more and more particles with higher and higher spins into the theory by introducing CDD poles, just as one may introduce more and more complex terms into a Lagrangian.

However, it is possible to show, using conditions (10), that this is not so. The single spectral functions of degree greater than one and all coefficients of the residual polynomial of degree greater than zero are completely determined by the double spectral function and the conditions (10).

To make this clear, suppose that there are two different amplitudes with the same double spectral function. Their difference satisfies (10), and is expressed by

$$\Delta A(s,t,u) = \sum_{p=0}^{M} t^{p} \frac{s^{M}}{\pi} \int \frac{\sigma_{p}(s')ds'}{(s'-s)(s'M)} + P_{s,t,u} + \sum_{p,q=0}^{L} t^{p} s^{q} \sigma_{p,q}, \quad (11)$$

#### HIGH-ENERGY PROPERTIES

The idea is now to prove that all of these terms must obey conditions (10) individually, in other words that there cannot be any cancellation between different terms. The details of the proof will not be discussed but the principles will be outlined.

Let us take different directions in the (s,t,u) plane, corresponding to different values of the angle  $\theta$ , and show that an expression like (11) cannot satisfy (10b) for 3M + 2L + 1 different values of  $\cos \theta$  unless

 $\sigma_{p}(s) < C s^{-p+3/4} \ln^{3/2} s$ ,

$$\sigma_{p,q} = 0$$
 except  $\sigma_{00}$ .

In that case, if s is fixed negative and t variable, the largest term which contributes to the asymptotic behaviour is

$$t^{M}\int \frac{\sigma_{M}(s')ds'}{(s'-s)}$$
.

This term violates (10a) if M > 1 and therefore must vanish on the negative real s axis and therefore it vanishes everywhere by analytic continuation. Thus we prove that M = 1 and L = 0 in Eq. (11).

In the same way, this method allows in principle to compute the single spectral functions of degree greater than one from the double spectral function. This is in practice very difficult to carry out because of the analytic continuation mentioned above.

Even in principle, it appears very difficult to prove that the partial waves obtained by this method will satisfy unitarity. At any rate, it is sufficiently demonstrated that particles with spin greater than one cannot be elementary, in the sense that one cannot introduce arbitrary CDD poles for the higher waves (j > 1) as there is no N/D equation in that case. In that sense, we shall say that all particles with spin greater than one are "dyna mical". It is interesting to note that the condition just obtained looks quite similar to the old-fashion "renormalizability" condition. The connection may be deeper in the sense that these two conditions both reflect the fact that, unless very pecular cancellations take place, unitarity is strongly violated at high energies if one introduces a priori high spin particles into the theory. Now the paradox is complete: we have proved that the behaviour of the amplitude in the physical regions is completely different from that in the "spectral" regions: we have an upper bound in the physical region due to unitarity and we have a larger lower bound in the unphysical region as a result of poles of particles with high spin.

How this be reconciled with the analyticity properties ? There must be some kind of oscillation of the amplitude in the spectral region so that the dispersion integrals expressing the amplitude in the physical region do not in fact behave at all like their integrands, but increase more slowly as a result of cancellations inside the integral.

A very natural kind of function with just such a behaviour is, for example:

$$A(s,t) = \beta(t)s^{\alpha(t)}$$
(12)

where  $\alpha(t)$  is real, less than one for t < 0, has a cut for t > 4 and is such that Re  $\alpha(t)$  stays bounded in the cut plane;  $\beta(t)$  is any function satisfying a dispersion relation in the cut t plane.

Such a function would indeed resolve all paradoxes above. It may be shown (the reader may do this as an exercise) that the Gribov paradox, although holding for any real  $\alpha$ , ceases to hold as soon as  $\alpha$  is non-real. This is precisely the result of the cancellations introduced by the oscillations described by phase  $[s^{\alpha(t)}] = \text{Im } \alpha \text{ Ins.}$ 

### 3. THE INTRODUCTION OF REGGE POLES

Having been compelled to consider amplitudes of the form (12) Chew looked around and found Regge's paper [2] which predicted an asymptotic behaviour of precisely this form in potential scattering with a momentum transfer and t energy variable. This coincidence was very striking because, to reach the form (12) from the relativistic theory, we constantly used the crossing symmetry or equivalently, unitarity in all three channels, which is very specific of the relativistic theory. On the other hand, one may argue that after all it is not so surprising, as the unitarity equation reads much the same for potential scattering and for the elastic regions of the relativistic problem. At any rate, it was very intriguing, and still is, to see whether or not the Regge poles have a logical place in the framework of S-matrix theory. It is very difficult now to expose as nobody yet has produced anything very convincing.

Let me start by describing the way people agree to choose the "best" interpolation.

## 3.1. Definition of the partial-wave amplitude

In the theory of scattering by superposition of Yukawa potentials (see Regge's lectures<sup>\*</sup>) the amplitude  $a(\ell, q^2)$  has the following properties. It is holomorphic for Re  $\ell > N$ . It decreases exponentially with Re  $\ell$  and increases at most like a polynomial with Im  $\ell$ . Furthermore, it is unitary all along the real axis, even for non-integer points.

If we start from the Mandelstam representation, we have a dispersion relation in  $\cos \theta$ , which we can write as

$$A(q^2, \cos \theta) = \frac{1}{\pi} \cos^N \theta \int \frac{A_t(q^2, x) dx}{x^N(x - \cos \theta)} + Polynomial.$$
(13)

We have expressed now the amplitude A and its absorptive parts in  $\cos \theta$ , A<sub>t</sub>, in terms of  $q^2 = (s-4)/4$  and  $\cos \theta = 1 + 2t/(s-4)$ .

The integral extends on the real axis, somewhere outside of [-1, +1]. The partial wave computed from this is, for integer  $\ell$ :

$$\mathbf{Q}_{\boldsymbol{\ell}}(\mathbf{q}^2) = \frac{1}{2\pi} \int_{-1}^{+1} \mathrm{d}\cos\theta \ \mathbf{P}_{\boldsymbol{\ell}} \ (\cos\theta) \left[ \cos^{\mathbf{N}}\theta \int \frac{\mathrm{A}_{\mathbf{t}}(\mathbf{q}^2, \mathbf{x})\mathrm{d}\mathbf{x}}{\mathbf{x}^{\mathbf{N}}(\mathbf{x}-\cos\theta)} + \mathbf{P}_{\mathbf{N}-1} \ (\cos\theta) \right]. \tag{14}$$

Let us compute the partial waves for  $\ell > N$  only.

<sup>\*</sup> These proceedings.

Then we can integrate:

$$\frac{1}{2} \int_{-1}^{+1} \operatorname{dcos} \theta \, \operatorname{P}_{\boldsymbol{\ell}} \left( \cos \theta \right) \, \frac{\cos^{N} \theta}{x^{N} (x - \cos \theta)} = \operatorname{Q}_{\boldsymbol{\ell}} (x) \tag{15}$$

where  $Q_{\ell}(x)$  is that Legendre function of second kind which is real for x > 1 real and we get

$$a_{g}(q^{2}) = \frac{1}{\pi} \int dx A_{t}(q^{2}, x)Q_{g}(x).$$
 (16)

This integral converges, as  $Q_{\rho}(x) \approx 1/(2x)^{\ell+1}$  for large x.

We note at this point that equations (14) and (16) are equivalent only for integer values of  $\ell$ . Furthermore we note that in general  $A_t(q^2, x)$  gives contributions to (16) from the side of x > 1 and from the side of x < -1, thereby introducing terms which behave like  $e^{i\pi \ell}$  which do not satisfy our conditions. If however, we introduce the following functions:

$$a_{\ell}^{\pm}(q^2) = \frac{1}{\pi} \int_{x>1} dx \ Q_{\ell}(x) \left[ A_t(q^2, x) \pm A_t(q^2, -x) \right], \tag{17}$$

we get an asymptotic behaviour in the half-plane Re l > N which is exactly what we want: polynomial at most in Im l, and exponentially decaying with Re l. The physical  $a_l$  is equal to either  $a^+$  or  $a^-$  according to whether l is even or odd.

Now we may introduce a theorem [3] which is very useful for the following.

#### 3.2. Carlson's Theorem

Let f(z) be regular and of the form  $0(e^{a |\operatorname{Im} z| + \beta \operatorname{Re} z})$  in Re  $z \ge 0$ ,  $\alpha$  and  $\beta$  real,  $\alpha < \pi$ ; let f(z) = 0 for z = 0, 1, 2... Then f(z) is identically zero.

Proof

We can write the Cauchy theorem (Re z > 0) for the regular function:

$$\frac{e^{\lambda z}f(z)}{\sin \pi z} = -\frac{1}{2\pi i} \int_{0-i\infty}^{0+i\infty} \frac{e^{\lambda z} f(x) dx}{\sin \pi x (x-z)} \text{ for } : \frac{|\operatorname{Im} \lambda| < \pi - \alpha}{\operatorname{Re} \lambda < -\beta}.$$

Both sides are analytic in the whole strip  $|\text{Im }\lambda| < \pi - \alpha$  and the equation holds there. But on the real  $\lambda$  axis, the right hand side is bounded; the left-hand side can only be bounded if f(z) = 0.

It is clear that this theorem guarantees the uniqueness of the interpolation  $a_{\ell}^{\pm}$  that we have defined, which satisfies very comfortably the conditions of the theorem, as  $Q_{\ell}(x) \approx 1/(x + \sqrt{x^2 - 1})^{\ell+1}$  for large  $\ell$ .

This theorem is also useful to prove that, for the regions of energy where the partial wave is unitary (elastic region), the interpolations  $a_{l}^{\dagger}$  are unitary each on the real l axis. To show this for  $a_{l}^{\dagger}$ , let us write l=2z+2Nand build:
$$a_{2z+2N}^{+} - (a_{2z^{*}+2N}^{+})^{*} - 2i\sqrt{\frac{s-4}{s}} a_{2z+2N}^{+} (a_{2z^{*}+2N}^{+})^{*} \equiv f(z).$$

f(z) vanishes for every integer value of z, as at that point at takes on a physical value at an even angular momentum. On the other hand f(z) is regular and satisfies Carlson's asymptotic condition and is therefore identically zero, at satisfies unitarity in the complex half-plane everywhere in the sense that

$$a_{\ell}^{+} - (a_{\ell^{*}}^{+})^{*} = 2 i \sqrt{(s-4)/s} a_{\ell}^{+} (a_{\ell^{*}}^{+})^{*}.$$
 (18)

The reasoning is the same for a; and leads to the same result.

The reader may show as an excercise that if one is to take only one interpolation, valid for both even and odd partial waves, for example  $(a^++a^-)/2 + e^{i\pi t} (a^+-a^-)/2$ , Carlson's theorem does not apply any more to prove unitarity and in fact the amplitude thus obtained is not unitary in general.

We have so far established a number of properties which are quite interesting in the sense that they remind us strongly of the potential scattering case. Notice also that if one has the Schrödinger equation with an exchange potential, one obtains twice the Regge behaviour: once with the even partial waves and an effective potential which is the sum of the direct and exchange parts, and once with the odd partial waves with the difference. Therefore in that case one also obtains two distinct interpolations  $a_{t}^{*}$  and  $a_{t}^{*}$  with the same properties.

# 3.3. Connection between asymptotic behaviour in $\cos \theta$ and singularities in the *l*-plane,

We have not yet reached the interesting part of the  $\ell$ -plane, in the sense that we are still on the right of any Regge pole (if there is any) in the region where Eq. (17) converges.

Indeed, if A displays a behaviour like (cos  $\theta$ )<sup> $\alpha$ </sup>, the integral (17) only converges for Re  $l > \text{Re } \alpha$ .

If however A  $(q^2, \cos \theta) \approx \beta(q^2) P_{\alpha(q^2)}(\cos \theta) + 0[(\cos \theta)^{\alpha'}]$  where  $\operatorname{Re} \alpha' < \operatorname{Re} \alpha_i$ then we may analytically continue the integral by writing:

$$\int_{1}^{+\infty} P_{\alpha}(\mathbf{x}) Q_{\ell}(\mathbf{x}) d\mathbf{x} = 1/(\ell - \alpha)(\ell + \alpha + 1)$$
(19)

and continuing this term by its exact expression, and the remainder converges further to Re  $\ell > \text{Re } \alpha'$ . Therefore we may again get Regge poles as a consequence of the behaviour (12).

Incidentally, it might help to see what kind of singularities other asymptotic behaviours may lead to.Consider for example

$$A(q^2, \cos \theta) \approx \cos^{\alpha} \theta \ln^{\beta} \cos \theta + 0 [(\cos \theta)^{\alpha'}].$$

Write

$$\cos^{\alpha}\theta \ln^{\beta}\cos\theta = \frac{\beta!}{2\pi i} \int_{\alpha'}^{\alpha'} \frac{\cos^{\zeta}\theta d\zeta}{(\zeta-\alpha)^{\beta+1}} + 0 \left[ (\cos\theta)^{\alpha'} \right],$$

#### HIGH-ENERGY PROPERTIES

the integral being taken around  $\alpha$ . We might as well replace  $\cos^{\xi} \theta$  by  $P_{\epsilon}(\cos \theta)$  and insert into (17) and (19) thus getting the leading singularity

$$[\beta!/(2\alpha+1)][1/(\ell-\alpha)^{\beta+1}].$$

This singularity for  $\beta$  integer negative becomes of logarithmic type. We may remark then that the power in  $\cos \theta$  will determine the location of the singularity, whereas the nature of the singularity will depend upon the departures from a simple power behaviour. It is therefore to be expected that any attempt to determine the nature of the singularity by using Eq. (17) is very delicate and it becomes dubious whether it does not at the same time determine the exact location of the singularity.

On the other hand, if one knows by other ways that there are only poles, then the analytic continuation of Eq. (17) is fairly possible: identify the poles by the asymptotic behaviour, and subtract them out. This has been done in practical calculations [4], in particular in potential scattering [5] where one knows that there are only poles.

#### 3.4. Bardacki's method

Recently, BARDACKI [6] has completed some very interesting work which is probably the first step towards a proof of the existence of Regge poles in relativistic S-matrix theory. His basic idea is the following: we assume that the overall number of subtractions for the Mandelstam representation is finite, N. Therefore, for any  $q^2$ ,  $a_{\ell}^{\pm}(q^2)$  is regular in the half plane Re  $\ell > N$ . On the other hand, we have seen that for s real and negative, the maximum power admissible for  $\cos \theta$  was one (unitarity in the crossed channel). It is very easy to see that, in fact, this holds also in an infinitesimal neighbourhood of the negative s axis. We therefore have another domain of regularity s negative, Re  $\ell > 1$ . We may take the holomorphy envelope of these two domains which provides a larger domain of holomorphy for  $Q_{\ell}(q^2)$ .

It turns out that the calculation is fairly trivial. If one makes a conformal mapping to map the s-plane cut from  $-\infty$  to 0 and from 4 to  $+\infty$  onto a strip:

 $S = 2 + 2 \sin z$ , -1 < Re z < 1.

We can almost use the tube theorem, saying, not rigorously, that we have analyticity in the region:

$$-1 < \text{Re} z < 1$$
,  $\text{Re} l > N$ ,  
Re  $z = -1$ ,  $\text{Re} l > 1$ .

We use the tube theorem, taking the convex hull of the base of the tube:

Re 
$$\ell > (N+1)/2 + [(N-1)/2]$$
Re z, or  
Re  $\ell > (N+1)/2 + [(N-1)/2]$ Re arc sin (s-2)/2. (20)

M. FROISSART

This is not quite rigorous because Re z = -1 is not a domain. However, it may be made rigorous. The extension of the domain of holomorphy stops there and one cannot go further. There is, however, a way to extend the analytic properties, but not holomorphy, only meromorphy. This consists in taking exactly into account the two-body unitarity as far as it is valid. It is known that Schwarz's reflection principle allows one to continue through the two-body elastic cut analytically or, alternatively, to write down a function of the partial-wave amplitude which does not have the two-body cut.

To be more specific, consider the partial wave at threshold copying Eq.(17) in the form

$$a_{p}(q^{2}) = \int_{x_{0}>1}^{\infty} Q_{p}(x) A_{t}(q^{2}, x) dx.$$
 (21)

We should keep in mind that  $x = 1 + t/2q^2$ . Only large values of x will contribute near threshold. Below threshold, at  $q^2 = |q|^2 e^{i\pi}$ ,  $A_t(q^2, x)$  is real, and therefore the phase of  $a_{\ell}(q^2)$  is that of  $Q_{\ell}(x) = Q_{\ell}[1+t/(2|q|^2e^{i\pi})]$ . It is  $\pm \exp i\pi \ell$ . Above threshold  $q^2 = |q|^2$ ,  $Q_{\ell}(q^2)$  is unitary, so that Im  $1/[q_{\ell}(q^2)] = q/\sqrt{1+q^2}$ . This gives us the whole set of rules to continue  $a_{\ell}(q^2)$  around q = 0 any number of times. The construction of the function

$$R_{g}(q^{2}) = q^{2\ell} / a_{g}(q^{2}) + 2i q^{2\ell+1} / [1 + \exp(2i\pi\ell)] \sqrt{q^{2} + 1}.$$
 (22)

so that it turns out to be real for both  $q^2 = \pm |q|^2$  is left to the reader. Furthermore, it is bounded at  $q^2 = 0$ , because

$$a_{\ell}(q^2) \approx q^{2\ell} \int A_t(q^2, x) dt/t^{\ell+1}$$
, (Re  $\ell > N$ ).

It is therefore regular at the origin as a function of  $q^2$ . At any rate it is meromorphic wherever  $a_l(q^2)$  is.

Now if we assume (which is nearly rigorous) that the rules for completion of meromorphy domains are the same as for holomorphy domains, we can play the same game as before except that the initial domain has a cut starting from the first inelastic threshold (somewhere between 4 and 16), say 16. Then we get, for  $a_{i}(q^{2})$  the meromorphy domain as defined by

$$\operatorname{Re}(\ell) > (N+1)/2 + (N-1)/2 \operatorname{Re}[\operatorname{arc} \sin(s-8)/8].$$
 (23)

We see clearly that we are prevented from going further only by our lack of ability: we do not know how to eliminate the further cuts on the real axis. It is conceivable that someone who could master the 4-body unitarity condition could carry on the programme up to the 6-body cut, and so on. At any rate, it is comforting to see a domain of meromorphy which is larger than the domain of holomorphy, as this introduces a kind of proof which depends very little on Eq. (17) as far as the nature of the singularities is concerned. However, it might very well turn out that the 4-body cut introduces other kinds of singularities in the  $\ell$ -plane and that the reason that potential scattering has only poles is precisely the absence of inelastic contributions. This is of course an open question. If however, one makes <u>ad hoc</u> hypotheses on the inelastic contributions, for example, if one assumes [7] analyticity properties of the absorption coefficient  $\eta(\ell, q^2)$ , then it is possible to carry out the reasoning with threshold of infinity, thus getting meromorphy for Re  $\ell > 1$ , but it looks a little like assuming what one wants to prove. Another interesting try has been made recently by MANDELSTAM [8], in which he studies a problem where the kinematics are relativistic, the potential energy independent and where there are no inelastic processes. He then succeeds in proving that for a potential weak enough, the Regge-Sommerfeld-Watson formula is applicable down to Re  $\ell = 0$ , without using the unitarity condition in the crossed channel.

### 4. DISCUSSION OF CHEW'S HYPOTHESIS

We have seen in the last section how one might think of establishing the existence of Regge poles in S-matrix theory and that a long way still lies in front of us. However, CHEW [9] was bold enough to overcome this lack of logical support and to assume that the only singularities lying in the  $\ell$ -plane are poles and that the partial waves were given even for small  $\ell$  by the analytic continuation of  $a_{\ell}(q^2)$  as defined by [17].

Let us examine how this hypothesis solves and helps to understand the paradoxes encountered in the first section.

Gribov's paradox is now very clear. We have seen that a behaviour of the form  $\cos^{\alpha}\theta \ln^{\beta}(\cos \theta)$  brings in a singularity in the  $\ell$ -plane at  $\ell = \alpha$ , of the kind  $(\ell - \alpha)^{-(\beta+1)}$ . The content of Gribov's paradox is that no such singularity may lie on the real axis, where  $a_{\ell}$  is bounded by the unitarity condition unless  $\beta < -1$ , whatever the real value of  $\alpha$  is. But, of course, we assume now that the Regge poles move and if  $\alpha(q^2)$  is the position of the pole, according to Eq. (22) near the threshold:

$$R_{\alpha}(q^2) = 2iq^{2\alpha+1}[1 + \exp(2i\pi\alpha)]\sqrt{1+q^2}.$$

The solution of this equation,  $\alpha$ , moves out of the real axis just at threshold with an imaginary part [7] of the order of  $q^{2\alpha_0+1}$ ,  $\alpha_0$  being  $\alpha(q^2=0)$ . This is exactly what we need to avoid Gribov's paradox.

Similarly, it is now easier to see through the complexity of the dependence of single spectral functions upon the double spectral functions. For negative real s, the partial waves obtained without subtractions  $(\ell > 1)$ are indeed the analytic continuation from the region of Re  $\ell > N$ . Therefore, if this analytic continuation is unique when one analytically continues them to positive values of s, they are still the analytic continuation of  $a_t$  as defined by Eq. (17) and therefore are unitary by virtue of Eq. (18). If there are only poles, the analytic continuation is unique and therefore Chew's hypothesis explains the kind of magic which takes place here very well. Furthermore, it may be much easier to continue analytically in  $\ell$  rather than in s, as the continuation path may be shorter, and we have seen that the analytic continuation in l is relatively easy when there are only poles which one can separate out. If this connection is true, we see that the second part of Chew's hypothesis is forced upon us by unitarity in the crossed channels for intermediate partial waves ( $2 \le \ell \le N$ ), and therefore it is very natural to extend it to the S and P-waves.

### 4.1. Possible range of coupling constants

Let us make a little philosophical digression at this point which may illustrate the possible depth of Chew's hypothesis pretty well. Let us consider that, as in potential theory, the stronger the interactions are, the larger are the values of angular momenta of Regge poles. This is of course without proof of any kind. If, however, one admits this postulate as well as Chew's hypothesis, one is faced with the following situation: the interactions cannot be stronger than they are in the physical world, as this would correspond to amplitudes increasing like  $S^{\alpha}$ ,  $\alpha > 1$  in the physical region, which contradicts the unitarity condition. Chew called this circumstance "saturation of unitarity". It seems that the interactions in nature are "as strong as possible" On the other hand, can they be weaker? Perhaps, but not vanishingly small, since, according to Chew's hypothesis, if one wants to have one particle, one has to bring at least one Regge pole up to zero. The free-field theory in particular does not satisfy Chew's hypothesis, as its scattering amplitude has no poles and therefore no stable particles. It looks thus as if there was a finite range of interactions possible. If one is very optimistic, one may even hope that there is only one theory possible by this system, but this becomes science-fiction.

### 4.2. Accumulation of Regge poles

It has been pointed out by Gribov and Pomeranchuk and independently by Wilson that sometimes the Regge poles cluster around some accumulation points. They have used this fact to derive a lower bound on the asymptotic behaviour of cross-sections.

The first case [10] of such an occurrence is essentially kinematic and arises [11] also in potential scattering [5]. We can easily derive it from Eq.(22) in the neighbourhood of the threshold q = 0. The equation of the Regge poles reads

$$R_{n}(q^{2}) = 2i q^{2\alpha+1} [1 + \exp(2i\pi\alpha)] \sqrt{q^{2} + 1}.$$
(24)

This equation has an infinite number of solutions near =  $-\frac{1}{2}$ : this is best seen by taking the logarithm :

$$\ln R_{\alpha}(q^{2}) + \ln \left[\frac{1 + \exp(2i\pi\alpha)}{2i}\right] + \frac{1}{2} \ln[q^{2} + 1] = (2\alpha + 1) \ln q + 2 \min \pi,$$
  
$$\ln R + \ln[-\pi(\alpha + \frac{1}{2})] = (2\alpha + 1) \ln q + 2 \min \pi + 0(\alpha + \frac{1}{2}).$$
 (25)

We thus have an infinite number of poles labeled by m, going to  $-\frac{1}{2}$  roughly like  $-1/2 + 2m i \pi / \ln q$ , neglecting a factor of the order  $\ln |\ln q|$ .

This result leads to a prediction concerning the behaviour of  $A(q^2, \cos \theta)$  for  $q^2 = 0$ ,  $\cos \theta \rightarrow \infty$ :  $A(0, \cos \theta)$  cannot fall off faster than  $(\cos \theta)^{-1/2}$ . It is to be expected that such a behaviour will take place at every threshold, at  $\ell = -\frac{1}{2}$  for two-body thresholds, possibly at other values of  $\ell$  for many-body thresholds, as it depends upon the phase-space threshold behaviour.

Another point of accumulation of Regge poles [10] is a consequence of a very special feature of relativistic theory, i.e. the existence of a double spectral function at negative energy.

Consider the partial-wave amplitude as defined by Eq.(17). For  $q^{2}=e^{i\pi|q|t}$  near zero,  $a_{\ell}^{\pm}(q^{2})$  has a constant phase  $\pm e^{i\pi\ell}$ , that of  $Q_{\ell}(1+t/2q^{2})$ . For  $q^{2}<-t_{0}/4$ , a cut appears as a result of the coincidence of the limit of integration  $x_{0} \approx 1 + t_{0}/2q^{2}$  with -1, which is a branch point for  $Q_{\ell}$ .

The imaginary part of  $b_{\rho}(q^2) = q^{-2\ell}Q_{\rho}(q^2)$  above this cut is

$$\operatorname{Im} \mathbf{b}_{\ell}(\mathbf{q}^{2}+\mathbf{i}\epsilon) = -\frac{1}{\pi |\mathbf{q}|^{2}} \int_{1}^{\mathbf{x}_{0}} \operatorname{Im}[\mathbf{A}_{t}(\mathbf{q}^{2}+\mathbf{i}\epsilon, \mathbf{x}-\mathbf{i}\epsilon)\mathbf{Q}_{\ell}(\mathbf{x}-\mathbf{i}\epsilon) \mathbf{e}^{-\mathbf{i}\pi\ell}].$$
(26)

This is, in general

$$\operatorname{Im} \mathbf{b}_{\ell}(\mathbf{q}^{2} + i\epsilon) = -\frac{1}{\pi |\mathbf{q}|^{2}\ell} \int_{-1}^{\mathbf{x}_{0}} \operatorname{Im} \mathbf{A}_{t}(\mathbf{q}^{2} + i\epsilon, \mathbf{x} - i\epsilon) \operatorname{Re}[\mathbf{Q}_{\ell}(\mathbf{x} - i\epsilon)e^{-i\pi\ell}] d\mathbf{x} \\ -\frac{1}{\pi |\mathbf{q}|^{2}\ell} \int_{-1}^{\mathbf{x}_{0}} \operatorname{Re} \mathbf{A}_{t}(\mathbf{q}^{2} + i\epsilon, \mathbf{x} - i\epsilon) \operatorname{Im}[\mathbf{Q}_{\ell}(\mathbf{x} - i\epsilon)e^{-i\pi\ell}] d\mathbf{x} \\ \operatorname{Im} \mathbf{b}_{\ell}(\mathbf{q}^{2}) = \frac{1}{\pi |\mathbf{q}|^{2}} \int_{-1}^{\mathbf{x}_{0}} \rho_{t, \mathbf{u}}(\mathbf{q}^{2}, \mathbf{x}) \mathbf{G}_{\ell}(\mathbf{x}) d\mathbf{x} - \frac{1}{\pi |\mathbf{q}|^{2}\ell} \int_{-1}^{\mathbf{x}_{0}} \operatorname{Re} \mathbf{A}_{t}(\mathbf{q}^{2}, \mathbf{x}) \mathbf{P}_{\ell}(-\mathbf{x}) d\mathbf{x}.$$
(27)

where  $\rho_{t,u}(q^2, x)$  is the spectral function which lies in t > 0, u > 0, and where

$$Q_{\ell}(\mathbf{x}) \equiv \operatorname{Re} Q_{\ell}(\mathbf{x}) \quad , \quad -1 < \mathbf{x} < 1.$$

The second term is very quiet and, indeed, it is an entire function of  $\ell$ . It is the only one which exists in potential scattering and its nice analytic properties have been used by MANDELSTAM [8] in a recent study where he describes a model of relativistic theory which does not exhibit crossing symmetry. Mandelstam proves there that Chew's hypothesis is verified.

The first term, however, is not regular, but has the poles of  $Q_{\ell}(-x)$  which are at every negative integer  $\ell$ . In particular, the first pole at  $\ell = -1$  is very unlikely to vanish, as its residue is

$$\frac{1}{\pi}\int \rho_{t,u}(q^2, x)dx.$$

This can be checked in practice by putting the proper threshold behaviour of  $\rho_{t,u}(q^2, \mathbf{x})$  in every particular case of interest. Let us simply assume that the residue is not zero. (In any case, all residues cannot be simultaneously zero, as this would imply  $\rho_{t,u} \equiv 0$ , because of the completeness of Legendre polynomials which are the residues of  $Q_{\ell}(-\mathbf{x})$ .

Consider now the function

$$f(q^2) = \lim_{\ell \to -1} (\ell+1) b_{\ell}(q^2).$$

It has a non-zero left-hand cut, but, if  $a_{\ell}(q^2)$  is meromorphic down to  $\ell = -1$ ,  $a_{\ell}(q^2)$  is bounded by unitarity on the real positive  $q^2$  axis  $f(q^2) = 0$  there. This is a contradiction and proves that  $a_{\ell}(q^2)$  cannot be meromorphic along the real  $\ell$  axis down to  $\ell = -1$ .

A possible explanation of this phenomenon was furnished by Gribov. As  $\ell$  goes to -1, the importance of the left-hand cut increases. This has the effect in many instances of pulling poles out of the right hand cut ("bound states") in order to counteract the strong left-hand cut. Gribov suggests that more and more of these poles come out as  $\ell \rightarrow -1$ , until their residue distribution exactly cancels the left hand cut of  $f(q^2)$  at the limit  $\ell = -1$ . This implies an accumulation point of Regge poles around  $\ell = -1$ , each of which attains -1 only when  $q^2$  is infinite. Notice that in potential scattering with a regular potential, the Regge poles go to negative integers at infinite energies. If Gribov's mechanism is right, the occurrence of the "third" spectral function would only mean that an infinite number of Regge poles reach each negative integer. Obviously this reasoning only applies to the first nonvanishing pole of Eq. (27), but it may be expected that the result holds for all non-vanishing poles.

In all cases, however, we see that it is impossible for the amplitude to fall off faster than 1/s as we must have a singularity at least at l = -1. This should be experimentally checked.

### 5. CONNECTION WITH THE PHYSICAL WORLD [12]

There are basically two kinds of immediate tests of the whole Regge pole story. The first approach consists in looking in one channel at the Regge poles of the same channel going through physical values of the spin, or nearby, thus producing stable or unstable particles. The second approach consists in studying the asymptotic behaviour of the amplitudes in one physical region, thus getting information on the Regge poles of crossed channels. It is obvious that we cannot get complete information on Regge poles by these methods, but we may get enough to decide whether or not the Regge poles have anything to do with nature.

### 5.1. First approach: physical *l*

Consider a well defined channel, that is a well-defined set of quantum numbers, baryon number, charge, parity, strangeness and isotopic spin. In this channel, the S-matrix will be considered expressible in terms of the total angular momentum j and any other set of variables. We assume that, as a function of j, it is meromorphic down to Re j = 0 and that this analytic continuation furnishes the right value of the partial waves. Of course, we know already that even for 2-body amplitudes, it is not possible to define one amplitude, but rather two, according to Eq. (17). We thus assume that these two are enough and that every physical S-matrix element is either equal to the value of the interpolation by the S<sup>+</sup> matrix or by the S<sup>-</sup> matrix, it being understood that angular momenta differing by two are related to the same interpolation. The sign put in superscript will be called, following Gell-Mann, the signature.

In each channel, then, we may order all stable particles and all resonances according to their signature. Then we could expect these states to belong to the same Regge trajectory, or at least to belong to a finite number (smaller than the number of states) of Regge trajectories. The first attempt in this direction was made by CHEW and FRAUTSCHI [9]. They made a diagram of all then known particles with the squared mass in abscissa and the spin in ordinate. Only one pair of particles could be fitted: the nucleon  $P_{1/2}$  and the third nucleon resonance  $F_{5/2}$ . This corresponds to an average slope  $\partial \alpha / \partial s$  of  $1/50 \, m_{\pi}^2$ . This, quite remarkably, fits with a formula of potential scattering which expresses  $\partial \alpha / \partial s$  as  $R_{av}^2 / 4(2\alpha + 1)$ , where  $R_{av}$  is some average radius of the wave function. If we take it to be  $1/2 \, m_{\pi}$ , we get the result. This, of course, should not be taken too seriously as we are in the relativistic region. However, this figure of  $1/50 \, m_{\pi}^2$  should be retained as we shall encounter it many times.

For example, the possibility has been mentioned at the 1962 Geneva Conference of the existence of a resonance at 1920 MeV, B = 1 S = 0, I = 3/2. If the other quantum numbers turn out to be correct, this could correspond to the same Regge pole as the well-known (3,3) resonance. In the same way, the excited hyperon of mass 1815, which appears to have 1 = 0, could be the same pole as the  $\Lambda$ . These two cases would correspond to an average  $\partial \alpha / \partial s$  of  $1/50 m_{\pi}^2$  in the same way.

This is about all the information we can get from this first point of view and is pretty meager. However, the spectroscopy of high-energy resonances is a science in full bloom and the number of pairs associated to the same Regge poles may increase beyond expectation in a few years.

### 5.2. Second approach: asymptotic properties of cross-sections

The study of the asymptotic properties of the cross-sections at fixed momentum transfer as a function of the energy can also help checking the Regge pole hypothesis. This has to be done in a fairly indirect fashion, as we have seen that it is very difficult to determine from the asymptotic behaviour whether one has to do with poles or with other singularities. However, a number of non-trivial predictions can be made and checked against experiment.

The total cross-sections, being given by the optical theorem as the imaginary part of the amplitude up to some kinematical factor, are a very convenient tool. It should be possible to express them in the form:

$$\sigma_{\text{tot}}(s) \approx (1/s) \sum_{K} \beta_{\alpha}(0) P_{\alpha_{K}(0)}(1+s/2), \qquad (28)$$

the summation being carried out over all Regge poles having the appropriate quantum numbers, that is the quantum numbers of the particles which could be exchanged in the scattering process. At this point a very tempting assumption can be made, that of factorization [13].

The idea is the following. Consider a matrix M, function of some parameters  $\{\lambda\}$ . If this matrix is meromorphic in  $\{\lambda\}$ ; the poles are most likely to be simple and their residues to be of rank one in the following sense.

If we consider the inverse matrix  $N = M^{-1}$ , Det N has a zero at the pole and this zero is most likely to be simple, i.e. we may vary the elements of N by small amounts related by only one condition and still keep a simple zero. If we wanted to keep a double zero, we could only vary the elements of N by small amounts related by 2 conditions and so on. If the zero is simple N has only one eigenvalue zero and therefore M has only one eigenvalue infinite or, what is the same, the rank of the residue is one.

If this is accepted, then, we find that the S-matrix, as expressed as a function of complex j, is most likely to have residues which are of rank one.

This implies that the factors  $\beta$  in an expression like (28) may be written as follows.

Assume that the reaction under consideration is among particles a and  $\boldsymbol{b}$  . Then we have:

$$\beta_{\alpha}(0) = \beta_{K}^{(a)}(0) \cdot \beta_{K}^{(b)}(0)$$

This has very strong experimental implications, for example, if we assume that the leading term in (28) corresponds to

 $\alpha_1(0) = 1$  ("Pomeranchuk pole") (29)

which leads to constant cross-sections at infinity, then

$$\sigma_{\text{tot}}(a+a) \cdot \sigma_{\text{tot}}(b+b) = [\sigma_{\text{tot}}(a+b)]^2.$$
(30)

No such relations has yet been experimentally checked, as they always necessitate targets which are difficult to prepare. However, it is possible that in the future cross-sections like  $\sigma_{tot}(\pi + \pi)$  might be measured by some indirect way: extrapolation or the like.

It should also be possible to go farther than that and estimate the next terms of Eq. (28). One gets into trouble here. Take, for example, the case of (p, p) and (p,  $\overline{p}$ ) and (p,  $\overline{p}$ ) scattering. The total cross-sections look as if they were going down slowly towards their limit, the difference decreasing like S<sup>-0.5</sup>. However,  $\sigma_{pp}$  is much nearer to it than  $\sigma_{pp}$ . This is very nice and we hope that it could be the influence of the Regge pole of the  $\rho$  resonance. However, this  $\rho$  resonance (or the  $\omega$  resonance), because of its quantum numbers, only contributes to he difference  $\sigma_{pp} - p_{pp}$ . Therefore one needs another, as yet unknown, Regge pole which has about the same  $\alpha$  and  $\beta$  and which has the proper quantum numbers so that it contributes to the sum  $\sigma_{p\overline{p}}+\sigma_{pp}$ .

#### 5.3. Non-forward scattering

If this last pole exists, one may wonder why it does not correspond to any known particle. This is also true of the dominant ("Pomeranchuk") pole. In fact, the signature of the Pomeranchuk pole is + and therefore it should go through 0 for some negative value of t where a particle should appear. This has been investigated by Gell-Mann and, though not understood in full detail, the situation is pretty well clarified.

The idea, which has been checked by Gell-Mann in a 3-body model, is that for every integer j, there are two different families of Legendre functions which become completely independent. There are those with singularities and those without. The Legendre functions without singularities are connected with the representations of the rotation group. The others may also be connected with the rotation group, but they do not form a basis for a representation because of their singularities.

As a parenthesis, Wigner has shown what the representations of the Poincaré group look like for imaginary mass<sup>\*</sup>. The difference lies in the fact that, for imaginary mass, the relevant surface is not a sphere, but a hyperboloid, and the conditions for the absence of singularities on the hyperboloid are quite different from those for the absence of singularities on a sphere.

For example l, all Legendre functions have singularities on the sphere. When one follows a Regge pole, as a function of l and reaches an integer value of j, one expects the relevant "wave functions", whatever that means precisely, either to keep their singularities on the sphere, or to lose them. In the first case, one will not get any particle or resonance with that spin and this is what happens in the case of the "ghost" of the Pomeranchuk pole at j = 0. In the second case, it will furnish an honest particle which can be seen.

Thus, it is getting fairly difficult to trace the Regge poles in their own channel, one may miss them fairly frequently. The behaviour indicated by Eq.(12)  $A(s,t) = \beta(t)A^{\alpha(t)}$  which leads to a differential elastic cross-section of the form

$$d\sigma^{el}(s,t)/dt \approx |\beta(t)|^2 s^{2[\alpha(t)-1]}$$
(31)

has been experimentally checked, or at least, that it is not incompatible with experiment.

5.4. Classical picture of high-energy scattering - the puzzle of heavy nuclei

Equation (31) can be interpreted classically, as at very-high energies the wave length of the particles is much smaller than any of the dimensions involved in (31). It is therefore tempting to do so. We may rewrite (31), putting  $2\partial \alpha/\partial t = a$ .

$$d\sigma^{el}/dt \approx |\beta(t)|^2 \exp[-\alpha |t| \ln s].$$
(32)

The pattern is that of a shrinking diffraction peak. This corresponds to an increasing size of the target. However, the total cross-section being constant, we end up with a target which blows up like a puff of smoke, as the energy increases, becoming bigger and thinner.

This is a very striking feature of this whole analysis. One may start wondering what happens when the target is a heavy nucleus. It is known that the scattering of a high-energy proton by a heavy nucleus is essentially proportional to the area of the nucleus, therefore going like  $A^{2/3}$ , and is essentially constant up to cosmic ray energies.

But what if all the nucleons inside the nucleus start blowing up, thus becoming more and more transparent? Gell-Mann and Udgaonkar have pro-

<sup>\*</sup> These proceedings.

#### M. FROISSART

posed such a model and they show that at very high energies, the crosssection should be proportional to A, rather than  $A^{2/3}$ , as there is no screening effect any more. The transition of one set of cross-sections to the other should take place very slowly, as the increase in size of the nuclei, and finally, we end up with a cross-section which tends towards its limit as  $1/\ln s$ , which leads to a cut in the  $\ell$ -plane.

Another possibility is interesting to investigate. Let us use the factorization hypothesis in equation (32). We get for the amplitude

A(s, t) = is 
$$\beta^{(a)}(t)\beta^{(\ell)}(t)\exp\left(-\frac{a}{2}|t|\ln s\right)$$
. (33)

We now consider that this is pure diffraction scattering, which occurs with a very weak absorption over a large surface. We can therefore trace back the absorption density  $\rho(\vec{b})$  as a function of the impact parameter  $\vec{b}: \rho(\vec{b})$ is the two dimensional Fourrier transform of A(s,t) as expressed in terms of the two-dimensional transverse momentum transfer.

The product (31) is transformed into a convolution by this Fourrier transformation:

$$\rho(\overrightarrow{\mathbf{b}}) = \rho^{(\mathbf{a})}(\overrightarrow{\mathbf{b}}) * \rho^{(\mathbf{b})}(\overrightarrow{\mathbf{b}}) * (2\pi/a \ln s) \exp(-b^2/2 a \ln s).$$
(34)

Now it seems that this way of writing  $\rho(\vec{b})$  is fairly natural and represents a part involving the target and only the target, a part involving the incident particle and only it and a part involving the Pomeramchuk pole and only it. All these parts could be replaced by another of a similar nature and it would only describe another physical phenomenon.

The classical interpretation of (34) is obvious:  $\rho^{(a)}(\vec{b})$  and  $\rho^{(b)}(\vec{b})$  represents the net probability of emitting or absorbing a Pomeranchuk pole at a place  $\vec{b}$ , integrated along the line of flight, of particles a and b repectively.

The expression  $[\pi/a \ln s] \exp[-b^2/2 a \ln s]$  is the probability, again integrated along the line of flight, for a Pomeranchuk pole emitted at the origin, to be absorbed at a distance b from the origin.

We may also think that in fact, all these probabilities should be sperically symmetrically distributed; it is an easy matter then to compute the 3-dimensional distributions out of the integrated ones (Abel's problem).

In this case, however, the puzzle of the heavy nuclei disappears, as only the Pomeranchuk pole blows up and thins out. The screening effect still takes place inside  $\rho_{(a)}$  and  $\rho_{(b)}$  and the cross-section goes like  $A^{2/3}$ , even asymptotically.

In conclusion, one should bear in mind the amount of guesses and conjectures which have been used in this whole study. This is a very unscientific situation, in which the bases are so far away from the prediction of experiments that there is no such thing as a decisive experiment to test this or that basic postulate. It is therefore pretty frail and it would be in many ways a miracle if all this is still true in 10 years from now.

### REFERENCES

[1] CHEW, G.F., "S-matrix theory of strong interactions", Benjamin, New York.

[2] REGGE, T., Nuovo Cim. 18 (1960) 947.

[3] TITCHMARSH, E.C., "Theory of function", Oxford (1939) 185.

[4] IGI, K., University of California, Berkeley, preprint.

[5] AHMADRADCH, A., BURKE, P. and TATE, C., UCR, preprint.

[6] BARDACKI, K., University of Minnesota, preprint,

[7] BARUT, A. O. and ZWANZIGER, D., University of California, Berkeley, preprint,

[8] MANDELSTAM, S., University of Birmingham, preprint.

[9] CHEW, G.F. and FRAUTSCHL, S., Phys. Rev. Letters 7 (1961) 394; 8 (1962) 41.

[10] GRIBOV, V. N. and POMERANCHUK, LY., Inst. of Theor. and Exp. Phys., Moscow, preprint No.91. [11] WILSON, K., private communication.

[12] COCCONI and DRELL, Gen. Conf. on High Energy Physics, to be published (1962).

[13] GELL-MANN, M., Phys. Rev. Letters 8 (1962) 263;

GRIBOV, V. N. and POMERANCHUK, LY., Zh. Exp. Theor. Phys. 42 (1962).

### REGGE POLES AND STRIP APPROXIMATION

### S. MANDELSTAM

### DEPARTMENT OF MATHEMATICAL PHYSICS, UNIVERSITY OF BIRMINGHAM, BIRMINGHAM, UNITED KINGDOM

### 1. INTRODUCTION

My aim in these lectures is to outline an approximation scheme for calculating scattering amplitudes in dispersion theory by the use of which one would hope to overcome the difficulties associated with previous approximation schemes. At the moment a fully consistent set of equations has not been written down, but it is hoped that the materials for a solution of the problem are at hand. We shall see, probably in the next lecture, that the concept of "Regge poles" will play an important part in the analysis. In fact, it was in this connection that they were originally introduced into elementary particle physics.

First let me outline why we were unable to get consistent equations by the previous approach, used for instance by Chew and me in the pion-pion problem[1]. (This approach has also been treated in a paper by CINI and FUBINI [2]. One started with the double-dispersion representation [3],

$$A(s,t) = \frac{1}{\pi} \int ds' dt' \frac{A_{13}(s',t')}{(s'-s)(t'-t)} + \frac{1}{\pi} \int du' dt' \frac{A_{23}(u',t')}{(u'-u)(t'-t)} + \frac{1}{\pi} \int ds' du' \frac{A_{12}(s',u')}{(s'-s)(u'-u)}.$$
(1)

For the purpose of this lecture we have taken the case of neutral scalar particles with equal mass, when the variables s, t and u are related by the equation

$$s + t + u = 4\mu^2$$
.

We have not written the subtraction terms explicitly, but it is understood that such terms may and in fact will be present. The essence of the old approach was to assume that the scattering amplitude at low energies was dominated by the nearest singularities. Accordingly, one neglected terms where "one" was large. In the first approximation, one neglected contributions which began at the inelastic threshold.

In pion-pion scattering, this amounted to neglecting the double-spectral functions completely. The reason is that it is impossible to draw a diagram where the  $\pi\pi$  scattering, both in the s and t- channels, took place through a two-pion intermediate state. The processes with the lowest intermediate states were in fact as in Fig. 1.



Graphs for pion-pion scattering with the lowest intermediate state.

The first diagram goes through a two-pion state in the s-channel but through a four-pion state in the t-channel, the second through a four-pion state in the s-channel but through a two-pion state in the t-channel. Accordingly, the double-spectral function starts at a high threshold in at least one of the variables s and t, and must consequently be neglected.

We should emphasize that the neglect of the double-spectral function is purely due to the absence of a three-pion vertex. If there were a three-pion vertex, the following process (Fig. 2) would contribute, and the double-



Fig. 2 Pion-pion scattering through a three-pion vertex

spectral function could not be neglected as all double-spectral functions are neglected, the representation would consist purely of subtraction terms, and would appear as follows

$$A(s,t) = \frac{1}{\pi} \int \frac{f_s(s') ds'}{s' - s} + \frac{1}{\pi} P_1 \left(1 + \frac{2t}{s - 4\mu^2}\right) \int \frac{f_p(s') ds'}{(s' - s)} + crossed terms + \lambda \cdot (2)$$

We have taken the case of two subtractions in the t-variable, so that there will be a constant term and a term linear in t. We have re-grouped them into a constant term and a term involving the factor  $P_1 [1+2t/(s-4\mu^2)] = 1 + 2t/(s-4\mu^2)$  as these terms correspond to S- and P-waves. Thus, if P-waves are important, as they are in practice in  $\pi$ - $\pi$  scattering, one would expect to have to perform at least two subtractions in t. We have been a little careless in writing (2), as P-waves cannot occur in neutral (pseudo) scalar pion-pion scattering but, since we are really interested in charged pions we shall ignore this.

When Chew and I attempted to solve the problem, we arrived at singular integral equations which did not have a unique solution. The difficulty was due to the fact that the diagram (Fig. 3) represents the exchange of a P-wave pion pair.



Fig. 3 Exchange of a P-wave pion pair.

Now the exchange of a P-wave system corresponds to a very singular potential. If one were solving the problem by any other method, the singularity would be rounded off by the fact that a composite system such as a pion pair has an extension in space. Dispersion theory operates in terms of the S-matrix, however, and concepts such as spatial extension do not enter <u>directly</u> into the theory. In fact, in the simplest approximation as Chew and I treated it, the exchange of a composite system is treated on exactly the same footing as the exchange of an elementary particle, and leads to singular equations.

The difficulty actually arises from the factor  $1 + 2t/(s-4\mu^2)$  in the second term of (2), which approaches infinity with infinite t. Now, the function A (s,t) in addition to representing direct pion-pion scattering, also represents crossed pion-pion scattering, and now being the energy and s the momentum transfer. The amplitude for crossed pion-pion scattering then approaches infinity with the energy, and such a behaviour can lead one into conflict with the unitarity condition.

### 2. CALCULATION OF THE DOUBLE SPECTRAL FUNCTION

In the face of these difficulties, a much more ambitious approximation scheme was suggested independently by CHEW and FRAUTSCHI, McCAULEY, TER-MARTIROSYAN and WILSON [3]. The proposal was essentially to calculate all that one can with neglect of multi-particle states in the unitarity condition. Our next problem will therefore be to investigate how the doublespectral function may be calculated.

Let us suppose for the moment that we know the double-spectral function  $A_{23}$ . In practice, we do not know it in advance, of course, but must calculate it by means of an iteration procedure. We may then re-write (1) as

$$A(s,t) = \frac{1}{\pi} \int_{4\mu^{2}}^{\infty} dt' \frac{V_{3}(s,t)}{(t'-t)} + \frac{1}{\pi} \int_{4\mu^{2}}^{\infty} du' \frac{V_{2}(s,u')}{(u'-u)} + \frac{1}{\pi} \int ds' dt' \frac{A_{13}(s,t')}{(s'-s)(t'-t)} + \frac{1}{\pi} \int ds' du' \frac{A_{12}(s',u')}{(s'-s)(u'-u)} + subtraction terms involving s + \lambda.$$
(3)

In this equation, the first two integrals come from two sources:

- (i) Substraction terms in t and u;
- (ii) The second term of (1).

It is shown in [1] how the latter term can be written in the form of the first two terms of (3). If we have subtractions, the integrals in (3) will really have a more complicated form. This is also explained in [1]. Such complications are inessential and we shall ignore them here. We shall explicitly exclude more than one subtraction in each variable, however. For the moment we are interested in the calculation of  $A_{13}$  and  $A_{12}$  from unitarity when  $V_3$  and  $V_2$  are known. Later we shall have something to say on the iteration procedure for calculating  $V_3$  and  $V_2$ . We shall formulate all our equations in terms of neutral pion-pion scattering. Generalization to problems with spin, isotopic spin and unequal mass can be made, and they do not change the essential features of the calculation.

From (3), we can write a dispersion relation in the momentum transfer (for fixed s):

A (s, t) = 
$$\frac{1}{\pi} = \int_{4\mu^2}^{\infty} dt' \frac{A_3(s, t')}{(t'-t)} + \frac{1}{\pi} = \int_{4\mu^2}^{\infty} du' \frac{A_2(s, u')}{(u'-u)}$$
 (4)

where

$$A_3(s,t) = V_3(s,t) + \frac{1}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{A_{13}(s',t)}{(s'-s)}$$
 (5a)

$$A_2(s, u) = V_2(s, u) + \frac{1}{\pi} \int_{4_{11}2}^{\infty} ds' \frac{A_{12}(s, u)}{(s' - s)}$$
 (5b)

We now insert (4) into the unitarity relation, so as to obtain A, in terms of the spectral functions  $A_3$ ,  $A_2$  instead of in terms of A. The unitarity relation is

$$A_{1}(s,t) = \frac{q}{32\pi^{2}W} \int_{-1}^{1} dz_{1} \int_{0}^{2\pi} d\varphi \quad A^{*} \{s,t(z_{1})\} A \{s,t(z_{1} - \sqrt{(1-z^{2})(1-z_{1}^{2})\cos\varphi}\}$$
(6)

The notation should be fairly evident. The symbol z denotes the cosine of the ongle of scattering from the initial to the intermediate state, and  $z_1$  the cosine of the angle from the initial to the final state. The symbol  $\varphi$  is the azimuthal angle between the initial and the intermediate state, measured from the plane of the initial and the final state, and the integral is thus over all intermediate states. The cosine angle between the intermediate and the final states will thus be

$$z z_1 - \sqrt{(1 - z^2)(1 - z_1^2)} \cos \varphi$$
.

The variable z will be related to the momentum transger t by the formula

$$z = 1 + t/2q^2$$
 (7a)

while the expressions t  $(z_1)$  and t  $(z_2, -\sqrt{(1-z^2)(1-z_1^2)}\cos\varphi)$  in (6) indicate that t is to be expressed in terms of the cosine of the angle of scattering by the formula

$$t(z_1) = 2q^2(z_1 - 1)$$
 (7b)

$$t \left( z z_1 - \sqrt{(1 - z^2)(1 - z_1^2)} \cos \varphi \right) = 2q^2 \left( z z_1 - \sqrt{(1 - z^2)(1 - z_1^2)} \cos \varphi - 1 \right)$$
(7c)

In all these formulae, t is the centre-of-mass momentum given by

$$q^2 = \frac{1}{4} (s - 4\mu^2)$$
 (8a)

and W the centre-of-mass energy  $s^{\frac{1}{2}}$ .

At this point we shall first simplify the calculations by assuming that the second term of (4) is absent. We can then insert (4) into (6) and, on expressing t ( $z_1$ ) and t ( $zz_1 - \sqrt{(1-z^2)(1-z_1^2)} \cos \Phi$ ) by (7c), we arrive at the equation

$$A_{1}(s,t) = \frac{q}{32\pi^{2}W} \int_{-1}^{1} dz_{1} \int_{0}^{2\pi} d\phi \frac{1}{\pi} \int_{4\mu^{2}}^{\infty} dt' \frac{A_{3}(s,t')}{t'-2q^{2}(z_{1}-1)}$$

$$\times \int_{4\mu^{2}}^{\infty} dt'' \frac{A_{3}(s,t'')}{t''-2q^{2}(z_{1}-\sqrt{(1-z^{2})(1-z_{1}^{2})\cos\phi}-1)}.$$
(9)

The integrations over z and  $\varphi$  on the one hand, and t' and t" on the other hand, can now be interchanged. As the variables z and  $\varphi$  only occur in the denominators, the integrations over these can be performed. After expressing z in terms of t by (7a), the result is

$$A_{1}(s,t) = \frac{1}{32\pi^{3} qW} \int d\sigma' d\sigma'' A_{3}(s,t') A_{3}(s,t'') \frac{1}{\{K(q^{2};t,t',t'')\}^{\frac{1}{2}}}$$

$$\times \log \frac{\alpha (q^2; t, t', t'') + \{K (q^2, t, t', t'')\}^{\frac{1}{2}}}{\alpha (q^2; t, t', t'') - \{K (q^2, t, t', t'')\}^{\frac{1}{2}}}$$
(10)

where

$$K(q^2; t, t', t'') = t^2 + t'^2 + t''^2 - 2(tt' + tt'' + t't'') - tt't'' /q^2$$
 (11a)

$$\alpha (q^2, t, t', t'') = t - t' - t'' - t' t'' / 2q^2$$
(11b)

Eq.(10) gives A<sub>1</sub> in terms of A<sub>3</sub>. The equation as it stands is not particularly useful; however, it can easily be rewritten as an equation for A<sub>13</sub> in terms of A<sub>3</sub>. To see this, we observe that

$$\frac{1}{\{K(q^{2};t,t',t'')\}^{\frac{1}{2}}} \log \frac{\alpha (q^{2};t,t',t'') + \{K(q^{2};t,t',t'')\}^{\frac{1}{2}}}{\alpha (q^{2};t,t',t'') - \{K(q^{2};t,t',t'')\}^{\frac{1}{2}}}$$
$$= 2 \int \frac{dt'''}{t'''-t} \frac{1}{\{K(q^{2};t,t',t''')\}^{\frac{1}{2}}}$$
(12a)

where the integral is taken over the region.

$$(t^{m})^{\frac{1}{2}} > (t')^{\frac{1}{2}} + (t^{n})^{\frac{1}{2}}$$
  
 $K > 0$  (12b)

Eq.(12) may be derived by observing that the logarithm is analytic except for a cut along the real axis when the inequalities (12b) are satisfied, the discontinuity across the cut being  $2\pi K^{-\frac{1}{2}}$ . On substituting (12) into (10), we arrive at the equation

A, (s, t) = 
$$\frac{1}{\pi} \int \frac{dt'''}{t'''-t} \frac{1}{16\pi^2 qW} \int dt' dt'' \frac{1}{\{K(q^2; t, t', t'')\}^{\frac{1}{2}}}$$
  
  $\times A_3^*(s, t') A_3(s, t'')$  (13a)

where the integrals of t' and t'' ar  $\cdot$  taken over the region for which.

$$(t^{m})^{\frac{1}{2}} > (t')^{\frac{1}{2}} + (t^{n})^{\frac{1}{2}}$$
  
K > 0 . (13b)

We can now compare Eq.(13a) with the dispersion relation for  $A_1$  (s,t)

A<sub>1</sub> (s,t) = 
$$\frac{1}{\pi} \int dt' \frac{A_{13}(s,t')}{(t'-t)}$$
 (14)

(The integral over  $A_{23}(s, u)$  does not contribute when the second term of (4) is neglected). From Eq. (14) we observe at once that we can identify the integral. over t' and t" in Eq. (13a) with A<sub>13</sub>:

$$A_{13}(s,t) = \frac{1}{16\pi^2 qW} \int dt' dt'' \frac{1}{\{K(q^2;t,t',t'')\}^{\frac{1}{2}}} A_3(s,t') A_3(s,t''), \quad (15)$$

for

$$(t)^{\frac{1}{2}} > (t')^{\frac{1}{2}} + (t'')^{\frac{1}{2}}$$
, K > 0

otherwise

 $A_{13}(s,t) = 0$ ,

Eq.(14) is the unitarity equation for the double-spectral function which we require [3].

We now have two equations, Eqs. (5a) and (15), between A<sub>3</sub> and A<sub>13</sub>. One is linear, the other quadratic. Because of the limitations on the range of integration in Eq. (15) it turns out that one can obtain A<sub>13</sub> and A<sub>3</sub> without solving an integral equation. To see how this can be done, we observe that A<sub>13</sub>(s, t) will be zero if  $t < 4\mu^2$ . As the integration in Eq. (15) is taken only over the region  $t^{\frac{1}{2}} > t'^{\frac{1}{2}} + t''^{\frac{1}{2}}$ , it follows that

$$A_{13}(s,t) = 0$$
,  $t < 16\mu^2$ . (16a)

Thus, from Eq.(5a)

$$A_3(s,t) = V_3(s,t), \quad t \le 16\mu^2.$$
 (16b)

And  $A_3(s, t)$  will thus be known for this range of t.

Next, we observe from the inequality  $t^{\frac{1}{2}} > t'^{\frac{1}{2}} + t''^{\frac{1}{2}}$  that, if  $t < 36\mu^2$ , t' and t'' will both be less than  $16\mu^2$  (since both are greater than  $4\mu^2$ ). However,  $A_3(s,t)$  is known for  $t < 16\mu^2$ , so that  $A_{13}(s,t)$  can be calculated for  $t < 36\mu^2$ . Using the dispersion relation (4),  $A_3(s,t)$  can then be calculated for  $t < 36\mu^2$ .

The process of using successively Eqs. (15) and (4) may now be continued indefinitely. In the next stage, for instance,  $A_{13}(s, t)$  can be calculated from Eq.(15) for  $t < \sigma 4\mu^2$  if  $A_3$  (s, t) is known for  $t < 36\mu^2$ .  $A_3(s, t)$  can then be calculated for  $t < 64\mu^2$  from Eq.(4). We can thus construct the double-spectral function for successively larger ranges of t, and can reach any given value of t in a finite number of steps. The elastic unitarity and analyticity properties thus provide us with a means of calculating the double spectral function.

We may remark that a similar equation can be derived for calculating the double spectral function for a superposition of Yakuwa potentials, as was shown by BLANKENBECLER, GOLDBERGER, KHURI and TREIMAN [4]. There are two differences:

#### S. MANDELSTAM

- (i) The factor 1/W in Eq. 15 is absent and the numerical factors are different;
- (ii) The function V (s, t) is known in advance. It depends on t alone and is given by

$$U(t) = \frac{1}{\pi} \int dt' \frac{N-(t')}{(t'-t)}$$
(17)

where U (t) is the potential. A superposition of Yukawa potentials can always be written in the form (17). In the potential case the determination of the double-spectral function provides a complete solution of the problem, in the relativistic case it does not as  $V_3$  (s, t) is not known in advance and must be determined from the solution. Because of the similarities between the equations for the potential and relativistic theories, CHEW and FRAUTSCHI [5] have dubbed the function  $A_3$  (s, t) a "generalized potential". We should emphasize however, that the innocent-looking factor 1/W in the relativistic case is sufficient to destroy the equivalence between the present method and any Schroedinger-like equation, and one is forced to solve the problem using the procedure just outlined.

We have thus far simplified the problem by omitting the second term in (2). When such a term is present a similar procedure can be used, but the equations corresponding to (15) are slightly more complicated. They are:

$$A_{13}(s,t) = \frac{1}{16\pi^2 qW} \left[ \int_{R} dt' dt'' \frac{1}{\{K(q^2;t,t',t'')\}^{\frac{1}{2}}} A_3^{\frac{*}{3}}(s,t') A_3(s,t'') \right]$$

+ 
$$\int_{\mathbf{R}} du' du'' \frac{1}{\{K(q^2, t, u', u'')\}^{\frac{1}{2}}} A_2^*(s, u') A_2^*(s, u'') \Big],$$
 (18a)

$$A_{12}(s, u) = \frac{1}{16\pi^2 qW} \int_{R} dt' du' \frac{1}{\{K(q^2; u, t', u'')\}^{\frac{1}{2}}} \left[ A_3^*(s, t') A_2(s, u'') + A_2^*(s, u') A_3(s, t') \right].$$
(18b)

The sub-script R indicates that the integral is to be taken only over the region for which the last three arguments of K satisfy the inequality in Eq.(15).

We shall now outline very briefly the iteration procedure suggested by the authors named at the beginning of the lecture for calculating  $V_3$  and  $V_2$ . We shall not go into details, both because the scheme will probably be discussed in other lectures, and also because, as it stands at the moment, it does not appear to be free of divergences and will probably have to be modified. The iteration scheme is based on the crossing relation, which takes the form (for neutral pion-pion scattering)

$$A_{13}(s,t) = A_{13}(t,s) = A_{12}(s,t) = A_{12}(t,s) = A_{23}(s,t) = A_{23}(t,s).$$
 (19)

Now the function  $A_{13}$  calculated according to the iteration procedure is certainly not symmetric in its arguments. We shall therefore define

$$A_{13}(s,t) = A_{13el}(s,t) + A_{13in}(s,t)$$
 (20a)

where

$$A_{13in}(s,t) = A_{13el}(t,s).$$
 (20b)

The procedure outlined above gives  $A_{13e1}(s, t)$ , and corresponds to diagrams such as Fig. 4 (a). To maintain the crossing relation one must then include



Fig.4



diagrams corresponding to Fig. 4(b) as well, and they will correspond to the spectral function  $A_{13}$  in inelastic processes as they take into account to a certain extent, in fact, as will probably be discussed in Frautschi's lectures, they are taken into account in the "physical approximation". For the moment, however, we simply remark that we must include them in order to maintain the crossing relation.

Thus, in the iteration scheme, we would define (from Eq. (20b))

$$A_{131n}^{(n)}(s,t) = A_{13e1}^{(n-1)}(t,s),$$
 (21a)

$$A_{12in}^{(n)}(s,u) = A_{12el}^{(n-1)}(s,u),$$
 (21b)

and, from (19):

$$A_{12}^{(n)}(t, u) = A_{13e1}^{(n-1)}(u, t) + A_{12e1}^{(n-1)}(t, s).$$
 (21c)

The quantities V<sub>3</sub> (s, t) and V<sub>2</sub>(s, u) are obtained by inserting  $A_{13}$ ,  $A_{12}$  and  $A_{23}$  in the dispersion relations for  $A_3$  and  $A_2$ 

### S. MANDELSTAM

$$V_{3}^{(n)}(s,t) = \int ds' \frac{A_{13in}^{(n)}(s',t)}{(s'-s)} + \int du' \frac{A_{23in}^{(n)}(t,u')}{(u'-u)}$$

+ subtraction terms, (22a)

$$V\binom{n}{2}(s,\iota) = \int ds' \frac{A^{(n)}_{12}(s',u)}{(s'-s)} + \int dt' \frac{A^{(n)}_{23}(t',u)}{(t'-t)}$$

+ subtraction terms. (22b)

One obtains the subtraction terms (if there are any) by solving the Swave equations by the N/D method; the connection between the subtraction terms and the S-waves is outlined in [1]. The method given above can then be used to calculate  $A_{13\,el}^{(n)}(s,t)$  and  $A_{12el}^{(n)}(s,t)$ , and we thus have an iteration procedure which provides a solution of the problem on the assumption that it converges, of course.

### 3. THE SCATTERING AMPLITUDE IN THE CASE OF SUBTRACTIONS.

In the previous section we discussed the construction of the double spectral function  $A_{13}$  and the single spectral function  $A_3$  from unitarity. (Of course, in the general case, we also calculate  $A_{12}$  and  $A_2$ ). We pointed out that, in the absence of subtractions, one could now find the scattering amplitude simply by using the dispersion relation in the momentum transfer:

A (s, t) = 
$$\frac{1}{\pi} \int dt' \frac{A_3(s, t')}{(t' - t)}$$
 (23)

However, this is only true if there are no subtractions in the t dispersion relation. Now we want to discuss the question: what happens if there are subtractions? Can we still get the scattering amplitude by this procedure, knowing  $A_{13}$  and  $A_3$ ?

Just from ordinary, naive, common-or-garden dispersion theory, it does not look as though we can. There are no subtractions if  $A_3 \rightarrow 0$  as  $t \rightarrow \infty$ . However suppose we only have the weaker condition  $A_3/t \rightarrow 0$  as  $t \rightarrow \infty$ . We then need one subtraction, and have to write the equation as

$$A(s,t) = A(s,t_0) + \frac{t-t_0}{\pi} \int dt' \frac{A_3(s,t')}{(t'-t_0)(t'-t)}.$$
 (24)

Now you see that we don't know the scattering amplitude if we know  $A_3$  - we also have to know A (s, t<sub>0</sub>). This doesn't depend on the momentum transfer or the angle so it just corresponds to the S-wave. Therefore, if we have one subtraction, it means that the S-waves are apparently not determined by the double spectral function, but have to be calculated separately - for instance by the N/D method.

Similarly, if A behaves like t at infinity, so that only  $A/t^{2-+}0$ , we need two subtractions and we have to calculate both the S and P-waves separately - they are not determined by the double spectral function.

Now to answer the question: how does A actually go to zero in the problem of interest? We know what it does in perturbation theory. For potential scattering in perturbation theory,  $A(s,t) \rightarrow 0$  as  $t \rightarrow \infty$ , and in the relativistic case it goes to a constant. If the potential is sufficiently small, we can use perturbation theory, and everything is then determined by the spectral function A<sub>3</sub>. However, suppose we increase the strength of the potential, till we get an S-wave bound state. Then A will have a term 1/(s - sB), which is constant as  $t \rightarrow \infty$ . Similarly, if we increase the potential up to the point where we get a bound P-state, then we will have a term

$$(1 + t/2a^2)/(s - s_B).$$
 (25)

The numerator is just the first Legendre polynomial of the scattering angle. And you see that when the potential reaches this strength,  $A \rightarrow \infty$ , as  $t \rightarrow \infty$ , and we need two subtractions. So, as one increases the strength of the potential, the asymptotic behaviour gets worse and worse and, without any further information than we have already put in, one has to perform more and more subtractions.

Now, this, although it is not obviously wrong, does sound a bit paradoxical, because it would be funny if the double spectral function determined everything until the potential reached a certain strength, and then suddenly at this strength of the potential we lost information and couldn't get the Swaves from the double spectral function, and then when the potential reached another strength we lost still more information and couldn't get the P-waves from the spectral functions, and so on. I think that most people would consider this a rather implausible, although not necessarily ridiculous, situation. Now this itself is not so serious, but we see that, in the relativistic case, once  $A \rightarrow \infty$  at infinite t, then in the crossed reaction  $A \rightarrow \infty$  at infinite energy, and we get those troubles I was speaking about yesterday. So for the relativistic case, it is vital to analyse this asymptotic behaviour in more detail to see whether we can get rid of this trouble. As a matter of fact, that is really why I am going to the trouble of doing all this complicated procedure, instead of using the simpler procedure that we used in earlier calculations.

The solution to this problem of the asymptotic behaviour was solved completely by REGGE [6] in the potential theory - this is where he comes into the picture - and he showed that one can, in fact, get rid of this paradox, and that it is possible to get the whole scattering amplitude from the spectral functions  $A_3$  and  $A_{13}$ , even in the case where we have subtractions.

As I have just said, the problem was originally solved by the potential theory, but it can also be solved - and we get the same solution - for the relativistic case that I have just been discussing. What I have to say now will be adequate both for the potential case and the relativistic case. Now let me be careful what we are interested in. We are interested in the construction of the double spectral function and scattering amplitude from the successive procedure which I outlined yesterday which makes use of the

elastic unitarity approximation. The question whether the exact scattering amplitude in field theory has the properties that I am going to describe, is very much more complicated, and we can only apply conjectures at the moment. You are certainly going to hear a lot more about it in other lectures at this Seminar. But for the moment we are interested in the question of whether the functions constructed according to our approximation scheme have certain asymptotic properties, because we want to use these asymptotic properties in solving these equations, and for that we don't need to apply conjectures - everything has now been proved.

The essence of the Regge analysis is to look at everything in the complex  $\ell$  - plane,  $\ell$  being the angular momentum. So this is a new analytic continuation.

The only physical values of  $\ell$  are the positive integers: at  $\ell = 0$ , we have S-waves, at  $\ell = 1$  we have P-waves, at  $\ell = 2$  D-waves, and so on. Now what



Fig. 5 The complex *l*-plane

Regge showed was that this function A(s, l), which is equal to the S-wave at l = 0, the P-wave at l = 1, the D-wave at l = 2, can be continued analytically both to non-integral and complex l to the right of the line Re  $l = -\frac{1}{2}$ . Actually, in potential theory, one can get to the left of this line, but in field theory it seems a bit difficult to do so, so I think I will keep my analysis to what happens to the right of this line Re $l = -\frac{1}{2}$ . Our scattering amplitude is meromorphic in this region. There may be poles in the upper half plane, and these are in fact the Regge poles (Fig. 5).

We now want to use these properties to try and get the asymptotic behaviour of our scattering amplitude as a function of t, the momentum transfer. So let us use the ordinary partial wave expansion

$$A(s,t) = \epsilon (2\ell + 1) A_{\mu}(s) P_{\mu}(z), \quad z = 1 + t / 2q^2.$$
 (26)

What Regge did was to replace this sum by an integral over a contour like that in Fig. 4, of the following expression:

$$A = \int_{C} d\ell \frac{(2\ell + 1) A(s, 1) P_{\ell}(z)}{\sin(\pi)}$$
 (27)

Here  $\ell$  is not restricted to an integer any more. The contour must not enclose any of the poles of A  $(s, \ell)$ , so that the only singularities of the integrand are given by sin  $(\pi \ell) = 0$ , which of course are just the positive integers.

So evaluating the integral by the residue theorem, we get just the partial wave expansion. We put  $P_{\ell}$  (-z) instead of  $P_{\ell}$  (z) in order to cancel the alternation of sign of the residues of sin  $(\pi \ell)$  between the even and odd integers.  $(P_{\ell} (z) = (-1)^{\ell} P_{\ell} (z)$  for integer  $\ell$ ).

The next thing one does is to deform the contour of integration until it goes along the line  $\operatorname{Re} \ell = -\frac{1}{2}$ . However, in doing so we have to cross these poles of A (s,  $\ell$ ), so we must add

$$\Sigma (2\alpha + 1)\beta P\alpha (z) / \sin \pi \alpha.$$
 (28)

Now let me call the positions of these poles  $\alpha_1$ ,  $\alpha_2$  and so on, and the residues at each of these poles we will call  $\beta$ .  $\alpha$  and  $\beta$  will depend on the energy, so the equation becomes

$$A(s,z) = \int_{\operatorname{Re} \ell} d\ell \frac{(z\ell+1) A_{\ell}(s) P_{\ell}(-z)}{\sin(\pi \ell)} + \sum_{n} \frac{(2\alpha_{n}(s)+1) \beta_{n}(s) P_{\alpha n}(s)^{(z)}}{\sin(\pi \alpha_{n}(s))}$$
(29)

We can use this formula to find at once the asymptotic behaviour of A as a function of z. The reason is that we know that

$$P_{\alpha}(z) \sim z^{\alpha}$$
, as  $z \rightarrow \infty$ .

Now in the first (integral) term, the real part of l is  $-\frac{1}{2}$ , so this part goes down like  $|z|^{-\frac{1}{2}}$  at infinite z. So we will forget about that, since it goes down very nicely. Anything that goes to zero, we are not interested in. The pole terms, however, behave like  $z^{\alpha}$  as  $z \rightarrow \infty$ . In particular, the pole that dominates is the one that has the largest real part. So the asymptotic behaviour will be  $\sim z^{\alpha_1}$  where  $\alpha_1$  is the pole furthest to the right.

This gives the results in principle, but in order really to be able to see what is going on, we have to know how the function  $\alpha$  depends on s, so let me take the  $\ell$ -plane again. What normally \* happens is that for sufficiently large, negative values of the energy, the poles all lie to the left of the line  $\operatorname{Re} \ell = -\frac{1}{2}$ , so that we just do not see them., As the energy increases, the poles move to the right along the real axis. Now it may happen that, at some energy  $s_0 < 4\mu^2$ , before the threshold, one of the poles passes through  $\ell = 0$ . At that point the scattering amplitude becomes infinite, it therefore has a pole as a function of s, at s = s\_0. At  $\ell = 0$ , P = 1, so the residue at the pole does not involve the angle at all, and what we have is a bound S-state. As we continue to increase the energy, it may happen that, for a very strong potential, the pole actually passes through  $\ell = 1$  before we reach the threshold s =  $4\mu^2$ . Again we have a bound state, because  $\sin(\pi \ell)$  becomes zero. However, the residue is now proportional to  $P_1(z)$ , so we have a bound P-state. In general we have bound states at those values of s, for which  $\alpha$  (s) = a positive integer.

<sup>\*</sup> In relativistic theory, when one solves the N/D equations, they will sometimes give poles with rather unphysical properties ("ghosts"), which may be due to the failure of the approximation theory, I will ignore these.

Suppose that we have now got to the threshold  $s = 4\mu^2$ . Then what happens as we continue to increase s is that the poles move out into the complex plane. They go a certain distance further to the right, but eventually come back again, and when s is sufficiently large the poles disappear to the left of the line  $\operatorname{Re} \ell = -\frac{1}{2}$ .





Trajectory of a Regge pole in the complex l-plane

Now for positive kinetic energy, seeing that the poles are complex, they never pass through a positive integer, so we don't get bound states. However, it may happen, as in the case I have drawn, that a pole passes <u>near</u> a positive integer. Then  $\sin(\pi \ell)$  will be very small, and the scattering amplitude, although it does not get infinite, gets very large. So when the pole passes <u>near</u>  $\ell = 1$  we get not a P-wave bound state but a P-wave resonance.

Well one can draw this in a different way if one likes: suppose one just plots Re  $\alpha$  against s (Fig. 7). The line at the bottom is Re  $\alpha = -\frac{1}{2}$ . We don't know what is going on below this. The vertical line is the threshold s =  $4\mu^2$ .



A Regge trajectory: Re a as function of s

The points where the curve passes through integers are bound states to the left of this line, and resonances to the right of it. These curves are sometimes called Regge trajectories. For potential scattering, we have an  $\alpha$  for each value of the radial quantum number. All that the Regge trajectory then does is to interpolate between the known bound states, like a Bohr angular momentum plot turned sideways.

Let us turn back to this question of how we can find the scattering amplitude from the spectral functions if we need subtractions, without introducing any optional quantities. We can do this, given the fact that the scattering amplitude satisfies the Regge formula. The reason is the following: we know the analytic properties of  $P_{\alpha}(-z)$  in the z plane. It is analytic in z, except for a cut along the real axis from z = 1 onwards. The discontinuity across this cut is  $P_{\alpha}(z) \sin(\pi \alpha)$ . In particular for  $\alpha$  an integer,  $P_{n}(z)$  is analytic all the way, because this discontinuity is then equal to zero. If therefore the scattering amplitude satisfies the Regge formula, the spectral function  $A_{3}$ , which is the discontinuity as a function of z (or as a function of t, which is the same thing), will satisfy the formula

$$A_3(s,t) = A_{3B}(s,t) + \Sigma \left(2 \frac{\alpha}{n} (s) + 1\right) \beta_n(s) P_{\alpha_n(s)}(z).$$
 (30)

The first term, which I will call the background term, comes from taking the discontinuity of the integral. We can get the discontinuity of the pole terms from the discontinuity of  $P_{\alpha_n}$  (z), because this is the only place in them where the momentum transfer is involved.

Now, remember our problem is to calculate A, given A<sub>3</sub>. We cannot do this by putting A<sub>3</sub> into the dispersion relation for A, because we have subtractions. However, the "background" term goes down like  $t^{-\frac{1}{2}}$ . So, for the background term, we can find A from A<sub>3</sub> by using the dispersion relation without subtractions. Now, if we know A<sub>3</sub> numerically, then we can separate it into Regge pole terms and the "background" term by equation (30), and find the  $\alpha$ 's and  $\beta$ 's of the Regge pole terms. This is practical numerically [7]. And therefore all one needs to do is to put the  $\alpha$ 's and  $\beta$ 's into Eq. on (29) and one has obtained the whole scattering amplitude from the spectral function without introducing any arbitrary subtractions.

All right then, so this is how we get over this apparent paradox of not being able to calculate the scattering amplitude from the spectral functions. We see that, once we know the Regge formula, we can calculate the scattering amplitude from the spectral functions, even when we have lots of subtractions, and therefore we do not lose information when the asymptotic behaviour gets worse and worse, as we increase the strength of the potential or the strength of the coupling. Now let us go on to the second point: can this get us over our difficulties of bad asymptotic behaviour, which gave us singular integral equations in the previous scheme? Now you remember that in the old method we found that if we only had large S-waves then things go like a constant at large t, which is all right, we do not mind that sort of behaviour. However, if we have large P-waves as well, then things go proportional to t at large t, and similarly, if we have large S, P and D-waves, then things go proportional to  $t^2$ . Now what we have in the Regge formula is slightly different. According to the Regge formula, the asymptotic behaviour as  $t \rightarrow \infty$  depends on s. At those points where we have an S-wave

bound state,  $\alpha = 0$ , and the asymptotic behaviour is a constant just as before. Similarly, the asymptotic behaviour is still proportional to t at a P-wave bound state. However, even if we have a P-wave resonance or bound state. the asymptotic behaviour is no longer proportional to t everywhere. Now you know that the trouble resulted from the fact that, if the direct reaction has a bad asymptotic behaviour as a function of t. which does not matter. then the crossed reaction would have a bad asymptotic behaviour as a function of the energy, which does matter. However, the interchange of s and t only takes us into the physical region of the crossed reaction if s is negative. because in the crossed reaction t is the energy, s the momentum transfer. and for a physical reaction the momentum transfer is always negative. The energy is always positive. So we only expect to get into trouble if we have a bad asymptotic behaviour as  $t \rightarrow \infty$ , where s is negative. Therefore, in a Regge curve like Fig. 6, so long as we keep  $\alpha < 1$  when s is negative, we would not expect to get into trouble, even though  $\alpha > 1$  when s is positive, so that we could get P-wave resonances, and resonances of any higher angular momentum. So it is this dependence of  $\alpha$  on s which can probably get us over the difficulty that the old procedure led us into. And it is, as a matter of fact, the equivalent of the spreading out of the wave function of a composite system in space that one gets if one uses any method other than dispersion relations.

I may say that the only way  $\gamma$  can depend on s is if the spectral function oscillates. (In this case you observe that the spectral function does oscillate, because we have  $t^{\alpha}(s)$ , with  $\alpha$  complex for positive s, and a number to a complex power is an oscillatory function). This follows from the ordinary dispersion relation in the energy

A (s, t) = 
$$\frac{1}{\pi} \int ds' \frac{A_1(s', t)}{(s' - s)}$$
 (31)

Now suppose for a certain value of s',  $A_1$  had bad asymptotic behaviour as a function of t, and suppose there were no oscillations, so that there could be no cancellation in sign. Then, if one performs the integral, the expression on the left will have the same bad asymptotic behaviour, whatever the value of s. So, if our spectral function does not oscillate, we cannot have an asymptotic behaviour as a function of t which depends on s. However, if  $A_1$  does oscillate, this bad asymptotic behaviour may cancel out in the dispersion relation, and we can have the asymptotic behaviour depending on s, which is what does happen in the Regge formula.

Now you may think that this construction of the spectral function by the method of successive approximations, and subsequently isolating these Regge pole terms to get  $\alpha$ 's and  $\beta$ 's, is something rather complicated However, it has actually been carried out in model calculations by BURKE and TATE [7] well, not for the complete relativistic case, but for the relativistic case where they assumed  $V_{2,3}$  known, and they also did it for the potential case. The calculation is almost the same, and doing it for the potential case, they find that the results agree with the direct calculations of the Regge trajectories from the Schroedinger equation, which was made both by them and by LOVELACE at Imperial College [8]. So it looks as if this method is feasible. However, at the moment one does not have a consistent set of integral equations for the problem [9]. The difficulty comes from the fact that if we have an input with Regge asymptotic behaviour in s, and use the unitarity equation in the s channel, we obtain an output with still worse asymptotic behaviour. The situation has not been clarified, that is all I can say at the moment.

In order to conclude this lecture, let me now go on to the conjectures, which have not been made till now, that the exact scattering amplitude also has a Regge asymptotic behaviour, and see what experimental consequences that will lead to.

We shall now use crossing symmetry, so that s and t are interchanged, and we shall assume that A(s, t) behaves like  $\beta(t) s^{\alpha}(t)$  as  $s \to \infty$ . And we now conjecture that this is true of the exact scattering amplitude, not only the scattering amplitude which is constructed from the strip approximation. Let us use this formula to analyse what happens in the diffraction peak region, where s is large and t is small and negative. This was first done independently, I think, by CHEW, by FRAUTSCHI, GELL-MANN and ZACHARIASEN, and by LOVELACE [10]. First of all, we shall take it as an experimental fact that the cross-section is constant for large values of the energy.

It follows from the optical theorem that A(s, t) will go like s, as  $s \rightarrow \infty$ , at t = 0, which is the forward direction. Such a result does not follow from the Regge analysis, one has to put this in. We thus observe that  $\alpha = 1$  at t = 0, and we can rewrite the Regge asymptotic behaviour as

$$A(s,t) \approx i\beta(t)s e^{-\eta(t) \log s}$$
(32)

 $\eta$  is a decreasing function of t, so that as we go into the physical region (negative t), the scattering amplitude falls off, which is what one expects it to do. However, it does not fall off in the same way as one would expect in the optical model, for two reasons.

Firstly, it follows from the double dispersion relation that  $\eta$  must be analytic near t = 0, so we can put  $\eta \approx \gamma t$ , and we see that the scattering amplitude will go down exponentially as a function of t. In the optical model, if one assumes that the diffracting object has a Yukawa shape, which is a reasonable thing to do, the diffraction peak would go down much more slowly than exponentially, it would go down like an inverse power. Experimentally one definitely finds an exponential type of behaviour, rather than anything like an inverse power behaviour.

The second thing is that the width of the diffraction peak depends on the energy, because you could say that the width of the diffraction peak is essentially that value of t where A(s, t) reaches some given value, and you see that the bigger the value of log s, the less distance you will have to go in t in order to reach any particular value. The width of the diffraction peak therefore shrinks logarithmically as the energy s is increased. Now, if that was the only thing, one would not be so surprised, because a shrinking diffraction peak corresponds to an increase in the size of the diffracting object. The bigger the diffracting object, the smaller the diffraction peak. So one could say that the peripheral regions of the nucleon were just getting more and more effective as the energy was being increased. However, the thing that is very surprising from any classical sort of analysis is that the total oross-section is remaining constant at the same time, so not only is the nucleon getting bigger and bigger, but the inner part is getting more and more transparent at the same time, in order to keep the total cross-section constant. Such a feature is in conflict with any sort of physical visualization by means of an optical model, so if the Regge pole model is right then the optical model is wrong. And the experimentalists - I am not quite sure just under how much brain-washing - say they see a narrowing of the diffraction peak [11]. That would therefore mean that the optical model visualization is bad, and that there is an essential truth in this method of visualization. It does not prove that the conjecture of applying the Regge pole formula to the exact scattering amplitude is necessarily right. For instance, if the AMATI-FUBINI-STANGHELLINI analysis [12] is correct, and there are Regge cuts, in addition to Regge poles, then I think one could still fit the results to present experiments. So I would say that the experiments show that a Regge formula, or something of a similar sort which is rather more complicated, is correct.

In doing the unitarity condition, obviously, different quantum numbers do not get mixed up, so we get different <u>uncrossed</u> Regge trajectories associated with different quantum numbers. In terms of the crossed process, where the Regge asymptotic behaviour is in s, these correspond to different quantum numbers being exchanged. Pure diffraction scattering can't exchange any quantum number, so the Regge trajectory which produces the diffraction scattering, and goes through the value 1 at t = 0, must have the quantum numbers of the vacuum. Gell-Mann has called the object that gets exchanged a Pomeranchon, because if this Regge trajectory dominates, the Pomeranchuk theorem is valid.

Now, of course, for other kinds of elastic scattering, there will be other trajectories [13] coming below this one, and GELL-MANN, FRAUTSCHI and ZACHARIASEN [10] have proposed experiments to look at these lower trajectories, I do not think I need go into them, because we are sure to hear a lot more about that in further lectures.

But let me end by referring very briefly to a further conjecture made by CHEW and FRAUTSCHI [10] that fits in here. Remember, when we were speaking about the potential theory, I said that, in writing all these dispersion relations and double dispersion relations down, we might be prepared to include the S-wave subtraction explicitly. We do not want to include higher subtractions explicitly, because that would give much trouble with the unitarity equation. Thus, in addition to these double dispersion terms, we could have terms like

$$\int \frac{\mathrm{ds'f}(s')}{(s'-s)}$$
(33)

which do not depend on t. If one were doing the calculation in such a case, one would get all the angular momentum states other than the S-wave by integrating over the double spectral function, but for the S-wave one would conclude by performing an N/D calculation to find the function f. We thus observe that the asymptotic behaviour of the scattering amplitude consists of a Regge term  $t^{\alpha}(s)$ , coming from the double spectral function, plus another term which is asymptotically just a constant. In the Regge term the asymptotic behaviour does depend on s, in this other term it does not. The point is that, when we do have a subtraction, there is extra information which we can introduce. For instance, we might put into the calculation a pole a/(s - sp), with two constants. The constants a and sp have to be known beforehand, they are not given to us by the theory. And if we put in the pole like this, then in order to have consistency at least in the approximation schemes that have been tried up till now we would have to do a subtraction in t. We would have to calculate the S-waves by the N/D technique, they would not be given from the double spectral function, and the asymptotic behaviour would be given by the sum of two terms, one which does depend on s and one which does not.

Now the question is often raised whether there is really a distinction between elementary particles and non-elementary particles, and I do not think it is one that one can really answer definitely. There will probably always be conflicting views until we have a complete theory, and I rather think that, if we do ever get a complete theory, it is not going to make any distinction between elementary particles and non-elementary particles. But, if we do not have a complete theory at the moment, there may be some particles that one cannot get from the calculations - that one has to put in at the beginning - which provisionally one would call elementary particles, and some other particles which one can calculate, which one would not call elementary particles. And generally one would expect to have to introduce the masses and coupling constants for elementary particles, but to be able to get the masses and coupling constants for non-elementary particles in principle from the calculations.

When we put in the subtraction term from the beginning, we actually have to put in the mass sp - the position of the pole - and the coupling constant a. So one may therefore take the viewpoint that terms like this, where the asymptotic behaviour does not depend on the energy, correspond to elementary particles, whereas terms where the asymptotic behaviour does depend on the energy correspond to bound states. If we subtract P-waves or higher angular momentum waves, it is going to give us the old trouble again, and therefore, from this way of looking at it, we can only have Swave elementary particles, not P-wave and higher angular momenta. In other words, we get the same results we get from the renormalization theory studied by perturbation methods. And this again fits in with what I said about this s-dependence of the asymptotic behaviour corresponding to the spreading out of particles in space, because the elementary particle, which one naively supposes at least to have something in the middle which is not spread out in space, does not have this s-dependence of  $\alpha$  in the asymptotic behaviour. The proposal made by Chew and Frautschi is, in fact, that, even at the present moment, there are no particles that one has to put in and call elementary, but that all particles correspond to the points where these Regge trajectories pass through the positive integers, and all resonances to the points when they pass near the positive integers. Fairly detailed graphs have been drawn up with these Regge trajectories for all the different quantum numbers, and the various resonances have been put in. I think at the moment the number of resonances is not yet so much greater than the number of quantum states that one would ascribe much significance to this fit, and I do not think that the authors claim that one should. But at least it is interesting to see how, if we assume that it is correct, the various particles could be fitted into the Regge scheme.

For those of you whose family keep on asking you what sort of work you are doing, there is an article by Chew, Gell-Mann and Rosenfeld which is going to appear in the Scientific American, so you will be able to get them to read that. There, all these Regge trajectories are drawn in a nice coloured diagram.

### REFERENCES

- [1] CHEW, G.F. and MANDELSTAM, S., Phys. Rev. 119 (1960) 467.
- [2] CINI, M. and FUBINI, S., Ann. Phys. 10 (1960) 352.
- [3] CHEW, G.F. and FRAUTSCHI, S.C., Phys. Rev. <u>123</u> (1961) 1478; Phys. Rev. Lett. <u>5</u> (1960) 580;
   MANDELSTAM, S., Phys. Rev. <u>112</u> (1958) 1344; TER-MARTIROSYAN, K.A., JETP <u>39</u> (1960) 827;
   WILSON, K., Harvard preprint (1960).
- [4] BLANKENBECLER, R., GOLDBERGER, M., KHURI, N. and TREIMAN, S., Ann. Phys. 10 (1960) 62.
- [5] CHEW, G.F. and FRAUTSCHI, S.C., Phys. Rev. 124 (1961) 264.
- [6] REGGE, T., Nuovo Cimento 14 (1959) 951; 18 (1960) 947.
- [7] BURKE, P.G. and TATE, C., Berkeley preprint (1962).
- [8] AHMADZADEH, BURKE and TATE, Berkeley preprint (1962); LOVELACE, C. and MASSON, D. London preprint (1962); Nuovo Cim. to be published.
- [9] CHEW, G.F., FRAUTSCHI, S.C. and MANDELSTAM, S., Phys. Rev. (1962)
- [10] CHEW, G.F. and FRAUTSCHI, S.C., Phys. Rev. Lett. <u>7</u> (1961) 394; <u>8</u> (1962) 41; GRIBOV, V. N., JETP <u>4</u> (1961) 667; LOVELACE, C., London preprint (1961); Nuovo Cim. in press; FRAUTSCHI, U, S.C., GELL-MANN, M. and ZACHARIASEN, F., Cal. Tech. preprint (1961); Phys. Rev. to be published.
- [11] DIDDENS, A. N. et al., CERN preprint (1962).
- [12] AMATI, D., FUBINI, S. and STANGHELLINI, A., Phys. Lett. 1 (1962) 29.
- [13] BLANKENBECLER, R., GOLDBERGER, M.L., Phys. Rev. 126 (1962) 766.

## THE ONE-PION EXCHANGE MODEL AND ITS RELATION TO REGGE POLE EXCHANGE

### S. FRAUTSCHI

### NEWMAN LABORATORY OF NUCLEAR STUDIES, CORNELL UNIVERSITY, ITHACA, N.Y., UNITED STATES OF AMERICA

### 1. INTRODUCTION

Soon after the 30 GeV machine came into operation the one-pion exchange model [1] was proposed. It was the first dynamical model which provided some means of explaining phenomena observed at these high energies and it has even been extended up to cosmic ray energies. In this talk the main features of the model will be reviewed and some of its limitations and relationships with Regge poles will be discussed.

### 2. MAIN FEATURES OF THE MODEL AND ITS LIMITATIONS

The importance of one-pion exchange can be seen by considering peripheral collisions. In a semi-classical picture, the peripheral interaction is a glancing collision where the incident particle only interacts with the outer region of the target and receives a small momentum transfer. In this situation the long-range forces are the most important and in strong interactions the one-pion exchange provides the longest range force. Further more, the peripheral collisions seem to give a significant fraction of the total cross-section. This is because strong interaction cross-sections are of order

$$\tau_{\rm tot.} \sim \pi R^2$$

where R is the pion Compton wavelength. Cross-sections of such large magnitude are sensitive to the outer parts of the target and therefore to one-pion exchange. Another attractive feature is that one-pion exchange is relatively easy to analyse theoretically.

There are, however, a number of limitations to the application of this model:

1) A rather trivial limitation when electromagnetic interactions are also present is that the Coulomb force is of longer range. This gives a pole at zero momentum transfer (Fig. 1) and so it is necessary to avoid the resulting peak in the forward direction. At high energies the Coulomb peak is confined to a very small angles and presents no difficulty in practice.

2) A more important limitation is due to multi-particle exchange contributions, which are also present. For small momentum transfer the single

<sup>\*</sup> Text based on notes by A. P. Contagouris and G. C. Oades.



Position of singularities in the complex t-plane

pion pole may dominate the scattering, but for larger momentum transfer the two- and more-pion exchange contributions are at a comparable distanc (Fig. 1) and there is no longer any reason to neglect them.

3) Even for small momentum transfer the multi-particle contribution could be more important if the discontinuity across the cut is large enough. In terms of potentials, the one-pion pole gives a term of the form

$$g e^{-m_{\pi} r} / r$$
 (1)

while the multi-particle continuum gives:

$$\int_{4m_r^2}^{\infty} d\mu^2 g(\mu^2) \frac{e}{r} \frac{e}{r}; \qquad (2)$$

if  $g(\mu)$  is large enough, this term can be more important than (1). An example of this situation is provided by the diffraction peak in elastic scattering. To study the diffraction peak it is necessary to consider the absorptive part of the amplitude in the forward direction; the optical theorem gives

$$\operatorname{Im} f^{\mathbf{e}\mathbf{l}}(\mathbf{t} = \mathbf{o}) = \sum_{n} f^{\mathbf{e}} \rho f_{n}$$
(3)

where  $f_n$  is the amplitude for transition between the initial (or final) state and an intermediate state n and  $\rho_n$  is the phase space factor associated with this state n. In terms of diagrams Eq. (3) can be represented as shown in Fig. 2.



It is seen that at least two pions must be exchanged and therefore diffraction is associated with a shorter range term. There is a large contribution because all terms add coherently.

Consider as an example the process  $p + p \rightarrow p + n + \pi^+$ , which has been studied by SELLERI and FERRARI [2, 3, 4]. A typical one-pion exchange diagram is shown in Fig. 3.



Fig.3

A one-pion exchange contribution to  $p + p \rightarrow p + n + \pi^+$ 

This diagram gives a contribution to the cross-section of the form:

$$d\sigma/dt \alpha \left[g^2 t/(t - m_{\pi}^2)^2\right] \sigma(n^+ p \rightarrow \pi^+ p).$$
(4)

The factor t results from the negative parity of the pion. Other one-pion exchange diagrams give similar terms and also interference contributions. Selleri and Ferrari compared the final expression with experimental data between 1 and 3 GeV; they found good agreement provided they introduced a one-parameter cut-off for large momentum transfers. This is certainly a reasonable approach since the model is not expected to be valid at large momentum transfers, as pointed out above.

The last example was a particularly favourable case, since the pionnucleon coupling constant and the low-energy cross-section were known. In general, less information is available. In this case, certain tests can be applied to the experimental data to see if the model is working. First, there is a specific prediction for the momentum transfer variation which should be satisfied at least for small momentum transfers. A second type of test has been proposed by YANG and TREIMAN [5]. In the left-hand side of Fig.3 consider the frame in which the incoming proton is at rest ( $\vec{p} = 0$ ). A spin zero pion with momentum  $\vec{q}$  is then scattered on a stationary proton; therefore the final state should have azimuthal symmetry about q. If the data show an isotropic distribution for rotation of the final 3-momentum about  $\vec{q}$ , they are consistent with the exchange of a single pion.

In a case when all proposed tests are satisfied so that the validity of the model is likely, it is possible to use the data in many ways. For example, in the case of  $p + p \rightarrow p + n + \pi^+$ , the data could be used to obtain the coupling constant g if the pion-nucleon cross-section is known. Alternatively, if the coupling constant is known, then the data could be used to obtain  $\sigma(\pi^+ p \rightarrow \pi^+ p)$ . It should be noted, however, that some extrapolation is always required. Either the experimental results must be continued to the pion pole  $t = m^2$  so that physical values for  $\sigma(\pi^+ p \rightarrow \pi^+ p)$  can be used; or else the physical values for  $\sigma(\pi^+ p \rightarrow \pi^+ p)$  must be continued to the region  $t \leq 0$  where the experiment  $p + p \rightarrow p + n + \pi^+$  is conducted. Also, note that the tests made on the data refer to necessary but not sufficient conditions for the validity of the model.

As has been already stressed, with increasing momentum transfer corrections become necessary even in favourable cases. FERRARI and SELLERI [3] continue the physical  $\pi$  - N cross-section by use of Chew-Low theory; this is justified since they are still in the region of the 3-3 resonance even
for larger t. They also introduce [4] a pion form factor  $F_{\pi}(t)$  and express the cross-section in the form:

$$d\sigma/dt \propto g^2 F_{\pi}^2(t) \left[ t/(t-m_{\pi}^2)^2 \right] \sigma(\pi^+ p \to \pi^+ p) \cdot$$
 (5)

Then they obtain  $F_{\pi}$  (t) by fitting the experimental data,

Of course, states with other quantum numbers, such as  $\rho$  and  $\omega$ , can also be exchanged. As a first approximation one might hope to describe the scattering as an exchange of only a few such objects, each being treated with a form factor as was pion exchange. Formula (5) represents the optimistic limit in which only one object, the pion, is exchanged.

Suppose now that this procedure is applied at higher energies ( $\geq 2 \text{ GeV}$ ). For simplicity, consider a case where only a pion and a  $\rho$  are exchanged. The amplitude can then be represented by

$$f = P_o(\cos \theta_t) G_{\pi}(t) + P_1(\cos \theta_t) G_{\rho}(t)$$
(6)

where in terms of Mandelstam variables

$$\cos \theta_{\rm f} = -1 - s/2q_{\rm f}^2 . \tag{7}$$

The first term of (6) is due to the exchange of a pion (spin zero) and the second to that of a  $\rho$  - meson (spin one). Eq. (7) shows, then, that for large s the asymptotic contributions to the amplitude are correspondingly  $\sim s^{0}$  and  $\sim s^{1}$ . To test these predictions we can use the available data on protonproton scattering in the region 1-4 GeV (Brookhaven) and above 12 GeV (CERN)[6]. These data are derived from experiments in which fast outgoing protons of varying energy E' are detected at fixed angle for a given incoming proton energy E. A typical cross-section is shown in Fig. 4. Apart from the



Fig.4

High energy proton-proton differential cross-section

elastic peak (1), a number of other peaks are superimposed on the inelastic continuum. The positions of these peaks seem to be quite stable and are consistent with the interpretation that (2) corresponds to production of the P<sub>3/2</sub>, T = 3/2 isobar, (3) to the D<sub>3/2</sub>, T = 1/2 isobar and (4) to the F<sub>3/2</sub>, T = 1/2 isobar. Now, as the energy rises for fixed (high) momentum transfer, the most striking feature is that the peak (2) disappears quickly, while (3)

#### THE ONE-PION EXCHANGE MODEL

and (4) decrease only slowly. The observed decrease at  $|t| \ge 1 \text{ GeV}^2$  is found to be clearly more rapid than the predictions of a one-pion exchange model (asymptotic behaviour  $\sim s^0$ ) [6]. Exchange of spin one (asympt, behaviour  $\sim s^1$ ) or any finite number of higher spin particles will only make matters worse.

## 3. RELATIONSHIPS OF THE MODEL WITH REGGE POLES

An alternative way to handle the scattering is provided by the Regge pole hypothesis, according to which Regge poles of spin  $J_i(t)$  are exchanged. This has the following features at high energies:

(1) The maximum spin exchanged at each t is finite, since the Regge pole is a kind of bound state resulting from forces of finite range in the crossed channel. Exchange of the other, lower spin, terms can be ignored at sufficiently high energy, leaving a simple expression for the amplitude growing as  $s^{Jmax(t)}$ .

(2) The spin of each Regge pole varies with t, decreasing with increasing momentum transfer in the physical region of the process.

From the  $s^{J(t)}$  growth of the amplitude it follows that exchange of a Regge pole with spin J (t) contributes to the differential cross-section an asymptotic term of the form

$$\frac{d\sigma}{dt} = F(t) \left(\frac{S}{2m^2}\right)^{2J(t)-2}.$$
(8)

Now, the proton-proton CERN experiments above 12 GeV establish the following upper limit in the cross-section for production of the 33 isobar:

$$(d\sigma/d\Omega_{lab})_{33} \leq 0.05 (d\sigma/d\Omega_{lab})_{elastic}$$

If this is combined with Eq.(8) and the p-p scattering data at lower energies ( $\leq 4 \text{ GeV}$ ), the upper limit  $J_{max}(t)$  for the spin of the exchanged pole can be determined [6] (Fig. 5)



The maximum exchanged spin consistent with 3-3 production and the Pomeranchuk Regge trajectory

It can be seen now that the decrease of J(t) for increasing physical |t| and the possibility of having J(t) < 0 at sufficiently high momentum transfer may easily provide results compatible with  $J_{max}(t)$  of Fig. 4. This is clearly not

possible if the amplitude A behaves asymptotically like  $\sim s^0$  (exchange of a particle of constant J(t) = 0), let alone if A  $\sim s^n$  (n  $\ge 1$ ).

The production of the second and third pion-nucleon isobar can be studies in the same spirit. In this case the results are compatible with a spin varying as  $J_p(t)$  of Fig. 5. Notice that one pion exchange (J=0) does not dominate even at small |t|.

The difference in the spin of the exchanged Regge object for production of the  $P_{3/2}$  isobar and for production of the  $D_{3/2}$  or  $F_{5/2}$  isobar can be easily understood from Fig. 6. In the first case the isospin of the exchanged parti-



Fig.6

Diagrams for the exchange of the leading Regge term in p-p T = 3/2 and T = 1/2 isobar production and elastic scattering

cle has to be T = 1; in the case of  $D_{3/2}$  or  $F_{5/2}$  production, as in the case of elastic scattering, T = 0 is also present and therefore the "Pomeranchuk" (or "vacuum") trajectory can be exchanged.

At this point a possible application of the  $D_{3/2}$  or  $F_{5/2}$  production to future very high energy accelerators may be mentioned. These accelerators will provide fast protons but their usefulness will be extended if collisions of the protons with a target produce a good secondary beam of fast pions. Cosmic ray data indicate the existence of such events, but a more quantitative estimate is desirable. Now, the functions J(t) and f(t) of formula (8) for  $D_{3/2}$  and  $F_{5/2}$  isobar production can be determined at present machine energies; then the formula can be extrapolated to very high energies. The calculation carried out in [6] along these lines indicates that a significant flux of pions with energy up to 2/3 of the original proton energy should be expected.

#### 4. CONCLUSIONS

To summarize the situation in elastic or nearly-elastic events at high energy: We have presented some reasons why large momentum transfer events are better understood in terms of Regge pole exchange than in terms of exchange of a few particles like the  $\pi$  and  $\rho$ . As stated earlier, there was no strong reason for the one-pion to work in this region anyway. At low momentum transfer, if there is a Regge trajectory with higher J(t) than the pion trajectory, then this will dominate the scattering at sufficiently high energy. If there is no higher trajectory, or at intermediate energies where the factor S<sup>J(t)</sup> is not yet dominant, exchange of the pion Regge trajectory can control the scattering. In this case there is a correspondence with the one-pion exchange model at low momentum transfer where J<sub> $\pi$ </sub> (t)  $\approx$ 0 and, in fact, this correspondence is exact in the limit  $t \rightarrow m_{\pi}^2$ . From the work of Selleri and Ferrari, production of the 3-3 resonance in p - p scattering at 1 to 3 GeV is an example of this latter case.

Up to now, only a small class of inelastic processes has been discussed. I shall close with a few comments about our fragmentary understanding of more highly inelastic events. Many particle production, including energies above 30 BeV, has also been considered in terms of one-pion exchange. In place of the  $p\pi^+$  final state in Fig. 2, for example, one allows all available final states for  $p\pi^+$  scattering to emerge from the vertex on the left side. The resulting sum over amplitudes and phase space has an effect equivalent to a single amplitude growing like s<sup>1</sup> even though a spin zero particle was exchanged. In this way one recovers the possibility mentioned earlier, that one-pion exchange may give an appreciable part of the total cross-section.



One-pion exchange with many particle production

When one attempts to study the total cross-section, however, a problem arises. The factor s which has just been introduced has an effect similar to that associated with a spin-one exchange, which is known in general to introduce divergences. In fact, when the contributions of one-pion exchange terms are integrated over to obtain the total cross-section, it diverges logarithmically and some cut-off must be introduced. In the elastic case a precise formula is provided by the Regge pole hypothesis; as t falls below zero, J(t) decreases and the divergences are avoided in a natural way. In the inelastic case the mechanism of the cut-off is not understood but the data indicate the presence of strong damping at high momentum transfers so a cutoff is certainly present.



Multi-particle production resulting from one-pion exchange

One pion exchange with many particle production relates the amplitude for high energy pp scattering, for example, to a product of amplitudes for high energy  $\pi p$  scattering (Fig. 7). A further step has been taken by AMATI <u>et al.</u> [7], GOEBE [8], and by F. and G. SALZMAN [9], who break down the amplitude into the product of several one-pion exchange terms (Fig. 8). This step has the great advantage of reducing a high energy amplitude to a product of low energy amplitudes. It will be discussed in more detail in Fubini's lectures; a related approach which puts less emphasis on pion exchange is presented in my lecture on highly inelastic processes.

#### REFERENCES

- [1] A good review of developments up to 1961, with many references, is provided by DRELL, S. D., Rev. Mod. Phys., 33 (1961) 458. A more recent review is that of FERRARI, E. and SELLERI, F., CERN Report 2845/TH 243.
- [2] SELLERI, F., Phys. Rev. Letters, 6 (1961) 64.
- [3] FERRARI, E. and SELLERI, F. Nuovo Cimento, 21 (1961) 1028.
- [4] FERRARI, E. and SELLERI, F., Phys. Rev. Letters, 7 (1961) 387.
- [5] YANG, C.N. and TREIMAN. S.B., Phys Rev. Letters, 8 (1962) 140.
- [6] CONTOGOURIS, A.P., FRAUTSCHI, S.C. and WONG, H.S., (submitted to Physical Review)
- [7] AMATI, D., FUBINI, S., STANGHELLINI, A., TONIN, M., Nuovo Cimento, 22 (1961) 569.
- [8] GOEBEL, C., Proc. Int. Conf. on Theoretical Aspects of Very High-Energy Phenomena. CERN (1961) 353.
- [9] SALZMAN, F. and SALZMAN, G., Phys. Rev. 120 (1960); 599 121 (1961) 1541.

# REGGE POLES AND DRELL'S PRODUCTION OF HIGH-ENERGY PARTICLES\*

## A.P. CONTOGOURIS NUCLEAR RESEARCH CENTRE "DEMOCRITOS", ATHENS, GREECE

About two years ago DRELL [1] proposed a simple mechanism for the production of collimated beams of high-energy particles by photons. In the case of pion production, a photon of high-energy k incident, e.g., on a nucleon target, produces an energetic pion of momentum q essentially in the forward direction. If, in the spirit of peripheral collisions, the process is approxi-



Diagrams for the general production amplitude  $\gamma + N \rightarrow \pi + (n \text{ particles})$ and for the one-pion exchange contribution to this amplitude

mated by the one-pion exchange diagram (Fig. 1), the differential crosssection will be of the form

$$d\sigma_{\pi} \sim e^2 q^2 \sin^2 \theta_q \left[ \sigma_{\pi * N} (k - \omega_q) / (t - \mu^2)^2 \right] d^3 q \qquad (1)$$

where -t is the square of the invariant momentum transfer,  $\mu$  the pion mass,  $\theta_q$  the pion production angle and  $\sigma_{\pi^*N}(\mathbf{k} - \omega_q)$  the total cross-section for the process N+ (virtual pion)  $\rightarrow$  (N particles) at energy  $\mathbf{k} - \omega_q$ . For small momentum transfers the propagator  $|\mathbf{t} - \mu^2|^{-1}$  is of the order of  $1/\mu^2$  and this explains the importance of this diagram relative to others with more massive intermediate states. For high-energy  $\omega \sim \mathbf{k} \gg \mu$  and small angles

$$\left|\mathbf{t}-\boldsymbol{\mu}^{2}\right|\simeq\boldsymbol{\mu}^{2}(\mathbf{k}/\boldsymbol{\omega}_{q})\left[1+(\theta_{q}\boldsymbol{\omega}_{q}/\boldsymbol{\mu})^{2}\right].$$
(2)

Therefore, most of the pions will be produced in a forward cone of opening

$$\theta_{\mathbf{q}} \approx \mu / \omega_{\mathbf{q}}$$
(3)

\* The present work was done in co-operation with T. KINOSHITA and F. HADJIOANNOU.

For small |t|, the pion is nearly real and the approximation

$$\sigma_{\pi * N} \approx \sigma_{\pi N}$$

seems to be well justified. In this way and for k = 25 GeV,  $\omega_q = 20 \text{ GeV}$ ,  $\theta_q \cong 1^\circ$ Drell finds that his model enhances by a factor of 10 the predictions of the statistical model for pion production by an incident proton beam on a hydrogen target. This is not surprising because for nucleon-initiated processes, although one avoids the fine structure constant, the statistical model predicts that very high-energy secondaries emerge in only a very small fraction of the collisions.

Similar models could be used to make at least qualitative predictions for the production of energetic kaons or (anti-)baryons in the forward directions (Fig. 2). For the last case in particular we are far away from the pole of the exchanged particle. However, the cross-section for, e.g., antineutron production at k = 25 GeV,  $\omega_q = 20 \text{ GeV}$  and  $\theta_q \simeq 3^\circ$  is predicted to be greater than that for the statistical model for p-p collisions by a factor of 50.



Fig. 2

Diagrams for production of energetic charged kaons or antineutrons according to Drell's model

In the present work Drell's production is re-examined under the assumption that the exchanged object is not an elementary particle (i.e. a particle of constant spin), but a number of Regge poles [2]. For chargedpion production the exchanged object must have isospin one. Therefore, among recognized Regge trajectories, the  $\pi$  and the  $\rho$  are expected to contribute. Then, the amplitude for production of pions by high-energy photons is expected to be of the form\*

$$\mathbf{A}(t,\cos\theta_{t}) \simeq \mathbf{b}_{\pi}(t) \frac{1 + \exp[-i\pi\alpha_{\pi}]}{\sin\pi\alpha_{\pi}} \mathbf{P}_{\alpha_{\pi}}(-\cos\theta_{t}) + \mathbf{b}_{\rho}(t) \frac{1 - \exp[-i\pi\alpha_{\rho}]}{\sin\pi\alpha_{\rho}} \mathbf{P}_{\alpha_{\rho}}(-\cos\theta_{t})$$
(4)

<sup>\*</sup> In the present, simple-minded, treatment the asymptotic form of the amplitude is taken to be essentially the same as in potential scattering. In a better treatment, the amplitude, e.g., for  $\gamma N \rightarrow \pi N$  is at first analysed in its 4 invariant components; next, the partial wave analysis is carried out and finally the Sommerfeld-Watson transform is applied to each of the 4 partial wave series. Note that this procedure satisfies gauge invariance in each step.

where  $\alpha_{\pi} = \alpha_{\pi}(t)$  and  $\alpha_{\rho} = \alpha_{\rho}(t)$  are the Regge trajectories corresponding to the pion and the  $\rho$ ,  $1 \pm \exp(-i\pi\alpha)$  are factors taking into account the signature of the trajectories[3] and  $\theta_t$  is the scattering angle in the t-channel. At high energies

$$P_{\alpha}(\cos\theta_{t}) \sim (\cos\theta_{t})^{\alpha} \sim s^{\alpha}$$
(5)

where s is the square of the total barycentric energy. Therefore, the asymptotic form of the amplitude is expected to be

$$A(s,t) \sim \frac{\beta_{\pi}(t)}{\sin \left[\pi \alpha_{\pi}(t)/2\right]} \left(\frac{s}{s_{0}}\right)^{\alpha_{\pi}(t)} + \frac{\beta_{\rho}(t)}{\cos \left[\pi \alpha_{\rho}(t)/2\right]} \left(\frac{s}{s_{0}}\right)^{\alpha_{\rho}(t)}$$
(6)

Here,  $s_0$  is an arbitrary constant with dimensions of mass squared. In the subsequent calculations,  $s_0 = 2M^2$ , where M = nucleon mass, will be used; this is strongly suggested by a number of successful fits to the experimental data for p-p and  $\pi$ -p scattering.

For small |t| the  $\rho$ -trajectory seems to lie above the pion-trajectory (Fig. 3). In the range below 30 GeV, however, there are at least two reasons to expect that the main contribution to the production cross-section comes from the first term of Eq. (6).



(a) It is  $\alpha_{\pi}(\mu^2) = 0$  and therefore, for small |t|, the denominator  $\sin[\pi\alpha_{\pi}(t)/2]$  is very small; in contrast, the denominator  $\cos(\pi\alpha_{\rho}(t)/2)$  is of order 1.

(b) The residue function  $\beta_{\pi}$  is essentially proportional to  $e \times g_{\pi NN}$ ;  $\beta_{\rho}$  to  $g_{\gamma\pi\rho} \times g_{\rho NN}$ . Now, the analysis of the isovector part of the nucleon electromagnetic form-factors indicates that  $g_{\rho NN}$  is smaller than  $g_{\pi NN}$  by one order of magnitude. Furthermore, for reasons presented at the end of this lecture, the effective coupling constant  $g_{\gamma\pi\rho}$  is very small.

To make more detailed statements about the relative importance of the  $\pi$  and  $\rho$  contributions it will be assumed that the structure of  $\beta_{\pi}(t)$  and  $\beta_{\rho}(t)$  is essentially that of conventional field theory. In this case, the dif-



ferential cross-section for the process  $\gamma + \rho \rightarrow \pi^+$  (n particles) through a  $\rho$ -particle exchange may be written for high energies in the form:

$$d\sigma_{\rho} \sim g_{\gamma \pi \rho}^{2} k q^{3} \sin^{2} \theta_{q} \left[ \sigma_{\rho \star N} (k - \omega_{q}) / (t - m_{\rho}^{2})^{2} \right] d^{3}g$$
(7)

with notation similar to that of Eq. (1). Then, for  $t \simeq -\mu^2$ ,  $g_{\pi\rho}^2 = e^2/4 \mu^2$  and slopes of the  $\pi$ - and  $\rho$ -trajectories equal to  $1/50 \mu^2$  one finds:

$$\left(\frac{\mathrm{d}\sigma_{\rho}}{\mathrm{d}\sigma_{\pi}}\right) \times \frac{\sigma_{\pi^{*}N}}{\sigma_{\rho^{*}N}} \simeq \begin{cases} \left(\frac{\mathrm{s}}{20\mathrm{M}^{2}}\right)^{2} & \text{if } \pi \text{ and } \rho \text{ have constant spins,} \\ \frac{\mathrm{s}}{65\mathrm{M}^{2}} & \text{if } \pi \text{ and } \rho \text{ are Regge poles.} \end{cases}$$
(8)

For small momentum transfers the following conclusions are then readily derived. (a) In both cases, with increasing s the  $\rho$ -contribution becomes more and more important; this is easily understood from the fact that in both theories the  $\rho$  has a higher spin than the pion. (b) The value of s necessary for  $d\sigma_{\rho} \simeq d\sigma_{\pi}$  is much higher if the exchanged objects are Regge poles rather than elementary particles; this is also reasonable because in the region of small |t|, the transformation of the elementary pion and  $\rho$ -meson into Regge objects reduces the spin of the  $\rho$  much more than the spin of the pion (Fig. 3).

For the total cross-section  $\sigma_{\rho*N}$  there is no experimental information at present \*. Therefore, for a rough calculation of the effect of this transformation, the  $\rho$ -contribution will be neglected. Information about the pion Regge trajectory can be obtained from [4]. In this work, the Brookhaven p-p scattering data for production of the 3-3 pion-nucleon isobar have been analysed and a relatively unique determination of the  $\alpha_{\pi}(t)$  was possible. It is found that, within experimental error

$$\alpha_{\pi}(t) = (t - \mu^2) \times \alpha'_{\pi}(0), -0.4(\text{GeV})^2 \le t < 0$$
 (9)

with slope:

$$1/(35\mu^2) \lesssim \alpha'_{\pi}(0) \leq 1/(25\mu^2).$$

With  $\alpha_{\pi}(0) = 1/(30 \,\mu^2)$  and for k = 25 GeV, q = 20 GeV,  $\theta_q = 1$ , the Drell cross-section is reduced by a factor of 8; the decrease is due to the factor

$$(s/s_0)^{a_{\pi(t)}}$$

In the case of production of anti-neutrons with q = 20 GeV at an angle  $\theta_q = 3^\circ$  by photons of energy k = 25 GeV, use of a straight line with slope

<sup>\*</sup> A calculation for  $\gamma + N \rightarrow \pi + N^*$ , where N\* is the T=J=3/2 pion-nucleon isobar, is now under way. In this case, information about  $\delta_{\rho} \approx_N$  can be obtained from the high-energy cross-section of the process  $\pi + N \rightarrow \pi + N^*$ .

 $\sim 1/50\mu^2$  for the nucleon Regge trajectory reduces the Drell cross-section by a factor of the order of 100. This is because the distance of the nucleon pole t = M<sup>2</sup> from the physical t is great and therefore  $\alpha_N(t)$  is significantly smaller than the physical spin  $\frac{1}{2}$ .

For exchange of Regge objects, the effective angle opening of the forward cone, in which the production of energetic particles takes place, will be determined from the factor  $(s/s_0)^{\alpha(t)}$  rather than the propagator  $(t-m^2)^{-1}$ . For a given s one may, then, expect to have a significant flux of pions for t such that

$$\left[ (s/s_0)^{2a(t)} / (s/s_0)^{2a(|t| \min)} \right] \sim 1/e$$
(11)

where  $|t|_{\min}$  is the square of the minimum momentum transfer (for an inelastic process  $|t|_{\min} \neq 0$ , in general). For pion production and with a pion Regge trajectory of slope  $1/30\mu^2$  it follows that

$$\theta_{\rm q} \sim \left[15\mu^2/k\omega_{\rm q}\ln{({\rm s}/{\rm s}_0)}\right]^{1/2}.$$
 (12)

For k = 25 GeV and  $\omega_q = 20$  GeV;  $\theta_q = 0.75^\circ \sim \mu / \omega_q$ , i.e. of the same order of Drell production. For anti-neutron production and the same k,  $\omega_q$ :  $\theta_q \sim \frac{1}{4}M/\omega_q$ .

I will conclude with a short discussion of the magnitude of the effective coupling constant  $g_{\gamma\pi\rho}$ . This quantity appears in the photoproduction of pions on nucleons; therefore, one might expect to obtain some information about its order of magnitude by analysing the experimental data for  $\gamma N \rightarrow \pi N$ . This was first done by J. S. BALL [5] and subsequently by a number of people [6, 7]. The conclusion is that, under proper normalization,  $g_{\gamma\pi\rho}$  is of the order of e  $\times \mu$  /M. Notice that this value is in agreement with a perturbation calculation in which the ( $\gamma$ -  $3\pi$ ) coupling takes place through a nucleon-antinucleon loop (Fig. 4, C). The reason that this coupling takes place through



Diagrams for the  $(\gamma \cdot 3\pi)$  interaction; C<sub>1</sub> is excluded by the hypothesis of minimal electromagnetic interactions, C<sub>3</sub> is excluded by G-invariance

a particle loop is probably connected with the principle of minimal electromagnetic interactions [8]. According to this, a photon can be directly coupled only to a charged pair of particles. As a result, elementary couplings of the form  $C_1$  have to be excluded and the vertex ( $\gamma - 3\pi$ ) has to be further analysed. However, any analysis of the form  $\gamma \rightarrow \text{pion pair} \rightarrow 3$  pions (as in  $C_3$ ) is excluded by G-parity conservation. Therefore, intermediate pairs of heavier particles  $(N-\overline{N} \text{ or } k-\overline{k})$  must be present. In this way, however, the effective coupling constant is reduced by (roughly) the ratio  $\mu/M$ , where M = mass of intermediate heavy particle.

Finally, the assertion that the strength of the vertex  $(\gamma - 3\pi)$  is not an elementary coupling constant (but can be calculated in terms of elementary couplings, e.g. of the  $\pi$ -N and  $\gamma$ -N interactions) is also in agreement with the considerations of Mandelstam and Chew in connection with quadrilinear products of arbitrary fields. According to these, the impossibility of an arbitrary constant (like  $g_{\gamma\pi\rho}$ ) in the dispersion theoretical approach coincides with the non-renormalizability of the corresponding quadrilinear interaction. Now, the only local couplings for the vertex  $(\gamma - 3\pi)$  satisfying relativistic invariance and all basic symmetry principles are of the derivative type

$$\left(g_{\gamma \pi \rho} \epsilon^{k \lambda \mu \nu} A_k \frac{\delta \psi_1}{\delta x^{\lambda}} \frac{\delta \psi_2}{\delta x^{\mu}} \frac{\delta \psi_3}{\delta x^{\nu}}\right)$$

which are known to be non-renormalizable and therefore should not introduce any new constants. In contrast, the  $(4\pi)$ -vertex coupling  $\lambda \phi$ , which is renormalizable, is indeed known to introduce an arbitrary constant  $\lambda$  in the  $\pi$ - $\pi$  dispersion relations.

#### REFERENCES

- [1] DRELL, S. D., Phys. Rev. Letters 5 (1960) 278.
- [2] CHEW, G.F. and FRAUTSCHI, S.C., Phys. Rev. Letters <u>7</u> (1961) 394, and <u>8</u> (1962) 41; GRIBOV, V.N., J. Exptl. Theoret. Phys. (USSR) <u>41</u> (1961) 667.
- [3] FRAUTSCHI, S.C., GELL-MANN, M. and ZACHARIASEN, F., Phys. Rev. 126 (1962) 2204.
- [4] CONTOGOURIS, A. P., FRAUTSCHI, S. C. and WONG, HOW-SEN, submitted to Phys. Rev.
- [5] BALL, J. S., Phys. Rev. Letters 5 (1960) 73; Phys. Rev. 124 (1961) 2014.
- [6] FUJII, Y. and KAWAGUCHI, M., Progr. Theor. Phys. 26 (1961) 519.
- [7] de TOLLIS, B. and VERGANELAKIS, A., Nuovo Cimento 22 (1961) 406.
- [8] CONTOGOURIS, A. P., Nuovo Cimento 25 (1962) 104.

# BOOK IV

# COMPOUND SYSTEMS

# PART I

DYNAMICAL INVESTIGATIONS OF  $\pi N$  AND NN SYSTEMS

# UNITARITY AND PRODUCTION AMPLITUDES (APPLICATION TO $\pi$ N HIGHER RESONANCES)

#### B.W. LEE

## DEPARTMENT OF PHYSICS, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, UNITED STATES OF AMERICA

I would like to discuss a model of higher resonances in the  $\pi N$  scattering that has been worked out extensively in the past year, and for which still further works are being done. In this model which I am about to discuss, we consider the three-particle intermediate states  $N + 2\pi$  in which the two pions are strongly correlated in the T = J = 1 state as an unstable particle we now call the  $\rho$  meson [1, 2]. As this new channel becomes energetically open, the  $\rho N$  channel becomes coupled to the  $\pi N$  channel. In fact the  $\pi N$  system can make a virtual transition to the  $\rho N$  channel even below the  $\rho N$  threshold. In our model, we assume a strong coupling between the  $\pi N$  and  $\rho N$  channels, eventually developing a resonance either above or below the  $\rho N$  threshold. This model is very similar to, but significantly different in one respect from, the model suggested by Dalitz and Miller for the  $Y_1^*$ . The analogy of the Dalitz-Miller model in our problem would be to assume a strong force between the o and N particles to sustain a bound state which would be stable, were it not for the "weak" coupling between the  $\rho N$  and  $\pi N$  channels. In our model, however, it is precisely the coupling of the two channels that is responsible for the resonance. There is very little, if any, difference between these two models phenomenologically, but at a deeper dynamical level, there is a difference in outlook.

The reason that the model we are considering is capable of accounting for the higher resonances is seen as follows. Let  $L_J$  be the "orbital" angular momentum and the total angular momentum of the  $\pi N$  system and  $I_J$  the same for the  $\rho N$  system. Note that there is no change in the intrinsic parity in the reaction  $N + \pi \rightarrow N + \rho$ . Therefore

| $\frac{\pi N}{L_{J}}$ | <u>e</u><br>1    | $\frac{\rho N}{I_J}$ |  |
|-----------------------|------------------|----------------------|--|
| S <sub>1/2</sub>      | $S_{1/2}$        | D <sub>1/2</sub>     |  |
| $P_{1/2}$             | $P_{1/2}$        |                      |  |
| P <sub>3/2</sub>      | $P_{3/2}$        | $F_{3/2}$            |  |
| D <sub>3/2</sub>      | S <sub>3/2</sub> | D <sub>3/2</sub>     |  |
| D <sub>5/2</sub>      | D <sub>5/2</sub> | $G_{5/2}$            |  |
| F <sub>5/2</sub>      | $P_{5/2}$        | F <sub>5/2</sub>     |  |

Now as the  $\rho$ N channel becomes energetically open, the I = 0 (L<sub>I</sub> = D<sub>3/2</sub>, S<sub>1/2</sub>) state will be excited first, then the I = 1 (F<sub>5/2</sub>, P<sub>3/2</sub>, P<sub>1/2</sub>) state, and

so on. Since the statistical weight is proportional to (2 J + 1), we expect that the first resonance is predominantly in the state  $L_J = D_{3/2}$  and the second one in the  $L_J = F_{5/2}$ ,  $P_{3/2}$ ,  $P_{1/2}$  states, which of course agree with the experimental findings. The isotopic dependance is determined by the specific "primary" interaction we choose, and we will discuss it when the appropriate moment comes.

Our formal approach will be based on the unitarity and analyticity of the relevant amplitudes. Actually the analytic properties of the production and three-particle-to-three-particle scattering amplitudes are only scantly known, and we shall proceed by assuming a particular diagram as giving the main contribution to the left hand cuts of the production amplitude. On the unitarity relations of the coupled processes, I shall rely heavily on the recent work of Ball, Frazer and Nauenberg.

While for the scattering amplitude the unitarity relation

$$\mathbf{T} - \mathbf{T}^* = 2\pi \mathbf{i} \mathbf{T} \rho \mathbf{T}^* \tag{1}$$

gives directly the discontinuity across the right hand cut, this is not the case in general for the production and 3-particle-to-3-particle amplitudes. Let us denote by subscript 1 the  $\pi N$  channel, and by 2 the  $\rho N$  channel. Then various elements of the T-matrix are defined as

$$T_{11} (s_{+}) = \sqrt{2 k_{0}'} \sqrt{p_{0}'/m} \langle \pi (k') N (p')^{out} | J_{N}^{\dagger} (0) | \pi (k) \rangle \sqrt{2 k_{0}} u(p),$$

$$T_{21} (s_{+}, \sigma_{+}) = \sqrt{4 k_{10}' k_{20}'} \sqrt{q_{0}'/m} \langle \pi (k_{1}') \pi (k_{2}') N (q')^{out} | J_{N}^{\dagger} (0) | \pi (k) \rangle \sqrt{2 k_{0}} u(p), \quad (2)$$

$$\cdots$$

$$T_{22} (s_{+}, \sigma_{+}', \sigma_{+}) = \sqrt{4 k_{10} k_{20}'} \sqrt{q_{0}'/m} \langle \pi (k_{1}') \pi (k_{2}') N (q')^{out} | J_{N}^{\dagger} (0) | \pi (k_{1}) \pi (k_{2})^{in} \rangle$$

$$\times \sqrt{4 k_{10} k_{20}} u(q).$$

In the above definitions of the T-matrix elements, we have deliberately contracted the nucleon rather than the pion operators so as to keep the two pions in the bra or ket together. To specify the kinematics of the reactions  $\pi + N \iff \pi + N$ ,  $\pi + N \iff \pi + \pi + N$ , and  $\pi + \pi + N \iff \pi + \pi + N$  we need to specify 2, 5 and 8 variables, respectively. We specify the total energy of the system in the centre of mass which is common in three processes:

$$s = -(p + k)^{2} = -(p' + k')^{2}$$
$$= -(q + k_{1} + k_{2})^{2} = -(q' + k_{1}' + k_{2}')^{2}.$$
(3)

For three-particle states, we will denote by  $\sigma$  the total energy square of the two pion system in its own c.m.:

$$\sigma = - (k_1 + k_2)^2$$
  

$$\sigma' = - (k_1' + k_2')^2.$$
(4)

The other remaining variables are appropriately chosen angles.

Suppressing the angular variables, we deduce the discontinuity of the  $T_{ii}$  across the real s-axis,  $s \ge (m + \mu)^2$  as

$$1/2i [T_{11} (s_{+}) - T_{11} (s_{-})] = \Sigma_{2} T_{11} (s_{+})\rho_{1} (s) T_{11} (s_{-}) + \Sigma_{3,\sigma} T_{12} (s_{+,\sigma_{+}})\rho_{2} (s,\sigma) T_{21} (s_{-,\sigma_{-}}), 1/2i [T_{12} (s_{+,\sigma}) - T_{12} (s_{-,\sigma})] = \Sigma_{2} T_{11} (s_{+})\rho_{1} (s) T_{12} (s_{-}) + \Sigma_{3,\sigma'} T_{12} (s_{+,\sigma'_{+}})\rho_{2} (s,\sigma) T_{22}^{c} (s_{-,\sigma'_{+},\sigma),$$
(5)

$$\frac{1}{2i} [T_{22}^{c}(s_{+}, \sigma', \sigma) - T_{22}^{c}(s_{-}, \sigma', \sigma) = \sum_{2} T_{21}(s_{+}, \sigma')\rho_{1}(s) T_{12}(s_{-}, \sigma) + \sum_{3, \sigma''} T_{22}^{c}(s_{+}, \sigma', \sigma'')\rho_{0}(s, \sigma'') T_{22}^{c}(s_{-}, \sigma'', \sigma)$$

where  $\Sigma_2$  is the two-particle phase space integral for fixed s:

$$\Sigma_2 \propto \int_{-1}^{1} d\cos\theta \int_{-1}^{2\pi} d\varphi$$

and  $\rho_1$  (s) is the phase space factor of the  $\pi N$  system :

$$\rho_{\rm I}({\rm s}) \equiv \rho_{\rm I}({\rm s};{\rm m},\mu) \propto \frac{{\rm P}}{\sqrt{{\rm s}}} \theta \left({\rm s} - ({\rm m}+\mu)^2\right) = \frac{\left[{\rm s} - ({\rm m}+\mu)^2\right]^{1/2} \left[{\rm s} - ({\rm m}-\mu)^2\right]^{1/2}}{2{\rm s}} \theta \left({\rm s} - ({\rm m}+\mu)^2\right).$$

 $\Sigma_{3,\sigma}$  is the three-particle phase space integral including the integration over the continuous mass variable  $\sigma$ , and  $\rho_2(s,\sigma)$  is the phase space factor for the  $\rho N$  system:

$$\rho_{2}(\mathbf{s},\sigma) \propto \rho_{1}(\mathbf{s};\mathbf{m},\sqrt{\sigma}) \rho_{1}(\sigma;\mu,\mu) \theta(\sqrt{\mathbf{s}}-\mathbf{m}-\sqrt{\sigma}) \theta(\sigma-4\mu^{2}) \theta(\mathbf{s}-(\mathbf{m}+\mu)^{2}).$$

Eqs. (5) are derivable (at least heuristically!) using the L.S.Z. formalism. It must be emphasized that Eqs. (5) are not the unitarity relations, albeit they are intimately related to the latter. Before exhibiting the connection between those two, we wish to note the topological structure of the  $T_{22}^c$  in Eqs. (5). From the definition of  $T_{22}^c$  in Eq. (2), we see that

$$S_{22} = \frac{q_0}{m} \,\delta(\vec{q} - \vec{q}') \,S_{\pi\pi} + (2\pi)^4 \,i\,\delta(q + k_1 + k_2 - q' - k_1' - k_2') \,\ldots T_{22}^c$$
*factors*

so that T<sub>22</sub> is defined as

$$T_{22}^{c} = T_{22} - \frac{q_0}{m} \delta(\vec{q} - \vec{q}') T_{\pi\pi}$$

That is to say, the "connected" amplitude excludes the disconnected group in which the nucleon is non-interacting (Fig. 1).

To get rid of the complicating angular dependence in Eqs.(5), it is convenient to decompose the amplitudes into partial waves. The decomposition of the elastic amplitude is well-known, so I shall not elaborate upon this.



Fig. 1

To decompose the production amplitude  $T_{12}$  we first go to the  $2\pi$  centre-ofmass system and choose the  $\frac{2}{7}$ -axis along the direction of the q (see Fig. 2).



Fig. 2

In this frame, we project out a particular angular momentum state  $\ell$  of the  $2\pi$ -system with the quantization axis along the  $\tilde{z}$ -axis. Let  $\ell_{\tilde{z}}$  be the projection of  $\ell$  onto the  $\tilde{z}$ -axis. Now we Lorentz-transform the system to the total centre of mass. Since the Lorentz transformation is along the  $\tilde{z}$ -axis with the velocity

 $\mathbf{v}=\left|\vec{\mathbf{q}}\right|/\sqrt{\left|\vec{\mathbf{q}}\right|^{2}+\sigma}$ 

the projection  $\ell_{\vec{s}}$  remains invariant and  $\ell_{\vec{s}}$  acquires the role of the helicity of the  $2\pi$  system. As the two-pion system is now equivalent to a particle of mass  $\sqrt{\sigma}$ , spin  $\ell$ , helicity  $\ell_{\vec{s}}$  as far as kinematics is concerned, the decomposition into definite  $(J,\pi)$  states follows in the standard manner of Jacob and Wick. The three-particles amplitude  $T_{22}$  may be decomposed in a similar manner. If we assume, as we shall do, that only one particular  $\ell$  dominates, and that the mass-distribution in  $\sqrt{\sigma}$  is sharply peaked near  $\sigma = \sigma_r$ , the picture of an unstable particle can be naturally incorporated into our scheme. From now on we shall retain only the  $\ell = 1$  amplitudes in our considerations.

Once the amplitudes  $T_{ij}$  are decomposed into partial waves, the angular integrations in Eqs. (5) can be performed trivially. For given quantum numbers  $J, \pi$ , we obtain

$$T_{11}^{J\pi}(s_{+}) - T_{11}^{J\pi}(s_{-}) = 2i T_{11}^{J\pi}(s_{+}) \rho_{1}(s) T_{11}^{J\pi}(s_{-}) + 2i \sum_{\alpha} \int_{4\mu^{2}}^{(\sqrt{s} m)^{2}} d\sigma T_{12}^{J\pi}(s_{+}, \sigma_{+}) \rho_{2}(s, \sigma) T_{21}^{J\pi}(s_{-}, \sigma_{-}),$$
(6)

where, for  $\ell = 1$ ,  $\alpha = 1$ , 2 or 3 is the polarization index corresponding to two transverse, one longitudinal polarizations.

At this juncture let us clarify the connection between the discontinuity Eq. (6) and the unitarity relations. For  $T_{12}$ , for example, the unitarity relation asserts that (suppressing  $J, \pi, \Sigma, \alpha$  hereafter)

$$T_{12}(s_{+},\sigma_{+}) - T_{12}^{*}(s_{-},\sigma_{-}) = 2i T_{11}^{*}(s) \rho_{1}(s) T_{12}(s,\sigma) + 2i \int d\sigma' T_{12}^{*}(s,\sigma') \rho_{2}(s,\sigma') T_{22}(s,\sigma',\sigma).$$
(7)

We assume  $T_{ij}^{*}(s, x, --) = T_{ij}(s^{*}, x^{*}, ---)$ , i.e.,  $T_{ij}$  are real analytic functions in the energy variables. We are unable to give a rigorous, general proof for this, but this is true in perturbation theory as far as can be ascertained. Then we may write Eq. (7) as

$$[T_{12}(s_{+},\sigma_{+}) - T_{12}(s_{-},\sigma_{+})] + [T_{12}(s_{-},\sigma_{+}) - T_{12}(s_{-},\sigma_{-})]$$
  
= 2i T\_{11}(s-i\epsilon)  $\rho_{l}(s)$  T\_{11}(s+i\epsilon) + 2i  $\int d\sigma' T_{12}(s_{-},\sigma'_{-})\rho_{2}(s,\sigma)$  T\_{22}(s\_{+},\sigma'\_{+},\sigma\_{+}). (8)

Now we note that

$$T_{12}(s, \sigma_{+}) - [T_{12}(s, \sigma_{-}) = 2i e^{i\delta(\sigma)} \sin \delta(\sigma) T_{12}(s_{-}, \sigma_{-})$$
$$\approx 2i \int d\sigma'(T_{22} - T_{22}^{c}) \rho_{2}(s, \sigma') T_{12}(s_{-}, \sigma_{-}).$$
(9)

Therefore, Eqs. (8) and (9) imply the second (not written out) line of Eq. (6). What it means may become clearer if you consider a particular diagram (see Fig. 3).



When we consider the imaginary part, there are contributions from partitions 2 and 3 of the diagram as well as that from the partition 1. In computing the "absorptive" part, however, the contributions from 2 and 3 should not be included (since  $\sigma$  is fixed). It may be further remarked that the partitions 2 and 3 give rise to terms of the form

$$\int d\sigma' T_{12}(s_{-}, \sigma_{-}) \rho_{2}(s, \sigma') T_{22}^{D}(s_{+}, \sigma'_{+}, \sigma)$$

where  $T_{22}^{D}$  is the excluded disconnected part

$$T_{22}^{D} = T_{22} - T_{22}^{c}$$
.

So much for the unitarity aspect of the problem, let us now look at the dynamics. We would like to take into account the longest range "force" that contributes to the process  $\pi + N \iff \rho + N$ . One-pion exchange is possible between the nucleon and the meson (see Fig. 4).



Fig. 4

In fact, the reason why we have singled out the N +  $\rho$  intermediate state, but neglected other states such as N +  $\omega$  (n) is,that this is the only state consisting of one nucleon and one unstable meson that can be reached from the N +  $\pi$  state through one-pion exchange. The matrix element for Fig. 4 is

$$\overline{u}(q) \chi u(p) \text{ ig } \frac{1}{t-\mu^2} F_{\pi\pi}(\sigma, \cdots)$$
 (10)

where t = -  $(p-q)^2$ , and  $F_{\pi\pi}$  is

$$\begin{split} \mathbf{F}_{\pi\pi} \left( \sigma, \cdots \right) &\propto \ 3 \ \sqrt{\frac{\sigma}{\sigma - 4\mu^2}} \ \mathbf{e}^{\mathbf{i}\delta(\sigma)} \ \sin \ \delta(\sigma) \ \mathbf{P}_1 \left( \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}} \right) \Big|_{2\pi. \ \mathrm{c. \ m.}} \\ &\equiv \ \mathbf{f}_{\pi\pi}(\sigma) \ 3 \ \mathbf{P}_1 \left( \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}} \right) \Big|_{2\pi. \ \mathrm{c. \ m.}} \end{split}$$

When we make a partial wave projection of (10) we get two branch lines, one extending from s=0 to  $-\infty$ , the other one between the two branch points  $s^{\pm}$ , given by

$$s^{\pm}(\sigma) = m^2 + \sigma/2 \pm \frac{\sqrt{\sigma}}{2\mu} [(4m^2 - \mu^2)(4\mu^2 - \sigma)]^{1/2}$$

If we give a small positive imaginary part to  $\sigma: \sigma \to \sigma + i\eta$ ,  $\eta > 0$ , and increase  $\sigma$  from a certain small value, the branch points  $s^{\pm}(\sigma)$  move as indicated in Fig. 5.



Fig. 5

That is, the projection of the one-pion exchange amplitude develops an anomalous singularity at  $\sigma = 2\mu^2(1 + \mu/2m)$ , a complex singularity at  $\sigma = 4\mu^2$ . If, instead, we give a small imaginary part to  $\sigma$ , the loci of the singularities in Fig.5 are reflected about the real axis.

The amplitudes  $T_{ij}\,$  contain some kinematical cuts. We define the new set of amplitudes  $M_{\,ii}\,$  by

$$M_{ij} = \frac{1}{\sqrt{g_i}} T_{ij} \frac{1}{\sqrt{g_j}}$$

where the  $g_i$  are factors proportional to  $p_i^{2Li}$ ,  $p_i$  being the magnitude of the channel momentum,  $L_i$  the lowest channel orbital angular momentum ( $L_i = L$  for i = 1,  $L_i = I$  for i = 2, where L and I are defined as before). On defining new quantities  $\rho_1^L(s)$ ,  $\rho_2^I(s,\sigma)$  by

UNITARITY AND PRODUCTION AMPLITUDES

$$\begin{split} \rho_1^{\mathrm{L}}(\mathbf{s}) &= \mathbf{g}_1 \rho_1(\mathbf{s}) \sim \left[ p^{2\,\mathrm{L}+1} \mathbf{a} \mathbf{s} \quad \mathbf{s} \rightarrow (m+\mu)^2 \right], \\ \rho_2^{\mathrm{I}}(\mathbf{s},\sigma) &= \mathbf{g}_2 \rho_2(\mathbf{s},\sigma) \sim \left[ q(\mathbf{s},\sigma)^{2\,\mathrm{I}+1} \mathbf{a} \mathbf{s} \quad \mathbf{s} \rightarrow (m+2\mu)^2 \right], \end{split}$$

we can write the discontinuity conditions, Eq. (6), in the form

$$M_{11}^{J\pi}(s_{+}) - M_{11}^{J\pi}(s_{-}) = 2i M_{11}^{J\pi}(s_{-})\rho_{1}^{L} M_{11}(s_{+}) + 2i \Sigma \int d\sigma M_{12}^{J\pi}(s_{-}, \sigma_{-})\rho_{2}^{L}(s, \sigma) M_{21}^{J\pi}(s_{+}, \sigma_{+}).$$
(11)

The factors  $\rho_1^L$  and  $\rho_2^I$  express the combined effect of the variation of the available phase space and the centrifugal barrier. As a consequence, the dominant energy dependence of the amplitudes near thresholds is removed from  $M_{ii}$ : the  $M_{ii}$  are approximately constant near the thresholds.

We can now write down the dispersion equations for the  $M_{ij}$ . The"input" amplitude  $B_{21}(s, \sigma)$  has a spectral representation of the form

$$\mathbf{B}_{21}(\mathbf{s},\sigma) = \frac{\mathbf{f}_{\pi\pi}(\sigma)}{\pi} \left[ \int_{s^+}^{s^-} \frac{\mathrm{d}s'}{\mathbf{s}'-\mathbf{s}} \alpha(\mathbf{s}',\sigma) + \int_{-\infty}^{0} \frac{\mathrm{d}s'}{\mathbf{s}'-\mathbf{s}} \beta(\mathbf{s}',\sigma) \right] \,.$$

If we fix the value of  $\sigma$  below  $2\mu^2(1 + \mu/2m)$ , we can write down the dispersion equation for  $M_{21}(s, \sigma)$ :

$$\mathbf{M}_{21}(s,\sigma) = \mathbf{B}_{21}(s,\sigma) + \frac{1}{\pi} \int_{(m+\mu)^2}^{\infty} \frac{\mathrm{d}s'}{s'-s} \mathbf{M}_{21}(s') \rho_1(s') \mathbf{M}_{11}(s')$$

$$+\frac{1}{\pi}\int_{(m+2\mu)^2}^{\infty}\frac{\mathrm{d}s'}{s'-s}\cdots$$
 (12)

Let us now consider the analytic continuation of Eq. (11) in the mass  $\sigma$ . As we have seen,  $M_{21}$  will develop an anomalous singularity, and we must deform the contour in the second term of (12) to avoid the intruding cut of  $M_{21}(s'_4, \sigma)$ .

The amplitude  $M_{21}(s', \sigma)$  in the integrand must be continued to the second sheet through the two-particle unitarity cut:

$$\begin{split} \mathbf{M}_{21}(\mathbf{s}_{*},\sigma) &= \mathbf{M}_{21}^{\mathrm{II}}(\mathbf{s}_{*},\sigma) \\ &= \mathbf{M}_{21}(\mathbf{s}_{*},\sigma)[1+2\mathrm{i}\rho_{1}\left(\mathbf{s}\right)\mathbf{M}_{11}(\mathbf{s}_{*},\sigma)]^{-1} \\ \rho_{1}^{\mathrm{II}}(\mathbf{s}-\mathrm{i}\epsilon) &= \rho_{1}(\mathbf{s}+\mathrm{i}\epsilon) = -\rho_{1}(\mathbf{s}-\mathrm{i}\epsilon) \; . \end{split}$$

Therefore the continuation of Eq. (11) in  $\sigma$  now gives

$$M_{21}(s,\sigma) = B_{21}(s,\sigma) + \frac{1}{\pi} \int_{(m^+\mu)^2}^{\infty} \frac{ds'}{s'-s} \dots + \frac{1}{\pi} \int_{s^+(\sigma)}^{(m^+\mu)^2} \frac{ds'}{s'-s} \operatorname{disc} M_{21}^{II}(s') \rho_1(s') M_{11}^{II}(s').$$
(13)

One can show that  $M_{21}(s, \sigma)$  given by Eq. (13) does not satisfy the discontinuity condition in  $\sigma$ , namely

$$M_{21}(s,\sigma_{+}) - M_{21}(s,\sigma_{-}) = 2if_{\pi\pi}(\sigma_{+})\rho(\sigma) M_{21}(s,\sigma_{-});\rho(\sigma) = \sqrt{\frac{\sigma - 4\mu^{2}}{\sigma}}.$$
 (14)

Eq. (14) means that

$$\underset{\sigma}{\text{disc}} \left[ \frac{M_{21}(s,\sigma+i\epsilon)}{f_{\pi\pi}(\sigma+i\epsilon)} \right] = \sigma,$$

but because of the unsymmetrical (in the s-plane) complex anomalous singularity, Eq.(13) gives

•

$$\begin{aligned} \operatorname{disc}_{o} \left[ \frac{M_{21}(s, \sigma + i\epsilon)}{f_{\pi\pi}(\sigma + i\epsilon)} \right] &= \frac{1}{\pi} \int_{s^{+}}^{(m^{+}\mu)^{2}} \frac{\mathrm{d}s'}{s' - s} \left[ \operatorname{disc} \frac{M_{21}^{\mathrm{II}}(s', \sigma_{+})}{f_{\pi\pi}(\sigma_{+})} \right] \rho_{1}^{\mathrm{II}}(s') \, \mathrm{M}_{11}(s') \\ &+ \frac{1}{\pi} \int_{(m^{+}\mu)^{2}}^{s^{-}} \frac{\mathrm{d}s'}{s' - s} \left[ \operatorname{disc} \frac{M_{21}(s', \sigma_{-})}{f_{\pi\pi}(\sigma_{-})} \right] \rho_{1}(s') \, \mathrm{M}_{11}^{\mathrm{II}}(s') \\ &\# 0. \end{aligned}$$

Ball, Frazer and Nauenberg noted that the diagram shown in Fig.6 has a cut from s+ to s-, corresponding to the nuclear line indicated by arrow on the mass shell.



The contribution from the branch cut from  $s^+$  to  $s^-$  of this diagram can be shown to be

$$\frac{1}{\pi\rho(\sigma)} \left[ \int_{s^+}^{(m^+\mu)^2} \frac{\mathrm{d}s'}{s'-s} \alpha(s',\sigma) \rho_1^{\mathrm{II}}(s') \operatorname{M}_{11}^{\mathrm{II}}(s') + \int_{(m^+\mu)^2}^{s^-} \frac{\mathrm{d}s'}{s'-s} \alpha(s',\sigma) \rho_1(s') \operatorname{M}_{11}(s') \right]$$

When we add this term to Eq. (13) and manipulate a little bit, we obtain

$$\frac{\mathbf{M}_{21}(\mathbf{s},\sigma)}{\mathbf{f}_{\pi\pi}(\sigma)} = \frac{\mathbf{B}_{21}(\mathbf{s},\sigma)}{\mathbf{f}_{\pi\pi}(\sigma)} + \frac{1}{\pi\rho(\sigma)\mathbf{f}_{\pi\pi}^{\kappa}(\sigma)} \int_{\mathbf{s}^{+}(\sigma)}^{(\mathbf{m}+\mu)^{2}} \frac{\mathbf{d}\mathbf{s}'}{\mathbf{s}'-\mathbf{s}'} \alpha(\mathbf{s}',\sigma) \rho_{1}^{\mathrm{II}}(\mathbf{s}') \mathbf{M}_{11}(\mathbf{s}') + \frac{1}{\pi\rho(\sigma)\mathbf{f}_{\pi\pi}^{\sigma}} \int_{(\mathbf{m}^{+}\mu)^{2}}^{\mathbf{s}^{-}(\sigma)} \frac{\mathbf{d}\mathbf{s}'}{\mathbf{s}'-\mathbf{s}} \alpha(\mathbf{s}',\sigma)\rho_{1}(\mathbf{s}') \mathbf{M}_{11}(\mathbf{s}') + \text{unitary contributions. (15)}$$

By virtue of the relations

$$\begin{aligned} \alpha^*(\mathbf{s},\sigma) &= -\alpha(\mathbf{s}^*,\sigma^*), \\ \frac{1}{f_{\pi\pi}(\sigma)} - \frac{1}{f_{\pi\pi}^*(\sigma)} &= -2i\rho(\sigma), \end{aligned}$$

the amplitude  $M_{21}(s,\sigma)$  satisfies Eq. (15).

The above consideration may be understood better in terms of diagrams. Consider, for example, a Cutkosky diagram of Fig. 7.



The discontinuity in  $\sigma$  corresponds to the partition of the diagram along line 1, and this must be written in the form (see Eq.14)

$$2if_{\pi\pi}(\sigma) \rho(\sigma) M_{12}(s, \sigma_{-})$$

Therefore,  $M_{21}$  must include the diagram shown in Fig.8.





In this model both the elastic amplitude  $M_{11}$  and the 3 particle amplitude  $M_{22}$  are driven by the inelastic process, which in turn is generated by the one-pion exchange mechanism and the unitarity. Schematically the whole coupled processes can be shown as in Fig.9.



The non-linear set of equations (11) and (15) can be solved by the matrix N/D method. Of course a slight modification is necessary to accommodate the complex anomalous singularities. An important result is that, while individual elements  $D_{ij}$  do have complex singularities, the determinant of D does not.

Moreover we observe that

$$S = \rho^{1/2} D(s - i\epsilon) D^{-1}(s + i\epsilon) \rho^{-1/2}$$

so that

$$\ln \det \mathbf{D}(\mathbf{s}'_{-}) - \ln \det \mathbf{D}(\mathbf{s}_{+}) = \ln \det \mathbf{S} = \sum \delta_{a}(\mathbf{s})$$

and from this and the fact that det D has only the normal unitarity cut, the multichannel Levinson theorem follows:

$$\sum_{a} \left\{ \delta_{a}(\mathbf{m} + \mu) - \delta_{a}(\infty) \right\} = \pi (\mathbf{n}_{\text{bound}} - \mathbf{n}_{c.D.D}) ,$$

Instead of demonstrating the N/D solution, let us be content with the semiphenomenological K-matrix solution, to glean the nature of the more elaborate solution. Let us adopt the matrix notation and write

$$\mathbf{M} = \mathbf{K} + \mathbf{i}\mathbf{K} \cdot \boldsymbol{\rho}^{\mathbf{J}\pi} - \mathbf{M} \tag{16}$$

where the rows and columns of the matrices are labelled by the discrete channel indeces i and the continuous  $\boldsymbol{\sigma}$ 

$$(M) = \begin{pmatrix} M_{11} \cdots & M_{2\sigma,1} \cdots \\ \vdots & & \\ M_{1,2\sigma}, & M_{2\sigma,2\sigma'} \\ \vdots & & \end{pmatrix} \text{ etc.}$$

and the matrix  $\rho$  is diagonal

$$\left(\rho^{\mathbf{J}\,\boldsymbol{\pi}}\right) = \left(\begin{array}{cc}\rho_{\mathbf{1}}^{\mathbf{L}} & 0 & 0\\ 0 & \rho_{\mathbf{2}}^{\mathbf{I}}(\mathbf{s},\boldsymbol{\sigma})\\ 0 & \end{array}\right)$$

We write

$$\begin{split} \mathbf{m}_{11} &= \mathbf{M}_{11} \\ \mathbf{m}_{12}(\mathbf{s}, \sigma) &= \mathbf{M}_{12}(\mathbf{s}, \sigma) \mathbf{f}_{\pi\pi}^{-1}(\sigma) \\ \mathbf{m}_{22}(\mathbf{s}, \sigma', \sigma) &= \mathbf{f}_{\pi\pi}^{-1}(\sigma') \mathbf{M}_{22}(\mathbf{s}, \sigma', \sigma) \mathbf{f}_{\pi\pi}^{-1}(\sigma) \end{split}$$

Now we make an essential approximation that  $f_{\pi\pi}(\sigma)$  is sharply peaked at  $\sigma = \sigma_r$ , and, in the spirit of the steepest descent approximation,

$$m_{12}(s,\sigma) \simeq m_{12}(s,\sigma_r) \equiv m_{12}(s)$$
, etc.

Likewise we define  $\kappa_{ij}$  by removing the sharp dependance of  $\kappa_{ij}$  on the mass distribution  $\sigma$ . On defining the 3 particle factor  $\rho_2^{-1}$  corrected for the final interaction

$$\overline{\rho}_{2}^{I}(s) = \int_{1-2}^{(\sqrt{s}-m)^{2}} d\sigma \left| \mathbf{f}_{\pi\pi}(\sigma) \right|^{2} \rho_{2}(s,\sigma) .$$

We obtain a reduced  $2 \times 2$  matrix relation between m and  $\kappa$ :

$$m_{11}(s) = \kappa_{11}(s) + i \kappa_{11}(s) \rho_1^{L}(s) m_{11}(s) + i \kappa_{12} \overline{\rho}_2^{L} m_{21'}$$
(17)

In order that the  $M_{ij}$  satisfy the relation, Eq. (12), it is necessary and sufficient that  $\kappa$  be real symmetric; the  $M_{ij}$  can be written as

$$m = (1 - i\overline{\rho} \cdot \kappa)^{-1} \kappa ,$$

the condition for resonance is

or

$$0 = \operatorname{Re} \left[1 + \rho_{1}^{L} \,\overline{\rho}_{2}^{I} \left(\kappa_{12} \,\kappa_{21} - \kappa_{11} \,\kappa_{22}\right) - i \,\rho_{1}^{L} \,\kappa_{11} - i \,\overline{\rho}_{2}^{I} \,\kappa_{22}\right] \,. \tag{18}$$

Now in the range  $(m + \mu)^2 \leq s \leq (m + \sqrt{\sigma_r})^2 \rho_1$  is purely real, while  $\overline{\rho_2}^I$  is predominantly imaginary, so that Eq.(18) reduces approximately to

 $\left| + \left| \overline{\rho}_{2}^{\mathrm{I}} \right| \kappa_{22} \simeq 0 \right| . \tag{18}$ 

A necessary condition for resonance below the  $\rho N$  threshold is then

 $\kappa_{22} < 0.$ 

A detailed calculation shows that  $\kappa_{22}$  is in fact negative both in the  $D_{3/2}$ , and  $F_{5/2}$  states (T = 1/2) for the interaction considered (one-pion exchange), and the ratio of the  $\kappa_{22}$  in the T = 1/2 state to that in the T = 3/2 is  $\simeq 4$ :1(see Fig.10). Since  $\left| \overline{\rho}_{2}^{I} \right| \simeq \left| \overline{\rho} \right|^{2I+1}$ , we see that, assuming  $\kappa_{22}^{I=0} \simeq \kappa_{22}^{I=1}$ , the  $D_{3/2}$  resonance lies below the  $F_{5/2}$  resonance in the T = 1/2 channel.

A crude calculation made by Cook and myself may be summarized in the following graph (Fig. 11).



#### REFERENCÉS

[1] COOK, L.F. and LEE, B.W., Phys. Rev., to be published.

[2] BALL, J.S., FRAZER, W.R. and NAUENBERG, M., to be published.

# BOOK IV

# COMPOUND SYSTEMS

PART II

COMPOUND PARTICLE MODELS

## COMPOUND PARTICLE MODELS \*

## W. THIRRING INSTITUTE FOR THEORETICAL PHYSICS, UNIVERSITY OF VIENNA, VIENNA, AUSTRIA

#### 1. INTRODUCTION

In these lectures I would like to talk on bound states and resonances in quantum field theory [1]. I will assume that you are familiar with this problem in potential scattering and will investigate similar problems in field theory. The difficulty is that the only systematic method of calculation in field theory is the perturbation theory. If you look at the analytical properties of single graphs the poles and singularities which correspond to bound states and resonances do not appear. Therefore one has to do something better and I will consider a particular model, the so-called ZACHARIASEN model [2] which is essentially the summing up of the chain-diagrams (Fig. 1a and 1b).



Typical chain diagrams

Typical ladder diagrams

In principle one could think also of the summing up of ladder diagrams (Fig. 2). They are, however, much more complicated and we will at first consider the chain.

The summing up of the chain leads to a unitary S-matrix which is by no means trivial, since one has taken out only special types of diagrams from the whole perturbation expansion. It also satisfies analytically, but not crossing-symmetry.

The model can either be considered as a prescription to select a certain type of diagram in ordinary perturbation theory, or in the framework of

\* Text based on notes, by K. Chadan, F. Hadjioannou, A.N. Mitra, H. Mitter, H. Pietschmann, T.K. Radha, E. van der Spuy.

dispersion theory, as a restriction on certain intermediate states, or as the exact solution of a Lagrangian field theory, in which the pairs of particles forming the bubble in Fig. 1a occur as a field with continuous mass spectrum [3]. The scattering amplitude T depends in this model only on one variable or, in other words, we are dealing only with the interaction of particles in a special angular momentum state the moment of which we take to be 0. The only possible justification of the approximation, which this model represents, comes from the field theoretic treatment of the many-body problem, where the bubble-summation leads to physically significant results (plasma oscillations of an electron gas). We shall discuss it in some detail since it reflects several features which are conjectured for a full-fledged field theory.

#### 2. THE SCATTERING AMPLITUDE

In the model we have two particles, one represented by  $\psi$  (solid line) with mass  $\frac{1}{2}$ , the other represented by  $\phi$  (broken line) with mass  $\mu_0$ , both with spin 0 (the generalization to spin  $\frac{1}{2}$  for  $\psi$  will be considered later). In the language of a Lagrangian formalism we have to consider the interaction term:

$$L' = -\lambda_0 \psi^4 + g_0 \phi \psi^2$$
 (1)

(note that a negative  $\lambda_0$  means attraction, a positive one repulsion) and have to sum up the diagrams shown in Fig. 2:

$$\texttt{X} = \texttt{X} + \texttt{Y} +$$

#### Fig. 2

$$T(s) = \lambda_0 + \frac{g_0^2}{s - \mu_0^2} + \lambda_0 \Delta(s) \lambda_0 + \frac{g_0^2}{s - \mu_0^2} \Delta(s) \lambda_0 + \lambda_0 \Delta(s) \frac{g_0^2}{s - \mu_0^2} + \frac{g_0^2}{s - \mu_0^2} \Delta \frac{g_0^2}{s - \mu_0^2}$$

 $\Delta$  (s) is an abbreviation for the divergent bubble; which is in co-ordinate space the product of two scalar Feynman propagators with the same argument. This product can be written as a weighted integral over single propagators [4]. In momentum-space this means:

$$\Delta(\mathbf{s}) = -\int_{1}^{\infty} \frac{Wds'}{s'-s+i\epsilon} \quad \text{with} \quad W = \frac{1}{16\pi} \sqrt{\frac{s'-s}{s'}} \quad (3), (4)$$

which is logarithmically divergent. We extract the singular part by making one subtraction as s = 0:

#### COMPOUND PARTICLE MODELS

$$\Delta(\mathbf{s}) = \Delta(0) + \Delta(\mathbf{s}) - \Delta(0) = \Delta(0) - \mathbf{s} \int_{1}^{\infty} \frac{\mathbf{W} \mathrm{ds'}}{\mathbf{s'}(\mathbf{s'} - \mathbf{s})} \quad . \tag{5}$$

 $\Delta(0) = -\overline{B}$  is an infinite constant and the remaining integral is convergant (note that  $\overline{B}$  is positive).

It can be seen immediately that (2) is a geometrical series which can be summed up to yield:

$$T(s) = \left[\lambda_{0} + g_{0}^{2} / (s - \mu_{0}^{2})\right] \left\{ 1 - \left[\lambda_{0} + g_{0}^{2} / (s - \mu_{0}^{2})\right] \Delta(s) \right\}^{-1}$$
(6)

Introducing (5) this can be written in the form:

$$T(s) = \overline{\lambda} / [1 + \overline{\lambda} s \int_{1}^{\infty} \frac{ds'W}{s'(s'-s)} + \frac{sR}{\overline{\mu}^{2}(s-\overline{\mu}^{2})}] = \overline{\lambda} / \overline{D}(s)$$
(7)

where:

for

$$\overline{\lambda} = T(0) = \lambda_0 / [1 + B\lambda_0 + g_0^2 / (\lambda_0 \mu_0^2 - g_0^2)]$$

$$R = -\overline{\lambda} g_0^2 / \lambda_0^2; \quad \overline{\mu}^2 = \mu_0^2 - g_0^2 / \lambda_0.$$
(8)

The form (7) shows explicitly that  $T \rightarrow \lambda_0$  as  $s \rightarrow \infty$ .

#### 3. POLES OF T(s)

We now want to investigate the explicit form of T(s), specially its poles, which will give us information on bound states and resonances. We therefore evaluate the integral occurring in  $\overline{D}(s)$ :

$$A \equiv s \int_{1}^{\infty} \frac{ds' \sqrt{1 - 1/s'}}{s'(s' - s)} = 2 - \sqrt{\frac{s - 1}{s}} \ln \frac{1 + \sqrt{s/(s - 1)}}{1 - \sqrt{s/(s - 1)}}$$
(9)  

$$s \rightarrow -\infty \qquad A \rightarrow 2 - \ln 2|s|$$
  

$$s \rightarrow 0 \qquad A \rightarrow \frac{2}{3} s(s + 1)$$
  

$$s \rightarrow 1 \qquad A \rightarrow 2 - i\pi \sqrt{s - 1} - 2(s - 1)$$
  

$$s \rightarrow +\infty \qquad A \rightarrow 2 - \ln 2s - i\pi$$

It should be noticed, that for s > 1, A becomes complex, since the denominators have to be taken with small imaginary parts in order to exhibit the properties of Feynman propagators. The real part has a cusp for s = 1.

We now discuss the zeros of D and take for granted, that a zero for s < 0 means a ghost, for 0 < s < 1 means a bound state and for s > 1 means



Re A(s) from Eq. (9)

a resonance. We shall call resonance everything where the phase shift goes through 90°. This will, however, be discussed in more detail later on. We first consider the case where there is no  $\phi$ -field and hence  $g_0^2 = R = 0$ . It can be seen from the first part of Eq. (8) that, if  $\lambda$  is to be finite,  $\lambda_0$  must approach 0 from negative values, i.e. for  $B \rightarrow \infty$  we have to start with weak attractive interaction.

The situation for negative  $\overline{\lambda}$  is plotted in Fig. 4 for two typical cases. Since Re D has a cusp at s = 1 there can be either no bound-state and no resonance  $(\overline{\lambda}_1)$  or one bound state and one resonance  $(\overline{\lambda}_2)$ . This situation is familiar



Fig. 4

a) Re  $\overline{D}$  for  $g_0 = 0$  for two typical negative values of  $\overline{\lambda}$ . b) Phase-shift for the same cases

from s-wave attractive potential scattering (for p-waves the situation is different. There one can have two resonances and one bound state). If  $\overline{\lambda}$  is bigger than zero, we will get a ghost and a resonance, as is seen from Fig. 5. We will therefore not consider this case.

The situation is similar to potential scattering with weak attractive potential, where the phase shift starts negative, unless a bound state has already been formed, in which case it changes its sign.

a)

b)



Re  $\overline{D}$  for  $g_0 = 0$  and positive  $\overline{\lambda}$  (occurrence of a ghost)

Another possibility is  $R \neq 0$ , in which case we also restrict  $\lambda < 0$ , (otherwise we have again a ghost), which implies also R > 0 (compare (8)). Because of the additional term in  $\overline{D}(s)$  the real part of  $\overline{D}$  has to change sign, since it eventually goes to  $-\infty$  at  $s = \overline{\mu^2}$ . We have again plotted two typical cases (Fig. 6a). One gets either one resonance ( $\lambda_1$ ) or one bound state ( $\lambda_2$ ) alone,



Re  $\overline{D}$  for  $g_0 = 0$  for two typical negative values of  $\overline{\lambda}$ .



(or with two resonances), depending on the magnitude of  $\lambda$ . ( $\lambda_1 < \lambda_2 < 0$ ). Finally we investigate the connection with the phase shift

tg 
$$\delta$$
 (s) = (16/ $\overline{\lambda}$ ).  $\sqrt{s(s - 1) \operatorname{Re} \overline{D}}$  (s). (10)

The corresponding plots are (4b) and (6b). The first case again resembles potential scattering. In the second case, we get a slight generalization of the Levinson theorem: in its usual form, it connects the number of bound states to phase shift difference, while here we have [5]

$$\delta(\infty) - \delta(0) = \pi.$$
 (Number of bare particles -  
number of physical particles) (11)

where physical particle means a discrete point in the mass spectrum, irrespective of whether there is a bare particle associated with this quantum number or not, i.e. all bound states are counted as physical particles. This can easily be seen from (4b) and (6b).

### 4. RENORMALIZATION AND DEFINITION OF COUPLING CONSTANTS

At first we will discuss the coupling constants attached to the bound states. We will use the usual definition, which states that the coupling constant is the residue of the corresponding pole of the T-matrix.

We rewrite T in the original unsubtracted form

$$T(s) = \lambda_0 \left( s - \overline{\mu}^2 \right) / \left[ s - \mu_0^2 - \Delta \lambda_0 \left( s - \overline{\mu}^2 \right) \right] = \lambda_0 \left( s - \overline{\mu}^2 \right) / D(s).$$
(12)

( $\Delta$  is the original expression (3)).

Since we are dealing with bound states only, we suppose

$$D(\mu^2) = 0$$
 for  $0 < \mu^2 < 1$  (13)

(this has to be considered as the definition of the physical mass  $\mu$ ).

It is then better to do the subtraction at the point  $\mu^2$  so that we now have the infinite constant

$$B = -\Delta(\mu^2) \rightarrow +\infty \tag{14}$$

(13) reads then

$$\mu^2 - \mu_0^2 + B\lambda_0 \left(\mu^2 - \overline{\mu}^2\right) = 0.$$
 (15)

We now subtract (15) from the denominator of (12) and obtain

$$D(s) - D(\mu^{2}) = D(s) = (s - \mu^{2}) \left[ 1 + \lambda_{0} \int_{1}^{\infty} \frac{ds'W}{(s' - \mu^{2})} \cdot \frac{s' - \mu^{2}}{(s' - s)} \right]$$
(16)

$$= (\mathbf{s} - \mu^2) [1 + \lambda_0 (\mathbf{B} + (\mu^2 - \overline{\mu}^2)\mathbf{I}] + \lambda_0 (\mathbf{s} - \mu^2) \int_{1}^{\infty} \frac{\mathrm{d}\mathbf{s}' \mathbf{W} (\mathbf{s} - \overline{\mu}^2)}{(\mathbf{s}' - \mu^2)^2 (\mathbf{s}' - \mathbf{s})}$$

where

$$I = \frac{d\Delta(\mu^2)}{d\mu^2} = \int_{1}^{\infty} \frac{Wds'}{(s' - \mu^2)} = \frac{1}{16\pi^2 \mu^2} \left[ \frac{\operatorname{arc} tg\sqrt{\mu^2/(1 - \mu^2)}}{\sqrt{\mu^2(1 - \mu^2)}} - 1 \right]$$
(17)

The limiting cases are

I 
$$\rightarrow$$
 (2/3) (1/16  $\pi^2$ ) for  $\mu \rightarrow 0$ 

and

$$1 \rightarrow 1/(32\pi \sqrt{1-\mu^2})$$
 for  $\mu \rightarrow 1$ 

Defining now a new renormalized

~

$$\lambda = \lambda_0 / \left\{ 1 + \left[ B(\mu^2 - \overline{\mu}^2) \right] \lambda_0 \right\} .$$
 (18)

We can write  ${\bf T}$  in a form which exhibits explicitly the singularity as well as the residue:

$$T(s) = \lambda (s - \overline{\mu}^2) / (s - \mu^2) \left[ 1 + \lambda (s - \mu^2) \right].$$
 (19)

The renormalized coupling constant for the bound state is, as already mentioned, defined by

$$\Gamma = g/(s - \mu^2)$$
 if  $s \rightarrow \mu^2$ .

Hence

$$g^2 = \lambda(\mu^2 - \overline{\mu}^2) \tag{21}$$

and one sees that  $\overline{\mu}^2$  can be expressed in the same manner by the unrenormalized and renormalized quantities

$$\overline{\mu}^2 = \mu - g_0^2 / \lambda_0 = \mu^2 - g^2 / \lambda$$
.

From this one can infer several interesting points. The first provides us with limits for g. We have

$$1/g^{2} = I + (1/g_{0}^{2}) \left[ (\overline{\mu}^{2} - \mu_{0}^{2})/(\overline{\mu}^{2} - \mu^{2}) \right]^{2} \geq I.$$
 (22)

In the case of a pure bound state the equality sign is valid, (case will be shown below), so that in general  $g^2$  is not arbitrary but restricted, its maximum value being  $\frac{1}{\Gamma}$ , and is attained for a bound particle. Furthermore  $\lambda$  is again restricted to be negative otherwise T(s) has a role for  $s < \mu^2$ . From (21) it then follows that  $\overline{\mu}^2 > \mu^2$ . After noticing these restrictions on the renormalized quantities, we ask whether the whole procedure is consistent. Imposing the condition that all renormalized quantities  $(g, \mu, \lambda)$  are finite and only B tends to infinity, we end up with a positive  $g_0^2$  tending to zero

$$g_0^2 = g^2 (1 - Ig^2)/(1 - \lambda B - Iq^2) \longrightarrow 0^+$$

and in addition we have

$$\mu_0^2 = \mu^2 - (g^2/\lambda) + (g^2/\lambda)(1 - Ig^2)/(1 - \lambda B - Ig^2) \rightarrow \tilde{\mu}^2$$

$$\lambda_0 = \lambda / (1 - \lambda B - Iq^2) \rightarrow 0^-.$$
(23)

#### 5. SPECIAL CASES

Let us consider some special limiting cases of the above: where either  $\lambda_0$  or  $g_0 = 0$ . In both cases  $\overline{\mu}^2$  tends to infinity and the T matrix (19) in terms of the proper renormalized quantities then reads as follows:

$$T(s) = g^{2} / \left\{ (s - \mu^{2}) \left[ 1 + (s - \mu^{2}) g^{2} \int_{1}^{\infty} \frac{ds' W}{(s' - \mu^{2})(s' - s)} \right] \right\}.$$
 (24)

A.  $\lambda_0 = 0$  (and  $\overline{\mu}^2 = \infty$ ), that is  $L' = g_0 \phi \psi^2$  and the renormalized mass value  $\mu^2$  will hence be that of an elementary particle of this field. One finds:

and

$$\frac{1/g^2}{\mu_0^2} = I(\mu) + \frac{1/g_0^2}{\mu_0^2}$$
(25)  
$$\mu_0^2 = \mu^2 + g_0^2 B$$

so that  $\mu_0^2$  tends to infinity as B tends to infinity. Thus we have to start with an infinite mass  $\mu_0^2$  and use an infinite mass renormalization to obtain the physical mass  $\mu^2$ .

B.  $g_0 = 0$  or  $L' = -\lambda_0 \psi^4$ , so that we have no  $\phi$  field but have the possibility of bound states of the  $\psi$  field due to the point interaction. Now there is no question of mass renormalization; one finds:

$$g^2 = 1/I(\mu)$$
. (26)

Formally this has some of the features of the previous case with  $g_0 \rightarrow \infty$ . This paradox need not be surprising because if we consider the  $\phi$  propagator:

$$g_0^2 / (s - \mu_0^2)$$

and let  $\mu_0^2$  tend to infinity with  $g_0^2$ , we can forget about the s dependence and  $g_0^2/(s - \mu_0^2)$  tends to  $\lambda_0$  a constant and the propagator shrinks to a point which is the present case.

Now from (26) and (17):

$$\frac{g^2}{4\pi} = \frac{1}{4\pi I(s)} = \begin{cases} 6\pi & \text{for } \mu \to 0^+ \\ \\ 8\sqrt{1-\mu^2} & \text{for } \mu \to 1. \end{cases}$$
(27)

The first case is that of strong binding; in this limit a strong coupling  $(g^2/4\pi = 6\pi)$  is obtained irrespective of the coupling constant  $\lambda_0$  to start with. This may suggest that strong interactions arise because the participating mesons are compound particles.

The latter case corresponds to the weak binding limit. This limit was also studied in Landau's consideration of elementary quantum field theory [6]. From the consideration of the coupling constant in relation to the asymptotic behaviour of the wave functions Landau deduced:

$$g^2/4\pi = 8\sqrt{2\epsilon (M_1 + M_2)}$$
 (28)

where  $\epsilon$  represents the binding energy and  $M_{1,2}$  are the masses. In our case  $M_1 = M_2 = \frac{1}{2}$ ,  $\epsilon = 1 - \mu$  so that we get exactly the Landau formula.

If we go to values of  $\mu^2 > 1$  the particle becomes unstable; formally, however, the formulae apply also in this case.  $\overline{D}$  (s) now develops an imaginary part and does not vanish in the physical sheet. (Compare (7)).

The condition for a resonance at  $s = \mu^2$  now has to be defined as:

$$\operatorname{Re} \overline{D}(\mu^2) = 0, \tag{29}$$

and consequently one only subtracts  $\operatorname{Re}\overline{D}(\mu^2)$ , (compare (16) for the bound state). Hence the definition (14) for the infinite constant B has to be changed to:

$$B = Re \int_{1}^{\infty} \frac{ds' W}{s' - \mu^2 + i\epsilon} = B \int_{1}^{\infty} \frac{ds' W}{s' - \mu^2},$$
 (30)

and similarly for I(s):

$$I(s) = \int_{1}^{\infty} \frac{ds' W}{(s' - \mu^2 \pm i\epsilon)^2},$$
 (31)

where the  $\pm i\epsilon$  shall indicate that one has to take the integration path once above and once below the singularity and then average. (Which is a sort of generalization of the principal value for higher powers of the denominator).

The coupling constant can now most conveniently be found by means of

Re 
$$1/T \rightarrow (s - \mu^2)/g^2$$
 for  $s \rightarrow \mu^2$ . (32)

It has the following significance in terms of the resonance width  $\Gamma$  (compare also (10)):

1
Re 
$$(1/T) = (1/16\pi) \cdot \sqrt{(s-1)/s} \cdot \cot \delta = (1/16\pi) \cdot \sqrt{(s-1)/s} \cdot (s-\mu^2)/\Gamma$$
(33)

which is a relativistic Breit - Wigner formula. Here

$$\Gamma = (g^2 / 16\pi) \cdot \sqrt{(\mu^2 - 1)/\mu^2}.$$
 (34)

#### 6. ANALYTICAL PROPERTIES AND RIEMANNIAN SHEETS

Now we investigate the analytical properties of T(s) and observe (one sees easily that Im  $T^{-1}(x + iy) \neq 0$  unless y = 0) that it has not poles in the complex s-plane unless one continues through the branch-line s = 1 to  $\infty$ . Even this branch-line is only a consequence of the choice of the variable s and one can get rid of it by considering T as a function of  $g = \sqrt{s - 1}$  which is half the momentum in the centre-of-mass system.

$$\frac{16\pi^2}{T} = \frac{16\pi^2}{\lambda} + s \int_{1}^{\infty} \frac{ds'\sqrt{1 - 1/s'}}{s'(s' - s)} + \frac{s - R}{s - \overline{\mu}^2}$$
$$= C - \frac{q}{\sqrt{1 + q^2}} \ln \frac{q + \sqrt{1 + q^2}}{q - \sqrt{1 + q^2}} + \frac{\overline{R}(q^2 + 1)}{q^2 + 1 - \overline{\mu}^2}, \quad (35)$$

where

$$C = 2 + 16\pi^2 / \overline{\lambda}$$
 (36)

is a convenient abbreviation for subsequent discussions and

 $\overline{R} = -16 \pi^2 g_0^2 / \overline{\mu}^2 \lambda_0$ .

The integral over s' in (35) behaves like  $\sqrt{s-1}$  for  $s \rightarrow 1$  as can be seen from (9) whereas no such root is present in the second, q-dependent, part. The mapping of the complex s-plane into the complex q-plane is shown in Fig. 7. The crosses are two complex conjugate points in the s-plane which are thus mapped into two points symmetrical with respect to the imaginary q-axis.

From (35) one can further obtain the relations

$$T (s^*) = T^* (s),$$
  
 $T (-q^*) = T^* (q).$ 
(37)

,The fact that T(s) has no singularities in the physical s-plane means that there are no singularities in the upper q-plane; except for possible bound-states which correspond to poles on the positive imaginary q-axis between O and i.



Fig.7

Mapping of the complex s-plane in the complex q-plane

In the second s-sheet (the lower half of the q-plane) we have

$$(T^{II} (s))^{-1} = (T^{I} (s))^{-1} + 2\pi i \sqrt{(s-1)/s}$$
(38)  
$$T (-q)^{-1} = T(q)^{-1} + 2\pi i q / \sqrt{1+q^2}.$$

or

This means an additional cut from 
$$-\infty to 0$$
 in the second s-sheet, corresponding to a branch-line extending from  $q = -i to -i^{\infty}$  in the lower half q-plane. These singularities are shown in Figs. 8 and 9.



Fig.8

Singularities in the s-plane

Because of the logarithmic character of T<sup>-1</sup> there are an infinite number of additional Rieman<sup>-</sup>ian sheets and the problem thus arises how to continue to the various sheets. If, in the second s-sheet, one does not cross the additional cut from  $-\infty$  to 0, one does not touch any cut in the q-plane and therefore reaches the first sheet again by crossing the main cut from 1 to  $\infty$ . A possible path of this type is shown in Fig. 10a. On the other hand, by crossing the  $-\infty$  -0 cut in s, one crosses the q-cut and hence steps down to the third s-sheet when one crosses the main cut. (See Fig. 10b).



Singularities in the q-plane



Fig. 10a

Particular singularities in the s, q-planes



Fig. 10b

Particular singularities in the s, y-planes

# 7. SPECIAL CASES

As before we now consider the two typical cases in the q-plane. A.  $g_0 = 0$  and hence  $\overline{R} = 0$ . (no  $\phi$ -field). To find the poles of T(q) corresponding to bound states, and therefore lying on the imaginary axis, put:

q = ik.

As can be seen from (35), the condition for such a pole turns out to be

$$C = \frac{\pi k}{\sqrt{1 - k^2}} - \frac{2k}{\sqrt{1 - k^2}} \arctan \frac{k}{\sqrt{1 - k^2}} \begin{cases} = 2 \text{ for } k = 1 \\ -\frac{2\pi k}{1 - k^2} \text{ for } k \rightarrow -1. \end{cases}$$
(39)

These two limits 2 and  $-\infty$  for C correspond respectively to  $\overline{\lambda} \rightarrow -\infty$  and  $0^-$ , as the cases of very strong and weak binding. Therefore as we turn on the interaction, the pole moves from q = -i to +i, along the imaginary axis.

As  $q \rightarrow 0$  the S-matrix assumes the simple form:

$$S = (q + iC/\pi)/(-q + iC/\pi)$$

where C now determines the scattering length.

For q outside the domain -i to +i on the imaginary axis the pole would correspond to a ghost, which could occur if  $\overline{\lambda}$  were not negative. B.  $\lambda_0 = 0$ ,  $\overline{R} \ge 0$  (presence of an elementary particle corresponding to  $\phi$ ). Here one pole moves as before but just somewhat displaced. We have now, however, two other poles due to the last term in (35) which equals  $R(q^2 + 1)$  when  $\overline{\mu}^2$  tends to infinity. It turns out that the signs of the imaginary parts of the logarithm and this latter term are opposite only in the lower half plane and thus the additional poles have to be located there. They move as indicated in Fig.9, corresponding to the  $\phi$ -particle which becomes less and less stable when one turns on the interaction.

## 8. **DISPERSION RELATION**

Because of its analytic properties one has the simple dispersion relation for T(s) which has no left hand cut because our model has no crossing symmetry:

$$T(s) = \frac{g^2}{s \cdot \mu^2} + \int_{1}^{\infty} \frac{ds' W}{s' \cdot s} |T(s')|^2.$$
 (40)

It has been mentioned in the introduction that the dispersion relation together with elastic unitarity provides us with a different starting point for our model.

#### 9. PROPAGATOR IN THE ZACHARIASEN MODEL

In the previous sections we investigated the S-matrix in the Zachariasen model. Now, I would like to investigate some other field theoretical quanti-

#### W. THIRRING

ties like the propagator and form-factor in the same model. Now, one can make a complete field theory out of the Zachariasen model. Therefore, one would expect that those features of the propagator and form-factors which one could deduce from general principles of field theory should also hold in the Zachariasen model. However, there is one feature in the Zachariasen model which violates the general principles of field theory: there is no crossing symmetry in the S-matrix. This can be traced back to a failure of satisfying the asymptotic condition. However, this does not disturb our finding of the propagator and form-factors; and for these quantities the model provides an interesting illustration of various general conjectures and general theorems about the propagator and form factors. Before going to these quantities, let us make a short digression on the failure of the asymptotic condition.

One can formulate the Zachariasen model in the following way: one takes as the basic diagrams of the model shown in Fig. 11.



## Fig. 11

#### Basic diagrams

The pairs of particles can be represented by a field with a continuous mass distribution. This corresponds to a quadratic Hamiltonian which can be diagonalized exactly. Now the question is why this field theory does not satisfy the asymptotic condition. The reason is that the particles always occur in pairs. Therefore, a single particle cannot be projected out. This means that the single-particle states are not coupled in the Zachariasen model. Yet what we are interested in is the scattering of these particles and there is no crossing symmetry in the S-matrix because of the lack of the asymptotic condition. If the asymptotic condition were true one would have automatically the crossing symmetry as it is obvious from the definition of the S-matrix in terms of the asymptotic fields in the L.S.Z. formalism.

Because of this also we cannot trace Regge pole trajectories. We do not have any momentum-transfer dependence in the S-matrix and therefore there is only one angular momentum involved.

To calculate the propagator, let us start as in the old-fashioned way by summing diagrams. These are shown in Fig. 12.

> ---- + --- - - - - - - - - - - + ----Fig. 12

Diagram for the propagator calculations

Actually, a partial summation of these has been done a long time ago by Dyson who showed that the complete unrenormalized propagator has the form:

$$\Delta_{\rm u} = 1/({\rm s} - \mu_0^2 + \pi). \tag{41}$$

 $\pi$  is the proper self-energy part and is the sum of the diagrams, represented in Fig. 13.



Fig. 13

### Diagrams related to the proper self energy part

i.e., the diagrams which cannot be separated in two parts by cutting a dask n line. This is again a geometrical series and we can sum up the whole series. We get:

$$\pi = g_0^2 \Delta / (1 - \lambda_0 \Delta), \qquad (42)$$

and

$$\Delta_{\rm u} = 1/[s - \mu_0^2 + g^2 \Delta / (1 - \lambda_0 \Delta)] = (1 - \lambda_0 \Delta) / [(s - \mu_0^2)(1 - \lambda_0 \Delta) + g^2 \Delta].$$
(43)

Let us now study the poles and various other properties of this propagator. To do this we first write  $\Delta_u$  in the following form:

$$\Delta_{\rm u} = \left[ \frac{1}{({\rm s} - \overline{\mu}^2)} \right] \left[ \frac{1 - \lambda_0 \Delta + ({\rm s} - \mu^2)}{({\rm s} - \overline{\mu}^2)} - \frac{({\rm s} - \mu^2)}{({\rm s} - \overline{\mu}^2)} \right] / \left[ \frac{({\rm s} - \mu_0^2)}{({\rm s} - \overline{\mu}^2)} - \frac{\lambda_0 \Delta}{({\rm s} - \overline{\mu}^2)} \right]$$
  
=  $\frac{1}{({\rm s} - \overline{\mu}^2)} + \frac{(\mu_0^2 - \overline{\mu}^2)}{({\rm s} - \overline{\mu}^2)^2} + \frac{({\rm s} - \overline{\mu}^2)^2}{({\rm s} - \overline{\mu}^2)^2} - \frac{\lambda_0}{({\rm s} -$ 

We see therefore that this contains T(s) and other known factors; so we know essentially the poles of the propagator. They are identical with those of the T-matrix. If we introduce the function D, introduced before, formula (15), we get:

$$\Delta_{\rm u} = 1/({\rm s} - \overline{\mu}^2) + (\mu_0^2 - \overline{\mu}^2) \lambda / ({\rm s} - \overline{\mu}^2) ({\rm s} - \mu^2) \lambda_0 D({\rm s}),$$

$$D(\mu^2) \approx 1.$$
(45)

This expression is valid provided there is a stable particle  $\mu^2 < 1$ . Therefore,  $\Delta_u$  has a pole at  $s = \mu^2$ . One might think that there is also one pole at  $s = \overline{\mu}$ , but this is only apparent because  $T(\overline{\mu}^2) = 0$ . There is only one pole corresponding to one stable particle and we can renormalize the expression ' according to the usual prescription:

$$\Delta_{u} \xrightarrow[s \to \mu^2]{} Z_3 / (s - \mu^2).$$
(46)

This gives:

$$Z_{3} = (\lambda/\lambda_{0}) \left[ (\mu_{0}^{2} - \overline{\mu}^{2}) / (\mu^{2} - \overline{\mu}^{2}) \right] = 1 - Ig^{2} = D(\overline{\mu}^{2})$$
(47)

where I is given by (17).

As we have seen before, one has always

 $g^2 < 1/I$ .

except in the case of a compound particle  $(g_0 = 0)$ . Therefore, we see that, in general:

$$0 < Z_3 < 1$$
,

which means that the theory is a good field theory. The renormalized propagator defined by:

$$\Delta_{\pi} = \Delta_{\rm M}/Z_3 \tag{48}$$

is given by:

$$\Delta_{\pi} = [1/(s - \overline{\mu}^2)] [1/D(\overline{\mu}^2)] + (\mu^2 - \overline{\mu}^2)/(s - \overline{\mu}^2)(s - \mu^2), \quad (49)$$

or, in a better form:

$$\Delta \pi = [1/(s - \mu^2)] [1/D(s)] + [1/(s - \mu^2)] [1/D(\overline{\mu}^2) - 1/D(s)].$$
(50)

As we shall see later, this last form of the renormalized propagator is appropriate for the introduction of the spectral representation.

The value  $g^2 = 1/I$  is obtained in the case of a compound particle. In this case  $Z_3 = 0$ .

If one starts with  $\lambda_0 = 0$  and  $g_0^2 \neq 0$ , one gets:

$$I = g^2 / g_0^2 . (51)$$

A bound particle can also be considered as the limit where one has only  $g_0^2$  and  $\mu_0^2 \rightarrow \infty$ , which means  $g_0^2 / (s - \mu_0^2) \rightarrow \lambda_0$ .



Fig. 13a

Diagrams related to  $g_0^2/(s - \mu_0^2) \rightarrow \lambda_0$ 

In this limit  $Z_3 = 0$  and this is clear if we remember the physical significance of  $Z_3$ . Let us assume that there is a vacuum in the theory and apply to it

466

the Heisenberg operator  $\phi(0)$ . This generates a state which we represent by 1):

$$\phi(0) \mid 0 \geq \approx \mid 1)$$

up to a normalization factor. This state is neither a physical one-particle state nor a bare one-particle state. Let us call it the undressed one-particle state. Then  $Z_3$  is just the probability on finding a physical one-particle state in the undressed one-particle state. Now, this undressed particle is not an eigenstate of the energy but has a mass distribution and one can show that this mass distribution is centred around the bare mass  $\mu_0$ . In fact, the bare mass is just the average of this mass distribution. However, the physical mass is  $\mu$ . If we keep  $\mu^2$  fixed and let  $\mu_0^2 \rightarrow \infty$ , the probability of finding the state  $|1\rangle$  in the state  $|1\rangle$  becomes smaller and smaller and therefore  $Z_3 \rightarrow 0$ .

As far as the bound particle is concerned, one can also argue in the following way. If we start with  $g_0 = 0$ , the field  $\phi$  is no longer coupled to the field  $\psi$  and therefore does not come into the game. However, we can consider  $\psi^2$  (pairs of particles) instead of  $\phi$ . In this way we can also define a propagator of  $\psi^2$  rather than of  $\phi$  and see if it is possible to define a reasonable renormalization constant in this way. This means that we consider only the diagrams shown in Fig. 14.

 $\bigcirc + \bigcirc + \bigcirc + \bigcirc +$ 

#### Fig. 14

Diagrams related to the case  $g_0 = 0$ 

In this case we find:

$$\Delta_{\rm u} = \Delta / (1 - \lambda_0 \Delta) \tag{52}$$

and again one can look for the poles and define  $Z_3$ . We get:

$$Z_3 = g^2$$
 (53)

which is no longer less than unity. This fact is not in contradiction with any fundamental principle of the Q.F.T. because the bound unity for  $Z_3$  was derived from the canonical commutation relations for the field  $\phi$ . Now,  $\psi^2$  does not satisfy similar canonical commutation relations and therefore  $Z_3$  is no longer bounded by one.

# 10. SPECTRAL REPRESENTATION OF THE PROPAGATOR

After this digression about  $Z_3$ , let us see what the spectral representation of the propagator looks like. First, because of the analytic properties of D(s), D<sup>-1</sup>(s) can be written in the form:

$$D^{-1}(s) = 1 - g^2 \int_{1}^{\infty} \frac{ds'}{s' - s} \frac{(s - \mu^2)(s' - \mu^2)}{(s' - \mu^2)^2 |D(s')|^2}.$$
 (54)

By inserting this we get the spectral representation of  $\Delta_{\pi}(s)$ . This is of interest if we want to see what happens if there is no stable particle but only a resonance. The question is what happens to the pole. In fact, one would conjecture the following: in the case of one stable particle the spectral function looks like the curves shown in Fig.15.



Fig. 15

Spectral function

If the particle becomes unstable, then the continuum moves down, goes below  $\mu^2$  and the  $\delta$ -function disappears. However, a bump will be left on the continuum and the width of it will be related to the lifetime of the unstable particle. That this actually happens in our theory can be seen if we go back to T(s) and

$$\Delta_{11} = \frac{1}{(s - \overline{\mu}^2)} + (\mu_0^2 - \overline{\mu}^2) T(s) / (s - \overline{\mu}^2) \lambda_0.$$

In this form we see that what happens to the pole in the propagator is exactly the same that happens to the pole in T(s): the pole moves to the second sheet of the Riemann surface (the unphysical sheet) and its effects show up by a strong peak in the spectral function.

# 11. FORM-FACTORS IN THE ZACHARIASEN MODEL

The form factor of the pion is defined by

$$F(s) = \langle 2\pi | j | 0 \rangle = g \langle 2\pi | \psi^2 | 0 \rangle;$$
 (55)

As shown in the diagram one has

$$F_{(s)} \land + \land + - = T_{(s)}$$

This is very closely related to the S-matrix. In fact, if we stick to F(s), the following pieces

$$F(s) | \wedge + \wedge = T(s),$$

we reproduce the T-matrix. In other words, we have

$$F(s)[\lambda_0 + g_0^2 / (s - \mu_0^2)] = g T(s).$$
(56)

Therefore

$$F(s) = gT(s)/\lambda_0 (s - \mu_0^2) = gT(s)(s - \mu_0^2)/(s - \overline{\mu}^2).$$
 (57)

Remembering that  $T(\mu^2) = 0$ , the last equation shows that the poles of F(s) are just those of T(s). From this fact immediately one draws the conclusion that, if there is a resonance, it manifests also in the form-factor; i.e. a bump on the spectral representation of F(s).

# 12. MORE REALISTIC MODELS

Till now we have worked only with scalar particles. This means that we can get only an S-wave resonance. Indeed this summing of the bubbles gives just the S-wave dominant solution of Chew and Mandelstam. But we know that such a resonance has not been found in nature. To get a p-wave resonance one can try to generalize the theory. For instance, one can produce a p-wave resonance by introducing a vector particle.

Another generalization is to assume that  $\psi$  corresponds to nucleons and see whether one can produce a bound-state of nucleon-antinucleon by summing up the chain diagrams shown in Fig. 16.



#### Fig. 16

Chain diagrams to be summed when y corresponds to nucleons

In both cases, one meets with divergences. In fact, the degree of divergency is increased by one compared to the simple case we had before. Whereas we had before one infinite constant, here everything diverges by one power more and therefore everything is more ambiguous in the model. In fact, if  $\psi$  is a spinor, the calculations go more or less in the same way as before. For instance, if we have a  $\gamma_5$  interaction we have to take the trace around the bubbles. In this case the square root we had before goes to

$$W \longrightarrow (1/4\pi^2) \sqrt{1 - 1/s} (s + 1/4)$$
(58)

Therefore, the expression I diverges logarithmically. If we use a cut-off  $\Lambda,$  then

$$I \sim \log \Lambda$$

This has been done by Nambu who gets

$$g^2 = 2\pi \sqrt{2} / \sqrt{\log \Lambda}$$
.

If we identify the bound object with the pion, we obtain the pion nucleon coupling constant. Actually, it turns out that the same kind of formula also holds if we sum up not the bubbles but the ladder diagrams shown in Fig. 17.



#### Fig. 17

Ladder diagrams

However, this is a more difficult problem and what one can do is to sum this up in the extreme relativistic limit where the masses involved are neglected compared with moments which are involved. In fact, one uses the Bethe-Salpeter equation which has a very simple spectral representation in the relativistic limit.

The problem of finding whether or not the pion is composed of nucleonantinucleon can be solved in the following way. As we have seen before, there is a distinction in our theory between the case where there is an elementary particle behind it and the case where there is no elementary particle behind it. This just goes via the Levinson theorem. In particular, if we have no bare particle, then we can have for  $R_eT^{-1}$  a behaviour like Fig. 4.

If there is a bare particle,  $R_eT^{-1}$  has a behaviour similar to Fig.6. If there is an additional resonance then the phase shift goes back to 90° as shown in Fig.4b.

So, what one can do is to calculate the phase-shift supposing the pion is a composite particle. Then one looks where the phase-shift goes to 90° and sees whether one can find it experimentally. This sounds nice theoretically, but in practice it does not seem feasible.

If we sum the chain diagrams, after obtaining the renormalization constant and the mass of the pion the rest is then cut off independent and finite. So one can calculate the place where the phase-shift goes to  $90^{\circ}$  and the width of it if we call it a resonance. Numerically, it turns out that the mass is:

$$M = 2 M_{D} + 940 MeV.$$

 $M_p$  is the mass of the proton and the width is 450 MeV. This means a very broad S-wave nucleon-antinucleon resonance. This is physically very hard

to observe because in this energy region the S-wave cross-section is very small and also there are inelastic channels and they will obscure the picture further.

Let us now make some remarks about the influence of the inelastic channels on the whole structure of the theory. For simplicity, let us assume first that  $g_0 = 0$  and keep only the direct coupling. We assume that there are two kinds of particles:

A-particle : \_\_\_\_ mass  $\frac{1}{2}$ B-particle : \_\_\_\_ mass m

The interactions are of the type shown in Fig. 18,



Fig. 18

Types of interactions

and the basic diagrams are those shown in Fig. 19, and similarly for the exchange scattering  $T_{AB}$  and for  $T_{BB}$ . Now, we are

$$T_{\mathsf{A}\mathsf{A}}=\times+\times\!\!\times\!\!\times+\times\!\!\times\!\!\times$$

Fig. 19

Basic diagrams

dealing once again with geometrical series which can be summed up rather easily. If we use matrix notations  $(2 \times 2 \text{ matrices})$  then the coupling could be written in the form:

 $\lambda = \begin{pmatrix} \lambda_{AA} & \lambda_{AB} \\ \\ \lambda_{BA} & \lambda_{BB} \end{pmatrix}$ (60)

and  $\Delta$ , the sum of the bubbles:

in the form:

$$\Delta = \begin{pmatrix} \Delta_{\mathbf{A}} & 0 \\ 0 & \Delta_{\mathbf{B}} \end{pmatrix}$$
(61)

Then, the T-matrix is given by:

$$T = \lambda + \lambda \Delta T. \tag{62}$$

 $\Delta$  is diagonal, but  $\lambda$  has non-diagonal elements, so that T is non-diagonal. We can easily solve this equation and the solution is:

$$T^{-1} = \lambda^{-1} - \Delta. \tag{63}$$

Again, we have a rather simple expression for the T-matrix and the analysis goes as before.

This model in itself is interesting to study, for instance, the cusps in one cross-section at the threshold of a second channel.

Let me make a last remark as to what happens to the Levinson theorem if we have a two-channel reaction. For instance, let us suppose that the diagram shown in Fig. 20



```
Diagram corresponding to nucleon-antinucleon annihilation into two pions
```

corresponds to the annihilation of nucleon-antinucleon into two pions and let us assume that we have also these kinds of inelastic contributions. How does this change our conclusion that the phase-shift goes back at 90° at certain energy? It turns out that in this model the Levinson theorem still holds in an analogous form to the case we had before. Here, however, the phaseshifts are complex but the real part goes back to 90°. So what one has to do is to look at Re $\delta$  and see if it goes back to 90°. Practically, this is very difficult because the elastic cross-section is a small part of  $\sigma_{\text{total}}$ . If there were a small bump in  $\sigma_{\text{elastic}}$  it will be overshadowed by  $\sigma_{\text{inelastic}}$ .

One can say that, in principle, in the framework of field theory there may be an exact criterion to distinguish whether a particle is elementary or composite. However, in practice, it will take quite a long time until one can really make this test experimentally.

# 13. GENERALIZATION TO MORE REALISTIC CASES

Up to now we have been confined to the Zachariasen model which neglects the crossing symmetry as well as the contribution from the inelastic channels. To incorporate crossing symmetry also we have to include the left hand out

472

so that the general dispersion relation for the partial wave scattering amplitude T(s) reads as

$$T(s) = \left\{ \begin{array}{c} 0 & \infty \\ \int + \int \\ -\infty & 4\mu^2 \end{array} \right\} ds' \quad \frac{\operatorname{Im} T(s' + i\epsilon)}{\pi(s' - s)}$$

where  $+i\epsilon$  indicates that we are integrating above the real axis. The lower limit of integration  $4\mu^2$  indicates the physical threshold (as we had in the Zachariasen model). However, superimposed on it we will have inelastic thresholds corresponding to branch cuts starting at s. The unitarity condition in the form we had so far holds only in the elastic region, i.e.  $s_i < s < 4\mu^2$ ,

$$\operatorname{Im} \mathbf{T}(\mathbf{s} + \mathbf{i}\epsilon) = \rho \left(\mathbf{s} + \mathbf{i}\epsilon\right) |\mathbf{T}|^2 \tag{65}$$

where  $\rho$ , the phase space factor is given by

$$\rho(s) = \sqrt{(s - 4\mu^2)/s}$$
 (66)

Above the inelastic threshold the situation becomes more complicated and has been dealt with by B. Lee (see his lecture notes). We therefore confine ourselves to the region below production threshold. Cf. also [9]. In writing the dispersion relation (64), we have assumed certain analyticity properties which we have not proved. But this follows in particular from the Mandelstam representation, and we shall not bother about it here. Again T is considered only as a function of s which holds for any partial wave amplitude. However the unitarity relation (65) holds only for S waves but the generalization can easily be done. The relations (37) follow from the dispersion relation (64). The analytical continuation through the branch cut in the S plane extensively dealt with in lecture II can be done only between  $4\mu^2$  and s;  $\rho$  (s) has two branch points at s =  $4\mu^2$  and s = 0, and as before we locate the branch cut from  $-\infty$  to 0 and  $4\mu^2$  to  $+\infty$ . It then follows  $\rho$  (s) =  $-[\rho(s)]^*$  since in this region  $\rho$  (s) is purely imaginary.

From (65) we obtain

$$\operatorname{Im} T^{-1} (s + i\epsilon) = -\operatorname{Im} T(s + i\epsilon) / |T|^2 = -\rho(s + i\epsilon) = \rho(s - i\epsilon) \quad (67)$$

and therefore get the analytic continuation to the second sheet of the inverse amplitude.

$$T_{I}^{-1}(s+i\epsilon) = T_{I}^{-1}(s-i\epsilon) + 2i\rho (s-i\epsilon) = T_{II}^{-1}(s-i\epsilon)$$
(68)

or if we invert this relation (68) we obtain

$$T_{\rm II} = T_{\rm I} / (1 + 2i\rho T_{\rm I}).$$
 (69)

Again we see there may be poles in the second sheet due to the vanishing of the denominator in (69)

(i.e.) 
$$1 + 2i\rho T = 0$$

and these poles correspond to resonances. The location of the branch cuts in the s and q plane are given in Fig. 21.



Fig. 21

Location of branch cuts in the s and q-plane

Concluding this section, we list a few formulae decomposing T into dispersive and absorptive parts [10]:

and inverting we get

.

$$a = TT_{II}$$
(71)  
$$d \approx T_{II} (1 + i\rho T).$$

The advantage of this decomposition is that even in the s plane the elastic branch cut is absorbed into the phase space factor  $\rho$  (s).

In particular in the elastic region

(i.e.) 
$$4\mu^2 < s < s_i$$
  
we have  $a = \frac{1}{\rho} \text{ Im } T$  (72)  
 $d = \text{ Re } T$ .

The partial wave S matrix now takes the simple form

$$S = 1 + 2i \rho T_{I} = T_{I} / T_{II} = T(q) / T(-q).$$
(73)

Comparing (69) with (73) we see that the resonances correspond to the zeros of the S matrix (See also [11])

474

# 14. FORM FACTORS

We will now investigate whether the same resonances show up in the form factors also. We will distinguish the two cases with and without anomalous threshold.

### A. No anomalous threshold

The graphical representation for the form factor is shown in Fig. 22 where the solid line represents the particle whose scattering we have con-





Graphical representation of the form factor

sidered before. The connection between the scattering amplitude and the form factor can be seen from Fig. 23



Fig. 23

Decomposition of the form factor

where we have decomposed the bubble taking out the lowest intermediate state. This immediately implies that there is a branch cut starting at  $4\mu^2$ . Hence we obtain

$$\mathbf{F}(\mathbf{s}) = 1 + \frac{\mathbf{s}}{\pi} \int_{-\frac{1}{2}}^{\infty} d\mathbf{s}' \frac{\mathrm{Im} \mathbf{F}(\mathbf{s}' + i\epsilon)}{\mathbf{s}'(\mathbf{s}' - \mathbf{s})}$$

This formula again implies the analyticity and boundedness properties of F(s) which is also normalized to unity at s = 0.

To apply the dispersion relation (74) we require some explicit statements about the imaginary part of F(s). In the Zachariasen model we saw that it is directly related to the scattering amplitude. We will now show that a similar relation holds also here. By the general Lehmann - Symanzik- Zimmermann technique [12].

F(s) is related to  $\langle \pi | j | \pi \rangle$  or to  $\langle 0 | j | 2\pi_{in} \rangle$ . Applying the time reversal operation (assuming the current j has definite properties under this operation) we obtain

$$F(s) \sim < 0 |j| 2\pi_{in} > = [<0|j| 2\pi_{out} > ]^*.$$
 (75)

Since the in- and out- states are related by the S-matrix and since  $< 0 | j^*$  is a state of definite angular momentum (implying a diagonalized S-matrix) we obtain

$$\mathbf{F}(\mathbf{s}) = \mathbf{F}^*(\mathbf{s}) e^{2\mathbf{i}\boldsymbol{\delta}}$$
(76)

which leads to

$$Im F = \sin \delta e^{-id} F = \rho T^*F.$$
(77)

Because of the simple diagonalization of the S-matrix this holds only in the elastic region. All quantities in (77) have to be taken on the upper lip of the branch cut and because of (68) and (69) we have

Im F = 
$$\rho FT/(1 + 2i\rho T)$$
. (78)

Since we now know the imaginary part of F, we can continue F analytically into the second sheet by

$$F_{II} = F_{I} - 2iIm F_{I} = F(1-2i\rho T^{*})$$
 (79)

and because of (78) and the unitarity of the S-matrix we end up with

$$F_{II} = F_{I} / (1 + 2i\rho T).$$
 (80)

Therefore in the second sheet F has the same poles as T, corresponding to the same resonances.

Similar considerations can be made in the q-plane.

476

# B. ANOMALOUS THRESHOLDS [13]

In this more complicated case we have to consider a special contribution to the graph of Fig. 22 which leads to the anomalous branch point (Fig. 24). This is the form factor of the particle m and as a physical example we consider the form factor of the  $\Sigma$ ; i.e.  $\mu \leftrightarrow \pi$ ,  $M \leftrightarrow \Lambda$  and  $m \leftrightarrow \Sigma$ .





Graph, leading to anomalous branch point

The anomalous thresholds appear only for special relations among the masses M, m and  $\mu$  and to investigate this problem we first consider the diagram in Fig. 25,



Fig. 25

Scattering diagram, leading to anomalous threshold

where the four vectors  $\mathbf{p}_i$  and  $\mathbf{q}_i$  are related by energy-momentum conservation.

We choose the usual three invariant variables

$$s = (q_1 + q_2)^2$$
  

$$t = (p_1 + q_1)^2$$
  

$$t = (p_1 + q_2)^2$$
(81)

In the centre-of-mass system the energy-momentum vectors are given by

$$q_{1} = \begin{pmatrix} -\sqrt{q^{12} + \mu^{2}} \\ -\overline{q^{1}} \end{pmatrix} \qquad q_{2} = \begin{pmatrix} \sqrt{q^{12} + \mu^{2}} \\ \overline{q^{1}} \end{pmatrix}$$

$$p_{1} = \begin{pmatrix} -\sqrt{q^{2} + \mu^{2}} \\ -\overline{q} \end{pmatrix} \qquad p_{2} = \begin{pmatrix} \sqrt{q^{2} + m^{2}} \\ -\overline{q} \end{pmatrix}$$

$$p_{2} = \begin{pmatrix} \sqrt{q^{2} + m^{2}} \\ -\overline{q} \end{pmatrix}$$

$$p_{3} = \begin{pmatrix} \sqrt{q^{2} + m^{2}} \\ -\overline{q} \end{pmatrix}$$

$$p_{4} = \begin{pmatrix} \sqrt{q^{2} + m^{2}} \\ -\overline{q} \end{pmatrix}$$

$$p_{5} = \begin{pmatrix} \sqrt{q^{2} + m^{2}} \\ -\overline{q} \end{pmatrix}$$

$$p_{6} = \begin{pmatrix} \sqrt{q^{2} + m^{2}} \\ -\overline{q} \end{pmatrix}$$

and the following relations hold:

$$s = 4 (q^{12} + \mu^{2}) = 4 (q^{2} + m^{2})$$
  

$$t = -q^{2} - q^{12} + 2 qq^{1} \cos \theta$$
  

$$\overline{t} = -q^{2} - q^{12} - 2 q_{1} \cos \theta$$
(83)

where  $\theta$  is the scattering angle.

In the usual way, the poles correspond to t or  $\overline{t} = M^2$ . However, if we project out a partial wave in the s-channel, that is

$$T_{\boldsymbol{\ell}}(s) = \int_{-1}^{1} d\eta P_{\boldsymbol{\ell}}(\eta) T(s, \eta) \qquad \eta \equiv \cos \theta$$
(84)

the pole is converted into a branch-line with the end points corresponding to  $\cos \theta = \pm 1$ . From (83) we thus get the branch points at

 $(q \pm q')^2 = -M^2$ . (85)

Solving (83) and (85) we get

$$q'_{a} = \pm i \frac{m^{2} + M^{2} - \mu^{2}}{2M}$$
 (86)

or

$$s = -\frac{1}{M^2} \{m^2 - (M + \mu)^2\} \{m^2 - (M - \mu)^2\} = g(m^2)$$
(87)

where g is, of course, a function of M and  $\mu$  also. If we plot it, however, as a function of m<sup>2</sup>, we obtain the location of the branch-points as shown in Fig. 26.

The points  $(M + \mu)^2$  and  $(M - \mu)^2$  correspond to  $\Sigma$  and  $\Lambda$  respectively, becoming instable with the decay processes  $\Sigma \rightarrow \Lambda + \pi$  and  $\Lambda \rightarrow \Sigma + \pi$ . Between these two points all particles are stable. We can now draw the branch-cuts in the q'-plane for two typical cases (Fig. 27).

where the two branch-lines approach each other when  $m^2 = M^2 + \mu^2$ . It is clear how this looks in the s-plane for  $m^2 = M^2 + \mu^2$  the left branch cut of the first and second sheet meet each other at the threshold of the right-hand





 $g(m^2)$  - Eq.(87)- as a function of  $m^2$ .





Branch-cuts in q'-plane for two typical cases

cut. For  $m^2 > M + \mu^2$  the lower branch point moves out into the physical sheet, leading to an anomalous threshold. The situation is indicated in Fig. 28.



Fig. 28

Anomalous threshold in the s-plane

For the form factor the situation is entirely similar except there is no left-hand cut in the physical sheet.

We therefore have to modify the lower limit of the integral in equation (74) from  $4\mu^2$  to  $g(m^2)$ .

It is well-known that in the configuration space a dispersion relation corresponds just to the superposition of Yukawa potentials. (In the centreof-mass system).

$$f(x) = \int_{g}^{\infty} ds \frac{\mathrm{ImF}(s) \varrho^{-|\bar{x}| f_s}}{4\pi |\bar{x}|} .$$
 (88)

Hence  $g(m^2)$  provides us with the Yukawa potential of longest range, given by

$$\frac{e^{-|\mathbf{x}| \cdot \mathbf{g}}}{4\pi |\mathbf{x}|}$$

where  $|x| \sqrt{g}$  can be calculated from (87) to be

$$\left|\overline{x}\sqrt{g}\right| = \left|\overline{x}\right| \cdot 2(M+\mu)/M\sqrt{2\epsilon M\mu/(M+\mu)} = 2r 2\epsilon M$$

where  $\epsilon$  is the binding energy, defined by

$$m = M + \mu - \epsilon$$

and  $r_e$  is the relative co-ordinate, say of the  $\Lambda \pi$  system forming the  $\epsilon$ .  $M_r$  is the reduced mass. We therefore conclude, that physically an anomalous threshold corresponds to a spreading of the compound system which is greater then the Compton wave-length of the particles into which it can decay virtually.

### 15. CONCLUSION

We have been considering only a particular model and a slight extension, where one removes some of the restrictions. It is, of course, to be hoped that all this can one day directly be deduced from axiomatic field theory, but up to now it was not possible, and one has to be content to illustrate the situation by simple examples. Nevertheless, these examples show that the situation in field theory ties on directly to the situation one has in potential scattering.

### REFERENCES

[1] THIRRING, W., Nuovo Cimento 23 (1962) 1064.

[2] ZACHARIASEN, F., Phys. Rev. 121 (1961) 1851.

[3] GELL-MANN, M. and ZACHARIASEN, F., Phys. Rev. 124 (1961) 953.

[4] THIRRING, W., Phys. Rev. to appear in Vol. 126.

- [5] THIRRING, W., The Principles of Quantum Electrodynamics, New York, 1958, App. II.
- [6] IDA, M., Progr. of Theor. Phys. 20 (1958) 979.
- [7] LANDAU, JETP 12 (1961) 1294.

.

- [8] BJORKEN, J. D., Phys. Rev. Lett. 4 (1960) 473.
- [9] OEHME, R., Herzegnovi-Lectures (1961).
- [10] OEHME, R., The Compound Structure of Elementary Particles, Celebration Volume for Heisenberg's 60th Birthday (1961).
- [11] LEVY, M., Nuovo Cimento 13 (1959) 115.
- [12] LEHMANN, H., SYMANZIK, K., ZIMMERMANN, W., Nuovo Cimento 1 (1955) 205 and 6 (1957) 319.
- [13] KARPLUS, R., SOMMERFIELD, C. and WICHMAN, E., Phys. Rev. 111 (1958) 1187.

# BOOK V

# VERY HIGH ENERGIES

PART I

STRONG INTERACTIONS

# VERY HIGH-ENERGÝ PHENOMENA

# S. HAYAKAWA PHYSICAL INSTITUTE, NAGOYA UNIVERSITY, CHIKUSA-KU, NAGOYA, JAPAN

# 1. INTRODUCTION

Nowadays protons of energies up to 30 GeV are available with accelerators and their interactions have been investigated in considerable detail. At higher energies cosmic rays provide information on high energy phenomena. By the term "very high energy phenomena" one usually means those phenomena which take place in interactions of cosmic rays of very high energies, higher than the energy reached by existing accelerators.

Many features of very high energy phenomena defined as above are not always characteristic of very high energy interactions but are rather closely connected with those which are seen in the GeV region. This is because the energies of particles participating in an interaction may turn out to be rather low in an appropriate coordinate system. Therefore the knowledge obtained at accelerator energies is indispensable in understanding very high energy phenomena.

In practice, however, the attitude towards the study of very high energy phenomena with cosmic rays is considerably different from that with accelerators. This is due mainly to the fact that events to be obtained with cosmic rays are so few that one has inevitably to restrict oneself to qualitative understanding. This is on one hand a drawback of cosmic ray research, but on the other hand it is an advantage in introducing rather brave imagination, which may play the role of a patrol to find problems to be investigated more quantitatively.

Moreover, there is an important difference in technical problems; with accelerators the energy and the intensity of incident particles are well known, whereas with cosmic rays, their determination is the central part of experiment. Much progress has been made in the technique of determining the energy of a particle in cosmic rays, and its results are now found to be useful in the observation of secondary particles produced by a beam from an accelerator.

Experimental information on high energy interactions in cosmic rays is most directly obtained by observing jet showers with photographic plates. The terminology widely used for very high energy phenomena stems mostly from that defined by POWELL and his collaborators [1] in their course of study with photographic plates. We therefore begin with defining such technical terms in conjunction with the description of characteristic phenomena to be observed. If the energy goes still higher, one can hardly observe the primary particles which initiate the interactions, because they are very few, but one may observe secondary particles with a detector covering a large area. Since an ultra-high energy particle produces an enormous number of particles in successive interactions and these particles are scattered over a large area, a detector to observe them is able to catch primary particles falling on this area. A bundle of such particles is called the extensive air shower (EAS), because it is a shower developed in the atmosphere and the lateral distribution of particles contained therein extends as far as 1 km. Interactions at energies higher than  $10^{15}$  eV can now be studied only through EAS. Some discussions are, therefore, added on the properties of EAS in connection with interactions at ultra-high energies.

One of the most characteristic features of very high energy interactions has been regarded as the multiple production of particles. As this has not been easily described in terms of conventional quantum field theory, various methods have been put forward on the basis of approximate treatments of quantum field theory and by analogy to hydrodynamics and thermodynamics. These theories are so instructive in their ways of thinking that they may be regarded as implying the future development of high energy theory. However, a number of critical opinions have been raised rather strongly against these theories. At the present stage, it seems to be important to keep the open minded attitude and to distinguish between those which are essential and those which are not.

A more sound way of attacking very high energy phenomena has been brought into fashion in recent years. This is to assume a model on the basis of essential points extracted from experiment. Once a particular model is adopted, there hold simple kinematical relations and through them a number of well-defined quantities inherent to the model are obtained by a careful analysis of experiment. Some of these quantities can be related to those which are found in an accelerator energy region and they may be put in a framework of the present high energy theory.

Our discussions in this lecture will proceed along the line described above. Two previous articles by ROZENTHAL and CERNAVSKY [2] and by KOBA and TAKAGI [4] published in 1954 and 1959 respectively will be repeatedly referred to without mentioning the authors. Unlike these previous reviews we purposely minimize references; only those which help the reader's understanding and some of the latest ones are collected.

#### 2. QUANTITIES OBSERVED BY EXPERIMENT

High energy events characterized by multiple production have been observed by means of counter hodoscopes, cloud chambers and photographic plates, and also indirectly through extensive air showers, particularly the photons, nucleons and muons of high energies therein. Among them, observations with photographic plates are most direct and fruitful and one usually uses the terminology inherent in photographic observations.

### 2.1. Classification of produced particles

In a photographic emulsion the multiple particles produced at a high energy interaction look like a shower emerging from a point at which the interaction is regarded to have taken place. The particles produced are divided into two groups according to their appearance; one consists of collimated thin tracks and the other of grey and black tracks which are less anisotropic. Particles belonging to the former group are called shower particles, which are of relativistic energies and which consist mainly of charged pions, whereas those belonging to the latter are called heavy particles which are of non-relativistic energies and which consist mostly of protons arising from the disintegration of a target nucleus. The numbers of particles of the respective groups are designated by ns and Nh. Normally one characterizes a jet shower by  $N_h + n_{s,x}$ , where x stands for p, n or  $\alpha$ indicating that the primary particle is charged, neutral or an  $\alpha$ -particle. For example, an event of  $2 + 16_p$  means that  $N_h = 2$ ,  $n_s = 16$  and the primary particle is charged, probably a proton. n, gives one a rough idea of the multiplicity while N<sub>h</sub> indicates the approximate number of nucleons taking part in the interaction. If N<sub>h</sub> is large, say larger than three, one regards the event as due to multiple collisions with nucleons in a target nucleus, so that features of the collision with a single nucleon are more or less masked. Consequently, one is more interested in events of small  $N_b$  which may be due to the collisions with hydrogen nuclei or to the peripheral collisions with heavy nuclei.

If the incident energy is as high as  $10^{13}$  eV or higher, even the head-on collision with a heavy nucleus often gives small N<sub>h</sub> and reveals essential features of the collision with a single nucleon. At such a high energy most of the shower particles are emitted within a narrow cone like a jet. In most cases the shower particles can be distinctly divided into two groups, one belonging to a very narrow cone and the other to a diffuse cone. They may be respectively regarded as particles emitted in forward and backward directions in the centre of mass system.

In addition to charged particles, one often observes cascade showers initiated by high energy photons which are regarded as the decay products of neutral pions. The observation of cascade showers is an advantage in identifying pions among secondary particles, whereas charged particles may contain kaons and baryons in addition to pions. These particles, heavier than pions, are collectively called X-particles. Neutral particles which do not immediately decay are X-particles. The neutral X-particles produced by a primary collision can be detected by observing secondary charged particles, though they are sometimes confused with those produced by uncorrelated neutral particles. Observing n<sub>s</sub> and the number of neutral pions,  $n(\pi^0)$ , and assuming the charge independence for the production of pions,  $n(\pi^0) = (\frac{1}{2}) n(\pi^{\pm})$ , one can derive the number of charged X-particles as

$$n(X^{\pm}) = n_{s} - n(\pi^{\pm}) = n_{s} - 2n(\pi^{0}), \qquad (2.1)$$

The relative abundances thus obtained are roughly

$$n(\pi^{\pm}): n(\pi^{0}): n(X^{\pm}): n(X^{0}) \simeq 1: 1/2: 1/4: 1/4,$$
 (2.2)

nearly independent of energy for the nucleon-nucleon collisions above 10 GeV. By experiments with accelerators, most of the X-particles are found to be kaons and so may be at higher energies. Since the positive-negative ratio of kaons is large at accelerator energies, few kaons are produced in pairs but most of them are associated with the production of hyperons. Indeed, the relative abundances of secondary particles produced by proton-proton collisions at about  $30 \,\, {\rm GeV}$  are found to be

$$n(\pi^+): n(K^+): n(K^-): n(\overline{p}) \simeq 1: 0.15: 0.05: 0.01.$$
 (2.3)

If this ratio were to hold also at high energies, we would expect a considerable number of hyperons among X-particles.

# 2.2. Determination of energy

Only in a few cases can the energy of an incident particle in cosmic rays be determined by its magnetic rigidity or by multiple scattering. In most cases the primary energy has to be deduced from measurable quantities, say, the angles of emission of secondary particles, with full use of kinematical relations. This approach is reliable only when the secondary particles are numerous and they are emitted symmetrically in forward and backward directions in the centre-of-mass system (CMS). In fact, the symmetric nature is verified in most cases and asymmetric events are usually ascribed to the incorrect assignment of CMS by regarding the pion-nucleon or the nucleon-nucleus collisions as the nucleon-nucleon ones, although the asymmetric production may be possible also by the nucleon-nucleon collision.

Let us consider a collision of two particles, A and B, with respective masses  $M_A$  and  $M_B$ . In the laboratory system (LS) A is an incident particle of total energy  $E_A$  and momentum  $\vec{P}_A$ , whereas B is a target particle at rest.  $E_A$  and  $\vec{P}_A$  form a 4-momentum  $\underline{P}_A = (E_A, \vec{P}_A)$  and its absolute square is expressed as

$$\underline{\mathbf{P}}_{\mathbf{A}} \cdot \underline{\mathbf{P}}_{\mathbf{A}} = \mathbf{E}_{\mathbf{A}}^2 - \mathbf{P}_{\mathbf{A}}^2 = \mathbf{M}_{\mathbf{A}}^2, \qquad (2.4)$$

where the light velocity c is put equal to unity as usual. Evaluating the absolute square of the sum of two 4-vectors,  $(\underline{P}_A + \underline{P}_B) \cdot (\underline{P}_A + \underline{P}_B)$ , which is invariant under the Lorentz transformation, one in CMS and the other in LS, we obtain

$$(\underline{\mathbf{P}}_{\mathbf{A}} + \underline{\mathbf{P}}_{\mathbf{B}}) \cdot (\underline{\mathbf{P}}_{\mathbf{A}} + \underline{\mathbf{P}}_{\mathbf{B}}) = (\underline{\mathbf{E}}_{\mathbf{A}}^{*} + \underline{\mathbf{E}}_{\mathbf{B}}^{*})^{2} = (\underline{\mathbf{E}}_{\mathbf{A}} + \underline{\mathbf{M}}_{\mathbf{B}})^{2} - \underline{\mathbf{P}}_{\mathbf{A}}^{2}, \qquad (2.5)$$

where the asterisk indicates a quantity in CMS.

Noting that the momentum of the total system vanishes and the energy thereof is  $E^* = E_A^* + E_B^*$ , the velocity,  $\beta_c$ , and the Lorentz factor,  $\gamma_c = (1 - \beta_c^2)^{-\frac{1}{2}}$ , for the Lorentz transformation between CMS and LS are given by:

$$\mathbf{P}_{\mathbf{A}} = \beta_{\mathbf{c}} \gamma_{\mathbf{c}} \mathbf{E}^{*}, \qquad \mathbf{E}_{\mathbf{A}} + \mathbf{M}_{\mathbf{B}} = \gamma_{\mathbf{c}} \mathbf{E}^{*}. \tag{2.6}$$

Thus we obtain '

$$\beta_{c} = P_{A} / (E_{A} + M_{B}), \quad \gamma_{c} = (E_{A} + M_{B}) (2 E_{A} M_{B} + M_{A}^{2} + M_{B}^{2})^{-\frac{1}{2}}.$$
 (2.7)

At extremely high energies,  $E_A \gg M_A$ ,  $M_B$ , these are reduced to

VERY HIGH-ENERGY PHENOMENA

$$\beta_c \simeq 1$$
,  $\gamma_c \simeq (E_A/2 M_B)^{\frac{1}{2}}$  (2.8)

Since the dependence on  $M_A$  becomes weak as energy increases, the mass of an incident particle does not much affect kinematical relations. On the other hand the target mass is so important that the possibility of the collision with a cluster of nucleons has to be examined cautiously. Moreover, the multiple collisions with nucleons in a target nucleus bring about complexity.

As a result of the collision many particles are produced, one of which is emitted at angle  $\theta$  with energy E and momentum P. E and P may be expressed as

$$\mathbf{E} = \gamma \mathbf{M}, \qquad \mathbf{P} = \beta' \gamma \mathbf{M}, \qquad (2.9)$$

where M indicates the mass of the emitted particle. According to the Lorentz transformation the energy and the emission angle in LS are given respectively by

$$\gamma = \gamma_c \gamma^* (1 + \beta_c \beta^* \cos \theta^*), \qquad (2.10)$$

$$\tan \theta = \frac{\sin \theta^*}{\gamma_c \left(\cos \theta^{*+}\beta_c / \beta^*\right)}.$$
 (2.11)

where the asterisk again indicates the corresponding quantities in CMS. The transverse momentum defined by  $\pm$ 

$$P_{T} = P \sin \theta = P^{*} \sin \theta^{*} \qquad (2.12)$$

is invariant under the Lorentz transformation and is, therefore, found to be a useful quantity for describing high energy collisions.

It must be noted from (2.11) that the angle of emission in LS is limited for  $\beta^* < \beta_c$  as

$$[(\gamma^2 - \gamma^{*2})/(\gamma^2 - 1)]^{1/2} \leq \cos \theta \leq 1 \text{ for } \beta^* < \beta_c, \qquad (2.13)$$

while any angle is allowed for  $\beta^* \ge \beta_c$ . If  $\beta^* < \beta_c$  holds for any secondary particle, therefore, all particles in CMS are emitted forward. This seems to be the case at incident energies above  $10^{11}$  eV. Then the energy of a particle takes double values for given  $\theta$ :

$$\gamma_{\pm} (\theta) = \{\gamma^{\pm} \pm \beta_{c} [\gamma^{\pm} 2 - \gamma_{c}^{2} (1 - \beta_{c} \cos^{2} \theta)]^{\frac{1}{2}} \} / \gamma_{c} (1 - \beta_{c}^{2} \cos^{2} \theta)$$
  
for  $\beta^{\pm} < \beta_{c}$ . (2.14)

This causes some ambiguity in analysing high energy jets.

The energy-angular distribution in LS,  $f(E, \theta)dE d\Omega$ , where  $d\Omega$  is the solid angle element, is related to that in CMS,  $f^{*}(E^{*}, \theta^{*})dE^{*} d\Omega^{*}$ , by taking account of an invariant relation

489

$$dP/E = P^2 dP d\Omega/dE = P dE d\Omega = P^* dE^* d\Omega^*$$
(2.15)

 $\mathbf{as}$ 

$$f(\mathbf{E}, \theta) = f^*(\mathbf{E}^*, \theta^*) d\mathbf{E}^* d\Omega^*/d\mathbf{E} d\Omega = f^*(\mathbf{E}^*, \theta^*) \mathbf{P}/\mathbf{P}^*.$$
(2.16)

Since  $f(\mathbf{E}, \theta)$  has two branches corresponding to the double values of  $\gamma_{\pm}(\theta)$ , it is considerably complicated to obtain explicit forms of the energy and the angular distributions in LS. In some special cases, however, simpler relations can be obtained.

(i) <u>F-plot</u>. If  $\beta^*$  is not too small so that  $1 + (\beta_c/\beta^*)\cos\theta^* \ge 0$ ,  $\theta$  is an increasing function of  $\theta^*$ . Therefore, one can find the integrated angular distribution function in LS uniquely corresponding to that in CMS:

$$\mathbf{F}(\boldsymbol{\theta}) = \int_{\mathbf{d}\Omega}^{\boldsymbol{\theta}} \int_{\mathbf{d}E} \mathbf{f}(\mathbf{E}, \boldsymbol{\theta}) = \int_{\mathbf{d}\Omega}^{\boldsymbol{\theta}^{*}} \int_{\mathbf{d}E} \mathbf{E}^{*} \mathbf{f}^{*}(\mathbf{E}^{*}, \boldsymbol{\theta}^{*}). \quad (2.17)$$

F ( $\theta$ ) is proportional to the number of particles emitted at angles smaller than  $\theta$  and also to that at angles smaller than  $\theta^*$  in CMS. If the angular distribution in CMS is isotropic, the last expression of (2.17) simply gives the solid angle which is proportional to  $1 - \cos \theta^* \propto \sin^2(\theta^*/2)$ . If F ( $\theta$ ) is normalized so that F ( $\pi$ ) = 1, we have

$$F(\theta) = \sin^2(\theta^*/2) \text{ or } F(\theta) / [1 - F(\theta)] = \tan^2(\theta^*/2).$$
 (2.18)

This is now compared with (2.11). This is rewritten as

$$\gamma_{c}^{2} \tan^{2} \theta^{2} = \tan^{2} \left(\theta^{*}/2\right) \{1/[1+(\beta_{c}/\beta^{*}-1)/2 \cos^{2}(\theta^{*}/2)]\}^{2}$$
$$\approx F(\theta)/[1-F(\theta)]. \qquad (2.19)$$

The last expression comes out if  $\beta^* \approx \beta_c$ . Plotting F/(1-F) against  $\tan \theta$  in the log-log scale, one obtains a straight line with slope two, but with a slight deviation due to the factor in the square bracket in (2.19).

$$2 \log \tan \theta = -2 \log \gamma_c + \log \frac{F(\theta)}{1-(\theta)} - 2 \log \left[1 + \frac{(\beta_c / \beta^* - 1)}{2 \cos^2(\theta^* / 2)}\right]$$
(2.20)

A deviation from the straight line indicates the presence of anisotropy. For example, if particles are concentrated in forward and backward directions,  $F(\theta)$  is constant within a certain interval of  $\theta$  and the curve thus plotted splits into two parts. This method of obtaining the angular distribution in CMS is called the F-plot and was first applied by DULLER and WALKER[3] in their analysis of high-energy interactions observed with a cloud chamber.

An example of the F-plot is shown in Fig.1. The jet shower of  $2+16_p$  consists of a narrow cone and eight shower particles of considerable emission angles. This is revealed in the F-plot as the separation into two branches, one representing forward and the other backward particles in CMS.

490



Fig. 1

F-plot of a 2 + 16p event, observed by P. L. Jain. Phys. Rev. 125 (1962) 679

(ii) <u>Median angle method</u>. At the median angle of the angular distribution in LS,  $\theta_1$ , we have  $F(\theta_1) = \frac{1}{2}$  and

$$\tan\theta_{\frac{1}{2}} = \beta^* / \beta_c \gamma_c \approx 1 / \gamma_c. \tag{2.21}$$

This might be regarded as a direct method for obtaining the incident energy, but in practice the choice of the median angle is not always free from ambiguity. It is rather difficult to determine the median angle from a conventional plot of the angular distribution. The median angle determined from the symmetry point of the F-plot seems to be more reliable.

(iii) Log-tan  $\theta$  plot. Since (2.19) holds for each outgoing particle designated with suffix i, points of log tan  $\theta_i$  may be plotted on a horizontal line. The distribution of these points shows the angular distribution in CMS through log tan  $\theta_i = -\log \gamma_c + \log \tan \theta_i^*/2 - \log [1 + (\beta_c/\beta_i^* - 1)/2 \cos^2(\theta^*/2)]$ . (2.22) If the last term is negligible compared with the others, the angular distribution symmetric in forward and backward directions in CMS results in the symmetric distribution. Hence the centre of mass of the distribution of log tan  $\theta_i$  gives the Lorentz factor by reference to the angles of emission of N particles observed as .

$$-\log \gamma_{c} = (1/N) \sum_{i=1}^{\infty} \log \tan \theta_{i} + \log C, \qquad (2.23)$$

since  $\Sigma$  log tan  $(\theta_i^*/2)$  vanishes owing to symmetry. An energy independent term log C represents effects of the other terms and C  $\approx$  1.4 is found experimentally. This method has been used extensively since its use by CASTAGNOLI et al. [5].

A measure of the angular distribution may be obtained from the dispersion of log tan  $\theta_i$  as (6)

$$\sigma = \langle (\log \tan \theta - \overline{\log \tan \theta})^2 \rangle^{\frac{1}{2}}$$
 (2.24)

For an isotropic distribution  $\sigma = 0.39$  and  $\sigma$  increases as the angular distribution becomes anisotropic. In order to obtain the shape of the angular distribution, one has to calculate higher order moments. The fourth moment is required for examining whether the distribution consists of two peaks or not.



Fig. 2

Log-tan plot of an event obtained by R. G. Glasser, D. M. Hashkin and M. Schein, Phys. Rev. <u>99</u> (1955) 1555, and analysed by K. Niu, Nuovo Cimento 10 (1958) 994

A typical example of the log-tan  $\theta$  plot is shown in Fig.2. In the jet shower of  $\gamma_c \approx 100$ , 15 secondary particles are clearly separated into two groups.

(iv) Energy determination based on the fire-ball method. Both the F-plot and the log-tan plot suggest the concentration of secondary particles in the forward and backward directions. This characteristic feature may be interpreted in such a way that two lumps are formed by a high-energy collision and they subsequently decay into many secondary particles with rather low energies in the respective co-ordinate system of the lumps. These lumps may be regarded as highly excited states or fire-balls [6].

If the fire-ball model is taken seriously, one may evaluate  $\gamma_c$  by going one step further. Since forward and backward particles are supposed to be well separated, the summation of (2.22) is now taken separately for each group:

$$-\log \gamma_{f} = (1/n_{f}) \sum_{i < f} \log \tan \theta_{i} - (1/n_{f}) \sum_{i < f} \log \tan (\theta_{i}^{+}/2),$$
  
$$-\log \gamma_{b} = (1/n_{b}) \sum_{i < b} \log \tan \theta_{j} - (1/n_{b}) \sum_{j < b} \log \tan (\theta_{j}^{+}/2),$$

where  $\theta^+$  is the angle of emission in the system of a fire-ball. If particles are emitted isotropically from the fire-ball, the last terms in the respective equations vanish.

From this one obtains

$$\gamma_{\rm c} = \sqrt{\gamma_{\rm f} \gamma_{\rm b}} \,. \tag{2.25}$$

The values of  $\gamma_c$  thus obtained are found to be in essential agreement with those by (2.23), as they should be.

(v) Energy dissipated by secondary particles. The methods described above are based on the angular distribution of charged particles produced, but nearly one third of secondary particles are neutral and they may carry away a significant part of the incident energy. A number of events have thus far been found, such that a single neutral particle possesses an energy far greater than others. In such cases, the primary energy estimated by reference to charged secondary particles may be in error, even in the order of magnitude. In fact, there have been those cases in which the energy of a secondary particle determined by its interaction is found to be greater than the primary energy estimated by one of the above methods.

The energy of a charged secondary particle may be obtained by a direct measurement of the properties of its track, say, the scattering of grains formed by the particle, if its energy is low and the track is long enough for the scattering measurement. In most cases, however, one is not so lucky as to obtain such tracks, and moreover, even if such tracks are obtained, the energies carried away by these particles ordinarily consist in a minor portion of the total energy available for all secondary particles.

Among secondary particles a nuclear active particle of long lifetime may produce a nuclear interaction. If it is charged, its energy is roughly estimated from its angle of emission, as will be described in Section 2.4, and this is compared with an energy estimate based on the secondary interaction. This comparison checks the reliability of the primary energy estimated by the above means. For neutral particles, the secondary interactions provide not only the sole method of obtaining their energies but also a means of knowing their existence.

Short-lived particles usually decay before their interactions. This is the case for neutral pions. A neutral pion decays into two photons which subsequently produce electron pairs and eventually develop into cascade showers. The number of electrons contained in a very high energy cascade shower is so large that the shower may be detected by visual inspection of photographic emulsions or sensitive X-ray films through which the shower has penetrated. The energy of a neutral pion is estimated by reference to the properties of the cascade shower and high energy electro-magnetic interactions; an observation of neutral pions has an advantage over that of charged particles because of the feasibility of their energy determination. The methods of energy determination will be described in the next sub-section.

2.3. Determination of energies of neutral pions and photons

The neutral pion decays into two photons with a probability of 99% and into a photon and a pair of electrons with a probability of about 1%. The angle between the photons is given by

$$\theta_{2\gamma} = m_{\pi} c^2 (E_1 E_2)^{\frac{1}{2}} = (m_{\pi} c^2 / E_{\pi}) (\sqrt{n} + 1 / \sqrt{n}), n = E_1 / E_2,$$
 (2.26)

where  $m_{\pi}$  and  $E_{\pi}$  are the mass and the energy of the neutral pion and  $E_1$  and  $E_2$  are the energies of respective photons. According to (2.26), the pion energy can be determined in principle by measuring either the energies of the respective photons or their ratio and the angle between the two photons. In practice, however, these measurements are not always easy.

(i) Emulsion chamber. Let us consider an example in which  $E_{\pi} = 10^{12} \text{ eV}$ and n = 1. Then  $\theta_{2\gamma} \approx 1.4 \times 10^{-4}$  results in the separation of only  $4\mu$  after traversing the mean conversion length, about 3 cm in ordinary nuclear emulsions. The separation of  $4\mu$  is slightly greater than the minimum detectable distance but is as large as the mean lateral scattering of the conversion electrons to be suffered in one cascade length. Since the mean lateral spread of electrons is also inversely proportional to energy, it is usually difficult to distinguish two cascade showers arising from the decay photons. Moreover, several charged particles fall in this region, so that the position of the pair creation is difficult to observe.

It is, therefore, advisable to use a light material such as graphite as a shower producer, through which decay photons penetrate without conversion, and to put a heavy converter such as lead underneath, so that the pair creation takes place after the photons are sufficiently separated from each other. The whole apparatus thus consists of carbon layers as the producers of nuclear collisions, lead plates for developing cascade showers and nuclear emulsions sandwiched between them. This apparatus is called the emulsion chamber and is extensively used for the study of jets [7].

Having obtained the separated cascade showers, one can determine the energies of photons initiating these showers by reference to the cascade theory [8]. On account of the theoretical accuracy of the cascade theory and the experimental feasibility of picking up electron tracks, it is recommended to count only those electron tracks which are located near the shower core, say within  $50\mu$ , and which are consequently of high energies. The number of such electron tracks against the thickness of matter gives a shower transition curve to be compared with the theoretical one for a given initial energy.  $E_1$  and  $E_2$  thus obtained are substituted in (2.26), giving the opening angle  $\theta_{2\gamma}$ . This is compared with the observed value of  $\theta_{2\gamma}$ , thus giving a check of the procedure of energy determination.

In applying the cascade theory one has to keep the following remark in mind. As the energy of an electron increases, the electric field of the electron becomes so strong, due to the relativistic effect, that it can interact with atoms lying nearby, while it undergoes the radiative transition. The modulation effect caused by the nearby atoms cuts down the transition probability in the same way as in the narrowing effect of the resonance. Consequently the mean distance for bremsstrahlung and pair creation increases with the energy of an electron or a photon, depending on the density of a medium and the energy distribution of secondary particles. This is called the LANDAU-POMERANCHUK effect [9] and has to be taken into account for energies higher than  $10^{13}$  eV in a dense medium. There are some examples which might be regarded as anomalous if the above effect were not taken into account. In such events one of the decay photons initiates a cascade shower after several cascade units and the development of the cascade is slow compared with the ordinary one. However, this is found to be in reasonable agreement with what is expected from the Landau-Pomeranchuk effect.

(ii) Observation of electron pairs. In some cases individual electron pairs can be observed. The energy of a photon initiating the pair may be obtained by the direct measurement of the energies of the pair electrons or by the opening angle at low energies, while such methods are not applicable at high energies. If the energy of a photon is as high as  $10^{12}$  eV, a pair of electrons are so close that the electric fields of the two electrons give rise to an interference effect that diminishes the ionization probabilities due to two independent electrons. This may be called the ČUDAKOV effect, as a quantitative analysis of this effect was first made by him [10]. The diminution in the grain density can be seen in the first several hundred microns of the tracks of a pair of electrons.

(iii) <u>Calorimeter</u>. An opposite extreme to the above is the calorimetric method [11], by which the total energy of electrons dissipated for ionization

is measured with an ionization chamber or a scintillator. The Čerenkov detector may also be regarded as a calorimeter, because the total track length of electrons measured therewith is proportional to the total energy dissipated.

It must be noted that in the calorimetric method one often observes the energy dissipated by all secondary particles produced by a high energy interaction rather than the energy of a cascade shower alone. Therefore, the calorimeter is better used for the purpose described in Section 2.2.(v).

(iv) <u>Electron number</u>. The number of electrons at a given age of cascade shower is a unique function of the primary energy. The age of a shower can be obtained, in principle, from the lateral distribution of electrons. Hence, the observation of electrons at one section of a shower would make it possible to deduce the primary energy. This method was applied for obtaining the primary energy of an EAS, but it could give only the order of magnitude because the EAS is a result of complicated nuclear and electromagnetic processes. Even in a pure electronic cascade the number of electrons and the lateral distribution are so sensitive to the energies of electrons to be observed that these quantities alone can hardly give a reliable value of the primary energy. Nevertheless, the number of electrons is a useful quantity to give a rough idea of the primary energy.

## 2.4. Transverse momentum

Among various properties of secondary particles the transverse momentum is one of the most important quantities, because it is invariant under the Lorentz transformation on the one hand and is nearly independent of primary and secondary energies on the other hand, as was first noticed by NISHIMURA [7].

Observing neutral pions produced by cosmic rays with the emulsion chamber described in Section 2.3. (i), the transverse momenta of neutral pions were found to be confined in a narrow range for primary energies of  $10^{12}$  to  $10^{14}$  eV and secondary energies of  $5 \times 10^{10}$  eV to  $3 \times 10^{12}$  eV. The distribution of transverse momenta, P<sub>T</sub> has a peak at about 400 MeV/c and does not extend beyond 800 MeV/c. In nearly the same energy region the transverse momenta of charged particles were observed by emulsion stacks and their distribution was in essential agreement with that of neutral pions, although several particles were found to have P<sub>T</sub> greater than 1 GeV/c [12]. The average values of the transverse momenta thus obtained are

$$\langle P_T \rangle_{\pi^0} = 325 \pm 20 \text{ MeV/c}, \quad \langle P_T \rangle_{\text{charge}} = 520 \pm 160 \text{ MeV/c}.$$
 (2.27)

The same feature was observed also at accelerator energies\*.Observing the proton-proton collisions at 23 GeV with a hydrogen bubble chamber, the transverse momentum distribution was found to be practically independent of the kind of the secondary particles and to be expressed as (13):

<sup>\*</sup> A slow increase of  $P_T$  with energy was reported at the International Conference on High Energy Physics, Geneva, 1962.
$$f(P_T) 2\pi P_T dP_T = \exp(-P_T / P_0) 2\pi P_T dP_T,$$
 (2.28)

with

$$\langle P_T \rangle = 2P_0 = 340 \text{ MeV/c.}$$
 (2.29)

The fact that the transverse momentum is small and is practically independent of other properties provides useful means of analysing very high energy phenomena. Firstly, an approximate value of the longitudinal momentum and consequently the absolute value of momentum of a secondary particle can be obtained from its angle of emission,  $\theta$ , as

$$P_{I} = P_{T} \cot \theta \approx P_{T} / \theta \approx P_{I}$$
(2.30)

in which the last two expressions hold for small  $\theta$ . Since P increases with primary energy, while  $P_T$  remains constant, the energy imparted to secondary particles tends to be concentrated into the longitudinal mode as the primary energy increases.

Secondly, the momentum distribution can be connected with the angular distribution in a simple way. Since the transverse momentum distribution is independent of primary energy and of longitudinal momentum, (2.28) can be factored out in the momentum - angle distribution as:

$$f(P_T) 2 \pi P_T dP_T \cdot g(P_L / P_F) dP_L$$
,

where  $P_E$  is a scaling factor which depends on primary energy E and  $g(P_L/P_E)$ , is the momentum distribution integrated over angles.

Taking into account (2.30) and the solid angle element

$$d\Omega = 2\pi \sin\theta \ d\theta \approx 2\pi P_T \ dP_T / P_L^2$$
,

we can express the momentum - angle distribution as .

$$f(P_{T}) 2 \pi P_{T} dP_{T} \cdot g\left(\frac{P_{L}}{P_{E}}\right) dP_{L} = f\left(\theta P_{E} \cdot \frac{P_{L}}{P_{E}}\right) g\left(\frac{P_{L}}{P_{E}}\right)^{2} P_{E}^{2} dP_{L} d\Omega.$$
(2.31)

This implies that the momentum distribution at one angle can predict that at other angles by reference to the functional form of f. The functional form of  $g(P_L/P_E)$  can be obtained by measuring the momenta of all secondary particles irrespective of their angles. Therefore, an observation either at a given angle or without regard to emission angles is sufficient for obtaining the momentum - angle distribution, provided that the distribution of transverse momenta is known.

Thirdly, the smallness of transverse momentum allows one to regard the whole collision process as one-dimensional. In the first approximation the transverse components of momenta taking part in the collision process are neglected. Then kinematical relations are considerably simplified, as will be shown in the following sub-section.

496

# 2.5. Momentum transfer

The absolute square of 4-momentum transfer is also an important invariant quantity. In the collision

$$A + B \longrightarrow A' + B' + (i)_f + (j)_b$$

where A' and B' are respectively regarded as the recoil particles of A and B, and (i)<sub>f</sub> and (j)<sub>b</sub> the particles produced in the forward and backward directions in CMS respectively, the 4-momentum transfer is defined by

$$\underline{\Delta}_{\mathbf{A}} = \underline{\mathbf{P}}_{\mathbf{A}} - \underline{\mathbf{P}}_{\mathbf{A}'} - \sum_{i \leq f} \underline{\mathbf{P}}_{i}, \quad \underline{\Delta}_{\mathbf{B}} = \underline{\mathbf{P}}_{\mathbf{B}} - \underline{\mathbf{P}}_{\mathbf{B}'} - \sum_{j \leq b} \underline{\mathbf{P}}_{j}. \quad (2.32)$$

The 4-momentum changes of the incident particles are written as

$$\underline{\Delta}'_{\mathbf{A}} = \underline{\mathbf{P}}_{\mathbf{A}} - \underline{\mathbf{P}}_{\mathbf{A}'}, \quad \underline{\Delta}'_{\mathbf{B}} = \underline{\mathbf{P}}_{\mathbf{B}} - \underline{\mathbf{P}}'_{\mathbf{B}'}. \tag{2.33}$$

By reference to LS the absolute square of  $\underline{\Delta}'_{\mathbf{B}}$  is obtained as

$$\underline{\Delta}'_{\mathbf{B}'} \cdot \underline{\Delta}'_{\mathbf{B}'} = -2M_{\mathbf{B}}(E_{\mathbf{B}'} - M_{\mathbf{B}}), \qquad (2.34)$$

which is proportional to the kinetic energy of the recoil particle B'. It is also convenient to introduce

$$\underline{\mathbf{P}}_{\mathbf{f}} = \sum_{\mathbf{i} < \mathbf{f}} \mathbf{P}_{\mathbf{i}} , \quad \underline{\mathbf{P}}_{\mathbf{b}} = \sum_{\mathbf{j} < \mathbf{b}} \underline{\mathbf{P}}_{\mathbf{j}} .$$
(2.35)

If the produced particles form two fire-balls, their masses are given by

$$\mathcal{M}_{f}^{2} = \underline{P}_{f} \cdot \underline{P}_{f}, \quad \mathcal{M}_{b}^{2} = \underline{P}_{b} \cdot \underline{P}_{b}.$$
 (2.36)

If this is not the case,  $\mathcal{M}_{f}$  and  $\mathcal{M}_{b}$  should be understood as the absolute values of  $\underline{P}_{f}$  and  $\underline{P}_{b}$  respectively.

The conservation of energy leads to  $\underline{\Delta}_A + \underline{\Delta}_B = 0$ . In the symmetric collision, we have in CMS  $\overline{\Delta}_A^* + \overline{\Delta}_B^* = 0$ . In this case, therefore, the time components of  $\underline{\Delta}_A$  and  $\underline{\Delta}_B$  vanish in CMS. This results in

$$\underline{\Delta}_{\mathbf{A}} \cdot \underline{\Delta}_{\mathbf{A}} = \underline{\Delta}_{\mathbf{B}} \cdot \underline{\Delta}_{\mathbf{B}} = -\underline{\Delta}_{\mathbf{A}} \cdot \underline{\Delta}_{\mathbf{B}} = -\overline{\Delta}_{\mathbf{A}}^{*} \cdot \overline{\Delta}_{\mathbf{A}}^{*} \equiv -\Delta^{2} \cdot \mathbf{A}^{2} \cdot$$

Utilizing (2.37), (2.34) and (2.36), we obtain from (2.32)

$$-\Delta^{2} = -2M_{B}(E'_{B} - M_{B}) + \mathcal{M}^{2}_{b} - 2\underline{\Delta'}_{B}, \underline{P}_{b}$$
$$= 2(E_{b} - M_{B})(E'_{B} - M_{B}) + \mathcal{M}^{2}_{b} - \overline{2P}_{B'}, \overline{P}_{b}. \qquad (2.38)$$

Since all quantities in the last expressions are those of relatively low energy particles emitted in a wide angular region, they may be measured rather easily, thus providing a means to evaluate the 4-momentum transfer. Incid ntally, the recoil nucleon of kinetic energy  $E'_B - M_B$  is usually identified with a gre, track emitted at a large angle, and the value of  $(-\Delta'_B \cdot \Delta'_B)^{1/2}$  is thus estimated by NIU [6] to be of the order of a few GeV/c. Another method of obtaining the value of  $\Delta^2$  is to refer to the square of  $\Delta_{\mathbf{B}} + \underline{\mathbf{P}}_{\mathbf{b}} = \underline{\Delta}'_{\mathbf{B}}$ :

$$-\Delta^2 + \mathcal{M}_b^2 - 2 \overline{\Delta}_B^* \cdot \overline{P}_b^* = \underline{\Delta}_{B'} \cdot \underline{\Delta}_{B'} = -2 M_B (E'_B - M_B).$$

On account of the smallness of the transverse momentum,  $\vec{\Delta_B}^*$ .  $\vec{P_b}^*$  can be approximated as  $\Delta P_b^*$ , where  $P_b^* = |\vec{P_b}|$ . Thus we obtain

$$\Delta \approx (\mathbf{P}_{b}^{*2} + \mathcal{M}_{b}^{2} + 2\mathbf{M}_{B}(\mathbf{E}_{B}' - \mathbf{M}_{B})^{1/2} - \mathbf{P}_{b}^{*}.$$
(2.39)

If  $P_b^{*2}$  is much greater than other terms in the right hand side,  $\Delta$  is further approximated as:

$$\Delta \approx \left[ \mathcal{M}_{\mathbf{b}}^{2} + 2\mathbf{M}_{\mathbf{B}}(\mathbf{E}_{\mathbf{B}}' - \mathbf{M}_{\mathbf{B}}) \right] / 2\mathbf{P}_{\mathbf{b}}^{*}.$$
(2.39')

This implies that  $\Delta$  is a rather small quantity. However, this relation is not too useful, because  $P_b^*$  therein is not directly measured.

The third method seems to be most directly connected with experiment. On account of the energy-momentum conservation,  $\Delta_A$  and  $\Delta_B$  defined in (2.32) are related as  $\Delta_A + \Delta_B = 0$ . Therefore, (2.37) may be expressed as

$$-\Delta^2 = -\underline{\Delta}_{\mathbf{A}} \cdot \underline{\Delta}_{\mathbf{B}} = \overline{\Delta}_{\mathbf{A}} \cdot \overline{\Delta}_{\mathbf{B}} = \Delta_{\mathbf{A}_{\mathbf{0}}} \Delta_{\mathbf{B}_{\mathbf{0}}} = \Delta_{\mathbf{0}}^2 - \overline{\Delta}^2,$$

where  $\Delta_{0} = \Delta_{A_{0}} = -\Delta_{B_{0}}$  is the time component of  $\Delta$  and  $\overline{\Delta} = \overline{\Delta}_{A} = -\overline{\Delta}_{B}$ .

Denoting the longitudinal and transverse components of  $\overline{\Delta}$  as  $\Delta_{L}$  and  $\Delta_{T}$  respectively, we can express the 4-momentum transfer as

$$\Delta^{2} = \Delta_{A_{0}} \Delta_{B_{0}} - \Delta_{AL} \Delta_{BL} - \Delta_{AT} \Delta_{BT}$$
$$= (\Delta_{A_{0}} - \Delta_{AL}) (\Delta_{B_{0}} + \Delta_{BL}) + \Delta_{T}^{2}$$
(2.37')

In the first term of the third expression, the energies and momenta of secondary particles appear in the following combinations:

$$\begin{split} \mathbf{E}_{i} - \mathbf{P}_{iL} &= \left(\mathbf{P}_{iL}^{2} + \mathbf{P}_{iT}^{2} + \mathbf{M}_{i}^{2}\right)^{1/2} - \mathbf{P}_{iL} \approx \mathbf{P}_{iT}^{2} + \mathbf{M}_{i}^{2}/2 \, \mathbf{P}_{iL} \\ &= \left(\mathbf{P}_{iT} / 2\right) \left(1 + \mathbf{M}_{i}^{2} / \mathbf{P}_{iT}^{2}\right) \tan \theta_{i} \,, \\ \mathbf{E}_{j} + \mathbf{P}_{jL} &= \left(\mathbf{P}_{jL}^{2} + \mathbf{P}_{jT}^{2} + \mathbf{M}_{j}^{2}\right)^{1/2} + \mathbf{P}_{jL} \approx 2 \, \mathbf{P}_{jL} \left(\mathbf{P}_{jT}^{2} + \mathbf{M}_{j}^{2} / 2 \, \mathbf{P}_{jI} \right) \\ &= 2 \mathbf{P}_{jT} \left[\cot \theta_{j} + \left(1 + \mathbf{M}_{j}^{2} / 2 \, \mathbf{P}_{jT}^{2}\right) \tan \theta_{j}\right] \end{split}$$

Here we have assumed that all secondary particles have such high energies that  $P_L >> P_T$ , M. On account of that  $P_T$  is independent of particle and energy, (2.37') is thus expressed as:

$$\Delta^2 = \mathbf{P}_{\mathbf{T}}^2 \left[ \sum_{\mathbf{i}} \left( 1 + \frac{\mathbf{M}_{\mathbf{f}}^2}{\mathbf{P}_{\mathbf{T}}^2} \right) \tan \theta_{\mathbf{i}} - \frac{\mathbf{M}_{\mathbf{A}}^2}{\mathbf{E}_{\mathbf{A}} \mathbf{P}_{\mathbf{T}}} \right] \left[ \sum_{\mathbf{j}} \left\{ \cot \theta_{\mathbf{j}} + \left( 1 + \frac{\mathbf{M}_{\mathbf{j}}^2}{2 \mathbf{P}_{\mathbf{T}}^2} \right) \tan \theta_{\mathbf{j}} \right\} - \frac{\mathbf{M}_{\mathbf{B}}}{2 \mathbf{P}_{\mathbf{T}}} \right] + \Delta_{\mathbf{T}}^2$$

In LS the angle of emission is so small that the term containing tan  $\theta_j$  can be neglected in comparison with cot  $\theta_j$ . Quantities concerning two colliding particles may also be negligible.  $\Delta_T^2$  can be approximated as

 $\Delta_T^2 \approx \, {\bf n}_i \, {\bf n}_j \, {\bf P}_T^2$  ,

where  $n_i$  and  $n_j$  represent the numbers of secondary particles in the respective groups. Finally we obtain an approximate expression of  $\Delta^2$  as

$$\Delta^2 \approx P_T^2 \left[ \left( \sum_i \left( 1 + M_i / P_T^2 \right) \tan \theta_i \right) \left( \sum_j \cot \theta_j \right) + n_i n_j \right]$$
(2.40)

The right hand side contains only those quantities which can be directly observed.

The expression (2.40) depends on how the two groups are divided. In the symmetric case the value of  $\Delta^2$  thus obtained should reach a minimum. This provides a method of separating the forward and backward groups, independent of the F-plot and of the log-tan plot [14]. The value of  $\Delta$  thus derived is found to distribute around 1 GeV/c.

In obtaining the value of  $\Delta$  from (2.40), one has to keep in mind that one usually observes either charged particles or neutral pions. Therefore, not all secondary particles are included in evaluating the right hand side of (2.40). Since a large contribution to  $\Sigma$  tan  $\theta_i \Sigma$  cot  $\theta_j$  comes from a particle of the smallest  $\theta_j$  and one of the largest  $\theta_i$ , the absence of such particles causes  $\Delta$  to be unterestimated.

# 2.6. Multiplicity and inelasticity

The above discussions indicate that properties of individual events can hardly be determined with sufficient reliability but may be obtained only statistically. Thus one can speak about a certain property of the high energy interaction in a rather wide energy range. Properties nearly independent of energy, such as the transverse momenta and the ratio of secondary particles, may be obtained with more reliability than those dependent on energy.

One of the most important energy dependent properties is the multiplicity of secondary particles, most of which are pions. Observations with photographic emulsions give us a multiplicity law:

$$n \simeq 1.8 E_{A}^{1/4},$$
 (2.41)

where the primary energy  $E_A$  is measured in GeV. As emphasized above, not only  $E_A$  means an average value but also n is obtained as an average multiplicity of observed values of n which are subject to considerable fluctuations, say  $\Delta n \simeq 0.5$  n. Although the  $\frac{1}{4}$ -power law (2.41) has been widely accepted because of the reputable theory of FERMI [15], it merely expresses a general trend valid for coarse-grained observations. More detailed analyses claim the  $\frac{1}{2}$ -power law below 100 GeV or the stepwise increase of multiplicity. Nevertheless, (2.41) is found at least to be a good empirical law, provided that  $E_A$  is taken as an average value of primary energies ranging over a factor of one hundred or so.

In determining the multiplicity, one usually refers either to the number of charged particles or to that of neutral pions. Knowing the relative abundances of these secondary particles, one can obtain the total number of secondary particles.

A rather slow increase of the multiplicity with energy seems to hold even at energies higher than  $10^{14}$  eV, at which the EAS is the main source of information. One of its consequences is the division of an incident energy into many particles, so that no single particle has a great energy comparable to the incident one. Otherwise a neutral pion of large energy would produce a huge cascade shower with its maximum at a low altitude. Since the altitude at which an EAS has the maximum size is rather high and is not sensitive to the primary energy, the energy of a single neutral pion increases only very slowly or may level off as the primary energy increases.

If this were to hold for any secondary particle, the energy of an EAS transported to sea level would be much smaller than that which we observe. Moreover, there is an indication that nuclear active particles of considerable energies are contained in an EAS even at sea level. This suggests that a substantial part of the incident energy is carried away by a nuclear active particle, possibly a nucleon having survived after a nuclear collision. This means that the nuclear collision at ultra-high energies is rather elastic, in spite of the fact that many particles are created by this collision. This tendency is observed at energies below  $10^{14}$  eV with photographic emulsions and is found to hold even at accelerator energies.

The quantity which represents the elastic nature is called the inelasticity, defined by the ratio of the energy imparted to all secondary particles but the direct descendants of the incident particles to the energy available for the production of the secondary particles.

$$K = \left(\sum_{f} E_{i}^{*} + \sum_{b} E_{j}^{*}\right) / (E_{A}^{*} + E_{B}^{*} - M_{A} - M_{B}).$$
(2.42)

In the symmetric collision we can write

$$K = (E_{A}^{*} - E_{A}^{*}) / (E_{A}^{*} - M_{A}) = (E_{B}^{*} - E_{B}^{*}) / (E_{B}^{*} - M_{B})$$
(2.42')

The value of K fluctuates from one event to another, say from 0.01 to 1, but in most cases lies between 0.1 and 0.8. The average value is found as about  $\frac{1}{3}$  nearly independent of energy, but claim is often raised against the energy independence.

#### 3. VARIOUS THEORETICAL ASPECTS

It has occasionally been thought that the multiple production may reveal novel features characteristic to a future theory which will be valid beyond the applicability of quantum field theory. In spite of much effort towards this direction, no indication has yet been found to prove the evidence definitely against the present theory, although no satisfactory explanation of observed phenomena can be given within the framework of the existing theory. Both those who are in favour of preserving the present theory and those who are not, seem to agree on a point of view that any future theory must have some correspondence to the present one, quantum field theory. As is well known, quantum field theory has classical correspondence in two respects; one is a wave picture and the other is a particle picture. A future theory might come out of the correspondence of either of these classical pictures without going through quantum field theory. Putting aside the problem of future theory, we describe here various theoretical aspects which seem to show up at least one aspect of the truth.

These aspects may be classified into three, being based on wave,quantum and particle pictures. In quantum theory wave and particle are mutually complementary concepts; in other words, the phase of a wave and the number of particles cannot be simultaneously determined. If the phase is well defined and does not change in the course of interactions, one can no longer speak of the number of particles. In such a case the particle picture is of little use and one describes the system of interest in terms of the wave picture. In the opposite case, the phase correlation is so strong that the system concerned may be described as the average over phase values. Consequently one may speak of particle. In quantum theory an interaction results in a finite value of phase change and many waves of different phase values turn into a particle due to the interference of these phases. Thus particles are created through an interaction.

These three pictures are closely related to each other and no theory insists on one picture alone. Both the wave and particle pictures can be regarded as extreme cases of the quantum picture; hence three pictures are differentiated only by the point of emphasis. Keeping this remark in mind, we classify various aspects into these three pictures if possible and intermediate ones if not possible.

## 3.1. Wave picture

The description of a system in terms of linear fields brings about a trivial result. Some sort of non-linearity is necessary for causing the change of an amplitude. The amplitude after a collision is obtained by solving the wave equation of non-linearity, and its Fourier transform  $f(\vec{p})$  gives one the number of particles with momentum  $\vec{p}$  in  $d\vec{p}$  as

$$n\left(\vec{p}\right)\vec{dp} = f\left(\vec{p}\right)^{2}\vec{dp}/E, \qquad (3.1)$$

where E is the total energy corresponding to p. Although the relation (2.1) is based on quantum theory, a procedure of getting  $f(\overline{p})$  is entirely classical. The quantum nature is implied in the interpretation that  $n(\overline{p})$  is a mean particle density around which an actual one fluctuates.

Since the non-linear equation is familiar in hydrodynamics, hydrodynamical terminology is often used in the multiple production. A large amplitude corresponding to many particles may be regarded as due to a shock wave which is associated with a finite value of amplitude in hydrodynamics. Mathematically, however, any non-linearity does not always give rise to the shock wave. The shock wave takes place for a quasi-linear

equation in which the non-linearity appears in a term of the highest derivative, such as the transport term in the Euler equation of motion, but not for a semi-linear equation in which the coefficient of the highest derivative term is a constant or a known function. The latter is the case in the non-linearity arising from the Fermi interaction of four fields. Nevertheless, the semilinearity may be responsible for the multiple production, because energy is distributed into many modes by semi-linear coupling. For example, a semi-linear equation  $\Box \varphi - \mu^2 \varphi - g \varphi^3 = 0$  results in the 1/p spectrum of produced particles.

Turbulence is another concept often used. The non-linearity of the transport term in hydrodynamics is known to give rise to turbulence and the Kolmogoroff spectrum of turbulence is taken as analogy to the energy spectrum of created particles. In hydrodynamics, however, an energy contained in a turbulent unit increases with wave length, in contrast to the inverse relation between energy and wave length in quantum mechanics. This shows that a hydrodynamical interpretation is no more than analogy and merely helps one's intuition.

A serious attempt at a non-linear wave theory has been made by HEISENBERG [16]. His theory starts with a Lagrangian:

$$L = \ell^{-4} \left[ 1 + \ell^4 \{ \{ \partial \phi / \partial x_{,,} \}^2 + \mu^2 \sigma^2 \}^{1/2} \right]$$
(3.2)

where  $\mu$  is the pion mass and  $\ell$ a constant with the dimension of length in units of  $c = \frac{1}{5} = 1$ . If  $\ell$  is small, (3.2.) is reduced to the conventional Lagrangian for the Klein-Gordon particle. Since two colliding particles at a high energy can be regarded as very thin discs in the centre-of-mass system, the whole process may be approximated by the one-dimensional expansion, starting from a thin disc resulting from the crash of two particles. Then the wave function  $\varphi$  is expressed as a function of  $s = t^2 - x^2$  and obeys the equation

$$4 d/ds (s \phi') + \mu^2 \phi = 8 \ell^4 s(\phi')^2 (\phi + \mu^2 \phi) / (1 + \ell^4 \mu^2 \phi^2)$$
(3.3)

The non-linear term causes the energy dissipation at high momenta, so that the spectrum is expressed as  $dp/p^2$  for large p. At low p, however, the dissipation is unappreciable and a wave behaves like free expansion. In the low energy region, where the spectrum is given approximate by dp/p, a substantial part of energy is contained below a critical energy  $p_c$ . Consequently the multiplicity is given for a primary energy  $E_A^*$  in CMS by

$$n \simeq (E_{A}^{*}/p_{c}) ln(p_{c}/\mu).$$
 (3.4)

Since  $p_c$  is not much larger than  $\mu$ , most of the particles are emitted with low energies and consequently with low transverse momenta.

The power spectrum derived above is analogous to that obtained for isotropic turbulence. The reason for obtaining the power spectrum is similar in both cases, the energy flowing from small p to large p in hydrodynamics, whereas from large p to small p in pion field; one should not regard the turbulent pion fluid as more than analogy.

The multiplicity (3.4) depends too strongly on the incident energy and the inelasticity can hardly be taken into account. Heisenberg tried to get rid of these defects with the aid of an <u>ad hoc</u> assumption not inherent in his wave field theory. If one regards the pion cloud around a nucleon as a disc, in which the pion field strength decreases as  $\exp(-\mu r)$  as the radial distance r increases. Hence the momentum exchanged for an impact parameter r may be reduced to  $p_A^* \exp(-\mu r)$ . Now  $\exp(-\mu r)$  may exhibit the inelasticity and  $E_A^*$  in (3.4) may have to be replaced by  $p_A^* \exp(-\mu r)$ . Although a critical argument is raised against this reasoning [2], this should be regarded as a mere argument of plausibility somehow like the hydrodynamical analogy.

The characteristic feature of Heisenberg's theory seems to lie in that the energy contained in high frequency components dissipates into low frequency components and then the waves thereof fly away. Until one comes to the last stage, the phase of a wave sustains its initial value but its amplitude changes due to the non-linearity. This may be interpreted in terms of quantum theory in the following way. Due to a sudden collision, most of the energy is exchanged between high frequency components and low frequency waves are knocked-out keeping their forms. A strong interaction between waves results in distributing an energy concentrated in a few components over many components, so that most of the components eventually possess low frequencies. This leads to the quantum picture described in what follows.

# 3.2. Quantum picture

A quantum mechanical theory of multiple production as described just above was worked out by MIYAZIMA and TOMONAGA [17] with their intermediate coupling theory. In this theory are introduced the inertia which supresses the excitation of high frequency components and the damping effect which is responsible for dissipating energy contained therein. Hence a nucleon excited by a collision is de-excited by emitting pions stepwise to lower levels, thus leading to the multiple production. This process is analogous to the radiation of very low energy photons by a charged particle deflected in a Coulomb field.

The radiative process results in the well-known infra-red divergence, if it is treated by the lowest order perturbation but gives a convergent result if all higher order terms are taken into account. The higher order effect can be calculated if the recoil of the radiating particle is neglected. The same procedure is applicable to the pion production but each term does not suffer from the infra-red divergence owing to the finite pion mass. Along this line LEWIS, OPPENHEIMER and WOUTHUYSEN [18] worked out a theory of the multiple production essentially based on perturbation theory. Putting the square of a matrix for one pion production as A, the probability of emitting n pions is given by

$$W_{n} \propto \int (A^{n}/n^{13}) \prod_{i} (\vec{dp}_{i}/E_{i}) \simeq (A\overline{p}^{2})^{n}/n^{1}, \qquad (3.5)$$

where  $\overline{p}$  is the maximum momentum of pions emitted. The probability reaches a maximum at :

$$\tilde{n} \cong (A_{\overline{p}})^{2/3} \propto (KE_{A}^{*})^{2/3} \propto E_{A}^{1/3}$$
(3.6)

in which the second relation is obtained on account of  $\overline{p} \propto E_A^*$ .

The quantum mechanical treatment above is based on the following assumptions: (i) The collision time is so short that the scattering of two colliding nucleons and the emission of pions can be separated. (ii) The momenta of both real and virtual pions are so small that the recoil effect of nucleons is negligible. These assumptions result in the angular distribution essentially dependent on nucleon-nucleon scattering and the multiplicity determined by the phase space volume. Since scattered nucleons carry away most of the initial angular momentum, the angular distribution of pions in the centreof-mass system is rather isotropic. In such a case the whole phase space is nearly equally occupied, thus leading to the  $E_A^{1/3}$  law given in (3.6). If, however, the peripheral collision is effective, the result is much different from the prediction of the original form of the Low theory but may be more or less similar to the bremsstrahlung by electromagnetic interactions.

Although we have called the above theory as quantum mechanical, this neglects a characteristic point of quantum theory, that is, the interference effect. If the interference is taken into account, each pion cannot be treated independently but the correlation of emission processes becomes important. In the limit of extremely strong correlation, pions created would behave as if they formed a cloud or a fire-ball and the number of particles could be defined by taking the average over phase.

#### 3.3. Particle picture

The particle picture is thus based on the strong interaction between created pions. This has led FERMI [15] to assume the thermodynamical equilibrium of pions in a volume  $\Omega$ . The total energy,  $2E_A^*$ , contained in  $\Omega$  is distributed over many modes according to the Planck law and consequently the energy density is given by the Stefan-Boltzmann law as

$$2 E_{A}^{*}/\Omega \propto T^{4}. \tag{3.7}$$

Here T corresponds to the temperature in units of the Boltzmann constant being equal to unity and may represent the average energy of a pion. Hence the most probable number of pions produced is

$$\tilde{\mathbf{n}} \propto \mathbf{E}_{\mathbf{A}}^{\mathbf{*}} / \mathbf{T} \propto \Omega \mathbf{T}^{\mathbf{3}} \propto \mathbf{E}_{\mathbf{A}}^{\mathbf{*3/4}} \Omega^{1/4}. \tag{3.8}$$

Taking  $\Omega$  as the Lorentz-contracted volume:

$$\Omega = (M/E_{A}^{*}) \Omega_{0} \tag{3.9}$$

where M is the nucleon mass and  $\Omega_0$  an energy independent volume, we obtain the multiplicity law

$$\bar{n} \propto E_A^{*1/2} \Omega_0^{1/4} \propto E_A^{1/4}$$
 (3.10)

the well-known  $\frac{1}{4}$  power law.

# VERY HIGH-ENERGY PHENOMENA

The thermodynamical theory may be compared with the quantum mechanical theory in the following way. The squared matrix element A/E in (3.5) is now proportional to  $\Omega/V$ , the probability of finding a particle in  $\Omega$ , where V is the normalization volume. In obtaining the most probable multiplicity, however, there arises a difference between these two. The difference corresponds to that between the Rayleigh-Jeans law and the Planck law of radiation.

In the application of the Planck law the assumptions are implicitly made that the density of pions is well defined and the pions are free. The latter assumption might be regarded as contradicting the condition of thermal equilibrium, according to which a sufficient amount of energy should be exchanged between particles. However, this is not always a contradiction, because there may be such an interaction between particles that gives a large collision cross-section but a small value of potential energy. The first assumption seems to be subject to serious criticism because the particle density of bosons is known to be not positive definite and the thickness of  $\Omega$  is smaller than the wave length of a particle with energy T. These points are related to an unsatisfactory consequence of this theory, which is a large value of the transverse momentum of about T in disagreement with the observed value. Moreover, there is no room to introduce the inelasticity in Fermi's theory.

The first of these two disadvantages may be avoided by assuming the local equilibrium alone. LANDAU [19] thus introduced macroscopic quantities as functions of space and time co-ordinates. The introduction of the velocity field, the energy density and the pressure allows one to construct a fluid dynamics of ordinary form but the conservation of the particle number, because the particle density, cannot be introduced macroscopically. The behaviour of the fluid may be described in terms of the Navier-Stokes equation, the continuity condition of energy-momentum flow and the equation of state which relates the local pressure p to the energy density  $\epsilon$  as

$$\mathbf{p} = \epsilon / 3. \tag{3.11}$$

The equation of state results from the black body radiation with vanishing chemical potential. On the same footing, therefore, the local temperature T and the entropy density s can be defined and they are related as:

$$\epsilon \propto T^4$$
,  $s \propto T^3$ . (3.12)

Now we can draw a picture of a whole collision process as follows. The collision of two nucleons forms a fluid of disc shape with thickness  $\Delta$ , in which the fluid is heated to a high temperature as in Fermi's theory due to a shock wave originated by a violent impact. Hence the initial values of the temperature and the energy density  $\epsilon_0$  are essentially the same in Fermi's case. Thereafter the fluid expands in the direction of impact and the energy density density decreases as

$$\epsilon (\mathbf{x}, \mathbf{t}) = \epsilon_0 \exp\left[-\frac{4}{3}(\eta + \tau - \sqrt{\tau \eta})\right].$$

where  $\tau = \ln(t/\Delta)$  and  $\eta = \ln[(t - x)/\Delta]$ . This indicates that the energy is

505

concentrated near the front and the maximum of entropy density also lies there, though the latter is less concentrated. Since the entropy is turned out to be proportional to the number of particles, this results in that a small number of particles are emitted with large energy. In addition to the onedimensional expansion, the lateral expansion takes place due to the pressure. However, the pressure given by (3.11) is rather weak, so that the transverse momentum is expected to be small.

In the course of the expansion the entropy seems to change very little, because the transport mean free path is so short in a high density fluid of strong interactions that the dissipation of energy is negligible. Therefore, the initial value of entropy is kept essentially constant, until the mean free path becomes as large as the dimension of the fluid expanded. When the fluid reaches this stage, fluid elements can fly away as fragments and each fragment can be regarded as a free particle. Thus one can define the particle number. Since this is proportional to the entropy, the multiplicity may be evaluated by reference to the initial value of entropy as

$$n \propto S = s \Omega \propto \epsilon_0^{3/4} \Omega = E_\lambda^{*3/4} \Omega^{1/4} \propto E_\lambda^{*1/2}.$$
(3.14)

This is identical with the multiplicity law given by (3.10).

Since the particle can be defined only in the final stage, its properties are determined by conditions therein. Since the longitudinal fluid velocity is much larger than the transverse one and the temperature is low enough to give a small thermal velocity, particles are emitted preferentially in forward and backward directions. The temperature can be estimated on account of the smallness of the transverse momentum as low as  $\mu$ , where  $\mu$ is the pion mass. The same value can also be inferred from the pion-kaon ratio, because the number of particles with mass M emitted at temperature T will be proportional to  $M^{3/2}exp(-M/T)$ . As has been seen above, Landau's theory seems to explain the essential features of the multiple production but not the inelasticity.

The most striking success of Landau's theory seems to lie in such a respect that it is able to account for the small transverse momentum and consequently the angular distribution peaked in forward and backward directions, though historically Nishimura's idea on the smallness of transverse momenta was motivated by Landau's theory. The peaked distribution in the hydrodynamical theory of Landau stems from the concentration of energy near the fronts of an expanding fluid. If this is idealized, the two front parts may be regarded as fire-balls, each of which looks like a particle of large mass.

#### 3.4. Fire-ball model

Prior to Landau's theory, a fire-ball model was proposed by TAKAGI [20] as a modification of Fermi's theory. The collision of two nucleons does not lead to the formation of one fire-ball, but leaves each nucleon excited to energy  $\mathcal{M}$ , which subsequently decays into pions. Then Fermi's theory is applied to each fire-ball and consequently the multiplicity and the average energy of emitted pions in the fire-ball system are given respectively by:

# VERY HIGH-ENERGY PHENOMENA

$$n \simeq 2 m^{3/4}, \quad T \simeq m^{1/4}$$
 (3.15)

in which the  $\mathfrak{M}^{3/4}$  dependence comes out of an energy independent volume  $\Omega$ . Even in a completely inelastic collision, therefore, this gives a multiplicity law n  $\propto E_A^{*3/4} \propto E_A^{3/8}$ , different from Fermi's, and small but weakly energy dependent transverse momenta.

Careful analyses of experiments suggested the indication for the twofire-ball model [6], in which two fire-balls fly out with considerable energy of translation. If only two fire-balls are produced with mass  $\mathcal{M}$  and energy  $\gamma_f^* \mathcal{M}$ , containing an incident nucleon in each of them, there holds

$$\mathbf{E}_{\mathbf{A}}^{*} = \gamma_{c} \mathbf{M} = \gamma_{f}^{*} \mathcal{M}. \tag{3.16}$$

If the momentum transfer is small and independent of the incident energy, a straight forward calculation gives:

$$\gamma_f^* \propto \gamma_c^{1/2}, \quad \mathcal{W} \propto \gamma_c^{1/2}.$$
 (3.17)

If the final temperature is independent of the fire-ball mass as in the case of Landau's theory, the  $\gamma_c^{1/2}$  multiplicity law is obtained and the transverse momenta are as small as T.

The elasticity of the collision can be taken into account by letting an incident nucleon fly away without being amalgamated in the fire-ball. In this case, too, the same assumptions as above give an essentially identical result. The inelasticity is given by

$$K = \gamma_{\epsilon}^{*} \mathscr{U} / (\gamma_{c} - 1) M, \qquad (3.18)$$

which is independent of the incident energy.

A more recent analysis of experiments has suggested that not only the number of fire-balls produced may be two, four or six but also  $\mathcal{M} \simeq 2M$  and  $\gamma_f^*$  has discrete values [21]. Then (3.16.) is modified as

$$\gamma_{c}M = \sum_{i} \gamma_{i}^{*} 2M + \text{recoil energy}$$
 (3.19)

with  $\gamma_1^* \approx 1.5$ ,  $\gamma_2^* \approx 7.5$  and  $\gamma_3^* \sim 25$  and so forth. If this is taken for granted, there exist threshold energies for the production of two, four and six fireballs at

$$\gamma_c(1) \approx 4$$
,  $\gamma_c(2) \approx 19$ ,  $\gamma_c(3) \sim 69$ . (3.20)

If the average energy of pions emitted from a fire-ball is about  $2\mu$ , the multiplicity jumps at respective thresholds to  $2(M/\mu)$ ,  $4(M/\mu)$ ,  $6(M/\mu)$  and so forth. If one takes the multiplicity averaged over a wide energy range, this would correspond to the  $\gamma_c^{1/2}$  law for  $\gamma_c > 5$ , while it may well be approximated by the  $\gamma_c$  law at low energies. Corresponding to the stepwise increase in the energy imparted to pions, the inelasticity decreases to the threshold energy at which new fire-balls begin to be created as:

507

$$K = \sum_{i} 2 M \gamma_{i}^{*} / (\gamma_{c} - 1) M. \qquad (3.21)$$

This gives a multiplicity law expressed as

$$n \simeq 2K(M/\mu) (\sum_{i} \gamma_{i}^{*}) (\gamma_{c} - 1),$$
 (3.22)

which is consistent with the one observed by KANEKO and OKAZAKI [22]. In their analysis it has been shown that both n and K fluctuate considerably but n/K does not. If one plots n/K against the incident energy, therefore, one can find a curve connecting the plotted experimental points rather more easily than in the  $n - \gamma_c$  plot.

# 4. CONSEQUENCES OF FIRE-BALL MODELS

As briefly discussed in Section 3.4., the fire ball model can be described by simple kinematical relations, so that its consequences are readily predictable. By comparison with experimental results one can examine how good a particular model is. Depending on the experimental data on which they are based and also on the way of analysing the data, a number of authors have proposed a variety of-fire ball models. These may be classified into two groups; one may be called the two-fire-ball model proposed by three groups of authors in reference [6] independently, and the other the multi-fireball model by HASEGAWA [21]. Experimental evidences for or against these models will be discussed in more detail.

## 4.1. Two-fire-ball model

The two-fire-ball model is based on the essential equality of transverse and longitudinal momenta of secondary particles in appropriate co-ordinate systems which fly away forward and backward in CMS with nearly equal speed. It is quite natural to interpret this in such a way that pions produced by a collision form two lumps that fly away in opposite directions and subsequently decay into many pions. Such a lump may be regarded as a particle of short life-time. Then the discussions given in Section 2.5 are valid with  $\mathcal{M}_{\rm f}$  and  $\mathcal{M}_{\rm b}$  independent of the momenta of produced particles.

The angle of emission of a fire ball may be defined by

$$\log \tan \theta_{f} \equiv \frac{1}{n} \sum_{f} \log \tan \theta_{i}, \ \log \tan \frac{\theta_{f}^{*}}{2} = \frac{1}{n} \sum_{f} \log \tan \frac{\theta_{i}^{*}}{2}.$$
(4.1)

The discussions in Section 2.2, (iii) and (iv) yield

$$\gamma_{\rm f} = 1/\tan \theta_{\rm f}, \ \gamma_{\rm c} = \tan \frac{\theta_{\rm f}^*}{2}/\tan \theta_{\rm f} = 1/\tan \theta_{\rm i}^{}$$

The Lorentz factor in the centre-of-mass system is given, analogous to the derivation of (2.23), as

$$\gamma_{f}^{*} \approx \frac{1}{\tan \theta_{f}^{*}} \approx \frac{1 - \tan^{2}(\theta_{f}^{*}/2)}{2 \tan(\theta_{f}^{*}/2)} = \frac{1}{2} \left( \frac{\tan \theta_{1}}{\tan_{f} \theta} - \frac{\tan \theta_{f}}{\tan \theta_{1}} \right), \quad (4.2)$$

thus being expressed only by observable quantities.

508

Having obtained the fire-ball system, one can derive the longitudinal momenta of secondary particles in this system. In Fig.3 the distribution of the longitudinal momenta are compared with that of transverse momenta.



Fig. 3

Distribution of transverse and longitudinal momenta in the fire ball system, analysed by K. NIU [6]

These two momentum components are all small, so that low energy pions are emitted nearly isotropically from the fire-ball. This seems to suggest that the pion emission is analogous to the thermal radiation of a hot sphere. In fact, the energy spectrum of the pions in the fire-ball system is well represented by the Planck law with temperature of the order of  $\mu$ , although



Fig. 4



the energy spectrum appears to be a little flatter for the highest energy event indicated by X, as shown in Fig.4. Therefore, the mass of a fire ball can be given by

$$\mathfrak{M}_{f} = \frac{1}{2} \cdot n \langle E^{\dagger} \rangle \approx \frac{3}{4} n_{s} P_{T}$$
(4.3)

The mass of the fire-ball can be related to the momentum transfer by (2.39') as

$$\Delta \approx \left[\mathcal{M}_{f}^{2} + 2 \operatorname{M}_{B}(\operatorname{E}_{B}' - \operatorname{M}_{B})\right] / 2 \operatorname{P}_{f}^{*} \approx \left(\mathcal{M}_{f} / 2 \gamma_{f}^{*}\right) + \operatorname{M}_{B}(\operatorname{E}_{B}' - \operatorname{M}_{B}) / \gamma_{f}^{*} \mathcal{M}_{f} \qquad (4.4)$$

Since  $\Delta$  lies around 1 GeV/c nearly independent of energy, as briefly mentioned in Section 3.4, and the last term in (4.4) is relatively small at high energies, this would indicate  $\mathfrak{M}_{f}^{*} \propto \gamma_{f}^{*}$ , provided that  $\mathfrak{M}_{f}$  and  $\gamma_{f}^{*}$  are independent of  $\Delta$ . Actually, however, the independence seems to hold only for small  $\mathfrak{M}_{f}$  and  $\gamma_{f}^{*}$ , as can be seen in Fig. 5. At large  $\mathfrak{M}_{f}$  and  $\gamma_{f}^{*}$ , it may be seen that  $\mathfrak{M}_{f}^{*} \propto \Delta$  and  $\gamma_{f}^{*} \propto 1/\Delta$  respectively. It seems more likely to be  $\gamma_{f}^{*} \mathfrak{M}_{f}^{*} \sim \text{constant}$ , which means the independence of the fire-ball energy in



Dependences of the fire ball mass and its energy on the momentum transfer

CMS upon  $\Delta$  and  $\gamma_c$ . If this is the case, the inelasticity will have to decrease with increasing energy. Such a trend could hardly be incorporated with various observed facts in cosmic rays. Since the two-fire-ball model is based mainly on experimental data for primary energies between 10<sup>11</sup> eV and 10<sup>13</sup> eV, discussions on the energy dependence will need more data covering a wider range of energies.

## 4.2. Multi-fire-ball model

The multi-fire-ball model has been proposed by HASEGAWA [21] on the basis of more than one hundred jets. Although they include many events of large  $N_h$ , the forward and backward groups can be assigned in most of the cases except seven. Some typical examples of the log-tan plots are shown in Fig.6. By inspection of the log-tan plots one can assign not only two groups



Examples of log-tan plots. Fire balls are identified by horizontal bars and their Lorentz factors are shown unterneath :

- (a) J. Duthi, C. Fisher, P. H. Fowler, A. Kaddours, D. H. Perkins and K. Pinkau, Proc. Moscow Conf. on Cosmic Rays I (1959) 35
- (b), (c) A. G. Barkow, B. Chamney, D. H. Hashkin, P. L. Jain, E. Lohrman, M. W. Tencher and M. Schein, Phys. Rev. 122 (1961) 617

but also four groups. In a few cases six groups may be seen, but they are either produced by an  $\alpha$  particle or associated with a considerable number of heavy tracks, so that they could be due to plural collisions in nuclei.

In order to see the reality of four groups, the F-plots are made for respective groups, as shown by an example in Fig. 7. From the discussions given in Section 2.2 (iv), there holds a relation for the k-th fire-ball:







where  $\theta_i^k$  is the emission angle of the i-th particle in the rest system of the k-th fire-ball. Since (2.18) holds for tan ( $\theta_i^K/2$ ), the F-plot is possible for those particles which are supposed to belong to the k-th fire-ball. Fig. 7 indicates that the F-plots of four groups are well in agreement with each other. Even if the F-plots are made for all possible fire-balls produced in different events, plotted points are concentrated along a curve, as shown in Fig.8. The F-plot shown here is much steeper than that expected for the isotropic distribution and may indicate a  $\sin^2 \theta^+$  distribution in the fire-ball system.



Fig.8

F-plots for fire balls produced in different events. Curves indicate the theoretical distributions,  $\sin^2(\theta^K)d(\cos\theta^K)$ , for  $\beta_C/\beta_f^+ = 1$ . 1 and 1. 3. Dotted line shows the isotropic distribution for  $\beta_C/\beta_f = 1$ .

However, some caution is necessary, because the velocity of particles emitted in the fire-ball system is not high enough for neglecting the square bracket factor in (2.19). Indeed, a slight change of  $\beta_c/\beta^+$  gives rise to a considerable shift of the expected curve for large  $\theta$ , but not for small  $\theta$ . Since most of plotted points lie lower than the isotropic line at small  $\theta$ , one may be allowed to conclude that the probability for particles emitted along the direction of flight of a fire-ball is relatively low. It may be worth while remarking that the anisotropic distribution indicates a non-vanishing spin of the fire-ball.

#### VERY HIGH-ENERGY PHENOMENA



Distribution of  $\gamma_c \tan \theta_i$  and  $\gamma_K \tan \theta_K$ , based on 18 events in each of which four fire-balls are identified. Smooth curves are the distributions excepted for the  $\sin^2 \theta_f^{\dagger}$  law

The existence of four fire-balls is illustrated in Fig.9 by the log  $\gamma_c \tan \theta$  plot for all secondary particles observed in 18 such events. In the Figure particles belonging to the backward fire-balls are superposed by turning over their log  $\gamma_c \tan \theta$  distribution with respect to the median point. The distribution is found to consist of two peaks reasonably separated from one another. In the same Figure the log  $\gamma_K \tan \theta_K$  distribution of fire balls is shown in the same manner. This consists of two evidently separated parts. From their respective parts the mean values of  $\gamma_K$  can be obtained.

The log  $\gamma_c$  tan  $\theta$  distributions of particles emitted from such representative fire-balls are shown by two curves based on the sin  $2\theta^+$  law. Again one can see the fair agreement between expected and observed distributions. Since the distribution based on the sin  $2\theta^+$  law is narrower than that on the isotropic law, the separation of two peaks would not be so distinct on the latter assumption.

If the multi-fire-ball model is taken for granted, there follow a number of interesting consequences.

(i) The mass of a fire-ball. If fire-balls are assigned as above, the number of tracks emitted from a fire-ball can be measured. The distribution of track numbers is limited in a rather narrow range and is strongly peaked at about four, as shown in Fig.10. On the basis of 377 fire-balls observed, the average number of tracks and the standard deviation are shown by

$$n_f (charged) = 4.1 \pm 1.2.$$
 (4.6)

The distribution is found to be independent of primary energy, the direction



Fig. 10



of fire-balls and the velocity thereof. Including neutrical particles, we may regard the average number of particles emitted from a fire-ball,

$$\langle n_f \rangle \approx 6,$$
 (4.6')

as a characteristic feature of the fire-ball.

This leads us to the mass value of a fire-ball.

$$\langle \mathcal{M}_{\rm f} \rangle \approx 2 \, \mathrm{M}_{\rm n}$$
 (4.7)

where  $M_n$  is the nucleon mass, on account of the average pion energy of about 2  $M_\pi$  in the fire-ball system. This suggests that the fire-ball in the multi-fire-ball model is more like a particle than a strongly coupled system in the two-fire-ball model. This may be considered as an excited state which behaves like a nucleon-antinucleon system or an excited state of a boson.

(ii) The velocities of fire-balls. The Lorentz factor of a fire-ball given in (4.2.) can be reduced, on account of:

$$\tan(\theta_{\rm f}^{*}/2) = \tan[(\pi + \theta_{\rm b}^{*})/2] = -\cot(\theta_{\rm b}^{*}/2),$$
  
$$\gamma_{\rm f}^{*} = (1/2\gamma_{\rm c}) \left[\cot(\theta_{\rm f}^{*}/2) + \cot(\theta_{\rm b}^{*}/2)\right] = 1/2\gamma_{\rm c} (\gamma_{\rm f} + \gamma_{\rm b})$$
(4.8)

to

for a pair of symmetric fire-balls. If  $\gamma_f / \gamma_c$  is obtained in CMS, therefore,  $\gamma_f^* = \gamma_b^*$  can be calculated from (4.8) with  $\gamma_f \gamma_b = \gamma_c^2$ .



Correlation between the Lorents factors of fire-balls and primary particles

In Fig.11 the correlation between  $\gamma_f / \gamma_c$  and  $\gamma_c$  is shown for forward fire-balls. It may be seen that the distribution of points consists of three groups:

| Group I :   | $\gamma_{\rm f}/\gamma_{\rm c} \approx 2 \sim 3$ , | $\gamma_{\rm c}\gtrsim~5$ ,   |       |
|-------------|--|-------------------------------|-------|
| Group II :  | $\gamma_f / \gamma_c \approx 10 \simeq 20$ ,       | $\gamma_{\rm c}~\gtrsim~20$ , | (4.9) |
| Group III : | $\gamma_{\rm f}/\gamma_{\rm c}\gtrsim30$ ,         | $\gamma_{\rm c}~\gtrsim~70$ , |       |

Events with two fire-balls are all found to belong to Group I, while those with four fire-balls are such cases in which slow fire-balls belong to Group I and fast ones to Group II. Group III consists of those events which seem to have six fire-balls; in such an event three pairs of fire-balls belong to three respective groups.

The separation into these groups can be seen from Fig.12, in which the distribution of  $\gamma_f / \gamma_c$  is shown. Two distinct peaks are found to correspond to Groups I and II, but the existence of Group III is as obscure as the existence of events of six fire-balls. From this the average Lorentz factors of fire-balls in respective groups are obtained as

$$\langle \gamma_{\rm f}^* \rangle \approx 1.5, 8, 45,$$
 (4.10)

the last one being not definite yet.



Fig. 12 Distribution of the Lorentz factors of-fire balls

The threshold energies for producing two and four fire-balls with energies are 2  $M_n \sum \gamma_f^*$ , being given in (4.10.), are obtained as

 $\gamma_c$  (threshold)  $\approx$  4, 20,...., (4.11)

(4.10) and (4.11) are in good agreement with the observed  $\gamma_f - \gamma_c$  relation shown in Fig.11 and (4.9).

The above analysis shows that the velocity of a fire-ball has discrete values and the transition from two to four fire-balls takes place at a threshold energy, at which the average multiplicity also jumps from 12 to 24.

(iii) <u>Inelasticity</u>. A direct consequence of (i) and (ii) is the energy dependence of inelasticity. The definition of inelasticity given in (2.42\*) is now written as

$$K = \left(\sum_{K} \gamma_{K}^{*} \cdot 2 M_{n}\right) / 2 (\gamma_{c} - 1) M_{n} = \left[2 \sum_{K c t} \gamma_{K}^{*}\right] / (\gamma_{c} - 1),$$
 (4.12)

where the summation in the last expression runs only over forward fireballs and is restricted by (4.11). Since  $\gamma_K^*$  is constant, the value of K decreases with increasing primary energy until it reaches a threshold of producing more fire-balls. This trend can be seen from Fig.13.



#### Fig. 13

#### Energy dependence of inelasticity

The energy dependence of K results in highly elastic collisions slightly below the threshold and in highly inelastic ones slightly above it. Therefore, the accumulation of nucleons with several hundred GeV and the deficiency of them at about  $10^{11}$  eV are expected in cosmic rays in the low atmosphere, provided that the cross-sections are independent of energy, but no such evidence has yet been observed.

(iv) <u>Momentum transfer</u>. Since we know that both  $\mathcal{M}_{f}$  and  $\gamma_{f}^{*}$  are constant, the mass of a fire-ball given in (2.39.) is related to momentum transfer as

$$\mathcal{M}_{f}^{2} = \Delta^{2} + 2\Delta P_{f}^{*} - \Delta \tag{4.13}$$

in the case of two fire-balls, again neglecting the transverse momentum of a fire ball.

If four fire-balls are produced, we can introduce another momentum transfer equal to the difference of 4-momenta of two fire-balls produced in the same direction:

$$\Delta_{f} \equiv \underline{P}_{1} - \underline{P}_{2}, \qquad (4.14)$$

In defining  $\Delta_f$  as in (4.14) we may assume that the fire-balls are formed by pion-pion interactions, as shown in Fig.14.

Then there holds:

$$\underline{\Delta}_f = \underline{\Delta} + \underline{P}_1 \,. \tag{4.15}$$

By replacing  $\Delta$  by  $\Delta_f$ , we get the same result as in (4.13):

$$\mathfrak{M}_{1}^{2} = \Delta^{2} + 2 \Delta \mathbf{P}_{1}^{*} - \Delta_{f}^{2},$$
 (4.16)

where  $\Delta_f^2 = -\Delta_f \cdot \Delta_f$ .



Diagrams for the 4-fire-ball model

Concerning the second vertex from which  ${\rm P}_{\!_2}$  comes out, we have

$$\mathcal{M}_{2}^{2} = \underline{\Delta}^{\prime} \cdot \underline{\Delta}^{\prime} - 2 \mathcal{P}_{2} \cdot \underline{\lambda}_{f} - \underline{\lambda}_{f} \cdot \underline{\lambda}_{f}$$
$$\Delta_{f}^{2} - 2 \left( \mathbf{E}_{1}^{*} \mathbf{E}_{2}^{*} - \overrightarrow{\mathbf{P}_{1}^{*}} \cdot \mathbf{P}_{2}^{*} \right) + 2 \overrightarrow{\Delta}^{*} \overrightarrow{\mathbf{P}_{2}^{*}} - \underline{\Delta}^{2} \qquad (4.17)$$

Eliminating  $\Delta_f^2$  on account of (4.16), we obtain

=

$$\mathfrak{M}_{1}^{2} + \mathfrak{M}_{2}^{2} = -2 \gamma_{1}^{*} \gamma_{2}^{*} (1 - \beta_{1}^{*} \beta_{2}^{*}) \mathfrak{M}_{1} \mathfrak{M}_{2}^{*} + \Delta^{2} - \Delta^{2}$$

$$+ 2 \Delta (\beta_{2}^{*} \gamma_{2}^{*} \mathfrak{M}_{2}^{*} + \beta_{1}^{*} \gamma_{1}^{*} \mathfrak{M}_{1}).$$

$$(4.17')$$

 ${\Delta'}^2$  in (4.13) and (4.17) can be expressed as a function of inelasticity K, defined in (2.42'). In the approximation of high incident energy and small transverse momenta:

$$\Delta^{2} = (\vec{P}_{A}^{*} - \vec{P}_{A'}^{*})^{2} - (E_{A}^{*} - E_{A'}^{*})^{2} \approx (E_{A}^{*} - E_{A'}^{*})^{2} M_{A}^{2} / E_{A}^{*} E_{A'}^{*}$$
$$= [K^{2} / (1 - K)] [(1 - 1/\gamma_{c})^{2} / (1 + K/\gamma_{c}) M_{A}^{2}$$
(4.18)

This indicates that the value of  $\Delta'$  is of the order of  $M_A$ , unless the collision is extremely inelastic. For small K, the momentum transfer  $\Delta'$  is essentially proportional to inelasticity. If K decreases with increasing energy, as was indicated by (4.12),  $\Delta'$  also decreases nearly inversely proportional to  $\gamma_c$ . If the cross-section decreases with increasing  $\Delta'$ , the cross-section times inelasticity is a quantity slowly varying with energy. If this is the case, the energy spectrum of nuclear active particles behaves smoothly, as is currently believed.

Comparing (4.13) and (4.16), one may conclude that  $\Delta_{\rm f}$  is of the same order of magnitude as  $\Delta'_{\rm f}$  because the mass and the velocity of a fire-ball is the same for the production of two fire balls and for that of four fire-balls. Neglecting  $\Delta_{\rm f}^2 - \Delta'^2$  in (4.17), we can easily see that properties (i) and (ii) are explained from (4.16) and (4.17) for  $\Delta \sim M_{\rm n}$ . In fact, the distribution of  $\Delta$  obtained by means of (2.40) indicates this feature, as shown in Fig.15.

#### VERY HIGH-ENERGY PHENOMENA



Fig. 15

Distribution of 4-momentum transfer. The value of  $\Delta$  is obtained from the minimum value of (2.41)

The above considerations allow us to discuss the characteristic features of the fire ball model further in detail. From (4.13), (4.16) and (4.17) we see that the production of fire-balls is possibly only for  $\Delta$  greater than a certain critical value. The incident momentum could be shared by an outgoing nucleon, a fire-ball and the transferred momentum. In the two-fireball model the first one receives a constant fraction and the momentum of the second one increases as  $\gamma_1^{1/2}$ . In the multi-fire-ball model the momentum of the outgoing nucleon increases with incident energy because of the decrease of inelasticity, whereas the other two are kept constant. However, this trend cannot continue to higher energy in the multi-fire ball model. At a certain value of the incident energy, new fire-balls are created. Since they have large momenta, the value of  $\triangle$  does not appreciably change but still lies near M<sub>n</sub>. Summarizing the above, we may express the most remarkable feature of the multi-fire-ball model in such a way that the squared 4-momentum of any internal line is of the order of  $-M_n^2$ , while that of any external line is of the order of  $+M_n^2$ .

Whether the high energy pion is produced through an excited baryon or indicates something new can be judged by observing its transverse momentum. One experiment [23] suggests a low value consistent with (2.27), whereas the other [24] claims a higher value. It is too early to draw any conclusion, but this is an exceedingly interesting problem for the future investigation.

# 5. REMARKS ON COLLISIONS ABOVE 10<sup>14</sup> eV

Although little direct evidence has been available for nuclear interactions above  $10^{14}$  eV, there seems to be some indication that some new features appear in this extremely high energy region. We have seen from Fig.4 that the energy spectrum of produced particles becomes flatter as the energy of an incident particle increases. In a few  $10^{15}$  eV events observed with a large emulsion chamber, it has been found that a single neutral pion has a much greater energy than other secondary particles [23]. Similar events have also been observed with an air shower detector [24].

Such high energy particles cannot be the members produced from a fire-ball, because of their very high longitudinal momenta. They may belong to a new fire-ball of much higher speed than others, as this motivated the

multi-fire-ball model of Hasegawa. It is however rather plausible to assume that they are the decay products of excited baryons [23]; an incident nucleon is left after the collision in a state not only of high translational energy but also of some excited energy, as has been seen from the resonances recently verified by experiments with accelerators. The importance of the hyperons and the excited baryons with high translational energies has been suggested by PETERS [25] independently of the above direct evidence but on the basis of other evidence, such as the hyperon production rate and the resonances in the accelerator energy region as well as the high energy part of the spectrum in cosmic rays.

#### REFERENCES

- POWELL, C. F., FOWLER, P. H. and PERKINS, D. H., "The study of elementary particles by the photographic method", Pergamon Press (1959).
- [2] ROZENTAL, I.L. and CERNAVSKY, D.S., Usp. Fiz. Nauk. 52 (1954) 185, Fortschr. Phys. 4 (1956) 560.
- [3] DULLER, N. M. and WALKER, W. D., Phys. Rev. <u>93</u> (1954) 215.
- [4] KOBA, Z. and TAKAGI, S., Fortschr. Phys. 7 (1959) 1.
- [5] CASTAGNOLI, C., CORTINI, G., FRANZINETTI, C., MANFREDINI, A. and MORENO, D., Nuovo, Cimento <u>10</u> (1953) 1539.
- [6] CIOK, P., COGHEN, T., GERULA, J., HOLYNSKI, R., JURAH, A., MIESOWICZ, M., SAINEWSKA, T. and STANISZ, O., Nuvo Cimento <u>8</u> (1958) 166, <u>10</u> (1958) 741; COCCONI, G., Phys. Rev. <u>111</u> (1958) 1699; NIU, K., Nuvo Cimento <u>10</u> (1958) 994.
- [7] MINAKAWA, O., NISHIMURA, Y., TSUZUKI, M., YAMANOUCHI, H., AIZU, H., HASEGAWA, H., ISHU, Y., TOKUNAGA, S., FUJIMOTO, Y., HASEGAWA, S., NISHIMURA, J., NIU, K., NISHIKAWA, K., IMAEDA, K. and KAZUNO, M., Suppl. Nuvo Cimento <u>8</u> (1958) 761: <u>11</u> (1959) 125.
- [8] KAMATA, K. and NISHIMURA, J., Prog. Theor. Phys. Suppl. 6 (1958) 93.
- [9] LANDAU, L. D. and POMERANCHUK, I. Ya., Doklady Akad. Nauk. USSR, <u>92</u> (1953) 535, 735; MIGDAL, A.B., Phys. Rev. <u>103</u> (1956) 1811.
- [10] CUDAKOV, A.E., Izv. Akad. Nauk. USSR, 19 (1955) 651.
- [11] BRIKKER, S. L., GRIGOROV, N. L., KONGRATEVA, M. A., PODGURSKAYA, A. V., SABELEVA, A. L., SOBINYAKOV, V. A. and SHESTOPEROB, V. Ya., Suppl. Nuvo Cimento <u>8</u> (1958) 1733.
- [12] EDWARDS, B., LOSTY, J., PERKINS, D.H., PINKAU, K. and REYNOLDS, J., Phil. Mag. 3 (1958) 237.
- [13] MORRISON, D. R. O., Proc. Int. Conf. on Theoretical Aspects of Very High-Energy Phenomena, CERN, 61-22 (1961) 153.
- [14] HASEGAWA, S. and YOKOI, K., International Conference on High Energy Physics, CERN (1962).
- [15] FERMI, E., Prog. Theor. Phys. 5 (1950) 570.
- [16] HEISENBERG, W., Z. Phys. 133 (1952) 65.
- [17] MIYAZIMA, T. and TOMONAGA, S., Sci. Pap. I. P. C. R. <u>40</u> (1942) 21; reprinted in Prog. Theor. Phys. Suppl. No. 2 (1955) 21.
- [18] LEWIS, H. W., OPPENHEIMER, J. R. and WONTHUYSEN, S. A., Phys. Rev. 73 (1949) 127.
- [19] LANDAU, L. D., Izv. Akad. Nauk. USSR, 17 (1953) 57.
- [20] TAKAGI, S., Prog. Theor. Phys. 7 (1952) 123.
- [21] HASEGAWA, S., Prog. Theor. Phys. <u>26</u> (1961) 150; International Conference on High Energy Physics, CERN (1962).
- [22] KANEKO, S. and OKAZAKI, S., Nuvo Cimento 8 (1958) 521.
- [23] AKASHI, M. et al., International Conference on Cosmic Rays, Kyoto (1961); International Conference on High Energy Physics, CERN (1962).

\* · ·

- [24] MIURA, L, ODA, M. and TANAKA, Y., ibid.
- [25] PETERS, B., ibid.

# A MODEL FOR HIGH-ENERGY NUCLEON-NUCLEUS COLLISIONS

# J. WERLE INSTITUTE FOR NUCLEAR RESEARCH, WARSAW, POLAND

# 1. EXPERIMENTAL FACTS IN VERY HIGH ENERGY NUCLEON-NUCLEUS COLLISIONS

I would like to tell you briefly about some interesting experimental facts observed in very high energy nucleon-nucleus collisions. Then, I want to present you some single theoretical model aiming at a plausible explanation of the mentioned experimental data. By very high energy I mean the region extending from highest accelerator energies of ~10 GeV to the not too rare cosmic ray energy of, say, ~10<sup>5</sup> GeV. A typical high energy nucleon-nucleus collision can be distinguished from more elementary nucleon-nucleon or pion nucleon collisions by the presence of a number of black tracks of non-relativistic particles. What the emulsion people actually see is drawn schematically on Fig. 1.



#### Fig. 1

Typical high energy nucleon-nucleus collision

The thick black lines refer to non-relativistic particles, the dotted lines refer to relativistic particles. At high energies of the incident particle the relativistic secondaries are emitted in a rather narrow cone whereas the non-relativistic ones are distributed almost isotropically. (Throughout this paper, I am always speaking of the situation in the laboratory system). Major part of non-relativistic particles forming black tracks consists of nucleons whereas the relativistic particles are mainly mesons. However, the experimentalists have found also heavier fragments of nuclear matter among the particles responsible for black tracks. Some number of light nuclei d, t, He, Li, Be, etc. could be identified. There is no evidence for the existence of such heavy fragments among the relativistic particles. These striking differences between the structures of relativistic and the non-relativistic parts of secondaries emitted in the same collision imply completely different mechanisms for these two phenomena.

In fact one can explain these differences by assuming two distinct stages of the very high energy nucleon-nucleus collision. The first stage, which may be called "knock off", consists of a single or a few elementary collisions between the incident nucleon and those nucleons of the nucleus which just happen to stand in its way. Some secondary pionnucleon collisions may be also responsible for the structure of the relativistic cone. It seems that in many cases the properties of the narrow jets of relativistic particles emerging from high energy nucleon-nucleus collisions are very similar to those emitted in nucleon-nucleon collisions. I shall not speak of the properties of relativistic particles since this has already been done in some detail by Prof. Hayakawa. I will be mainly concerned with the struck nucleus or rather with what remained of it after knocking off the relativistic jets.

Here, the situation can depend very much on the energy of the incident nucleon, size of the struck nucleus, impact parameter and so on. E. g. at lower incident energies the scattering or emission angles can be large enough to start in the nucleus an internuclear cascade which can blow off a great portion of the nucleus. For higher energies it is rather justified to assume that the only result of the first "knock off" stage is, as far as the struck nucleus is concerned, a hole or a tunnel bored through the nucleus by the incident nucleon and the relativistic secondaries. The existence of such a tunnel, in which (Fig. 2) the smaller opening angle the greater is the incident energy can be justified by small binding energy of the nucleons in the nucleus in comparison to the very large energy of the incident nucleon



Fig. 2 Tunnel mechanism

and also by the very short duration of the "knock off" stage. The remaining recoil nucleus with a whole bored through it is, of course, highly excited. The excitation energy is quickly redistributed according to the laws of statistical thermodynamics (as it is believed). The excitation energy is usually high enough to justify the use of the simple statistical model of the nucleus by Fermi and to characterize this excited piece of nuclear matter by some temperature T. The second stage of the process which is supposed to be responsible for the omission of non-relativistic particles consists of the redistribution of energy till some kind of thermodynamic equilibrium characterized by the temperature T is reached. This stage ends with the evaporation of some number of nucleons or heavier fragments which carry off the excess of energy and allow the remaining nucleus to come to some metastable or even stable state. I should like to point out that there are substantial differences between the use of the statistical model for the excited nucleus and those applications of it to elementary nucleon-nucleon collision which were described by Prof. Hayakawa.

#### A MODEL FOR HIGH-ENERGY COLLISIONS

Using the simple statistical model of the nucleus whose state is characterized by a very small number of parameters like temperature and total mass we can make certain predictions. If this model is correct one should expect isotropic emission of evaporated particles in the rest system of the recoil nucleus irrespective of their mass and energy. We can also calculate the energy distribution of particles of given mass as well as the frequency of emitting particles with different masses provided that the value of the excitation temperature T is known and roughly the same for all cases under consideration. The result of such estimations done with some reasonable assumptions about the value of T shows that for example the frequency of emission should drop quickly with the increasing mass of the emitted particles.

How do these predictions compare with the experiment? If one considers the angular distribution of all black tracks without trying to distinguish their masses one finds roughly isotropic distribution in the rest system of the recoil nucleus. In the laboratory system, there is, of course, some tendency for the forward emission, which can be understood as a purely kinematic effect of the finite recoil momentum. In other words during the first stage of boring the tunnel through the nucleus some amout of linear momentum must be transferred to the nucleus. It seems however, that sometimes the experimental values of this recoil momentum which are necessary to obtain isotropic distribution of black tracks exceed significantly the theoretically expected limits. This is, however, not a very serious disagreement.

More interesting phenomena come into light if we start to differentiate the emitted non-relativistic particles recording to their mass and energy. It has been found by several laboratories that the number of heavier fragments definitely exceeds the values found from the simple statistical model of evaporation. There is also some definite excess of energy carried off by these heavier fragments when compared with the values expected on the basis of the statistical model by Fermi. The third interesting deviation from the statistical model is the angular distribution of the heavier fragments which indicates preferential emission into the forward hemisphere with a maximum at some angles  $\theta \neq 0$  but rather with  $20^{\circ} < \theta < 80^{\circ}$ .

We have, therefore to explain three deviations from the statistical model of the non-relativistic particles emission in high-energy nucleon-nucleus collisions:

- a) excess in number of heavier fragments,
- b) excess in energy of heavier fragments,
- c) preferential angle of emission  $\theta \neq 0$ .

# 2. THEORETICAL EXPLANATION

Before going over to the proposed theoretical explanation of these facts I should like to stress that the experimental situation is by no means transparent. In spite of the fact that the first discovery of these deviations was made by Perkins quite long ago the existing statistics of such events is still not sufficient to tell us what are the angular and energy distributions of nonrelativistic particles at different but fixed energies of the indicent nucleon. We also do not know very much about the dependence of these functions on the type of the struck nucleus, mass of the fragment etc. To have a more transparent picture one should have much more numerous statistics of events differentiated according to: 1) energy of the incident nucleon  $E_i$ , 2) type of the struck nucleus A, 3) mass of the emitted fragment  $M_f$ , 4) excitation energy  $\epsilon$  of the recoil nucleus, 5) energy of the emitted fragment  $E_f$ , 6) angle of emission  $\theta_f$  etc, etc. In the present experimental situation data referring to quite different values of the above experimental parameters are being put together which makes the theoretical analysis quite difficult and obscures the whole picture. In spite of the fact that we may wait a long time for more transparent experimental data we can extract already now some important results and try to make a theory which would be able to make some predictions.

Although the figures given by several laboratories are varying and cannot be simply compared for reasons stated above it seems that the very existence of the three deviations is beyond doubt and requires some theoretical explanation. One may first think that because of these deviations one must reject the simple statistical model completely. However, it seems that it works rather reasonably well as far as emission of nucleons is concerned. Therefore, it will be rather better to look for some additional mechanism which would be chiefly responsible for the emission of heavier fragments with the observed properties. In other words we shall assume that between the very quick "knock off" stage and the comparatively slow evaporation stage describing the emission of most nucleons we have to do with some process which immediately follows the "knock off" stage and ceases to work before the evaporation takes place. We may call this intermediate stage fragmentation. Thus we have to distinguish probably not two but three stages:

- 1) knock off of the relativistic particles,
- 2) fragmentation of the recoil nucleus,
- 3) redistribution of energy and evaporation of the recoil nucleus.

Just the second stage would be responsible for the creation of most heavier fragments with rather small contribution to the emission of nucleons described chiefly by the statistical evaporation model. To produce heavier fragments with observed excess energies and emitted at some preferential angles we need some long range forces which would be able to act coherently upon larger pieces of the nucleus without destroying the bounds between the nucleons in the emitted fragments.

The first thought is to make the internuclear cascade responsible for this phenomenon. However, the internuclear cascade of angles wide enough can evolve only at lower incident energies  $E_i$ . It would not work in the high energy region where there cannot be other cascade apart from the narrow jets of relativistic particles. But even at low energies one may rather expect that the internuclear cascade has the tendency of breaking everything into single nucleons than to act coherently upon larger pieces of nuclear matter. Calculations basing on the internuclear cascade as a possible mechanism of fragmentation show that we cannot fit the experimental data unless we assume the existence of some new long range forces.

Let us, therefore, start from another point. Let us look for a possible source of long range forces in the nucleon-nucleus scattering. (By "long range" I mean here a range exceeding several times the well known range  $\frac{1}{\mu}$  of nuclear forces). It seems to me that such a possibility exists due to the peculiar behaviour of the meson field accompanying a non-uniformly moving nucleon.

Let us consider the equation for the meson field  $\varphi$  created by some given source  $\rho(\vec{x}, t)$ 

$$(\Box - \mu^2) \oplus (\vec{\mathbf{x}}, t) = -4\pi \rho(\vec{\mathbf{x}}, t).$$
(1)

If the source function is time independent (static) the solution we are usually interested in can be obtained by means of the static Greens function

$$g_0(\vec{x}, \vec{x}^d) = |\vec{x} \cdot \vec{x}'|^{-1} \exp\{-\mu |\vec{x} \cdot \vec{x}'|^d\}.$$
 (2)

For the case of point source we obtain the well known Yukawa potential with well defined range of forces equal to  $\frac{1}{\mu}$ :

In the more general case of time dependent sources one can solve the equation (1) by means of similar Green's functions if one writes  $\rho$  and  $\phi$  in the form of Fourrier integrals

$$\rho(\vec{\mathbf{x}}, t) = \int_{-\infty}^{+\infty} \rho(\vec{\mathbf{x}}, \omega) e^{i\omega t} d\omega,$$
  
$$\varphi(\vec{\mathbf{x}}, t) = \int_{-\infty}^{+\infty} \varphi(\vec{\mathbf{x}}, \omega) e^{i\omega t} d\omega,$$

and after inserting into (1) compares the coefficients at the same frequencies. One obtains in this way

$$[\Delta - (\mu^2 - \omega^2)] \varphi (\vec{\mathbf{x}}, \omega) = -4\pi \rho(\vec{\mathbf{x}}, \omega). \qquad (1.a)$$

The spherically symmetric Green's function for this equation has the form similar to (2) but the exponential is changed

$$g_{\omega}(\vec{x},\vec{x}') = |\vec{x} \cdot \vec{x}'|^{-1} \exp\{-\sqrt{\mu^2 - \omega^2} |\vec{x} \cdot \vec{x}'|\}.$$
 (3)

One must of course, be careful with taking the proper sign of the square root in (3). We see that for  $\omega^2$  increasing from 0 to  $\mu^2$  the range of forces transmitted by the  $\omega$  component of the source is increasing from the minimum static value  $\frac{1}{\mu}$  to infinity. For  $\omega^2 > \mu^2$  we have to do with wave type solutions because in this region  $\sqrt{\mu^2 \omega^2}$  is imaginary. This corresponds, of course, to the presence of real mesons propagated in accordance with the Green's function  $|\mathbf{x}-\mathbf{x}'|^{-1}\exp i\mathbf{k}|\mathbf{x}-\mathbf{x}'|$ . The complete solution of the equation (1) can be now put into the form

$$\varphi(\vec{\mathbf{x}}, \mathbf{t}) = \int \int d\omega d_3 \mathbf{x}' \ \rho(\vec{\mathbf{x}}, \omega) e^{i\omega \mathbf{t}} \frac{e^{-\sqrt{\mu^2 - \omega^2} |\vec{\mathbf{x}} - \vec{\mathbf{x}}'|}}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|}$$
(4)

We see that the components of the source corresponding to given frequency  $\omega$  are multiplied by exponential factors of a range increasing rapidly with increasing  $|\omega|$ . The formula (4) is quite general and valid for any kind of sources. However, one cannot say much more about the general case since the result of the integration over  $\omega$ , i.e. the interference effect between different  $\omega$ 's, cannot be predicted in the general case. One may, however, expect that in some cases we may be able to observe a realistic increase in the range of the mesonic field.

Let us come back to the case of high energy nucleon-nucleus collision. I am suggesting that the fragmentation stage is the result of the coherent action of the mesonic field accompanying the incident nucleon and distorted by its collisions with the other nucleons in the tunnel. The question immediately arises: can this distorted mesonic field have the proper range and shape necessary to describe the experimental data? The answer is not easy as the result of the calculation may depend in a very crucial way on the assumptions about  $\rho$ , i.e. on the time development of the "knock off" stage.

In order to see if there is any chance of describing the qualitative features of the fragmentation stage we have made in Warsaw some simple assumptions about  $\rho$  trying, however, to fit the free parameters to the experimental material in possession of the Warsaw cosmic ray group. Thus we have assumed that due to high energies the nucleon can be well localized and correspondingly one can justify the use of the "classical" source function

$$\rho(\vec{\mathbf{x}}, t) \sim \delta_3(\vec{\mathbf{x}} - \vec{\xi}(t))$$

with some prescribed law of motion given by the function  $\overline{\xi}$  (t). We have assumed that the motion of the incident nucleon is uniform up to the point of the collision uniformly decelerated during the passage through the nucleus and again uniform after leaving the nucleus. The plot of the assumed nucleon velocity is on Fig. 3.



Assumed velocity versus time plot

The free parameters like initial and final velocities, duration of the collision (resulting from the assumed size of the nucleus and central character of the collision) were taken from the experimental set up. Kaniowski, a student from Warsaw, has made preliminary calculations looking just for the shape of the mesonic field accompanying such a uniformly decelerated nucleon at a time shortly before it leaves the nucleus. The result of these calculations seems to be very encouraging. Fig. 4 shows the schematic plot of the



Fig. 4

Schematic plot of the shape of the curves  $\varphi = \text{const.}$  as a function of z and the distance  $\pi$  from the z-axis

shape of the curves (actually surfaces)  $\varphi = \text{const}$  as function of z and the distance  $r = \sqrt{x^2 + y^2}$  from the z axis. On the right the corresponding plot for uniform motion is shown. For uniformly decelerated motion we have found that the meson field gets distorted into the right direction.

In fact some kind of shock wave is created. The "wings" of this shock wave extend as far as 4-5 fermis. For the assumed values of the free parameters the angle  $\theta_f = 26.5^\circ$  which fits quite well to the experimental value of the Warsaw cosmic ray group of ~30°. Of course, these results are very preliminary. We are now doing more elaborate calculation on an electronic computer aiming at a more detailed knowledge of the shape, time development of the shock wave, the energy carried by its "wings" etc. Unfortunately, I have not got the result of these more extensive calculation as yet.

# STRONG DAMPING OF LARGE MOMENTUM TRANSFERS, AND ITS CONSEQUENCES FOR HIGH-ENERGY INELASTIC PROCESSES

#### S. FRAUTSCHI

# NEWMAN LABORATORY OF NUCLEAR STUDIES, CORNELL UNIVERSITY, ITHACA, N.Y., UNITED STATES OF AMERICA

# I. INTRODUCTION

Recently some progress has been made towards understanding the properties of cosmic ray events in terms of properties of strong interactions at lower energies. For example, the one-pion exchange model was applied

# 

Fig.1

A highly inelastic event leading to several "clumps" by means of repeated one-pion exchange

to peripheral collisions and then generalized by the SALZMANS [1], GOEBEL [2] and AMATI et al. [3] to a repeated one-pion exchange model which leads to several "clumps" of particles in the final state (Fig.1). In this generalized approach a very high energy process is reduced to a product of factors, each representing production of one of the clumps at much lower energy where the interactions are better understood.

Meanwhile elastic proton-proton scattering [4] at accelerator energies has been found to decrease exponentially with increasing momentum transfer |t|. Over part of the range of experiments, especially at  $|t| \leq 1 (\text{GeV})^2$ , the observed behaviour may be explained by the exchange of a single dominant Regge pole [5-8], but the exponential falloff persists at larger |t| where the detailed mechanism is not understood.

In the present approach we shall assume, without attempting to understand the underlying reasons or formalism, that the exponential damping of large momentum transfers is a general characteristic of high-energy amplitudes. The rate of damping will be taken from the existing elastic protonproton results [4] and applied to inelastic p + p and  $\pi + p$  events. We also employ a breakdown into low-energy clumps as in the work of the SALZMANs [1], GOEBEL [2] and AMATI et al. [3], whose approach and results we follow in many respects. No restricition is made to one-pion exchange between clumps, however.

Observed features [9, 10], such as "fireballs" and constant transverse momentum of secondary particles, come out in a natural way, with reasonable magnitudes. The relation of these properties of cosmic rays to small momen-

#### S. FRAUTSCHI

tum transfers has already been noticed by cosmic ray experts [10], so the present approach serves especially to emphasize that momentum transfers are comparably small at machine energies and at higher energies. One result of the present approach is that a definite conception of the fireball, as distinguished from individual particles, emerges. This picture will be discussed in detail, especially in the energy region  $10^3 - 10^5$  GeV where most of the data on fireballs has been obtained.

#### II. THE ASSUMPTION ON MOMENTUM TRANSFER DEPENDENCE

At -1 (GeV)<sup>2</sup>< t < 0, elastic proton-proton scattering decreases exponentially with increasing |t| and the width of the exponential peak decreases slowly as the energy rises. The data are consistent with the formula [5,8]

$$\frac{\mathrm{d}\sigma(\mathbf{s},\mathbf{t})}{\mathrm{d}\mathbf{t}} = f(\mathbf{t}) \left(\frac{\mathbf{s}}{2 \ \mathbf{M}^2}\right)^2 \overset{\alpha(\mathbf{t}) - 2}{\sim} f(\mathbf{t}) e^{-2|\mathbf{t}| \alpha' \ln(\mathbf{s}/2\mathbf{M}^2)}, \quad (II.1)$$

where s is the square of the centre of mass energy, M is the nucleon mass, and  $\alpha$  (t) is the spin of the dominant Regge trajectory, rising from about  $\alpha = 0$  at t = -1 (GeV)<sup>2</sup> to  $\alpha = 1$  at t = 0.

At larger -t,  $\alpha$  (t) seems to stabilize in the region  $0.5 < \alpha < 0$ , with large errors. If this is true, the factor exp  $[-2|t|\alpha'\ln(s/2M^2)]$  decreases no further; nevertheless  $d\sigma/dt$  still falls with increasing -t at approximately the rate  $10^{t/M^2} = \exp[2.3 t/M^2]$ , rather independent of energy [4]. The reason for this behaviour is not known.

Our assumption will be that any high-energy amplitude decreases at least as fast as  $\exp [1.1 t/M^2]$ . This is taken directly from the square root of the elastic proton-proton cross-section. If (II.1) is appropriate, the amplitude may decrease faster. For the dominant inelastic processes, however, a simple kinematic analysis shows that the reactions are not in the asymptotic region where (II.1) is valid.

Actually there are two momentum transfers in elastic scattering, the "direct" transfer t and the "exchange" transfer u. They are related by the constraint  $s + t + u = \Sigma M_i^2$ . The distance to the nearest singularity ( $t = u^2$  at small  $|t\rangle$  is therefore the same in either variable;  $|t - u^2| = |u - \Sigma M_i^2 + u^2 + s|$ . Thus our assumption can be formulated more generally: in each variable the amplitude falls off exponentially as the distance of the variable from the nearest singularity increases. Naturally it is most convenient to use t at small |t|, for then the nearest singularity lies at a small mass fixed independently of s. In the inelastic case where many momentum transfers can be defined, we shall again find it convenient to use a small one.

#### III. HOW LARGE ARE THE CLUMPS?

To appreciate the effect of the assumption made in Section II, consider Fig. 2 for the reaction  $A + B \rightarrow C + D$ , where C and D are arbitrary clumps



.

The reaction  $A + B \rightarrow C + D$ 

of particles with energies  $M_3$  and  $M_4$  respectively in their own centres of mass. Define  $s = (p_1 + p_2)^2$  and  $t = (p_3 - p_1)^2$  in the usual way. In the centre-of-mass system of the entire reaction, C and D each emerge with momentum  $p_T$  transverse to the initial direction of motion. The relation between t and  $p_T$  is given by

$$t = -p_T^2 + \frac{1}{4s} \{ [M_3^2 - M_4^2 + M_1^2 - M_2^2]^2 - [M_3^2 + M_4^2 - M_1^2 - M_2^2 + 2p_T^2]^2 \}$$
(III.1)

One sees that -t grows directly with  $p_T^2$ . Thus our assumption of exp  $[1.1/M^2]$  falloff implies exp  $[-1.1 p_T^2/M^2]$  falloff with increasing transverse momentum. In fact, from (III.1) it is clear that the experimental absence of large pT directly implies that large |t| are absent. The momentum transfer is somewhat less sensitive to the masses of clumps at high energy s, and the exponential falloff tends to restrict the masses only when they grow at least as fast as  $M_3^2 M_4^2 \sim s$ .

Before discussing further the dynamical limitation on clump size, we need to agree on a definite way to assign the various particles in a complicated final state to clumps. Consider the centre-of-mass frame for the reaction  $A + B \rightarrow$  many particles. Now clump C will be defined to consist of all particles which go forward in the centre of mass, and clump D will be defined to consist of all particles which go backward. This definition yields a relatively small momentum transfer and coincides with the natural experimental division into forward and backward groups.

There are various ways to categorize the exchange that occurs between (A, C) and (B, D) in Fig. 2. It can be described as a one-pion exchange, plus a two-pion exchange, plus an NN exchange, and so forth. Or it can be described as the exchange of 'a succession of Regge poles. In any case the complete amplitude factors into a product of terms:

(1) The amplitude for A + exchanged object  $E \rightarrow C$ . (Of course, the amplitude must be continued from the physical square mass of E to a negative square mass.)

(2) A factor involving only E.

(3) The amplitude for  $B + E \rightarrow D$ .

The next step is to take amplitude (1) or (3) and again break the final state into two groups of particles. For example (Fig. 3) in the centre-of-mass of (1) we include forward-moving particles in group 5, with energy  $M_5$  in its own rest frame, and backward-moving particles in group 6.

There are now four groups of particles in the final state, and these groups could be sub-divided further to the point where each clump contains


Fig.3

Breakdown of  $A + B \rightarrow C + D$  into 4 clumps

only one particle. But we shall carry the subdivision only down to the point where each clump contains a couple of GeV. At this point it is possible to make some qualitative estimate of what will happen without reducing the energy further, and our assumption on exponential damping of large momentum transfers cannot be used at lower energies. The question then is: how many subdivisions are required before each clump is reduced to a couple of GeV ? If there were no dynamical restrictions, the energies  $M_3$  and  $M_4$ of clumps C and D in their own rest frames could take up all the available centre-of-mass energy  $\sqrt{s}$ , leaving no relative kinetic energy for the clumps. In this case many subdivisions would be required to reduce the clumps to low masses. However,  $M_3^2$   $M_4^2$  would then grow as  $s^2$  and -t would grow as s, and here the dynamical assumption of section II which damps large momentum transfers becomes relevant.

Consider first  $A + B \rightarrow C + D(Fig. 2)$ . The cross-section can be expressed in terms of the cross-sections  $\sigma_{AEC}(t; M_3^2)$  for  $A + E \rightarrow C$  [E has  $m^2 = t$ ] and  $\sigma_{BED}(t; M_4^2)$  for  $B + E \rightarrow D$  by a slightly modified form of the Salzman relation [1]:

$$\frac{\partial^{3}\sigma}{\partial t \ \partial M_{3}^{2}\partial M_{4}^{2}} = \frac{1}{2(2\pi)^{3} p_{iL}^{2} M^{2}} \left[\sigma_{AEC}(t; M_{3}^{2})q_{A}M^{3}\right] F(s, t, s_{1}, s_{2})\left[\sigma_{BED}(t; M_{4}^{2})q_{B}M_{4}\right]$$
(III.2)

where  $p_{iL}$  is the momentum of A in the lab [rest frame of B],  $q_c$  is the momentum of A in the centre of mass of the reaction  $A + E \rightarrow C$ , and  $q_B$  is the momentum of B in the centre of mass of  $B + E \rightarrow D$ . The factor F(t) refers to the system exchanged; it is  $(t-m_{\pi}^2)^{-2}$  in one-pion exchange and exponentially decreasing in our case. At high energies, with  $M_1$  and  $M_2$  fixed, (III.2) simplifies to

$$\frac{\partial^3 \sigma}{\partial t \partial M_2^2 \partial M_4^2} = \frac{M_3^2 M_4^2}{2 (2 \pi)^3 s^2} \sigma_{AEC} F \sigma_{BED}.$$
 (III.3)

The cross-sections  $\sigma_{AEC}$  and  $\sigma_{BED}$  are expected to remain approximately constant as  $M_3^2$  and  $M_4^2$  respectively increase. Our method is too weak to understand the t dependence of  $\sigma_{AEC} \sigma_{BED}$  or the dependence of F on s, s<sub>1</sub> and s<sub>2</sub>, but provided none of these factors increase exponentially, it is clear that large |t| are restrained by F ~ exp (-2.3 |t|), and this is sufficient

~

#### STRONG DAMPING

to establish that  $M_3^2 M_4^2 \leq s M^{*2}$  where  $M^*$  is of order 1 GeV\*. The next step is to break clump C down into subgroups with masses  $M_5$  and  $M_6$ . The centre-of-mass energy squared for the reaction  $A + E \rightarrow C$  is  $M_3^2$ , and the limitation on momentum transfer leads in this case to  $M_5^2 M_6^2 \leq M_3^2 M^{*2}$ . Similarly the breakdown of clump D leads to  $M_7^2 M_8^2 \leq M_4^2 M^{*2}$ . Altogether one has  $M_5 M_6 M_7 M_8 \leq M^{*3} s^{\frac{1}{2}}$ .

For example, if the lap energy of a proton-proton collision is  $10^4$ GeV, then s =  $2 \times 10^4$  GeV<sup>2</sup> and  $M_5M_6M_7M_8 \lesssim 1.4 \times 100 \times M^{*3}$ . If we take  $M^{*2} = (2.3)^{-1}$  GeV<sup>2</sup>[the value for which exp [-2.3 | t | ] becomes exp (-1)], then  $M_5M_6M_7M_8 \lesssim 40$  GeV<sup>4</sup>. In case each split was symmetric,  $M_5 = M_6 = M_7 = M_8 \leq 2.5$  GeV and all 4 clumps have reached the low-energy region where one can make plausible guesses about them without further reductions. Of course non-symmetric splits are also allowed, and in extreme cases larger clumps would require more than two successive reductions at  $10^4$  GeV.

### **IV. FIREBALLS**

Let us discuss in more detail the 4 clumps obtained in proton-proton reactions at lab energies of  $10^4$  GeV. Although we have only obtained a maximum size, the experiments suggest that this maximum size is about normal; and we shall confine the discussion to the case where the maximum is attained without attempting to discover why it is usually attained. In the centre of mass one will see a fast clump moving forward along the original direction of A (remember that  $p_T$  must be small for a clump) and another moving backward along the original direction of C, each followed by a slower clump moving along the same line (Fig. 4).



#### Motion of clumps in centre of mass of reaction $A + B \rightarrow C + D$

The damping of large momentum transfers between clumps suggests the dominance of long-range forces, and on this basis one expects that systems of baryon number zero will normally be exchanged between the clumps. Each of the two fast clumps (5 and 8) then carries baryon number one since the incoming particles A and C were baryons, and the two slow clumps (6 and 7) carry baryon number zero. In accordance with cosmic ray terminology the clumps with baryon number one will be called nucleon isobars, and the clumps with baryon number zero will be called "fireballs".

How many fireballs are there in general? We have adopted the procedure of subdividing until reduced scattering events are obtained, each at a rel-

<sup>\*</sup> It might be objected that, as -t becomes very large and far from the nearest singularity at positive  $t = \mu^2$ , it may approach the nearest singularity at negative  $t = \Sigma M_i^2 - s - \mu^2$ , and the cross-section may rise again. This possibility is excluded by the definition of t as  $(p_0 - p_A)^2$  where  $\overrightarrow{p_c}$  and  $\overrightarrow{p_A}$  are both in the forward hemisphere in the centre of mass.

#### S. FRAUTSCHI

atively low energy. Because of the low energies involved, each of these scatterings produces a relatively isotropic final state in its own centre of mass, though still peaked somewhat forward and backward. It is these relatively isotropic final states which are called nucleon isobars or fireballs. Now as the overall energy of the reaction is increased, the centre-of-mass energy of each "fireball" and "isobar" slowly increases, and each of them becomes more strongly peaked forward and backward. Above some clump mass, of order 5 GeV for the clumps with nucleon number one and perhaps lower for the fireballs, it becomes meaningful to split the clump again into its forward and backward components, each of which has a mass between 1 and 2 GeV and is relatively isotropic again. In summary the mass of fireball always lies between extremes of order 1 and 5 GeV, and as the overall energy of the reaction increases fireballs swell into dumbbell shapes and divide rather than grow beyond their proper sizes [11]. The process is illustrated in Fig. 5\*.



The growth of clumps along the original direction of motion in the centre of mass, as the energy increases

Although the number of fireballs increases with energy, the increase is slow. At 30 GeV lab energy, proton-proton scattering leads to two "isobars" and no fireballs. At  $10^4$  GeV lab energy two fireballs have also developed. At  $10^8$  GeV a symmetric split-up leads to 6 fireballs, each with a mass of about 2.3 GeV. In general n clumps are obtained with repeated applications of the formula

$$M_3^2 M_4^2 / M^{*2} = s$$
 (IV.1)

leading to

<sup>\*</sup>It should be mentioned, however, that there is some evidence for fireballs emitting secondaries into a disk pattern peaked perpendicular to the incoming direction, rather than into a dumbbell pattern (e.g. Ref.[10]). Further evidence on this point should be of great importance for the consistency of the multiple fireball picture.

#### STRONG DAMPING

where each of the  $M_i$  is a fireball or isobar mass, not greater than 5 GeV. Taking an average square mass for the fireballs and isobars, one finds

n = -[ln (s/M\*<sup>2</sup>)] /[ln 
$$(\widetilde{M_i^2}/M*^2)]$$
 (IV.3)

so the multiplicity of isotropic clumps increases as ln s.

Asymmetric as well as symmetric split-ups occur, since only the combination  $M_3^2$   $M_4^2$  enters into Eq. (III.1) when  $M_3^2$  and  $M_4^2$  are each much greater than 1 GeV<sup>2</sup>. The asymmetry is especially noticeable when it occurs in the first split-up, leading to a depletion of secondaries in one hemisphere in the centre of mass. Suppose this happens at 10<sup>4</sup> GeV lab energy, and  $M_3^2$  is large while  $M_4^2$  is only a few GeV<sup>2</sup>. Then in the backward hemisphere a nuclear isobar, or perhaps only a single nucleon emerges, while in the forward hemisphere  $M_3^2 = M*^2s/M_4^2$  can be split into 4 clumps if  $M_4^2 = 1 \text{ GeV}^2$  (single nucleon), or 3 to 4 clumps if  $M_4^2$  = several GeV<sup>2</sup> (nucleon isobar). The general nature of the derivation showing that fireball multiplicity rises as ln s ensures that asymmetric split-ups lead to similar multiplicities.

From the foregoing description it is clear that the nucleon isobars and fireballs have a similar origin in the present model. The masses of fireballs and isobars are sufficiently low relative to the nucleon mass, however, to lead to certain differences, and one of these is the multiplicity of particles emitted from the fireball or isobar. Consider the mass 2.5 GeV, for example. A state with this mass and baryon number one is expected to contain one nucleon and one or two pions. A state with this mass and baryon number zero is expected to contain three or four particles which are most likely  $\pi$ ,  $\rho$ ,  $\omega$  or n. The decay of the  $\rho$ ,  $\omega$  or n then leads to a final state with about six pions. This is what happens, for example, in the final state of  $p\overline{p}$  annihilation. Thus the fireballs produce pions much more copiously than the nucleon isobars. For an incident lab energy of 10<sup>4</sup> GeV each isobar emits one or two pions and each fireball about six, or a total of about 15 pions.

The logarithmic growth with s of fireball and clump multiplicity Eq. (IV.3) indicates that particle multiplicity increases as lns at large s where the fireball picture is applicable [3], since an average fireball emits about the same number of particles whatever the original s is. Actually the rate of increase in pp scattering from 30 GeV to  $10^4$  GeV in the lab is somewhat enhanced because the two fireballs which appear in this energy region provide more particles than the two "nucleon isobars" which were already present at 30 GeV. For example, we expect the total number of particles, N, to increase from about 5 to 17 as the energy rises from 30 GeV (2 nucleon isobars) to  $10^4$  GeV (15 pions + 2 nucleons) whereas the form N = a ln s would give a rise from about 5 to 13 in this interval. This makes our predictions fairly compatible with the data even though the observed multiplicity is traditionally represented as growing at a rate N ~ s<sup>‡</sup> over much of this region [9, 10].

Everything that has been said for proton-proton scattering would also hold for pion-pion scattering, with one of the outside nucleon isobars replaced by a fireball. As a by-product at 30 GeV where only two clumps are typically formed, one expects a somewhat higher particle multiplicity in  $\pi$  N reactions than in NN reactions because one of the clumps contains only pions in the former case.

Since events of arbitrarily high energy reduce to products of events at several GeV, most of the secondaries are pions, and K mesons and baryons will be produced in ratios similar to those found at a few GeV.

#### V. TRANSVERSE MOMENTA

One of the most persistent phenomena in high-energy and cosmic-ray physics involves the transverse momentum distribution of inelastic secondaries: for any incident energy, the distribution is peaked around  $p_T \sim 0.4$  GeV/c. At accelerator energies the tail of large  $p_T$  has also been studied quantitatively [12] and is found to fall off exponentially, consistent with exp (- $p_T/0.2$ ) [ $p_T$  in GeV/c].

The kinematical dependence of t on  $p_T^2$  (Eq. III.1), together with exponential damping of large |t|, damps the transverse momentum of each clump as exp (-2.3  $p_T^2$ ). As a consequence each fireball or nucleon isobar moves approximately along the line of flight of the particles which initiated the reaction. Then the transverse momentum of each <u>particle</u> has a component due to the motion of its clump (shared among several particles and therefore small), plus the motion of the particle relative to the clump centre of mass. The later contribution refers to a reaction of only a few GeV, so the transverse momentum of individual particles reduces approximately to the low-energy figure no matter what the incoming energy is. Large transverse momenta are strongly damped by dynamical factors, and further damped at a few GeV by competition among the particles in a clump for phase space.

What does this model have to say about the transverse momenta of different kinds of secondaries:  $\pi$ , K, N? Distinctions can appear only in the last stage where a clump is broken down into several particles, and this involves reactions at a few GeV. Here the dynamical damping of crosssections at large momentum transfer is probably of order (t - M<sup>2</sup>)<sup>-2</sup>, where M is the exchanged mass, rather than exponential. The exchanged mass is greater for production of K's and baryons than for pion production, so the dynamical damping of large p<sub>T</sub> relative to small p<sub>T</sub> may be weaker for K's and baryons. There are also phase space factors to consider, and these strongly inhibit the transverse momentum of any particle from becoming very large, especially in the fireballs because they have more particles than the "nucleon isobars". At accelerator energies pp scattering does not yet produce fireballs, and the strange particles which require a large mass exchange may well have larger p<sub>T</sub> than pions have.

### VI. INELASTICITY

As a measure of the distribution of energy in the final states, cosmicray physicists [9,10] define the inelasticity K

#### STRONG DAMPING

$$K = \frac{Centre-of-mass energy of secondaries}{Centre-of-mass energy of initial state}.$$
 (VI.1)

The "secondaries" are defined as all particles in the final state except for the fastest particle in each hemisphere.

Let us first consider the contribution of a fireball to K, in the energy range where there are only two fireballs. In section III the relation  $M_2^2 M_2^2 M_2^2 M_2^2 \sim M^{*6}$ s was established for this energy range. A particularly low inelasticity is found when the reaction is symmetric in the centre of mass  $[M_5 = M_8, M_6 = M_7]$  and the "nucleon isobar" is simply one nucleon  $[M_5]$ = M]. Then the fireball mass  $M_6$  is proportional to  $s^{\frac{1}{4}}$ , as compared with the total centre-of-mass energy  $s^{\frac{1}{2}}$ , so the inelasticity K can fall off as rapidly as s<sup>4</sup> if the fireball moves only slowly in the centre-of-mass frame (a possibility which is consistent with our conditions). Larger inelasticities are also possible, especially when the nucleon isobar is larger and emits pions. The result depends somewhat on the detection method; for example, only charged secondaries may be detected, and then the question is whether the fast nucleon isobar in the lab emits more than one charged particle. If it does, the inelasticity can easily be 0.5 or greater. At somewhat higher energies where 4 fireballs appear, the inelasticity can be low if the original fireballs have grown large and split in two, or large if the original nucleon isobar has grown and split in two [11]. At all energies, then, the inelasticity will have a broad spread. The average is essentially controlled by the fraction of energy the fastest nucleon isobar shares with pions that get counted as "secondaries". The composition of the nucleon isobar is not very energydependent, so the average should be approximately energy-independent [3]. The inelasticity for  $\pi$  N events should behave similarly, but the average should be higher because there are more fast pions.

### VII. REGGE POLES

The author began this study of highly inelastic events with the hope that exchange of a dominant Regge trajectory would lead to a simple formula like (II.I). This worked in the case of elastic or nearly elastic scattering [5-8], where the amplitude at fixed t and large s was dominated by a term proportional to  $P_{\alpha}$  (cos  $\theta_t$ ) ~ (cos  $\theta_t$ )<sup> $\alpha$ </sup>. As s increased, cos  $\theta_t$  grew as

$$\cos \theta_{\rm r} = -1 - 2 \, {\rm s} / \, ({\rm t} - 4 \, {\rm M}^2) \,,$$
 (VII.1)

taking equal masses as an example, and the amplitude grew as  $s^{\alpha}$ . Now if all four masses are unequal in the process  $A + B \rightarrow C + D$ , (VII.1) is replaced by

$$\cos\theta_{t} = \frac{-\{t^{2} + t(2s - M_{1}^{2} - M_{2}^{2} - M_{3}^{2} - M_{4}^{2}\} + (M_{1}^{2} - M_{3}^{2})(M_{2}^{2} - M_{4}^{2})\}}{\sqrt{t - (M_{1} - M_{3})^{2}}\sqrt{t - (M_{1} + M_{3})^{2}}\sqrt{t - (M_{2} - M_{4})^{2}}\sqrt{t - (M_{2} + M_{4})^{2}}}.$$
(VII.2)

We are interested in the cosmic-ray case discussed in the previous sections, where  $M_3^2$   $M_4^2 = M^{*2}s$  and  $M_1$ ,  $M_2$  can be neglected, leaving

$$\cos \theta_t \sim -2 \text{ ts}/M_3^2 M_4^2 \sim -2 \text{ t}/M^{*2}$$
. (VII.3)

at large s and fixed t. So  $\cos \theta_t$  does not increase with s, and (II.1) cannot be used.

Nevertheless it would be desirable to have a Regge pole formalism applicable to highly inelastic events; it might help to put the very simple considerations of the preceding sections on a more adequate basis. The author does not know how to do this but would like to call attention to a few problems which come up [13].

To begin with, recall the one-pion exchange model for  $A + B \rightarrow C + D$ . The amplitude is written as the product of

(a) the amplitude for A + exchanged  $\pi \rightarrow C$ ,

(b) the pion propagator,

(c) the amplitude for  $B + \pi \rightarrow D$ .

So far the unknown amplitude for  $A + B \rightarrow C + D$  has simply been reduced to a product of unknown amplitudes. The next step is to calculate  $A + \pi \rightarrow C$ , which is done by expressing this amplitude as another one-pion exchange. The process is repeated until one has a product of low-energy amplitudes. The incoming objects in these amplitudes (except for the original particles A and B) have spin zero, and their masses are in many cases continued from  $t = m^2$  to negative t.

Now if the pion lies on a Regge trajectory, the one-pion exchange procedure still applies for  $t = m^2$ , and a natural extension is to exchange the pion Regge trajectory (or to be more complete, the sum over all trajectories) at  $t \neq m^2$ . The original amplitude can still be factored [14, 15] into the products of (a) the amplitude for A + exchanged trajectory  $E \rightarrow C$ , (b) a term involving only the Regge pole, (c) the amplitude for  $B + E \rightarrow D$ . Let us assume that the amplitude for A +  $E \rightarrow C$ , for example, can be expressed in terms of another Regge pole exchange. One again obtains a product of low-energy amplitudes. This time the incoming objects in the low-energy amplitudes (except for A and B) are Regge poles which not only have masses continued to  $m^2 = t$  where t may be negative, but also have spins continued to noninteger values which vary with t. In order to construct a theory of repeated Regge pole exchange, then, it will be necessary to construct a theory of amplitudes in which some of the external objects are Regge poles.

Suppose that all this can be done, and consider the case where  $s/M_3^2M_4^2$  grows and cos  $\theta_t$  becomes large. The exchange of a Regge pole with  $\alpha_i$  (t) between clumps C and D gives a factor

$$\left(\frac{-2 t s}{M_3^2 M_4^2}\right)^{\alpha(t)}; \qquad (VII.4)$$

the exchange of a pole with  $\alpha_j(t^1)$  between clumps 5 and 6 (Fig. 3) gives a factor  $(-2t^1M_3^2/M_5^2 M_6^2)^{\alpha_j}(t^1)$ , and so forth. The first factor (VII.4) con-

tributes to F in the Salzman formula (III.2), which may however contain further dependence on t,  $M_3^2 = s_1$ , and  $M_4^2 = s_2$ . We would like to end with the point that until this further dependence is known one has no idea whether the highest  $\alpha$  dominates even the contribution to  $d\sigma/dt$  from large  $\cos \theta_t$ , for  $d\sigma/dt$  involves an integration  $\int \int dM_3^2 dM_4^2$  and the factor (VII.4) suppresses large  $M_2^2$  and  $M_4^2$  when it favours large s.

#### REFERENCES

- [1] SALZMAN, F. and SALZMAN, G., Phys. Rev. <u>120</u> (1960) 599 and <u>121</u> (1961) 1541.
- [2] GOEBEL, C., Proceedings of the 1961 International Conference on Theoretical Aspects of Very High-Energy Phenomena at CERN, (1961) 353.
- [3] AMATI, D., FUBINI, S., STANGHELLINI, A. and TONIN, M., Nuovo Cimento 22 (1961) 569.
- [4] COCCONI, G., DIDDENS, A.N., LILLETHUN, E., MANNING, G., TAYLOR, A.E., WALKER, T.G. and WETHERELL, A.M., Phys. Rev. Letters, 7 (1961) 450. Some more recent points at larger momentum transfer are given in the report of G. COCCONI to the International Conference on High Energy Physics, CERN (1962).
- [5] GRIBOV, V.N., J. Exptl. Theoret. Phys. (USSR) <u>41</u> (1961) 667 (translation: Sov. Phys. JETP <u>14</u> (1961) 478)).
- [6] LOVELACE, C., Imperial College preprint.
- [7] BLANKENBECLER, R., and GOLDBERGER, M.L., Phys. Rev. 126 (1962) 766.
- [8] CHEW, G.F. and FRAUTSCHI, S.C., Phys. Rev. Letters 7 (1961) 394.
- [9] For an excellent summary of data up to 1961 see PERKINS, D.H., Proc. Int. Conf. on Theoretical Aspects of Very High-Energy Phenomena, CERN (1961).
- [10] For a survey of data and its interpretation as of 1963 see HAYAKAWA, S., Trieste Seminar on Theoretical Physics (1962).
- [11] Some observational evidence for events with more than 2 fireballs is argued by HASEGAWA, S., Prog. Theor. Phys. 26 (1961) 150.
- [12] MORRISSON, D.R.O., Proc. Int. Conf. on Theoretical Aspects of Very High-Energy Phenomena, CERN (1961) 153.
- [13] These questions have been investigated independently by K. Ter-Marterosyan (reported by V. Gribov at the International Conference on High Energy Physics, CERN (1962)).
- [14] GELL-MANN, M., Phys. Rev. Letters 8 (1962) 263.
- [15] GRIBOV, V.N. and POMERANCHUK, I.Ya., Phys. Rev. Letters 8 (1962) 343.

# BOOK V

# VERY HIGH ENERGIES

## PART II

ELECTROMAGNETIC INTERACTIONS

# METHODS FOR ACCURATE CALCULATIONS IN HIGH-ENERGY QUANTUM ELECTRODYNAMICS

#### K.E. ERICSSON

### INSTITUTE OF THEORETICAL PHYSICS, UPSALA, SWEDEN

In this paper "quantum electrodynamics" (QED) will be used in the sense of a closed theory of point-like photons and electrons. Muons could then easily be included. We make the usual assumption that the perturbation expansion of renormalized QED gives at least an asymptotic expression of the exact theory, i.e. that the sum over a few terms in the beginning of the perturbation series is a good approximation of the exact theory.

We expect QED in this sense to break down at small distances, i.e. at large momentum transfers, because of structure effects resulting from other kinds of interaction, primarily the interactions of the electromagnetic field with the current of strongly interacting particles. This will first show up as vacuum polarization through mesons. On the other hand we have no reason to believe that the fundamental theory of electrodynamics, i.e. the theory of a massless vector field interacting with a conserved current, will break down.

We shall discuss methods for summing part of the perturbation expansion for QED:

- (i) Summation of a certain class of Feynman diagrams, the infrared divergent diagrams;
- (ii) Summation to all orders  $\sin \alpha$  of a certain class of terms that are dominant at high momentum transfers, at least in the lowest orders of the perturbation expansion (terms of the kind  $[\alpha \ln(q^2/m^2)]^n$  where q is the momentum transfer).

A combination of the two types of summation gives a very accurate expression for the quantity to be calculated.

The result thus obtained may be useful if very accurate high-energy experiments can be performed in the future, in order to investigate at what distances structure effects show up. One may also hope that it will somehow turn out to be possible to include structure effects (through form factors) in the summation procedure.

We shall discuss the following topics here:

- (a) The importance of the mechanism of low-energy radiation;
- (b) The summation (i);
- (c) The renormalization group which, together with a result in (a), will be used to perform;
- (d) The summation (ii);
- (e) Combination of (i) and (ii) and the consequences.

(a) When one tries to investigate high-energy QED one finds oneself in the somewhat paradoxical situation that one has to understand the mechanism of low-energy radiation before one can get any results for the highenergy processes. This is, however, not so surprising for the following reason.

#### K.E. ERICSSON

The electron mass should be negligible at high energies. Let us therefore put m = 0 in the evaluation of closed electron loops. Then we obtain infrared divergences very similar to those that occur as a result of the zero mass of the photon. The divergences thus obtained cancel when multiple creation of soft massless electron pairs is taken into account, just as the photon divergences cancel when multiple photon emission is taken into account. The mechanism of this cancellation has been demonstrated in a paper by NAKANISHI [5]. Eq. (3.7) in Nakanishi's paper says that the degree of divergence of a diagram does not exceed D where

$$D = 2n + n' + 1 - 4\nu$$
 (1)

 $k_j$  are independent integration variables (j = 1, 2, ...,  $\nu$ ) (soft-particle momenta)

n = the number of boson propagators containing only  $k_i$ 's.

n' = the number of fermion propagators containing only  $k_i$ 's.

1 = the number of propagators of the type

$$[(p + \Sigma c_i k_i)^2 + m^2 - i\epsilon]^{-1}$$
, where  $p^2 + m^2 = 0$ .

From this follows that the only infrared divergent diagrams are those in which the soft particles are connected with external electron lines. Hence self-energy diagrams for photons off the mass shell do not lead to infrared divergences in the limit of vanishing electron mass. This fact will be used later on.

(b) Let M be the invariant amplitude of a process  $|i\rangle \rightarrow |f\rangle$  in which r electrons (and positrons) with momenta  $p_1, \ldots, p_r$  and charges  $Q_1, \ldots, Q_r$  participate:

$$< f | S - 1 | i > = \delta(P_f - P_i)M.$$
<sup>(2)</sup>

Let us define  $\epsilon_1, \ldots, \epsilon_r$  such that  $\epsilon_i = 1(-1)$  if the *i*th particle is outgoing (incoming).

In general M contains infrared divergences. But if we take into account that along with our basic process  $|i\rangle \rightarrow |f\rangle$  soft photons may be emitted such that their total energy does not exceed a certain value  $\Delta E$ , we shall obtain a finite result. We assume that

$$\Delta E/q < < 1, \tag{3}$$

where q is a typical energy in the process. (We shall assume that if we have a high-energy process, q is the momentum transfer and all momenta  $p_i$  are of the same order of magnitude as q. This can usually be achieved for non-forward scattering with an isotopic  $\Delta E$  in the c.m.s.). We may then neglect recoil effects from the soft photons. Let

$$M(\underline{k}_1, e_1; \ldots; \underline{k}_m, e_m)$$
(4)

be the invariant amplitude for emission of m soft photons (with momenta

 $k_j$  and polarizations  $e_j$ ) along with the process  $|i \rangle \rightarrow |f \rangle$ . Then the physically interesting quantity is

$$\mathbf{P}(\Delta \mathbf{E}) = \sum_{m=0}^{\infty} \frac{1}{m!} \int d^{3}\mathbf{k}_{1} \dots d^{3}\mathbf{k}_{m} \int d\mathbf{x} \,\delta(\sum_{j=1}^{m} \mathbf{k}_{j0} - \mathbf{x}) \\ |\mathbf{k}_{j}| \leq \Delta \mathbf{E} \\ \mathbf{k}_{j0} = \sqrt{\frac{\mathbf{k}_{j}^{2} + \lambda^{2}}{\mathbf{k}_{m}^{2} + \mathbf{k}_{m}^{2}}} \\ \times \sum_{pol} |\mathbf{M}(\underline{\mathbf{k}}_{1}, \mathbf{e}_{1}; \dots; \underline{\mathbf{k}}_{m}, \mathbf{e}_{m})|^{2}.$$
(5)

Here a small fictitious photon mass  $\lambda$  has been introduced. It is understood that the limit  $\lambda \rightarrow 0$  is to be taken in the final result. The x-integral guarantees that  $\Sigma k_{j0} \leq \Delta E$ . The factor 1/m! arises from the loosening of an ordering restriction of the kind

$$\mathbf{k}_{10} \geq \mathbf{k}_{20} \geq \dots \geq \mathbf{k}_{m0} \quad . \tag{6}$$

We shall now exhibit the  $\lambda$ -dependence and the  $k_j$ -dependence of  $M(\underline{k}_1, e_1; ...; \underline{k}_m, e_m)$ . We are interested only in the singular (logarithmic)  $\lambda$ -dependence and we neglect terms in  $k_j$  leading to linear  $\Delta E$ -dependence. Then it is sufficient to consider only the coupling of soft photons to external electron lines.

Now let n soft photons (with momenta and polarizations  $\underline{k}_j$ ,  $\underline{e}_j$ ) be coupled to the ith external electron line (Fig. 1). Evaluation of this gives for  $\epsilon_i = 1$   $(\bar{u}(p_i)\hat{M}_i = \hat{M})$ 



$$(eQ_{i})^{n} \sum_{\substack{(i_{1},\dots,i_{n}) \\ (i_{1},\dots,i_{n}) \\ j = 1}} \overline{u}(p_{i}) \frac{\gamma^{\mu_{j_{1}}} (ip_{i} - m)\gamma^{\mu_{j_{2}}} (ip_{i} - m) \dots \gamma^{\mu_{j_{n}}} (ip_{i} - m)}{2k_{j_{1}} \cdot p_{i} [2(k_{j_{1}} + k_{j_{2}})p_{i}] \dots [2(k_{j_{1}} + \dots + k_{j_{n}})p_{i}]} \hat{M}_{i}$$

$$= \prod_{j=1}^{n} (ieQ_{i}p_{i}^{\mu_{j}}) \sum_{\substack{perm \\ (i_{1},\dots,i_{n}) \\ (i_{1},\dots,i_{n})}} \frac{1}{(k_{j_{1}} \cdot p_{i})[(k_{j_{1}} + k_{j_{2}})p_{i}] \dots [(k_{j_{1}} + \dots + k_{j_{n}})p_{i}]} \hat{M}$$

$$= \prod_{j=1}^{n} \left( \frac{ieQ_{i}p_{i}^{\mu_{j}}}{k_{j} \cdot p_{i}} \right) \hat{M}.$$
(7)

When  $\epsilon_i$  is included the result is

$$\prod_{j=1}^{n} \left( \frac{i e Q_i p_i^{\mu_j}}{k_j \cdot \epsilon_i p_i} \right) \hat{\mathbf{M}},$$
(8)

which means that the photons are emitted independently of each other by the ith external electron line and that the emission of the jth photon is (except for photon factors) connected with a factor

$$(ieQ_i p_i^{\mu_j})/(k_j \epsilon_i p_j).$$
(9)

Adding the possibilities for emission from all external electron lines we get the emission factor for one soft photon:

$$(2\pi)^{\frac{3}{2}} s^{\mu_j}(k_j), \quad k_j^2 \approx 0$$
, (10)

where

$$\mathbf{s}^{\mu}(\mathbf{k}) = \frac{ie}{(2\pi)^{3}} \sum_{i=1}^{r} \frac{2\mathbf{Q}_{i}\mathbf{p}_{i}^{\mu}}{\mathbf{k}^{2} + 2\mathbf{k}\cdot\epsilon_{i}\mathbf{p}_{i}} \quad .$$
(11)

When photon factors,  $i(2\pi)^{-\frac{3}{2}} e_{\mu} (2k_0)^{-\frac{1}{2}}$  for a real and  $i(2\pi)^{-4} (k^2 + \lambda^2)^{-1}$  for a virtual photon, are included we get the emission factor

$$(i/\sqrt{2k_{j0}})e_{j}s(k_{j})$$
(12)

for a real soft photon characterized by  $k_{\rm j},\ e_{\rm j},\ whereas$  a virtual soft photon gives a factor

$$-\frac{1}{2\pi i} \int \frac{d^4 k}{k^2 + \lambda^2} s(k) \cdot s(-k) .$$
(13)
soft
region

Summation over all possible numbers of soft virtual photons gives the result

$$\sum_{\ell=0}^{\infty} \frac{1}{\ell! 2^{\ell}} \left[ -\frac{1}{2\pi i} \int \frac{d^4 k}{k^2 + \lambda^2} s(k) s(-k) \right]^{\ell} = \exp \left[ -\frac{1}{4\pi i} \int \frac{d^4 k}{k^2 + \lambda^2} s(k) s(-k) \right].$$
(14)

The combinatorial factor  $(\ell!2^{\ell})^{-1}$  has the following origin. The virtual photons have been treated as if they were distinguishable and as if their emission and absorption were distinguishable processes. As a result of this in  $[\ldots]^{\ell}$  each diagram has been counted  $\ell! 2^{\ell}$  times.

The expression (14) contains the whole infrared contribution from virtual photons. If the integration is extended over the whole k-space then (14) changes only by a finite factor. Hence we can define the "non-infrared invariant amplitude"  $\overline{M}$  (finite when  $\lambda \rightarrow 0$ ) through

$$M = e^{-\frac{1}{2}A} \overline{M}$$
(15)

where

$$A = \frac{1}{2\pi i} \int \frac{d^4 k}{k^2 + \lambda^2} s(k) \cdot s(-k).$$
 (16)

The amplitude for emission of m soft photons  $M(\underline{k_1}, e_1; \ldots; \underline{k}_m, e_m)$  is obtained from M through multiplication by m emission factors (12):

$$\mathbf{M}(\underline{\mathbf{k}}_{1}, \mathbf{e}_{1}; \ldots; \underline{\mathbf{k}}_{m}, \mathbf{e}_{m}) = \prod_{j=1}^{m} \left( \frac{\mathrm{is}(\mathbf{k}_{j}) \cdot \mathbf{e}_{j}}{\sqrt{2\mathbf{k}_{j0}}} \right) \mathrm{e}^{-\frac{1}{2}A} \overline{\mathbf{M}} .$$
(17)

Using the sum rule  $\sum_{pol} e^{\mu}e^{\nu} = g^{\mu\nu}$  we get

$$\sum_{\text{pol}} \left| \mathbf{M}(\underline{\mathbf{k}}_{1}, \mathbf{e}_{1}; \ldots; \underline{\mathbf{k}}_{m}, \mathbf{e}_{m}) \right|^{2} = \prod_{j=1}^{m} \left[ \frac{1}{2k_{j0}} \mathbf{s}(\mathbf{k}_{j}) \cdot \mathbf{s}^{*}(\mathbf{k}_{j}) \right] e^{-\operatorname{Re}\left\{ A \right\}} \left| \overline{\mathbf{M}} \right|^{2}$$
(18)

which inserted in (5) gives (we introduce the Fourier transform for the  $\delta$ -function)

$$\begin{split} \mathbf{P}(\Delta \mathbf{E}) &= \sum_{m=0}^{\infty} \frac{1}{m!} \int d^{3} \mathbf{k}_{1} \dots d^{3} \mathbf{k}_{m} \int_{0}^{\Delta \mathbf{E}} d\mathbf{x} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\mathbf{y} e^{i\mathbf{y}(\sum_{j=1}^{M} k_{j0} - \mathbf{x})} \\ &\times \prod_{j=1}^{m} \left[ \frac{1}{2k_{j0}} \mathbf{s}(\mathbf{k}_{j}) \cdot \mathbf{s}^{*}(\mathbf{k}_{j}) \right] e^{-\operatorname{Re}\{A\}} \mid \overline{\mathbf{M}} \mid ^{2} \\ &= \frac{1}{2\pi} \int_{0}^{\Delta \mathbf{E}} d\mathbf{x} \int_{-\infty}^{\infty} d\mathbf{y} e^{-i\mathbf{y}\mathbf{x}} \sum_{m=0}^{\infty} \frac{1}{|\frac{\mathbf{k}}{\mathbf{k}}| \leq \Delta \mathbf{E}} \\ &\quad \mathbf{k}_{0} = \sqrt{\frac{\mathbf{k}}{\mathbf{k}}^{2} + \lambda^{2}} \end{split}$$

or

$$P(\Delta E) = be^{G(\Delta E)} | \overline{M} |^2$$

where

$$\mathbf{b} = \frac{1}{2\pi} \int_{0}^{\Delta E} d\mathbf{x} \int_{-\infty}^{\infty} d\mathbf{y} e^{-i\mathbf{y}\mathbf{x}} \exp\left[ \int_{\mathbf{k}_{0}^{\infty} |\underline{\mathbf{k}}| \le \Delta E} d\mathbf{x}_{0}^{2} \mathbf{s}(\mathbf{k}) \cdot \mathbf{s}^{*}(\mathbf{k}) (e^{i\mathbf{y}\mathbf{k}_{0}} - 1) \right]$$

$$\mathbf{G}(\Delta \mathbf{E}) = \lim_{\lambda \to 0} \left[ \int_{|\underline{\mathbf{k}}| \le \Delta E} d^{3} \mathbf{k} \mathbf{s}(\mathbf{k}) \cdot \mathbf{s}^{*}(\mathbf{k}) - \operatorname{Re} \{\mathbf{A}\} \right].$$

$$(20)$$

We have put  $\lambda = 0$  in b, because b is finite in this limit. If one substitutes  $x \to x \Delta E$ ;  $y \to y/\Delta E$ ;  $k \to k\Delta E$  and remembers  $s(k) \ll k_0^{-1}$  in the soft region then  $\Delta E$  becomes replaced by 1 in the expression for b. Hence b is independent of  $\Delta E$  and the whole  $\Delta E$ -dependence is contained in  $G(\Delta E)$ .

Details of the evaluation of b and  $G(\Delta E)$  can be obtained from Refs.[4] and [12]. Let us only state the results here:

b = 
$$e^{-\gamma G} / \Gamma(1 + G) = 1 - (\pi^2 / 12)G^2 + ...; \gamma$$
 = Euler's const.  
 $e^{-G(\Delta E)} = (\Delta E / g)^G e^B$ 
(21)

where

$$G = -\frac{\alpha}{\pi} \left[ r + \sum_{\substack{1 \le i < j \le r \\ 1 \le i < j \le r \\ }} \frac{\epsilon_i Q_i \epsilon_j Q_i}{G_{ij}} \ln\left(\frac{1 + G_{ij}}{1 - G_{ij}}\right) \right];$$

$$G_{ij} = \left[ 1 - \left(\frac{m^2}{p_i p_j}\right)^2 \right]^{\frac{1}{2}}$$

$$B = \frac{\alpha}{\pi} \left[ r + \frac{m^2}{q^2} \sum_{i=1}^r h\left(\frac{p_i}{q}\right) - 2\sum_{\substack{1 \le i < j \le r \\ 1 \le i < j \le r \\ }} Q_i Q_j \frac{p_i p_j}{q^2} \operatorname{Re} \left\{ \int_0^1 dxh\left(\frac{\epsilon_i p_i x - \epsilon_j p_j (1 - x)}{q}\right) \right\} \right]$$

$$h(p) = \frac{1}{p^2} \left[ \frac{1}{2} \ln\left(\frac{-p^2}{4}\right) + \int_0^1 dy \frac{p_0^2}{p_0^2 - p^2 y^2} \right]$$
(22)

(19)

We see that G is Lorentz-invariant and that the dependence on the coordinate system (in which  $\Delta E$  is isotropic) is contained entirely in B.

In the high-energy limit terms of order  $m^2/q^2$  can be neglected. Then

$$C = \frac{\alpha}{\pi} \left[ r \ln q^2 / m^2 - r - \sum_{1 \le i < j \le r} \epsilon_i Q_i \epsilon_j Q_j \ln \left(\frac{2p_i p_j}{q^2}\right)^2 \right]$$

$$B = \frac{\alpha}{\pi} \left[ r - 2 \sum_{1 \le i < j \le r} Q_i Q_j \frac{p_i p_j}{q^2} \operatorname{Re} \left\{ \int_0^1 dx \ln \left(\frac{\epsilon_i p_i x - \epsilon_j p_j (1 - x)}{q}\right) \right\} \right].$$
(23)

A word about renormalization: The virtual soft photons contribute only to the spurious charge renormalization. However, by a suitable gauge transformation, one can achieve Z = 1 for this renormalization, and a gauge transformation does not effect the interaction of the external line current  $s_{\mu}(k)$  with soft photons, since k s(k) vanishes for soft k (see Eq.(11)). The above discussion is therefore done in the renormalized theory.

(c) We are going to make use of the fact that QED is a renormalizable theory. We shall derive some equations that express the two kinds of charge renormalization in a convenient way. Let us use q as momentum variable and  $\lambda$  as normalization momentum ( $\lambda^2 > 0$ , space-like<sup>\*</sup>). We shall consider the function d<sub>0</sub> (q<sup>2</sup>/m<sup>2</sup>, e<sup>2</sup><sub>0</sub>) which occurs in the photon propagation function (e<sub>0</sub> is the observed electron charge)

$$D_{\mu\nu}(q) = \frac{d_0(q^2/m^2, e_0^2)}{q^2 - i\epsilon} \left(g_{\mu\nu} - \frac{q_{\mu}q_{\nu}}{q^2}\right), \quad (24)$$

and an unspecified renormalization function  $X_0(q^2/m^2, e_0^2)$ . Let us introduce the general functions  $d(q^2/\lambda^2, m^2/\lambda^2, e^2)$  and  $X(q^2/\lambda^2, m^2/\lambda^2, e^2)$  normalized in the following way:

$$d(1, m^2/\lambda^2, e^2) = 1$$
,  
X(1, m^2/\lambda^2, e^2) = 1. (25)

The ordinary functions  $d_0$  and  $X_0$  are obtained in the limit of zero  $\lambda^2$  :

$$\begin{split} \lim_{\lambda^2 \to 0} d(q^2/\lambda^2, \ m^2/\lambda^2, \ e^2) &= d_0(q^2/m^2, \ e^2_0), \\ \lim_{\lambda^2 \to 0} X(q^2/\lambda^2, \ m^2/\lambda^2, \ e^2) &= X_0(q^2/m^2, \ e^2_0). \end{split}$$
(26)

Under renormalization, d,  $e^2$  and X have the following transformation properties:

\* This condition ensures real charge renormalization (see Eq. (29)).

$$d(q^{2}/\lambda_{2}^{2}, m^{2}/\lambda_{2}^{2}, e_{2}^{2}) = Z_{3} d(q^{2}/\lambda_{1}^{2}, m^{2}/\lambda_{1}^{2}, e_{1}^{2}),$$

$$e_{2}^{2} = Z_{3}^{-1} e_{1}^{2},$$

$$X(q^{2}/\lambda_{2}^{2}, m^{2}/\lambda_{2}^{2}, e_{2}^{2}) = Z' X(q^{2}/\lambda_{1}^{2}, m^{2}/\lambda_{1}^{2}, e_{1}^{2})$$
(27)

where the renormalization constants 
$$Z_3$$
 and  $Z'$  do not depend on  $q^2$ , and  $Z'$  is a combination of  $Z_3$  and  $Z$  of the type  $Z_3^a Z^b$  or  $Z_3^a |Z|^b$ .

We can eliminate  $Z_3$  and Z' in (27) and use (25) to obtain the functional equations:

$$d(q^{2}/\lambda_{2}^{2}, m^{2}/\lambda_{2}^{2}, e_{2}^{2}) = \frac{d(q^{2}/\lambda_{1}^{2}, m^{2}/\lambda_{1}^{2}, e_{1}^{2})}{d(\lambda_{2}^{2}/\lambda_{1}^{2}, m^{2}/\lambda_{1}^{2}, e_{1}^{2})},$$

$$e_{2}^{2} = e_{1}^{2}d(\lambda_{2}^{2}/\lambda_{1}^{2}, m^{2}/\lambda_{1}^{2}, e_{1}^{2}),$$

$$X(q^{2}/\lambda_{2}^{2}, m^{2}/\lambda_{2}^{2}, e_{2}^{2}) = \frac{X(q^{2}/\lambda_{1}^{2}, m^{2}/\lambda_{1}^{2}, e_{1}^{2})}{X(\lambda_{2}^{2}/\lambda_{1}^{2}, m^{2}/\lambda_{1}^{2}, e_{1}^{2})}.$$
(28)

In the limit  $\lambda_1^2 \to 0$  with  $q^2 \,/ \lambda_2^2$  = x,  $m^2 \,/ \lambda_2^2$  = y and  $e_2^2$  =  $e^2$  we obtain

$$d(x, y, e^{2}) = \frac{d_{0}(x/y, e^{2}_{0})}{d_{0}(1/y, e^{2}_{0})},$$

$$e^{2} = e^{2}_{0}d(1/y, e^{2}_{0}),$$

$$X(x, y, e^{2}) = \frac{X_{0}(x/y, e^{2}_{0})}{X_{0}(1/y, e^{2}_{0})}.$$
(29)

From (28) the following Lie differential equations are obtained:

$$\frac{\partial d(x, y, e^2)}{\partial x} = \frac{d(x, y, e^2)}{x} \left[ \frac{\partial}{\partial \xi} d(\xi, y/x, e^2 d(x, y, e^2)) \right]_{\xi = 1} (30)$$

$$\frac{\partial \ln X(x, y, e^2)}{\partial x} = \frac{1}{x} \left[ \frac{\partial}{\partial \xi} \ln X(\xi, y/x, e^2 d(x, y, e^2)) \right]_{\xi = 1} .$$

The first Eq.(30) is obtained by substituting in (28)  $q^2/\lambda_1^2 = x$ ,  $m^2/\lambda_1^2 = y$ ,  $\lambda_2^2/\lambda_1^2 = t$  and  $e_1^2 = e^2$ . In the resulting equation

$$d(x, y, e^2) = d(t, y, e^2)d(\frac{x}{t}, \frac{y}{t}, e^2d(t, y, e^2))$$

one then takes the partial derivative with respect to x and sets t = x. The first Eq.(30) results. The second equation is obtained in the same way.

Inserting (29) in (30) and changing x/y into x, we get

$$\frac{\partial d_0(x, e_0^2)}{\partial x} = \frac{d_0(x, e_0^2)}{x} \left[ \frac{\partial}{\partial \xi} d(\xi, 1/x, e_0^2 d_0(x, e_0^2)) \right]_{\xi=1}, \quad (31)$$

$$\frac{\partial \ln X_0(x, e_0^2)}{\partial x} = \frac{1}{x} \left[ \frac{\partial}{\partial \xi} \ln X(\xi, 1/x, e_0^2 d_0(x, e_0^2)) \right]_{\xi=1}.$$

### K. E. ERICSSON

(d) From the concluding comment in (a) it follows that  $d(x, y, e^2)$  is independent of y for  $x \ge 1 >> y$ , i.e.  $q^2 \ge \lambda^2 >> m^2$ . The corresponding is true also for X(x, y,  $e^2$ ) provided the perturbation expansion of X<sub>0</sub>  $(q^2/m^2, e_0^2)$  for large  $q^2/m^2$  can be written in the form\*

$$X_{0}(q^{2}/m^{2}, e_{0}^{2}) = \sum_{\substack{i,j=0\\i$$

i.e. provided that

$$\frac{\alpha}{\pi} \ln \frac{q^2}{m^2} \tag{33}$$

is the effective expansion parameter \*\* of  $X_0(q^2/m^2, e^2)$ . The proof that  $X(x, y, e^2)$  ceases to depend on y for  $x \ge 1 >> y$  when (32) is fulfilled will be omitted here but will be given in a later paper [31].

Now (29) and (31) can be simplified by omission of the second variable in d and X and we get

$$d(\mathbf{x}, \mathbf{e}_{0}^{2}) = \frac{d_{0}(\mathbf{x}/\mathbf{y}, \mathbf{e}_{0}^{2})}{d_{0}(1/\mathbf{y}, \mathbf{e}_{0}^{2})} ,$$

$$\mathbf{e}^{2} = \mathbf{e}_{0}^{2} d_{0}(1/\mathbf{y}, \mathbf{e}_{0}^{2}),$$

$$X(\mathbf{x}, \mathbf{e}^{2}) = \frac{X_{0}(\mathbf{x}/\mathbf{y}, \mathbf{e}_{0}^{2})}{X_{0}(1/\mathbf{y}, \mathbf{e}_{0}^{2})} ,$$
(34)

and

$$\frac{\partial d_{0}(x, e_{0}^{2})}{\partial x} = \frac{d_{0}(x, e_{0}^{2})}{x} \varphi(e_{0}^{2} d_{0}(x, e_{0}^{2})),$$

$$\frac{\partial \ln X_{0}(x, e_{0}^{2})}{\partial x} = \frac{1}{x} \psi(e_{0}^{2} d_{0}(x, e_{0}^{2})),$$
(35)

where

$$\varphi(z) = \left[\frac{\partial}{\partial \xi} d(\xi, z)\right]_{\xi=1},$$
  

$$\psi(z) = \left[\frac{\partial}{\partial \xi} \ln X(\xi, z)\right]_{\xi=1}.$$
(36)

For constant  $e_0^2$  we can write (35) in the form

<sup>\*</sup> Feynman diagrams lead to Spence functions which for large arguments behave like integer powers of logarithms.

**<sup>\*\*</sup>** For a discussion of the expansion parameter in QED see Ref.[11]. If Nakanishi's result mentioned in (a) is applied one obtains (33) instead of  $(\alpha/n) \ln^{4/3} (q^2/m^2)$ .

$$d \ln x = \frac{dz}{z\varphi(z)} ,$$

$$z = e_0^2 d_0 (x, e_0^2), \qquad (37)$$

$$d \ln X_0 (x, e_0^2) = \frac{dz}{z\varphi(z)} \psi(z)$$

and integrate

$$\ln x = \int_{e_0^2 d_0(1, e_0^2)}^{e_0^2 d_0(x, e_0^2)} \frac{dz}{z\varphi(z)} ,$$

$$\ln X_0(x, e_0^2) = \ln X_0(1, e_0^2) + \int_{e_0^2 d_0(1, e_0^2)}^{e_0^2 d_0(x, e_0^2)} \frac{dz}{z\varphi(z)} \psi(z) ,$$
(38)

which, according to (34), gives for d and X

.

$$\ln x = \int_{e^2}^{e^2 d(x, e^2)} \frac{dz}{z\varphi(z)} ,$$

$$\ln X(x, e^2) = \int_{e^2}^{e^2 d(x, e^2)} \frac{dz}{z\varphi(z)} \psi(z) .$$
(39)

If the following perturbation expansions are used for the integrands in (38)

$$\frac{1}{z\phi(z)} = \frac{1}{z^2} \sum_{i=0}^{\infty} \alpha_i z^i ,$$

$$\frac{\psi(z)}{z\phi(z)} = \frac{1}{z} \sum_{i=0}^{\infty} \beta_i z^i ,$$
(40)

the integration yields

$$\ln x = -\alpha_0 e_0^{-2} [(d(x))^{-1} - (d(1))^{-1}] + \alpha_1 [\ln d(x) - \ln d(1)] + \sum_{n=1}^{\infty} \frac{\alpha_{n+1}}{n} e_0^{2n} [(d(x))^n - (d(1))^n] , \qquad (41)$$
$$\ln X(x) = \ln X(1) + \beta_0 [\ln d(x) - \ln d(1)] + \sum_{n=1}^{\infty} \frac{\beta_n}{n} e_0^{2n} [(d(x))^n - (d(1))^n],$$

where x>> 1 and  $e_0^2\,\ln x \ll 4\pi^2\,.$ 

Here the notation d(x) and X(x) has been used instead of  $d_0(x, e_0^2)$  and  $X_0(x, e_0^2)$ . The formulae (41) give us improved perturbation expansions

$$d(q^{2}/m^{2}) = \sum_{n=0}^{\infty} e_{0}^{2n} d^{(n)} (e_{0}^{2} \ln q^{2}/m^{2}),$$
  

$$X(q^{2}/m^{2}) = \sum_{n=0}^{\infty} e_{0}^{2n} X^{(n)} (e_{0}^{2} \ln q^{2}/m^{2})$$
(42)

with an effective expansion parameter of order  $\alpha$  instead of  $(\alpha/\pi)\ln(q^2/m^2)$ . The calculations of radiative corrections for large  $q^2/m^2$  are now reduced to the determination of the constants  $\alpha_0$ ,  $\alpha_1$ , ..., and  $\beta_0$ ,  $\beta_1$ , ..., that appeared in (40). The knowledge of  $\alpha_0$ ,  $\alpha_1$ , ...,  $\alpha_n$  and  $\beta_0$ ,  $\beta_1$ , ...,  $\beta_n$  is sufficient for the determination of  $d^{(0)}$ ,  $d^{(1)}$ , ...,  $d^{(n)}$  and  $X^{(0)}$ ,  $X^{(1)}$ , ...,  $X^{(n)}$ .

In practice, when  $X(q^2/m^2)$  is the S matrix element for a process with high momentum transfer  $q^2$ , one usually knows only the first order radiative corrections, i.e. one knows only  $\beta_0$ . So one can derive only  $X_0(e_0^2 \ln q^2/m^2)$ . From (41) we get if we approximate d(1) by 1

$$d^{(0)} (q^2/m^2) = (1 - \frac{1}{\alpha_0} e_0^2 \ln \frac{q^2}{m^2})^{-1} ,$$

$$X^{(0)} (q^2/m^2) = X(1)(1 - \frac{1}{\alpha_0} e_0^2 \ln \frac{q^2}{m^2})^{-\theta_0} .$$
(43)

We have thus performed a summation over the "maximally logarithmic terms". The expression for  $d^{(0)}(q^2/m^2)$  was obtained in this way by BOGOLIUBOV and SHIRKOV [8].

(e) The summation procedure of (b) and (d) can now be combined. We assume that in the c.m.s. the conditions

$$m << \Delta E << q$$

$$\frac{\Delta p(\theta, \phi)}{\Delta E} \approx 1$$
(44)

are fulfilled, where  $\Delta p(\theta, \phi)$  is the momentum resolution in the direction  $(\theta, \phi)$ . Let  $\sigma_0$  be the cross-section to lowest order in  $\alpha$  of a process satisfying (44), and  $\sigma_1$  the cross-section with first order radiative corrections included:

$$\sigma_1 = \sigma_0 \left\{ 1 + \frac{\alpha}{\pi} \left[ \left( c_1 \ln \frac{q^2}{m^2} + c_2 \right) \ln \frac{\Delta E}{q} + c_3 \ln \frac{q^2}{m^2} + c_4 + 0 \left( \frac{\Delta E}{q} \right) \right] \right\}.$$
(45)

Here  $c_1$ ,  $c_2$ ,  $c_3$  and  $c_4$  are quantities of order unity. They are in general functions of  $\Delta p/\Delta E$  and  $p_i/q$  ( $p_i$  being the momenta of the particles participating in the process).

We take X to be the ratio between the exact cross-section and the crosssection to lowest order in  $\alpha$ :

$$X = \sigma / \sigma_0 .. \tag{46}$$

Combining the summation over soft photon effects with the summation over maximally logarithmic terms and inserting the value of  $\alpha_0$ ,

$$\alpha_0 = 12\pi^2$$
, (47)

we arrive at the following formula:

$$\sigma = \sigma_0 \left[ \left( 1 + \frac{\alpha}{\pi} c_4 \right) \exp\left\{ \left[ \frac{\alpha}{\pi} \left( c_1 \ln \frac{q^2}{m^2} + c_2 \right) - 3c_1 \left( \ln \left( 1 - \frac{\alpha}{3\pi} \ln \frac{q^2}{m^2} \right) \right) + \frac{\alpha}{3\pi} \ln \frac{q^2}{m^2} \right) \right] \ln \frac{\Delta E}{q} - 3c_3 \ln \left( 1 - \frac{\alpha}{3\pi} \ln \frac{q^2}{m^2} \right) \right\} + 0 \left( \frac{\alpha}{\pi} \frac{\Delta E}{q} \right) + 0 \left( \frac{\alpha^2}{\pi^2} \ln \frac{q^2}{m^2} \right) \right].$$

$$(48)$$

This formula also contains the combined emission of photons and electron pairs with a total energy less than  $\Delta E$ . For details of derivation and interpretation of (48) see Ref.[13].

One interesting consequence of the summation procedure in (d) is that maximally logarithmic terms in radiative corrections to electromagnetic scattering cross-sections appear only in one-photon exchange. Calculations of first order radiative corrections show that only vertex diagrams and vacuum polarization diagrams contribute terms of the type  $(\alpha/\pi)\ln(q^2/m^2)$ . Thus in terms of diagrams we have (Fig. 2) with logarithmic accuracy to order  $\alpha^2$  (we can use an infrared cut-off which gets replaced by  $\Delta E$  in the expression for the cross-section), i.e. within this accuracy only one-photon



exchange contributes. But if a summation over maximally logarithmic terms is performed according to (43), it is clear that Fig. 2 still holds after this summation. Thus the correction because of the exchange of n photons with  $n \ge 2$  does not contribute any maximally logarithmic terms. It is at most of the relative order of magnitude

$$(\alpha/\pi)[\alpha/\pi)\ln(q^2/m^2)]^{n-2}$$
 (49)

One has, however, to be careful in the neighbourhood of maximal momentum transfer (backward scattering in the c.m.s.) because in this region the twophoton exchange contributes terms which in the limit m = 0 have a singular angle dependence.

#### REFERENCES

- [1] SCHWINGER, J., Phys. Rev. 76 (1949) 790.
- [2] YENNIE, D. R. and SUURA, H., Phys. Rev. 105 (1957) 1378.
- [3] LOMON, E. L., Phys. Rev. 113 (1959) 726.

#### K. E. ERICSSON

- [4] YENNIE, D. R., FRAUTSCHI, S. C. and SUURA, H., Ann. Phys. 13 (1961) 379.
- [5] NAKANISHI, N., Prog. theor. Phys. 19 (1958) 159.
- [6] STUECKELBERG, E. C. G. and PETERMAN, A., Helv. Phys. Acta <u>24</u> (1951) 317; Helv. Phys. Acta <u>26</u> (1953) 499.
- [7] GELL-MANN, M. and LOW, F.E., Phys. Rev. 95 (1954) 1300.
- [8] BOGOLIUBOV, N. N. and SHIRKOV, D. V., Introduction to the theory of quantized fields, New York and London (1959) sections 42 - 44.
- [9] TSAI, Y.S., Phys. Rev. 120 (1960) 269.
- [10] ERIKSSON, K.E., Nuovo Cim. 19 (1961) 1029.
- [11] ERIKSSON, K.E. and PETERMAN, A., Phys. Rev. Letters 5 (1960) 444.
- [12] ERIKSSON, K.E., Nuovo Cim. 19 (1961) 1010.
- [13] ERIKSSON, K.E., Nuovo Cim., to be published.

# SOFT-PHOTON CONTRIBUTION TO ELECTRO-DYNAMICAL CROSS-SECTIONS AT VERY HIGH ENERGIES

## G. FURLAN UNIVERSITY OF TRIESTE, TRIESTE, ITALY

The aim of this paper is to discuss some characteristic features of the infrared divergence phenomena such as the soft photon contribution to the radiative corrections in the domain of very high energies. The discussion will procede as follows: a typical electrodynamic process such as the electron-positron annihilation into photons will be considered and the results of a complete perturbation calculation of order  $\alpha^3$  will be discussed briefly. Then an attempt will be made to generalize the rather interesting suggestions deriving from that calculation to every order of  $\alpha$ .

ERIKSSON, in his lecture [5], has explained what is meant by infrared divergence and how the soft photon contribution works in its elimination. Let us briefly recall and apply those considerations to the annihilation process.

Let us try to calculate the higher order corrections in  $\alpha$  to the Born approximation represented by the simple graph in Fig. 1.



Fig.1

As is well known, the relevant corrections are obtained by inserting one internal photon line into the above skeleton graph in all possible ways. This situation is shown in Fig.2 in terms of Feynmann's diagrams. (Obviously there are as many as graphs deriving from the exchange of the two final photons  $k_1 \leftrightarrow k_2$ .)

All these terms diverge when we integrate over the energy of the virtual photon down to the limit zero and we find something like this:

In order to avoid this divergence we regularize the above integrals by ascribing a fictious, non-zero mass  $\Lambda$  to the photon so that we finally get the  $\Lambda$ -dependent (but Lorentz-invariant) result:

$$d\sigma_{\nu}^{(\alpha^{3})} = d\sigma_{0}^{(\alpha^{2})} \left\{ 1 + (\alpha/\pi) \left[ F(\gamma) \ln (\Lambda/m) + f_{1}(\gamma, \theta) \right] \right\}$$
(1)

where  $d\sigma_{\nu}^{(\alpha^3)}$  is the differential cross-section for the annihilation process corrected by virtual photons and  $d\sigma_0^{(\alpha^2)}$  is the corresponding quantity calculated in Born approximation  $\gamma = \epsilon_+/m$ , and  $\theta$  is the scattering angle. Note



the isotropic coefficient  $F(\gamma)$  of the  $\Lambda$  dependent part. This fact can be easily understood by direct inspection of the Feynmann's graphs but we will not go into the details here.

Equation (1) is not a physically meaningful result owing to the  $\Lambda$  dependence. In order to remove it, the consideration is made that a scattering process (in a broad sense) can be never considered as purely elastic. In our case, for instance, we cannot ensure experimentally that the annihilation of the pair leads to a final state with two photons only. In fact there is always an inelastic contribution because of the emission of some supplementary real photons. They are not detected if their total energy is less than an upper limit  $\Delta E$  which can be taken as the resolving power of the experiment. However, although not experimentally distinguishable from the fundamental process, this sort of background effect cannot be omitted. So we have to add the cross-section for the emission of one additive photon (to  $\alpha_3$  order), with an energy under the threshold of detection  $\Delta E$ , to Eq. (1). It is possible to eliminate the infrared divergence, taking into account that the emission probability of soft photons diverges in the limit  $\omega \rightarrow 0$ . Usually the procedure followed in the calculation is to consider the inelastic contribution from real soft photons whose energy is less than a given quantity  $\Delta \ll m$ , in a particular reference frame.

In this way we get the cross-section for annihilation into three photons, one of which with energy  $\leq \Delta \ll m$ 

$$d\sigma_{\text{soft}}^{(\alpha^3)} = d\sigma_0^{(\alpha^2)} (\alpha/\pi) [F(\gamma) \ln(2\Delta/\Lambda) + \varphi(\gamma)]$$
$$= d\sigma_0^{(\alpha^2)} (\alpha/\pi) [F(\gamma) \ln(\Delta/\gamma\Lambda) + \varphi(\gamma)].$$
(2)

In order to calculate Eq. (2) we have taken into account only the infrared part of the above cross-section that corresponds to the soft photon emission from external electron lines only. This causes the appearing of a contribution only  $\gamma$  dependent.

In the C.M.S. system we get

$$\varphi(\gamma) = 2 \ln^2 (2\gamma) - \pi^2/3,$$
  
 $F(\gamma) = 2[2 \ln(2\gamma) - 1].$ 
(3)

By combination of Eqs. (1) and (2) we get the  $\Lambda$  independent result

SOFT-PHOTON CONTRIBUTION

$$d\sigma_{v+s}^{(\alpha^3)} = d\sigma_0 \left\{ 1 + \frac{\alpha}{\pi} \left[ \phi(\gamma) + F(\gamma) \ln \frac{\Delta}{m\gamma} + f_1(\gamma, \theta) \right] \right\}.$$
 (4)

While the introduction of soft photons only is sufficient to cancel out the infrared divergence it is not entirely realistic. In fact at high energies the actual resolving power  $\Delta E$  of the experimental device does not fulfill the condition  $\Delta E \ll m$  and generally has an angular dependence. So in order to compare the theoretical calculations with the experimental results one ought also to take into account the photons (not detected) in the range  $\Delta \rightarrow \Delta E$ . They are usually called hard photons. If we want a result independent of  $\Delta E$  we can go to the limit of no resolving power by allowing the energy of the additive photon to reach the maximum value given by the conservation laws. In so doing the total correction is the sum of the virtual plus the inelastic part and we might call it a radiative correction in a broader sense.

Going back to our particular process this pattern of thought gives as a final result the cross-section for annihilation of a pair into two and three photons up to  $\alpha^3$  order.

The cross-section for annihilation of a pair into three hard photons (with energies larger than  $\Delta$ ) is

$$d\sigma_{hard}^{(\alpha^3)} = d\sigma_0(\alpha/\pi) [G(\gamma) \ln (m\gamma/\Delta) + g_1(\gamma, \theta)] .$$
 (5)

It is verified that  $G(\gamma) = F(\gamma)$  so the combination of Eqs. (5) and (4) gives

$$d\sigma^{(\alpha^{s})} = d\sigma_{0} \{1 + (\alpha/\pi) [\varphi(\gamma) + f_{1}(\gamma, \theta) + g_{1}(\gamma, \theta)]\}$$
$$= d\sigma_{0} \{1 + \delta(\gamma, \theta)\}.$$
(6)

In Eq. (6)  $\varphi$ ,  $g_1$ ,  $f_1$  represent the soft photon, the virtual photon and the hard photon contribution to the total correction  $\delta$ , respectively. (Really this division is rather arbitrary and not unique depending on the used regularization procedure and not invariant owing to the not covariant definition of soft photons.)

For computational reasons it is simpler to discuss the integrated correction  $\delta(\gamma) = \int d\Omega \delta(\gamma, \theta)$ , or the total cross-section for annihilation into photons up to  $\alpha^3$  order.

$$\sigma = \sigma_0 \left\{ 1 + (\alpha/\pi) \left[ \varphi(\gamma) + f(\gamma) + g(\gamma) \right] \right\} = \sigma_0 \left\{ 1 + \delta(\gamma) \right\}.$$
(7)

In this case it is possible to write the final result in the extreme relativistic case

$$\sigma_0 = (\pi r_0^2 / 2\gamma^2) [2 \ln(2\gamma) - 1].$$
(8)

$$\delta(\gamma) = (\alpha/12\pi) \left[ 8\ln^2(2\gamma) - 2\ln(2\gamma) + 4\pi^2 - 13 + \frac{11 - 5\pi^2}{2\ln(2\gamma) - 1} \right], \tag{9}$$

 $\gamma \gg 1$  (C.M.S. energy).



Fig. 3

The relevant point in Eq. (9) is the ln  $(2\gamma)$  behaviour of the total correction. If in order to understand its origin we perform the usual division between virtual soft and hard photon contribution, we find in the C.M.S.:

$$\sigma_{\mathbf{v}+\Delta} = \sigma_0 \{1 + (\alpha/12\pi) [8\ln^2(2\gamma) + 22\ln(2\gamma) + \ldots + \mathbf{F}(\gamma)\ln(\Delta/m\gamma)]\}$$
  
$$\sigma_{\text{hard}} = \sigma_0(\alpha/12\pi) [-24\ln(2\gamma) + \ldots + \mathbf{F}(\gamma)\ln(m\gamma/\Delta)],$$

so that we can ascribe the  $\ln^2(2\gamma)$  contribution to the virtual plus soft photon part. Hard photons do not contribute to the dominant behaviour in the very high energy case, so that, from this point of view, we can forget their existence. To get a further insight into the  $\ln^2(2\gamma)$  derivation, let us go back to the differential cross-section of Eq. (4):

$$d\sigma_{v+s} = d\sigma_0 \{1 + \delta_{v+s} (\gamma, \theta, \Delta)\}.$$

Some general previsions can be made on the high energy behaviour of  $\delta_{v+s}$ . A quite general theorem by ERIKSSON and PETERMANN [2] states that for large values of the momentum transfer (in the C. M. S.),  $q^2 \gg m^2$ ,  $\delta_{v+s}$ behaves at most like  $(\alpha/\pi)\ln(q^2/m^2)$  to order  $\alpha^n$ . More precisely, the first correction can be written in the form

$$\delta_{v+s} = (\alpha/\pi) \left[ c_1 \ln(q^2/m^2) \ln(\Delta/E) + c_2 \ln(q^2/m^2) + c_3 \ln(\Delta/E) + c_n \right].$$

The validity of the Petermann theorem can also be verified in our case. This means that, being the soft photon part isotropic and always  $\approx \ln^2(2\gamma)$ , the contribution of the virtual photon for large momentum transfer is such as to compensate those  $\ln^2(2\gamma)$  terms. Let us look at another boundary condition, the region of small momentum transfer, where

$$q^2 = (p_1 - k_1)^2 = -m^2 + 2m^2\gamma^2 (1 - \beta \cos \theta) \sim m^2 \gamma^2 \theta$$

or

$$q^2 \leq m^2$$
,  $\theta \leq 1/\gamma$ .

In this situation we find that the virtual photon contribution is small (no  $\ln^2$  (2  $\gamma$ ) terms) so that there is no more compensation and the  $\ln^2$  (2  $\gamma$ ) from the soft part is still present and dominant. The subsequent integration and the addition of the hard photons do not cancel the  $\ln^2$  (2  $\gamma$ ).

Obviously these are considerations whose validity is limited to the explicit  $\alpha^3$  calculation. Let us try to generalize those results: let us consider, for instance, the  $\alpha^n$  situation (cross-section to  $\alpha^{n+2}$  order). The correction



Fig. 4

is the result of n virtual photons, n - 1 virtual + one soft, .... n soft photons. We can make the following assumption based on the ideas of our previous calculation: if we limit ourselves to the small momentum transfer region (in S. C. M.) the virtual photon contribution is small, i.e. not of the  $\alpha^n \ln^{2n} (2 \gamma)$  type but at most of  $\alpha^n \ln^{2n-1} (2 \gamma)$  type. So the relevant term in the asymptotic limit comes practically from the graph with n soft photons. In this way the result is obtained at one and looks like

$$d\sigma^{(n+2)} \simeq d\sigma_0(\alpha/\pi)^n (1/m!) [\varphi(\gamma) + F(\gamma) \ln (\Delta/m\gamma)]^n$$
(10)

corresponding to the graph in Fig. 3.

If we sum up, using Eq. (3):

$$d\sigma \sim d\sigma \sum_{0}^{\infty} (\alpha/\pi)^{n} (1/n!) [\varphi(\gamma) + F(\gamma) \ln (\Delta/m\gamma)]^{n}$$
  
= 
$$d\sigma_{0} e^{\frac{\alpha}{\pi} \varphi(\gamma)} e^{(\alpha/\pi) F(\gamma) \ln \Delta/m\gamma}$$
  
= 
$$d\sigma_{0} e^{\frac{2\alpha}{\pi} \ell n^{2}(2\gamma)} (\Delta/m\gamma) (4\alpha/\pi) \ln (2\gamma)$$
(11)

This is only s rough evaluation but it can give an idea of the philosophy we shall follow. Obviously a more rigorous derivation is possible.

To this purpose we will use in a slightly modified form the general result by ERIKSSON [3]. That is

$$P(\Delta) = \frac{e^{-\gamma c}}{\Gamma(1+c)} \quad \left(\frac{\Delta}{m\gamma}\right)^{c} e^{B+A} \left(\frac{m}{\Lambda}\right)^{c} |M|^{2}$$
(12)

where the symbols are those of Eriksson except A which is defined as

$$\mathbf{A} = \frac{\alpha}{\pi} \left\{ -\sum_{i} \mathbf{Q}_{i}^{2} + 2\sum_{i < j} \operatorname{Re} \int_{0}^{1} g[\mathbf{x}(\mathbf{p}_{i} \boldsymbol{\epsilon}_{i} + \mathbf{p}_{j} \boldsymbol{\epsilon}_{j}) - \boldsymbol{\epsilon}_{j} \mathbf{p}_{j}] \, \mathrm{dx} \right\}$$

(for a full understanding of the notations see [3].

 $|\mathbf{M}|^2$  is the squared matrix element corrected by all the virtual photons. It is infrared divergent but we can regularize it with the fictitious photon mass  $\Delta$ . Furthermore, the general theory of infrared divergences allows us to say that the dependence of  $\Lambda$  cancel out with  $(\mathbf{m}/\Lambda)^c$ . Eq. (12) is a correct result, though a not covariant one, owing to the presence of  $\Lambda$ . Let us consider the annihilation case where an explicit calculation gives:

A = 
$$(2\alpha/\pi) \ln^2(2\gamma)$$
,  
B =  $(4\alpha/\pi) \ln (2\gamma)$ , (13)  
C =  $(4\alpha/\pi) \ln (2\gamma)$ .

Note again the characteristic isotropy of the soft photon part for the annihilation process. Thus the angular dependence is contained in  $|\mathbf{M}|^2$ . If we assume the validity of the Eriksson and Petermann theorem for  $q^2 \gg m^2$   $(\theta \sim \pi/z)$ ,  $|\mathbf{M}|^2$  has to behave like  $e^{-2\alpha/\pi \ln^2(2\gamma)}$  in order to compensate the soft photon part. What happens for  $q^2 \sim 0$ ? If we put

$$(m/\Lambda)^{c} |M|^{2} = |M_{0}|^{2} \chi (p_{i}, k_{i}),$$

our assumption is that for  $q^2 \rightarrow 0$ , X (pi, ki) goes to a constant or, more

Fig. 5

generally, it behaves in a simple logarithmic and not a  $(\log)^2$  manner. In this way, neglecting all the logarithmic terms, the result is

$$d\sigma \sim d\sigma_0 e^{(2\alpha/\pi) \ln^2 (2\gamma) (\Delta/m\gamma) (4\alpha/\pi) \ln(2\gamma)}$$

which is Eq. (11). So our general conclusion is that it is possible in forward annihilation to discriminate in a very clear out way between the virtual and soft photon contribution. Really this result may seem a rather academic one owing to the  $\Delta$  term. So our perturbative result could lead us to another hypothesis, i.e. that the hard photons do not contribute  $\ln^2 (2\gamma)$  terms. Consequently we eliminate the  $\Delta$  dependence by adding the hard photons while the dominant behaviour remains still

$$d\sigma \sim d\sigma_0 e^{(2\alpha/\pi) \ln^2(2\gamma)}$$
(14)

with  $d\sigma$  forward (or nearly forward) cross-section for annihilation into photons.

It is necessary to find out if our hypotheses are verified. The problem is not difficult for the virtual photon contribution. We all know that there are two classes of diagrams which contribute to M. They can be represented by the reducible graphs typified by Fig. 4 and the irreducible ones typified by Fig. 5. Though many photons exchange, as in Fig. 6.



As long as we confine ourselves to the logarithmic terms, it is possible to show by direct inspection<sup>(\*)</sup> of the matrix elements to every order of  $\alpha$ , that our first hypothesis works very well for both classes, that is

 $|\mathbf{M}|^2 (\mathbf{m}/\Lambda)^c \xrightarrow[q^2 \to 0]{} |\mathbf{M}_0|^2.$ 

So Eq. (11) is correct.

The hard photon part is more complicated to hand. The problem is under study and we hope to be able to prove that Eq. (14) is also true soon.

#### REFERENCES

- [1] ANDREASSI, G., CALUCCI, G., CAZZOLA, P., FURLAN, G. and PERESSUTTI, G., to be published in Phys. Rev.
- [2] ERIKSSON, K.E. and PETERMAN, A., Phys. Rev. Letters 5 (1960) 444
- [3] ERIKSSON, K.E., Nuovo Cim. 19 (1961) 1010.
- [4] SUDAKOV, V.V., JETP 3, (1956) 65.
- [5] ERIKSSON, K.E., these proceedings.

<sup>\*</sup> By using for instance, the method of SUDAKOV [4].

# INFRARED DIVERGENCE PHENOMENA, HIGH-ENERGY PROCESSES, AND REGGE POLES

## S. FRAUTSCHI NEWMAN LABORATORY OF NUCLEAR STUDIES, CORNELL UNIVERSITY, ITHACA, N.Y., UNITED STATES OF AMERICA

### 1. INTRODUCTION

Infrared divergence phenomena are already well known from semiclassical arguments. For example, suppose an electron in motion is deflected due to its interaction with a potential. The Lorentz-contracted proper field of the electron will be altered by the collision, and the change in the proper field will be emitted as electromagnetic radiation. For sufficiently long wavelengths (kR < < 1, where k is the wave number and R is a dimension of the scattering region), the radiation can be calculated without knowledge of the details of the trajectory in the scattering region. It depends only on the initial and final momenta of the electron and the direction in which the radiation is observed (assuming the electron suffers no time delay in the scattering region). As is well known, the energy emitted per unit frequency is constant in this limit. Making the transcription to the photon description, it is clear that the number of photons emitted per unit frequency range is inversely proportional to the frequency; i.e., the photon spectrum is of the form dk/k, which diverges as  $k \rightarrow 0$ . This is the infrared divergence for real photons.

The angular distribution can also be understood by the semiclassical argument. In the extreme relativistic limit, the proper fields will be Lorentz-contracted in a small region near the plane perpendicular to the direction of motion of the charge and moving along with the charge. This leads to a strong peaking of the radiation parallel either to the incident or final direction of motion.

The essential idea for understanding the problems posed by infrared divergences was introduced by BLOCK and NORDSIECK [1] in 1937. They pointed out that in any practical experiment involving charged particles it is impossible to specify completely the final state of the system. Because individual photons can be emitted with arbitrarily small energies, it is always possible that some photons will escape detection. In fact, they showed that the probability that only a finite number of photons will escape detection is precisely zero; this is due to the infrared divergence associated with soft virtual photons. On the other hand, when the cross-section is summed over all final states compatible with the detection arrangement, including all possible undetected photons, a nonvanishing result is obtained. In fact, the observed cross-section is very nearly the cross-section that would be obtained if all radiative corrections were ignored. This is the well known cancellation between the real and virtual infrared divergences.

#### S. FRAUTSCHI

## 2. SEMICLASSICAL PHENOMENA, HIGH-ENERGY PROCESSES AND REGGE POLES

Now I want to begin by describing how these semiclassical phenomena emerge from quantum electrodynamics. A well-defined separation into infrared terms and "shorter wavelength" terms can be made, and the infrared terms calculated explicitly to all orders. This infrared terms turn out to have special significance in high energy processes, and among other things they contain a Regge - like behaviour. Up to the discussion of the Regge behaviour, I shall follow the paper of YENNIE, FRAUTSCHI and SUURA [2], where references to some of the alternative treatments of the subject [3] can be found.



A representation of the matrix element Ma.



Some of the ways in which an additional virtual photon can be inserted in Fig. 1

Consider a process in which an electron scatters from a state of 4momentum p into one of p'. Let  $M_0$  be the matrix element corresponding to any set of Feynman diagrams (Fig. 1.). Add one virtual photon to  $M_0$ , in the manner indicated in Fig. 2. That part of the contribution which diverges at small k can be represented by  $M_0 \alpha$  B where B is the gauge-invariant expression

$$B = \frac{i}{(2\pi)^3} \int \frac{d^4k}{k^2 - \lambda^2} \left( \frac{2p'_{\mu} - k_{\mu}}{2p'_{\mu} - k^2} - \frac{2p_{\mu} - k_{\mu}}{2p_{\mu} - k - k^2} \right)^2$$
(1)

and  $\lambda$  is a fictitious photon mass. At small k the integral has the form dk/k, characteristic of the infrared divergence.

Now add a second virtual photon in the same way - inserting it only into the outside lines of the previous diagrams. Symmetrize the two virtual photons, introducing a factor  $\frac{1}{2}$ ! to prevent double counting. One obtains  $M_0$ ( $\alpha$  B)<sup>2</sup>/2! plus other terms. Some of these other terms also have an infrared divergence, but a careful check reveals that they cancel when all other ways of introducing two virtual photons into  $M_0$  are considered [2]. The same property is found in higher orders, so the addition of arbitrary numbers of virtual photons to  $M_0$  yields the series

$$M_0 \sum_{n=0}^{\infty} (\alpha B)^n / (n !) = M_0 \exp (\alpha B)$$
(2)

plus terms M' which have no infrared divergence. We can proceed to treat M' in the same way as  $M_0$ , and eventually we find that the entire matrix element has a common factor exp ( $\alpha$  B).

Since infrared effects are "long range", it is not surprising that the corresponding Feynman diagrams involve virtual photons emitted from external electron lines (Fig. 3). This provides electron propagators which are nearly real at small k and can spread far out into space. The exponential form (2) is also reasonable. Emission and absorption of low energy, low momentum photons do not appreciably disturb the motion of the electron; this means that such photons are emitted and absorbed independently, resulting in a Poisson distribution ( $\alpha B$ )<sup>n</sup> /(n!).



Fig.3

A typical infrared diagram involving several virtual photons.

Equation (2) leads to a cross-section  $d\sigma_0$ , proportional to exp (2  $\alpha$  B). We must add to this cross-section the cross-section for emission of an undetected real photon, with energy bounded by  $K_m$  in order to escape detection. The infrared terms are associated with diagrams in which the real photon



Fig.4

An additional real photon, inserted into M<sub>n</sub> in these ways, gives an infrared-divergent contribution.

is emitted from the external lines (Fig.4), and one obtains essentially  $2 \alpha \widetilde{B} d\sigma_0$  where  $2 \alpha \widetilde{B}$  is given by a product of phase space and squared-matrix element:

$$\widetilde{B} = -\frac{1}{8\pi^2} \int_0^{K_m} \frac{d^3 k}{(k^2 + \lambda^2)^{1/2}} \left(\frac{p'_{\mu}}{p' \cdot k} - \frac{p_{\mu}}{p \cdot k}\right)^2$$
(3)

(for correction factors arising from a more careful treatment of energy conservation, see Ref. [2]).

At small k the integral again has the form dk/k. At high energy, where  $|\vec{p}| \sim E$ , etc, we see that the photon tends to emerge along the initial or final electron direction, as expected from the semiclassical argument.

The cross-section for emission of two undetected real photons must also be added. As in the case of virtual photons, this gives  $(2 \alpha \widetilde{B})^2 (2!)^{-1}$ 

 $d\sigma_0, \ plus \ terms \ which \ have no infrared divergence. The sum over all undetected real photons gives$ 

$$d\sigma_0 \sum_{n=0}^{\infty} (2 \alpha \widetilde{B})^n / (n!) = d\sigma_0 \exp(2 \alpha \widetilde{B})$$
(4)

and the observed cross-section is proportional to exp 2  $\alpha$  (B +  $\widetilde{B}$ ).

It is well known, and can easily be verified explicitly from Eqs (1) and (3), that the infrared divergence in the lowest order real and virtual photon radiative corrections  $2 \alpha$  (B +  $\tilde{B}$ ) cancel, leaving a finite result. Since the higher order infrared terms simply raise  $2 \alpha$  (B +  $\tilde{B}$ ) to an exponential, the cancellation holds in all orders. One also finds that B approaches -  $\infty$  as  $\lambda \rightarrow 0$ , ensuring that the cross-section for emission of no undetected real photons vanishes.

An attractive feature of these results is that the factor  $\exp 2\alpha$  (B + B) is known in general, independent of the details of short-range interactions in the matrix element. In the case of electron scattering from a nuclear target, for example,  $\exp 2\alpha$  (B + B) has the same form whether the target is left in its ground state or an excited state.

Another interesting aspect of the infrared factor can be seen from its form at large electron energies and momentum transfers  $(E > m, E' > m^2, p, p' >> m^2)$ . If  $E/K_m$  is large the leading (double logarithm) term is

$$\exp\left[-(\alpha / \pi) \ln (2 \text{ p. p}' / \text{m}^2) \ln (\text{EE}' / \text{K}^2 \text{m})\right].$$
 (5)

Here the small denominators which allow the virtual particles in infrared terms to travel far out from the target have been integrated over to give large logarithmic factors. Shorter wavelength photons are associated with at most one small denominator, and give at most single logarithms. Thus the infrared terms tend to provide the dominant radiative correction at high energy.

This result means that while the power of  $\alpha$  provides a good index to the size of a radiative correction, the "range" of the correction should also receive some attention; long-range effects should be treated with special care.

Consider, for example, corrections of order  $\alpha$  to electron-electron scattering. TSAI [4] has considered a clashing beam arrangement in which the two scattered electrons are detected in coincidence with good angular resolution  $\Delta \theta$  but virtually no energy resolution ( $\Delta E \sim E$ ). It is clear that if a photon is emitted parallel to either final electron,  $K_m$  is then of order E. However, if it is emitted perpendicular to the direction of the final electrons,  $K_m$  is much smaller and is determined by the angular resolution ( $\Delta E \sim E\Delta \theta$ ). Thus  $K_m(\theta)$  has a very strong angular dependence. An incorrect treatment of this angular dependence would change the double log term (5) by several per cent; the experimental energy resolution has to be treated carefully before it becomes worth while to calculate shorter range corrections of the same order in  $\alpha$  but with no logarithms.

A numerical example will illustrate the related point that  $\alpha^2 \ln^4$  terms can be at least as important as shorter range terms of order  $\alpha$ . If E is 500 MeV, K' = 5 MeV, E'  $\simeq$  500 MeV, p. p' = E<sup>2</sup>, then the  $\alpha^2 \ln^4$  term obtained by expanding (5) contributes + 3%. Another aspect of the infrared terms can be seen by recasting them as functions of the 4 - momentum transfer  $t = (p-p')^2 = 2m^2 - 2p$ , p'. The virtual photon factor exp (2  $\alpha$  B) behaves like a form factor for electron scattering and depends only on t. But the real photon factor exp (2  $\alpha$   $\widetilde{B}$ ) introduces energy dependence as well (we let E = E' now for simplicity), and (5) has the form:

$$\exp\left[-\frac{2\alpha}{\pi}\ln\left(\frac{2m^2-t}{m^2}\right)\ln\frac{E}{K_m}\right] = \left(\frac{E}{K_m}\right)^{\frac{2\alpha}{\pi}}\ln\left(\frac{2m^2-t}{m^2}\right).$$
 (5.a)

Since the differential cross-section is proportional to this exponential,(5. a) resembles the formula

$$d\sigma/dt \sim E^{2 J(t)-2}$$
(6)

expressing the exchange of a Regge pole with spin J(t) at high energies [5].

Do we expect to find a state of spin J(t) in the t channel? Consider Fig. 5, a typical infrared diagram as seen in the t channel where the incoming particles are an electron with 4 - momentum and a positron with - p'. The Fig. is obviously a ladder diagram with multiple exchange of photons. It would not be surprising if such diagrams gave some hint of the Bohr or positronium states which cluster just below threshold at t =  $4m^2$ . The angular momenta of the positronium states (ignoring decay into photons) increase through the integers from J = 0 to J =  $\infty$  as threshold is approached, according to the Bohr formula

$$g_{f}^{2}/2\mu = (-\mu c^{2} \alpha^{2}/2(n + J + 1))^{2}$$
(7)

where  $\mu$  is the reduced mass and  $g_t$  the momentum of the electron in the centre-of-mass. It is known [6] that a Regge pole with spin J(t) interpolates smoothly between the integer J, still following Eq.(7), for each n. The Regge pole with highest J at each energy level corresponds to n = 0, and with the specializations  $\mu = m/2$ , c = 1,  $4q_t^2 = t - 4m^2$ , appropriate to our case, one finds from (7):

$$J(t) = -1 + m \alpha / \sqrt{4m^2 - t}.$$
 (8)

Before the infrared factor is compared with this result it must be recalculated, since the form (5.a) is valid only at large |t|. (Note that even the


threshold for the t channel is wrong in (5.a). One finds that near t =  $4m^2$  the leading term in the infrared factor is

$$\exp\left[\frac{2 \alpha m}{\sqrt{4m^2 - t}} \ln \frac{E}{K_m}\right] = \left(\frac{E}{K_m}\right) \frac{2m \alpha}{\sqrt{4m^2 - t}}$$
(9)

The other factors in the cross-section give much smaller powers of energy, which are nearly constant in the region of the Bohr levels. Thus when a cross-section containing the infrared factor (9) is interpreted according to the Regge form  $d\sigma/dt \sim E^{2J(t)-2}$ , it gives essentially the Bohr trajectory (8) at  $t \sim 4 \text{ m}^2$  to within a constant of order one.

#### 3. CONCLUSION

In conclusion I should mention several peculiar features:

- Although the infrared factor essentially contains the <u>power</u> associated with the Bohr trajectory, it does not contain the <u>poles</u> associated with the Bohr levels. After all, we have only taken the lowest order radiative correction, which has no bound state poles, and raised it to an exponential, which does not introduce further singularities. A fuller treatment of the scattering would be required to obtain the poles [7].
- (2) Eq. (5b) can alternatively be written in a form appropriate for large |t|,

$$\exp\left[-\frac{2\alpha}{\pi}\ln\left(\frac{2m^2-t}{m^2}\right)\ln\frac{E}{K_m}\right] = \left(\frac{2m^2-t}{m^2}\right)^{\frac{2\alpha}{\pi}-\ln\frac{2}{K_m}},$$
 (5.b)

giving a power whose rate of variation depends on the experimental energy resolution.

(3) The power of lab energy in (9) can be identified with 2J(t)-2, where J(t) refers to the Bohr trajectory, only if  $K_m$  is energy-independent in the laboratory.

#### REFERENCES

- [1] BLOCK, F. and NORDSIECK, A., Phys. Rev. 52 (1937) 54.
- [2] YENNIE, FRAUTSCHI and SUURA, Annals of Physics 13 (1961) 379.
- [3] Another paper which especially close to reference [2] is ERIKSON, K.E., Nuovo Cimento 19 (1961) 1010.
- [4] TSAI, Y.S., Phys. Rev. 120 (1960) 269.
- [5] Eg. (5) can also be derived by the charge renormalization group method, as Erikson shows in his report to this Seminar. The existence of t-dependent powers of energy in expressions obtained by the renormalization group method was discovered by E.L. Feinberg and reported by N.N. Bogoliubov at the International Conference on High Energy Physics, CERN (1962).
- [6] SINGH, V., 'Analyticity in the Complex Angular Momentum Plane of the Coulomb Scattering Amplitude', UCRL-9972 (to be published in Physical Review).
- [7] In fact the full treatment of ladder diagrams, given in the lectures of Fubini and Lec and Sawyer, does give Regge poles.

### EXPERIMENTS WITH ELECTRON COLLIDING BEAMS

## R. GATTO

#### NATIONAL LABORATORY OF FRASCATI, FRASCATI, ITALY

#### INTRODUCTION

The two subjects on which I shall talk,  $e^+e^-$  colliding beam experiments and  $p\bar{p}$  annihilation into leptons, though pertaining to quite different experimental projects, have a common theoretical background that makes it convenient to discuss them together.

Electron-positron colliding beams constitute the largest project now under development at the Frascati National Laboratory. A first part of the project, the construction of a small ring (called AdA from the Italian Anello di Accumulazione) has now been completed, and the relevant experiments  $(e^+e^- \text{ annihilations at energies up to ~450 MeV}$  in the centre-of-mass system) are being carried out using for the injection the linear accelerator at Orsay.

The second but more ambitious part of the project consists in the development of one or more larger storage rings, up to energies of  $\sim 1.5$  GeV for each of the two colliding beams (i.e. 3 GeV total centre-of-mass energy). This second part of the project is called Adone (Adone is the Italian for "Adonis" but can also mean "larger AdA").

The pp annihilation experiments are being carried out on the protonsynchrotron of the European Organization for Nuclear Research (CERN) by Conversi, Farley, Müller and Zichichi, using an intense beam of antiprotons.

#### THE FRASCATI STORAGE RING PROJECT

A description of AdA can be found in a paper presented by TOUSCHEK at the CERN Conference on high energies in 1961 [1], and also in a note by BERNARDINI, CORAZZA, GHIGO and TOUSCHEK [2]. The Adone project is discussed by AMMAN, BERNARDINI, GATTO, GHIGO and TOUSCHEK in a preliminary form [3] and, at a more advanced stage, by AMMAN, BASSETTI, BERNARDINI, CORAZZA, MASSAROTTI, MANGO, PELLEGRINI, PLACIDI, PUGLIGI and TAZZIOLI [4].

The use of storage rings was first proposed by O'NEILL [5], and an experiment on electron-electron colliding beams is being carried out at Stanford.

A storage ring project like Adone consists roughly of an injector (together with an injection system), a doughnut kept at high vacuum, a magnet and a radiofrequency system. The injector in Adone will be a high-energy linear accelerator for electrons and positrons. Its energy depends on the energy that one wants to attain in the storage ring and must also be large if a large positron intensity is needed. For instance, it is intended in Adone

#### R. GATTO

to use a linear accelerator of  $\sim 500$  MeV for a ring of 750 MeV, and a linear accelerator of ~800 MeV for a ring of 1.5 GeV. The electrons and positrons, after being accelerated by the linear accelerator, are injected, in many pulses, through an optical system into the doughnut. During the accumulation time the magnetic field is kept constant. Then it is varied until the beams reach the desired final energy. The energy variation is provided by the radiofrequency system, whose main purpose is to compensate for the energy losses due to radiation. The stored electrons and positrons circulate on the same orbit in opposite directions and one can look at their annihilation products when they collide. In spite of the high vacuum, collisions also occur with the residual gas in the doughnut, and this effect contributes to produce the "finite life-time" of the beam, which can however be made of the order of ten hours or more by going to relatively high vacuum ( $\sim 10^{-9}$  to r). Electronpositron collisions take place only in a few regions of the doughnut, as a result of fact that the electrons and positrons proceed grouped in packets, whose total number is given by the ratio of the radiofrequency to the beam frequency. If the cross-section for a given type of event is  $\sigma$ , the number of events per second that one observes will be Lo where L depends on the characteristics of the ring and is called luminosity. The luminosity L is expressed by a simple formula in terms of the parameters of the ring, and it is essentially proportional to the product  $s_{+} s_{-}$  of the positron and electron transverse densities (densities on a surface transverse to the beam direction). These densities cannot unfortunately be made arbitrarily high. Their values are limited up to smax which depends on the so-called space-charge effects. In fact, the force acting on one electron (or positron) in the beam is also dependent on the action of the other electrons and positrons. For



Perturbation theory cross-section for  $e^+ + e^- \rightarrow 2$ 

and order of magnitude,  $s_{max}$  is expected in Adone to be of the order of  $10^{11}$  - $10^{12}$  particles per cm<sup>2</sup>. The cross-sections in perturbation theory decrease rapidly with energy. In Fig.1, we report the perturbation theory cross-sections for  $e^+ + e^- \rightarrow 2\gamma$ . In Fig.2, we report the perturbation theory



Differential cross-section for  $e^+ + e^- \rightarrow e^+ + e^-$ 

cross-sections for  $e^+ + e^- \rightarrow e^+ + e^-$ , and in Fig. 3, we have plotted the perturbation theory cross-section for  $e^+ + e^- \rightarrow \mu^+ + \mu^-$ , together with the perturbation theory cross-sections for  $e^+ + e^- \rightarrow \pi^+ + \pi^-$ ,  $e^+ + e^- \rightarrow p + \bar{p}$ , and  $e^{+} + e^{-} \rightarrow K + \overline{K}$ . The perturbation theory cross-sections for these last three reactions may well be wrong by orders of magnitude, since any effect of strong interactions is entirely neglected. In Fig. 3, the cross-sections are plotted against E/m, where 2E = total centre-of-mass energy and m is the mass of the final particle produced ( $\mu$ -meson, pion, K-meson, nucleon). In Fig. 4, the perturbation theory cross-section is also reported for the mode of annihilation  $e^{\dagger} + e^{-} \rightarrow B + \overline{B}$ , where B is a vector meson, on the assumption that it has no anomalous moment. In spite of the general unreliability of perturbation theory estimates, especially when strong interacting particles are produced, it is safe to require that the luminosity L be of the order of 10<sup>33</sup> cm<sup>2</sup> per hour in order to have a vast range of experimental uses and a reasonably fast counting rate. To attain such a luminosity one needs a large current intensity. If one defines the intensity as the ratio between the total charge of the beam and the revolution period, the required values are about 100 mA for both positrons and electrons.

# POSSIBLE EXPERIMENTS WITH COLLIDING BEAMS OF ELECTRONS AND POSITRONS

Theoretical discussions of electron-positron colliding beam experiments have already been given, A general discussion will not be given in these



TOTAL CROSS-SECTION IN PERTURBATION THEORY

Fig. 3

Total cross-sections in perturbation theory where 2E = total centre-of-mass energy and m = mass of the produced particles



TOTAL CROSS-SECTION FOR e+e-B+B-



R. GATTO

Fig. 4

Total cross-section for  $e^+ + e^- \rightarrow B^+ + B^-$  where  $m_B = mass$  of B in GeV and 2E = total centre-of-mass energy

notes, and I shall refer to previous papers [6]. I shall only mention here a few general points that have been examined

#### 1. Tests of quantum electrodynamics

The reactions  $e^{t} + e^{-} \rightarrow 2\gamma$ ,  $e^{t} + e^{-} \rightarrow e^{t} + e^{-}$  and  $e^{t} + e^{-} \rightarrow \mu^{t} + \mu^{-}$  can conveniently be used to test quantum electrodynamics at small distances (the last equation will also test the muon structure). A reliable calculation of the radiative corrections is essential for the interpretation of the above reactions. Calculations of the radiative corrections have been carried out at Frascati and at Trieste by Da PRATO, MOSCO and PUTZOLU [7, 8] and by BUDINI and FURLAN [9].

Following the conventional way of modifying electrodynamics one finds that at a beam energy as low as E = 250 MeV (i.e. 500 MeV total centre-ofmass energy) a measurement of  $e^+ + e^- \rightarrow 2\gamma$  with a 7% accuracy carried out at 60° - 90° can test electrodynamics up to distances of ~0.2 fermi. A measurement of  $e^+ + e^- \rightarrow e^+ + e^-$  around 90° at an energy  $E \sim 300$  MeV with a 10% accuracy is sufficient to test electrodynamics up to distances of ~0.1 fermi. Similarly, a measurement with a 10% accuracy of  $e^+ + e^- \rightarrow \mu^+ + \mu^$ at  $E \sim 300$  MeV would also test electrodynamics (or muon structure) to ~0.1 fermi. Tests of electrodynamics can also be conveniently carried out with more complicated reactions such as  $e^+ + e^- \rightarrow \mu^+ + \mu^- + \gamma$  which are still easily detectable but have low counting rates.

#### 2. Annihilation into strong interacting particles

The relevant possibility offered by such experiments is that of measuring the electromagnetic form factors of strong interacting particles for time-like values of the momentum transfer. As long as the reaction goes through the one-photon channel, pairs of zero-spin bosons are produced in p states and pairs of spin 1/2 fermions in  ${}^{3}S_{1}$  and  ${}^{3}D_{1}$  states. As a general remark let me stress one point which is peculiar to  $e^+$  -  $e^-$  experiments and makes their interpretation more direct than for the corresponding electron-scattering experiments. In an electron-scattering experiment, such as  $e^{+} + p \rightarrow e^{+} + p$ , at a given energy the form factors are taken for different values of their argument corresponding to each different scattering angle. On the other hand, in an annihilation experiment such as  $e^+ + e^- \rightarrow p + \bar{p}$ , at a given energy the form factors are taken at one given value of their argument corresponding to the energy in the experiment. As a result of this, angular distributions are predicted at most up to one parameter (in e  $+ e \rightarrow$  $p + \bar{p}$  it is the ratio of the magnetic to the charge-form-factor) and there is no need at all for good angular resolutions. I shall not expand here on the topic of annihilation into strong interacting particles, especially as some of the aspects of the discussion will reappear among the special topics I am going to discuss in these notes.

#### 3. Annihilation into pairs of vector mesons

Unless some damping due to rapidly decreasing form factors or to other possible mechanisms occurs, the mode of annihilation

where B and  $\overline{B}$  are suggested moderately weakly interacting vector bosons, is expected to be a dominant one. Its cross-section in perturbation theory, on the assumption of no anomalous magnetic moment or electric quadrupole moment, is reported in Fig. 4. Comparing this with the other perturbation theory results, one sees that annihilation into vector mesons might well be rather frequent, in spite of the many uncertainties of the electromagnetic properties of this particle. If neutral intermediate vector mesons exist they would also appear as resonances, for instance in  $e^{\dagger} + e^{-} \rightarrow B^{0} \rightarrow \mu^{\dagger} + \mu^{-}$ . We have calculated that such resonant interaction, if it existed, even after averaging over the experimental energy resolution, could very well compete with the direct electromagnetic process.

For further considerations we again refer to the published papers quoted in reference [6].

In the following I shall only discuss a few points on the subject of  $e^{\dagger} - e^{-1}$  colliding beams that have been clarified by recent developments. Then I shall talk mostly of  $p + \bar{p}$  annihilation experiments.

#### THE ONE-PHOTON CHANNEL

In this section I shall briefly review the main theoretical notions for the analysis of a reaction

$$e^+ + e^- \rightarrow \text{final state}$$
 (1)

assuming that it mainly proceeds through a graph of the kind shown in Fig.5.



Fig.5

I shall also discuss some aspects of the problem of radiative corrections to such a graph. The S-matrix element for reaction (1), assumed to proceed through a graph shown in Fig.5, can be written in the form

$$\langle f | S | e^{+} e^{-} \rangle = \frac{2\pi e}{k^2} (v \gamma_{\nu} u) \langle f | j_{\nu}(0) | 0 \rangle \delta(q_{+} + q_{-} - q_{f})$$
 (2)

where f denotes the final state of total 4-momentum  $q_f$ ,  $k = q_+ + q_-$  is the 4-momentum of the virtual photon,  $q_+$  and  $q_-$  are the  $e^+$  and  $e^-$  momentum

respectively, v and u are Dirac spinors and  $j_{\nu}$  is the electromagnetic current operator. In the centre-of-mass system, which is indeed the laboratory system in colliding beam experiments, one has

$$k^{2} = (q_{+} + q_{-})^{2} = -4 E^{2}$$
 (3)

where E is the energy of  $e^+$  (or of  $e^-$ ). The electromagnetic current  $j_{\nu}(x)$  has to satisfy a conservation equation  $\delta j_{\mu}(x)/\delta x_{\mu} = 0$  from which

$$k_{\nu} \langle f | j_{\nu}(0) | 0 \rangle = 0$$
 (4)

In the centre-of-mass system Eq. (4) takes the form

$$k_{4} < f | j_{\nu}(0) | 0 > = 2i E < f | j_{4}(0) | 0 > = 0.$$
(5)

Thus if we define

$$J_{\nu} = (2\pi)^{\frac{3n}{2}} \langle f | j_{\nu}(0) | 0 \rangle$$
(6)

where we have introduced a factor  $(2\pi)^{3n}$  for normalization purposes and where the number of the final particles) we conclude from Eq. (5) that in the centreof-mass system  $J_{\nu} = (\vec{J}, 0)$  where  $\vec{J}$  is a three-dimensional vector. The vector  $\vec{J}$  can also be decomposed as  $\vec{J} = \vec{J}^{(S)} + \vec{J}^{(V)}$  where the superscripts S and V refer respectively to scalar or vector in isotopic-spin space. It thus follows that the final state must have the following quantum numbers: total angular momentum J = 1, parity P = -1, charge conjugation number C = -1, isotopic spin T = 0 or 1. From the S-matrix element considered in Eq.(2) and from Eq.(6) it follows that the total cross-section for unpolarized initial and final particles is given by

$$\sigma = \frac{(2\pi)^{5-3n}\alpha}{16\mathrm{E}^4} \int (\mathrm{d}^3 f_1) \dots (\mathrm{d}^3 f_n) \,\delta \left(\mathrm{E}_{\mathrm{f}} - 2\mathrm{E}\right) \delta_3(\vec{q}_{\mathrm{f}}) \,\mathrm{T}_{\mathrm{mn}} \,\sum_{\mathrm{f}} \mathrm{R}_{\mathrm{mn}} \tag{7}$$

where  $\alpha = e^2/4\pi = 1/137$ ,  $f_i$  is the momentum of the final ith particle,  $T_{mn} = 1/2 (i_m i_n - \delta_{mn})$  where  $\vec{i}$  is a unit vector pointing along the direction of the incoming positron and  $R_{mn} = -J_m J_n^*$ . The summation  $\sum_f$  is over the final spin state.

When radiative corrections are included one must add to the graph of Fig. 5 other graphs such as those shown in Figs. 6 and 7.

The graph shown in Fig. 7 cannot be expressed in terms of the vertex indicated indicated in Fig. 8 (suitably modified for the inclusion of radiative corrections).

For the two-photon channel most of the general considerations valid for the one-photon channel (such as its typical selection rules) do not apply. The dominant contribution from the two-photon channel is expected to arise



Fig.8

from its interference with one-photon graph shown in Fig. 5. This interference is of order e in the cross-section. At this point there is a very simple and general result that can be useful for the interpretation of the experiments. For an experiment which does not distinguish between a final state and its charge conjugate (such as a total cross-section measurement, or any measurement which treats symmetrically the produced charged particles) such an interference term, between the one-photon graph of Fig.5 and the two-photon graph of Fig.8, does not contribute. Such a theorem is a general consequence of charge-conjugation invariance and can be proved in many ways. A simple proof is the following. Consider a transition from the initial state i to a final state f. We call  $S_A$  that part of the S-matrix corresponding to the one-photon channel and  $S_B$  that part corresponding to the two-photon channel. The interference term we are considering between the one-photon and the two-photon channel is of the form

$$\operatorname{Re}_{f} \leq i |S_{A}| f \geq f |S_{B}| i \geq \operatorname{Re} \leq i |S_{A}P_{f}S_{B}| i \rangle$$
(8)

where  $P_f = \frac{r}{f} |f\rangle \langle f|$  is a projection operator into those final states f which are selected by the experiment. We assume that the set of states f is invariant under charge conjugation. So we can write

$$\mathcal{C}^{-1}S_{A}P_{f}S_{B}\mathcal{C} = S_{A}P_{f}S_{B}$$
(9)

where  $\mathcal{E}$  is the charge conjugation operator. Next we note that if we split i ) into a part even under  $\mathcal{E}$  and a part odd under  $\mathcal{E}$ 

$$|\mathbf{i}\rangle = |\mathbf{i}_{+}\rangle + |\mathbf{i}_{-}\rangle, \tag{10}$$

we have  $S_B |i\rangle = S_B |i_+\rangle$  and  $S_A |i\rangle = S_A |i_-\rangle$ . Thus Eq.(8) can be written as

$$\operatorname{Re} \langle i_{-} | S_{A} P_{f} S_{B} | i_{+} \rangle = \operatorname{Re} \langle i_{-} | \mathcal{C}^{-1} S_{A} P_{f} S_{B} \mathcal{C} | i_{+} \rangle$$
$$= \operatorname{Re} \langle i_{-} | S_{A} P_{f} S_{B} | i_{+} \rangle = 0$$
(11)

which shows the vanishing of such interference term on the assumption that  $P_f$  is even under %. Therefore, considerations, valid for the one-photon channel hold including terms  $e^6$ , as long as only symmetric experiments are performed.

Still on the subject of radiative corrections there is another feature of electron-positron colliding beams which deserves to be mentioned. The exploration of the limits of validity of quantum electrodynamics can be carried out rigorously only up to energies such that the virtual effects of interacting particles can be neglected. As soon as the breakdown effects to be observed become of the same order as the effects resulting from virtual strong interacting particles a much more complicated theoretical analysis is required, which does not have, so far, the degree of the typical reliability of calculation with electrodynamics. In colliding beam experiments the virtual effects of strong interacting particles come in such a form that they are related to total cross-section measurements carried out with the same colliding beam system. There is therefore, in this respect, no need for a theoretical treatment of strong interactions. Let us consider the modification to the photon propagator arising from virtual strong interacting particles. The modified photon propagator can be written as

$$D_{\mu\nu}^{1}(k^{2}) = \frac{\delta_{\mu\nu}}{k^{2} - i\epsilon} + \frac{k^{2}\delta_{\mu\nu} - k_{\mu}k_{\nu}}{k^{2}} \cdot \frac{\overline{\Pi}(0) - \overline{\Pi}(k^{2}) - i\pi\Pi(k^{2})}{k^{2} - i\epsilon}$$
(12)

where

$$\Pi(\mathbf{k}^{2}) = -\frac{(2\pi)^{3}}{3\mathbf{k}^{2}} \sum_{\mathbf{p}_{z}=\mathbf{k}} \langle 0 | \mathbf{j}_{v}(0) | z \rangle \langle z | \mathbf{j}_{v}(0) | 0 \rangle$$
(13)

and

$$\overline{\Pi} (k^2) = P \int_0^\infty \frac{\pi (-a) da}{k^2 + a} .$$
 (14)

In Eq. (13),  $j_{\nu}$  is the current operator and the sum is extended over all physical states with total four-momentum  $p_z = k$ . The photon propagator in Eq.(12) is defined as

$$D_{\mu\nu}^{1} (x - x') = i \langle 0 | P [A_{\mu}(x) A_{\nu}(x)] | 0 \rangle$$
(15)

where P is the chronological product and  $A_{\mu}$  is the electromagnetic field. Formally Eq.(12) cannot be written as

$$D_{\mu\nu}^{1}(k^{2}) = \frac{\delta_{\mu\nu}}{k^{2}} + \left(\delta_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^{2}}\right) \int_{0}^{\infty} \frac{da \Pi (-a)}{a (k^{2} + a - i\epsilon)}.$$
 (16)

It is easy to see that the experimentally measured cross-sections for processes  $e^+ + e^- \rightarrow \gamma \rightarrow F$  where F denotes a group of final states, is directly related to the contribution to Eq. (13) from the group of state F among the states on which the summation is extended. In this way one can calculate the modifications of the photon propagator resulting from virtual strong interacting particles, directly from measured values of cross-sections.

Let us note that in Eq. (13) the matrix elements  $\langle 0 | j_{\nu}(0) | z \rangle$  of the current  $j_{\nu}$  are proportional to  $J_{\nu}$  as defined from Eq.(6). In fact, the total cross-section for annihilations leading to the final states F, can be written, in the centre-of-mass system, as

$$\sigma_{\rm F} ({\rm E}) = -\frac{(2\pi)^{3} \alpha}{16 {\rm E}^{4}} \quad {\rm T}_{{\rm inn}} \sum_{p_{\rm z}=k} \langle 0 | j_{\rm m}(0) | z \rangle \langle z | j_{\rm n}(0) | 0 \rangle.$$
(17)

This is again Eq. (7) in a different notation. The sum occuring in Eq.(13) is slightly different from that in Eq.(17). In Eq.(13) there occurs a scalar product of the two matrix elements of  $j_{\nu}$ , while in Eq. (17) the two spaces indices m and n are different. However one can use gauge invariance to relate the two expressions. One has

$$(2\pi)^{3} \Sigma_{F} \langle 0 | j_{\mu}(0) | z \rangle \langle z | j_{\nu}(0) | 0 \rangle = \Pi_{F} (k^{2}) (k_{\mu}k_{\nu} - k^{2} \delta_{\mu\nu})$$
(18)  
$$p_{Z} = k$$

where  $\Pi_F(k^2)$  denotes the contribution to  $\Pi(k^2)$  from the group of intermediate states F. If we substitute Eq.(18) into Eq.(17), we obtain the direct connection we were looking for, namely

$$\sigma_{\rm F}({\rm E}) = (\pi^2 \alpha / {\rm E}^2) \Pi_{\rm F} (-4 {\rm E}^2) . \qquad (19)$$

In Eq.(16) for  $D^1_{\mu\nu}$  (k<sup>2</sup>) we can now substitute  $\Pi_F(-4E^2)$  from Eq.(19) to get a formal expression for the photon propagator modified for the virtual contribution of the states F

$$D_{\mu\nu}(k) = \frac{\delta_{\mu\nu}}{k^2} + \left(\delta_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k}\right) \frac{2}{\pi^2\alpha} \int_0^\infty \frac{EdE \sigma_F(E)}{k^2 + 4E^2 - i\epsilon}.$$
 (20)

In Eq. (20)  $\sigma_F$  is at least of order  $\alpha^2$  so that the modifications are usually very small. In some exceptional cases, however, they become rather big. Such a situation occurs, for instance, at energies corresponding to the  $\omega^0$ mass. The cross-section  $\sigma_F$  (E) for processes that go through a  $\omega^0$  intermediate state is very sharply peaked around the  $\omega^0$  mass. The processes that can go through  $\omega^0$  are those in which the final isotopic spin is zero,apart

#### ELECTRON COLLIDING BEAMS

from isotopic-spin mixing effects that are indeed quite large (see the discussion of such resonant contributions in the next section). The effect of virtual  $\omega$  brings about noticeable modifications also in processes such as  $e^+ + e^- \rightarrow e^+ + e^- \rightarrow \mu^+ + \mu^-$  in which no strong interacting particles are present in both initial and final state. Such effects become difficult to detect in experiments with rough energy resolution, but they become very apparent as soon as the energy resolution is of the same order as the lifetime of the resonant state. Such a situation is probably realizable with  $e^+ - e^-$  colliding beams, where it should be possible to reach energy resolutions of the order of 100 keV without very great technical difficulties [10, 11].

## THE RESONANT CONTRIBUTION FROM $\rho^0$ AND $\omega^0$ .

We have seen that states with J = 1, P = -1, C = -1, T = 0 or 1, can be coupled to  $e^+ e^-$  in the one-photon channel. The vector mesons  $\rho^0$  and  $\omega^0$  that have recently been discovered have the right quantum numbers to couple in the one-photon channel. They may give rise to resonant contributions through graphs of the general kind as shown in Fig.9.



Fig.9

Recently there has been some information about the couplings of  $\rho^0$  and  $\omega^0$ . One is thus able to estimate the cross-sections for processes going through intermediate  $\rho^0$  and  $\omega^0$ . I shall first review here the general strategy that one follows to determine the couplings of  $\rho^0$  and  $\omega^0$ . Of course much of the work is based on theoretical models (mainly on the so-called  $\rho$ -dominant model by Gell-Mann) and therefore some of the conclusion may well turn out to be not very accurate. We believe, however, that the qualitative features of such an approach are essentially right and that the main conclusions that we derive (such as the almost unbelievably large values of the crosssections through intermediate  $\omega^0$ ) are essentially independent of the model. In reference [12] are listed some of the papers that are relevant for the problem of the determination of the couplings of  $\rho^0$  and  $\omega^0$ . I shall now describe, without going into details, the strategy that leads to a determination of the coupling constants.

(1)  $\omega^0 \rightarrow \pi^+ + \pi^- + \pi^0$  decay.

In the  $\rho$ -dominant model one describes such a decay through the graph shown in Fig. 10. One needs therefore two coupling constants:  $f_{\omega\rho\pi}$  and  $\gamma_{\rho\pi\pi}$ .

(2)  $\rho \rightarrow \pi + \pi$  decay



The width  $\Gamma$  for the decay which is shown in Fig.11, is expressed in terms of  $\gamma_{0\pi\pi}$ . Inserting for  $\Gamma$  a value ~100 MeV one gets

$$\gamma_{\rho\pi\pi} / 4 \pi \simeq 1 / 2 . \tag{21}$$

$$-\frac{\pi}{\beta} - \frac{\gamma}{\beta} \beta^{\mu\pi\pi}$$
Fig. 11

(3) Pion form factor

The  $\gamma - \rho$  coupling constant is called  $em_{\rho}^2/2 \gamma_{\rho}$ , where  $m_{\rho}$  is the  $\rho$ -mass. To determine  $\gamma_{\rho}$ , one writes down the expression for the pion form as given in terms of the graph shown in Fig. 12:

eF (k<sup>2</sup>) = (em<sub> $\rho$ </sub><sup>2</sup> /  $\gamma_{\rho}$ ) [1/(m<sub> $\rho$ </sub><sup>2</sup> + k<sup>2</sup>)]  $\gamma_{\rho\pi\pi}$ .



Fig. 12

From F(0) = 1, one gets

 $\gamma_{\rho\pi\pi} / \gamma_{\rho} = 1.$  (22)

From (21) and (22) one has

$$\gamma_{\rho}^{2}/4\pi \simeq 1/2.$$
 (23)

It should be noted that one would also derive the relation  $\gamma_{\rho NN} / \gamma_{\rho} = 1$  from the nucleon isovector form factor, by a similar procedure. In general, as stressed by Gell-Mann and Sakurai, it is required that the coupling of  $\rho$  be universal:  $\gamma_{\rho\pi\pi} = \gamma_{\rho NN} = \gamma_{\rho KK}$ . (4)  $\omega \rightarrow \pi + \gamma$  decay



For this decay mode (Fig.13), one still needs  $f_{\omega\rho\pi}$  .  $\gamma_{\rho}$  is known by Eq.(23). However, the branching ratio between  $\omega \rightarrow \pi^0 + \gamma$  and  $\omega \rightarrow 3\pi$  can already be predicted at this stage. (5)  $\pi^0 \rightarrow \gamma + \gamma$  decay



Fig. 14

The coupling constant  $\gamma - \omega$  is defined as  $\operatorname{em}_{\omega}^2/2\gamma_{\omega}\sqrt{3}$  (see Fig.14). One needs  $\gamma_{\omega}$  and  $f_{\omega\rho\pi}$ . From unitary symmetry one has

$$\gamma_{\omega} = \gamma_{\rho}$$
 (24)

This is the reason why the factor  $\sqrt{3}$  was inserted in the definition of the  $\omega$  -  $\gamma$  coupling constant. Then inserting a value for the  $\pi^0$  lifetime one determines  $f_{\omega\rho\pi}$  . Inserting  $~\Gamma_{\!\pi}\sim$  3 eV, one finds  $f_{\omega\rho\pi}$  and, from such a value and from Eq. (21), one derives

$$\Gamma_{\rm e} \simeq 0.4 \, {\rm MeV}.$$
 (25)

The determination of the coupling constants is now complete. For instance, one can compute the rate for  $\omega 
ightarrow \pi + \pi$  through the decay shown in Fig. 15., for  $\omega \to e^+ + e^-$  (or  $\mu^+ + \mu^-$ ) through the decay shown in Fig. 16



and for similar decay modes for  $\rho$ . We recall that two main assumptions have been made: the  $\rho$ -dominant model and unitary symmetry, in obtaining Eq. (24). The colliding beam experiments, that I am going to describe, will provide a check of such assumptions.

Let us first consider the contribution from  $\rho^0$  intermediate states. The reaction  $e^+ + e^- \rightarrow \pi^+ + \pi^-$  can go through  $\rho^0$  excitation and consequent decay. The resonant cross-section is

$$\sigma_{r+s} \simeq 3.4 \times 10^{-31} \,\mathrm{cm}^2$$
. (26)

To compare it with a typical colliding beam cross-section, we consider  $e^+ + e^- \rightarrow \mu^+ + \mu^-$ , the cross-section of which is

$$\sigma \cong (1/3) \alpha^2 \pi \chi^2, \qquad (27)$$

as soon as the energy is such that one can neglect the  $\mu$ -mass. The crosssection (26) is about 1.7 times larger than that for annihilation into  $2\mu$  at the same energy.

One may also wonder whether the contribution of intermediate  $\rho^0$  states can become important in the electromagnetic processes  $e^+ + e^- \rightarrow e^+ + e^$ and  $e^+ + e^- \rightarrow \mu^+ + \mu^-$ . In fact, by a rough estimate one can convince oneself that this will not be the case. A detailed calculation was carried out by BROWN and COLOGERO showing that the effects are negligible [13].

Now let us consider the contribution from  $\omega^0$  intermediate states. The very important feature here is the very narrow width of  $\omega^0$ . We have estimated that it should be of the order of 0.4 MeV, according to Eq. (25). The energy resolution with  $e^+ + e^-$  colliding beams may easily be pushed down to a small fraction of a MeV. Under such conditions the  $\omega^0$  will show up in a very apparent way, and its will in fact originate the dominant contributions to the cross-sections at energies around its rest mass. We consider the following three processes

$$e^+ + e^- \rightarrow \omega^0 \rightarrow \pi^+ + \pi^- + \pi^0, \qquad (28)$$

$$e^{+} + e^{-} \rightarrow \omega^{0} \rightarrow \pi^{0} + \gamma, \qquad (29)$$

$$\mathbf{e}^{+} + \mathbf{e}^{-} \rightarrow \omega^{0} \rightarrow \pi^{+} + \pi^{-}, \qquad (30)$$

all proceeding through intermediate  $\omega^0$ . Making use of the information derived before on the couplings of  $\omega^0$ , we get for the total cross-sections of the reactions (28), (29) and (30), at the resonant energy:

$$\sigma_{\rm r} (3\pi) = 7 \times 10^{-29} \,{\rm cm}^2, \tag{31}$$

$$\sigma_{\rm r} (\pi^0 \gamma) = 4 \times 10^{-30} \ {\rm cm}^2, \tag{32}$$

$$\sigma_r (2\pi) = 0.8 \times 10^{-30} \text{ cm}^2.$$
 (33)

These values of the cross-sections are almost unbelievably large. We compare them with the cross-section for  $e^+ + e^- \rightarrow \mu^+ + \mu^-$  as given by the formula  $\sigma(2\mu) \simeq 1/3 \pi \alpha^2 \lambda^2$ . Note however that this formula does not hold at an energy near the  $\omega^0$  mass, just because there is also an amplitude arising from  $e^+ + e^- \rightarrow \omega^0 \rightarrow \mu^+ + \mu^-$  which interferes with the purely electro dy-

#### ELECTRON COLLIDING BEAMS

namic amplitude for  $e^+ + e^- \rightarrow \mu^+ + \mu^-$ , as discussed in the previous section. These two amplitudes are of the same order of magnitude. We find that:  $\sigma_r (3\pi)$ , as given by Eq. (31), is 390 times larger than  $1/3 \pi \alpha^2 \lambda^2$ ;  $\sigma_r (\pi^0 \gamma)$ is 22 times larger;  $\sigma_r (2\pi)$  is 4.7 times larger. It may also be relevant to stress that  $e^+ + e^- \rightarrow \pi^0 + \gamma$  is of higher electromagnetic order than  $e^+ + e^- \rightarrow \mu^+ + \mu^-$ . In fact, the processes (28), (29) and (30) are of order  $e^4$ ,  $e^6$  and  $e^8$  respectively. The decay of  $\omega^0$  in (29) occurs in fact by violating isotopicspin selection rules since the isospin of  $\omega$  is T = 0 and that of the final state must be T = 1. This illustrates how peculiar the situation becomes at that energy. Again, we want to stress the interest of an observation of the interference effect of  $e^+ + e^- \rightarrow \omega^0 \rightarrow \mu^+ + \mu^-$  with the lowest order electromagnetic amplitude.

# PROTON-ANTIPROTON ANNIHILATION INTO LEPTONS AND INTO VECTOR MESONS

I shall talk now on an experiment which is closely related to the colliding beam experiments we have been discussing, and which is being carried out at CERN by CONVERSI, FARLEY, MÜLLER and ZICHICHI [14]. The experiment consists in measuring the cross-section for:

$$p + \bar{p} \rightarrow e^{\dagger} + e^{-}. \tag{34}$$

The possibility of measuring the cross-sections for

$$p + \bar{p} \rightarrow \mu^{+} + \mu^{-} \tag{35}$$

and

$$p + \bar{p} \rightarrow B + \bar{B},$$
 (36)

where B is the suggested intermediate vector boson, is also being examined by the same group. These experiments were discussed in detail in a paper by ZICHICHI, BERMAN, CABIBBO and GATTO [15].

Let us first discuss the  $p + \tilde{p} \rightarrow e^+ + e^-$  process. It goes through the graph shown in Fig. 17 and is essentially the inverse process of  $e^+ + e^- \rightarrow$ 

 $p + \bar{p}$ . Any consideration valid for  $e^+ + e^- \rightarrow p + p$  also applies, with the relevant changes, to  $p + \bar{p} \rightarrow e^+ + e^-$ . The proton electromagnetic vertex is explored in the experiment for timelike values of the momentum transfer. In the equal notations the proton electromagnetic vertex is given by



$$\overline{u}(p) \{ F_1(k^2) \gamma_{\mu} + [F_2(k^2)/2M] \sigma_{\mu\nu} k_{\nu} \} u(p)$$
(37)

where we have defined

$$\begin{aligned} \sigma_{\mu\nu} &= (1/2)(\gamma_{\mu} \gamma_{\nu} - \gamma_{\nu} \gamma_{\mu}), \\ F_{1}(0) &= e \\ F_{2}(0) &= e \mu_{\rho}. \end{aligned}$$

In Fig. 18 we consider the real  $k^2$  axis where  $k = q_+ + q_-$  is the virtualphoton momentum, and indicate the physical region of the electron-proton scattering experiments  $(k^2 > 0)$ , the physical region for  $e^{\dagger} + e^{-} \rightarrow p + \bar{p}$  or  $p + \bar{p} \rightarrow e^{\dagger} + e^{-} (k^2 < -4M^2)$ , and the absorptive region  $(k^2 < -4\mu^2)$ . The form factors are complex only in the absorptive region.



Fig. 18

The cross-section for the process (34) is given in the centre-of-mass system by

$$\frac{\mathrm{d}\sigma\left(p\bar{p}\to\bar{e}^{\dagger}\bar{e}^{-}\right)}{\mathrm{d}\left(\cos\theta_{c}\right)} = \frac{\pi}{8} \frac{\alpha^{2}}{\mathrm{E}\sqrt{\mathrm{E}^{2}-\mathrm{M}^{2}}} \left[ \left| \mathrm{F}_{1}+\mathrm{F}_{2}\right|^{2} (1+\cos^{2}\theta_{c}) + \left| \frac{\mathrm{M}}{\mathrm{E}}\mathrm{F}_{1}+\frac{\mathrm{E}}{\mathrm{M}}\mathrm{F}_{2}\right|^{2} \sin\theta_{c} \right]$$
(38)

where  $F_1$  and  $F_2$  are taken at  $k^2 = -4E^2$ , E being the  $\bar{p}$  energy in the centreof-mass system and  $\theta_c$  the centre-of-mass angle. With the form factors  $G_1 = F_1 + F_2$  and  $G_2 = F_1 + (E/M)^2 F_2$ , Eq. (38) can be written as

$$\frac{\mathrm{d}\sigma\left(p\bar{p}\to e^{+}e^{-}\right)}{\mathrm{d}\left(\cos\theta_{c}\right)} = \frac{\pi}{8} \frac{\alpha^{2}}{\mathrm{E}\sqrt{\mathrm{E}^{2}-\mathrm{M}^{2}}} \left[ \left| \mathrm{G}_{1} \right|^{2} (1+\cos\theta_{c}) + \left(\frac{\mathrm{M}}{\mathrm{E}}\right)^{2} \left| \mathrm{G}_{2} \right|^{2} \sin\theta_{c} \right].$$
(39)

Numerical estimates of what the cross-section might be have very little value since nothing is known on the form factors in the timelike region. For example, we can consider two models:

(i) pointlike proton model

In this case  $F_1 = e$  and  $F_2 = 1.79 e$ ;

(ii) extrapolation of a fit to spacelike experiments [16] In this model, we have the following form factors:

$$F_1 = e[1 - 1.18 k^2 / (k^2 + 30 \mu^2)]$$
 and  $F_2 = 1.79 e[1 - 1.59 k^2 / (k^2 + 30 \mu^2)]$ .

#### ELECTRON COLLIDING BEAMS



Total cross-section for  $p + \overline{p} \rightarrow e^+ + e^-$  in units of 0. 75 × 10<sup>-\$1</sup> cm<sup>2</sup>

In Fig. 19 we have reported the values which one obtains in this way for  $\sigma_{tot}$  ( $p\bar{p} \rightarrow e^+e^-$ ), in units of  $0.75 \times 10 \text{ cm}^2$ . The upper curve 1 is for point-like proton with  $\mu_p = 1.79$ , the lower curve 2 is for model (ii).

With luck, from the angular distribution, one may measure  $|G_1|^2$  and  $|G_2|^2$ . However, in the physical region of the experiment the form factors are complex. This fact produces a polarization effect that does not occur in electron-proton scattering, in the physical region of which the form factors are real. Suppose you can dispose of polarized antiproton beams. If they collide on unpolarized protons and annihilate into  $e^+ + e^-$ , the cross-section is not the same for all  $\varphi$ , but there is an azimuthal effect, depending on the cosine of the angle that the normal to the production plane forms with the antiproton polarization vector. This effect is proportional, of course, to the sine of the phase difference between the two form factors  $F_1$  and  $F_2$  (or  $G_1$  and  $G_2$ ). The cross-section is given by

$$\frac{d\sigma}{d(\cos\theta_c)} = \frac{d\sigma}{d(\cos\theta_c)} \bigg|_{unp} + \frac{M}{E} \operatorname{Im} (G_1^* G_2) \bigg| \sin 2\theta_c \bigg| (\vec{P} \cdot \vec{n})$$
(40)

where  $\vec{P}$  is the antiproton polarization vector and  $\vec{n}$  the normal to the production plane. If the experiment is, instead, carried out with unpolarized antiprotons on a polarized-proton target, Eq. (40) holds again, except for changing the + sign into a - sign (this follows in general from the TCP theorem). If we write

$$G_1 \approx |G_1| e^{i\delta_1} ; \qquad (41)$$

$$G_2 = |G_2| e^{i\delta_2} , (42)$$

the experiment would inform us on  $\sin(\delta_1 - \delta_2)$ . However, some independent knowledge on the phases can be gained by the use of dispersion relations. It is clear that dispersion relations, giving an equation connecting the real part to the imaginary part of a form factor, can also be interpreted as giving a connection of the phase to the modulus. In fact an equation, such as

Re G (k<sup>2</sup>) = 
$$\frac{1}{\pi} P \int_{4\mu^2}^{\infty} \frac{dm^2 \operatorname{Im} G(m^2)}{(m^2 + k^2)}$$
, (43)

can also be interpreted as an integral equation for the phase  $\delta(k^2)$ , once  $\left|G(k^2)\right|$  is known:

$$|G(k^{2})|\cos\delta(k^{2}) = \frac{1}{\pi} P \int_{4\mu^{2}}^{\infty} dm^{2} \frac{|G(m^{2})|\sin\delta(m^{2})|}{(m^{2}+k^{2})}$$
(44)

Eq. (44) has inconvenient mathematical features. A most convenient way of using such an information has been suggested by FREUND and KUMMER [17]. Consider a function  $f(z) = |f(z)| \exp[i\delta(z)]$  which (a) is analytic in the cut plane z (Fig. 20), the cut going from a to  $\infty$ ; (b) has no complex zeros; and (c) satisfies  $f(z^*) = f^*(z)$ .



Fig. 20

Suppose you know |f(z)| on the whole real axis. Because of the condition (c), |f(z)| has no discontinuity; in fact  $|f(x+i\epsilon)| - |f(x-i\epsilon)| = |f(x+i\epsilon)| - |f^*(x+i\epsilon)| = 0$ .

What can be said on  $\delta(z)$ ? To this purpose we consider  $\log + (z) \approx \log |f(z)| + i\delta(z)$ . We use a subtracted Cauchy relation  $(z_0 \text{ is the point of subtraction})$ , taking the contour indicated in Fig.21. We have explicitly



Fig. 21

used our restriction (b) on the zeros of f(z). Taking now z and  $z_0$  real, and assuming that the integral on the circle can be neglected, we obtain

$$\log f(\mathbf{x}) - \log f(\mathbf{x}_0) = \frac{\mathbf{x} - \mathbf{x}_0}{2\pi i} \int_{-\infty}^{+\infty} \frac{\log f(\mathbf{x}') \, d\mathbf{x}'}{(\mathbf{x}' + i\epsilon - \mathbf{x}) (\mathbf{x}' + i\epsilon - \mathbf{x}_0)} \cdot$$

Now we use the identity

$$(x - x_0) \frac{1}{(x' - x + i\epsilon)} \cdot \frac{1}{(x' - x_0 + i\epsilon)} = (x - x_0) P \frac{1}{(x' - x)} \cdot \frac{1}{(x' - x_0)}$$

and, by separating the real part from the imaginary part, we find

$$\delta(x) - \delta(x_0) = -\frac{x - x_0}{2\pi} P \int_{-\infty}^{+\infty} \frac{\log |f(x')| dx'}{(x' - x) (x' - x_0)}$$
(45)

which is our desired equation, and also

$$\log |f(x)| - \log |f(x_0)| = \frac{x - x_0}{2\pi} P \int_0^{+\infty} \frac{\delta(x') dx'}{(x' - x)(x' - x_0)}$$
(46)

Let us apply Eq.(45) to  $G_1 = |G_1| \exp(i\delta_1)$ . It is convenient to subtract at  $k^2 = 0$ . We obtain

$$\delta_2(k^2) = -\frac{k^2}{2\pi} P \int_{-\infty}^{+\infty} \frac{\log |G_2(-m^2)| dm^2}{(k^2 + m^2) m^2}$$
(47)

How can one use Eq. (47)? Let us split the integration region into three parts:

$$\int_{-\infty}^{+\infty} = \int_{-\infty}^{-4M^2} + \int_{-4M^2}^{0} + \int_{0}^{+\infty}$$

In the first region one can insert the measured values of  $G_2(k^2)$  (from  $p+\tilde{p} \rightarrow e^++e^-$  or  $e^++e^- \rightarrow p+\tilde{p}$ ). In the second one can only make guesses; fortunately resonances seem to dominate in that region. In the third region one again inserts experimental values (from  $e+p \rightarrow e+p$ ). The guesses are not quite arbitrary. In fact one must find  $\delta(k^2)$  such that: (a) it must be real away from the absorptive region, i.e. for all  $k^2 - 4\mu^2$ ; (b)  $\delta_2(k^2) - \delta_2(k^2)$  must fit the values measured in the polarization experiment described before.

In the expressions (38) and (39) terms of the order  $(m_e/M)^2$ , where  $m_e$  is the electron mass and M is the nucleon mass have safely been neglected. If one wants to take into account the muon mass in the process  $p + \tilde{p} \rightarrow \mu^+ + \mu^-$ , one has simply to introduce a factor  $\beta_{\mu}$  (velocity of final  $\mu$  in the centre-of-mass system) in front of Eq. (38) or (39) and make the following substitution

$$\begin{aligned} (1+\cos^2\theta_{\rm c}) &\to (2-\beta_{\mu}^2\sin^2\theta_{\rm c})\,,\\ \\ &\sin^2\theta_{\rm c} \to (1-\beta_{\mu}^2\cos^2\theta_{\rm c})\,. \end{aligned}$$

For the total cross-sections one then finds

$$\frac{\sigma_{\rm t} \left( p \,\bar{p} \to \mu \,\mu \right)}{\sigma_{\rm t} \left( p \,\bar{p} \to e \,e \right)} = \frac{1}{2} \,\beta_{\mu} \left( 3 - \beta_{\mu}^2 \right) = 1 - \left( \frac{3}{8} \right) \left( \frac{m_{\mu}}{E} \right)^4 + 0 \left( \left( \frac{m_{\mu}}{E} \right)^6 \right). \tag{48}$$

This branching ratio is almost exactly equal to one. More important than these kinematic corrections are the radiative corrections. Their calcula-

tion is being carried out at Frascati. For the evaluation of them, the remark we made before concerning the interference effects between the twophoton and the one-photon channels comes out to be very relevant and simplifying. The  $2\mu$  versus 2e ratio in pp annihilation offers a very suitable way of measuring a possible high-energy breakdown of electrodynamics or a possible muon structure. The momentum transfers are always larger than 2M and they are the largest so far considered in such experiments. Furthermore, they are time-like and thus they provide essentially different information that provided space-like experiments.

The last topic I am going to review here is the mode of annihilation  $p+\bar{p} \rightarrow B+\bar{B}$  where B is a vector meson as suggested for mediating weak interactions.



The process occurs through graph shown in Fig. 22. The most general form of the electromagnetic vertex, for a spin-one boson is, on invariance grounds

$$J_{\mu} = G_{2} (\epsilon_{1} \epsilon_{2}) p_{\mu} + (G_{1} + \mu G_{2} + \epsilon G_{3}) [(\epsilon_{1} q) \epsilon_{2\mu} - (\epsilon_{2} q) \epsilon_{1\mu}]$$
$$+ \epsilon G_{3} m_{B}^{-2} [(q \epsilon_{1}) (q \epsilon_{2}) - 1/2 q^{2} (\epsilon_{1} \epsilon_{2})] p_{\mu}$$
(49)

where p is the difference of the final four-momenta of B and  $\overline{B}$ ,  $\epsilon_1$  and  $\epsilon_2$  are the polarization vectors of B and  $\overline{B}$ ,  $m_B$  is the mass of B,  $\mu + \epsilon$  is a possible anomalous magnetic moment of B and  $2\epsilon$  a possible anomalous electric quadrupole moment. The form factors  $G_1$ ,  $G_2$  and  $G_3$  depend on the squared momentum transfer  $q^2$ .

We also define the bilinear combinations

$$\begin{aligned} \mathbf{\hat{R}} &= (1/2) \left| \mathbf{G}_{1} + \mu \, \mathbf{G}_{2} + \epsilon \, \mathbf{G}_{3} \right|^{2} \, (\mathbf{E}/m_{B})^{2} \,, \\ \mathbf{S} &= (1/2) \left| \mathbf{G}_{1} + 2 \, (\mathbf{E}/m_{B})^{2} \, \epsilon \, \mathbf{G}_{3} \right|^{2} + (1/4) \, \mathbf{G}_{1} + 2 \, (\mathbf{E}/m_{B})^{2} \, \mu \, \mathbf{G}_{2} \, \Big|^{2} \,. \end{aligned}$$

The general expression for the cross-section is given, in the centreof-mass system,

$$\frac{d\delta(p\bar{p}\to B\bar{B})}{d(\cos\theta)} = \frac{\pi\alpha^2}{2EP} \beta_B^3 \left\{ R(A+B) + SA + (S-R) (B-A) \cos^2\theta \right\}, \quad (50)$$

$$\delta_{\rm T} ({\rm B}\bar{{\rm B}}) = \frac{\pi \alpha^2}{3 {\rm E} {\rm P}} \beta_{\rm B}^3 (2{\rm A} + {\rm B}) (2{\rm R} + {\rm S}) .$$
 (51)

In Eqs. (50) and (51)  $\beta_B$  is the velocity of B; A =  $(1/2) |F_1 + F_2|^2$  and B =  $(1/2) |(M/E) F_1 + (E/M) F_2|^2$  are exactly the same combinations of the nucleon form factors which determine the angular distribution of  $p + \tilde{p} \rightarrow e^+ + e^-$ .

#### ELECTRON COLLIDING BEAMS

Similarly, 2A + B also determines the total cross-section for  $p + \bar{p} \rightarrow e^+ + e^-$ . One thus finds for the ratio of  $B\bar{B}$  annihilation to  $e^+e^-$  annihilation

$$\mathbf{b} = \sigma_{\mathbf{f}} \ (\mathbf{p}\bar{\mathbf{p}} \to \mathbf{B}\bar{\mathbf{B}}) \ / \sigma_{\mathbf{f}} \ (\mathbf{p}\bar{\mathbf{p}} \to \mathbf{e}^{+}\mathbf{e}^{-}) = \beta_{\mathbf{B}}^{3} \ (2\mathbf{R} + \mathbf{S}) \ . \tag{52}$$

Eq. (52) holds in the most general case, and is still valid if the antiprotons are at rest.

If B has no anomalous moments and constant form factors, b is simply:  $b = \beta_B^3 [(3/4) + (E/m_B)^2]$ . In Fig. 23 this branching ratio is reported versus  $E/m_B$ . Of course E must always be larger than the nucleon mass. One sees that annihilation into a pair of intermediate mesons is favoured with respect to annihilation into e<sup>+</sup>e<sup>-</sup> or  $\mu^+\mu^-$  already for a centre-of-mass energy larger than 1.5 m<sub>B</sub>, provided B has no anomalous electromagnetic properties. In Fig. 23 we have also reported b for  $\mu = +1$  and  $\mu = -1$ ,  $\epsilon = 0$  and constant form factors. Once B is produced according to Eq. (36) it will





decay rapidly (in about  $10^{-17}$ s) into its disintegration products ( $2\pi$ ,  $3\pi$ ,  $\pi$  +K,  $\mu + \nu$ ,  $e + \nu$ , etc.). The annihilation events will exhibit definite angular correlations and in some cases they will be of the kind:

$$p + \bar{p} \rightarrow B^+ + B^- \rightarrow (\mu + \nu) + (\pi + \pi),$$

$$p + \bar{p} \rightarrow B^+ + B^- \rightarrow (\mu + \nu) + (e + \nu),$$

$$p + \bar{p} \rightarrow B^+ + B^- \rightarrow (K^0 + \pi^+) + (\pi^- + \pi^0)$$
 etc.,

which should allow the identification of B. Branching ratios among the various decay modes of B have recently been discussed by BERNSTEIN and FEINBERG [18].

To show that Eq. (49) is the most general form of the vertex  $J_{\mu}$  we note that there are four independent four-vectors, out of which  $J_{\mu}$  must be formed. They are  $q = p_1 + p_2$ ,  $p = p_1 - p_2$ ,  $\epsilon_1$  and  $\epsilon_2$ , where  $p_1$  and  $p_2$  are the final momenta of B and B. It can easily be checked that

$$p^2 + q^2 = -4 m_B^2$$

and so, one can form the following independent scalars

$$q^2$$
,  $(\epsilon_1 \cdot k)$ ,  $(\epsilon_2 \cdot k)$ , and  $(\epsilon_1 \cdot \epsilon_2)$ .

Then, observing that  $J_{\mu}$  must be linear in both  $\epsilon_1$  and  $\epsilon_2$ , one can write in general:

$$\begin{aligned} J_{\mu} &= q_{\mu} \left[ a(k^{2}) (\epsilon_{1}\epsilon_{2}) + b(k^{2}) (\epsilon_{1}k) (\epsilon_{2}k) \right] \\ &+ p_{\mu} \left[ c(q^{2}) (\epsilon_{1}\epsilon_{2}) + d(q^{2}) (\epsilon_{1}q) (\epsilon_{2}q) \right] \\ &+ \epsilon_{1\mu} (\epsilon_{2}k) e(k^{2}) + \epsilon_{2\mu} (\epsilon_{1}k) f(k^{2}). \end{aligned}$$

One has now to impose:

 $k_u J_u = 0$ 

which gives

$$Q(k^2) = 0,$$
  
 $b(k^2)k^2 + c(k^2) + f(k^2) = 0.$ 

However,  $J_{\mu} \rightarrow -J_{\mu}$  when  $p \rightarrow -p$  and  $\epsilon_1 \rightarrow \epsilon_2$  (because of charge conjugation invariance). Therefore e = -f and consequently also b = 0. Using all these conditions and by a redefinition of the form factors one gets the general expression (49) given above.

#### REFERENCES

- [1] TOUSCHEK, B., Proceedings of the Conference on High Energies, CERN (1961).
- [2] BERNARDINI, C., CORAZZA, G.F., GHIGO, G. and TOUSCHEK, B., Nuovo Cimento 18 (1960) 1293.
- [3] AMMAN, F., BERNARDINI, C., GATTO, R., GHIGO, G. and TOUSCHEK, B., LNF 61/5, Frascati National Laboratories, (January 27, 1961).
- [4] AMMAN, F., BASSETTI, M., BERNARDINI, M., CORAZZA, G.F., MASSAROTTI, A., MANGO, L., PELLEGRINI, C., PLACIDI, M., PUGLISI, M., TAZZIOLI, F., in Ricerca Scientifica (to be published).
- [5] O'NEILL, G. K., Proceedings of the International Conference on High Energy Accelerator and Instrumentation, CERN (1959) 125.
- [6] GATTO, R., Proceedings of the Conference on High Energy Physics, CERN(1961); CABIBBO, N. and GATTO, R., Phys. Rev. Lett. <u>4</u> (1960), 313; Phys. Rev. <u>124</u> (1961) 1577; GATTO, R, Proceedings of the Aix-en-Provence Conference on Elementary Particles (1961).
- [7] PUTZOLU, G., Nuovo Cimento 20 (1961) 542.
- [8] DA PRATO, MOSCO and PUTZOLU (to be published)
- [9] BUDINI and FURLAN, Phys. Rev. (to be published)
- [10] DRELL, S., Annals of Physics 4 (1958) 75.
- [11] KÄLLEN, G., Helv. Phys. Acta 25 (1952) 417.
- [12] GELL-MANN, M. and ZACHARIASEN, F., Phys. Rev. <u>124</u> (1961) 953; NAMBU, Y. and SAKURAI, Phys. Rev. Lett. <u>7</u> (1961) 469; GELL-MANN, SHARP and WAGNER, Phys. Rev. Lett. <u>8</u> (1962) 261; GELL-MANN, M., Phys. Rev. <u>125</u> (1962) 1067.
- [13] BROWN, L. M. and COLOGERO, F., Phys. Rev. 120 (1960) 653.
- [14] CONVERSI, FARLEY, MÜLLER and ZICHICHI (private communication).
- [15] ZICHICHI, BERMAN, CABIBBO and GATTO, Nuovo Cimento 24 (1962) 170.
- [16] FUBINI, S., Proceedings of the Aix-en-Provence Conference on Elementary Particles (1961).
- [17] FREUND, P. and KUMMER, Nuovo Cimento 24 (1962) 1160.
- [18] BERNSTEIN, J. and FEINBERG, G., Proceedings of the Aix-en-Provence Conference on Elementary Particles (1961).

# BOOK V

# VERY HIGH ENERGIES

PART III

WEAK INTERACTIONS

.

# WEAK INTERACTIONS AT HIGH ENERGIES

#### A. PAIS

### INSTITUTE FOR ADVANCED STUDY, PRINCETON, N.J. UNITED STATES OF AMERICA

#### 1. INTRODUCTION

These lectures will be devoted to a review of some open questions in the theory of the weak interactions, in particular such questions which stem from our ignorance about their high energy behaviour. I shall not discuss many of the better established facts about weak interactions which can be found in various survey articles [1-8]. I shall further confine myself to a few selected topics in the theory of leptonic phenomena. An attempt will be made to concentrate on such problems which are least obscured by guesses about the high energy behaviour of strong interactions and the corresponding form factors. In this spirit we shall discuss some reactions (such as weak lepton-lepton scattering) which are rather outlandish from an immediate practical point of view. However, the theoretical study of such processes is especially suited to bring out some of the most interesting general questions in weak interaction theory.

Quite recently strong evidence has been found [9] for the fact that the neutrino  $(\nu_{\mu})$  which accompanies the  $\mu^{+}$  in  $\pi^{+}$ -decay is distinct from the neutrino  $(\nu_{e})$  which accompanies the positron in  $\beta^{+}$ -decay.  $\nu_{\mu}$  and  $\nu_{e}$  have the same helicity (negative; left-handed). The probable upper limit on the mass of the  $\nu_{\mu}$  is about 5 electron masses [10]. (The limit on the  $\nu_{e}$ -mass is 10<sup>4</sup> times smaller). In what follows we neglect any consequences of a possibly finite  $\nu_{\mu}$ -mass. (It would be surprising if this mass were non-zero). Thus we assume for both kinds of neutrinos the invariance under the transformation K<sup>+</sup>  $\rightarrow \mu^{+} + \nu_{e}$ . As a result, neutrinos can have no (induced) magnetic moment. However, there does exist a non-vanishing electric charge form factor for neutrinos [11].

Earlier it had been noted as a theoretical possibility [12] that in strange particle decays the  $\nu_{\mu}$  and  $\nu_{e}$  might change roles (so that  $K_{\mu^{2}}^{*}$  decay would be  $K^{*} \rightarrow \mu^{*} + \nu_{e}$ ). There is experimental evidence against this interchange [9]. In the following we assume throughout that there exist two distinct neutrinos and that in all processes  $\mu^{*}$  is paired with  $\nu_{\mu}$ ,  $e^{*}$  with  $\nu_{e}$ .

There exists a body of evidence [7, 8] in support of the so-called  $\mu$ -e universality. Prior to the two-neutrino discovery this principle could be stated as a substitutional invariance  $\mu \leftrightarrow e$  in all interactions;  $\mu$  and e are identically coupled and differ only in their mass. Now we must add: if we interchange  $\mu$  and e, the same need be done for  $\nu_{\mu}$  and  $\nu_{e}$ . If one believes that the  $\mu$ -e difference is a secular effect of some interaction, then it would be hard to believe that there would not also exist non-secular  $\mu$ -e distinctions. About this subject the last word has not been spoken by any means. It is going to be one of the main experimental problems for the future to find out how well this universality works in high energy leptonic processes. However, for the purposes of the present lectures,  $\mu$ -e universality will be defined in the way just mentioned.

In Sec. II we consider pure leptonic processes, such as  $\mu$ -decay and lepton-lepton scattering. The difficulties of the high frequency behaviour of the Fermi-interactions treated in lowest order are discussed as well as their modification if there exist intermediate vector bosons. Sec. III is devoted to a discussion of general aspects of weak radiative effects. Here the famous question is: If higher order corrections are small why is this so? A brief discussion is given of the high <u>versus</u> low cut off alternatives and of the difficulties connected with a power series expansion of these corrections. It is recalled that these problems present themselves whether or not there exist intermediate vector bosons.

In the now past one-neutrino days some prime examples for the discussion of higher order weak effects were the processes  $\mu \rightarrow e\gamma$ , 3e, etc. With the advent of the two neutrinos these questions are now happily solved they don't exist to any order. But also in the two-neutrino theory there remain reactions of interest for the study of higher order problems, in particular some of those which cannot occur in lowest order and yet are not strictly forbidden. Such processes are mentioned in Sec.IV, devoted to some speculations about invariance groups for lepton problems which may be relevant to higher order weak effects.

In Sec. V we review the general structure of the heavy particle currents as they enter in the weak interactions with particular reference to CP-in-variance and to a  $|\Delta \vec{T}| = 1$  rule for the strangeness conserving processes. Finally we discuss in Sec. VI the principle of local action of lepton currents which may be of interest for an experimental exploration of weak radiative corrections.

The general spirit of what follows is to take the higher order weak effects seriously. This is not done because one can guarantee that they will produce observable effects in the foreseeable future (although one cannot assert the opposite either). Rather, the recent developments have served to bring to focus long known theoretical questions which now seem more immediate than before. Some of these questions may turn out to be ultraviolet herrings - like some of the problems posed in the early days of quantum electrodynamics. But even a proof of this would mean a distinct advance.

The topics to be discussed are all in the domain of high energy leptonic interactions. It is essential to the reasoning that one can isolate one dynamical factor, the lepton current, which can be studied independently of strong interaction effects. An approach of this kind cannot be followed for high energy non-leptonic interactions and this is the theoretical reason why such phenomena have not attracted much attention. The experimental reason is, of course, that one deals with tremendous background problems. In a sense the first experiments in this area have already been done [13]. At one time it was interesting to go below the associated production thresholds and see if single production is at all appreciable. We can now look upon such attempts as non-leptonic weak interaction experiments. However, since that time the nature of the problems has changed. From the theoretical weak interaction point of view, associated production thresholds do no longer form a particular point of interest (but from the experimental side the question will get progressively harder if one passes these thresholds). It is well to state, in the face of the experimental complexities, some qualitative questions one would like to ask. What about the  $\Delta S = 1$  and  $|\Delta \vec{T}| = \frac{1}{2}$  rules at high energies? What about the interesting parity properties found in  $\Sigma$ -decays, are they just low energy dynamical accidents or is something more subtle going on [14]? The hyperon and  $K_{\pi 2}$ -decays, being all of the two-body kind, give us a small number of "points" about non leptonic weak interactions but not distributions as in the 3-body leptonic decays.  $\tau$ -decays give distributions of low Q-value only. It may well become necessary for the understanding of the non-leptonic weak interactions to face also the intricacies of high energy reactions in this domain.

It is appropriate to recall that several of the problems here discussed are "old" ones in the time scale of modern theoretical physics. Thus already in 1936, Heisenberg noted [15] that the n-th order weak interactions behave in the high energy region as (momentum)<sup>2n</sup>. This led him to speculate on the existence of a universal length [16]. I am told [17] that these considerations created a great stir when they were first presented at a Copenhagen Conference of that time. The first calculations on "weak" radiative corrections also were made in the mid-thirties. They were attempts to describe nuclear forces by lepton pairs [18]. These early explorations were all in the spirit of high cut offs (in the sense explained in Sec. III). Also the earliest calculation on high energy effects with low cut offs dates from this time, namely a study of the  $\beta$ -decay of fast protons with momentum transfer to a Coulomb field [19].

We have learned a lot more physics since then, but the high energy behaviour of weak interactions no doubt still has to yield most of its secrets.

I am indebted to Dr. G. Feinberg for stimulating discussions on many of the questions discussed here.

#### 2. PURE LEPTONIC PROCESSES

These are the phenomena where to our knowledge strong interactions do not enter. An example is  $\mu$ -decay. For this process the effective interaction is

$$L_{eff} = -\frac{G}{\sqrt{2}} j_{\lambda}^{(\mu)*}(\mathbf{x}) j_{\lambda}^{(e)}(\mathbf{x}) + h.c.,$$

$$j_{\lambda}^{(\ell)}(\mathbf{x}) = \bar{\mathcal{I}}(\mathbf{x})\gamma_{\lambda}(1+\gamma_{5})\gamma_{\ell}(\mathbf{x}), \quad \bar{\mathcal{I}} = \mathcal{I}^{\dagger}\gamma_{4} \qquad (2.1)$$

$$j_{\lambda}^{(\ell)*}(\mathbf{x}) = \eta_{\lambda} j_{\lambda}^{(\ell)\dagger}(\mathbf{x}), \quad \eta_{\lambda} = \begin{cases} +1 & \lambda = 1, 2, 3\\ -1 & \lambda = 4 \end{cases}$$

*t* denotes hermitian conjugate. We put  $\hbar = c = 1$  throughout. stands for e or  $\mu$ . We introduce from the start the distinction between  $\nu_{\mu}$  and  $\nu_{e}$ . (2.1) gives a good account of the decay  $\bar{\mu} \rightarrow \bar{e} + \bar{\nu}_{e} + \nu_{\mu}$  with  $G \approx 10^{-5} \, \mathrm{m}_{\mathrm{proton}}^{-2}$ . This coupling implies also the existence of the reactions

$$\bar{\nu}_{e} + e^{-} \rightarrow \bar{\nu}_{\mu} + \mu^{-} \qquad (2.2)$$

$$\nu_{\mu} + e^{-} \rightarrow \nu_{e} + \mu^{-} \tag{2.3}$$

and of the adjoint reactions  $\nu_e + e^+ \rightarrow \nu_\mu + \mu^+$ ,  $\bar{\nu}_\mu + e^+ \rightarrow \bar{\nu}_e + \mu^+$ . These reactions are in accord with two conservation principles.

(1) The conservation of leptons;

(2) The conservation of  $\mu$ -number [20].

These are additive laws for quantum numbers which may be assigned as follows:

| Particle       | Lepton number | µ-number |
|----------------|---------------|----------|
| eī             | 1             | 0        |
| ν <sub>e</sub> | 1             | 0        |
| μ-             | 1             | 1        |
| $ u_{\mu}$     | . 1           | 1        |

The quantum numbers for the corresponding anti-particles have opposite sign. An example of a forbidden reaction is

$$\nu_{\rm e} + {\rm e}^- \to \nu_{\mu} + \mu^-$$
 (2.4)

Remark. As long as weak interactions are treated only to lowest order, it is not wrong but a bit silly to introduce quantum numbers; what is allowed and what not is read off directly from the (effective) coupling. However, there has developed recently some interest in the possible observable higher order effects of weak interactions (see Sec. III). In this more general situation quantum numbers are helpful. Reaction (2.4) cannot go to any order in weak interactions.

We consider the cross-sections  $d\sigma_{\tilde{\nu}}$  and  $d\sigma_{\nu}$  for the reaction (2.2) and (2.3) respectively. Let  $\tilde{k}$  and  $\tilde{k}'$  be the initial and final 3-momentum respectively in the centre-of-momentum system and  $\theta$  the scattering angle. We have

$$d\sigma_{e} = (G^{2}/\pi) \left[ k^{2}/\omega^{2} \left( k^{\prime} + \omega_{\mu} \right) \right] \left( k \cos \theta + \omega_{e} \right) \left( k^{\prime} \cos \theta + \omega_{e} \right) d \cos \theta , \quad (2.5)$$

$$d\sigma_{\mu} = (G^2/\pi) \left[ (k' + \omega_{\mu})/\omega_e \right] k'^2 d\cos\theta, \qquad (2.6)$$

where  $\omega_e = (k^2 + m_e^2)^{1/2}$ ,  $\omega_{\mu} = (k^{12} + m_{\mu}^2)^{1/2}$ .  $\bar{\nu}$ - and  $\nu$ -scattering are therefore different. This is due to a (V, A) interference effect as we shall see later in more detail. This interference vanishes in the forward direction,  $d\sigma_{\bar{\nu}} (\theta = 0) = d\sigma_{\nu} (\theta = 0)$ .

Eq. (2.6) shows that the scattering (2.3) goes via J = 0 only. The unitarity limit for total J = 0 scattering here is  $\pi/2k^2$ . The Fermi coupling

(1) therefore cannot possibly be correct [6] when we surpass this limit which is reached (neglect  $m_e$  and  $m_{\mu}$ ) for

k = 
$$(\pi^2 / 8G^2)^{1/4} \simeq 300 \text{ GeV}$$
 (2.7)

corresponding to  $\sigma \sim G \sim 10^{-33} \text{ cm}^2$ .

What damps the cross-section at these extremely high energies? Effectively it must be a mechanism which introduces some non locality. There are two suggestions which by no means mutually exclude each other; both of them may well be necessary to get a consistent picture.

(1) The Fermi interactions damp themselves. Because the crosssections get so large at ultra high energies there is no reason to confine oneself to first order calculations, if one takes (2.1) seriously as a field theory coupling at such energies. The higher order terms will now also become important and one will guess largely on dimensional grounds at a damping factor [11]  $\sim (1+G^2k^4)^{-1}$ .

(2) The damping comes about by a physical mechanism that shows how the interaction (2.1) in itself is only an approximate description of the state of affairs which (no doubt) holds well at low frequencies. A natural guess here is [21, 22] that the effective weak interaction (2.1) is brought about by the coupling of the lepton current to an intermediate charged boson field  $W_{\lambda}$ 

L = 
$$-g W_{\lambda}^{*} (j_{\lambda}^{(e)} + j_{\lambda}^{(\mu)}) + h.c.$$
 (2.8)

. .

 ${\bf G}$  is then related to the dimensionless coupling constant g and to the boson mass  ${\bf m}$  by

. .

$$G^2 / \sqrt{2} = g^2 / m^2$$
 (2.9)

Continuing to neglect lepton masses, the differential cross-section (2.6) is now damped by a factor  $[1+2m^{-2}k^2(1+\cos\theta)]^{-2}$ . We have therefore the high energy limit

$$\sigma_{\nu} \sim G^2 m^2 \sim g^2/m^2$$
 . (2.10)

(Note that this expression blows up for  $m \rightarrow 0$  (for fixed g). This is as it should be - it is like the infrared catastrophe in electron-electron scattering treated to lowest order with neglect of the influence of soft radiation emission).

It should be noted that the expression (2.10) contains contributions from all J-values, not just from J = 0. If one projects out the J = 0 part of the amplitude in question to calculate the J = 0 scattering cross-section  $\sigma_0$ , the result is

$$\sigma_0 \sim (Gm^2/k^2) lg^2 k/m$$
 (2.11)

This shows that it is not enough to take the intermediate boson effect to lowest approximation in order to avoid the conflict with unitarity at all energies [23]. If one now also takes into account self-damping one finds [23]  $\sigma \sim \text{Gm}^2$  $\text{k}^{-2} \log^{-1} g^{-2}$  and that the partial wave amplitudes decrease fast enough.

The interaction (2.8) obviously satisfies the conservations of leptons and  $\mu$ -number. (2.4) remains forbidden. A new consequence of (2.8) is a second order (in g) coupling of the (e,  $\nu_e$ ) current to itself and likewise for ( $\mu$ ,  $\nu_{\mu}$ ). It follows that the reactions (2.2) and

$$v_e + e^- \rightarrow \bar{v}_e + e^-$$
 (2.12)

should have the same cross-sections  $\sim g^4$ , apart from lepton mass effects. The Fermi interaction (2.1) allows the reaction (2.12) to happen as a higher order effect only, so there this equality does not hold.

The cross-sections involved here are presumably always  $\leq 10^{-33}$  cm<sup>2</sup> and much smaller at low energies. It will be hard to observe them. But their study has taught us something about the high frequency behaviour of the weak interactions under conditions unobscured by strong coupling effects. By this the following is meant. If we study the scattering  $\nu_e + \mu \rightarrow$  $p + e^-$ , say, using a point interaction of the Fermi type, then we would also get a cross-section  $\sim k^2$  at high energies. However, the existence of strong couplings implies that the effective interaction for this process is certainly not a local interaction, as we shall discuss in considerable detail (Section V). Unlike the pure leptonic processes, we cannot use reactions involving strongly interacting particles to infer that unavoidably something new has to happen with weak interactions at high frequencies.

#### 3. WEAK RADIATIVE CORRECTIONS

Higher order effects due to weak interactions are of interest because two opposing trends are at work. On the one hand we get higher powers of G which tend to make these effects insignificant. On the other hand, we saw in Sec. II that the local Fermi couplings are quite singular at high frequencies. (We leave the intermediate boson idea aside for a little while.) In the calculation of weak radiative corrections we meet of course integrations over all virtual frequencies and the question arises if high momentum contributions could perhaps be significant even though higher powers of G enter.

As an example, consider the corrections to the reaction (2.2). Note first of all that the order of the lowest non vanishing correction depends on the presence or absence of the self couplings of the (e,  $\nu_e$ ) current and of the ( $\mu$ ,  $\nu_{\mu}$ ) current which we discussed in relation with Eq. (2.8). (The existence of such self couplings can of course be considered also in the absence of W-fields.) If these couplings are present, the lowest corrections are a result of the graphs in Fig.1. As a result the <u>ratio</u> of the correction term to the leading term in the cross-section is proportional to the dimensionless quantity  $G\Lambda^2$ , where  $\Lambda$  is a cut off momentum. If there is no self-coupling, the lowest correction is due to the graph in Fig.2. In this case the ratio just mentioned is proportional to  $(G\Lambda^2)^2$ . Similar consequences of self coupling effects enter in many problems [24].

In any case, we are faced with the important question, what should be our guess for the magnitude of  $G\Lambda^2$ , that is for the cut off. In a future theory this cut off must of course be connected with real physics. The interesting



Lowest corrections to (2, 2) in the presence of lepton current self-coupling.



Lowest corrections to (2.2) in the absence of lepton current self-coupling.

thing is that which ever way we shall eventually get out of the dilemma something new happens.

(a) The cut off is low - weak radiative corrections are small. In other words, weak interactions are indeed weak at all frequencies, much weaker than electromagnetic effects. Thus the guess is now that  $G\Lambda^2 \ll e^2$  or

$$\Lambda \leq 30 \text{ GeV}$$
. (3.1)

This characteristic energy is much lower than the "breakdown energy" (2.7) which we discussed earlier. Thus if we believe all weak radiative corrections to be small, the physical mechanism which causes the cut off must set in long before we come close to the unitarity conflict discussed in Sec. II.

(b) The cut off is high. Now the interest lies in the possibility that weak radiative corrections might show up directly.

Whether or not these corrections will turn out to be observable, it does not seem sensible to treat them by a series expansion in G because higher powers in G are connected with increasing degrees of singularity. For example, the self-damping effect of Fermi interactions mentioned after Eq. (2.7) would look silly if expanded in  $G^2$ .

Actually, there are some indications that the cut off is low, namely the smallness of the  $K_1^0 - K_2^0$  mass difference and the present limits on the existence of parity non-conserving nuclear interactions [25]. It should be remembered, however, that in such cases the cut off due to strong interaction form factors may play an important role too. The example of lepton-lepton scattering is just so interesting because of the absence of strong interaction effects, but they are also pretty unrealistic for practical purposes. In Sec. VI we shall come back to the question how one may attempt to detect phenomena due to weak radiative corrections which cannot be confused with strong coupling effects.

Under any circumstances the weak radiative corrections pose interesting theoretical problems. If they are small, why are they small? Are they uniformly small in very high energy (real) phenomena? In these notes we keep an open mind about these questions. As we go along we will mention at times some items which bear on these corrections.

We saw in Sec. II that an intermediate vector boson field (if it exists) provides a mechanism to damp to lowest order the cross-section for reactions like (2.2) and (2.3). However, it does not follow by any means that the weak radiative corrections are small as well. The qualitative reason is the following. The propagator for a virtual vector boson is given by

$$-i (\delta_{\mu\nu} + m^{-2} q_{\mu} q_{\nu}) / (q^2 + m^2 - i\epsilon) .$$

Here  $q_u$  is the four momentum transferred to the W-particle. When we calculate the cross-section for (2.3) to lowest order this propagator is sandwiched between free lepton spinors and the application of the Dirac equation to those spinors shows that the term  $m^{-2} q_{\mu} q_{\nu}$  gives a contribution  $\sim m_e m_{\mu} m^{-2}$  (which we actually neglected earlier). Thus in this case the propagator contributes a factor  $\sim k^{-2}$  to the matrix element, as we saw earlier.

If the propagator is not taken between free particle states, its order may be (momentum)<sup>0</sup> rather than (momentum)<sup>-2</sup>, unless some angular averaging (or a renormalization argument) reduces the order. Thus the question of weak radiative corrections remains critical even in the presence of a vector boson field.

In the language of field theory we can summarize the situation as follows. The four Fermi interaction is unrenormalizable, but so is the theory of a charged massive spin l field. (For a neutral field of this kind it was shown that the theory is renormalizable [26].) Whether or not there are Wfields, the theoretical study of weak radiative corrections is therefore important. In fact, they promise to be far more interesting than the electromagnetic corrections (for spin 0 and 1/2 interactions) just because the latter are much less singular. It has recently been shown by Lee [27] for electromagnetic interactions of charged massive vector mesons how one may attempt to obtain finite results by summing up the most singular parts of the higher order effects, assuming that there exists a finite limit as an effective cut off tends to  $\infty$ .

#### 4. LEPTONIC SPIN

If we are only interested in  $\mu$ -decay, it is possible to write down a more general interaction than Eq. (2.8), namely we could couple the electron- and  $\mu$ -currents with distinct constants  $g_e$  and  $g_{\mu}$  respectively, for in  $\mu$ -decay only the product  $g_e g_{\mu}$  enters. The equality  $g_e = g_{\mu} = g$  means that we have chosen an interaction which satisfies  $\mu$ -e universality (see Sec. I).

Actually, the recent discovery of the distinction between  $\nu_e$  and  $\nu_{\mu}$  invites speculation about the existence of some sort of "spin" for leptons - in some ways similar to the isotopic spin for strongly interacting particles. In such an approach one may attempt to look upon the  $\mu$ -e universality operation (see Sec. I) as a discrete element of some rotation group. Such a group can certainly manifest itself only where it makes sense to neglect the  $\mu$ -e mass difference (or correct for it only in the kinematics). This may not be such a bad approximation in certain high energy neutrino experiments.

It may be somewhat early to pursue this subject at great length. It is another of those questions which would become of considerable interest if higher order weak effects were to be relevant in practice. Nevertheless, let us briefly state the general nature of the problem. There are two ways of approach which one may contemplate.

1. One groups the particles in terms of two "spinors" as follows [28]:

$$\psi_{\rm e} = ({\rm e}_{\nu_{\rm e}}), \quad \psi_{\mu} = ({\rm \mu}_{\nu_{\mu}}).$$
 (4.1)

There is a Pauli-type spin operator  $\bar{g}$  acting on these spinors. The eigenvalues +1, -1 of  $\zeta_3$  refer to the upper and lower components of the spinors respectively. Consider structures of the type  $\bar{\psi}_e \bar{g} \psi_e$ ,  $\bar{\psi}_\mu \bar{g} \psi_\mu$ . (For a while we neglect all ordinary spin factors, etc..) It is evident that we need the 1-and 2-components of these vectors in  $\bar{g}$ -space to construct the currents  $j_{\lambda}^{(e)}$  and  $j_{\lambda}^{(\mu)}$ . If we wish to impose a rotational invariance with respect to this space, it follows that we should also necessarily have to reckon with the occurrence of neutral lepton currents. And if we couple the lepton currents to the currents of the strongly interacting particles so as to describe  $\beta$ -decay,  $\pi$ -decay, hyperon and K-decays etc., we run therefore into the problem that no neutral leptonic decays of strongly interacting particles seem to exist. To be more precise, there is no evidence for the presence of neutral currents in strangeness changing leptonic decays [29]. For strangeness conserving decays it could easily be possible for neutral lepton currents to exist and yet to escape detection [30]. ( $\pi^0 \rightarrow e^+ + e^-$  by this mechanism would be masked by electromagnetic processes,  $\pi^0 \rightarrow \nu + \overline{\nu}$  is hopeless.)

It does not seem fruitful, therefore, to introduce a  $\zeta$ -space invariance in the manner described. Note further that electromagnetic interactions also violate this invariance. Also one has to neglect the e- and the  $\mu$ -mass completely to be able to rotate at all.

2. One groups the particles as follows [23]:

$$\psi_{\ell} = \begin{pmatrix} e \\ \mu \end{pmatrix}, \quad \psi_{\nu} = \begin{pmatrix} \nu_{e} \\ \nu_{\mu} \end{pmatrix}. \tag{4.2}$$

There is a spin  $\overrightarrow{\rho}$ , the "leptonic" acting on these spinors and  $\rho_3 = +1$ , -1 again refers to the upper and lower components respectively. We observe the following:

(a) Rotational symmetry in  $\rho$ -space can be upheld if we neglect the  $\mu$ -e mass difference.

(b) The conservation of leptons is a consequence of the gauge invariance of the first kind in leptonic spin space. This gauge group together with the leptonic spin group forms the group U(2). The  $\mu$ -number introduced in Sec. II is the relevant eigenvalue of  $\frac{1}{2}(1-\rho_3)$ .  $\mu$ -e universality follows as the consequence of the invariance under the unitary operation S =  $i\rho_2$  (which is the analogue for this space of the charge conjugation operation).

(c) To construct the currents  $j_{\lambda}^{(e)} + j_{\lambda}^{(\mu)}$  we need the combination  $\psi_{\ell} \psi_{\nu}$  which is a scalar in leptonic spin space. No neutral lepton currents are necessary. (d) To get the electromagnetic interactions we need  $\psi_{\ell} \psi_{\ell}$  which is also a scalar in this space. Electromagnetism respects leptonic spin. (e) If we couple a  $\psi_{\ell} \psi_{\nu}$ -structure to the strongly interacting particle current, the latter should be scalar too with respect to leptonic spin. Thus one can consider all strongly interacting particles individually as leptonic spin scalars. (The situation would be more complex in this respect if  $\nu_e$  and  $\nu_{\mu}$  interchanged in strangeness violating decays, but as we said in the introduction, we assume that this interchange does not take place. Note further that the use of (4. 1) would also necessitate to assign to the heavy particles a leptonic kind of quantum number.)

We give one example of the consequences of leptonic spin invariance in pure leptonic processes. Until further notice (to be given shortly) we neglect all electromagnetic effects. It was noted in Sec. II that in lowest order the reactions (2. 2) and (2. 12) have equal cross-sections (apart from the  $\mu$ -e mass difference effects). Call these cross-sections  $\sigma_a$  and  $\sigma_b$  respectively. Next note that the reaction

$$\overline{\nu}_{\mu} + e^{-} \rightarrow \overline{\nu}_{\mu} + e^{-}$$
 (4.3)

is forbidden to lowest non vanishing order. But it is not forbidden rigorously as it satisfies conservation of leptons and of  $\mu$ -number. See Fig. 3a for a typical graph. Let  $\sigma_c$  denote the cross-section of this last reaction.



Fig. 3a

Reaction (4.3) as a second order weak effect.

If leptonic spin conservation applies, there exists a triangular inequality between  $\sigma_a$ ,  $\sigma_b$  and  $\sigma_c$  namely [23]

$$\sqrt{\sigma_{b}} + \sqrt{\sigma_{c}} \ge \sqrt{\sigma_{a}} \ge \sqrt{\sigma_{b}} - \sqrt{\sigma_{c}}$$
 (4.4)

This is proved by the same methods as are used in isotopic spin discussions of nucleon-nucleon (and anti-nucleon) scattering.

It is also possible to apply related considerations to reactions involving heavy particles, for example to  $\overline{\nu}_{\mu} + p \rightarrow \overline{l} + \overline{l} + \ell + n$ , where  $\ell$  = e or  $\mu$  and where such  $\ell$ -combinations are chosen that conserve  $\mu$ -number.

It will be clear from these examples that it may well take a long time before we will have proof of the validity of leptonic spin or related ideas. Perhaps more important than this invariance itself is the question what breaks it. What causes the  $\mu$ -e mass difference? Here we cannot hide our ignorance behind strong interactions, as is often done with such abandon for the heavy particle mass differences. The  $\mu$ -e difference is in fact our strongest present clue for the existence of something new at high frequencies [31]. Something clearly eludes us here.
#### WEAK INTERACTIONS AT HIGH ENERGIES

In the previous discussion one point has emerged that is more general than leptonic spin itself, namely the possibility that higher order weak interactions might manifest themselves through processes that are forbidden if weak interactions are taken to lowest order only. A class of such phenomena are those where a neutral lepton pair is produced. It is clear that if we only insist on conservation of leptons and of  $\mu$ -number, it is possible to have higher order processes of the kind

| $\mathbf{A} \rightarrow \mathbf{B} + \boldsymbol{\mu}^+ + \boldsymbol{\mu}^-,$ | (4,5) |
|--|-------|
| $A \rightarrow B + a^{\dagger} + a^{-}$  | (1 6) |

|   |                             | · · ,                   | (40)  |
|---|-----------------------------|-------------------------|-------|
| L | $\rightarrow$ B + $\nu_{o}$ | $+\overline{\nu}_{a}$ . | (4,7) |

$$\mathbf{A} \rightarrow \mathbf{B} + \nu_{\mu} + \overline{\nu}_{\mu} \,. \tag{4.8}$$

(Leptonic spin will relate rather than forbid such reactions.) Thus higher order weak interactions generate effective neutral lepton currents even if we assume that such currents do not appear in the primitive interactions.

An example of a reaction with a charged lepton pair is  $K_2^0 \rightarrow \mu^+ + \mu^-$ . This conceivable but not observed decay is just the one used earlier to find bounds on the coupling strength of a neutral lepton current. It can now also be used to set additional bounds on the cut off (see Sec. III) for weak radiative effects in strangeness changing processes.

Examples of neutrino pair production are  $K^+ \rightarrow \pi^+ + \nu_e + \bar{\nu}_e$  or  $\pi^+ + \nu_\mu + \bar{\nu}_\mu$ . (These would be hard to entangle from  $K^+ \rightarrow \pi^+ + 2\gamma$  without subsequent conversion.) In the spirit of Sec. III these reactions are mentioned here because their slow rate may be a further reflection of the cut off mechanism of the weak interactions.

We now ask what is the influence of electromagnetic effects on Eqs. (4, 3 - 8). The following should be observed:

(1) The reaction (4.3) can go as a first order weak interaction provided we use a virtual photon. The mechanism is shown in Fig. 3b. This is the type of graph associated with the electric neutrino form factor. Also the reaction (2.12) gets a similar contribution. As electromagnetism respects leptonic spin, the relation (4.4) remains valid.



Reaction (4.3) as a first order weak effect generated by the neutrino form factor.

(2) Also the reaction  $K_2^0 \rightarrow \mu^+ + \mu^-$  can proceed as a first order weak (non-leptonic) process, namely [32]  $K_2^0 \rightarrow 2\gamma \rightarrow \mu^+ + \mu^-$ .

This reaction, if ever found, does therefore not necessarily constitute evidence for (primitive) neutral lepton currents.

(3) Reactions like  $K^+ \rightarrow \pi^+ + \nu_e + \bar{\nu}_e$  can also go as a first order weak process via the electric neutrino form factor, namely  $K^+ \rightarrow \pi^+ + \gamma$ ,  $\gamma \rightarrow \nu_e + \bar{\nu}_e$ . (The first stage is allowed as a virtual process only.)

Finally we note that there is another way of introducing a  $\mu$ -quantum number [33], namely a " $\mu$ -parity" instead of the  $\mu$ -number introduced in Sec. II. The  $\mu$ -parity rules are incompatible with leptonic spin.

#### 5. THE STRUCTURE OF HEAVY PARTICLE CURRENTS

It is assumed in all current theories that the effective interaction for leptonic processes involving strongly interacting particles is of the form:

$$\mathcal{L}_{eff} = -[J_{\lambda}^{*}(x)\{j_{\lambda}^{(e)}(x) + j_{\lambda}^{(\mu)}(x)\} + J_{\lambda}(x)\{j_{\lambda}^{(e)*}(x) + j_{\lambda}^{(\mu)*}(x)\}].$$
(5.1)

 $J_{\lambda}(x)$  is the effective heavy particle current. The \*-notation is as in Eq. (2. 1). (5. 1) is in accordance with  $\mu$ -e universality.  $J_{\lambda}$  contains one part which refers to strangeness conserving processes ( $\Delta S = 0$ ) and a part for which  $\Delta S = 1$ . We like to think that  $J_{\lambda}$  does not have a  $\Delta S = 2$  part even though the direct evidence for this is meagre where leptonic processes are concerned.

 $J_{\lambda}$  contains bilinear baryon terms and bilinear meson terms (also linear meson terms). The general form of one of the baryon terms is, as it appears

in 
$$J_{\lambda}$$
:  $\overline{B}_1 O_{\lambda} B_2$ , in  $J_{\lambda}^*$ :  $\eta_{\lambda} \overline{B}_2 \overline{O}_{\lambda} B_1$ , (5.2)

where

$$\bar{O}_{\lambda} = \gamma_4 O_{\lambda}^{\dagger} \gamma_4$$

and where the most general form of the vector operator  $O_{\lambda}$  is [34]

$$O_{\lambda} = \frac{i}{\sqrt{2}} [\gamma_{\lambda} (g_{V} + g_{A} \gamma_{5}) + (\frac{\overline{\partial}}{\partial x_{\lambda}} - \frac{\overline{\partial}}{\partial x_{\lambda}}) (f_{V} + f_{A} \gamma_{5}) + (\frac{\overline{\partial}}{\partial x_{\lambda}} + \frac{\overline{\partial}}{\partial x_{\lambda}}) (h_{V} + h_{A} \gamma_{5})].$$
(5.3)

Therefore

$$\overline{O}_{\lambda} = \frac{i}{\sqrt{2}} \left[ \gamma_{\lambda} \left( g_{V}^{*} + g_{A}^{*} \gamma_{5} \right) + \left( \frac{\overline{\partial}}{\partial x_{\lambda}} - \frac{\overline{\partial}}{\partial x_{\lambda}} \right) \left( f_{V}^{*} - f_{A}^{*} \gamma_{5} \right) - \left( \frac{\overline{\partial}}{\partial x_{\lambda}} + \frac{\overline{\partial}}{\partial x_{\lambda}} \right) \left( h_{V}^{*} - h_{A}^{*} \gamma_{5} \right) \right].$$
(5.4)

The choice of  $(B_1, B_2)$  pairs is first of all dictated by charge conservation. For  $\Delta S = O$  we can take  $(B_1, B_2) = (n, p)$ ,  $(\Sigma^0, \Sigma^+)$ ,  $(\Sigma^-, \Sigma^0)$  etc., for  $\Delta S = 1$ ,  $(B_1, B_2) = (\Lambda, p)$ ,  $(\Sigma^- n)$ , etc. The six quantities  $g_V$ , ...,  $h_A$  are invariant operators, that is, they are functions of the  $\Box$  operator (space-time), or equivalently of the invariant momentum transfer (momentum space). As we are discussing an <u>effective</u> interaction, these six functions must therefore be considered to describe the results of the iteration of all strong interactions. They are the unknown form factors or structure functions. We have a set of six such functions for each  $(B_1, B_2)$  pair. High energy leptonic reactions will largely be a gathering of information about the behaviour of the structure functions for large momentum transfers.

Similar considerations apply to those meson terms which are bilinear in the K- and  $\pi$ -fields [35]. Let us further assume that the K-particles are pseudoscalar, as is now pretty definite.

Then the meson terms have definite parity (they are all vector structures) and their general form is, as they appear

in 
$$J_{\lambda}$$
:  $\overline{M}_1 O_{\lambda} M_2$ , in  $J_{\lambda}^*$ :  $\eta_{\lambda} \overline{M}_2 O_{\lambda} M_1$ , (5.5)

where

$$O_{\lambda} = i \left[ f_{V} \left( \frac{\overrightarrow{\partial}}{\partial x_{\lambda}} - \frac{\overleftarrow{\partial}}{\partial x_{\lambda}} \right) + h_{V} \left( \frac{\overrightarrow{\partial}}{\partial x_{\lambda}} + \frac{\overleftarrow{\partial}}{\partial x_{\lambda}} \right) \right], \qquad (5.6)$$

$$O_{\lambda} = i \left[ f_{V}^{*} \left( \frac{\overline{\partial}}{\partial x_{\lambda}} - \frac{\overline{\partial}}{\partial x_{\lambda}} \right) - h_{V}^{*} \left( \frac{\overline{\partial}}{\partial x_{\lambda}} + \frac{\overline{\partial}}{\partial x_{\lambda}} \right) \right].$$
 (5.7)

Thus, for example, the  $\Delta S = 1$  current contains  $(\bar{K}, \pi^0)$  terms (K is the field of the charged K particles) which involve the two structure functions of  $K_{\ell s}^{\dagger}$  decay. At present, the only direct experimental information on structure functions outside the non-relativistic domain stems in fact from the various  $K_{\ell s}$  models. It has been found that these functions vary quite slowly over the range covered by the spectra [36].

Form factors have been studied extensively within the framework of dispersion theory. We shall not review these calculations here, but rather concentrate on some general properties of structure functions. As we shall see presently, the number of independent structure functions becomes constrained when certain invariance arguments are used. Therefore, high energy experiments may provide tests of the validity in the high energy range of the symmetries in question. Before we discuss these problems, we first make two general remarks.

(1) The h-functions. These appear in conjunction with the sum of right and left derivatives which is the total derivative of the bilinear form on hand. By partial integration we can throw the total derivative over on the lepton terms. It follows from the application of the lepton Dirac equation that in any leptonic process the h-terms give contributions proportional to the lepton mass. Chances are therefore much better to observe such terms in  $\mu$ - than in e-processes. Generally, the neglect of the lepton mass in any given process implies that we ignore the role of h-functions. This is true for all baryon and all meson terms in (5. 1). From Eqs. (5. 5 - 7) we see that wherever it is appropriate to neglect lepton masses, the effective meson current involves one single form factor only. This circumstance is of particular interest for  $K_{e3}$ -decay, for example [37].

(2) The f-functions. These appear together with the difference of right and left derivative. In the non-relativistic limit we retain only  $\partial/\partial x_4$ -terms which give contributions proportional to the heavy particle mass in question. The best known example is  $\beta$ -decay,  $\Delta S = 0$  where we have:

$$(B_1, B_2) = (n, p) : O_{\lambda} = \gamma_{\lambda} (G_V - G_A \gamma_5), (N, R, )$$
 (5.8)

$$G_V = g_V(q^2 = 0) - 2mf_V(q^2 = 0) \approx 10^{-5} m^{-2}$$
, (5.9)

$$G_A = -g_A (q^2 = 0) \cong 1, 2 G_V.$$
 (5.10)

 $G_V$  and  $G_A$  are the Fermi and Gamow-Teller constants respectively.  $q^2$  is the invariant momentum transfer. For  $|\Delta S| = 1 \beta$ -decays, the non-relativistic approximation may not be so good. For the e-mode in  $\Lambda$ -decay we have momentum transfers up to 175 MeV, for example [38].

Next we consider some invariance arguments.

(a) CP-invariance. It is a sufficient condition for CPT-invariance that we have a local theory invariant under the proper Lorentz group. Of course, the local property refers to primitive interactions and not to the effective interaction under discussion. The experimental situation with regard to CPT invariance has been discussed elsewhere [39] and it has been noted that more experiments are needed to verify its validity. We assume that CPT-invariance holds so that CP- and T-invariance imply each other [40]. We ask for the implications of CP-invariance.

Under the CP-transformation

$$(CP): j_{\lambda}^{(\ell)}(\mathbf{x}) \longleftrightarrow j_{\lambda}^{(\ell)*}(\mathbf{x}).$$
(5.11)

For the heavy particle current we have:

(C) : 
$$\frac{\overline{\partial}}{\partial x_{\lambda}} - \frac{\overline{\partial}}{\partial x_{\lambda}}$$
, (5.12)

(P): 
$$\frac{\partial}{\partial x_{\lambda}} \rightarrow -\eta_{\lambda} \frac{\partial}{\partial x_{\lambda}}$$
, for both ( $\rightarrow$ ) and ( $\leftarrow$ ). (5.13)

It follows that CP-invariance implies that:

(CP): all f, g, h-functions are real. 
$$(5.14)$$

This is true for all baryon and meson terms.

(b)  $|\Delta T| = 1$  for  $\Delta S = 0$ . Next we consider isotopic spin arguments for which we must treat  $\Delta S = 0$  and 1 separately. This subsection is exclusively devoted to  $\Delta S = 0$  currents. We inquire about the behaviour of these currents under the charge symmetry operation. This was first done by WEINBERG [41].

To begin with we note that in such  $\Delta S = 0$  processes as neutron  $\beta$ -decay and  $\pi$ -decay we have  $\Delta T_3 = 1$  and also  $|\Delta T| = 1$ . One can imagine nuclear  $\beta$ -decays in which  $|\Delta T|$  could take different values, say 2. A current bilinear in nucleons cannot produce such a change, but there are other currents which could give such an effect, namely those bilinear in  $\Sigma$ 's or in  $\pi$ 's. It is most economical to assume that such currents are not there [42]. This is the origin of the  $|\Delta T| = 1$  rule which, it should be stressed, is a stronger statement than just the exclusion of  $|\Delta T| \neq 1$  terms in  $\Delta S = 0$  currents.  $|\Delta T| = 1$  rule. Not only do J and J\* (for  $\Delta S = 0$ ) each behave as components of an isovector, but they transform as components of the same isovector with the same phase relations as those which occur in the strong interactions.

#### WEAK INTERACTIONS AT HIGH ENERGIES

Thus J and J<sup>\*</sup> are isotopically related to each other (with suitable conventions) in the same way as  $\pi^+$  and  $\pi^-$ , not as  $\pi^+$  and  $-\pi^-$  or as a complex mixture of both. It should be noted that distinctions of this kind only make physical sense if the phase relations between  $\pi^+$  and  $\pi^-$  (and likewise between other particle pairs related by charge symmetry) have been defined by another part of the interaction. In the present case the phase relations are of course defined by the strong interactions themselves. The weak couplings compatible with the  $|\Delta \overline{T}| = 1$  rule are the first class couplings in the sense of WEINBERG [41].

The  $|\Delta \vec{T}| = 1$  rule has consequences of two kinds. First, the fact that other  $|\Delta \vec{T}|$  values than 1 are to be excluded affects specifically those terms in the currents which are bilinear in  $\Sigma$ , or in  $\pi$ , or in K. The pure  $\Sigma$ -part of  $J_{\lambda}$  is of the general form

$$(\Sigma, \Sigma) : \overline{\Sigma}^0 O_{\lambda} \Sigma^{\dagger} + \overline{\Sigma}^{\dagger} O_{\lambda}' \Sigma^0.$$
(5.15)

and our restriction means that.

$$O_{\lambda} = -O'_{\overline{\lambda}}.$$
 (5.16)

Eq. (5.16) similarly applies to  $\pi\pi$ - and to KK-terms.

Secondly, the specific connection between J and J<sup>\*</sup> implied by the  $|\Delta \vec{T}|=1$  rule has two kinds of consequences. First, for all baryon and mesonterms we have

$$\bar{O}_{\lambda} = O_{\lambda}.$$
 (5.17)

Hence

$$|\Delta T| = 1: g_A, g_V, f_V, h_A$$
 are real, (5.18)  
 $f_A, h_V$  are imaginary,

Next, consider the  $\Sigma$   $\Lambda$ -terms in  $J_{\lambda}$  which are of the general form:

$$\overline{\Sigma}^{*}O_{\lambda}\Lambda + \overline{\Lambda}O_{\lambda}^{'}\Sigma^{'}.$$
 (5.19)

Each of these terms separately behaves like an isovector. Because of the conditions on phase relations we have (compare with the strong  $\Sigma \Lambda \pi$  coupling!)

$$(\Sigma \Lambda) : O'_{\lambda} = O_{\lambda}.$$
 (5.20)

Eqs. (5, 17) and (5, 20) imply that, apart from phase space corrections [43]

$$R(\Sigma^{-} \rightarrow \Lambda + e^{-} + \nu)/R(\Sigma^{+} \rightarrow \Lambda + e^{+} + \nu) = 1.$$
 (5.21)

Other consequences of the  $|\Delta \vec{T}| = 1$  rule are to be found in inelastic neutrino processes. Consider for example [22]:

$$\nu_{\ell} + p \rightarrow \ell^{-} + p + \pi^{+} \qquad (R_{1})$$

$$\nu_{\ell} + n \rightarrow \ell^{-} + n + \pi^{+} \qquad (R_{2}) \qquad (5.22)$$

$$\nu_{\ell} + p \rightarrow \ell^{-} + p + \pi^{0} \qquad (R_{3})$$

The symbols in brackets denote the respective rates. The rule implies triangular inequalities like

$$\sqrt{R_2} + \sqrt{2R_3} \ge \sqrt{R_1}, \text{ etc.}$$
 (5.23)

It should be pointed out that the  $|\Delta \vec{T}| = 1$  rule may be considered as a condition on primitive leptonic weak interactions because strong interactions respect the rule. It is clear that the rule is violated by electromagnetic corrections (and weak radiative corrections). These have therefore tacitly been ignored in the foregoing.

(c) Combined CP-invariance and  $|\Delta \vec{T}| < 1$ ,  $\Delta S = 0$ . If both requirements are imposed it follows from Eqs. (5. 14) and (5. 18) that

$$f_A = h_V = 0$$
 (5.24)

for all oaryon and meson terms.

A consequence of  $h_v = 0$  is the absence of an induced scalar term in  $\mu$ -absorption by a nucleon.

(d) Conserved vector current, [21]  $\Delta S = 0$ . In this theory, CP-invariance and  $|\Delta T| = 1$  are incorporated. The basic idea is that the V-part of  $J_{\lambda}$ ,  $J_{\lambda}^{*}$ , ( $\Delta S = 0$ ) are proportional to the  $T_3 = +1$ , -1 components respectively of the isotopic spin current. This has two consequences. (1) These V-currents are conserved if we neglect electromagnetic effects. This explains the equality to a good approximation of the Fermi-constant in  $\beta$ -decay and the  $\mu$ -decay constant (absence of renormalization effects). It is easily verified that our V-currents satisfy  $\partial J_{\lambda}^{V}/\partial x_{\lambda} = 0$  and likewise for  $J_{\lambda}^{*V}$  as long as we neglect mass differences within any isotopic multiplet, because we may use (5, 24). (We may apply the free particle wave equation to all field operators occurring in  $J_{\lambda}$ ,  $J_{\lambda}^{*}$ .) Therefore, the fact that the V-current is conserved does not impose restrictions on the form factors stronger than the consequences of the (weaker) requirements of combined CP-invariance and the  $\left| \Delta \vec{T} \right| = 1$  rule. (2) The V-parts in question have V-structure functions which are proportional to those structure functions which occur in the  $T_3 = 0$ part of the isotopic spin current. That is, they are proportional to the corresponding electromagnetic isotopic vector form factors.

For the nucleon, for example, we have two form factors  $f_V$  and  $g_V$  (see also Eq. (5. 24)) and the conserved current theory says that these can be expressed as follows (m = nucleon mass):

$$g_{V} = G_{V} [F_{Q} + (\mu_{p} - \mu_{n})F_{M}],$$
  

$$f_{V} = [G_{V}/2m] (\mu_{p} - \mu_{n})F_{M}.$$
(5.25)

 $F_O$  and  $F_M$  are the isotopic vector electromagnetic form factors for charge

and magnetic moment, normalized to unity at  $q^2 = 0$ ,  $\mu_p$  and  $\mu_n$  are the proton and neutron moments in units e/2m. Thus Eq. (5. 25) allows us to use the information on the nucleon electromagnetic form factors in high energy lepton experiments [44]. In particular, if we neglect the lepton mass, we deal with only three form factors in the reaction  $\nu$  + nucleon  $\rightarrow \hat{\ell}$  +nucleon, namely  $f_V$ ,  $g_V$  and  $g_A$ . The first two are determined from high energy electronnucleon scattering via Eq. (5. 25), if the conserved current idea is correct. There remains  $g_A$  as the only unknown structure function.

Another example where the proportionality to electromagnetic form factors is useful is the decay  $\pi^- \rightarrow \pi^0 + e^- + \bar{\nu}_e$  which goes entirely via the V-current. This is practically a zero momentum transfer process so that the proportionality for the corresponding  $f_V$ , see Eq. (5.6), is as  $G_V/\sqrt{2}$  to e. Similarly for  $K^0 \rightarrow K^- + e^+ + \nu_e$  [45].

(e) Isotopic spin properties,  $|\Delta S| = 1$ . It has been suggested [46] that the  $|\Delta T| = 1/2$  rule for non-leptonic processes should also apply to leptonic reactions  $\Delta S = 1$ . This rule implies the validity of the rule [21]  $\Delta S/\Delta Q = +1$ (but not vice versa) and the latter seems to be violated [47]. There are certain theoretical ideas [48] which involve leptonic  $\Delta S = 1$  couplings which are not exclusively of the  $|\Delta T| = 1/2$  type, but these fall outside the scope of the present survey and we shall not discuss them here.

(f) "Overall current × current coupling". It has been suggested [21] thatall weak processes, leptonic and non-leptonic, follow from an effective interaction of the structure current times current. (In this scheme there appear also non-leptonic neutral currents.) The validity of this scheme (sometimes called universal Fermi interaction scheme) is tied to the applicability of a  $|\Delta T| = 1/2$  rule to both non-leptonic and leptonic interactions. In view of the preceding remarks it is not timely to discuss such proposals at the present stage of developments.

(g) Primitive interactions. We have exclusively dealt with effective interactions. One may ask about the structure of the primitive couplings which effectively lead to the currents here discussed. Suppose for example that we start from a theory with trilinear local couplings. Then the conserved vector current proposition implies that the primitive V-current for  $\Delta S = 0$  is

$$\frac{\mathrm{i}G_{\mathbf{V}}}{\sqrt{2}} \left[ \overline{p}\gamma_{\lambda}n + \overline{\Xi}^{0} \gamma_{\lambda} \Xi^{-} + \sqrt{2} \left( \overline{\Sigma}^{0} \gamma_{\lambda}\Sigma^{-} - \overline{\Sigma}^{+} \gamma_{\lambda}\Sigma^{0} \right) + \sqrt{2} \pi^{+} \left( \frac{\overline{\partial}}{\partial x_{\lambda}} - \frac{\overline{\partial}}{\partial x_{\lambda}} \right) \pi^{0} + K^{+} \left( \frac{\overline{\partial}}{\partial x_{\lambda}} - \frac{\overline{\partial}}{\partial x_{\lambda}} \right) K_{0} \right].$$

$$(5.26)$$

So far, the study of such "basic" interactions has not yielded any useful results. Still, Eq. (5.26) is at least interesting to look at, and to remind us that we are in need of arguments about the relative magnitude of the various terms in a current.

#### 6. LOCAL ACTION OF LEPTON CURRENTS

As we discussed in Section V, the most general form of the terms which enter in the heavy particle current is an operator of the type  $O_{\lambda}$  or  $\bar{O}_{\lambda}$  sandwiched between free fields. The finite distance character of these operators is the mathematical expression of the smearing out effects typical for strong coupling form factors. On the other hand, the lepton currents which appear

in (5. 1) are local, see their definition Eq. (2. 1). This expresses the absence of strong interactions for leptons. This point structure of  $j_{\lambda}^{(\ell)}(x)$  in Eqs. (2. 1) and (5. 1) is called the local action of lepton currents.

Schematically, the situation for a reaction of the type [49]

$$\nu + T \rightarrow F + \ell \tag{6.1}$$

is therefore as indicated in Fig. 4. The box represents the effects of strong interactions. T and F are attached at different points but the lepton pair



Fig. 4 The local action of lepton currents.

emerges at one point. The same picture applies, of course, also to decay reactions of the type

$$T \rightarrow F + \nu + \ell . \tag{6.2}$$

In Section III we surveyed some of the interesting theoretical questions connected with weak radiative corrections. In Eqs. (4.5 - 8) examples were given of reactions which are possible only via such higher order mechanisms if no neutral lepton currents exist. We now observe that if the weak radiative corrections play any observable role, deviations from the local action of lepton currents would be one possible way to find this out [50].

Consider for example the reactions

$$\overline{\nu}_{\theta} + p \rightarrow n + \ell^{+}$$
 (6.3)

$$\overline{\nu}_{\ell} + n \rightarrow p + \ell^{-}. \tag{6.4}$$

The character of the weak radiative corrections depends on the presence or absence of neutral lepton currents. If they are present (or absent) the lowest order corrections are as in Fig. 5 (or Fig. 6). (As was mentioned



Fig. 5

Lowest weak radiative correction to (6.3) in the presence of a neutral lepton current.



Lowest weak radiative corrections to (6.3) in the absence of a neutral lepton current.

in Section III one should not conclude from this that in the latter case the corrections are necessarily smaller than in the former.) In either case the leptons emerge from different points and we have an effective non-local action of the lepton current.

In order to judge whether non-local effects of this kind are present it is necessary to find out first what local action implies in practice. Before we turn to this question it seems worthwhile to observe the following.

(1) If deviations from local action are to turn up at all it is to be expected that the effect will be more manifest at high energies. In principle one can raise the question already for neutron  $\beta$ -decay but there, of course, one can not get much dynamical information anyway because the phase space is so small. The situation is more favourable for  $K_{\ell 3}$  decays (and hyperon  $\beta$ -decays). The local action problem was first raised in a study of these modes [37]. Still, even here the momentum transfers are not very high. (The same is true for reactions (4, 5 - 8).) Thus, high energy lepton reactions are the best place to look for such effects.

(2) If deviations from local action are ever found it will be of particular interest to know if they satisfy  $\mu$  -e universality.

(3) Also in the intermediate boson theories do we have the assumption of local action so that the whole question is independent of the existence of these bosons. To avoid confusion we note that in pure leptonic processes (Section II) an intermediate boson produces to lowest order a non-locality between two lepton pairs only [51], but not between the members of each pair.

(4) The local action problem can be raised independently of strong interaction form factors.

(5) The problem is also independent of the existence of neutral lepton currents. If the latter were to exist (which at the moment does not look plausible) one would certainly assume that they were local to lowest order, on the same grounds as for the charged lepton currents.

(6) It is a hard question whether weak radiative corrections are the only conceivable source for possible deviations from locality.

(7) The first high energy neutrino experiment has shown [9] that a sizeable fraction of the events is inelastic. It is therefore of interest to look for such implications of local action which are valid also if F in Eq. (6.1) represents an assembly of strongly interacting particles [52].

To get the results in their simplest form the choice of a co-ordinate system and of the variables in that system are important. We shall always work in the rest system of T. The various energy-momentum four vectors will be denoted as follows.  $P_{\lambda}^{(0)} = (0, im_0)$  for T,  $P_{\lambda}^{\ell} = (\vec{p}^{\ell}, i_{\omega})$  for  $\ell$ ,  $P_{\lambda}^{\nu} = (\vec{p}^{\nu}, ip^{\nu})$  for  $\nu$  and  $P_{\lambda} = (\vec{p}, iE)$  for F. Whenever F is an assembly of particles,  $P_{\lambda}^2 = -m^2$  is to be considered [52] as an independent variable along with  $p = [\vec{P}]$ . W will be either the differential cross-section for (6.1) or the decay distribution for (6.2), in either case summed over the lepton spin and over all intrinsic variables of F and T. Finally  $\theta$  will be the angle between  $\vec{p}^{\nu}$  and  $\vec{p}$ . Theorem I [53]. Apart from a given kinematic factor, the local action of the lepton current implies that W is a quadratic function in each of the three variables  $\cos \theta$ ,  $\omega$  and  $p_{\nu}$ :

K W (p, m, cos 
$$\theta$$
) =  $\alpha_0$  (p, m) +  $\alpha_1$  (p, m) cos  $\theta$  +  $\alpha_2$  (p, m) cos<sup>2</sup>  $\theta$ , (6.5)

K W (p, m, ω) = 
$$\beta_0$$
 (p, m) +  $\beta_1$  (p, m)ω +  $\beta_2$  (p, m)ω<sup>2</sup>, (6.6)

$$K''W''(p, m, p_{\nu}) = \gamma_0(p, m) + \gamma_1(p, m)p_{\nu} + \gamma_2(p, m)p^2.$$
(6.7)

Here the K<sup>•</sup>s denote the kinematic factors. The coefficients  $\alpha$ ,  $\beta$ , and  $\gamma$  depend on p and m only.

The proof of one of these relations implies that the other two hold as well because of [54]:

$$p = \pm (m^2 + m_0^2 - \mu^2 - 2m_0 E)/2(m_0 - E + p \cos \theta), \ \omega = \mp p_v + m_0 - E.(6.8)$$

We derive Eq. (6.6). The transition probability for the reactions (6.1) and (6.2) are found by taking the appropriate matrix element of the space integral of (5.1). Average the absolute square of the matrix element over the lepton spin. Because the lepton current is local, it follows that this average is of the form  $A_{\alpha\beta} P^{\nu}_{\alpha} P^{\rho}_{\beta}$ . Apart from a kinematic factor  $\omega^{-1} p^{-1}_{\nu}$ ,  $A_{\alpha\beta}$  depends on the heavy particle variables only and after performing all averages described above  $A_{\alpha\beta}$  depends on P and P<sup>0</sup> only. The dependence on the  $(l,\nu)$  variables is therefore as follows. Either we get terms containing  $P^{l}_{\alpha} P^{\nu}_{\alpha}$  which are independent of  $\omega$ ; or else we have to multiply each of the two factors:

$$P_{\alpha}^{\nu} P_{\alpha}^{\nu} = \pm m_{0} (E + \omega - m_{0}),$$

$$P_{\alpha} P_{\alpha}^{\nu} = \pm 1/2 [2m_{0}\omega + m^{2} - m_{0}^{2} - \mu^{2}]$$
(6.9)

with either of the factors;

$$P_{\alpha}^{0} P_{\alpha}^{\ell} = -m_{0}\omega$$
,  $P_{\alpha} P_{\alpha}^{\ell} = m_{0}(m_{0} - \omega - E) + 1/2(m^{2} + \mu^{2} - m_{0}^{2})$ ,

Under any circumstance we therefore get a quadratic function in  $\omega$ . The scalar coefficients still depend on the residual independent heavy particle variables p and m. This proves Eq. (6, 6).

The proof of these relations evidently does not depend on the validity of CP-invariance nor on any of the  $|\Delta T|$  -rules for weak interactions. It has been shown [52] that if one considers the reactions (6.1) and (6.2) with specified lepton helicity, one obtains an expression with five structure functions [55].

It should be noted that there exists one unavoidable deviation from local action, due to the electromagnetic coupling of l with either T or F or both [56]. This effect should be small, especially at high energies.

For the "elastic" processes (6. 3) and (6. 4) it is of course possible to express the coefficients in Eqs.(6. 5 - 7) in terms of the nucleon structure functions, using Eqs.(5.1 - 4) with  $(B_1, B_2) = (p, n)$ . Eq. (6.6) then takes the following explicit form [53]:

$$d\sigma = [\beta_2(p)\omega^2 + \beta_1(p)\omega + \beta_0(p)] (md^3 p) / [4\pi^2 \omega E_{p\nu} dE_{tot}]$$
(6.10)

where the three structure functions  $\beta$  are given by [57]

$$(1/2)\beta_{2} = |g_{A}|^{2} + |g_{V} - 2mf_{V}|^{2} + 2m(E - m)(|f_{V}|^{2} + |f_{A}|^{2}),$$

$$(1/2)\beta_{1} = (E - m)|g_{A} \pm g_{V}|^{2} - (\mu^{2}/2m)(|g_{A}|^{2} + |g_{V}|^{2})$$

$$+ 2m(E - m - \mu^{2}/2m)[(E + m)|f_{V}|^{2} + (E - m)|f_{A}|^{2} - 2Ref_{V}^{*}g_{V}]$$

$$- \mu^{2}Re[h_{V}^{*}g_{V} - (E + m)f_{V}^{*}h_{V} - (E - m)f_{A}^{*}h_{A}] \qquad (6.12)$$

$$\beta_{0} = (E - m)(E - m - \mu^{2}/2m)|g_{A} \pm g_{V}|^{2} + m(E - m + \mu^{2}/2m)$$

$$(|g_{A}|^{2} - |g_{V}|^{2}) - [2m^{2}(E - m) + (\mu^{2}/2m)(3E - m) - \mu^{4}/4m]$$

$$[(E + m)|f_{V}^{2}| + (E - m)(|f_{A}|^{2} - 2Ref_{V}^{*}g_{V}] + (\mu^{2}/2)(E - m + \mu^{2}/2m)$$

$$[(E + m)(h_{V}|^{2} + (E - m)|h_{A}|^{2}]$$

$$- \mu^{2}(E - m)Re[h_{V}^{*}g_{V} + h_{A}^{*}g_{A} - (E + m)f_{V}^{*}h_{V} - (E - m)f_{A}^{*}h_{A}]$$

$$+ (\mu 4/2m)Re[h_{V}^{*}g_{V} - h_{A}^{*}g_{A} - (E + m)f_{V}^{*}h_{V} - (E - m)f_{A}^{*}h_{A}] \qquad (6.13)$$

In Eqs. (6, 12 - 13) the upper and lower signs refer to the reactions (6, 4) and (6, 3) respectively. It follows from Eq. (6, 11) that the difference  $d\sigma_{\nu} - d\sigma_{\overline{\nu}}$  is (apart from a kinematic factor) a linear function in  $\omega_{\bullet}$  This difference depends on one combination of structure functions only, namely the V-A interference effect Re  $g_A^*$  g<sub>V</sub>. This is the same type of effect which we encountered in the comparison of Eqs. (2, 5) and (2, 6). It can be shown from quite general considerations that in these instances the difference between neutrino and anti-neutrino reactions must be due to interference between structure functions related to terms in the current of opposite parity [58]. The expressions for the  $\beta$ 's simplify if both CP-invariance and the  $|\Delta T| = 1$  rule hold, on account of Eq. (5, 24).

Eq. (6.7) takes the following form for the reactions (6.3 - 4). Replace in Eq. (6.10) the square bracket by  $\gamma_2(p)p_{\nu}^2 + \gamma_1(p)p_{\nu} + \gamma_0(p)$  with

$$\begin{split} \gamma_2(\mathbf{p}) &= \beta_2(\mathbf{p}), \quad \gamma_1(\mathbf{p}) = 2(\mathbf{m} - \mathbf{E})\beta_1(\mathbf{p}) + \beta_0(\mathbf{p}) \\ \gamma_2(\mathbf{p}) &= (\mathbf{m} - \mathbf{E})^2\beta_2(\mathbf{p}) + (\mathbf{m} - \mathbf{E})\beta_1(\mathbf{p}) + \beta_0(\mathbf{p}). \end{split}$$
(6.14)

This form is well suited to perform averages over a (known) incident neutrino spectrum for fixed p. After having done this, one can find one relation for fixed p between the three  $\gamma$ 's. The theorem implies that there should in general exist a linear relation between four such measurements (that is, done with four distinct spectra).

Equations similar to (6.10 - 14) can also be written down [59] for the reactions  $\nu$  + nucleon - hyperon +  $\ell$ .

Finally we examine the structures which arise when deviations from local action due to weak radiative effects are taken seriously. We shall maintain the view that the primitive interaction is due to a local lepton current of the (V, A) type coupled to something else (be it a boson field or a heavy particle current).

When we take into account only those non-local effects induced by the strong interactions in the heavy particle current, then it follows from Lorentz invariance and from the just mentioned structure of the primitive interaction that the effective interaction is of the form: (V, A) heavy particle source x (V, A) lepton source. This is no longer true if the weak radiative corrections are included as well. It is instructive to distinguish between two general classes of such radiative effects.

(a) Lepton-lepton weak radiative corrections. These are schematically indicated in Fig. 7. A lepton pair is produced in point interaction with the



General structure of lepton-lepton weak radiative corrections.

heavy particle source. The leptons then interact weakly with each other. In this special case the general interaction is still of the general form (5.1) where  $J_{\lambda}$  is still the general current discussed in Section V. But  $j_{\lambda}^{(\ell)}(\mathbf{x})$  is now no longer given by Eq. (2.1). Instead, we must also admit the presence of induced leptonic terms. Thus  $j_{\lambda}^{(\ell)}$  is now of the form:

$$j_{\lambda}^{(\ell)} = \bar{\ell} O_{\lambda} \nu, \qquad (6.15)$$

where  $O_{\lambda}$  is given by Eq. (5. 3) with structure functions g, f, h appropriate to induced weak effects. However, we have assumed throughout that all basic neutrino reactions are of the two-component type. Hence the primitive interaction is invariant under the  $\gamma_5$ -transformation:

$$\nu' = \gamma_5 \nu, \quad \overline{\nu}' = -\overline{\nu} \gamma_5 \tag{6.16}$$

and so, therefore, is the effective interaction. It follows that

$$(l, \nu)$$
 :  $g_A = g_V$ ,  $f_A = f_V$ ,  $h_A = h_V$ . (6.17)

Let us next look at the expression (6.15) in the zero lepton mass approximation which at high momentum transfers is certainly good for electrons and is not bad for  $\mu$ -mesons. In this approximation we have the additional invariance for

$$\mathcal{L}' = \gamma_5 \mathcal{L}, \qquad \bar{\mathcal{L}}' = -\bar{\mathcal{L}}\gamma_5. \qquad (6.18)$$

Whenever Eq. (6.18) applies we have

$$h_A = h_V = f_A = f_V = 0.$$
 (6.19)

The only structure function which then remains is  $g_{\nu}$  which may now depend on the invariant momentum transfer. This does not change the situation insofar as the dependence on the individual four momenta  $p_{\lambda}^{\nu}$  and  $p_{\lambda}^{\dagger}$  is concerned. Thus the arguments used in the proof of Theorem I apply here too. <u>Theorem II.</u> In the zero lepton mass approximation the equations (6, 5 - 7) of the local action theorem are also valid if lepton-lepton weak radiative corrections are included.

(b) Lepton-heavy particle weak radiative corrections. These involve combinations of interactions between either  $\nu$  or l and F or T. One example is drawn in Fig. 8. The effective interaction now contains in general the following kinds of terms:





A lepton-heavy particle weak radiative correction.

 $(\alpha)$  Scalar terms of the form:

$$S(x)\overline{\ell}(1+\gamma_5)\nu + h.c. \qquad (6.20)$$

where S is a scalar /pseudoscalar function of the heavy particle fields. In Eq. (6. 20) the strict  $\gamma_5$ -invariance (6. 16) has been taken into account. A

scalar structure function originating from the lepton part of (6.16) may be thought to be absorbed in S. It follows from Eq. (6.18) that the induced terms (6.20) are zero in the zero lepton mass approximation.

( $\beta$ ) Vector terms. Their discussion is identical with the one given above for lepton-lepton corrections.

 $(\gamma)$  Tensor terms in the effective interaction of the form

$$J_{\mu\nu}(\mathbf{x})\overline{l}[g_{T}\sigma_{\mu\nu} + f_{T}\gamma_{\mu}(\frac{\overline{\partial}}{\partial x_{\nu}} - \frac{\overline{\partial}}{\partial x_{\nu}}) + h_{T}\gamma_{\mu}(\frac{\overline{\partial}}{\partial x_{\nu}} + \frac{\overline{\partial}}{\partial x_{\nu}})](1 + \gamma_{5})\nu \qquad (6.21)$$

with three lepton structure functions  $g_T$ ,  $f_T$ ,  $h_{T^*}$ ,  $J_{\mu\nu}$  is a heavy particle tensor source. Note that the  $h_T$ -terms can be brought to the form (V, A) x(V, A) by a partial integration, so for this term Theorem II applies forthwith. Moreover, in the zero lepton mass approximation the g-term goes to zero as well, so that in this case only the  $f_T$ -terms survive. If we now decompose  $J_{\mu\nu}$  into its irreducible parts, its trace term does not contribute for zero lepton mass.

( $\delta$ ) In the same way one can discuss tensors of higher rank. In the zero lepton mass approximation we thus find that the effective interaction can be written generally as:

$$J_{\mu} \bar{l} \gamma_{\mu} (1 + \gamma_{5}) \nu + J_{\mu\nu} \bar{l} \gamma_{\mu} D_{\nu} (1 + \gamma_{5}) \nu + J_{\mu\nu\rho} \bar{l} \gamma_{\mu} D_{\nu} D_{\rho} (1 + \gamma_{5}) \nu + \dots + h.c.$$
(6.22)

where  $D_{\mu} = \frac{1}{\partial x_{\mu}} - \frac{1}{\partial x_{\mu}}$ . We have  $J_{\mu\mu} = 0$ ,  $J_{\mu\nu\rho} = J_{\mu\rho\nu}$ ,  $J_{\mu\mu\rho} = 0$  etc. Each successive term raises by two the maximum power of  $\cos \theta$ , or  $\omega$ , or  $p_{\gamma}$  which appears in the differential cross-section for any process of the type (6, 1).

#### REFERENCES

- WU. C.S., "History of B-decay", in "Beiträge zur Physik und Chemie des 20. Jahrhunderts", Vieweg. Braunschweig, 1958.
- [2] LEE, T.D., and YANG, C.N., Elementary Particles and Weak Interactions, Brookhaven National Laboratory BNL 443 (T-91) (1957).
- [3] GELL-MANN, M., and ROSENFELD, A., Ann. Rev. nucl. Science 7 (1957) (Annual Reviews Inc. Palo Alto, California 1957).
- [4] WU, C.S., "The neutrino", Pauli Memorial Volume, p. 249. Interscience Publishers Inc., New York, 1960.
- [5] OKUN, L., Ann. Rev. nucl. Science, 9 (1960).
- [6] Particle Physics Seminar, 1961, CERN Report 61-30 (1961).
- [7] PAIS, A., Report on Weak Interactions, 12th Solvay Congress (1961).
- [8] FEINBERG, G., Lectures on Weak Interactions, 1962, Brookhaven National Laboratories (to be published).
- [9] DANBY, G., et al., Phys. Rev. Letters 9 (1962) 36.
- [10] BAHCALL, J., and CURTIS, R., Nuovo Cimento, 21 (1961) 422.
- [11] BERMAN, S., Proc. International Conference on Very High Energy Phenomena, CERN Report 61-22(1961).
- [12] FEINBERG, G., GÜRSEY, F., and PAIS, A., Phys. Rev. Letters 7 (1961) 208.
- [13] GARWIN, R., Phys. Rev. <u>90</u> (1953) 274; BALANDIN, M., et al. JETP <u>29</u> (1955) 265; Soviet Physics JETP 2 (1956) 98.
- [14] For a survey see PAIS, A., Revs. mod. Phys. 33 (1961) 493.
- [15] HEISENBERG, W., Z. f. Phys. 101 (1936) 533.
- [16] HEISENBERG, W., Ann. der Physik 32 (1938) 20.
- [17] UHLENBECK, G.E., private communication.

- [18] WEIZSACKER, C., Z. f. Phys. 102 (1936) 572; FIERZ, M., Z. f. Phys. 104 (1936) 553.
- [19] NORDHEIM, G., NORDHEIM, L., OPPENHEIMER, J., and SERBER, R., Phys. Rev. 51 (1937) 1037.
- [20] See K. NISHIJIMA, Phys. Rev. 108 (1957) 907; SCHWINGER, J., Ann. Phys. 2 (1957) 407; BLUDMAN,
   S., Nuovo Cimento 9 (1958) 433; PONTECORVO, B., JETP 37 (1959) 1751; Soviet Physics JETP 10 (1959) 1236; FEINBERG, G., and WEINBERG, S., Phys. Rev. Letters, 6 (1961) 381.
- [21] FEYNMAN, R. and GELL-MANN, M., Phys. Rev. 109 (1958) 193.
- [22] LEE, T.D., and YANG, C.N., Phys. Rev. 119 (1960) 1410.
- [23] FEINBERG, G., and GÜRSEY, F., "Approximate Symmetries in the 2-neutrino Theory", preprint.
- [24] IOFFE, B., JETP <u>38</u> (1960); 1608; Soviet Physics JETP <u>11</u> (1960) 1158; PONTECORVO, B., Physics Letters 1 (1962) 287.
- [25] Theoretical discussions are given by R. BLIN-STOYLE, Phys. Rev. 118 (1960) 1605; 120 (1960) 181.
- [26] MATTHEWS, P.T., Phys. Rev. 76 (1949) 1254.
- [27] LEE, T.D., "Application of 5-limiting Process to Intermediate Bosons", preprint.
- [28] BLUDMAN, S., Nuovo Cimento 9 (1958) 433; LIPMANOV, E., JETP 37 (1959) 1054; Soviet Physics JETP 10 (1960) 750; RYAN, C., "A doublet scheme for leptons", Dublin, preprint.
- [29] Reference 22, sec. VIII.
- [30] BLUDMAN, S., Phys., Rev. 115 (1959) 468.
- [31] Some authors have attempted to relate the μ-e difference to an asymmetric coupling of leptons to a neutral boson. See KOBZAREV, I., and OKUN, L., JETP 41 (1961) 1205, Soviet Physics JETP 14 (1962) 859, where references to earlier literature are also found.
- [32] One may attempt to estimate such effects by S. DRELL's method, Nuovo Cimento 11 (1959) 694.
- [33] FEINBERG, G., and WEINBERG, S., See reference [20].
- [34] The factor  $i/\sqrt{2}$  is conventional. In the expression for  $J_{\lambda^2}$  a derivative with an arrow to the right (left) acts on **B** on  $B_2$  ( $B_1$ ) only. The B-operators in (5.2) and also the M-operators in (5.5) are to be considered as free fields.
- [35] Generally one should not introduce as separate terms in the currents such "mesons" which decay by strong interactions. Their presence should be expressed by the structure of those terms in the currents which correspond to the decay products of these resonances. (It may under circumstances be necessary to consider other than bilinear meson terms.) I am indebted to Dr. M. BEG for a discussion of this point.
- [36] ROE, B., et al. Phys. Rev. Letters 7 (1961) 346; DOBBS, J., et al., Phys. Rev. Letters, 8 (1962) 295;
   BROWN, J., et al. Phys. Rev. Letters 8 (1962) 450.
- [37] PAIS, A., and TREIMAN, S., Phys. Rev. 105 (1957) 1616.
- [38] The general θ-decay matrix element for the β-process is discussed by ALBRIGHT, C., Phys. Rev. <u>115</u> (1959) 750.
- [39] See references [7] and [8].
- [40] The validity of CP-invariance in leptonic K-decays has been questioned by SACHS, R., and TREIMAN, S., Phys. Rev. Letters, 8 (1962) 137.
- [41] WEINBERG, S., Phys. Rev. 112 (1958) 1375.
- [42] TREIMAN, S., Nuovo Cimento 15 (1960) 916.
- [43] See reference [41]. R denotes the rate. The phase space correction changes the right hand side of Eq. (5.21) to 1,57, see reference [22].
- [44] LEE. F.D., and YANG, C.N., Phys. Rev. Letters 4 (1960) 307; YAMAGUCHI, Y., Progr. Theor. Phys., 23 (1960) 1117: CABIBBO, N., and GATTO, R., Nuovo Cimento 15 (1960) 304; LEE, T.D., MACKSTEIN, P., and YANG, C.N., Phys. Rev. Letters 7 (1961) 429.
- [45] Implications of the conserved current theory for the B-decays of S<sup>±</sup> into A are discussed by CABIBBO, N., and GATTO, R. Nuovo Cimento 15 (1960) 159.
- [46] OKUBO, S., MARSHAK, R., and SUDARSHAN, E., Phys. Rev. Letters 2 (1959) 12.
- [47] ELY. R.P., <u>e. al.</u> Phys. Rev. Letters, <u>8</u> (1962) 132; ALEXANDER, G., <u>et al.</u> Phys. Rev. Letters, <u>9</u> (1962) 69; these papers deal with  $\Delta S/\Delta Q$  problems in K<sub>53</sub>-decays. BARBARO-GALTIERI, A., <u>et al.</u>, Phys. Rev. Letters <u>9</u> (1962) 26 report an example of  $\Sigma^+ \rightarrow \eta + e^+ + \nu$  decay which also violates  $\Delta S/\Delta Q = = -1$ .
- [48] PAIS, A., Nuovo Cimento <u>18</u> (1960) 1003; BEHRENDS, R.E., and SIRLIN, A., Phys. Rev. Letters <u>5</u> (1960) 476; 8 (1960) 221; Phys. Rev. 121 (1961) 324.
- [49] In Eqs.(6.1) and (6.2)  $\nu$ ,  $\ell$  denote any lepton pair allowed by all conservation laws.
- [50] I am indebted to Dr. G. FEINBERG for stimulating discussions on this point,
- [51] For a discussion of such non-local effects see LEE, T.D., and YANG, C.N., Phys. Rev. 108 (1957) 1611.

#### A. PAIS

- [52] This general approach was first followed by LEE, T.D., and YANG, C.N., Phys. Rev. 126 (1962) 2239.
- [53] PAIS, A., Phys. Rev. Letters 9 (1962) August 1; see also, GOURDIN, M., and MARTIN, A., CERN preprint, 3568/TH.261, 1962.
- [54] In Eqs.(6.8 9) the upper and lower signs refer to the reactions (6.2) and (6.1) respectively.
- [55] See also ref. [52], footnote 13.
- [56] See ref. [37], Fig. 1 (b).
- [57] These expressions are in general not all positive, as is the case for the structure functions in ref. [53].
- [58] See TOLHOEK, H., and DEGROOT, S., Physica, 17 (1951) 81 and ref. [41] Theorem 1.
- [59] Cf. AZIMOV, Ya., and SHEKHTER, V., JETP 41 (1961) 592, Soviet Physics JETP 14 (1962) 424 and also ref. [15].

# THE INTERACTIONS IN K<sub>u3</sub> AND K<sub>e3</sub> DECAY

# S. W. MC DOWELL BRAZILIAN CENTRE OF RESEARCH IN PHYSICS, RIO DE JANEIRO, BRAZIL

#### INTRODUCTION

An attractive feature common to all present theories of weak interactions is the assumption that they result from the coupling (either direct or via intermediate bosons) of vector-axial vector currents. The V-A coupling of leptonic currents and baryonic currents conserving strangeness successfully explains the characteristic properties of  $\beta$ -decay,  $\pi$ -decay and  $\mu$ -decay. On the other hand, the leptonic decay modes of K-mesons provide the best source of information on the nature of the currents carrying strangeness. From  $K_{\mu 2}$  one learns that there must exist an axial-vector current carrying strangeness; further, if the leptonic weak interaction is universal, one can explain the absence of the K<sub>e2</sub> mode by excluding a pseudo-scalar interaction. From  $K_{\mu3}$  and  $K_{e3}$  decay one can obtain information on scalar, vector and tensor coupling of strangeness non-conserving currents with leptons. In recent experiments on  $K_{e3}$  decay [1, 2] a good fitting of the pion spectrum was obtained with a constant vector form factor. Although regarding this result as strong evidence of pure vector coupling, one might still argue that other possibilities, such as a mixture of vector and tensor couplings with energy-dependent form factors, are not excluded.

I shall discuss here, in the first place, some properties of the transition amplitudes which are independent of the structure of form factors. They provide a test of universality and a criterion for unambiguous determination of the nature of the weak interaction involved in these processes. Secondly, I shall discuss the structure of form factors, by means of dispersion relations and introducing explicitly the effect of the K\*-resonance in the  $K\pi$ -interaction.

### 1. NATURE OF INTERACTION

The basic assumption involved in this discussion is that the lepton pair is locally produced through an interaction containing no derivative coupling (Fig. 1).

## 1.1. Kinematics

The kinematical configuration of the decay products depends on two scalar variables. I choose them as the total energy W of the lepton pair in its centreof-mass system and the angle  $\alpha$  between the direction of the neutrino and pion in that system. The advantage of this choice of variables will soon be-



K-decay kinematics in the centre-of-mass system of the lepton pair

come apparent (Fig.2). These variables are related to the energies  $E_{\pi},~E_{\ell},~E_{\nu}$  for decay at rest by

$$W^{2} = m_{K}^{2} + m_{\pi}^{2} - 2m_{K}E_{\pi}, \qquad (1)$$

$$\cos \alpha = [(\mathbf{E}_{\ell} - \mathbf{E}_{\nu})\mathbf{W}^{2} - \mathbf{m}_{\ell}^{2}(\mathbf{m}_{K} - \mathbf{E}_{\pi})] / (\mathbf{W}^{2} - \mathbf{m}_{\ell}^{2}) \mathbf{p}_{\pi}.$$
(2)

The range of values of W<sup>2</sup> is independent of  $\cos \alpha : [m_f^2, (m_K - m_\pi)^2]$ .

#### 1.2. Dynamics

The general form of the matrix element is

$$\mathbf{M}^{\mathrm{I}} = (4 \mathbf{p}_{\mathrm{K0}} \mathbf{p}_{\pi 0})^{\frac{1}{2}} \langle \pi | \mathbf{J}_{\alpha}^{\mathrm{I}}(0) | \mathbf{K} \rangle \overline{\mathbf{u}}_{\ell} \mathbf{0}^{\alpha} \mathbf{u}_{\nu} = \overline{\mathbf{u}}_{\ell} \mathfrak{M}^{\mathrm{I}} \mathbf{u}_{\nu}, \qquad (3)$$

$$\mathfrak{M} = \frac{1}{\mathfrak{m}_{K}} g_{s} + \frac{1}{\mathfrak{m}_{K}^{2}} \gamma_{\alpha} (g_{\nu} p_{K}^{\alpha} + g_{\nu}^{\prime} p_{\pi}^{\alpha}) + \frac{i}{\mathfrak{m}_{K}^{3}} g_{T} \sigma_{\alpha\beta} p_{K}^{\alpha} p_{\pi}^{\beta}.$$
(4)

The index I in (3) stands for the isospin of the current. The g's are form factors that, following our basic assumption, depend only on the pion energy  $E_{\pi}$ . They are real if time reversal invariance holds.

#### 1.3. Transition rate [3]

$$dT = \frac{1}{(4\pi m_K)^3} \rho(s, \cos \alpha) \frac{p_{\pi}}{m_K} (1 - \frac{m_\ell^2}{s}) ds d\cos \alpha, \qquad (5)$$

$$\rho = \frac{1}{4} \left( 1 - \frac{m_{\ell}^2}{s} \right) \left\{ s \left| \left( g_s + \frac{m_{\ell}}{m_K} f_0 \right) + \left( g_T - \frac{m_{\ell} m_K}{s} f_1 \right) \frac{p_{\pi}}{m_K} \cos \alpha \right|^2 + \left| f_1 - \frac{m_{\ell}}{m_K} g_T \right|^2 p_{\pi}^2 \sin^2 \alpha \right\},$$
(6)

where  $s = W^2$  and

$$f_{0} = (1/2) (g_{V} - g_{V}) + (1/2) (g_{V} + g_{V}) (m_{K}^{2} - m_{\pi}^{2})/s$$

$$f_{1} = g_{V} + g_{V}'.$$
(7)

One can write

$$\frac{dT}{ds \ d \ \cos \alpha} = \frac{1}{(4\pi \ m_{\kappa})^3} (a_0(s) + \sqrt{2} \ a_1(s) \ \cos \alpha + a_2(s) \ \cos 2\alpha).$$
(8)

#### 1.4. Universality

I shall first draw attention to the coefficient

$$a_{2}(s) = \frac{p_{\pi}^{3}}{8 m_{K}} (1 - \frac{m_{\ell}^{2}}{s})^{3} \left(\frac{s}{m_{K}^{2}} |g_{T}|^{2} - |f_{1}|^{2}\right)$$

which depends on the lepton mass only through the factor  $(1 - \frac{m_{\ell}^2}{s})^3$ . Hence if the couplings with muons and electrons are identical, then [3]

$$(\mathbf{s} - \mathbf{m}_{\mu}^{2})^{-3} \mathbf{a}_{2\mu}(\mathbf{s}) \equiv (\mathbf{s} - \mathbf{m}_{e}^{2})^{-3} \mathbf{a}_{2e}(\mathbf{s}).$$
 (9)

The verification of this identity provides a test of the universality of the weak interaction, independent of the structure and nature of form factors.

#### 1.5. Angular correlation

The simple structure of the angular correlation (8) stems from the fact that in its centre of mass system the lepton pair is produced in the singlet state for scalar coupling  $g_s$  and  $f_0$  and in a triplet for vector  $f_1$  and tensor  $g_T$ . In K e<sub>3</sub>, if the electron mass is neglected, the spins would be parallel and

#### S. W. MC DOWELL

anti-parallel, respectively, for vector and tensor coupling ( $s_z = 1$  and  $s_z = 0$ ). The advantage of our choice of variables is that the simple angular correlation is preserved after integration over the pion energy or any part of the pion spectrum. This angular correlation is particularly suitable for the identification of the type of coupling. We give in Table I a summary of values of  $\lambda_{1,2} = a_{1,2}/a_0$  for  $K_{e3}$  decay when the electron mass is neglected [3] in the angular correlation

$$(dT/d\cos\alpha) \sim 1 + \sqrt{2} \lambda_1 \cos\alpha + \lambda_2 \cos 2\alpha.$$
(10)

#### TABLE I

#### VALUE OF $\lambda$ IN THE ANGULAR CORRELATION

 $(dT/d \cos \alpha) \sim 1 + \sqrt{2} \lambda_1 \cos \alpha + \lambda_2 \cos 2\alpha$  for  $K_{e3}$  decay

|                | v  | s | Т | (VS) | (VT)       | (ST)       |
|----------------|----|---|---|------|------------|------------|
| λι             | 0  | Q | 0 | 0    | 0.         | <b>≠</b> 0 |
| λ <sub>2</sub> | -1 | 0 | 1 | <0   | <b>≠</b> 0 | >0         |

The absolute value of  $\lambda$  is always less than or equal to one. By measuring this angular correlation one obtains clear-cut discrimination of the type of coupling. In particular for pure vector coupling (dT/d cos  $\alpha$ ) ~ sin<sup>2</sup> $\alpha$ .

#### 1.6. Ke3 decay

In  $K_{e3}$  decay the contribution to the transition rate of terms that depend on the electron mass is practically negligible. The expressions for the pion and electron spectrum are greatly simplified by neglecting the electron mass.

(a) Pion spectrum

$$\frac{\mathrm{dT}}{\mathrm{dE}_{\pi}} = \frac{\mathrm{p}_{\pi}}{(4\pi\mathrm{m}_{\mathrm{K}})^{3}} \left( \mathbf{s} \left| \mathbf{g}_{\mathrm{s}} \right|^{2} + \frac{1}{3\mathrm{m}_{\mathrm{K}}^{2}} \mathrm{s} \mathrm{p}_{\pi}^{2} \left| \mathbf{g}_{\mathrm{T}} \right|^{2} + \frac{2}{3}\mathrm{p}_{\pi}^{2} \left| \mathbf{f}_{\mathrm{1}} \right|^{2} \right).$$
(11)

Since at the lower end of the pion spectrum  $p_{\pi}^2 = 0$  and at the upper end s = 0, these factors are expected to dominate the pion spectrum, giving specific features for each type of coupling. However, I would like to point out that a mixture of tensor and vector coupling cannot be distinguished from pure vector when the energy dependence of the form factors is taken into account.

A determination of the  $\pi^0$  spectrum in  $K_{e3}$  decay was carried out by BROWN et al. [2]. They measured the angular correlation of the two  $\gamma$ -rays from the decay of  $\pi^0$  produced in  $K_{e3}^+$  decay at rest. The frequency dis-



Ke3 decay at rest

tribution of events as a function of  $u = m_{\pi} \operatorname{cosec} (\theta/2)$  is related to the pion spectrum by

F (u) = 
$$\int_{u}^{W_{\pi}} \frac{u}{\sqrt{E_{\pi}^{12} - u^{2}}} \frac{1}{p_{\pi}^{!}} \frac{dT}{dE_{\pi}^{!}} dE_{\pi}^{!}$$
 (12)

where  $W_{\pi} = \frac{1}{2m_K} (m_K^2 + m_{\pi}^2 - m_{\ell}^2)$  is the maximum pion energy. Using for  $(dT/dE_{\pi})$  the expression given by (11) with constant form factors, they obtained a remarkably good fitting taking  $g_s = g_T = 0$ . One can also determine directly  $(dT/dE_{\pi})$  in terms of F(u). Multiplying (12) by  $1/\sqrt{u^2 - E_{\pi}^2}$  and integrating between  $E_{\pi}$  and  $W_{\pi}$ , one obtains, after interchanging the order of integrations on the right,

$$\int_{E_{\pi}}^{W_{\pi}} \frac{F(u)du}{\sqrt{u^2 - E_{\pi}^2}} = \frac{\pi}{2} \int_{E_{\pi}}^{W_{\pi}} \frac{1}{p'_{\pi}} \frac{dT}{dE'_{\pi}} dE'_{\pi}.$$

Hence,

$$\frac{1}{p_{\pi}} \frac{dT}{dE_{\pi}} = -\frac{2}{\pi} \frac{d}{dE_{\pi}} \int_{E_{\pi}}^{W_{\pi}} \frac{F(u) du}{\sqrt{u^2 - E_{\pi}^2}} = -\frac{2}{\pi} \int_{E_{\pi}}^{W_{\pi}} \frac{E_{\pi}}{\sqrt{u^2 - E_{\pi}^2}} \frac{d}{du} \left(\frac{F(u)}{u}\right) du.$$
(13)

This expression may be useful in analysing a similar experiment for  $K_{\mu3}$ . The pion spectrum for pure vector coupling in  $K_{e3}$  decay is as shown in Fig.4. The shape of the spectrum determines the vector form factor  $f_1(s)$ 





Pion spectrum for pure vector coupling in  $K_{e3}$  decay

which was found to be constant within experimental errors.

(b)Electron spectrum

Some definite predictions can be made about the electron spectrum if one assumes pure vector coupling [3, 4].

Let us introduce the variables

$$\eta = E (1 - E/m_{K}),$$
(14)
  
w = 2 E (W<sub>e</sub> - E)/(m<sub>K</sub> - 2 E),

where  $W_e = (1/2 m_K) (m_K^2 - m_{\pi}^2)$  is the maximum value of the electron energy E; w is positive in the physical region, vanishes at both ends of the spectrum (E = 0, E =  $W_e$ ) and has a maximum at  $E_0 = \frac{1}{2} (m_K - m_{\pi}) = 175$  MeV. The electron energy is a double-valued function of w:

$$\mathbf{E} = \frac{1}{2} \left( \mathbf{W}_{e} + \mathbf{w} \pm \left[ \left( \mathbf{W}_{\pi} - \mathbf{w} \right)^{2} - \mathbf{m}_{\pi}^{2} \right]^{\frac{1}{2}} \right).$$

Throughout this discussion the electron mass is being neglected. For pure vector coupling the spectrum  $dT/d\eta$  of  $\eta$  is given by

$$\frac{dT}{d\eta} = \frac{1}{(4 \pi m_K)^3} \int_0^{2m_K w} (w - \frac{s}{2m_K}) |f_1(s)|^2 ds$$
(15)

which is an increasing function of w. Hence the maximum of the spectrum is at the energy  $E_0$ , i. e.  $\eta = W_e/2$ , independently of the structure of the form factor (Fig. 5). For any combination (VS) or (VT) one can show that the maximum would be at a higher energy. If  $dT/d\eta$  is plotted against w, the two branches, corresponding to energies  $E < E_0$  and  $E > E_0$ , will coincide for pure vector coupling (Fig. 6); for any combination (VS) or (VT) the first branch always remains below the second one.

The common shape of the two branches in Fig.6 depends on the structure of the vector form factor  $f_1(s)$ . One can show that

$$|f_1(s)|^2 = \frac{(4\pi m_K)^3}{2 m_K} - \frac{d^2}{d w^2} \left(\frac{dT}{d\eta}\right)_{w=s/2m_K}.$$
 (16)

Since the form factor depends only on the second derivative of the electron spectrum, this spectrum does not provide a good determination of  $f_1(s)$ . Conversely, one can conclude that the electron spectrum is not very sensitive to the structure of the vector form factor.



Fig.5

Fig. 6

Ke3 decay plot for V-coupling

η spectrum for pure V-coupling





We have seen that from  $K_{e3}$  decay one can obtain the vector form factor  $f_1(s)$ , but  $f_0(s)$  has to be determined from  $K_{\mu3}$  decay. Here besides the energy



Fig.7

Energy dependence of longitudinal polarization of muons

spectra and angular correlation one has another source of information, namely, the longitudinal polarization of muons. In Fig. 7 this polarization is given for pure vector coupling and constant form factors, for different values of the ratio  $\xi = (g_V - g'_V)/(g_V + g'_V)$ .

### 2. FORM FACTORS

I turn now to our second topic, an investigation of the structure of form factors. I restrict myself to vector form factors.

One starts with the assumption that

t

#### S. W. MC DOWELL

 $f_+ = g_v + g'_v = f_1$  and  $f_- = g_v - g'_v$ 

are analytic functions of s with a cut from  $(m_{\kappa} + m_{\sigma})^2$  to  $+\infty$ . The discontinuity across this cut is given by the absorptive part of the matrix element  $(4 p_{\pi 0} p_{K0})^{\frac{1}{2}} \langle 0 | J_{\alpha} | \pi k \rangle$ . One can then write dispersion relations for f, and f, which might require subtractions, depending on the behaviour at infinity. I shall assume that the dispersion relation for f, needs one subtraction and that for f. no subtraction [5]. According to this assumption,  $f_{-}$  is entirely induced by strong interactions. One can show that the contribution to the absorptive amplitude of nucleon-anti-hyperon intermediate states is compatible with this assumption [5]. Another possible justification is to invoke some higher symmetry in strong interactions which would ensure the conservation of the current. Suppose that there is a higher symmetry in the strong interactions connected with the conservation of a current carrying strangeness. In the level where this possible underlying symmetry holds. the masses of non-strange and strange particles within the same multiplet would be equal, in particular the kaon and pion masses. In order to get the mass splitting some interactions have to be introduced that break symmetry. However, the existence of such higher symmetry would still lead to some definite consequences. When the interactions that break the symmetry are turned on and the mass differences are thereby introduced, the current carrying strangeness is no longer conserved. However,"in the limit where the mass differences within a multiplet are neglected, the current is conserved. In this limit the divergence of the current is zero". One speaks then of a partially conserved current. Now the matrix element of the divergence of the current between the states of a kaon and a pion is given by

$$(4 p_{\pi 0} p_{K 0})^{\frac{1}{2}} \delta^{\alpha} \langle \pi | J_{\alpha}(x) | K \rangle = \frac{1}{2 m_{K}^{2}} [f_{+} (m_{K}^{2} - m_{\pi}^{2}) + sf_{-}] e^{-i(p_{K} - p_{\pi}) \cdot x}, \quad (17)$$

and this must be zero when  $(m_K^2 - m_{\pi}^2) \rightarrow 0$ , which implies that

$$\lim_{(m_{K} - m_{\pi}) \to 0} f_{-}(s) = 0.$$
(18)

But at sufficiently high energies the mass differences can be neglected; therefore condition (18) leads to

$$\lim_{s \to \infty} f_{-}(s) = 0, \tag{19}$$

which is the assumption we have made. Therefore this assumption holds in a theory where there exists an underlying symmetry of strong interactions

#### THE INTERACTIONS IN K<sub>µ3</sub> AND K<sub>e3</sub> DECAY

that ensures the conservation of a current carrying strangeness. However, one should point out that from (19) the existence of a partially conserved current does not follow necessarily. A slightly different version of a partially conserved current hypothesis was proposed by BERNSTEIN and WEINBERG [5a]. They require the vanishing of (17) as  $s \rightarrow \infty$ . This requirement is stronger than (18).

Now we have the dispersion relations:

$$f_{+}(s) = f_{+}(0) + \frac{1}{\pi} \int_{0}^{\infty} \frac{s}{s'} \frac{ds'}{s'-s} \operatorname{Im} f_{+}(s'),$$

$$(m_{K} + m_{f'})^{2}$$

$$f_{-}(s) = \frac{1}{\pi} \int_{0}^{\infty} \frac{ds'}{s'-s} \operatorname{Im} f_{-}(s').$$

$$(20)$$

The form factor  $f_+$  corresponds to the state of the  $K\pi$  system with J = 1and  $f_0 = \frac{1}{2} (f_- + \frac{m_K^2 - m_\pi^2}{2} f_+)$  to the state with J = 0. Thus  $f_-$  is coupled to  $f_+$ by unitarity. Now I shall take into account in Im f only the contribution from the lowest massive intermediate state, which is a  $K\pi$ -state, and neglect the contributions from states of higher masses. In this way the relations (20), together with unitarity, become linear integral equations, which can be solved in terms of S and P wave  $K\pi$ -phase shifts [5];  $f_+$  depends only on  $\delta_1$  and  $f_0$  on  $\delta_0$ . One knows about the  $K\pi$ -interaction that it has a resonance  $K^*$  at W = 888 MeV with a width  $\Gamma \leq 50$  MeV in the state I =  $\frac{1}{2}$ , J = 0 or J = 1. The effect of the width is negligible in the region of energies we are interested in  $(0 < s < (m_K - m_\pi)^2)$ , and one can replace the resonant phase shift by a step function  $\theta$  (s -  $m_{K^*}^2/\pi$ ; the other phase shift will be neglected. The results of our calculations are equivalent to the Feynman diagrams shown in Fig. 8.



One obtains [6]:

(a) 
$$J_{K*} = 0$$
  
 $f_{+}(s) = f_{+}(0),$ 
(21)  
 $f_{-}(s) = \frac{m_{K}^{2} - m_{\pi}^{2}}{m_{K*}^{2} - s} \chi f_{+}(0),$ 
(21)  
(b)  $J_{K*} = 1$   
 $f_{+}(s) = \left(1 + \frac{\chi s}{m_{K*}^{2} - s}\right) f_{+}(0),$ 
(22)  
 $f_{-}(s) = \frac{m_{K}^{2} - m_{\pi}^{2}}{m_{K*}^{2} - s} \chi f_{+}(0).$ 

#### S. W. MC DOWELL

The parameter X measures the influence of  $K^*$  in the decay. The effect of an intermediate vector boson would be the same as of a vector  $K^*$  with X = 1. The above expressions should then be modified in the following way:

$$f_{+}(s) \rightarrow f_{+}(s) \frac{M^{2}}{M^{2} - s}; f_{-}(s) \rightarrow f_{-}(s) - f_{+}(s) \frac{m_{K}^{2} - m_{\pi}^{2}}{M^{2} - s}$$
 (23)

where M is the mass of the vector boson which has to be greater than the kaon mass  $M > m_{\chi^*}$ 

#### 3. EXPERIMENTAL RESULTS

The experimental value for the branching ratio  $(K_{\mu3}/K_{e3})$  is [7] R = 0.96 ± 0.15. Assuming universal vector interaction with constant form factors, the branching ratio is given in terms of  $\xi = f_{-}/f_{+}$  by [8]

$$R = 0.651 + 0.126 \xi + 0.019 \xi^2$$
(24)

which gives two possible values for  $\xi$ :

$$\xi = 1.85 \pm 0.8$$
 and  $\xi = -8.15 \pm 0.85$ .

In order to decide between the two solutions the muon spectrum was measured by two different groups, BROWN et al. [7] and DOBBS et al. [9]. Unfortunately their data are incompatible. The first group obtains the best fit with  $\xi = 1.46$  while the second one found  $\xi = -9$ . The pion spectrum in K<sub>µ3</sub> might provide a better determination of  $\xi$ .

In addition to the branching ratio one knows from the  $K_{e3}$  experiment [2, 7] that  $f_{+}(s)$  has very little energy dependence. It was found that

$$m_K^2 f'_{+}(0)/f_{+}(0) = 0.45 \pm 0.56.$$
 (25)

Let us compare these results with the theoretical predictions. If the spin of  $K^*$  is one then one,obtains

$$m_{K}^{2} f'_{+}(0) / f'_{-}(0) = -m_{K}^{2} / (m_{K}^{2} - m_{\pi}^{2}) = -1.1$$
 (26)

even if there exists an intermediate vector boson. Then

$$m_{K}^{2} f'_{+}(0)/f_{+}(0) = -1.1 (f_{-}(0)/f_{+}(0)).$$
 (27)

Taking into account (25) one can see that the value  $\xi = -9$  is inconsistent with this result. The other value  $\xi = 1.46$  is barely compatible.

For scalar K\* all the energy dependence of  $f_{+}(s)$  would come from the intermediate vector boson and is consistent with (25).

Another theoretical prediction valid in the absence of an intermediate vector boson is

$$m_{K}^{2} f'(0) / f(0) = (m_{K} / m_{K^{*}})^{2} = 0.3$$
 (28)

for both values of the K\* spin.

It should be pointed out that these predictions can be made only in so far as neglecting the contributions to the dispersion integrals of higher mass intermediate states is actually justified.

A last point I want to make very briefly is concerned with the isospin of the current. So far we have been dealing with a I = 1/2 current. There is now evidence for a mixture of I = 1/2 and I = 3/2 currents. Then the form factors for K<sup>+</sup>, K<sub>1</sub> and K<sub>2</sub> decay will not be the same. However, if the assumptions we have made hold for both currents and there exists no sizeable K $\pi$ -interaction in the I = 3/2, S and P states, then the structure of the form factors remain unchanged; in particular, one still obtains the relations (26) and (28) for all three decays.

#### REFERENCES

- [1] LUERS, D., MITRA, I.S., WILLIS, W.J. and YAMAMOTO, S., Phys. Rev. Letters 7 (1961) 265.
- [2] BROWN, J.L., KADYK, J.A., TRILLING, G.H., VAN DE WALLE, R.T., ROE, B.P. and SINCLAIR, D., Phys. Rev. Letters, <u>7</u> (1961) 423.
- [3] MacDOWELL, S.W., Annals of Physics 18 (1962) 171.
- [4] BOSTERLI, M. and GEFFEN, D.A., Phys. Rev. Letters 7 (1961) 203.
- [5] MacDOWELL, S.W., Phys. Rev. 116 (1959) 1047.
- [5a] BERNSTEIN, J. and WEINBERG, S., Phys. Rev. Letters 5 (1960) 481.
- [6] ACIOLI, J.L. and MacDOWELL, S.W., Nuovo Cimento 24 (1962) 606; CHEW, H., Phys. Rev. Letters 8 (1962) 297.
- [7] BROWN, J.L. et al., Phys. Rev. Letters 8 (1962) 450.
- [8] BRENE, N., EGARDT, L. and QVIST, B., Nuclear Phys. 22 (1961) 553.
- [9] DOBBS, J.M., LANDE, K., MANN, A.K., REIBEL, K., SCIULLI, F.J., UTO, H., WHITE, D.H. and YOUNG, K.K., Phys. Rev. Letters 8 (1962) 295.

#### BIBLIOGRAPHY

A complete list of references to earlier work is found in the review article: OKUN', L., Strange particles, Annual Review of Nuclear Science 9 (1959) 61.

# SEMINAR ON THEORETICAL PHYSICS

## HELD AT THE

# PALAZZINO TRIESTE, MIRAMARE, TRIESTE 16 JULY - 25 AUGUST 1962

## STAFF OF THE SEMINAR

| Director:  | A. SALAM      | Physics Department, Imperial<br>College of Science and Technology,<br>London, United Kingdom            |
|------------|---------------|---|
| Lecturers: | P. BUDINI     | Institute of Physics, University of<br>Trieste, Italy   |
|            | R. CAPPS      | Department of Physics, North-<br>western University, Evanston, Ill.,<br>United States of America        |
|            | M. FROISSART  | Service de physique mathématique,<br>Centre d'études nucléaires de Saclay,<br>France                    |
|            | S. FUBINI     | Institute of Physics, University of<br>Turin, Italy   |
|            | R. GATTO      | National Laboratory of Frascati,<br>Italy   |
|            | S. HAYAKAWA   | Physical Institute, Nagoya Univer-<br>sity, Japan   |
|            | S. MANDELSTAM | Department of Physics, University<br>of Birmingham, United Kingdom                                      |
|            | A. PAIS       | Institute for Advanced Studies,<br>Princeton, N.J., United States of<br>America                         |
|            | T. REGGE      | Institute of Physics, University of<br>Turin, Italy   |
|            | J.J. SAKURAI  | Enrico Fermi Institute for Nuclear<br>Studies, University of Chicago, Ill.,<br>United States of America |
|            | J. SCHWINGER  | Harvard University, Cambridge,<br>Mass., United States of America                                       |
|            | W. THIRRING   | Institute of Theoretical Physics,<br>University of Vienna, Austria                                      |
|            |               |   |

|                           | J. TIOMNO    | Brazilian Centre of Research in<br>Physics, Rio de Janeiro, Brazil                     |
|---------------------------|--------------|--|
| Lecturers:<br>(continued) | A. WIGHTMANN | Palmer Physical Laboratory,<br>Princeton University, N.J., United<br>States of America |
|                           | E. WIGNER    | Palmer Physical Laboratory,<br>Princeton University, N.J., United<br>States of America |

# SECRETARIAT OF THE SEMINAR

| Scientific              |                 |   |
|-------------------------|-----------------|---|
| Secretary:              | A.W. MCREYNOLDS | Division of Research and Laboratories, IAEA               |
| Assistant<br>Scientific |                 |   |
| Secretary:              | P. RENDI        | Division of Research and Laboratories, IAEA               |
| Editor:                 | A. HAMENDE      | Division of Scientific and Technical<br>Information, IAEA |
| Administrative          |                 |   |
| Secretary:              | H.H. STORHAUG   | Division of Scientific and Technical<br>Information, IAEA |

| Name               | Institution   | Nominating State<br>or Organization |
|--------------------|---|-------------------------------------|
| Alles, W.          | Institute of Physics, University of Bologna   | Italy                               |
| Andreassi, G.      | Institute of Physics, University of Trieste   | Italy                               |
| Atkinson, D.       | Physics Department, University College,<br>London   | United Kingdom                      |
| Balachandran, A.P. | Institute of Mathematical Sciences, Madras  | India                               |
| Ball, J.           | University of California, La Jolla, Calif.  | United States of America            |
| Bergia, S.         | National Institute of Nuclear Physics,<br>Section of Bologna  | Italy                               |
| Bollini, C.G.      | Departamento de Física, Facultad de<br>Ciencias Exactas y Naturales, Universidad<br>de Buenos Aires           | Argentína                           |
| Boulware, D.G.     | Harvard University, Cambridge, Mass.  | United States of America            |
| Brene, N.          | Institute of Theoretical Physics,<br>University of Copenhagen   | Denmark                             |
| Breitenlohner, P.  | Institute of Theoretical Physics,<br>University of Graz   | Austría                             |
| Brown, L.          | Physics Department, Imperial College<br>of Science and Technology, London                                     | Attended privately                  |
| Byers, N.          | Physics Department, University of California,<br>Los Angeles, Calif.  | United States of America            |
| Caianello, E.      | Institute of Theoretical Physics,<br>Mostra Oltremare, Naples   | Italy                               |
| Ceolin, C.         | Institute of Physics, University of Padua   | Italy                               |
| Chadan, K.         | Laboratoire de Physique théorique et hautes<br>énergies, Faculté des Sciences d'Orsay,<br>Université de Paris | Iran                                |
| Contogouris, A.    | Nuclear Research Centre "Democritos", Athens  | Greece                              |
| Cutkosky, R.       | Nordita, Copenhagen   | Nordita                             |
| El Kheshen         | Faculty of Science, University of Cairo   | United Arab Republic                |
| Enz, C.            | Institut de Physique de l'Université de<br>Neuchâtel  | Switzerland                         |
| Erkisson, K.E.     | Institute of Theoretical Physics, Upsala  | Sweden                              |

| Name                  | Institution   | Nominating State<br>or Organization |
|-----------------------|---|-------------------------------------|
| Fonda, L.             | Institute of Physics, University of Trieste   | Italy                               |
| Ferreira, E.H.        | Brazilian Centre of Research in Physics,<br>Rio de Janeiro  | Brazil                              |
| Frautschi, S.         | Newman Laboratory of Nuclear Studies,<br>Cornell University, Ithaca, N.Y.                           | United States of America            |
| Frenkel, A.           | Central Research Institute of Physics,<br>Budapest  | Hungary                             |
| Furlan, G.            | Institute of Physics, University of Trieste   | Italy                               |
| Galindo, A.           | División Física Teórica,<br>Junta de Energía Nuclear, Madrid  | Spain                               |
| Galzenati, E.         | Institute of Theoretical Physics,<br>Mostra Oltremare, Naples                                       | Italy                               |
| von Gehlen, G.        | Institute of Nuclear Theoretical Physics,<br>Karlsruhe  | Federal Republic of Germany         |
| Giambiaggi, J.J.      | Departamento de Física, Facultad de Ciencias<br>Exactas y Naturales, Universidad de<br>Buenos Aires | Argentina                           |
| Golberg, H.           | Israel Atomic Energy Commission, Rehovoth   | Israel                              |
| Goto, S.              | Rikkyo University, Tokyo  | Japan                               |
| Grashin, A.           | Institute of Theoretical and Experimental<br>Physics, Academy of Science of USSR, Moscow            | USSR                                |
| Greenberg, O.W.       | Department of Physics and Astronomy,<br>University of Maryland, College Park, Md.                   | United States of America            |
| Hadjioannou, F.       | Theoretical Division, European Organization for Nuclear Research, Geneva                            | Greece                              |
| Harun-ar-Rashid, A.M. | Pakistan Atomic Energy Commission, Dacca  | Pakistan                            |
| Henzi, R.             | Institut de Physique de l'Université<br>de Neuchâtel  | Switzerland                         |
| Höhler, G.            | Institute of Nuclear Theoretical Physics,<br>Karlsruhe  | Federal Republic of Germany         |
| Hunziker, W.          | Swiss Federal Institute of Technology, Zurich   | Switzerland                         |
| Isaev, P.             | Joint Institute for Nuclear Research, Dubna   | USSR                                |
| Jacob, M.             | Service de physique mathématique,<br>Centre d'études nucléaires de Saclay                           | France                              |

| Name            | Institution  | Nominating State<br>or Organization |
|-----------------|--|-------------------------------------|
| Jaksić, B.      | "Rudjer Bošković" Institute, Zagreb  | Yugoslavia                          |
| Johnson, K.     | Physics Department, Massachussetts Institute of Technology, Cambridge, Mass. | United States of America            |
| Krolikovski, W. | Institute for Nuclear Research,<br>Polish Academy of Sciences, Warsaw        | Poland                              |
| Lee, B.W.       | Department of Physics, University of<br>Pennsylvania, Philadelphia, Pa       | United States of America.           |
| Lovelace, C.    | Physics Department, Imperial College of<br>Science and Technology, London    | United Kingdom                      |
| Mahoux, G.      | Service de physique mathématique,<br>Centre d'études nucléaires de Saclay    | France                              |
| Mc Dowell, S.W. | Brazilian Centre of Research in Physics,<br>Rio de Janeiro                   | Brazil                              |
| Maki, Z.        | Department of Physics, Nagoya University                                     | Japan                               |
| Masson, D.      | Physics Department, Imperial College of Science<br>and Technology, London    | Canada                              |
| Maor, U.        | Physics Department, The Weizmann Institute<br>of Science, Rehovoth           | Israel                              |
| Mitra, A.N.     | University of Delhi, Delhi   | India                               |
| Mitter, H.      | Max Planck Institute of Physics, Munich                                      | Federal Republic of Germany         |
| Miyamoto, Y.    | Physics Department, Tokyo University<br>of Education                         | Japan                               |
| Morales, A.     | Theoretical Division, European Organization for Nuclear Research, Geneva     | CERN                                |
| Nielsen, P.W.   | Physics Institute, University of Aahrus                                      | Denmark                             |
| Nilsson, J.     | Theoretical Division, European Organization for Nuclear Research, Geneva     | CERN                                |
| Oades, G.       | Physics Department, University College, London                               | United Kingdom                      |
| Pagiola, E.     | Institute of Physics, University of Padua                                    | Italy                               |
| Perescutti, G.  | Institute of Physics, University of Trieste                                  | Italy                               |
| Pietschmann, H. | Institute of Theoretical Physics,<br>University of Vienna, Vienna            | Austria                             |
| Pócsik, G.      | Institute for Theoretical Physics, Budapest                                  | Hungary                             |

| Name             | Institution   | Nominating State<br>or Organization |
|------------------|---|-------------------------------------|
| Radha, T.K.      | Institute of Mathematical Sciences, Madras                                    | India                               |
| Rastall, P.      | Department of Physics, University of<br>British Columbia, Vancouver           | Canada                              |
| Reigner, J.      | Université Libre de Bruxelles, Bruxelles                                      | Belgium                             |
| Riaz-ud-din      | Physics Department, Panjab University, Lahore                                 | Pakistan                            |
| Rojo, O.,        | Escuela de Física y Matemáticas,<br>Universidad Central de Venezuela, Caracas | Venezuela                           |
| Rosen, B.        | Stevens Institute of Technology, Hoboken, N.J.                                | United States of America            |
| Ruegg, H.        | Institut de Physique théorique,<br>Université de Genève                       | Switzerland                         |
| Rothleiner, J.   | Institute of Theoretical Physics,<br>University of Heidelberg                 | Federal Republic of Germany         |
| Saavedra, I.     | Instituto de Física y Matemáticas,<br>Universidad de Chile, Santiago          | Chile                               |
| Sabry, A.A.      | University of Assyout   | United Arab Republic                |
| Sawyer, R.       | Department of Physics,<br>University of Wisconsin, Madison, Wisc.             | United States of America            |
| Soln, J.         | "Rudjer Bošković" Institute, Zagreb   | Yugoslavia                          |
| Van der Spuy, E. | Atomic Energy Board, Pretoria   | South Africa                        |
| Streater, R.     | Physics Department,<br>Imperial College of Science and<br>Technology, London  | United Kingdom                      |
| Stroffolini, R.  | Institute of Theoretical Physics, Mostra<br>Oltremare, Naples                 | Italy                               |
| Su-Ho, I.        | Atomic Energy Research Institute of the Republic of Korea, Seoul              | Korea                               |
| Theis, W.        | Institute of Theoretical Physics,<br>University of Hamburg                    | Federal Republic of Germany         |
| Verganelakis, A. | Institute of Physics, University of Rome                                      | Greece                              |
| Vymazal, M.      | Institute of Nuclear Research,<br>Czechoslovak Academy of Sciences, Prague    | Czechoslovak Socialist<br>Republic  |
| Wataghin, V.     | Institute of Physics, University of Turin                                     | Italy                               |
| Werle, J.        | Institute for Nuclear Research,<br>Polish Academy of Sciences, Warsaw         | Poland                              |

| Name           | Institution  | Nominating State<br>or Organization |
|----------------|--|-------------------------------------|
| Wolters, G.    | Zeeman Laboratory, Amsterdam   | Netherlands                         |
| Wouthuysen, S. | Institute of Theoretical Physics,<br>University of Amsterdam             | Netherlands                         |
| Zimerman, A.   | Theoretical Division, European Organization for Nuclear Research, Geneva | CERN                                |

# ANOTHER IAEA PUBLICATION ON THEORETICAL PHYSICS

#### SELECTED TOPICS IN NUCLEAR THEORY

500 p.  $(16 \times 24 \text{ cm})$  - STI/PUB/67 - United States and Canada; US \$10.00; elsewhere S 210, - (60 s. stg., NF 40, - DM 35, -)

A series of seven lectures presented at an International Summer School held 20 August - 9 September 1962 in the Low Tatra Mountains and organized by the Nuclear Research Institute of the Czechoslovak Academy of Sciences with the co-operation of the International Atomic Energy Agency. Contributions:

| N. AUSTERN         | Direct reactions  |
|--------------------|---|
| I.S. SHAPIRO       | Dispersion theory of direct nuclear reactions                               |
| J.P. ELLIOTT       | The nuclear shell model and its relation with other nuclear models          |
| A.de-SHALIT        | Electromagnetic properties of atomic nuclei                                 |
| V.G. SOLOVIEV      | Effect of superconducting pairing correlations on nuclear properties -      |
| S.T. BELYAEV       | Some aspects of collective properties of nuclei                             |
| H.A. TOLHOEK       | Weak interactions among nucleons and leptons                                |
| Scientific editor: | F. JANOUCH, Nuclear Research Institute,<br>Czechoslovak Academy of Sciences |

The above publication is obtainable from Sales Agents listed overleaf.

A complete Catalogue of all Agency publications will be gladly supplied by any of the Sales Agents or directly by the Editorial and Publications Section, International Atomic Energy Agency, Kärntner Ring, Vienna I, Austria.
## IAEA SALES AGENTS

ARGENTINA Editorial Sudamericana, S.A. Alsina 500 **Buenos** Aires AUSTRALIA Melbourne University Press 369, Lonsdale Street Melbourne, C.1 AUSTRIA Georg Fromme & Co. Spengergasse 39 Vienna V BELGIUM Office International de Librairie 30, avenue Mamix Brussels 5 BRAZIL Livraria Kosmos Editora Rua do Rosario, 135-137 Rio de Janeiro Agencia Expoente Oscar M. Silva Rua Xavier de Toledo, 140-1° Andar (Caixa Postal N° 5,614) São Paulo BURMA See under India BYELORUSSIAN SOVIET SOCIALIST REPUBLIC See under USSR CANADA The Queen's Printer Ottawa CEYLON See under India CHINA (Taiwan) Books and Scientific Supplies Service, Ltd., P.O. Box 83 Taipei DENMARK Ejnar Munksgaard Ltd. 6 Nörregade Copenhagen K ETHIOPIA G.P. Giannopoulos International Press Agency P.O. Box 120 Addis Ababa FINLAND Akateeminen Kirjakauppa Keskuskatu 2 Helsinki FRANCE and FRENCH UNION Masson et Cie, Editeurs 120, bd Saint-Germain Paris VIe

GERMANY, Federal Republic of R. Oldenbourg Rosenheimer Strasse 145 Munich 8 ICELAND Halldór Jónsson Mjóstræti 2 Reykjavík INDIA Orient Longmans Ltd. 17, Chittaranjan Ave. Calcutta 13 ISRAEL Heiliger and Co. 3 Nathan Strauss Street Jerusalem ITALY Agenzia Editoriale Internazionale Organizzazioni Universali (A.E.I.O.U.) Via Meravigli 16 Milan JAPAN Maruzen Company Ltd. 6. Tori Nichome Nihonbashi P.O. Box 605 Tokyo Central KOREA, Republic of The Eul-Yoo Publishing Co. 5, 2-ka Chong-ro Seoul MEXICO Librería Internacional Av. Sonora 206 Mexico 11, D.F. MONACO The British Library 30, bd des Moulins Monte Carlo MOROCCO Centre de diffusion documentaire du B.E.P.I. 8. rue Michaux-Bellaire (B.P. N° 211) Rabat NEPAL See under India NETHERLANDS N.V. Martinus Nijhoff Lange Voorhout 9 The Hague NEW ZEALAND Whitcombe & Tombs, Ltd. G.P.O. Box 1894 Wellington, C.1

NORWAY Johan Grundt Tanum Karl Johans gate 43 Oslo PAKISTAN Karachi Education Society Haroon Chambers South Napier Road P.O. Box No.4866 Karachi, 2 PARAGUAY Agencia de Librerías de Salvador Nizza Calle Pte. Franco No. 39-43 Asunción PERU Librería Internacional del Perú S. A. Boza 879 (Casilla 1417) Líma PHILIPPINES The Modern Book Company 508 Rizal Avenue Manila POLAND Ośrodek Rozpowszechniana Wydawnistw Naukowych Polska Akademia Nauk Palac Kultury i Nauki Warsaw PORTUGAL Livraria Rodrigues 186, Rua do Ouro, 188 Lisbon 2 SOUTH AFRICA Van Schaik's Bookstore (Pty) Ltd. Libri Building Church St. (P.O. Box 724) Pretoria

SPAIN Librería Bosch Ronda Universidad, 11 Barcelona SWEDEN C.E. Fritzes Kungl. Hovbokhandel Fredsgatan 2 Stockholm 16 SWITZERLAND Librairie Payot 40, rue du Marché Geneva SYRIA Georges N. Coussa Imm, Chanan rue Khan el-Harir (B.P. 779) Aleppo TURKEY Librairie Hachette 469, Istiklål Caddesi Beyoğlu, İstanbul UKRAINIAN SOVIET SOCIALIST REPUBLIC See under USSR UNION OF SOVIET SOCIALIST REPUBLICS Mezhdunarodnaya Kniga Kuznetsky Most, 18 Moscow G-200 UNITED KINGDOM OF GREAT BRITAIN AND NORTHERN IRELAND Her Majesty's Stationery Office P.O. Box 569 London, S.E.1 UNITED STATES OF AMERICA National Agency for International Publications, Inc. 801 Third Avenue New York 22, N.Y. YUGOSLAVIA Jugoslovenska Knjiga Teraziie 27 Belgrade

IAEA publications can also be purchased retail at the United Nations Bookshop at United Nations Headquarters, New York, at the news-stand at the Agency's Headquarters, Vienna, and at most conferences, symposia and seminars organized by the Agency.

Orders and inquiries from countries where sales agents have not yet been appointed may be sent to:

Distribution and Sales Unit, International Atomic Energy Agency, Kärntner Ring, Vienna I, Austria