CANONICAL APPROACH TO CONSTRUCTING CONSTANTS OF MOTION FOR NONLOCAL FIELD THEORIES

BY W. GARCZYŃSKI AND J. STELMACH

Institute of Theoretical Physics, University of Wrocław*

(Received August 8, 1983)

A general method of derivation of conservation laws for non-local field theories si presented. Differences in comparison with a local case are stressed. Two kinds of Lagrangians appearing in a non-local theory are examined. Canonical choice of constants of motion is made corresponding to the transformations from the confoımal and gauge groups.

PACS numbers: 03.50.Kk

1. Introduction

Motivating this work on the non-local field theory we would like to avoid repetition of various arguments against existing local theories, and also to avoid avoking belief in necessity to introduce somehow into a theory a parameter with dimensionality of length. Instead, we shall adopt rather pragmatic point of view saying that one encounters non--local interactions when working with a field theory on a lattice. Replacing lattice values of a field by its "quasicontinual representation" leads to non-local interactions, with definite given formfactors [1, 2]. One may say that regularizing field theory by putting it on a lattice one introduces nonlocality [3].

Another example of a non-local interaction of interest is obtained by solving the Gauss law constraint in a non-Abelian gauge field theory [4].

The very next step in dealing with a non-local interaction consists of finding constants of motion. This is precisely our problem which we discuss in this paper.

We follow, and develop it further, the variational approach to the problem outlined some time ago by Rzewuski [5]. Another approach was proposed by Pauli at the same time [6] and later by Marnelius [7].

Our derivation of conservation laws parallels Rzewuski's. Its novelty lies in a final selection of the constants of motion, out of an infinite family of them, by determining unspecified functionals appearing in Rzewuski's formulae. We believe that our derivation

* Address: Instytut Fizyki Teoretycznej, Uniwersytet Wrocławski, Cybulskiego 36, 50-205 Wrocław, Poland.

establishes the canonical way of constructing constants of motion for non-local field theories of a type considered throughout the paper. The difference in comparison with local case comes from the occurrence of two distinct Lagrangians in a theory – canonical and effective ones. Their mutual interplay is a characteristic feature of the method of constructing what we call – canonical set of constants of motion. They are of the same form as the Noether charges, and are expressed through the canonical Lagrangian while the effective one serves for derivation of Euler-Lagrange equations of motion.

2. *Basic definitions*

We consider a class of non-local theories specified by the action functional

$$S(\Omega) = \sum_{n=1}^{\infty} S^{(n)}(\Omega), \tag{2.1}$$

where $S^{(n)}(\Omega)$ is given by n-tuple integral over a region Ω restricted by the space-like surfaces $\sigma_1 < \sigma_2$ (σ_1 lies earlier than σ_2), and a time-like surface Σ. The number n of integrations is minimal in a sense that it can not be diminished by performing some integrations due to a presence of singular functions in the integrand. In other words we assume that the functions $L^{(n)}$ appearing in the expressions

$$S^{(n)}(\Omega) = \int_{\Omega} dx_1 \dots \int_{\Omega} dx_n L^{(n)}(x_1, \dots, x_n; \varphi, \partial\varphi),$$

$$dx = dx^0 dx^1 \dots dx^D = dx^0 d^D x \tag{2.2}$$

should be smooth, and symmetric under the permutations of their arguments, in the case one is dealing with a single multiplet φ of fields. We assume that x varies in the Minkowski space M_{D+1} of D spatial dimensions. Restriction of each variable x_k to the region Ω makes the $S^{(n)}(\Omega)$ a non-additive functional for $n > 1$. This kind of action functional was treated by Rzewuski (cf. [5]), while the case when only one variable is restricted to Ω was considered by Marnelius (cf. [7]).

For the sake of simplicity we restrict our considerations to the case when $L^{(n)}$ are homogeneous n-th order functionals of φ and $\partial\varphi$. Moreover, we assume that only local part of the action functional, represented by $S^{(1)}(\Omega)$, depends on derivatives of the fields while higher order action functionals are the n-th order monomials in the fields only

$$L^{(n)}(x_1, \dots, x_n; \varphi, \partial\varphi) = L^{(n)}(x_1, \dots, x_n)\varphi(x_1) \dots \varphi(x_n), \qquad n \geqslant 2 \tag{2.3}$$

and the formfactors $L^{(n)}(x_1, \dots, x_n)$ are smooth symmetric functions. More general cases may be treated without essential difficulties by the same method, which we shall illustrate on the above class of non-local interactions.

Before considering the variational principle let us identify the relevant Lagrange functionals corresponding to the action functional. Namely, we define the n-th order partial Lagrangians $L^{(n)}(x; \Omega)$ by the conditions

$$S^{(n)}(\Omega) = \int_{\Omega} dx L^{(n)}(x; \Omega), \qquad n = 1, 2, \dots . \tag{2.4}$$

Due to the symmetry of integrands in (2.2) it is possible to identify the Lagrangians

$$L^{(1)}(x;\Omega) \equiv L^{(1)}[x;\varphi(x),\partial\varphi(x)], \tag{2.5}$$

and for $n \geq 2$

$$L^{(n)}(x;\Omega) = \int_\Omega dx_1 \dots \int_\Omega dx_{n-1} L^{(n)}(x_1,\dots,x_{n-1},x)\varphi(x_1)\dots\varphi(x_{n-1})\varphi(x). \tag{2.6}$$

Having the partial Lagrangians we define two global ones: the Canonical Lagrangian $L(x;\Omega)$ given by

$$L(x;\Omega) \equiv \sum_{n=1}^{\infty} L^{(n)}(x;\Omega) \tag{2.7}$$

and the Effective Lagrangian $\mathscr{L}(x;\Omega)$

$$\mathscr{L}(x;\Omega) \equiv \sum_{n=1}^{\infty} nL^{(n)}(x;\Omega). \tag{2.8}$$

Clearly, they differ only when the non-locality occurs because of the multiplier n.

In order to formulate the variational principle we consider an infinitesimal variation of the variables x and fields, depending on some continuous parameters $\delta\omega_s$, $s = 1, \dots, N$.

$$x \to x' = x+\delta x,$$

$$\delta x^\mu = \delta^s x^\mu \delta\omega_s, \quad \mu = 0, 1, \dots, D, \quad s = 1, \dots, N, \tag{2.9}$$

$$\varphi(x) \to \varphi'(x') = \varphi(x)+\Delta\varphi(x), \tag{2.10}$$

$$\Delta\varphi(x) = \Delta^s\varphi(x)\delta\omega_s, \tag{2.11}$$

$$\delta\varphi(x) = \varphi'(x)-\varphi(x) = [\Delta^s\varphi(x)-\delta^s x^\mu\partial_\mu\varphi(x)]\delta\omega_s. \tag{2.12}$$

Following Rzewuski's method we find for the total variation of the action functional the expression

$$\Delta S(\Omega) = S'(\Omega')-S(\Omega) = \int_\Omega dx\left\{\left[\frac{\partial\mathscr{L}(x;\Omega)}{\partial\varphi(x)} - \partial_\mu\pi^\mu(x;\Omega)\right]\delta\varphi(x)\right.$$

$$\left. +\partial_\mu[\pi^\mu(x;\Omega)\delta\varphi(x)+\delta x^\mu\mathscr{L}(x;\Omega)]\right\}, \tag{2.13}$$

where the canonical momentum $\pi^\mu(x;\Omega)$ is

$$\pi^\mu(x;\Omega) \equiv \frac{\partial\mathscr{L}(x;\Omega)}{\partial\partial_\mu\varphi(x)} = \frac{\partial L^{(1)}(x;\Omega)}{\partial\partial_\mu\varphi(x)}. \tag{2.14}$$

Assuming that $\Delta S(\Omega)$ does not depend on the variation $\delta\varphi$ inside of the region Ω, (Stationary Action Principle), one gets the equation of motion for the fields

$$\frac{\partial\mathscr{L}(x;\Omega)}{\partial\varphi(x)} - \partial_\mu\pi^\mu(x;\Omega) = 0 \tag{2.15}$$

for x varying inside of Ω. Thus for any solution of the equation of motion one has the equality

$$\Delta S(\Omega) \doteq F[\sigma_2; \Omega] - F[\sigma_1; \Omega] + F[\Sigma; \Omega], \tag{2.16}$$

where the functional $F[\sigma; \Omega]$ is given by the formula

$$F[\sigma; \Omega] = \int_\sigma d\sigma_\mu(x)\,[\pi^\mu(x; \Omega)\delta\varphi(x) + \delta x^\mu \alpha(x; \Omega)] \equiv F^s[\sigma; \Omega]\delta\omega_s. \tag{2.17}$$

Here $d\sigma_\mu(x)$ is the oriented element of the surface σ around the point x. The dot above the equality sign means that it holds on a solution of the equations of motion.

Further considerations depend on specific choice of the domain Ω. We shall treat in detail two particular cases. First, when the region Ω covers the whole space-time M_{D+1}, and second when it consists of a box $\Omega = V \times T$, where T is a finite time interval.

3. *Conservation laws*

A. The case $\Omega = M_{D+1}$

Assume that the field equations hold throughout the whole space-time

$$\frac{\partial \mathscr{L}}{\partial \varphi} - \partial_\mu \pi^\mu = 0 \tag{3.1}$$

and define the charges as follows

$$Q^s[\sigma] \equiv -F^s[\sigma] = \int_\sigma d\sigma_\mu j^{\mu s}, \tag{3.2}$$

where the currents are given by the formula

$$j^{\mu s} \equiv \pi^\mu(\delta^s x^\nu \partial_\nu \varphi - \Delta^s \varphi) - \delta^s x^\mu \mathscr{L}. \tag{3.3}$$

The charges $Q^s[\sigma]$ depend, generally, on σ and thus are not conserved

$$\frac{\delta Q^s[\sigma]}{\delta \sigma(x)} = \partial_\mu j^{\mu s}(x) \equiv \Omega^s(x)$$

$$\doteq (\pi^\mu \partial_\nu \varphi - g^\mu{}_\nu \mathscr{L})\partial_\mu \delta^s x^\nu - \partial_\mu(\pi^\mu \Delta^s \varphi) - \delta^s x^\nu (\partial_\nu \mathscr{L})_{\text{ex}}. \tag{3.4}$$

The subindex "ex" means the derivative with respect to an explicit dependence of $\mathscr{L}(x)$ upon the x^ν variable. We use the Bjorken and Drell metric throughout the paper.

Integrating the last equalities over a region Ω_Σ bounded by the flat surfaces $x^0 = t_1$, $x^0 = t_2 > t_1$, and a time-like surface Σ, we obtain

$$\int dx \partial_\mu j^{\mu s} = Q^s_\Sigma(t_2) - Q^s_\Sigma(t_1) + Q^s(\Sigma) = \int_{\Omega_\Sigma} dx \Omega^s, \tag{3.5}$$

where

$$Q^s_\Sigma(t) = \int_{|\vec{x}| < \Sigma} d^D x j^{0s}(t, \vec{x}), \tag{3.6}$$

and

$$Q^s(\Sigma) = \int_\Sigma d\sigma_\mu j^{\mu s}. \tag{3.7}$$

Now, upon taking Σ to space infinity, and assuming that in this limit the charges $Q^s(\Sigma)$ vanish

$$\lim_{\Sigma \to \infty} Q^s(\Sigma) = 0, \quad s = 1, \ldots, N, \tag{3.8}$$

we get from (3.5) the equality

$$\int_{t_1}^{t_2} dx \partial_\mu j^{\mu s} = Q^s(t_2) - Q^s(t_1) = \int_{t_1}^{t_2} dx \Omega^s, \quad s = 1, \ldots, N. \tag{3.9}$$

For the symmetry transformations defined as follows

$$\partial_\mu j^{\mu s} \equiv \Omega^s = 0, \quad s = 1, \ldots, N, \tag{3.10}$$

the charges $Q^s(t)$ become familiar conserved Noether constants of motion, provided a theory is local since in this case the effective Lagrangian entering $j^{\mu s}$ coincides with the canonical Lagrangian appearing in the Noether charges. Generally, they are not conserved for arbitrary transformations (2.9)–(2.12) for which the functional Ω^s do not vanish. It is possible nevertheless, following Rzewuski, to construct conserved charges also in this case. Namely, it is clear that the following quantities are conserved on extremals

$$\tilde{Q}^s(t, \tau) \equiv Q^s(t) - \int_\tau^t dx \Omega^s + q^s(\tau), \tag{3.11}$$

provided that currents vanish sufficiently fast in the spatial directions

$$\int d^D x \partial_k j^{ks} = 0, \quad s = 1, \ldots, N \tag{3.12}$$

and $q^s(\tau)$ are arbitrary functionals of the fields φ, not depending on t, and τ is an arbitrary but fixed parameter

$$\frac{d}{dt} \tilde{Q}^s(t, \tau) \doteq 0, \quad s = 1, \ldots, N. \tag{3.13}$$

Thus we obtain an infinite family of conserved quantities corresponding to given transformations of variables x and fields φ. In the case of the symmetry transformations the charges $\tilde{Q}^s(t, \tau)$ coincide, in the local case, with the Noether charges, up to the undetermined functionals $q^s(\tau)$.

Since the tilded charges do not depend on the time we may simplify them by putting $t = \tau$,

$$Q^s(\tau) \equiv \tilde{Q}^s(t, \tau)|_{t=\tau} = Q^s(\tau) + q^s(\tau), \quad s = 1, \ldots, N. \tag{3.14}$$

The next task consists of selecting, out of this family of conserved quantities, some interesting ones by a proper choice of the functionals $q^s(\tau)$, and the parameter τ. Before going into the discussion of this question let us mention briefly the case when the domain Ω is restricted to a box V multiplied by a finite time interval T.

B. The case $\Omega = V\times T$

In the considered case the condition (3.8) is replaced by the requirement

$$Q^s(\Sigma) = 0, \qquad s = 1, \dots, N, \tag{3.15}$$

where the vanishing is due to the boundary conditions imposed on fields. The integration region Ω_Σ is in this case a part of the initial region Ω selected by the two flat surfaces $x^0 = t_1$ and $x^0 = t_2 > t_1$, intersecting Ω, $t_1, t_2 \in T$, and Σ is the corresponding part of a side surface of Ω. Adding the conditions

$$\int_V d^D x \partial_k j^{ks} = 0, \qquad s = 1, \dots, N. \tag{3.16}$$

one gets the following charges

$$Q^s(\tau) = \int_V d^D x j^{0s}(\tau, \vec{x}) + q_V^s(\tau), \qquad s = 1, \dots, N. \tag{3.17}$$

Thus one sees that both cases of an infinite and a finite region Ω can be treated simultaneously. One has to remember only that domain of the integration in the last case is a finite box.

The charges coincide with the Noether charges in the local case, when the currents are conserved. We shall demonstrate that it is possible to choose the undetermined functionals $q_V^s(\tau)$ in such a way as to get rid of the effective Lagrangian appearing in the currents, and to obtain a set of canonical constants of motion. To be specific we shall consider the transformations from the conformal and gauge group, and construct the relevant conserved quantities. We shall list briefly the main ingredients of the construction in order to fix the notations.

a) Translations:

$$\delta_T^\nu x^\mu = g^{\mu\nu}, \qquad \mu, \nu = 0, 1, \dots, D, \tag{3.18}$$

$$\Delta_T^\nu \varphi = 0, \tag{3.19}$$

$$j^{\mu\nu} \equiv T^{\mu\nu} = \pi^\mu \partial^\nu \varphi - g^{\mu\nu} \mathscr{L}, \tag{3.20}$$

$$\Omega_T^\nu \equiv \partial_\mu T^{\mu\nu} \doteq -(\partial^\nu \mathscr{L})_{\mathrm{ex}}, \tag{3.21}$$

$$P^\nu(x^0) = \int d^D x T^{0\nu}(x) + p^\nu(x^0). \tag{3.22}$$

b) Lorentz rotations:

$$\delta_L^{\lambda\varrho} x^\mu = g^{\mu\lambda} x^\varrho - g^{\mu\varrho} x^\lambda, \tag{3.23}$$

$$\Delta_L^{\lambda\varrho} \varphi = \Sigma^{\lambda\varrho} \varphi, \tag{3.24}$$

$$j^{\mu(\lambda\varrho)} \equiv M^{\mu(\lambda\varrho)} = x^\varrho T^{\mu\lambda} - x^\lambda T^{\mu\varrho} - \pi^\mu \Sigma^{\lambda\varrho} \varphi, \tag{3.25}$$

$$\partial_\mu M^{\mu(\lambda\varrho)} \equiv \Omega_L^{\lambda\varrho} \doteq (x^\lambda g^{\nu\varrho} - x^\varrho g^{\nu\lambda})(\partial_\nu \mathscr{L})_{\mathrm{ex}} + T^{\varrho\lambda} - T^{\lambda\varrho} - \partial_\mu(\pi^\mu \Sigma^{\lambda\varrho} \varphi), \tag{3.26}$$

$$M^{\lambda\varrho}(x^0) = \int d^D x M^{0(\lambda\varrho)}(x) + m^{\lambda\varrho}(x^0). \tag{3.27}$$

c) Dilatations:

$$\delta_D x^\mu = -x^\mu, \tag{3.28}$$

$$\Delta_D \varphi = d\varphi, \tag{3.29}$$

d – dimension of the field φ,

$$j_D^\mu \equiv \boldsymbol{D}^\mu = -x_\nu T^{\mu\nu} - \pi^\mu d\varphi, \tag{3.30}$$

$$\partial_\mu \boldsymbol{D}^\mu \equiv \Omega_D \doteq x^\nu(\partial_\nu \mathscr{L})_{ex} - \frac{\partial \mathscr{L}}{\partial \varphi} d\varphi - \pi^\nu(d+1)\partial_\nu\varphi + (D+1)\mathscr{L}, \tag{3.31}$$

$$\boldsymbol{D}(x^0) = \int d^D x \boldsymbol{D}^0(x) + d(x^0). \tag{3.32}$$

d) Special Conformal Transformations:

$$\delta^\nu_{conf} x^\mu = x^2 g^{\mu\nu} - 2x^\mu x^\nu, \tag{3.33}$$

$$\delta^\nu_{conf}\varphi = 2x_\mu(g^{\mu\nu}d - \Sigma^{\mu\nu})\varphi + (2x^\mu x^\nu - g^{\mu\nu}x^2)\partial_\mu\varphi, \tag{3.34}$$

$$j^{\mu\nu}_{conf} \equiv K^{\mu\nu} = -\pi^\mu[(2x^\nu x^\lambda - g^{\nu\lambda}x^2)\partial_\lambda\varphi + 2x^\nu d\cdot\varphi - 2x_\lambda\Sigma^{\lambda\nu}\varphi] + (2x^\mu x^\nu - g^{\mu\nu}x^2)\mathscr{L}, \tag{3.35}$$

$$\begin{aligned}\partial_\mu K^{\mu\nu} \equiv \Omega^\nu_{conf} &\doteq (2x^\nu x^\mu - g^{\nu\lambda}x^2)(\partial_\lambda\mathscr{L})_{ex} \\ &+ 2x_\lambda\left(\pi^\lambda\partial^\nu\varphi - \pi^\nu\partial^\lambda\varphi + \frac{\partial\mathscr{L}}{\partial\varphi}\Sigma^{\lambda\nu}\varphi + \pi^\varrho\Sigma^{\lambda\nu}\partial_\varrho\varphi\right) \\ &- 2x^\nu\left[\frac{\partial\mathscr{L}}{\partial\varphi}d\varphi - (D+1)\mathscr{L} + \pi^\lambda(d+1)\partial_\lambda\varphi\right] - 2(\pi^\nu d - \pi_\lambda\Sigma^{\lambda\nu})\varphi,\end{aligned} \tag{3.36}$$

$$K^\nu(x^0) = \int d^D x K^{0\nu}(x) + k^\nu(x^0). \tag{3.37}$$

e) Global Gauge Transformations:

$$\delta^a_G x^\mu = 0, \quad a = 1, \ldots, N, \tag{3.38}$$

$$\Delta^a_G\varphi = -N_a\varphi, \tag{3.39}$$

$$[N_a, N_b] = -f_{abc}N_c, \quad N_a = N_a^* = -N_a^T, \tag{3.40}$$

f_{abc} — real, antisymmetric,

$$j^\mu_a = \pi^\mu N_a\varphi, \tag{3.41}$$

$$\partial_\mu j^\mu_a \equiv \Omega_a = \partial_\mu(\pi^\mu N_a\varphi) \doteq \frac{\partial\mathscr{L}}{\partial\varphi}N_a\varphi + \pi^\mu N_a\partial_\mu\varphi, \tag{3.42}$$

$$Q_a(x^0) = \int d^D x j^0_a(x) + q_a(x^0), \quad a = 1, \ldots, N. \tag{3.43}$$

The quantities $p^\nu(x^0)$, $m^{\lambda\varrho}(x^0)$, $d(x^0)$, $k^\nu(x^0)$, $q_a(x^0)$ are so far unspecified functionals depending on fields. In the case of a field in a box all spatial integrations are restricted, and proper boundary conditions are imposed.

As always, the currents $j^{\mu s}$ are determined up to the gradients from some superpotentials. Replacing the currents by new ones

$$j^{\mu s} \to j'^{\mu s} = j^{\mu s} + \partial_\lambda X^{\lambda\mu s}, \tag{3.44}$$

$$X^{\lambda\mu s} = -X^{\mu\lambda s}, \tag{3.45}$$

and demanding that due to the proper behaviour of the superpotentials at the spatial infinity, or due to the proper boundary conditions in a case of a field in a box, the following conditions hold

$$\int d^D x \partial_k X^{k0s} = 0 \tag{3.46}$$

one gets the same charges from the new currents. Both currents have the same divergencies due to the antisymmetry of the superpotentials. A typical example of a superpotential is provided by the energy-momentum tensor $T^{\mu\nu}$ which may be replaced by the Belinfante tensor $\theta^{\mu\nu}$

$$\theta^{\mu\nu} \equiv T^{\mu\nu} - \partial_\lambda f^{\nu,\lambda\mu}, \tag{3.47}$$

where the superpotentials $f^{\nu,\lambda\mu}$ are

$$f^{\nu,\lambda\mu} = \tfrac{1}{2}(\pi^\nu \Sigma^{\lambda\mu} - \pi^\lambda \Sigma^{\mu\nu} - \pi^\mu \Sigma^{\nu\lambda}) = -f^{\nu,\mu\lambda}. \tag{3.48}$$

In the same way as in the conformally invariant local case one may "improve" the tensor $\theta^{\mu\nu}$ by adding another divergenceless term, and thus obtain simpler expressions for the currents $\boldsymbol{D}^\mu$ and $K^{\mu\nu}$, [8–10].

4. *Conservation laws in a theory which is local in the time variable*

Upon removal of a lattice from the time axis one obtains a theory which is local in time while remaining non-local in the spatial variables. In such a case it is possible to introduce the Hamilton formulation in the terms of the independent canonical variables

$$\varphi(x^0, \vec{x}) \equiv \varphi(x)$$

and

$$\pi(x^0, \vec{x}) \equiv \pi(x) \equiv \pi^0(x) \equiv \frac{\partial \mathscr{L}(x)}{\partial \dot{\varphi}(x)} \tag{4.1}$$

with x^0 – fixed. Taking for the Hamiltonian the functional

$$H(x^0) \equiv P^0(x^0) = \int d^D x(\pi\dot{\varphi} - \mathscr{L}) + p^0(x^0), \tag{4.2}$$

and imposing the Hamilton equations

$$\frac{dH(x^0)}{\delta\pi(x)} = \dot{\varphi}(x), \tag{4.3}$$

$$\frac{dH(x^0)}{\delta\varphi(x)} \doteq -\dot{\pi}(x), \tag{4.4}$$

where $\frac{d}{\delta\varphi}$ means a total functional derivative in contradistinction to the partial derivative $\frac{\delta}{\delta\varphi}$ which takes into account only explicit dependence of H upon the φ, one obtains the following equations on the functional

$$\frac{dp^0(x^0)}{\delta\pi(x)} = 0, \tag{4.5}$$

$$\frac{dp^0(x^0)}{\delta\varphi(x)} \doteq \sum_{n=2}^{\infty} n(n-1)\int d^Dx_1 \ldots \int d^Dx_{n-1} \left.\frac{\partial L^{(n)}(x_1, \ldots, x_{n-1}, x; \varphi)}{\partial\varphi(x)}\right|_{x_K{}^0 = x^0} \tag{4.6}$$

One can solve the equations rather easily taking into account the condition (2.3). In a general case one may use the general method for a restoration of a functional from its functional derivatives [11]. We obtain

$$p^0(x^0, \varphi) \doteq \int d^Dx[\mathscr{L}(x) - L(x)] + p^0(x^0, 0), \tag{4.7}$$

where $p^0(x^0, 0)$ is an additive constant, not depending functionally on fields, which we shall omit. Therefore, we get finally for the Hamiltonian the expression

$$H(x^0) = \int d^Dx[\pi(x)\dot{\varphi}(x) - L(x)] \equiv P^0_C(x^0). \tag{4.8}$$

Thus we arrived at the canonical form of the Hamiltonian corresponding to a given canonical Lagrangian L. Hinted by this result we select the unknown functionals in such a way as to get rid of the effective Lagrangian $\mathscr{L}$ from the charges. Resulting charges we call the canonical ones. Let us list them all in the same notations as before, and in the same sequence:

a)

$$T^{\mu\nu}_C = \pi^\mu\partial^\nu\varphi - g^{\mu\nu}L, \tag{4.9}$$

$$\partial_\mu T^{\mu\nu}_C \doteq -(\partial^\nu L)_{ex}, \tag{4.10}$$

$$P^\nu_C = \int d^Dx T^{0\nu}_C = \int d^Dx(\pi\partial^\nu\varphi - g^{0\nu}L). \tag{4.11}$$

b)

$$M^{\mu(\lambda\varrho)}_C = x^\varrho T^{\mu\lambda}_C - x^\lambda T^{\mu\varrho}_C - \pi^\mu\Sigma^{\lambda\varrho}\varphi, \tag{4.12}$$

$$\partial_\mu M^{\mu(\lambda\varrho)}_C \doteq (x^\lambda g^{\nu\varrho} - x^\varrho g^{\nu\lambda})(\partial_\nu L)_{ex} + T^{\varrho\lambda}_C - T^{\lambda\varrho}_C - \partial_\mu(\pi^\mu\Sigma^{\lambda\varrho}\varphi), \tag{4.13}$$

$$\begin{aligned} M^{\lambda\varrho}_C = \int d^Dx M^{0(\lambda\varrho)}_C = \int d^Dx[&x^\varrho(\pi\partial^\lambda\varphi - g^{0\lambda}L) \\ &- x^\lambda(\pi\partial^\varrho\varphi - g^{0\varrho}L) - \pi\Sigma^{\lambda\varrho}\varphi], \end{aligned} \tag{4.14}$$

$$[\Sigma^{\mu\nu}, \Sigma^{\lambda\varrho}] = g^{\nu\lambda}\Sigma^{\mu\varrho} + g^{\mu\lambda}\Sigma^{\nu\varrho} - g^{\mu\varrho}\Sigma^{\nu\lambda} - g^{\nu\varrho}\Sigma^{\mu\lambda}. \tag{4.15}$$

c)

$$D^\mu_C = -x_\nu T^{\mu\nu}_C - \pi^\mu d\varphi, \tag{4.16}$$

$$\partial_\mu D^\mu_C \doteq x^\nu(\partial_\nu L)_{ex} - \frac{\partial L}{\partial\varphi}d\varphi - \pi^\nu(d+1)\partial_\nu\varphi + (D+1)L, \tag{4.17}$$

$$D_C = \int d^Dx D^0_C = \int d^Dx[-\pi(x_\lambda\partial^\lambda\varphi + d\varphi) + x^0L], \tag{4.18}$$

d)

$$K_C^{\mu\nu} = T_{C\lambda}^{\mu}(g^{\nu\lambda}x^2 - 2x^\nu x^\lambda) + 2x_\lambda \pi^\mu(\Sigma^{\lambda\nu} - g^{\lambda\nu}d)\varphi \tag{4.19}$$

$$\partial_\mu K_C^{\mu\nu} \doteq (2x^\nu x^\lambda - g^{\nu\lambda}x^2)(\partial_\lambda L)_{ex}$$
$$+2x_\lambda\left(\pi^\lambda\partial^\nu\varphi - \pi^\nu\partial^\lambda\varphi + \frac{\partial L}{\partial\varphi}\Sigma^{\lambda\nu}\varphi + \pi^\varrho\Sigma^{\lambda\nu}\partial_\varrho\varphi\right)$$
$$-2x^\nu\left[\frac{\partial L}{\partial\varphi}d\varphi - (D+1)L + \pi^\lambda(d+1)\partial_\lambda\varphi\right] - 2(\pi^\nu d - \pi_\lambda\Sigma^{\lambda\nu})\varphi, \tag{4.20}$$

$$K_C^\nu = \int d^D x K_C^{0\nu} = \int d^D x\{\pi[(g^{\nu\lambda}x^2 - 2x^\nu x^\lambda)\partial_\lambda\varphi$$
$$+(2x_\lambda\Sigma^{\lambda\nu} - 2x^\nu d)\varphi] + (2x^\nu x^0 - g^{0\nu}x^2)L\}. \tag{4.21}$$

e)

$$j_{ac}^\mu = \pi^\mu N_a\varphi, \tag{4.22}$$

$$\partial_\mu j_{ac}^\mu \doteq \frac{\partial L}{\partial\varphi}N_a\varphi + \pi^\mu N_a\partial_\mu\varphi \doteq \partial_\mu(\pi^\mu N_a\varphi), \tag{4.23}$$

$$Q_{ac} = \int d^D x j_a^0 = \int d^D x \pi N_a\varphi. \tag{4.24}$$

Hence, the canonical charges are constructed exactly in the same way as the Noether charges in a local field theory. We advocate the above choice of charges also in a general case of non-local interactions, also in the time variable.

It is not difficult though rather tedious task to compute the Poisson brackets between various canonical charges. Namely defining the usual Poisson brackets

$$\{F, G\} = \sum_A \int d^D x\left(\frac{dF}{\delta\varphi^A(x)}\frac{dG}{\delta\pi_A(x)} - \frac{dF}{\delta\pi_A(x)}\frac{dG}{\delta\varphi^A(x)}\right)\bigg|_{x^0-\text{fix}}. \tag{4.25}$$

we find omitting the subscript C at the charges

$$\{P^\mu, \varphi(x)\} = -\partial^\mu\varphi(x) = \delta_T^\mu\varphi(x), \tag{4.26}$$

$$\{M^{\lambda\varrho}, \varphi(x)\} = (x^\lambda\partial^\varrho - x^\varrho\partial^\lambda + \Sigma^{\lambda\varrho})\varphi(x) = \delta_L^{\lambda\varrho}\varphi(x), \tag{4.27}$$

$$\{D, \varphi(x)\} = (d + x_\mu\partial^\mu)\varphi(x) = \delta_D\varphi(x), \tag{4.28}$$

$$\{K^\mu, \varphi(x)\} = (2x^\mu x^\nu - g^{\mu\nu}x^2)\partial_\nu\varphi(x) + 2x_\nu(g^{\mu\nu}d - \Sigma^{\nu\mu})\varphi(x) = \delta_{\text{conf}}^\mu\varphi(x), \tag{4.29}$$

$$\{Q_a, \varphi(x)\} = -N_a\varphi(x) = \delta_G^a\varphi(x). \tag{4.30}$$

As the examples of the Poisson brackets between the charges we shall mention the following ones

$$\{P^\mu, P^\nu\} \doteq \int d^D x(g^{0\nu}\partial_\sigma T^{\sigma\mu} - g^{0\mu}\partial_\sigma T^{\sigma\nu}) \doteq \int d^D x[g^{0\mu}(\partial^\nu\mathscr{L})_{ex} - g^{0\nu}(\partial^\mu\mathscr{L})_{ex}], \tag{4.31}$$

$$\{P^\nu, D\} \doteq -P^\nu - g^{0\nu}\int d^D x\partial_\mu D^\mu - x^0\int d^D x\partial_\mu T^{\mu\nu}, \tag{4.32}$$

$$\{P^{\nu}, Q_a\} \doteq -g^{0\nu} \int d^D x \partial_\mu j_a^\mu, \tag{4.33}$$

$$\{D, Q_a\} \doteq x^0 \int d^D x \partial_\mu j_a^\mu, \tag{4.34}$$

$$\{Q_a, Q_b\} = -f_{abc} Q_c. \tag{4.35}$$

Generally, besides usual terms one gets additional ones which vanish in the local limit.

5. *Conditions for a non-local theory to be invariant under the transformations from the conformal group*

Let the region of integration Ω covers the whole space-time. The total variation of the action functional under the infinitesimal transformations (2.9)–(2.12) is equal to

$$\Delta S = \int dx \delta L = \delta\omega_s \int dx \delta^s L, \tag{5.1}$$

where

$$\int dx \delta^s L = \int dx \left[\frac{\partial \mathscr{L}}{\partial \varphi} (\Delta^s \varphi - \delta^s x^\mu \partial_\mu \varphi) - \pi^\nu \partial_\nu (\Delta^s \varphi - \delta^s x^\mu \partial_\mu \varphi) \right]. \tag{5.2}$$

Assuming that terms of the form of a total divergence do not contribute to the action variation, we demand that remaining integral vanish. Thus we obtain the criteria for vanishing of the total variation of the action functional under the given transformation of fields and space-time variables. For instance, assuming that

$$\int dx \partial^\nu \mathscr{L} = 0, \tag{5.3}$$

we obtain as the condition for a theory to be translationally invariant

$$\int dx (\partial^\nu \mathscr{L})_{\text{ex}} = 0. \tag{5.4}$$

Similarly, if the integrals vanish

$$\int dx \partial_\mu [(g^{\mu\varrho} x^\lambda - g^{\mu\lambda} x^\varrho) \mathscr{L}] = 0, \tag{5.5}$$

then the condition for a theory to be Lorentz invariant is

$$\int dx [x^\lambda (\partial^\varrho \mathscr{L})_{\text{ex}} - x^\varrho (\partial^\lambda \mathscr{L})_{\text{ex}}]$$
$$= \int dx \left[\frac{\partial \mathscr{L}}{\partial \varphi} \Sigma^{\lambda\varrho} \varphi + \pi^\mu \Sigma^{\lambda\varrho} \partial_\mu \varphi + \pi^\lambda \partial^\varrho \varphi - \pi^\varrho \partial^\lambda \varphi \right]. \tag{5.6}$$

For a theory which in addition possesses Lorentz invariant formfactor, the left hand side vanishes. In the same way a theory satisfying the condition

$$\int dx \partial_\mu (x^\mu \mathscr{L}) = 0, \tag{5.7}$$

is dilatationally invariant if the following equality holds

$$\int dx \left[\frac{\partial \mathscr{L}}{\partial \varphi} d\varphi + \pi^\mu (d+1) \partial_\mu \varphi - (D+1) \mathscr{L} - x^\mu (\partial_\mu \mathscr{L})_{\text{ex}} \right] = 0. \tag{5.8}$$

Finally, if a theory is such that

$$\int dx\partial_\mu[(2x^\nu x^\mu - g^{\mu\nu}x^2)\mathscr{L}] = 0 \tag{5.9}$$

it is conformally invariant when the necessary and sufficient condition is fulfilled

$$\frac{1}{2}\int dx(2x^\nu x^\lambda - g^{\nu\lambda}x^2)(\partial_\lambda\mathscr{L})_{\mathrm{ex}} = \int dx\left\{x^\nu\left[\frac{\partial\mathscr{L}}{\partial\varphi}d\varphi + \pi^\lambda(d+1)\partial_\lambda\varphi - (D+1)\mathscr{L}\right]\right.$$
$$\left.-2x_\lambda\left[\frac{\partial\mathscr{L}}{\partial\varphi}\Sigma^{\lambda\nu}\varphi + \pi^\varrho\Sigma^{\lambda\nu}\partial_\varrho\varphi + \pi^\lambda\partial^\nu\varphi - \pi^\nu\partial^\lambda\varphi\right] + V^\nu\right\}, \tag{5.10}$$

where V^ν is so called a virial of the field

$$V^\nu = \pi_\lambda(g^{\lambda\nu}d - \Sigma^{\lambda\nu})\varphi. \tag{5.11}$$

In the local case the condition (5.10) simplifies for a theory which is in addition translationally and rotationally invariant. For non-local theories the integral form of the condition renders such a simplification impossible. The effective Lagrangian $\mathscr{L}$ rather than the canonical one L plays crucial role in the above symmetry considerations.

Applications of the formalism to the quasicontinual representation of a lattice field theory will be given elsewhere.

One of the authors (W.G.) thanks Professor C. Cronström for the useful discussion on related topics during the 18-th Winter School of Theoretical Physics at Karpacz, February 1981. Both authors thank Professor J. Rzewuski for several discussions on the subject.

REFERENCES

[1] W. Garczyński, J. Stelmach, *Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys.* **28**, 73 (1980).
[2] W. Garczyński, J. Stelmach, *J. Math. Phys.* **22**, 1106 (1981), and **23**, 1168 (1982).
[3] S. D. Drell, M. Weinstein, S. Yankielowicz, *Phys. Rev.* **D14**, 487 (1976).
[4] C. Cronström, *Quantization of Non-Abelian Gauge Theory I. Gauge Invariant Hamiltonian Formulation*, preprint HU-TFT-81-1, *Ann. Phys.* (to be published).
[5] J. Rzewuski, *Nuovo Cimento* **10**, 182 and 784 (1953), see also *Field Theory*, Vol. 1, Chap. 2, Illife, London 1958.
[6] W. Pauli, *Nuovo Cimento* **10**, 648 (1953).
[7] R. Marnelius, *Phys. Rev.* **D8**, 2472 (1973).
[8] G. Mack, A. Salam, *Ann. Phys.* (N.Y.) **53**, 174 (1969).
[9] C. G. Callan, Jr., S. Coleman, R. Jackiw, *Ann. Phys.* (N.Y.) **59**, 42 (1970).
[10] S. Coleman, R. Jackiw, *Ann. Phys.* (N.Y.) **67**, 552 (1971).
[11] W. Garczyński, J. Hanćkowiak, *J. Math. Phys.* **20**, 2427 (1979).